Chapter

1

Trigonometrical Ratios, Functions and Identities

Contents

The Great Indian Mathematician Aryabhatta

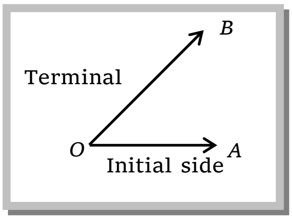
The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476 A.D.), Bhaskara I (600 A.D.) and Bhaskara (1114 A.D.) and Brahmagupta (598 A.D.) got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks has also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine function for angles more than 90º. A sixteenth century Malayalam work **Yuktibhasa** ***(period) contains a proof for the expansion of sin (A+B). Exact expressions for sines or cosines of 18º, 36º, 54º, 72º, etc. are given by Bhaskara II.***

Trigonometrical Ratios, Functions and Identities

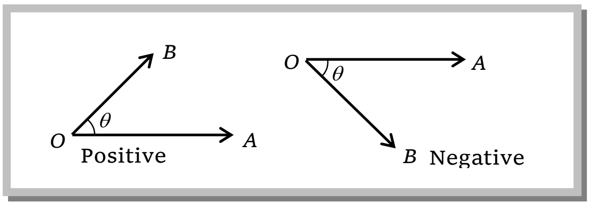
1.1 Definitions

(1) **Angle :** The motion of any revolving line in a plane from its initial position (initial side) to the final position (terminal side) is called angle. The end point O about which the line rotates is called the vertex of the angle.



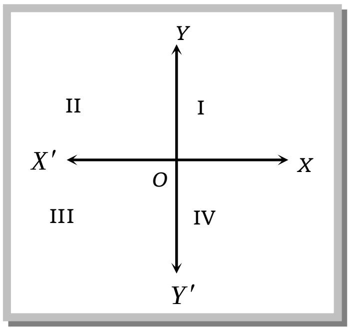
(2) **Measure of an angle :** The measure of an angle is the amount of rotation from the initial side to the terminal side.

(3) **Sense of an angle :** The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get the terminal side.



(4) **Right angle :** If the revolving ray starting from its initial position to final position describes one quarter of a circle. Then we say that the measure of the angle formed is a right angle.

(5) **Quadrants :** Let $X'OX$ and YOY' be two lines at right angles in the plane of the paper. These lines divide the plane of paper into four equal parts. Which are known as quadrants. The lines $X'OX$ and YOY' are known as x-axis and y-axis. These two lines taken together are known as the co-ordinate axes.



(6) **Angle in standard position :** An angle is said to be in standard position if its vertex concides with the origin O and the initial side concides with OX *i.e.*, the positive direction of x-axis.

(7) **Angle in a quadrant :** An angle is said to be in a particular quadrant if the terminal side of the angle in standard position lies in that quadrant.

(8) **Quadrant angle :** An angle is said to be a quadrant angle if the terminal side concides with one of the axes.

1.2 System of Measurement of Angles

There are three system for measuring angles

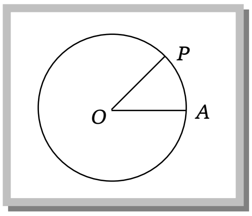
(1) **Sexagesimal or English system :** Here a right angle is divided into 90 equal parts known as degrees. Each degree is divided into 60 equal parts called minutes and each minute is further divided into 60 equal parts called seconds. Therefore, 1 right angle = 90 degree $(=90^o)$

$$1^o = 60 \text{ minutes } (=60')$$
$$1' = 60 \text{ second } (=60'')$$

(2) **Centesimal or French system :** It is also known as French system, here a right angle is divided into 100 equal parts called grades and each grade is divided into 100 equal parts, called minutes and each minute is further divided into 100 seconds. Therefore,

$$1 \text{ right angle} = 100 \text{ grades } (=100^g)$$
$$1 \text{ grade} = 100 \text{ minutes } (=100')$$
$$1 \text{ minute} = 100 \text{ seconds } (=100'')$$

(3) **Circular system :** In this system the unit of measurement is radian. One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



Consider a circle of radius r having centre at O. Let A be a point on the circle. Now cut off an arc AP whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOP$ is 1 radian $(=1^c)$.

1.3 Relation between Three Systems of Measurement of an Angle

Let D be the number of degrees, R be the number of radians and G be the number of grades in an angle θ.

Now, 90^o = 1 right angle $\Rightarrow 1^o = \frac{1}{90}$ right angle

$\Rightarrow$ $D^o = \frac{D}{90}$ right angles $\Rightarrow \theta = \frac{D}{90}$ right angles

........(i)

Again, π radians = 2 right angles $\Rightarrow$ 1 radian $= \frac{2}{\pi}$ right angles

$\Rightarrow$ R radians $= \frac{2R}{\pi}$ right angles $\Rightarrow \theta = \frac{2R}{\pi}$ right angles

........(ii)

and 100 grades = 1 right angle $\Rightarrow$ 1 grade $=\dfrac{1}{100}$ right angle

$\Rightarrow$ G grades $=\dfrac{G}{100}$ right angles $\Rightarrow \theta=\dfrac{G}{100}$ right angles

........(iii)

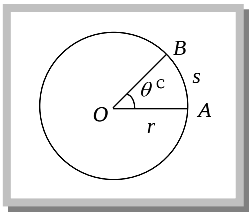
From (i), (ii) and (iii) we get, $\boxed{\dfrac{D}{90}=\dfrac{G}{100}=\dfrac{2R}{\pi}}$

This is the required relation between the three systems of measurement of an angle.

Note : ❑ One radian $=\dfrac{180^o}{\pi} \Rightarrow \pi$ radians $=180^o \Rightarrow$ 1 radian = $57^o\ 17'44.8'' \approx 57^o 17'45''$.

1.4 Relation between an Arc and an Angle

If s is the length of an arc of a circle of radius r, then the angle θ (in radians) subtended by this arc at the centre of the circle is given by $\theta=\dfrac{s}{r}$ or $\boxed{s=r\theta}$ *i.e.*, arc = radius × angle in radians



Sectorial area : Let OAB be a sector having central angle θ^C and radius r. Then area of the sector OAB is given by $\boxed{\dfrac{1}{2}r^2\theta}$.

Important Tips

☞ *The angle between two consecutive digits in a clock is* 30° (= *π/6 radians). The hour hand rotates through an angle of* 30° *in one hour.*

☞ *The minute hand rotate through an angle of* 6° *in one minute.*

Example: 1 The circular wire of radius 7 *cm* is cut and bend again into an arc of a circle of radius 12 *cm*. The angle subtended by an arc at the centre of the circle is **[Kerala (Engg.) 2002]**

(a) 50^o (b) 210^o (c) 100^o (d) 60^o

Solution: (b) Given the diameter of circular wire = 14 *cm*. Therefore length of wire = $14\pi\,cm$

Hence, required angle $=\dfrac{\text{Arc}}{\text{Radius}}=\dfrac{14\pi}{12}=\dfrac{7\pi}{6}$ radian $\Rightarrow \dfrac{7\pi}{6}\times\dfrac{180^o}{\pi}=210^o$.

Example: 2 The degree measure corresponding to the given radian $\left[\dfrac{2\pi}{15}\right]^c$

(a) 21° (b) 22° (c) 23° (d) 24°

Solution: (d) We have, π radians $=180^o$

$\therefore\ 1^c=\left[\dfrac{180}{\pi}\right]^o$; $\therefore\ \left[\dfrac{2\pi}{15}\right]^c=\left[\dfrac{2\pi}{15}\times\dfrac{180}{\pi}\right]^o=24^o$.

Example: 3 The angles of a quadrilateral are in A.P. and the greatest angle is 120°, the angles in radians are

(a) $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$ (b) $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{3}$ (c) $\frac{5\pi}{18}, \frac{8\pi}{18}, \frac{11\pi}{18}, \frac{12\pi}{18}$ (d) None of these

Solution: (a) Let the angles in degrees be $\alpha - 3\delta, \alpha - \delta, \alpha + \delta, \alpha + 3\delta$

Sum of the angles $= 4\alpha = 360^o$ $\therefore$ $\alpha = 90^o$

Also greatest angle $= \alpha + 3\delta = 120^o$, Hence, $3\delta = 120^o - \alpha = 120^o - 90^o = 30^o$ $\therefore$ $\delta = 10^o$

Hence the angles are $90^o - 30^o, 90^o - 10^o, 90^o + 10^o$ and $90^o + 30^o$

That is, the angles in degrees are $60^o, 80^o, 100^o$ and 120^o

$\therefore$ In terms of radians the angles are $60 \times \frac{\pi}{180}, 80 \times \frac{\pi}{180}, 100 \times \frac{\pi}{180}$ and $120 \times \frac{\pi}{180}$ that is $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}$ and $\frac{2\pi}{3}$.

Example: 4 The minute hand of a clock is 10 *cm* long. How far does the tip of the hand move in 20 *minutes*

(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$ (c) $\frac{30\pi}{3}$ (d) $\frac{40\pi}{3}$

Solution: (b) We know that the tip of the minute hand makes one complete round in one hour *i.e.* 60 *minutes* since the length of the hand is 10 *cm*. the distance moved by its tip in 60 *minutes* $= 2\pi \times 10 cm = 20\pi\ cm$

Hence the distance in 20 *minutes* $= \frac{20\pi}{60} \times 20 cm = \frac{20\pi}{3} cm$.

Example: 5 The angle subtended at the centre of radius 3 *metres* by the arc of length 1 *metre* is equal to **[UPSEAT 1973]**

(a) 20° (b) 60° (c) 1/3 radian (d) 3 radian

Solution: (c) Required angle = $\frac{\text{Arc}}{\text{radius}} = \frac{1}{3}$ radian .

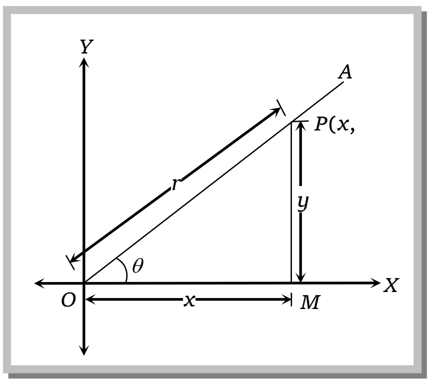
1.5 Trigonometrical Ratios or Functions

In the right angled triangle *OMP*, we have base = *OM* = *x*, perpendicular =*PM* = *y* and hypotenues = *OP* =*r*. We define the following trigonometric ratio which are also known as trigonometric function.

$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenues}} = \frac{y}{r}$ $\cos\theta = \frac{\text{Base}}{\text{Hypotenues}} = \frac{x}{r}$

$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}$ $\cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}$,

$\sec\theta = \frac{\text{Hypotenues}}{\text{Base}} = \frac{r}{x}$ $\text{cosec}\theta = \frac{\text{Hypotenues}}{\text{Perpendicular}} = \frac{r}{y}$



(1) Relation between trigonometric ratio (function)

(i) $\sin\theta.\text{cosec}\theta = 1$ (ii) $\tan\theta.\cot\theta = 1$

(iii) $\cos\theta.\sec\theta = 1$ (iv) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (v) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(2) Fundamental trigonometric identities

(i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $1 + \tan^2\theta = \sec^2\theta$ (iii) $1 + \cot^2\theta = \text{cosec}^2\theta$

Important Tips

- *If* $x = \sec\theta + \tan\theta$, *then* $\frac{1}{x} = \sec\theta - \tan\theta$.
- *If* $x = \text{coesc}\,\theta + \cot\theta$, *then* $\frac{1}{x} = \text{cosec}\,\theta - \cot\theta$.

(3) **Sign of trigonometrical ratios or functions :** Their signs depends on the quadrant in which the terminal side of the angle lies.

(i) **In first quadrant :** $x > 0, y > 0 \Rightarrow \sin\theta = \frac{y}{r} > 0, \cos\theta = \frac{x}{r} > 0, \tan\theta = \frac{y}{x} > 0$, $\text{cosec}\,\theta = \frac{r}{y} > 0$, $\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} > 0$. Thus, in the first quadrant all trigonometric functions are positive.

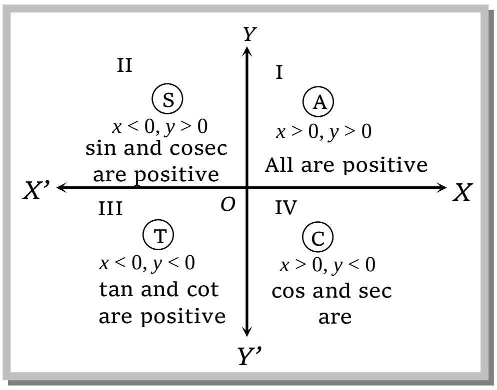
(ii) **In second quadrant :**

$x < 0, y > 0 \Rightarrow \sin\theta = \frac{y}{r} > 0, \cos\theta = \frac{x}{r} < 0, \tan\theta = \frac{y}{x} < 0, \text{cosec}\,\theta = \frac{r}{y} > 0$, $\sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} < 0$.

Thus, in the second quadrant sin and cosec function are positive and all others are negative.

(iii) **In third quadrant :** $x < 0, y < 0 \Rightarrow \sin\theta = \frac{y}{r} < 0, \cos\theta = \frac{x}{r} < 0, \tan\theta = \frac{y}{x} > 0, \text{cosec}\,\theta = \frac{r}{y} < 0$, $\sec\theta = \frac{r}{x} < 0$ and $\cot\theta = \frac{x}{y} > 0$. Thus, in the third quadrant all trigonometric functions are negative except tangent and cotangent.

(iv) **In fourth quadrant :** $x > 0, y < 0 \Rightarrow \sin\theta = \frac{y}{r} < 0$, $\cos\theta = \frac{x}{r} > 0$, $\tan\theta = \frac{y}{x} < 0$, $\text{cosec}\,\theta = \frac{r}{y} < 0$, $\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} < 0$ Thus, in the fourth quadrant all trigonometric functions are negative except cos and sec.

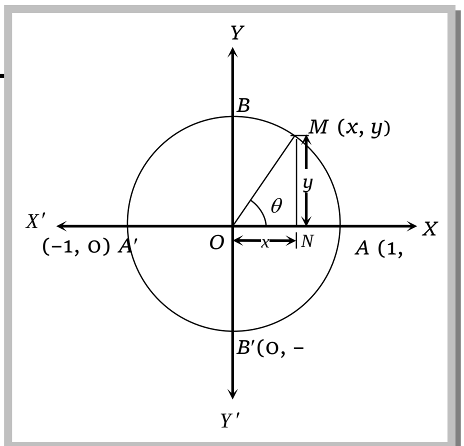


In brief : A crude aid to memorise the signs of trigonometrical ratio in different quadrant. "**A**dd **S**ugar **T**o **C**offee".

Important Tips

- *First determine the sign of the trigonometric function.*
- *If* θ *is measured from* $X'OX$ *i.e.,* $\{(\pi \pm \theta, 2\pi - \theta)\}$ *then retain the original name of the function.*
- *If* θ *is measured from* $Y'OY$ *i.e.,* $\left\{\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta\right\}$, *then change sine to cosine, cosine to sine, tangent to cotangent, cot to tan, sec to cosec and cosec to sec.*

(4) **Variations in values of trigonometric functions in different quadrants :** Let $X'OX$ and YOY' be the coordinate axes. Draw a circle with centre at origin O and radius unity.



Let $M(x,y)$ be a point on the circle such that $\angle AOM = \theta$ then $x = \cos\theta$ and $y = \sin\theta$; $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all values of θ.

II-Quadrant (S)	I-Quadrant (A)
$\sin\theta \to$ decreases from 1 to 0	$\sin\theta \to$ increases from 0 to 1
$\cos\theta \to$ decreases from 0 to – 1	$\cos\theta \to$ decreases from 1 to 0
$\tan\theta \to$ increases from – ∞ to 0	$\tan\theta \to$ increases from 0 to ∞
$\cot\theta \to$ decreases from 0 to – ∞	$\cot\theta \to$ decreases from ∞ to 0
$\sec\theta \to$ increases from – ∞ to – 1	$\sec\theta \to$ increases from 1 to ∞
$\text{cosec}\theta \to$ increases from 1 to ∞	$\text{cosec}\theta \to$ decreases from ∞ to 1
III-Quadrant (T)	**IV-Quadrant (C)**
$\sin\theta \to$ decreases from 0 to – 1	$\sin\theta \to$ increases from – 1 to 0
$\cos\theta \to$ increases from – 1 to 0	$\cos\theta \to$ increases from 0 to 1
$\tan\theta \to$ increases from 0 to ∞	$\tan\theta \to$ increases from – ∞ to 0
$\cot\theta \to$ decreases from ∞ to 0	$\cot\theta \to$ decreases from 0 to – ∞
$\sec\theta \to$ decreases from – 1 to – ∞	$\sec\theta \to$ decreases from ∞ to 1
$\text{cosec}\theta \to$ increases from – ∞ to – 1	$\text{cosec}\theta \to$ decreases from – 1 to – ∞

Note: ❑ $+\infty$ and – ∞ are two symbols. These are not real number. When we say that $\tan\theta$ increases from 0 to ∞ for as θ varies from 0 to $\frac{\pi}{2}$ it means that $\tan\theta$ increases in the interval $\left(0,\frac{\pi}{2}\right)$ and it attains large positive values as θ tends to $\frac{\pi}{2}$. Similarly for other trigonometric functions.

Example: 6 If $\sin\theta + \text{cosec}\,\theta = 2$, then $\sin^2\theta + \text{cosec}^2\theta =$ **[UPSEAT 2002; MP PET 1992; MNR 1990]**

(a) 1 (b) 4 (c) 2 (d) None of these

Solution: (c) $(\sin^2\theta + \text{cosec}^2\theta) = (\sin\theta + \text{cosec}\,\theta)^2 - 2\sin\theta.\text{cosec}\,\theta = 2^2 - 2 = 2$.

Example: 7 If $\sin\theta + \cos\theta = m$ and $\sec\theta + \text{cosec}\,\theta = n$, then $n(m+1)(m-1)$ equal to **[MP PET 1986]**

(a) m (b) n (c) $2m$ (d) $2n$

Solution: (c) $n(m^2 - 1) = (\sec\theta + \text{cosec}\,\theta).2\sin\theta.\cos\theta$ $[\because m^2 = 1 + 2\sin\theta.\cos\theta]$

$$= \frac{\sin\theta + \cos\theta}{\sin\theta.\cos\theta}.2\sin\theta.\cos\theta \quad = 2m\,.$$

Example: 8 If $\tan\theta = \frac{x\sin\phi}{1 - x\cos\phi}$ and $\tan\phi = \frac{y\sin\theta}{1 - y\cos\theta}$, then x/y equal to **[MP PET 1991]**

(a) $\frac{\sin\phi}{\sin\theta}$ (b) $\frac{\sin\theta}{\sin\phi}$ (c) $\frac{\sin\phi}{1-\cos\theta}$ (d) $\frac{\sin\theta}{1-\cos\phi}$

Solution: (b) $x\sin\phi = \tan\theta - x\cos\phi.\tan\theta$

$$x = \frac{\tan\theta}{\sin\phi + \cos\phi\tan\theta} = \frac{\sin\theta}{\sin\theta\cos\phi + \cos\theta\sin\phi}$$

Similarly, $y = \frac{\sin\phi}{\sin\theta\cos\phi + \cos\theta\sin\phi}$; $\therefore\ \frac{x}{y} = \frac{\sin\theta}{\sin\phi}$.

Example: 9 The equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is only possible when **[MP PET 1986; IIT 1996; Karnataka CET 1997; AMU 1987, 1991]**

(a) $x = y$ (b) $x < y$ (c) $x > y$ (d) None of these

Solution: (a) $\because\ \cos^2\theta \le 1 \ \Rightarrow\ \sec^2\theta = \frac{4xy}{(x+y)^2} \ge 1 \Rightarrow 4xy \ge (x+y)^2 \Rightarrow (x-y)^2 \le 0$

Which is possible only when $x = y$ $(\because x, y \in R)$

Example: 10 $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$ equals

(a) 0 (b) 1 (c) $\sec\theta - \tan\theta$ (d) $\sec\theta.\tan\theta$

Solution: (c) $\sqrt{\frac{(1-\sin\theta)^2}{(1-\sin^2\theta)}} = \frac{1-\sin\theta}{\cos\theta} = \sec\theta - \tan\theta$.

Example: 11 If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to **[Kerala (Engg.) 2002]**

(a) 110 (b) 191 (c) 80 (d) 194

Solution: (d) $\tan A + \cot A = 4 \ \Rightarrow\ \tan^2 A + \cot^2 A + 2\tan A\cot A = 16$

$\Rightarrow\ \tan^2 A + \cot^2 A = 14 \ \Rightarrow\ \tan^4 A + \cot^4 A + 2 = 196 \ \Rightarrow\ \tan^4 A + \cot^4 A = 194$.

Example: 12 If $\sin x + \cos x = \frac{1}{5}$, then $\tan 2x$ is **[UPSEAT 2003]**

(a) $\frac{25}{17}$ (b) $\frac{7}{25}$ (c) $\frac{25}{7}$ (d) $\frac{24}{7}$

Solution: (d) $\sin x + \cos x = \frac{1}{5} \ \Rightarrow\ \sin^2 x + \cos^2 x + 2\sin x\cos x = \frac{1}{25}$

$\sin 2x = -\frac{24}{25} \ \Rightarrow\ \cos 2x = -\frac{7}{25} \ \Rightarrow\ \tan 2x = \frac{24}{7}$.

Example: 13 If $\sin x = -\frac{24}{25}$, then the value of $\tan x$ is **[UPSEAT 2003]**

(a) $\frac{24}{25}$ (b) $\frac{-24}{7}$ (c) $\frac{25}{24}$ (d) None of these

Solution: (b) $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \left(\frac{-24}{25}\right)^2} = \frac{7}{25} \ \Rightarrow\ \tan x = \frac{\sin x}{\cos x} = \frac{-24}{7}$.

Example: 14 If $\tan\theta + \sec\theta = e^x$, then $\cos\theta$ equals **[AMU 2002]**

(a) $\frac{(e^x + e^{-x})}{2}$ (b) $\frac{2}{(e^x + e^{-x})}$ (c) $\frac{(e^x - e^{-x})}{2}$ (d) $\frac{(e^x - e^{-x})}{(e^x + e^{-x})}$

Solution: (b) $\tan\theta + \sec\theta = e^x$ ....(i)

$\therefore \sec\theta - \tan\theta = e^{-x}$ ....(ii)

From (i) and (ii), $\Rightarrow 2\sec\theta = e^x + e^{-x} \Rightarrow \cos\theta = \frac{2}{e^x + e^{-x}}$.

Example: 15 For $0 < \phi < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi, y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$,then

(a) $xyz = xz + y$ (b) $xyz = xy + z$ (c) $xyz = x + y + z$ (d) Both (b) and (c)

Solution: (d) From $s_\infty = \frac{a}{1-r}$

We get, $x = \frac{1}{1-\cos^2\phi} = \frac{1}{\sin^2\phi}$, $y = \frac{1}{1-\sin^2\phi} = \frac{1}{\cos^2\phi}$, $z = \frac{1}{1-\cos^2\phi \sin^2\phi} = \frac{1}{1-\frac{1}{xy}} = \frac{xy}{xy-1}$

$\Rightarrow xyz - z = xy \Rightarrow xyz = xy + z$ ......(i)

Also, $\frac{1}{x} + \frac{1}{y} = \cos^2\phi + \sin^2\phi = 1 \Rightarrow x + y = xy$; From (i), $xyz = x + y + z$.

Example: 16 If $P = \frac{2\sin\theta}{1+\sin\theta+\cos\theta}$ and $Q = \frac{\cos\theta}{1+\sin\theta}$, then **[MP PET 2001]**

(a) $PQ = 1$ (b) $\frac{Q}{P} = 1$ (c) $Q - P = 1$ (d) $Q + P = 1$

Solution: (d) $P + Q = \frac{2\sin\theta}{1+\sin\theta+\cos\theta} + \frac{\cos\theta}{1+\sin\theta}$

After solving, $P + Q = 1$.

Example: 17 The value of $6(\sin^6\theta + \cos^6\theta) - 9(\sin^4\theta + \cos^4\theta) + 4$ equals to **[MP PET 2001, 1997]**

(a) – 3 (b) 0 (c) 1 (d) 3

Solution: (c) $= 6(\sin^6\theta + \cos^6\theta) - 9(\sin^4\theta + \cos^4\theta) + 4$

$= 6[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 9[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta.\cos^2\theta] + 4$

$= 6[1 - 3\sin^2\theta\cos^2\theta] - 9[1 - 2\sin^2\theta\cos^2\theta] + 4 = 6 - 9 + 4 = 1$.

Example: 18 $\frac{\sin\theta}{1-\cot\theta} + \frac{\cos\theta}{1-\tan\theta}$ equals to **[Karnataka CET 1998]**

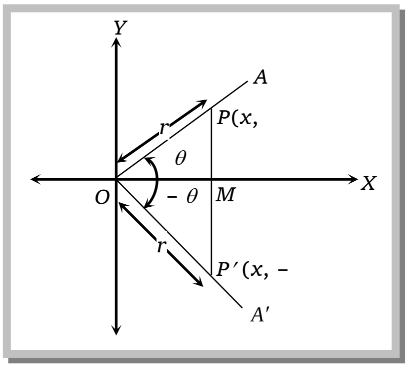
(a) 0 (b) 1 (c) $\cos\theta - \sin\theta$ (d) $\cos\theta + \sin\theta$

Solution: (d) $\frac{\sin\theta.\sin\theta}{\sin\theta(1-\cot\theta)} + \frac{\cos\theta.\cos\theta}{(1-\tan\theta)\cos\theta} = \frac{\sin^2\theta}{(\sin\theta-\cos\theta)} + \frac{\cos^2\theta}{(\cos\theta-\sin\theta)} = \frac{\cos^2\theta - \sin^2\theta}{(\cos\theta - \sin\theta)} = \cos\theta + \sin\theta$.

1.6 Trigonometrical Ratios of Allied Angles

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90^o.

(1) **Trigonometric ratios of $(-\theta)$:** Let a revolving ray starting from its initial position OX, trace out an angle $\angle XOA = \theta$. Let $P(x, y)$ be a point on OA such that $OP = r$. Draw $PM \perp$ from P on x-axis. Angle $\angle XOA' = -\theta$ in the clockwise sense. Let P' be a point on OA' such that $OP' = OP$. Clearly M and M' coincide and ΔOMP is congruent to $\Delta OMP'$ then P' are $(x, -y)$.



$$\sin(-\theta) = \frac{-y}{r} \Rightarrow \frac{-y}{r} = -\sin\theta\,; \quad \cos(-\theta) = \frac{x}{r} = \cos\theta\,; \quad \tan(-\theta) = \frac{-y}{x} = -\tan\theta$$

Taking the reciprocal of these trigonometric ratios;

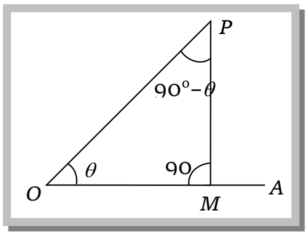
$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$, $\quad \sec(-\theta) = \sec\theta$ and $\cot(-\theta) = -\cot\theta$

Note : ❑ A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x in its domain.

❑ A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x in its domain.

❑ $\sin\theta, \tan\theta, \cot\theta, \operatorname{cosec}\theta$ are odd functions and $\cos\theta, \sec\theta$ are even functions.

(2) **Trigonometric function of $(90 - \theta)$:** Let the revolving line, starting from OA, trace out any acute angle AOP, equal to θ. From any point P, draw $PM \perp$ to OA. Three angles of a triangle are together equal to two right angles, and since OMP is a right angle, the sum of the two angles MOP and OPM is right angle. $\angle OPM = 90^o - \theta$



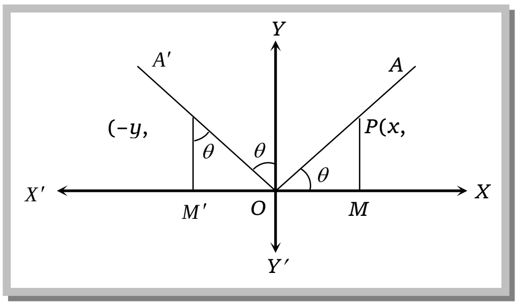
[When the angle OPM is consider, the line PM is the 'base' and MO is the 'perpendicular']

$$\sin(90^o - \theta) = \sin MPO = \frac{MO}{PO} = \cos AOP = \cos\theta\ , \qquad \cos(90^o - \theta) = \cos MPO = \frac{PM}{PO} = \sin AOP = \sin\theta$$

$$\tan(90^o - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot\theta\,, \qquad \cot(90^o - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan\theta$$

$$\operatorname{cosec}(90^o - \theta) = \operatorname{cosec}\ MPO = \frac{PO}{MO} = \sec AOP = \sec\theta\,, \quad \sec(90^o - \theta) = \sec MPO = \frac{PO}{PM} = \operatorname{cosec} AOP = \operatorname{cosec}\theta$$

(3) **Trigonometric function of $(90+\theta)$:** Let a revolving ray OA starting from its initial poisiton OX, trace out an angle $\angle XOA = \theta$ and let another revolving ray OA' starting from the same initial position OX, first trace out an angle θ. So as to coincide with OA and then it revolves through an angle of 90^o in anticlockwise direction to form an angle $\angle XOA' = 90^o + \theta$.



Let P and P' be points on OA and OA' respectively such that $OP = OP' = r$.

Draw perpendicular PM and PM' from P and P' respectively on OX. Let the coordinates of P be (x, y). Then $OM = x$ and $PM = y$ clearly, $OM' = PM = y$ and $P'M' = OM = x$.

So the coordinates of P' are $-y, x$

$$\sin(90 + \theta) = \frac{M'P'}{OP'} = \frac{x}{r} = \cos\theta, \qquad \cos(90 + \theta) = \frac{OM'}{OP'} = \frac{-y}{r} = -\sin\theta$$

$$\tan(90 + \theta) = \frac{M'P'}{OM'} = \frac{x}{-y} = \frac{-x}{y} = -\cot\theta\,, \cot(90 + \theta) = -\tan\theta\,, \sec(90 + \theta) = -\operatorname{cosec}\theta\,, \quad \operatorname{cosec}(90 + \theta) = \sec\theta$$

Allied angles → / Trigo. Ratio ↓	$(-\theta)$	$(90-\theta)$ or $\left(\frac{\pi}{2}-\theta\right)$	$(90+\theta)$ or $\left(\frac{\pi}{2}+\theta\right)$	$(180-\theta)$ or $(\pi-\theta)$	$(180+\theta)$ or $(\pi+\theta)$	$(270-\theta)$ or $\left(\frac{3\pi}{2}-\theta\right)$	$(270+\theta)$ or $\left(\frac{3\pi}{2}+\theta\right)$	$(360-\theta)$ or $(2\pi-\theta)$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$

Important Tips

- $\sin n\pi = 0, \quad \cos n\pi = (-1)^n$
- $\sin(n\pi+\theta) = (-1)^n \sin\theta$, $\cos(n\pi+\theta) = (-1)^n \cos\theta$
- $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$, if n is odd

  $= (-1)^{n/2} \sin\theta$, if n is even
- $\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$, if n is odd

  $= (-1)^{n/2} \cos\theta$, if n is even

1.7 Trigonometrical Ratios for Various Angles

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0	1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

1.8 Trigonometrical Ratios in terms of Each other

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\text{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\text{cosec}^2\theta-1}}{\text{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\text{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\text{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\text{cosec}\theta}{\sqrt{\text{cosec}^2\theta-1}}$
$\text{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\text{cosec}\theta$

Important Tips

☞ *Values for some standard angles*

$\sin 15^o = \cos 75^o = \frac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15^o = \sin 75^o = \frac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan 15^o = \cot 75^o = 2-\sqrt{3}$;

$\sin 18^o = \cos 72^o = \frac{\sqrt{5}-1}{4}$; $\cos 36^o = \sin 54^o = \frac{\sqrt{5}+1}{4}$; $\tan 75^o = \cot 15^o = 2+\sqrt{3}$

$\sin 22\frac{1}{2}^o = \cos 67\frac{1}{2}^o = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\cos 22\frac{1}{2}^o = \sin 67\frac{1}{2}^o = \frac{\sqrt{2+\sqrt{2}}}{2}$; $\cot\ 22\frac{1}{2}^o = \tan 67\frac{1}{2}^o = \sqrt{2}+1$

$\tan\ 22\frac{1}{2}^o = \cot 67\frac{1}{2}^o = \sqrt{2}-1$

Example: 19 $\sin 75^o =$ **[MNR 1979]**

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (c) $-\frac{\sqrt{3}-1}{2\sqrt{2}}$ (d) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

Solution: (b) $\sin 75^o = \sin(45^o + 30^o) = \sin 45^o \cos 30^o + \sin 30^o \cos 45^o = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{2}\times\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

Example: 20 The value of $\cos A - \sin A$, when $A = \frac{5\pi}{4}$ is **[MP PET 1990]**

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 0 (d) 1

Solution: (c) $\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} \Rightarrow -\cos\frac{\pi}{4} + \sin\frac{\pi}{4} \Rightarrow -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$.

Example: 21 $\tan A + \cot(180^o + A) + \cot(90^o + A) + \cot(360^o - A)$ equal to **[MP PET 1992]**

(a) 0 (b) $2\tan A$ (c) $2\cot A$ (d) $2(\tan A - \cot A)$

Solution: (a) $\tan A + \cot A + (-\tan A) + (-\cot A) = 0$.

Example: 22 The value of $\cos 15^o - \sin 15^o$ equal to **[UPSEAT 1975; MP PET 1994; MP PET 2002]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{-1}{\sqrt{2}}$ (d) Zero

Solution: (a) $\frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$.

Example: 23 $3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right) + \sin^4(3\pi+\alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right) + \sin^6(5\pi-\alpha)\right]$ **[IIT 1986]**

(a) 0 (b) 1 (c) 3 (d) $\sin 4\alpha + \sin 6\alpha$

Solution: (b) $= 3[(-\cos\alpha)^4 + (-\sin\alpha)^4] - 2[\cos^6\alpha + \sin^6\alpha]$

$= 3[(\cos^2\alpha + \sin^2\alpha)^2 - 2\sin^2\alpha\cos^2\alpha] - 2[(\cos^2\alpha + \sin^2\alpha)^3 - 3\cos^2\alpha\sin^2\alpha\ (\cos^2\alpha + \sin^2\alpha)]$

$= 3 - 6\sin^2\alpha\cos^2\alpha - 2 + 6\sin^2\alpha\cos^2\alpha = 1$.

Trick : Put $\alpha = 0, \frac{\pi}{2}$; then the value of expression remains constant *i.e.*, it is independent of α.

Example: 24 Which of the following number is rational **[IIT 1998]**

(a) $\sin 15^o$ (b) $\cos 15^o$ (c) $\sin 15^o . \cos 15^o$ (d) $\sin 15^o . \cos 75^o$

Solution: (c) $\sin 15^o = \sin(45^o - 30^o) = \frac{\sqrt{3}-1}{2\sqrt{2}}$ = irrational $\Rightarrow \cos 15^o = \cos(45^o - 30^o) = \frac{\sqrt{3}+1}{2\sqrt{2}}$ = irrational

$\sin 15^o . \cos 15^o = \frac{1}{2}(2 \sin 15^o \cos 15^o) = \frac{1}{2} \sin 30^o = \frac{1}{4}$ = rational

$\sin 15^o . \cos 75^o = \sin 15^o . \sin 15^o = \sin^2 15^o = \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2 = \frac{4-2\sqrt{3}}{8} =$ irrational.

Example: 25 If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 2$ is **[MP PET 2001]**

(a) 0 (b) 1 (c) -1 (d) 2

Solution: (c) Since $\sin x + \sin^2 x = 1 \Rightarrow \sin x = 1 - \sin^2 x = \cos^2 x$ ....(i)

From given expression, $\cos^6 x(\cos^6 x + 3\cos^4 x + 3\cos^2 x + 1) - 2 = \cos^6 x(\cos^2 x + 1)^3 - 2$

From (i) $\sin x = \cos^2 x$

$\therefore \sin^3 x(\sin x + 1)^3 - 2 = (\sin^2 x + \sin x)^3 - 2 = 1 - 2 = -1$.

Example: 26 If $4\sin\theta = 3\cos\theta$ then $\frac{\sec^2\theta}{4[1-\tan^2\theta]}$ equals to

(a) $\frac{25}{16}$ (b) $\frac{25}{28}$ (c) $\frac{1}{4}$ (d) 1

Solution: (b) Given $4\sin\theta = 3\cos\theta \Rightarrow \tan\theta = \frac{3}{4}$

The given expression is $\frac{\sec^2\theta}{4[1-\tan^2\theta]} = \frac{1+\tan^2\theta}{4(1-\tan^2\theta)} = \frac{1+\frac{9}{16}}{4\left(1-\frac{9}{16}\right)} = \frac{25}{28}$.

1.9 Formulae for the Trigonometric Ratios of Sum and Differences of Two Angles

(1) $\sin(A+B) = \sin A \cos B + \cos A \sin B$ (2) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(3) $\cos(A+B) = \cos A \cos B - \sin A \sin B$ (4) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(5) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ (6) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(7) $\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$ (8) $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

(9) $\sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(10) $\cos(A+B).\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(11) $\tan A \pm \tan B = \frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B} = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B} = \frac{\sin(A \pm B)}{\cos A . \cos B}$ $\left(A \neq n\pi + \frac{\pi}{2}, B \neq m\pi\right)$

(12) $\cot A \pm \cot B = \dfrac{\sin(B \pm A)}{\sin A.\sin B}$ $\left(A \neq n\pi, B \neq m\pi + \dfrac{\pi}{2}\right)$

1.10 Formulae for the Trigonometric Ratios of Sum and Differences of Three Angles

(1) $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

or sin $(A+B+C) = \cos A \cos B \cos C(\tan A + \tan B + \tan C - \tan A.\tan B.\tan C)$

(2) $\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

$\cos(A+B+C) = \cos A \cos B \cos C(1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

(3) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(4) $\cot(A+B+C) = \dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot A \cot B + \cot B \cot C + \cot C.\cot A - 1}$

In general;

(5) $\sin(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 ..... \cos A_n (S_1 - S_3 + S_5 - S_7 + ...)$

(6) $\cos(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 ... \cos A_n (1 - S_2 + S_4 - S_6 ....)$

(7) $\tan(A_1 + A_2 + ... + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}$

Where; $S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = The sum of the tangents of the separate angles.

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + ...$ = The sum of the tangents taken two at a time.

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ...$ = Sum of tangents three at a time, and so on.

If $A_1 = A_2 = ... = A_n = A$, then $S_1 = n \tan A$, $S_2 = {}^nC_2 \tan^2 A$, $S_3 = {}^nC_3 \tan^3 A, ...$

(8) $\sin nA = \cos^n A({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....)$

(9) $\cos nA = \cos^n A(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - ...)$

(10) $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ....}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A + ...}$

(11) $\sin nA + \cos nA = \cos^n A(1 + {}^nC_1 \tan A - {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A + {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A - {}^nC_6 \tan^6 A - .....)$

(12) $\sin nA - \cos nA = \cos^n A(-1 + {}^nC_1 \tan A + {}^nC_2 \tan^2 A - {}^nC_3 \tan^3 A - {}^nC_4 \tan^4 A + {}^nC_5 \tan^5 A + {}^nC_6 \tan^6 A ...)$

(13) $\sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta) = \dfrac{\sin\{\alpha + (n-1)(\beta/2)\}.\sin(n\beta/2)}{\sin(\beta/2)}$

(14) $\cos(\alpha)+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+....+\cos(\alpha+(n-1)\beta) = \dfrac{\cos\left\{\alpha+(n-1)\left(\dfrac{\beta}{2}\right)\right\}.\sin\left\{n\left(\dfrac{\beta}{2}\right)\right\}}{\sin\left(\dfrac{\beta}{2}\right)}$

1.11 Formulae to Transform the Product into Sum or Difference

(1) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(2) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(3) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(4) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Let $A+B=C$ and $A-B=D$

Then, $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$

Therefore, we find out the formulae to transform the sum or difference into product.

(5) $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(6) $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

(7) $\cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

(8) $\cos C-\cos D=2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}=-2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

Important Tips

- ☞ $\sin(60^o-\theta).\sin\theta\sin(60^o+\theta)=\dfrac{1}{4}\sin 3\theta$
- ☞ $\cos(60-\theta).\cos\theta\cos(60^o+\theta)=\dfrac{1}{4}\cos 3\theta$
- ☞ $\tan(60^o-\theta).\tan\theta\tan(60^o+\theta)=\tan 3\theta$
- ☞ $\cos A.\cos 2A.\cos 2^2A.\cos 2^3A.......\cos 2^{n-1}A=\dfrac{\sin 2^nA}{2^n\sin A}$, if $A=n\pi$

  = 1, if $A=2n\pi$

  = 1, if $A=(2n+1)\pi$

Example: 27 $\dfrac{\cos 12^o-\sin 12^o}{\cos 12^o+\sin 12^o}+\dfrac{\sin 147^o}{\cos 147^o}=$ **[MP PET 1991]**

(a) $2\tan 33^o$ (b) 1 (c) – 1 (d) 0

Solution: (d) $=\dfrac{1-\tan 12^o}{1+\tan 12^o}+\tan 147^o=\tan(45^o-12)+\tan(180^o-33^o)=\tan 33^o+(-\tan 33^o)=0$.

Example: 28 If $\sin\theta_1+\sin\theta_2+\sin\theta_3=3$, then $\cos\theta_1+\cos\theta_2+\cos\theta_3=$

(a) 3 (b) 2 (c) 1 (d) 0

Solution: (d) We know $|\sin\theta|\le 1$; So, each θ_1,θ_2 and θ_3 must be equal to $\pi/2$

$\therefore\quad \cos\theta_1+\cos\theta_2+\cos\theta_3=0$.

Example: 29 $\cos A+\cos(240^o+A)+\cos(240^o-A)=$ **[MP PET 1991]**

(a) $\cos A$ (b) 0 (c) $\sqrt{3}\sin A$ (d) $\sqrt{3}\cos A$

Solution: (b) $\cos A + [2\cos 240^o \cos A] = \cos A + 2(-\cos 60^o)\cos A$

$$= \cos A\left[1 - 2\left(\frac{1}{2}\right)\right] = 0 .$$

Example: 30 $\frac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} =$ **[MP PET 1993]**

(a) $\tan(A-B)$ (b) $\tan(A+B)$ (c) $\cot(A-B)$ (d) $\cot(A+B)$

Solution: (b) $\frac{2(\sin^2 A - \sin^2 B)}{2\sin A\cos A - 2\sin B\cos B} = \frac{2\sin(A+B).\sin(A-B)}{\sin 2A - \sin 2B} = \frac{2\sin(A+B)\sin(A-B)}{2\sin(A-B)\cos(A+B)} = \tan(A+B)$.

Example: 31 The expression $\cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A\cos B$ is

(a) Dependent on B (b) Dependent on A and B

(c) Dependent on A (d) Independent of A and B

Solution: (c) $\cos^2(A-B) + \cos^2 B - \cos(A-B)[\cos(A-B) + \cos(A+B)]$

$\Rightarrow \cos^2 B - \cos(A-B)\cos(A+B) \Rightarrow \cos^2 B - (\cos^2 A - \sin^2 B) = 1 - \cos^2 A$

Trick : Put $A = 90^o$ and 0° the value is $\sin^2 B + \cos^2 B = 1$ and 0 again put $B = 0^o$, 90° and the value is $\sin^2 A$ and $\sin^2 A$ means expression depends on A.

Example: 32 If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$ then $\alpha + \beta =$ **[IIT 1978]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) None of these

Solution: (b) We have $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$

$$\tan(\alpha+\beta) = \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{(m+1)}.\frac{1}{(2m+1)}} = \frac{2m^2 + m + m + 1}{2m^2 + m + 2m + 1 - m} \qquad \left[\because \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right]$$

$$= \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} = 1 \Rightarrow \tan(\alpha+\beta) = \tan\frac{\pi}{4}$$

Hence $\alpha + \beta = \frac{\pi}{4}$

Trick : As $\alpha + \beta$ is independent of m, therefore put $m = 1,$ then $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$.

Therefore, $\tan(\alpha+\beta) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1$, Hence $\alpha + \beta = \frac{\pi}{4}$ (Also check for other values of m)

Example: 33 If $\tan\theta - \cot\theta = a$ and $\sin\theta + \cos\theta = b$, then $(b^2 - 1)^2(a^2 + 4) =$ **[WB JEE 1979]**

(a) 2 (b) -4 (c) ± 4 (d) 4

Solution: (d) Given that $\tan\theta - \cot\theta = a$ ....(i) and $\sin\theta + \cos\theta = b$ ....(ii)

Now, $(b^2-1)^2(a^2+4) = \{(\sin\theta+\cos\theta)^2-1\}^2\{\tan\theta-\cot\theta)^2+4\}$

$$=[1+\sin 2\theta-1]^2[\tan^2\theta+\cot^2\theta-2+4]=\sin^2 2\theta(\text{cosec}^2\theta+\sec^2\theta)=4\sin^2\theta\cos^2\theta\left[\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right]=4$$

Trick : Obviously the value of expression $(b^2-1)^2(a^2+4)$ is independent of θ, therefore put any suitable value of θ. Let $\theta=45^o$, we get $a=0$, $b=\sqrt{2}$ so that $[(\sqrt{2})^2-1]^2(0^2+4)=4$.

Example: 34 If $\sin B = \frac{1}{5}\sin(2A+B)$, then $\frac{\tan(A+B)}{\tan A}=$

(a) $\frac{5}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{3}{5}$

Solution: (c) $\frac{\sin(2A+B)}{\sin B}=\frac{5}{1}$ by componendo and Dividendo. $\frac{\sin(2A+B)+\sin B}{\sin(2A+B)-\sin B}=\frac{5+1}{5-1}$

$\frac{2\sin(A+B).\cos A}{2\cos(A+B).\sin A}=\frac{6}{4} \Rightarrow \frac{\tan(A+B)}{\tan A}=\frac{3}{2}$.

Example: 35 $\frac{\sin 70^o+\cos 40^o}{\cos 70^o+\sin 40^o}=$ **[Karnataka CET 1986; MP PET 1999]**

(a) 1 (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\frac{1}{2}$

Solution: (c) $\frac{\sin 70^o+\cos 40^o}{\cos 70^o+\sin 40^o} = \frac{\sin 70^o+\sin 50^o}{\sin 20^o+\sin 40^o}=\frac{2\sin 60^o\cos 10^o}{2\sin 30^o\cos(-10^o)} = \frac{\sin 60^o}{\sin 30^o}=\frac{\sqrt{3}}{2}.\frac{2}{1}=\sqrt{3}$.

Example: 36 $\sin 47^o+\sin 61^o-\sin 11^o-\sin 25^o=$ **[EAMCET 2003; MP PET 2001]**

(a) $\sin 36^o$ (b) $\sin 7^o$ (c) $\cos 36^o$ (d) $\cos 7^o$

Solution: (d) $\sin 47^o+\sin 61^o-(\sin 11^o+\sin 25^o) = 2\sin 54^o.\cos 7^o-2\sin 18^o\cos 7^o$

$= 2\cos 7^o(\sin 54^o-\sin 18^o) = 2\cos 7^o.2\cos 36^o.\sin 18^o = 4.\cos 7^o.\frac{\sqrt{5}+1}{4}.\frac{\sqrt{5}-1}{4}=\cos 7^o$.

Example: 37 $\frac{\cos 10^o-\sin 10^o}{\cos 10^o+\sin 10^o}=$

(a) $\tan 55^o$ (b) $\cot 55^o$ (c) $-\tan 35^o$ (d) $-\cot 35^o$

Solution: (b) $\frac{\cos 10^o-\sin 10^o}{\cos 10^o+\sin 10^o}=\frac{1-\tan 10^o}{1+\tan 10^o}=\tan 35^o=\tan(90^o-35^o)=\cot 55^o$.

Example: 38 If $\tan(A+B)=p$ and $\tan(A-B)=q$ then the value of $\tan 2A=$

(a) $\frac{p+q}{p-q}$ (b) $\frac{p-q}{1+pq}$ (c) $\frac{1+pq}{1-p}$ (d) $\frac{p+q}{1-pq}$

Solution: (d) $2A=\{(A+B)+(A-B)\} \Rightarrow \tan 2A=\frac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B).\tan(A-B)} \Rightarrow \tan 2A=\frac{p+q}{1-pq}$

Example: 39 $\sin 163^o\cos 347^o+\sin 73^o\sin 167^o=$ **[MP PET 2000]**

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) None of these

Solution: (b) $\sin(90^o+73^o).\cos(360^o-13^o)+\sin 73^o.\sin(180^o-13^o)=\cos 73^o.\cos 13^o+\sin 73^o.\sin 13^o=\cos(73^o-13^o)=\cos 60^o=\frac{1}{2}$.

Example: 40 The value of $\cot 70^o+4\cos 70^o$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\frac{1}{2}$

Solution: (b) $\cot 70^o + 4\cos 70^o = \frac{\cos 70^o + 4\sin 70^o.\cos 70^o}{\sin 70^o} = \frac{\cos 70^o + 2\sin 140^o}{\sin 70^o}$

$= \frac{\cos 70^o + 2\sin(180 - 40^o)}{\sin 70^o} = \frac{\sin 20^o + \sin 40^o + \sin 40^o}{\sin 70^o}$

$= \frac{2\sin 30^o \cos 10^o + \sin 40^o}{\sin 70^o} = \frac{\sin 80^o + \sin 40^o}{\sin 70^o} = \frac{2\sin 60^o \cos 20^o}{\sin 70^o} = \sqrt{3}$.

Example: 41 If $\tan\alpha = (1+2^{-x})^{-1}, \tan\beta = (1+2^{x+1})^{-1}$, then $\alpha+\beta$ equals **[AMU 2002]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution: (b) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \Rightarrow \tan(\alpha+\beta) = \frac{\frac{1}{1+\frac{1}{2^x}} + \frac{1}{1+2^{x+1}}}{1 - \frac{1}{1+\frac{1}{2^x}}.\frac{1}{1+2^{x+1}}}$

$\Rightarrow \quad \tan(\alpha+\beta) = \frac{2^x + 2.2^{x+x} + 2^x + 1}{1 + 2^x + 2.2^x + 2.2^{x+x} - 2^x} \Rightarrow \tan(\alpha+\beta) = 1 = \tan\frac{\pi}{4} \Rightarrow \alpha+\beta = \frac{\pi}{4}$.

Example: 42 The value of $\frac{\tan 70^o - \tan 20^o}{\tan 50^o} =$ **[Karnataka CET 2003]**

(a) 1 (b) 2 (c) 3 (d) 0

Solution: (b) $\frac{\frac{\sin 70^o}{\cos 70^o} - \frac{\sin 20^o}{\cos 20^o}}{\frac{\sin 50^o}{\cos 50^o}} = \frac{\frac{\sin 70^o \cos 20^o - \cos 70^o \sin 20^o}{\cos 70^o.\cos 20^o}}{\frac{\sin 50^o}{\cos 50^o}} = \frac{2}{2} \times \frac{\sin(70^o - 20^o)\cos 50^o}{\cos 70^o.\cos 20^o.\sin 50^o} = \frac{2\sin 50^o.\cos 50^o}{2\cos 70^o \cos 20^o.\sin 50^o}$

$= \frac{2\cos 50^o}{\cos 90^o + \cos 50^o} = \frac{2\cos 50^o}{0 + \cos 50^o} = 2$.

1.12 Trigonometric Ratio of Multiple of an Angle

(1) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(2) $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$; where $A \neq (2n+1)\frac{\pi}{4}$.

(3) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

(4) $\sin 3A = 3\sin A - 4\sin^3 A = 4\sin(60^o - A).\sin A.\sin(60^o + A)$

(5) $\cos 3A = 4\cos^3 A - 3\cos A = 4\cos(60^o - A).\cos A.\cos(60^o + A)$

(6) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = \tan(60^o - A).\tan A.\tan(60^o + A)$, where $A \neq n\pi + \pi/6$

(7) $\sin 4\theta = 4\sin\theta.\cos^3\theta - 4\cos\theta\sin^3\theta$ (8) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

(9) $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(10) $\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$

(11) $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

1.13 Trigonometric Ratio of Sub-multiple of an Angle

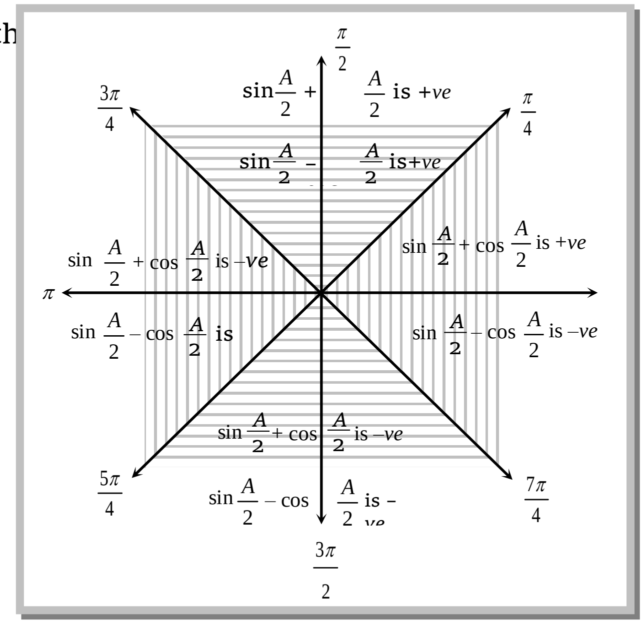
(1) $\left|\sin\frac{A}{2}+\cos\frac{A}{2}\right|=\sqrt{1+\sin A}$ or $\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}$ *i.e.*, $\begin{cases}+, \text{If } 2n\pi-\pi/4\le A/2\le 2n\pi+\frac{3\pi}{4}\\ -, \text{otherwise}\end{cases}$

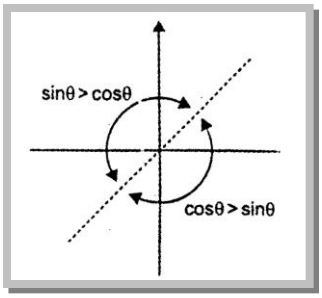
(2) $\left|\sin\frac{A}{2}-\cos\frac{A}{2}\right|=\sqrt{1-\sin A}$ or $(\sin\frac{A}{2}-\cos\frac{A}{2})=\pm\sqrt{1-\sin A}$ *i.e.*, $\begin{cases}+, \text{If } 2n\pi+\pi/4\le A/2\le 2n\pi+\frac{5\pi}{4}\\ -, \text{otherwise}\end{cases}$

(3) (i) $\tan\frac{A}{2}=\frac{\pm\sqrt{\tan^2 A+1}-1}{\tan A}=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{1-\cos A}{\sin A}$, where $A\ne(2n+1)\pi$

(ii) $\cot\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{1-\cos A}}=\frac{1+\cos A}{\sin A}$, where $A\ne 2n\pi$

The ambiguities of signs are removed by locating the quadrants in which $\frac{A}{2}$ lies or you can follow th





(4) $\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}$; where $A\ne(2n+1)\pi$ (5) $\cot^2\frac{A}{2}=\frac{1+\cos A}{1-\cos A}$; where $A\ne 2n\pi$

Important Tips

☞ *Any formula that gives the value of* $\sin\frac{A}{2}$ *in terms of sin A shall also give the value of sine of* $\frac{n\pi+(-1)^n A}{2}$.

☞ *Any formula that gives the value of* $\cos\frac{A}{2}$ *in terms of cos A shall also give the value of cos of* $\frac{2n\pi\pm A}{2}$.

☞ *Any formula that gives the value of* $\tan\frac{A}{2}$ *in terms of tan A shall also give the value of tan of* $\frac{n\pi\pm A}{2}$.

Example: 43 If $\sin\alpha=\frac{-3}{5}$ where $\pi<\alpha<\frac{3\pi}{2}$, then $\cos\frac{\alpha}{2}$ equal to **[MP PET 1998]**

(a) $\frac{1}{\sqrt{10}}$ (b) $-\frac{1}{\sqrt{10}}$ (c) $\frac{3}{\sqrt{10}}$ (d) $\frac{-3}{\sqrt{10}}$

Solution: (d) $\pi < \alpha < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4} \Rightarrow \cos\frac{\alpha}{2} = -ve \Rightarrow \therefore \cos\alpha = \frac{4}{5} \Rightarrow \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\frac{4}{5}}{2}} = -\sqrt{\frac{9}{10}} = \frac{-3}{\sqrt{10}}$.

Example: 44 $2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta)$ equal to **[UPSEAT 1993; IIT 1977]**

(a) $\sin 2\alpha$ (b) $\cos 2\beta$ (c) $\cos 2\alpha$ (d) $\sin 2\beta$

Solution: (c) Since $2\cos(\alpha+\beta) = 2\cos^2(\alpha+\beta) - 1$, $2\sin^2\beta = 1 - \cos 2\beta$ $= -\cos 2\beta + 2\cos(\alpha+\beta)[2\sin\alpha\sin\beta + \cos(\alpha+\beta)]$
$= -\cos 2\beta + 2\cos(\alpha+\beta).\cos(\alpha-\beta)$ $= -\cos 2\beta + \cos 2\alpha + \cos 2\beta = \cos 2\alpha$.

Example: 45 $\frac{\cot^2 15^o - 1}{\cot^2 15^o + 1}$ = **[MP PET 1998]**

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{3\sqrt{3}}{4}$ (d) $\sqrt{3}$

Solution: (b) $$\frac{1-\tan^2 15}{1+\tan^2 15} = \frac{1-[\tan(45^o-30^o)]^2}{1+[\tan(45^o-30^o)]^2} = \frac{1-\left[\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\right]^2}{1+\left[\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\right]^2} = \frac{1-\left[\frac{\sqrt{3}-1}{\sqrt{3}+1}\right]^2}{1+\left[\frac{\sqrt{3}-1}{\sqrt{3}+1}\right]^2} = \frac{[\sqrt{3}+1]^2-[\sqrt{3}-1]^2}{[\sqrt{3}+1]^2+[\sqrt{3}-1]^2} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

Trick : $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} \Rightarrow \frac{1-\tan^2 15^o}{1+\tan^2 15^o} = \cos 30^o = \frac{\sqrt{3}}{2}$.

Example: 46 If $\sin 6\theta = 32\cos^5\theta.\sin\theta - 32\cos^3\theta\sin\theta + 3x$, then x = **[EAMCET 2003]**

(a) $\cos\theta$ (b) $\cos 2\theta$ (c) $\sin\theta$ (d) $\sin 2\theta$

Solution: (d) $\sin 6\theta = 2\sin 3\theta.\cos 3\theta = 2[3\sin\theta - 4\sin^3\theta][4\cos^3\theta - 3\cos\theta]$

$= 24\sin\theta.\cos\theta(\sin^2\theta+\cos^2\theta) - 18\sin\theta\cos\theta - 32\sin^3\theta\cos^3\theta = 32\cos^5\theta.\sin\theta - 32\cos^3\theta.\sin\theta + 3\sin 2\theta$

On comparing, $x = \sin 2\theta$

Trick : Put $\theta = 0^o$, then $x = 0$. So, option (c) and (d) are correct.

Now put $\theta = 30^o$, then $x = \frac{\sqrt{3}}{2}$. Therefore, Only option (d) is correct.

Example: 47 If $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\cos\theta$, then $x^6 + x^{-6}$ = **[Karnataka CET 2003]**

(a) $2\cos 6\theta$ (b) $2\cos 12\theta$ (c) $2\cos 3\theta$ (d) $2\sin 3\theta$

Solution: (b) Given, $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\cos\theta$ ....(i)

On squaring both sides we get, $x + \frac{1}{x} + 2 = 4\cos^2\theta \Rightarrow x + \frac{1}{x} = 4\cos^2\theta - 2$

$\Rightarrow x + \frac{1}{x} = 2(2\cos^2\theta - 1) = 2\cos 2\theta$ ....(ii)

Again squaring both sides,

$x^2+\frac{1}{x^2}+2=4\cos^2 2\theta \Rightarrow x^2+\frac{1}{x^2}=4\cos^2 2\theta-2=2(2\cos^2 2\theta-1) \Rightarrow x^2+\frac{1}{x^2}=2\cos 4\theta$ ......(iii)

Now taking cube of both sides; $\left(x^2+\frac{1}{x^2}\right)^3=(2\cos 4\theta)^3 \Rightarrow x^6+\frac{1}{x^6}+3x^2.\frac{1}{x^2}\left(x^2+\frac{1}{x^2}\right)=8\cos^3 4\theta$

$\Rightarrow x^6+\frac{1}{x^6}+3(2\cos 4\theta) = 8\cos^3 4\theta \Rightarrow x^6+\frac{1}{x^6}=8\cos^3 4\theta-6\cos 4\theta$

$\Rightarrow x^6+\frac{1}{x^6}= 2(4\cos^3 4\theta-3\cos 4\theta)=2\cos 3(4\theta)=2\cos 12\theta$.

Example: 48 For $A = 133^o, 2\cos\frac{A}{2}$ is equal to

(a) $-\sqrt{1+\sin A}-\sqrt{1-\sin A}$

(b) $-\sqrt{1+\sin A}+\sqrt{1-\sin A}$

(c) $\sqrt{1+\sin A}-\sqrt{1-\sin A}$

(d) $\sqrt{1+\sin A}+\sqrt{1-\sin A}$

Solution: (c) For $A=133^o, \frac{A}{2}=66.5^o \Rightarrow \sin\frac{A}{2}>\cos\frac{A}{2}>0$

Hence, $\sqrt{1+\sin A}=\sin\frac{A}{2}+\cos\frac{A}{2}$ ......(i) and $\sqrt{1-\sin A}=\sin\frac{A}{2}-\cos\frac{A}{2}$ ......(ii)

Subtract (ii) from (i) we get, $2\cos\frac{A}{2}=\sqrt{1+\sin A}-\sqrt{1-\sin A}$.

Example: 49 If $2\tan A=3\tan B$, then $\frac{\sin 2B}{5-\cos 2B}$ is equal to **[AMU 2001]**

(a) $\tan A - \tan B$ (b) $\tan(A-B)$ (c) $\tan(A+B)$ (d) $\tan(A+2B)$

Solution: (b) $2\tan A=3\tan B \Rightarrow \tan A=\frac{3}{2}\tan B=\frac{3}{2}t$ (Let $\tan B=t$) $\Rightarrow \sin 2B=\frac{2t}{1+t^2}, \cos 2B=\frac{1-t^2}{1+t^2}$

$$\therefore \frac{\sin 2B}{5-\cos 2B}=\frac{\left(\frac{2t}{1+t^2}\right)}{5-\left(\frac{1-t^2}{1+t^2}\right)}=\frac{2t}{4+6t^2}=\frac{t}{2+3t^2}=\tan(A-B).$$

Example: 50 If $90^o<A<180^o$ and $\sin A=\frac{4}{5}$, then $\tan\frac{A}{2}$ is equal to **[AMU 2001]**

(a) $\frac{1}{2}$ (b) $\frac{3}{5}$ (c) $\frac{3}{2}$ (d) 2

Solution: (d) $\sin A=\frac{4}{5} \Rightarrow \tan A=-\frac{4}{3}$, $(90^o<A<180^o)$

$\tan A=\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$, Let $\tan\frac{A}{2}=P$

$\Rightarrow \quad \frac{-4}{3} = \frac{2P}{1-P^2} \Rightarrow 4P^2 - 6P - 4 = 0 \Rightarrow P = -\frac{1}{2}, 2 \Rightarrow P = -\frac{1}{2}$ (impossible)

So, $P = 2$ *i.e.*, $\tan\frac{A}{2} = 2$.

Example: 51 If $\tan\alpha = \frac{1}{7}$ and $\sin\beta = \frac{1}{\sqrt{10}}$, then $\tan(\alpha + 2\beta)$ is equal to

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

Solution: (a) $\tan\alpha = \frac{1}{7}, \sin\beta = \frac{1}{\sqrt{10}} \Rightarrow \tan\beta = \frac{1}{3} \Rightarrow \tan 2\beta = \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{3}{4}$

$\therefore \tan(\alpha + 2\beta) = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{3}{28}} = \frac{4+21}{25} = 1$

Example: 52 If $\tan\frac{\theta}{2} = t$, then $\frac{1-t^2}{1+t^2}$ is equal to

(a) $\cos\theta$ (b) $\sin\theta$ (c) $\sec\theta$ (d) $\cos 2\theta$

Solution: (a) $\frac{1-t^2}{1+t^2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ $(\because \tan\frac{\theta}{2} = t)$ $= \cos(2.\frac{\theta}{2}) = \cos\theta$.

Example: 53 The value of $\frac{\tan x}{\tan 3x}$ when ever defined never lie between **[Haryana CEE 1998; IIT 1992, 97]**

(a) $\frac{1}{3}$ and 3 (b) $\frac{1}{4}$ and 4 (c) $\frac{1}{5}$ and 5 (d) 5 and 6

Solution: (a) Let, $y = \frac{\tan x}{\tan 3x} = \frac{\tan x}{\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}}$

$y = \frac{1-3\tan^2 x}{3-\tan^2 x} = \frac{\frac{1}{3} - \tan^2 x}{1 - \frac{1}{3}\tan^2 x}$

Hence, y should never lie between $\frac{1}{3}$ and 3 whenever defined.

Example: 54 If $\tan\theta = t$, then $\tan 2\theta + \sec 2\theta$ equal to **[MP PET 1999]**

(a) $\frac{1+t}{1-t}$ (b) $\frac{1-t}{1+t}$ (c) $\frac{2t}{1-t}$ (d) $\frac{2t}{1+t}$

Solution: (a) $\tan 2\theta + \sec 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} + \frac{1+\tan^2\theta}{1-\tan^2\theta}$

Given $\tan\theta = t \Rightarrow \therefore \tan 2\theta + \sec 2\theta = \frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2} = \frac{2t+1+t^2}{1-t^2} = \frac{(t+1)^2}{1-t^2} = \frac{1+t}{1-t}$.

Example: 55 If $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$, then $\cos^2(\theta - \phi)$ equal to **[MP PET 2000]**

(a) $\frac{3}{8}$ (b) $\frac{5}{8}$ (c) $\frac{3}{4}$ (d) $\frac{5}{4}$

Solution: (b) Given, $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ ....(i) and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$ ....(ii)

Squaring and adding, $\therefore (\sin^2 2\theta + \cos^2 2\theta) + (\sin^2 2\phi + \cos^2 2\phi) + 2[\sin 2\theta . \sin 2\phi + \cos 2\theta . \cos 2\phi] = \frac{1}{4} + \frac{9}{4}$

$\Rightarrow \cos 2\theta . \cos 2\phi + \sin 2\theta . \sin 2\phi = \frac{1}{4} \Rightarrow \cos(2\theta - 2\phi) = \frac{1}{4} \Rightarrow \cos^2(\theta - \phi) = \frac{5}{8}$.

Example: 56 If $\tan x = \frac{b}{a}$, then $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$ equal to **[MP PET 1990, 2002]**

(a) $\frac{2\sin x}{\sqrt{\sin 2x}}$ (b) $\frac{2\cos x}{\sqrt{\cos 2x}}$ (c) $\frac{2\cos x}{\sqrt{\sin 2x}}$ (d) $\frac{2\sin x}{\sqrt{\cos 2x}}$

Solution: (b) Given $\tan x = \frac{b}{a} \Rightarrow \sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{1+b/a}{1-b/a}} + \sqrt{\frac{1-b/a}{1+b/a}} = \sqrt{\frac{1+\tan x}{1-\tan x}} + \sqrt{\frac{1-\tan x}{1+\tan x}} = \frac{2}{\sqrt{1-\tan^2 x}}$

Now, multiplying by $\sqrt{1+\tan^2 x}$ in $N'r$ and $D'r$ = $\frac{2}{\frac{\sqrt{1-\tan^2 x}}{\sqrt{1+\tan^2 x}} . \sqrt{1+\tan^2 x}} = \frac{2}{\sqrt{\cos 2x} . \sqrt{\sec^2 x}} = \frac{2\cos x}{\sqrt{\cos 2x}}$.

1.14 Maximum and Minimum Value of $a\cos\theta + b\sin\theta$

Let $a = r\cos\alpha$ ....(i) and $b = r\sin\alpha$ ....(ii)

Squaring and adding (i) and (ii), then $a^2 + b^2 = r^2$ or, $r = \sqrt{a^2 + b^2}$

$\therefore \quad a\sin\theta + b\cos\theta = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = r\sin(\theta + \alpha)$

But $-1 \le \sin\theta < 1$ So, $-1 \le \sin(\theta + \alpha) \le 1$; Then $-r \le r\sin(\theta + \alpha) \le r$

Hence, $-\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2 + b^2}$

Then the greatest and least values of $a\sin\theta + b\cos\theta$ are respectively $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$.

Note :
- $\sin^2 x + \text{cosec}^2 x \ge 2$, for every real *x*.
- $\cos^2 x + \sec^2 x \ge 2$, for every real *x*.
- $\tan^2 x + \cot^2 x \ge 2$, for every real *x*.

Important Tips

Use of Σ (Sigma) and Π (Pie) notation

☞ $\sin(A+B+C) = \Sigma \sin A \cos B \cos C - \Pi \sin A$, $\cos(A+B+C) = \Pi \cos A - \Sigma \cos A \sin B \sin C$,

$\tan(A+B+C) = \frac{\Sigma \tan A - \Pi \tan A}{1 - \Sigma \tan A \tan B}$. ($\because$ Σ *denotes summation*)

☞ $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha + 2\beta) + ...$ *n terms* ($\because$ Π *denotes product*)

$$= \frac{\sin[\alpha+(n-1)\beta/2]\sin[n\beta/2]}{\sin(\beta/2)} \quad \textit{or} \quad \sum_{r=1}^{n}\sin(A+\overline{r-1}B) = \frac{\sin\left(A+\frac{n-1}{2}B\right)\sin\frac{nB}{2}}{\sin\frac{B}{2}}.$$

☞ $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+.........n$ *terms* $= \frac{\cos[\alpha+(n-1)\beta/2]\sin[n\beta/2]}{\sin[\beta/2]}$ *or* $\sum_{r=1}^{n}\cos(A+\overline{r-1}B) = \frac{\cos\left(A+\frac{n-1}{2}B\right)\sin\frac{nB}{2}}{\sin\frac{B}{2}}$.

☞ $\sin A/2 \pm \cos A/2 = \sqrt{2}\sin[\pi/4 \pm A] = \sqrt{2}\cos[A \mp \pi/4]$.

☞ $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$.

☞ $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$.

☞ $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$.

Example: 57 If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy+yz+zx=$ **[EAMCET 1994]**

(a) –1 (b) 0 (c) 1 (d) 2

Solution: (b) We have $\frac{x}{1} = \frac{y}{-2} = \frac{z}{-2} = \lambda$ (say)

$\therefore \quad x=\lambda,\ y=-2\lambda, z=-2\lambda$; $\quad \therefore \quad xy+yz+zx = -2\lambda^2+4\lambda^2-2\lambda^2 = 0$

Example: 58 $\frac{\sec 8A - 1}{\sec 4A - 1}$ equal to **[MP PET 1995]**

(a) $\frac{\tan 2A}{\tan 8A}$ (b) $\frac{\tan 8A}{\tan 2A}$ (c) $\frac{\cot 8A}{\cot 2A}$ (d) None of these

Solution: (b) $\frac{1-\cos 8A}{\cos 8A}.\frac{\cos 4A}{1-\cos 4A} = \frac{2\sin^2 4A}{\cos 8A}.\frac{\cos 4A}{2\sin^2 2A} = \frac{2\sin 4.A\cos 4A.\sin 4A}{\cos 8A.2\sin^2 2A} = \frac{\sin 8A.2\sin 2A.\cos 2A}{\cos 8A.2\sin^2 2A} = \frac{\tan 8A}{\tan 2A}$.

Example: 59 If $\tan\theta = \frac{a}{b}$, then $\frac{\sin\theta}{\cos^8\theta}+\frac{\cos\theta}{\sin^8\theta}$ equal to **[WB JEE 1986]**

(a) $\pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8}+\frac{b}{a^8}\right]$ (b) $\pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8}-\frac{b}{a^8}\right]$

(c) $\pm\frac{(a^2-b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8}+\frac{b}{a^8}\right]$ (d) $\pm\frac{(a^2-b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8}-\frac{b}{a^8}\right]$

Solution: (a) Given , $\tan\theta = a/b \Rightarrow \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{b^2-a^2}{b^2+a^2}$

$$\sin\theta = \pm\frac{a}{\sqrt{a^2+b^2}}\ ;\ \cos\theta = \pm\frac{b}{\sqrt{a^2+b^2}}$$

$$\therefore \frac{\sin\theta}{\cos^8\theta}+\frac{\cos\theta}{\sin^8\theta} = \frac{\left(\frac{a}{\sqrt{a^2+b^2}}\right)}{\left(\frac{b}{\sqrt{a^2+b^2}}\right)^8}+\frac{\left(\frac{b}{\sqrt{a^2+b^2}}\right)}{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^8} = \frac{a(a^2+b^2)^4}{b^8(a^2+b^2)^{1/2}}+\frac{b(a^2+b^2)^4}{a^8(a^2+b^2)^{1/2}} = \pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8}+\frac{b}{a^8}\right].$$

Example: 60 The minimum value of $3\cos x + 4\sin x + 5 =$ **[UPSEAT 1991]**

(a) 5 (b) 9 (c) 7 (d) 0

Solution: (d) Minimum value of $3\cos x + 4\sin x \;= -\sqrt{3^2 + 4^2} = -5$

Minimum value of $3\cos x + 4\sin x + 5 = -5 + 5 = 0$.

Example: 61 The greatest and least value of $\sin x \cos x$ are **[UPSEAT 1975]**

(a) $1, -1$ (b) $\frac{1}{2}, -\frac{1}{2}$ (c) $\frac{1}{4}, -\frac{1}{4}$ (d) $2, -2$

Solution: (b) $\frac{1}{2}[2\sin x \cos x] = \frac{1}{2}\sin 2x$; $-1 \le \sin 2x \le 1$; $\frac{-1}{2} \le \frac{\sin 2x}{2} \le \frac{1}{2}$.

Example: 62 The value of $\sin\theta + \cos\theta$ will be greatest when **[UPSEAT 1977, 83; RPET 1995]**

(a) $\theta = 30^o$ (b) $\theta = 45^o$ (c) $\theta = 60^o$ (d) $\theta = 90^o$

Solution: (b) Let $f(x) = \sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \frac{\pi}{4})$

$-1 \le \sin(\theta + \frac{\pi}{4}) \le 1 \;\Rightarrow\; -\sqrt{2} \le \sqrt{2}\sin(\theta + \frac{\pi}{4}) \le \sqrt{2}$

If $f(x)$ is maximum then,

$\sin(\theta + \frac{\pi}{4}) = 1 = \sin\frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow \theta + \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$.

Example: 63 The maximum value of $\sin^2 x + 3\cos^2 x$ is **[Karnataka CET 2003]**

(a) 3 (b) 4 (c) 5 (d) 7

Solution: (b) $f(x) = 4\sin^2 x + 3\cos^2 x = \sin^2 x + 3$ and $0 \le |\sin x| \le 1$

$\therefore$ Maximum value of $\sin^2 x + 3\cos^2 x$ is 4.

Example: 64 If $A = \cos^2\theta + \sin^4\theta$, then for all values of θ **[UPSEAT 2001; IIT 1980; Roorkee 1992; EAMCET 1994]**

(a) $1 \le A \le 2$ (b) $\frac{13}{16} \le A \le 1$ (c) $\frac{3}{4} \le A \le \frac{13}{16}$ (d) $\frac{3}{4} \le A \le 1$

Solution: (d) $A = \cos^2 x + \sin^4 x \Rightarrow \cos^2\theta + \sin^2\theta \sin^2\theta$

$\Rightarrow A \le \cos^2\theta + \sin^2\theta$ $[\because \sin^2\theta \le 1]$ $\Rightarrow A \le 1$

Again $A = \cos^2\theta + \sin^4\theta = (1 - \sin^2\theta) + \sin^4\theta$

$$A = \left(\sin^2\theta - \frac{1}{2}\right)^3 + \frac{3}{4} \ge \frac{3}{4}$$

Hence, $\frac{3}{4} \le A \le 1$.

Example: 65 The value of $5\cos\theta + 3\cos(\theta + \frac{\pi}{3}) + 3$ lies between

(a) -4 and 4 (b) -4 and 6 (c) -4 and 8 (d) -4 and 10

Solution: (d) $5\cos\theta + 3\cos(\theta + \frac{\pi}{3}) + 3 \;=\; 5\cos\theta + 3[\cos\theta\cos\frac{\pi}{3} - \sin\theta.\sin\frac{\pi}{3}] + 3$

$= [5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta] + 3 \;=\; \left(\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta\right) + 3$

$\because\; -\sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} \le \left(\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta\right) \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}$

$-7 \le \left(\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta\right) \le +7$

$\therefore\; -7 + 3 \le \left(\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta\right) + 3 \le 7 + 3 \;\Rightarrow\; -4 \le \left(\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta\right) + 3 \le 10$

So, the value lies between – 4 and 10.

Example: 66 $\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\sin\frac{5\pi}{14}.\sin\frac{7\pi}{14}.\sin\frac{9\pi}{14}.\sin\frac{11\pi}{14}.\sin\frac{13\pi}{14}$ is equal to

(a) $\frac{1}{8}$ (b) $\frac{1}{16}$ (c) $\frac{1}{32}$ (d) $\frac{1}{64}$

Solution: (d) $\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\frac{\sin 5\pi}{14}.\sin\frac{7\pi}{14}.\sin\frac{9\pi}{14}.\sin\frac{11\pi}{14}.\sin\frac{13\pi}{14}$

$= \sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\sin\frac{5\pi}{14} \times 1 \times \sin\left(\pi - \frac{5\pi}{14}\right).\sin\left(\pi - \frac{3\pi}{14}\right).\sin\left(\pi - \frac{\pi}{14}\right) \;=\; \left[\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\sin\frac{5\pi}{14}.\sin\frac{7\pi}{14}\right]^2 = \frac{1}{64}$.

Example: 67 If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$ then $\tan\frac{\theta - \phi}{2}$ equal to

(a) $\sqrt{\frac{a^2 + b^2}{4 - a^2 - b^2}}$ (b) $\sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$ (c) $\sqrt{\frac{a^2 + b^2}{4 + a^2 + b^2}}$ (d) $\sqrt{\frac{4 + a^2 + b^2}{a^2 + b^2}}$

Soluton: (b) Given that, $\sin\theta + \sin\phi = a$ ....(i) and $\cos\theta + \cos\phi = b$ ......(ii)

Squaring, $\sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi = a^2$ and $\cos^2\theta + \cos^2\phi + 2\cos\theta\cos\phi = b^2$

Adding, $2 + 2(\sin\theta\sin\phi + \cos\theta.\cos\phi) = a^2 + b^2$

$\Rightarrow 2\cos(\theta - \phi) = a^2 + b^2 - 2 \;\Rightarrow\; \cos(\theta - \phi) = \frac{a^2 + b^2 - 2}{2} \;\Rightarrow\; \frac{1 - \tan^2\frac{(\theta - \phi)}{2}}{1 + \tan^2\frac{(\theta - \phi)}{2}} = \frac{a^2 + b^2 - 2}{2}$

$\Rightarrow (a^2 + b^2) + (a^2 + b^2)\tan^2\frac{\theta - \phi}{2} - 2 - 2\tan^2\frac{\theta - \phi}{2} = 2 - 2\tan^2\frac{\theta - \phi}{2}$

$\Rightarrow \frac{4 - a^2 - b^2}{a^2 + b^2} = \tan^2\frac{(\theta - \phi)}{2} \Rightarrow \tan\frac{(\theta - \phi)}{2} = \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$

Trick : Put $\theta = \frac{\pi}{2}, \phi = 0^o$, then $a = 1 = b$

$\therefore \tan\frac{\theta-\phi}{2} = 1$, which is given by (a) and (b).

Again putting $\theta = \frac{\pi}{4} = \phi$, we get $\tan\frac{\theta-\phi}{2} = 0$, which is given by (b).

Example: 68 The maximum value of $3\cos\theta + 4\sin\theta$ equal to **[MP PET 2002; UPSEAT 1990]**

(a) 3 (b) 4 (c) 5 (d) None of these

Solution: (c) Maximum value of $3\cos\theta + 4\sin\theta$ is $\sqrt{3^2+4^2} = 5$.

1.15 Conditional Trigonometrical Identitites

We have certain trigonometric identities. Like, $\sin^2\theta + \cos^2\theta = 1$ and $1 + \tan^2\theta = \sec^2\theta$ etc.

Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

If *A, B, C* denote the anlges of a triangle *ABC*, then the relation $A + B + C = \pi$ enables us to establish many important identities involving trigonometric ratios of these angles.

(1) If $A + B + C = \pi$, then $A + B = \pi - C$, $B + C = \pi - A$ and $C + A = \pi - B$.

(2) If $A + B + C = \pi$, then $\sin(A+B) = \sin(\pi - C) = \sin C$

Similarly, $\sin(B+C) = \sin(\pi - A) = \sin A$ and $\sin(C+A) = \sin(\pi - B) = \sin B$

(3) If $A + B + C = \pi$, then $\cos(A+B) = \cos(\pi - C) = -\cos C$

Similarly, $\cos(B+C) = \cos(\pi - A) = -\cos A$ and $\cos(C+A) = \cos(\pi - B) = -\cos B$

(4) If $A + B + C = \pi$, then $\tan(A+B) = \tan(\pi - C) = -\tan C$

Similarly, $\tan(B+C) = \tan(\pi - A) = -\tan A$ and $\tan(C+A) = \tan(\pi - B) = -\tan B$

(5) If $A+B+C = \pi$, then $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$ and $\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$ and $\frac{C+A}{2} = \frac{\pi}{2} - \frac{B}{2}$

$$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\left(\frac{C}{2}\right), \quad \cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\left(\frac{C}{2}\right),$$

$$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\left(\frac{C}{2}\right)$$

All problems on conditional identities are broadly divided into the following three types

1. **Identities involving sine and cosine of the multiple or sub-multiple of the angles involved**

Working Method

Step (i) **:** Use $C \pm D$ formulae.

Step (ii) **:** Use the given relation ($A + B + C = \pi$) in the expression obtained in step-(i) such that a factor can be taken common after using multiple angles formulae in the remaining term.

Step (iii) **:** Take the common factor outside.

Step (iv) **:** Again use the given relation ($A + B + C = \pi$) within the bracket in such a manner so that we can apply $C \pm D$ formulae.

Step (v) **:** Find the result according to the given options.

2. **Identities involving squares of sine and cosine of multiple or sub-multiples of the angles involved**

Working Method

Step (i) **:** Arrange the terms of the identity such that either $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B)$ or $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$ can be used.

Step (ii) **:** Take the common factor outside.

Step (iii) **:** Use the given relation $(A+B+C=\pi)$ within the bracket in such a manner so that we can apply $C \pm D$ formulae.

Step (iv) **:** Find the result according to the given options.

3. **Identities for tangent and cotangent of the angles**

Working Method

Step (i) **:** Express the sum of the two angles in terms of third angle by using the given relation $(A+B+C=\pi)$.

Step (ii) **:** Taking tangent or cotangent of the angles of both the sides.

Step (iii) **:** Use sum and difference formulae in the left hand side.

Step (iv) **:** Use cross multiplication in the expression obtained in the step (iii).

Step (v) **:** Arrange the terms as per the result required.

Example: 69 If $A+B+C=\pi$, then $\cos^2 A + \cos^2 B - \cos^2 C$ equal to

(a) $1-2\sin A\sin B\cos C$ (b) $1-2\cos A\cos B\sin C$

(c) $1+2\sin A\sin B\cos C$ (d) $1+2\cos A\cos B\sin C$

Solution: (a) $\cos^2 A+\cos^2 B-\cos^2 C=\cos^2 A+(1-\sin^2 B)-\cos^2 C$

$=1+[\cos^2 A-\sin^2 B]-\cos^2 C=1+\cos(A+B)\cos(A-B)-\cos^2 C$

$=1+\cos(\pi-C)\cos(A-B)-\cos^2 C$ $=1-\cos C[\cos(A-B)+\cos C]$

$=1-\cos C[\cos(A-B)+\cos\{\pi-(A+B)\}]$ $=1-\cos C[\cos(A-B)-\cos(A+B)]$

$=1-\cos C[2\sin A\sin B]=1-2\sin A\sin B\cos C$.

Example: 70 $\frac{\sin 2A+\sin 2B-\sin 2C}{\sin A+\sin B-\sin C}$ equal to

(a) $\frac{\cos A\cos B\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}$ (b) $\frac{\sin A\sin B\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$ (c) $-\frac{\cos A\cos B\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}$ (d) $-\frac{\sin A\sin B\cos C}{\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$

Solution: (a) $\frac{(\sin 2A+\sin 2B)-\sin 2C}{(\sin A+\sin B)-\sin C}=\frac{2\sin(A+B)\cos(A-B)-\sin 2C}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)-\sin C}=\frac{2\sin C\cos(A-B)-2\sin C\cos C}{2\sin\left(\frac{\pi-C}{2}\right)\cos\left(\frac{A-B}{2}\right)-2\sin\frac{C}{2}\cos\frac{C}{2}}$

$$=\frac{2\sin C[\cos(A-B)-\cos C]}{2\cos\frac{C}{2}\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}\right)\right]}\quad\left[\begin{array}{c}\because\ \sin C=2\sin\frac{C}{2}\cos\frac{C}{2}\\ \sin C/2=\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)=\cos\frac{(A+B)}{2}\end{array}\right]$$

$$=\frac{2\sin C[\cos(A-B)+\cos(A+B)]}{2\cos\frac{C}{2}\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}\right)\right]}=\frac{2\sin C[2\cos A\cos B]}{2\cos\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]}=\frac{\cos A\cos B\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}.$$

Trick : $\because$ $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$

and $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\Rightarrow\frac{\sin 2A+\sin 2B-\sin 2C}{\sin A+\sin B-\sin C}=\frac{\cos A\cos B\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}$.

Example: 71 If $\alpha+\beta+\gamma=2\pi$, then **[IIT 1979]**

(a) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$ (b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

(c) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$ (d) None of these

Solution: (a) We have $\alpha+\beta+\gamma=2\pi\Rightarrow\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\pi\ \Rightarrow\ \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)=\tan\pi=0$

$\Rightarrow\ \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}=0\ \Rightarrow\ \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}$

Example: 72 If $A+B+C=\pi$, then $\cos 2A+\cos 2B+\cos 2C$ equal to **[EAMCET 1982]**

(a) $1+4\cos A\cos B\sin C$ (b) $-1+4\sin A\sin B\cos C$ (c) $-1-4\cos A\cos B\cos C$ (d) None of these

Solution: (c) $\cos 2A+\cos 2B+\cos 2C=2\cos(A+B).\cos(A-B)+(2\cos^2 C-1)=-1-2\cos C.\cos(A-B)+2\cos^2 C$

$=-1-2\cos c[\cos(A-B)+\cos(A+B)]=-1-4\cos A.\cos B.\cos C$

Example: 73 If $A+B+C=180^o$, then $\frac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C-1}$ equal to

(a) $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ (b) $8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (c) $8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (d) $8\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solution: (b) $\frac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C-1}=\frac{2\sin(A+B).\cos(A-B)+2\sin C\cos C}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}-1}=\frac{2\sin C\cos(A-B)+2\cos C\sin C}{2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}}$

$$=\frac{2\sin C[\cos(A-B)+\cos(A+B)]}{2\sin\frac{C}{2}\left[\cos\frac{(A-B)}{2}-\cos\frac{(A+B)}{2}\right]}=\frac{4\sin A\sin B\sin C}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{4\times 2\sin\frac{A}{2}\times\cos\frac{A}{2}\times 2\sin\frac{B}{2}\cos\frac{B}{2}\times 2\sin\frac{C}{2}\cos\frac{C}{2}}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = 8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Example: 74 If $A+B+C=180^o$, then the value of $(\cot B+\cot C)(\cot C+\cot A)(\cot A+\cot B)$ will be

(a) $\sec A\sec B\sec C$ (b) $\text{cosec } A \text{ cosec } B \text{ cosec } C$ (c) $\tan A\tan B\tan C$ (d) 1

Solutio: (b) $\cot B+\cot C=\frac{\sin C\cos B+\sin B\cos C}{\sin B.\sin C}$ = $\frac{\sin(B+C)}{\sin B.\sin C}=\frac{\sin(180^o-A)}{\sin B.\sin C}=\frac{\sin A}{\sin B.\sin C}$

Similarly, $\cot C+\cot A=\frac{\sin B}{\sin C.\sin A}$ and $\cot A+\cot B=\frac{\sin C}{\sin A\sin B}$

Therefore, $(\cot B+\cot C)(\cot C+\cot A)(\cot A+\cot B)$

$$= \frac{\sin A}{\sin B.\sin C}.\frac{\sin B}{\sin C.\sin A}.\frac{\sin C}{\sin A\sin B}=\text{cosec}A.\text{cosec}B.\text{cosec}C.$$

Example: 75 If $A+B+C=180^o$, then the value of $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$ will be

(a) $2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ (b) $4\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ (c) $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ (d) $8\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

Solution: (c) $A+B+C=180^o$ $\therefore \frac{A}{2}+\frac{B}{2}=90^o-\frac{C}{2}$

$$\therefore \cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(90^o-\frac{C}{2}\right) \text{ or } \frac{\cot\frac{A}{2}.\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\tan\frac{C}{2}=\frac{1}{\cot\frac{C}{2}}$$

or $\left(\cot\frac{A}{2}.\cot\frac{B}{2}-1\right)\cot\frac{C}{2}=\cot\frac{B}{2}+\cot\frac{A}{2}$; $\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}=\cot\frac{C}{2}+\cot\frac{B}{2}+\cot\frac{A}{2}$

Example: 76 If A,B,C are angles of a triangle, then $\sin 2A+\sin 2B-\sin 2C$ is equal to **[MP PET 2003]**

(a) $4\sin A\cos B\cos C$ (b) $4\cos A$ (c) $4\sin A\cos A$ (d) $4\cos A\cos B\sin C$

Solution: (d) $\sin 2A+\sin 2B-\sin 2C=2\sin A\cos A+2\cos(B+C)\sin(B-C)$

$[\because A+B+C=\pi, B+C=\pi-A, \cos(B+C)=\cos(\pi-A), \cos(B+C)=-\cos A, \sin(B+C)=\sin A]$

$= 2\cos A[\sin A-\sin(B-C)]$ = $2\cos A[\sin(B+C)-\sin(B-C)]$ = $2\cos A.2\cos B.\sin C=4\cos A.\cos B.\sin C$

Trick: First put $A=B=C=60^o$, for these values. Options (a) and (b) satisfies the condition.

Now put $A=B=45^o$ and $C=90^o$, then only (d) satisfies.

Hence (d) is the answer.

Example: 77 In any triangle ABC $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$ is equal to **[MP PET 2003]**

(a) $1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (b) $1-2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(c) $1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ (d) $1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

Solution: (c) **Trick:** For $A=B=C=60^o$ only option (c) satisfies the condition.

Important Tips

☞ ***Method of componendo and dividendo***

If $\frac{p}{q}=\frac{a}{b}$*, then by componendo and dividendo*

We can write $\frac{p+q}{p-q}=\frac{a+b}{a-b}$ *or* $\frac{q+p}{q-p}=\frac{b+a}{b-a}$ *or* $\frac{p-q}{p+q}=\frac{a-b}{a+b}$ *or* $\frac{q-p}{q+p}=\frac{b-a}{b+a}$.

Example: 78 If $\tan\beta=\cos\theta.\tan\alpha$ then $\tan^2\frac{\theta}{2}$ equal to

(a) $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$ (b) $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$ (c) $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$ (d) $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}$

Solution: (a) The given relation is $\frac{\tan\alpha}{\tan\beta}=\frac{1}{\cos\theta}$

Applying componendo and dividendo rule, then

$$\Rightarrow \frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}=\frac{1-\cos\theta}{1+\cos\theta} \Rightarrow \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} \Rightarrow \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}=\tan^2\frac{\theta}{2}.$$

Example: 79 If $m\cos(\theta+\alpha)=n\cos(\theta-\alpha)$, then $\cot\theta\cot\alpha$ equal to

(a) $\frac{m+n}{m-n}$ (b) $\frac{m-n}{m+n}$ (c) $\frac{m+n}{n-m}$ (d) $\frac{n-m}{n+m}$

Solution: (c) $\frac{m}{n}=\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}$

By componendo and dividendo rule, $\frac{m+n}{m-n}=\frac{\cos(\theta-\alpha)+\cos(\theta+\alpha)}{\cos(\theta-\alpha)-\cos(\theta+\alpha)} \Rightarrow \frac{m+n}{m-n}=\frac{2\cos\theta\cos\alpha}{-2\sin\theta\sin\alpha}$

$\cot\theta\cos\alpha=\frac{m+n}{n-m}$.

Example: 80 If $\text{cosec}\,\theta=\frac{p+q}{p-q}$, then $\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=$ **[EAMCET 2001]**

(a) $\sqrt{\frac{p}{q}}$ (b) $\sqrt{\frac{q}{p}}$ (c) $\sqrt{pq}$ (d) pq

Solution: (b) Given, $\text{cosec}\,\theta=\frac{p+q}{p-q} \Rightarrow \frac{1}{\sin\theta}=\frac{p+q}{p-q}$,

Apply componendo and dividendo, $\frac{1+\sin\theta}{1-\sin\theta}=\frac{p+q+p-q}{p+q-p-q} \Rightarrow \left[\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right]^2=\frac{p}{q}$

$$\Rightarrow \left[\frac{1+\tan\theta/2}{1-\tan\theta/2}\right]^2=\frac{p}{q} \Rightarrow \tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{p}{q} \Rightarrow \cot^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{q}{p}$$

Note : $\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\sqrt{\frac{q}{p}}$ only if $\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)>0$.

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Assignment

Measurement of Angles, Trigonometrical Ratios , Function and

Basic Level

1. Which of the following relation is correct **[WB JEE 1991]**

(a) $\sin 1^o > \sin 1$ (b) $\sin 1 > \sin 1^o$ (c) $\sin 1 = \sin 1^o$ (d) $\frac{\pi}{180}\sin 1 = \sin 1^o$

2. The radius of the circle whose arc of length 15 *cm* makes an angle of 3/4 radian at the centre is

(a) 10 *cm* (b) 20 *cm* (c) $11\frac{1}{4}cm$ (d) $22\frac{1}{2}cm$

3. If $\tan\theta = \frac{-4}{3}$, then $\sin\theta =$ **[Orissa JEE 2002; IIT 1979]**

(a) $\frac{-4}{5}$ but not $\frac{4}{5}$ (b) $-\frac{4}{5}$ or $\frac{4}{5}$ (c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) None of these

4. If $f(x) = \cos^2 x + \sec^2 x$, then

(a) $f(x) < 1$ (b) $f(x) = 1$ (c) $1 < f(x) < 2$ (d) $f(x) \geq 2$

5. If $x = \sec\theta + \tan\theta$, then $x + \frac{1}{x} =$

(a) 1 (b) $2\sec\theta$ (c) 2 (d) $2\tan\theta$

6. If A lies in the second quadrant and 3 $\tan A + 4 = 0$ then the value of $2\cot A - 5\cos A + \sin A$ is equal to **[Harayana CEE 199**

(a) $\frac{-53}{10}$ (b) $\frac{-7}{10}$ (c) $\frac{7}{10}$ (d) $\frac{23}{10}$

7. $\tan 1^o \tan 2^o \tan 3^o \tan 4^o .... \tan 89^o =$ **[MP PET 1998, 2001]**

(a) 1 (b) 0 (c) ∞ (d) 1/2

8. The incorrect statement is

(a) $\sin\theta = -\frac{1}{5}$ (b) $\cos\theta = 1$ (c) $\sec\theta = \frac{1}{2}$ (d) $\tan\theta = 20$

9. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, then $\cos\theta + \sin\theta$ is equal to **[WB JEE 1988]**

(a) $\sqrt{2}\cos\theta$ (b) $\sqrt{2}\sin\theta$ (c) $2\cos\theta$ (d) $-\sqrt{2}\cos\theta$

10. If $\sec\theta + \tan\theta = p$, then $\tan\theta$ is equal to **[MP PET 1994]**

(a) $\frac{2p}{p^2 - 1}$ (b) $\frac{p^2 - 1}{2p}$ (c) $\frac{p^2 + 1}{2p}$ (d) $\frac{2p}{p^2 + 1}$

11. If $\sin\theta - \cos\theta = 1$, then $\sin\theta\cos\theta =$ **[Karnataka CET 1998]**

(a) 0 (b) 1 (c) 2 (d) 1/2

12. The value of $\cos 1^o \cos 2^o \cos 3^o ..... \cos 179^o$ is **[Karnataka CET 1999]**

(a) $\frac{1}{\sqrt{2}}$ (b) 0 (c) 1 (d) None of these

13. If $\tan\theta = +\frac{1}{\sqrt{5}}$ and θ lies in the 1st quadrant, then $\cos\theta$ is

(a) $\frac{1}{\sqrt{6}}$ (b) $-\frac{1}{\sqrt{6}}$ (c) $\frac{\sqrt{5}}{\sqrt{6}}$ (d) $-\frac{\sqrt{5}}{\sqrt{6}}$

14. If A lies in the third quadrant and $3\tan A - 4 = 0$, then $5\sin 2A + 3\sin A + 4\cos A =$ **[EAMCET 1994]**

(a) 0 (b) $\frac{-24}{5}$ (c) $\frac{24}{5}$ (d) $\frac{48}{5}$

15. $(\sec^2\theta - 1)(\text{cosec}^2\theta - 1) =$ **[Karnataka CET 1998]**

(a) 0 (b) 1 (c) $\sec\theta.\ \text{cosec}\,\theta$ (d) $\sin^2\theta - \cos^2\theta$

16. If $\tan\theta = \frac{20}{21}, \cos\theta$ will be **[MP PET 1994]**

(a) $\pm\frac{20}{41}$ (b) $\pm\frac{1}{21}$ (c) $\pm\frac{21}{29}$ (d) $\pm\frac{20}{21}$

17. If $\text{cosec}\,A + \cot A = \frac{11}{2}$, then $\tan A$ equal to **[Roorkee 1995]**

(a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{44}{117}$ (d) $\frac{117}{43}$

18. If $\sin\theta = \frac{24}{25}$ and θ lies in the second quadrant, then $\sec\theta + \tan\theta$ equal to **[MP PET 1997]**

(a) – 3 (b) – 5 (c) –7 (d) – 9

19. If $5\tan\theta = 4$, then $\frac{5\sin\theta - 3\cos\theta}{5\sin\theta + 2\cos\theta}$ equal to **[Karnataka CET 1998]**

(a) 0 (b) 1 (c) 1/6 (d) 6

20. $\frac{1+\cos\theta}{\sin^2\theta}$ equal to **[Karnataka CET 1998]**

(a) 0 (b) 1 (c) $\frac{1}{1-\cos\theta}$ (d) $\frac{1}{1+\cos\theta}$

21. The expression $\frac{1}{\tan A + \cot A}$ simplifies to **[SCRA 1999]**

(a) $\sec A \,\text{cosec}\, A$ (b) $\sin A \cos A$ (c) $\tan 2A$ (d) $\sin 2A$

22. If for real values of x, $\cos\theta = x + \frac{1}{x}$, then **[MP PET 1996]**

(a) θ is an acute angle (b) θ is a right angle (c) θ is an obtuse angle (d) No value of θ is possible

23. If $\sin x + \text{cosec}\, x = 2$, then $\sin^n x + \text{cosec}^n x$ is equal to **[UPSEAT 2002]**

(a) 2 (b) 2^n (c) 2^{n-1} (d) 2^{n-2}

Advance Level

24. One root of the equation $\cos x - x + \frac{1}{2} = 0$ lies in the interval

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{2}, 0\right]$ (c) $\left[\frac{\pi}{2}, \pi\right]$ (d) $\left[\pi, \frac{3\pi}{2}\right]$

25. If $\frac{2\sin\alpha}{\{1+\cos\alpha+\sin\alpha\}} = y$, then $\frac{\{1-\cos\alpha+\sin\alpha\}}{1+\sin\alpha} =$ **[BIT Ranchi 1996]**

(a) $\frac{1}{y}$ (b) y (c) $1 - y$ (d) $1 + y$

26. If $\sin\theta+\sin^2\theta+\sin^3\theta=1$, then $\cos^6\theta-4\cos^4\theta+8\cos^2\theta=$

(a) 4 (b) 2 (c) 1 (d) None of these

27. If θ and ϕ are angles in the 1st quadrant such that $\tan\theta=1/7$ and $\sin\phi=1/\sqrt{10}$. Then

(a) $\theta+2\phi=90^o$ (b) $\theta+2\phi=60^o$ (c) $\theta+2\phi=30^o$ (d) $\theta+2\phi=45^o$

28. The value of θ lying between 0 and $\pi/2$ and satisfying the equation $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix}=0$

[IIT 1988; MNR 1992; Kurukshetra CEE 1998; DCE 1996]

(a) $\frac{7\pi}{24}$ or $\frac{11\pi}{24}$ (b) $\frac{5\pi}{24}$ (c) $\frac{\pi}{24}$ (d) None of these

29. If $\frac{3\pi}{4}<\alpha<\pi$, then $\sqrt{\text{cosec}^2\alpha+2\cot\alpha}$ is equal to **[Pb. CET 2000, AMU 2001]**

(a) $1+\cot\alpha$ (b) $1-\cot\alpha$ (c) $-1-\cot\alpha$ (d) $-1+\cot\alpha$

30. If for all real values of x, $\frac{4x^2+1}{64x^2-96x.\sin\alpha+5}<\frac{1}{32}$, then α lies in the interval **[Roorkee 1998]**

(a) $\left(0,\frac{\pi}{3}\right)$ (b) $\left(\frac{\pi}{3},\frac{2\pi}{3}\right)$ (c) $\left(\frac{2\pi}{3},\pi\right)$ (d) $\left(\frac{4\pi}{3},\frac{5\pi}{3}\right)$

31. If $\tan\theta=\sqrt{\frac{3}{2}}$, then the sum of the infinite series $1+2(1-\cos\theta)+3(1-\cos\theta)^2+4(1-\cos\theta)^3+....\infty$ is

(a) $\frac{2}{3}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{5}{2\sqrt{2}}$ (d) $\frac{5}{2}$

32. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

(a) $\frac{3}{4}$ (b) $3\sqrt{3}$ (c) 3 (d) $\frac{3\sqrt{3}}{2}$

Trigonometrical Ratios of Allied Angles

Basic Level

33. If $x\sin 45^o\cos^2 60^o=\frac{\tan^2 60^o\,\text{cosec}\,30^o}{\sec 45^o\cot^2 30^o}$, then $x=$ **[Kerala (Engg.) 2002]**

(a) 2 (b) 4 (c) 8 (d) 16

34. $\cos A+\sin(270^o+A)-\sin(270^o-A)+\cos(180^o+A)=$ **[MP PET 1990]**

(a) –1 (b) 0 (c) 1 (d) None of these

35. $\sin(\pi+\theta)\sin(\pi-\theta)\text{cosec}^2\theta=$ **[EAMCET 1980]**

(a) 1 (b) –1 (c) $\sin\theta$ (d) $-\sin\theta$

36. The value of $\sin 600^o\cos 330^o+\cos 120^o\sin 150^o$ is **[MP PET 1994]**

(a) –1 (b) 1 (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

37. If $A = 130^o$ and $x = \sin A + \cos A$, then **[Karnataka CET 1989]**

(a) $x > 0$ (b) $x < 0$ (c) $x = 0$ (d) $x \le 0$

38. $\tan\theta \sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}-\theta\right) =$ **[EAMCET 1981]**

(a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) None of these

39. $\sin^2 5^o + \sin^2 10^o + \sin^2 15^o + ... + \sin^2 85^o + \sin^2 90^o =$ **[Karnataka CET 1999, 1995]**

(a) 7 (b) 8 (c) 9 (d) $9\frac{1}{2}$

40. Values of $\theta(0 < \theta < 360^o)$ satisfying $\text{cosec}\,\theta + 2 = 0$ are **[EAMCET 1994]**

(a) $210^o, 300^o$ (b) $240^o, 300^o$ (c) $210^o, 240^o$ (d) $210^o, 330^o$

41. The value of $\tan(-945^o)$ is **[MP PET 1997]**

(a) –1 (b) – 2 (c) – 3 (d) – 4

42. The value of $\frac{\cot 54^o}{\tan 36^o} + \frac{\tan 20^o}{\cot 70^o}$ is **[Karnataka CET 1999]**

(a) 2 (b) 3 (c) 1 (d) 0

43. $\tan 9^o - \tan 27^o - \tan 63^o + \tan 81^o =$ **[Roorkee 1989]**

(a) 1/2 (b) 2 (c) 4 (d) 8

44. $\cos 1^o + \cos 2^o + \cos 3^o + ... + \cos 180^o =$ **[Karnataka CET 2003]**

(a) 0 (b) 1 (c) –1 (d) 2

45. If $\tan(A - B) = 1$, $\sec(A + B) = \frac{2}{\sqrt{3}}$, then the smallest positive value of B is **[Kerala (Engg.) 2002]**

(a) $\frac{25}{24}\pi$ (b) $\frac{19}{24}\pi$ (c) $\frac{13}{24}\pi$ (d) $\frac{11}{24}\pi$

46. If $x = \sin 130^o \cos 80^o, y = \sin 80^o \cos 130^o, z = 1 + xy$, which one of the following is true **[AMU 1999]**

(a) $x > 0, y > 0, z > 0$ (b) $x > 0, y < 0, 0 < z < 1$ (c) $x > 0, y < 0, z > 1$ (d) $x < 0, y < 0, 0 < z < 1$

47. If $\alpha = 22^o 30'$, then $(1+\cos\alpha)(1+\cos 3\alpha)(1+\cos 5\alpha)(1+\cos 7\alpha)$ equals **[AMU 1999]**

(a) 1/8 (b) 1/4 (c) $\frac{1+\sqrt{2}}{2\sqrt{2}}$ (d) $\frac{\sqrt{2}-1}{\sqrt{2}+1}$

Trigonometrical Ratios of Sum & Difference of Two Angles, Tranformation of Product into Sum & Difference. Transformation of Sum & Difference into Product

Basic Level

48. If *A, B, C, D* are the angles of a cyclic quadrilateral then $\cos A + \cos B + \cos C + \cos D =$ **[IIT 1970]**

(a) $2(\cos A + \cos C)$ (b) $2(\cos A + \cos B)$ (c) $2(\cos A + \cos D)$ (d) 0

49. $\frac{\cos 17^o + \sin 17^o}{\cos 17^o - \sin 17^o}$ **[MP PET 1998 (Similar to EAMCET 1982)]**

(a) $\tan 62^o$ (b) $\tan 56^o$ (c) $\tan 54^o$ (d) $\tan 73^o$

50. $\cot(45^o + \theta)\cot(45^o - \theta) =$ **[MNR 1973]**

(a) –1 (b) 0 (c) 1 (d) ∞

51. $\tan 75^o - \cot 75^o =$ **[MNR 1982]**

(a) $2\sqrt{3}$ (b) $2+\sqrt{3}$ (c) $2-\sqrt{3}$ (d) None of these

52. $\sqrt{3}\,\text{cosec}\, 20^o - \sec 20^o =$ **[IIT 1988]**

(a) 2 (b) $\frac{2\sin 20^o}{\sin 40^o}$ (c) 4 (d) $\frac{4\sin 20^o}{\sin 40^o}$

53. $\sin 15^o + \cos 105^o =$ **[MP PET 1992]**

(a) 0 (b) $2\sin 15^o$ (c) $\cos 15^o + \sin 15^o$ (d) $\sin 15^o - \cos 15^o$

54. If $\cos(A+B) = \alpha\cos A\cos B + \beta\sin A\sin B,$ then $(\alpha, \beta) =$ **[MP PET 1992]**

(a) $(-1,-1)$ (b) $(-1,1)$ (c) $(1,-1)$ (d) $(1,1)$

55. $\cos^2\alpha + \cos^2(\alpha + 120^o) + \cos^2(\alpha - 120^o)$ is equal to **[MP PET 1993]**

(a) $\frac{3}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) 0

56. The value of $\cos 105^o + \sin 105^o$ is **[MNR 1975]**

(a) $\frac{1}{2}$ (b) 1 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

57. $\cos^2 48^o - \sin^2 12^o =$ **[MNR 1977]**

(a) $\frac{\sqrt{5}-1}{4}$ (b) $\frac{\sqrt{5}+1}{8}$ (c) $\frac{\sqrt{3}-1}{4}$ (d) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

58. $\sin 20^o \sin 40^o \sin 60^o \sin 80^o =$ **[MNR 1976, 1981]**

(a) $-3/16$ (b) $5/16$ (c) $3/16$ (d) $-5/16$

59. $\cos 20^o \cos 40^o \cos 80^o =$ **[MP PET 1989]**

(a) 1/2 (b) 1/4 (c) 1/6 (d) 1/8

60. $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} =$ **[IIT 1985]**

(a) 1/2 (b) 1/4 (c) 1/8 (d) 1/16

61. If $x = \cos 10^o \cos 20^o \cos 40^o$, then the value of x is **[Roorkee 1995]**

(a) $\frac{1}{4}\tan 10^o$ (b) $\frac{1}{8}\cot 10^o$ (c) $\frac{1}{8}\text{cosec}\, 10^o$ (d) $\frac{1}{8}\sec 10^o$

62. The value of $\cos 52^o + \cos 68^o + \cos 172^o$ is **[MP PET 1997]**

(a) 0 (b) 1 (c) 2 (d) $\frac{3}{2}$

63. $\cos 40^o + \cos 80^o + \cos 160^o + \cos 240^o =$ **[EAMCET 1996]**

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

64. $1 + \cos 56^o + \cos 58^o - \cos 66^o =$ **[IIT 1964]**

(a) $2\cos 28^o \cos 29^o \cos 33^o$ (b) $4\cos 28^o \cos 29^o \cos 33^o$ (c) $4\cos 28^o \cos 29^o \sin 33^o$ (d) $2\cos 28^o \cos 29^o \sin 33^o$

65. $\cos 15^o =$ **[MP PET 1998; MNR 1978]**

(a) $\sqrt{\frac{1+\cos 30^o}{2}}$ (b) $\sqrt{\frac{1-\cos 30^o}{2}}$ (c) $\pm\sqrt{\frac{1+\cos 30^o}{2}}$ (d) $\pm\sqrt{\frac{1-\cos 30^o}{2}}$

66. $\tan 5x \tan 3x \tan 2x =$ **[EAMCET 1991]**

(a) $\tan 5x - \tan 3x - \tan 2x$ (b) $\frac{\sin 5x - \sin 3x - \sin 2x}{\cos 5x - \cos 3x - \cos 2x}$ (c) 0 (d) None of these

67. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then $\cos 2\alpha + \cos 2\beta =$ **[EAMCET 1994]**

(a) $-2\sin(\alpha+\beta)$ (b) $-2\cos(\alpha+\beta)$ (c) $2\sin(\alpha+\beta)$ (d) $2\cos(\alpha+\beta)$

68. If $\tan A = -\frac{1}{2}$ and $\tan B = -\frac{1}{3}$, then $A+B$ = **[IIT 1967; UPSEAT 1987; MP PET 1989]**

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) None of these

69. If $\cos(A-B) = \frac{3}{5}$ and $\tan A \tan B = 2$, then **[MP PET 1997]**

(a) $\cos A \cos B = \frac{1}{5}$ (b) $\sin A \sin B = -\frac{2}{5}$ (c) $\cos A \cos B = -\frac{1}{5}$ (d) $\sin A \sin B = -\frac{1}{5}$

70. $\frac{\sin 3\theta - \cos 3\theta}{\sin\theta + \cos\theta} + 1$ =

(a) $2\sin 2\theta$ (b) $2\cos 2\theta$ (c) $\tan 2\theta$ (d) $\cot 2\theta$

71. $\tan 3A - \tan 2A - \tan A$ = **[MNR 1982; Pb. CET 1991]**

(a) $\tan 3A \tan 2A \tan A$ (b) $-\tan 3A \tan 2A \tan A$

(c) $\tan A \tan 2A - \tan 2A \tan 3A - \tan 3A \tan A$ (d) None of these

72. If $\cos A = m\cos B$, then **[UPSEAT 1990]**

(a) $\cot\frac{A+B}{2} = \frac{m+1}{m-1}\tan\frac{B-A}{2}$ (b) $\tan\frac{A+B}{2} = \frac{m+1}{m-1}\cot\frac{B-A}{2}$

(c) $\cot\frac{A+B}{2} = \frac{m+1}{m-1}\tan\frac{A-B}{2}$ (d) None of these

73. The value of $\cos 12^o + \cos 84^o + \cos 156^o + \cos 132^o$ is **[Kerala CEE 1993]**

(a) 1/2 (b) 1 (c) –1/2 (d) 1/8

74. $\tan 100^o + \tan 125^o + \tan 100^o \tan 125^o =$ **[DCE 1999]**

(a) 0 (b) $\frac{1}{2}$ (c) –1 (d) 1

75. If $\cos P = \frac{1}{7}$ and $\cos Q = \frac{13}{14}$ where P and Q both are acute angles. Then the value of $P - Q$ is

[Orissa JEE 2002; Karnataka CET 2002]

(a) 30^o (b) 60^o (c) 45^o (d) 75^o

76. If $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$, where A and B are positive acute angles, then $A+B=$

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

77. $\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right) =$ **[MNR 1984]**

(a) 1/2 (b) –1/2 (c) 1/4 (d) 1

78. $\sin 50^o - \sin 70^o + \sin 10^o =$ **[MNR 1979]**

(a) 1 (b) 0 (c) 1/2 (d) 2

79. If $\sin A = \sin B$ and $\cos A = \cos B$, then **[EAMCET 1994]**

(a) $\sin\frac{A-B}{2} = 0$ (b) $\sin\frac{A+B}{2} = 0$ (c) $\cos\frac{A-B}{2} = 0$ (d) $\cos(A+B) = 0$

80. $\sin 12^o \sin 48^o \sin 54^o =$ **[IIT 1982]**

(a) 1/16 (b) 1/32 (c) 1/8 (d) 1/4

81. If $(1+\tan\theta)(1+\tan\phi) = 2$, then $\theta + \phi =$ **[Karnataka CET 1993]**

(a) 30^o (b) 45^o (c) 60^o (d) 75^o

82. $\cos^2\left(\frac{\pi}{6}+\theta\right) - \sin^2\left(\frac{\pi}{6}-\theta\right) =$ **[EAMCET 2001]**

(a) $\frac{1}{2}\cos 2\theta$ (b) 0 (c) $-\frac{1}{2}\cos 2\theta$ (d) $\frac{1}{2}$

83. If $\sin\theta + \sin 2\theta + \sin 3\theta = \sin\alpha$ and $\cos\theta + \cos 2\theta + \cos 3\theta = \cos\alpha,$ then θ is equal to **[AMU 2001]**

(a) $\alpha/2$ (b) α (c) 2α (d) $\alpha/6$

84. $\cos\alpha.\sin(\beta-\gamma) + \cos\beta.\sin(\gamma-\alpha) + \cos\gamma.\sin(\alpha-\beta) =$ **[EAMCET 2003]**

(a) 0 (b) 1/2 (c) 1 (d) $4\cos\alpha\cos\beta\cos\gamma$

85. Given that $\cos\left(\frac{\alpha-\beta}{2}\right) = 2\cos\left(\frac{\alpha+\beta}{2}\right)$, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

Advance Level

86. If $\sin A + \sin B = C, \cos A + \cos B = D,$ then the value of $\sin(A+B) =$

(a) CD (b) $\frac{CD}{C^2+D^2}$ (c) $\frac{C^2+D^2}{2CD}$ (d) $\frac{2CD}{C^2+D^2}$

87. If $A + B = 225^o$, then $\frac{\cot A}{1+\cot A}.\frac{\cot B}{1+\cot B} =$ **[MNR 1974]**

(a) 1 (b) –1 (c) 0 (d) $\frac{1}{2}$

88. $\frac{1}{\sin 10^o} - \frac{\sqrt{3}}{\cos 10^o}$ **[IIT 1974]**

(a) 0 (b) 1 (c) 2 (d) 4

89. $\frac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta} =$ **[Roorkee 1973]**

(a) $\tan 3\theta$ (b) $\cot 3\theta$ (c) $\tan 6\theta$ (d) $\cot 6\theta$

90. If $\cos(\theta-\alpha), \cos\theta$ and $\cos(\theta+\alpha)$ are in H.P., then $\cos\theta\sec\frac{\alpha}{2}$ is equal to **[IIT 1997]**

(a) $\pm\sqrt{2}$ (b) $\pm\sqrt{3}$ (c) $\pm 1/\sqrt{2}$ (d) None of these

91. $\frac{\sin(B+A)+\cos(B-A)}{\sin(B-A)+\cos(B+A)}=$ **[Roorkee 1970; IIT 1966]**

(a) $\frac{\cos B+\sin B}{\cos B-\sin B}$ (b) $\frac{\cos A+\sin A}{\cos A-\sin A}$ (c) $\frac{\cos A-\sin A}{\cos A+\sin A}$ (d) None of these

92. If $\sin 2x = n\sin 2y$, then the value of $\frac{\tan(x+y)}{\tan(x-y)}$ is

(a) $\frac{n+1}{n-1}$ (b) $\frac{n-1}{n+1}$ (c) $\frac{1-n}{n+1}$ (d) $\frac{1+n}{1-n}$

93. If $3\sin\alpha = 5\sin\beta$, then $\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}=$ **[EAMCET 1996]**

(a) 1 (b) 2 (c) 3 (d) 4

94. If $\frac{\pi}{2}<\alpha<\pi, \pi<\beta<\frac{3\pi}{2}$, $\sin\alpha=\frac{15}{17}$ and $\tan\beta=\frac{12}{5}$, the value of $\sin(\beta-\alpha)$ is **[Roorkee 2000]**

(a) $\frac{-171}{221}$ (b) $\frac{-21}{221}$ (c) $\frac{21}{221}$ (d) $\frac{17}{221}$

95. $\cos^2 76^o+\cos^2 16^o-\cos 76^o\cos 16^o=$ **[EAMCET 2002]**

(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 0 (d) $\frac{3}{4}$

96. The value of $\cos^2\frac{\pi}{12}+\cos^2\frac{\pi}{4}+\cos^2\frac{5\pi}{12}$ is **[Karnataka CET 2002]**

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3+\sqrt{3}}{2}$ (d) $\frac{2}{3+\sqrt{3}}$

97. If angle θ be divided into two parts such that the tangents of one part is K times the tangent of the other and ϕ is their difference, then $\sin\theta=$

(a) $\frac{K+1}{K-1}\sin\phi$ (b) $\frac{K-1}{K+1}\sin\phi$ (c) $\frac{2K-1}{2K+1}\sin\phi$ (d) None of these

98. If $\tan\alpha, \tan\beta$ are the roots of the equation $x^2+px+q=0$ $(p\neq 0)$, then

(a) $\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)=q$ (b) $\tan(\alpha+\beta)=\frac{p}{q-1}$

(c) $\cos(\alpha+\beta)=1-q$ (d) $\sin(\alpha+\beta)=-p$

99. If $\tan\alpha$ equals the integral solution of the inequality $4x^2-16x+15<0$ and $\cos\beta$ equals to the slope of the bisector of first quadrant, then $\sin(\alpha+\beta)\sin(\alpha-\beta)$ is equal to

(a) $\frac{3}{5}$ (b) $\frac{-3}{5}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{4}{5}$

100. $\frac{\sqrt{2}-\sin\alpha-\cos\alpha}{\sin\alpha-\cos\alpha}=$ **[AMU 1999]**

(a) $\sec\left(\frac{\alpha}{2}-\frac{\pi}{8}\right)$ (b) $\cos\left(\frac{\pi}{8}-\frac{\alpha}{2}\right)$ (c) $\tan\left(\frac{\alpha}{2}-\frac{\pi}{8}\right)$ (d) $\cot\left(\frac{\alpha}{2}-\frac{\pi}{2}\right)$

101. The sum $S=\sin\theta+\sin 2\theta+.......+\sin n\theta$, equals **[AMU 2002]**

(a) $\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta/\sin\frac{\theta}{2}$

(b) $\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta/\sin\frac{\theta}{2}$

(c) $\sin\frac{1}{2}(n+1)\theta\cos\frac{1}{2}n\theta/\sin\frac{\theta}{2}$

(d) $\cos\frac{1}{2}(n+1)\theta\cos\frac{1}{2}n\theta/\sin\frac{\theta}{2}$

Trigonometrical Ratios of Multiple and Sub-multiple of an Angle

Basic Level

102. $2\cos^2\theta-2\sin^2\theta=1$, then $\theta=$ **[Karnataka CET 1998]**

(a) 15^o (b) 30^o (c) 45^o (d) 60^o

103. If $\cos A=\frac{3}{4}$, then $32\sin\frac{A}{2}\cos\frac{5}{2}A=$ **[EAMCET 1982]**

(a) $\sqrt{7}$ (b) $-\sqrt{7}$ (c) 7 (d) –7

104. $\cot x-\tan x=$ **[MP PET 1986]**

(a) $\cot 2x$ (b) $2\cot^2 x$ (c) $2\cot 2x$ (d) $\cot^2 2x$

105. $\cos^2 A(3-4\cos^2 A)^2+\sin^2 A(3-4\sin^2 A)^2=$

(a) $\cos 4A$ (b) $\sin 4A$ (c) 1 (d) None of these

106. $2\sin^2\beta+4\cos(\alpha+\beta)\sin\alpha\sin\beta+\cos 2(\alpha+\beta)=$ **[UPSEAT 1993]**

(a) $\sin 2\alpha$ (b) $\cos 2\beta$ (c) $\cos 2\alpha$ (d) $\sin 2\beta$

107. If $\tan A=\frac{1-\cos B}{\sin B}$, then the value of $\tan 2A$ in terms of $\tan B$

(a) $\tan 2A=\tan B$ (b) $\tan 2A=\tan^2 B$ (c) $\tan 2A=\tan^2 B+2\tan B$ (d) None of these

108. If $\tan A=\frac{1}{2}$, $\tan B=\frac{1}{3}$, then $\cos 2A=$ **[Karnataka CET 1986, 89]**

(a) $\sin B$ (b) $\sin 2B$ (c) $\sin 3B$ (d) None of these

109. If $a\cos 2\theta+b\sin 2\theta=c$ has α and β as its solution, then the value of $\tan\alpha+\tan\beta$ is **[Haryana CEE 1998]**

(a) $\frac{c+a}{2b}$ (b) $\frac{2b}{c+a}$ (c) $\frac{c-a}{2b}$ (d) $\frac{b}{c+a}$

110. $\frac{3\cos\theta+\cos 3\theta}{3\sin\theta-\sin 3\theta}$ is equal to **[EAMCET 1996]**

(a) $1+\cot^2\theta$ (b) $\cot^4\theta$ (c) $\cot^3\theta$ (d) $2\cot\theta$

111. $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}=$ **[Karnataka CET 1998]**

(a) 1 (b) –1 (c) 0 (d) 2

112. If $k = \sin\frac{\pi}{18}.\sin\frac{5\pi}{18}.\sin\frac{7\pi}{18}$, then the numerical value of k is **[IIT 1993; UPSEAT 1974]**

(a) 1/4 (b) 1/8 (c) 1/16 (d) None of these

113. $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} =$ **[MP PET 1998]**

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{8}$

114. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$, then the value of $\cos 3\theta$ is **[MP PET 2001]**

(a) $\frac{1}{8}\left(a^3+\frac{1}{a^3}\right)$ (b) $\frac{3}{2}\left(a+\frac{1}{a}\right)$ (c) $\frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$ (d) $\frac{1}{3}\left(a^3+\frac{1}{a^3}\right)$

115. $2\sin A\cos^3 A - 2\sin^3 A.\cos A =$ **[Roorkee 1975, Kerala (Engg.) 2002]**

(a) $\sin 4A$ (b) $\frac{1}{2}\sin 4A$ (c) $\frac{1}{4}\sin 4A$ (d) None of these

116. If $\cos A = \frac{3}{4}$, then 32 $\sin\left(\frac{A}{2}\right)\sin\left(\frac{5A}{2}\right) =$ **[DCE 1996]**

(a) 7 (b) 8 (c) 11 (d) None of these

117. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0, \pi/2 < \alpha < \pi,$ then $\sin 2\alpha$ is equal to **[UPSEAT 2001]**

(a) 24/25 (b) –24/25 (c) 13/18 (d) –13/18

Advance Level

118. $\tan 20^o \tan 40^o \tan 60^o \tan 80^o =$ **[IIT 1974]**

(a) 1 (b) 2 (c) 3 (d) $\sqrt{3}/2$

119. If $\cos\theta = \frac{3}{5}$ and $\cos\phi = \frac{4}{5}$, where θ and ϕ are positive acute angles, then $\cos\frac{\theta-\phi}{2} =$ **[MP PET 1988]**

(a) $\frac{7}{\sqrt{2}}$ (b) $\frac{7}{5\sqrt{2}}$ (c) $\frac{7}{\sqrt{5}}$ (d) $\frac{7}{2\sqrt{5}}$

120. If $\cos(\alpha+\beta) = \frac{4}{5}, \sin(\alpha-\beta) = \frac{5}{13}$ and α, β lie between 0 and $\frac{\pi}{4}$, then $\tan 2\alpha =$ **[IIT 1979; EAMCET 2002]**

(a) $\frac{16}{63}$ (b) $\frac{56}{33}$ (c) $\frac{28}{33}$ (d) None of these

121. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, then the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is equal to **[IIT 1984]**

(a) 1 (b) 2 (c) 0 (d) $3\cos\theta$

122. If $a\tan\theta = b$, then $a\cos 2\theta + b\sin 2\theta =$ **[EAMCET 1981, 82; MP PET 1996]**

(a) a (b) b (c) $-a$ (d) $-b$

123. If $2\sec 2\alpha = \tan\beta + \cot\beta$, then one of the values of $\alpha + \beta$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

124. If $\cos x + \cos y + \cos\alpha = 0$ and $\sin x + \sin y + \sin\alpha = 0$, then $\cot\left(\frac{x+y}{2}\right) =$ **[Karnataka CET 2001]**

(a) $\sin\alpha$ (b) $\cos\alpha$ (c) $\cot\alpha$ (d) $\sin\left(\frac{x+y}{2}\right)$

125. If $\sin 2\theta + \sin 2\phi = \frac{1}{2}$ and $\cos 2\theta + \cos 2\phi = \frac{3}{2}$, then $\cos^2(\theta - \phi) =$ **[MP PET 2000]**

(a) $\frac{3}{8}$ (b) $\frac{5}{8}$ (c) $\frac{3}{4}$ (d) $\frac{5}{4}$

126. If $(\sec\alpha + \tan\alpha)(\sec\beta + \tan\beta)(\sec\gamma + \tan\gamma) = \tan\alpha\ \tan\beta\ \tan\gamma$, then $(\sec\alpha - \tan\alpha)(\sec\beta - \tan\beta)(\sec\gamma - \tan\gamma) =$

[Haryana CEE 1998]

(a) $\cot\alpha\ \cot\beta\ \cot\gamma$ (b) $\tan\alpha\ \tan\beta\ \tan\gamma$ (c) $\cot\alpha + \cot\beta + \cot\gamma$ (d) $\tan\alpha + \tan\beta + \tan\gamma$

127. If $\cos 2B = \frac{\cos(A+C)}{\cos(A-C)}$, then $\tan A, \tan B, \tan C$ are in

(a) A.P (b) G.P (c) H.P (d) None of these

128. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) =$ **[IIT 1994; WB JEE 1992]**

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

129. If $\frac{\sin^4 A}{a} + \frac{\cos^4 A}{b} = \frac{1}{a+b}$, then the value of $\frac{\sin^8 A}{a^3} + \frac{\cos^8 A}{b^3}$ is equal to **[WB JEE 1971]**

(a) $\frac{1}{(a+b)^3}$ (b) $\frac{a^3 b^3}{(a+b)^3}$ (c) $\frac{a^2 b^2}{(a+b)^2}$ (d) None of these

130. $\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$ is equal to **[IIT 1966, 1975]**

(a) $\cot 7\frac{1}{2}^o$ (b) $\sin 7\frac{1}{2}^o$ (c) $\sin 15^o$ (d) $\cos 15^o$

131. If $\sin\beta$ is the geometric mean between $\sin\alpha$ and $\cos\alpha$, then $\cos 2\beta$ is equal to

(a) $2\sin^2\left(\frac{\pi}{4} - \alpha\right)$ (b) $2\cos^2\left(\frac{\pi}{4} - \alpha\right)$ (c) $2\cos^2\left(\frac{\pi}{4} + \alpha\right)$ (d) $2\sin^2\left(\frac{\pi}{4} + \alpha\right)$

132. The value of k, for which $(\cos x + \sin x)^2 + k\sin x\cos x - 1 = 0$ is an identity, is **[Kerala (Engg.) 2001]**

(a) –1 (b) – 2 (c) 0 (d) 1

133. If $\sin^3 x \sin 3x = \sum_{m=0}^{n} c_m \cos mx$ where $c_0, c_1, c_2, ..., c_n$ are constants and $c_n \neq 0$, then the value of n is

(a) 15 (b) 6 (c) 1 (d) 0

134. Let $0 < x < \frac{\pi}{4}$. Then $\sec 2x - \tan 2x =$ **[IIT 1994]**

(a) $\tan\left(x - \frac{\pi}{4}\right)$ (b) $\tan\left(\frac{\pi}{4} - x\right)$ (c) $\tan\left(x + \frac{\pi}{4}\right)$ (d) $\tan^2\left(x + \frac{\pi}{4}\right)$

135. If x is A.M. of $\tan\frac{\pi}{9}$ and $\tan\frac{5\pi}{18}$ and y is A.M. of $\tan\frac{\pi}{9}$ and $\tan\frac{7\pi}{18}$, then

(a) $x > y$ (b) $x = y$ (c) $2x = y$ (d) $x = 2y$

136. If $\cos^4\theta\sec^2\alpha, \frac{1}{2}$ and $\sin^4\theta\,\text{cosec}^2\alpha$ are in A.P. , then $\cos^8\theta\sec^6\alpha, \frac{1}{2}$ and $\sin^8\theta\,\text{cosec}^6\alpha$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

137. Let $f_n(\theta) = \tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta).......(1+\sec 2^n\theta)$. Then **[IIT Screening 1999; 2001]**

(a) $f_2\left(\frac{\pi}{16}\right) = 1$ (b) $f_3\left(\frac{\pi}{32}\right) = 1$ (c) $f_4\left(\frac{\pi}{64}\right) = 1$ (d) All of these

138. If A, B, C, D are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{A}{2} + 3\sin\frac{B}{2} + 2\sin\frac{C}{2} + \sin\frac{D}{2}$ is equal to

(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$ (c) $2\sqrt{k}$ (d) None of these

139. If α, β are different values of x satisfying $a\cos x + b\sin x = c$, then $\tan\left(\frac{\alpha+\beta}{2}\right) =$ **[Orissa JEE 2003; EAMCET 1986]**

(a) $a+b$ (b) $a-b$ (c) $\frac{b}{a}$ (d) $\frac{a}{b}$

Maximum and Minimum value

Basic Level

140. The maximum value of $a\cos x + b\sin x$ is **[MNR 1991; MP PET 1999]**

(a) $a+b$ (b) $a-b$ (c) $|a| + |b|$ (d) $(a^2+b^2)^{1/2}$

141. The minimum value of $\cos\theta + \sin\theta$ is **[MNR 1976]**

(a) 0 (b) $-\sqrt{2}$ (c) 1/2 (d) $\sqrt{2}$

142. The minimum value of $3\cos x + 4\sin x + 8$ is **[UPSEAT 1991]**

(a) 5 (b) 9 (c) 2 (d) 3

143. If θ is an acute angle and $\sin\theta = \frac{p-6}{8-p}$, then p must satisfy

(a) $6 \le p < 8$ (b) $6 \le p < 7$ (c) $3 \le p \le 4$ (d) $4 \le p < 7$

Advance Level

144. Maximum value of $\cos^2 x + \cos^2 y - \cos^2 z$ is

(a) 0 (b) 1 (c) 3 (d) 2

145. Let n be a positive integer such that $\sin\left(\frac{\pi}{2^n}\right) + \cos\left(\frac{\pi}{2^n}\right) = \frac{\sqrt{n}}{2}$, then

(a) $6 \le n \le 8$ (b) $4 < n \le 8$ (c) $4 \le n < 8$ (d) $4 < n < 8$

146. If $\alpha \in \left(0, \frac{\pi}{2}\right)$, then $\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}$ is always greater than or equal to **[IIT Screening 2003]**

(a) $2\tan\alpha$ (b) 1 (c) 2 (d) $\sec^2\alpha$

147. The maximum value of $(\cos\alpha_1).(\cos\alpha_2).....(\cos\alpha_n)$, under the restrictions, $0 \le \alpha_1, \alpha_2 ......\alpha_n \le \frac{\pi}{2}$ and $(\cot\alpha_1).(\cot\alpha_2)........(\cot\alpha_n) = 1$ is **[IIT Screening 2001]**

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2n}$ (d) 1

148. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then **[IIT Screening 2000]**

(a) $f(\theta) \ge 0$ only when $\theta \ge 0$ (b) $f(\theta) \le 0$ only when $\theta \le 0$

(c) $f(\theta) \ge 0$ for all real θ (d) None of these

149. The minimum value of $2^{\sin x} + 2^{\cos x}$ is

(a) 1 (b) 2 (c) $2^{-\frac{1}{\sqrt{2}}}$ (d) $2^{1-\frac{1}{\sqrt{2}}}$

Conditional Trigonometrical Identities

Basic Level

150. If $A + B + C = \pi$, then $\frac{\tan A + \tan B + \tan C}{\tan A.\tan B.\tan C} =$ **[EAMCET 1989]**

(a) 0 (b) 2 (c) 1 (d) –1

151. If $A + B + C = \pi$ and $\cos A = \cos B \cos C$, then $\tan B \tan C$ is equal to **[AMU 2001]**

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) $-\frac{1}{2}$

Advance Level

152. If $\alpha + \beta - \gamma = \pi$, then $\sin^2\alpha + \sin^2\beta - \sin^2\gamma =$ **[IIT 1980]**

(a) $2\sin\alpha\sin\beta\cos\gamma$ (b) $2\cos\alpha\cos\beta\cos\gamma$ (c) $2\sin\alpha\sin\beta\sin\gamma$ (d) None of these

153. If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C + 4\sin A \sin B \sin C =$ **[EAMCET 2003; 1989]**

(a) 0 (b) 1 (c) 2 (d) 3

154. If *A, B, C* are the angles of a triangle, then $\sin^2 A + \sin^2 B + \sin^2 C - 2\cos A \cos B \cos C$ = **[Karnataka CET 1989]**

(a) 1 (b) 2 (c) 3 (d) 4

155. If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan \alpha$ equals **[IIT Screening 2001]**

(a) $2(\tan \beta + \tan \gamma)$ (b) $\tan \beta + \tan \gamma$ (c) $\tan \beta + 2 \tan \gamma$ (d) $2 \tan \beta + \tan \gamma$

156. Let A, B and C are the angles of a plain triangle and $\tan \frac{A}{2} = \frac{1}{3}$, $\tan \frac{B}{2} = \frac{2}{3}$. Then $\tan \frac{C}{2}$ is equal to

(a) 7/9 (b) 2/9 (c) 1/3 (d) 2/3

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Answer Sheet

Trigonometrical Ratios, Functions and Assignment (Basic & Advance Level)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
b	b	b	d	b	d	a	c	a	b	a	b	c	a	b	c	c	c	c	c
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
b	d	a	a	b	a	d	a	c	b,d	d	c	c	b	b	a	a	d	d	d
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
a	a	c	c	b	b	a	d	a	c	a	c	a	c	a	d	b	c	d	d
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
b	a	d	c	a	a	b	b	a	a	a	a	c	d	b	d	c	b	a	c

81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
b	a	a	a	b	d	d	d	c	a	b	a	d	d	d	a	a	a, b	d	c
101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
a	b	b	c	c	c	a	b	b	c	d	b	d	c	b	c	b	c	b	b
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140
c	a	a	c	b	a	b	c	a	a	a, c	b	b	b	c	a	d	b	c	d
141	142	143	144	145	146	147	148	149	150	151	152	153	154	155	156				
b	d	b	d	b	a	a	c	d	c	b	a	b	b	c	a				