Chapter 5

State Space Analysis

CHAPTER HIGHLIGHTS

- *Basic Definitions*
- *State Space Representation*
- *Transfer Function to State Space Model*
- *Controllable Canonical Form*
- *Observable Canonical Form*
- *Diagonal Canonical Form*
- *Jordan Canonical Form*
- *Controllability*
- *Kalman's Test for Controllability*
- *Observability*
- *Kalman's Test for Observability*
- *Solution of State Equations*
- *Transfer Function*

Basic Definitions

System state Minimum information needed in order to completely determine the output of the system from a given moment provided the input is known from that moment.

System variable Any variable that responds to an input or initial conditions in a system

State variables The smallest set of linearly, independent system variables such that the values of the set members at time t_0 along with known forcing function completely determine the value of all system variables for all $t \geq t_0$.

State vector Vector whose elements are the state variables

State space n-dimensional space whose axes represent the state variables

State equations A set of n simultaneous, first-order differential equations with n variables, where n variables to be solved are the state variables.

State space representation

A state space representation of an LTI system has the general for x.

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) - Du(t)$$
$$x(t_0) = X_0 \rightarrow \text{initial conditions}$$

Where

$x(t)$	:	State vector (n dimensional)
$y(t)$	:	Output vector (P dimensional)
$u(t)$	:	Input or control vector (m dimensional)
A	:	Dynamic or system matrix ($n \times n$)
B	:	Input matrix ($n \times m$)
C	:	Output matrix ($p \times n$)
D	:	Feed forward (direct) matrix ($p \times m$)

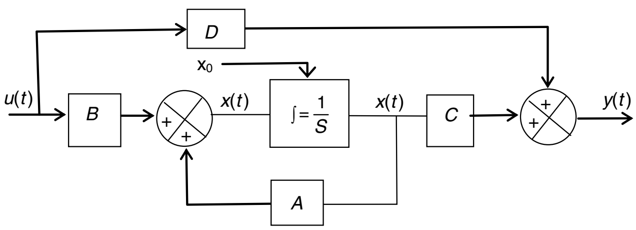


Figure 1 Block diagram representation of state space model

Advantages of State Space Analysis

1. It is applicable to multiple input and multiple output systems.
2. It is applicable to systems with non-zero initial conditions.
3. It is applicable to both linear time-invariant (LTI) and non-linear time-varying systems.
4. All the internal states can be determined.
5. It is more accurate than transfer function (TF).
6. It gives the information about controllability and observability.

State space representation

In order to select the state variables, we must follow the rules given here:

(a) A minimum number of state variables must be selected.

(b) They must be linearly independent.

The minimum number of state variables required equals the order of the differential equation describing the system. From the TF point of view, the order of the differential equation is the order of the denominator of the TF after cancelling common factors in the numerator and denominator.

A practical way to determine the number of state variables is to count the number of independent energy storage elements in the system (capacitor and inductors in electrical system and masses and springs in mechanical system).

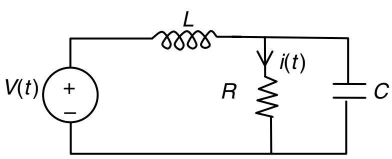
The following is the procedure for state representation of electrical network:

1. Write a simple derivative equation for each energy storage element (node equation for the node at which an inductor is connected and loop equation for the loop in which capacitor is connected in electrical network).
2. Solve for the each derivate term as a linear combination of the system variable and the input.
3. Each differentiated variable is selected as a state variable.
4. All other variables and output are written in terms of the state variables and the input.

Solved Examples

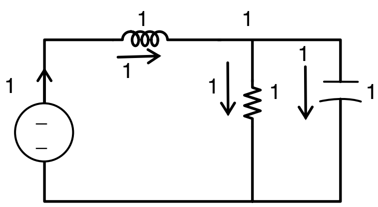
Example 1

Find the state space representation of the system shown in the figure if the output is the current through the resistor.



Solution

Step 1: Label all the branch variables in the network.



Step 2: Select the state variables. Write the derivative equations for all energy storage elements (L and C)

$$C.\frac{dv_c}{dt} = i_c; \; L.\frac{di_L}{dt} = V_L$$

Choose the differentiated quantities as the state variables (v_C and i_L).

Step 3: Write each differentiated term as a linear combination of system variables and the input.

$$C.\frac{dv_c}{dt} = i_C = f_1(v_C, i_C, V(t)) \tag{1}$$

$$L.\frac{di_L}{dt} = v_c = f_2(v_C, i_L, V(t)) \tag{2}$$

Applying node equation at V_1

$$i_C = -i_R + i_L = -\frac{1}{R}v_c + i_L \tag{3}$$

Applying loop equation in capacitor loop

$$V_L = -V_C + V(t) \tag{4}$$

From Eqs (1), (2), (3), and (4)

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L}v_C + \frac{1}{L}V(t)$$

$$\text{Output equation } i_R = \frac{1}{R}v_C$$

Step 4: Obtain state space representation in vector matrix form.

$$\begin{bmatrix} \dot{v}_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} v(t)$$

$$\begin{bmatrix} i_R \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix}$$

Transfer Function To State Space Model

This case corresponds to a linear system that can be represented as an nth-order differential equation with constant coefficient as follows:

$$\frac{d^n \; y(t)}{dt^{\,n}} + \frac{d^{n-1} \; y(t)}{dt^{n-1}} a_{n-1} + \frac{d^{n-2} \; y(t)}{dt^{\,n-2}} a_{n-2} + ... + a_1 \frac{dy(t)}{dt} + a_0$$
$$= b_o U(t)$$

The classical transfer function (TF) representation of the system is obtained by applying the laplace transform to the differential equation.

$$G(s) = \frac{Y(S)}{U(S)} = \frac{b_o}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + ... + a_1 s + a_0}$$

If the state space representation has to be obtained, a convenient way to select state variables is to choose the output $y(t)$ and its '$n - 1$' derivatives as the state variables. This is called phase variables choice.

The state space representation using the phase variable choice of the state variables is said to be in the controllable canonical form (CCF).

Controller canonical form of the given differential equation is as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & 0 \; 0 \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ b_o \end{bmatrix} U \quad Y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Consider a transfer function with polynomial in numerator:

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \ldots\ldots + a_{n-1}\dot{y} + a_n y = b_o u^n + b_1 u^{n-1} + \ldots\ldots + b_{n-1}\dot{u} + b_n u.$$

where U is the input and y is the output. The transfer function can be written as follows:

$$\frac{Y(S)}{U(S)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots . b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

State space representation of the system in controllable canonical form and observable canonical form are given as follows:

Controllable Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} U \quad y = [b_n - a_n b_o : b_{n-1} - a_{n-1} b_o : \ldots\ldots : b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_o u.$$

The controllable canonical form is important in discussing the pole placement approach to the control system design.

Observable Canonical Form

The following state–space representation is called an observable canonical form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1, b_0 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

System matrix in observable canonical form is the transpose of system matrix in controllable canonical form.

Diagonal Canonical Form

Transfer function with numerator polynomial can be written as follows:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots b_{n-1}s + b_n}{(s+p_1)(s+p_2)(s+p_3)\ldots(s+p_n)}$$

$$= b_0 + \frac{C_1}{s+p_1} + \frac{C_2}{s+P_2} + \ldots \frac{C_n}{s+P_n}$$

where $p_1, p_2, \ldots p_n$ are the locations of poles.

The diagonal canonical form of the state space representation of this system is given as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} -p_1 & . & . & . & . & . & . & . & 0 \\ 0 & -p_2 & . & . & . & . & . & . & 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 & & & & & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot\cdot \\ \cdot \\ 1 \end{bmatrix} u \quad y = [C_1 \; C_2 \ldots . C_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot\cdot \\ \cdot \\ x_n \end{bmatrix} + b_o u$$

Jordan Canonical Form

When the system involves multiple roots, the diagonal canonical form must be modified into Jordon canonical form.

For example, if there are three equal poles ($p_1 = P_2 = P_3$), the factored form $Y(s)/U(s)$ becomes

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{(s+P_1)^3} + \frac{C_2}{(s+p_1)^2} + \frac{C_3}{(s+p_1)} + \frac{C_4}{s+P_4} + \ldots + \frac{C_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 . \\ .\dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & 1 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & -p_1 & 1 & 1. & 0 & . & . & . & \\ 0 & 0 & -p_1 & 1 & 0 & . & . & . & 0 \\ - & - & - & - & & - & - & - & . \\ 0 & 0 & 0 & . & -p_4 & . & . & . & 0 \\ . & & & . \\ . & . & & . \\ . & & & . \\ . & . & & . \\ 0 & 0 & 0 & . & 0 & . & . & . & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 . \\ .x_4 \\ . \\ . \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ . \\ . \\ .. \\ . \\ 1 \end{bmatrix} u \quad y = [C_1 \; C_2 \ldots . C_n] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot\cdot \\ \cdot \\ x_n \end{bmatrix} + b_o u.$$

Eigen values of an n × n matrix A

The characteristic equation of a square matrix 'A' is given as follows:

$$|\lambda I - A| = 0$$

The values of λ which satisfy the characteristic equation are called eigen values of A matrix.

NOTES

1. The poles of the transfer function are given by the $|sI - A| = 0$. This function is the same equation as the characteristic equation of A. Therefore, we can conclude that the eigen values of the state model and the poles of the transfer function are the same.
2. Stability of the SISO (single input single output) system depends on the eigen values of system matrix in its state space model.

Controllability

A state $x(t)$ is said to be controllable at $t = t_0$ if there exists a piecewise continuous input $u(t)$ that will drive the state to any final state $x(t_f)$ for a finite time $(t_f - t_0) \geq 0$.

If every state $x(t_0)$ of the system is controllable in a finite time interval, the system is said to be completely controllable or simply, controllable.

Kalman's Test for Controllability

Consider a state space model:

$$\dot{x} = Ax + Bu$$

Then, controllability matrix

$$Q_c = [B:AB:A^2B : \ldots.:A^{n-1} B]$$

NOTES

1. A system is said to be controllable if the rank of Q_C is equal to the order of matrix A or $|Q_c| \neq 0$.
2. The number of uncontrollable states is computed as the difference between order of A matrix (n) and rank of Q_C matrix (P).

Observability

An LTI system is said to be observable if in given any input $u(t)$, there exists a finite time $t_f \geq t_0$ such that the knowledge of $u(t)$ for $t_0 \leq t, < f_f$, matrix A, B, C, and D and the output $y(t)$ for $t_0 \leq t < t_f$ are sufficient to determined $x(t_0)$.

If every state of the system is observable for a finite t_f, we say that the system is completely observable or simply, observable.

Kalman's Test for Observability

Consider a state space model:

$$\dot{x} = Ax + Bu \text{ and } y = Cx + Du$$

Then observability matrix

$$Q_0 = [C \quad CA \quad CA^2 \quad \ldots\ldots CA^{n-1}]^T$$

NOTES

1. The system is said to be observable if the rank of Q_0 is equal to the order of matrix A

   or $|Q_c| \neq 0$.
2. The number of unobservable states is computed as the difference between order of A matrix (n) and rank of Q_0 matrix (q).

Solution of State Equations

The state equation of a linear system is as follows:

$$\dot{x} = Ax + Bu; \qquad x(t_0) = x(0):$$

The solution for the above state equation is

$$x(t) = \underbrace{e^{At}\, x(0)}_{\text{Homogeneous Solution}} + \underbrace{\int_0^t e^{-A(t-\tau)} Bu(\tau)\, .d\tau}_{\text{Forced Solution}}$$

In the absence of the input to the system, the response of the system or solution of state equations with initial conditions alone is given as follows:

$$X(t) \; e^{At}\, x(0) = e^{At} x_0$$

It is observed that the initial state x_0 at $t = 0$ is driven to a state $x(t)$ at time 't'. This transition in state is carried out by the matrix exponential e^{At}. Due to this property, e^{At} is known as state transition matrix and is denoted by $\phi(t)$.

Properties of State Transition Matrix $\phi(t)$

The following are the properties of state transition matrix:

1. $\phi(0) = e^{A0} = I$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} \,.\, e^{At_2}$
   $= \phi(t_1).\, \phi(t_2) = \phi(t_2).\, \phi(t_1)$
4. $[\phi(t)]^n = [e^{At}]^n = e^{An}+ = \phi(nt)$
5. $\phi(t_1 - t)\,.\,\phi(t - t_2) = e^{A(t1-t)}\,.\,e^{A(t-t2)}$
   $= e^{A(t1-t2)} = \phi(t_1 - t_2)$

Transfer Function

Given the state space model of SISO system as

$$\dot{x} = Ax + Bu$$

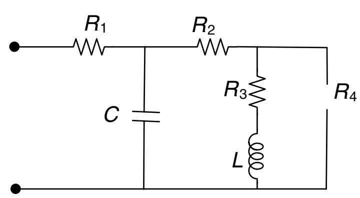
$$Y = Cx + Du$$

The transfer function of the system is

$$TF = C[sI - A]^{-1} B + D$$

Example 2

The maximum number of states required to describe the network as shown in figure is



(A) 1 (B) 2 (C) 3 (D) 4

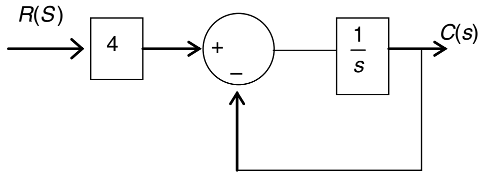
Solution

No. of energy storage elements $(L, C) = 2$

No. of states required to analysis $= 2$

Example 3

The matrix of any state space equation for the transfer function $\frac{C(s)}{R(s)}$ of the system shown in figure is



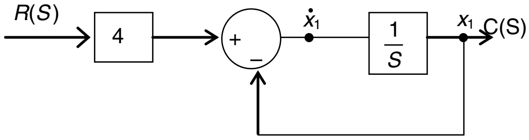
(A) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (B) $[-1]$

(C) $[4]$ (D) $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

Solution

Output of integrator is considered as state and number of integrators is equal to the number of states.

Number of states = 1



$$\dot{x}_1 = -x_1 + 4r(t)$$

$$[\dot{x}_1] = [-1]\,[x_1] + [4]\,[r(t)]$$

Matrix of state space equation is [–1].

Example 4

A system is described by the state equations as follows:

$\dot{x} = Ax + Bu$. the output is given by $y = Cx$

where $A = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

The transfer function $G(s)$ of the system is

(A) $\dfrac{s}{s^2+5s+7}$ (B) $\dfrac{s+7}{s^2+5s+7}$

(C) $\dfrac{1}{s^2+5s+7}$ (D) $\dfrac{s}{s^2+s+5}$

Solution

Transfer function of the state space model is

$$\text{TF} = \frac{Y(s)}{U(s)} = C[sI - A]^{-1}B + D$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{(s+1)(s+4)}\begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}$$

$$\text{TF} = -C[sI - A]^{-1}B = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{s^2+5s+7}$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+1-1 \\ 3+s+4 \end{bmatrix}}{s^2+5s+7}$$

$$= \frac{s+7}{s^2+5s+7}$$

Example 5

Given the homogeneous state space equations

$\dot{x} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix} x$, the steady-state value of $x_{ss} = \underset{t\to\infty}{Lt}\, x(t)$, given the initial state value of $x[0] = [10\ -10]^T$ is

(A) $x_{ss} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$ (B) $x_{ss} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$

(C) $x_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (D) $x_{ss} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$

Solution

Solution of homogenous equation

$$A\,\dot{x} = Ax$$

$$x(t) = e^{At}\,x(0)$$

$$e^{At} = L\left[[sI - A]^{-1}\right]$$

$$sI - A = \begin{bmatrix} s+4 & -1 \\ 0 & s+3 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{-1}{(s+4)(s+3)}\begin{bmatrix} s+3 & 1 \\ 0 & s+4 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s+4} & \dfrac{1}{(s+4)(s+3)} \\ 0 & \dfrac{1}{s+3} \end{bmatrix}$$

$$e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} e^{-4t} & e^{-3t} - e^{-4t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) = \begin{bmatrix} 10e^{-4t} + 10e^{-3t} + 10e^{-4t} \\ -10e^{-3t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 20e^{-4t} - 10e^{-3t} \\ -10^{-3t} \end{bmatrix}$$

$$x_{ss} = \underset{t\to\infty}{lt}\, x(t) = \begin{bmatrix} \underset{t\to\infty}{lt}\,(20e^{-4t} - 10e^{-3t}) \\ \underset{t\to\infty}{lt}\, -10e^{-3t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 6

A second-order system starts with an initial condition of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ without any external input. The state transition matrix for the system is given by $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix}$

The state of the system at end of 2 s is given by

(A) $10^{-3} \times \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (B) $10^{-3} \times \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ (D) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Solution

Solution of state equations without external input is

$$x(\text{t}) = e^{At} \times (0)$$

$$= \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$X(\text{t}) = \begin{bmatrix} 2e^{-3t} \\ 3e^{-4t} \end{bmatrix}$$

At $t = 2$ s, $x(2) = \begin{bmatrix} 2e^{-3\times 2} \\ 3e^{-4\times 2} \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} .10^{-3}$

Example 7

For a system with transfer function

$$G(s) = \frac{2s+4}{s^3+4s^2+9s+4}$$

The matrix A in the state space form $\dot{x} = Ax + Bu$ is equal to

(A) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -9 & -4 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 & 0 \\ -2 & -4 & 1 \\ -4 & -9 & -4 \end{bmatrix}$

Solution

From the standard controllable canonical form of the transfer function

$$\text{TF} = \frac{2s+4}{s^3+4s^2+9s+4}$$

State space representation

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

Example 8

For the system $\dot{x} = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ u, which of the following statements about the system is true?

(A) Controllable and stable
(B) Uncontrollable and stable
(C) Controllable and unstable
(D) Uncontrollable and unstable

Solution

For stability analysis, location of poles are the roots of characteristic equation.

$$|sI - A| = \begin{vmatrix} s-3 & -4 \\ 0 & s-5 \end{vmatrix} = (s-3)(s-5) = 0$$

$$s = 3 \text{ and } 5$$

Therefore, poles are located on RHS plane, system is unstable.

Controllability matrix $Q_C = [B \quad AB]$

$$[A \quad B] = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$Q_C = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow |Q_C| = 0$$

Therefore, the system is uncontrollable.

Exercises

Practice Problems I

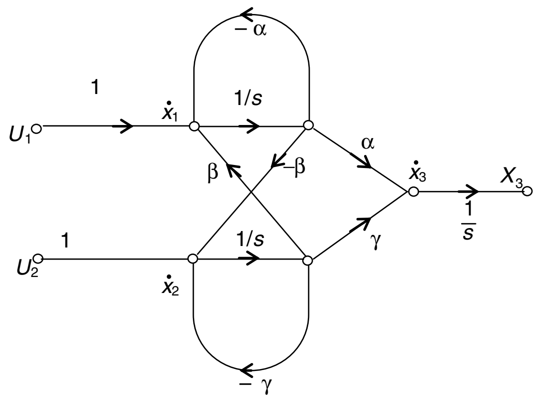
Direction for questions 1 to 15: Select the correct alternative from the given choices.

1. Given that $\dot{X} = AX$ for the system described by the differential equation $\ddot{y} + 2\dot{y} + 3y = 0$. The matrix A is

(A) $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ -1 & -3 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$

2. A signal flow graph of a system is given below:



If the state equation that corresponds to the above signal flow graph is $X = AX + BU$, the matrix A and B are

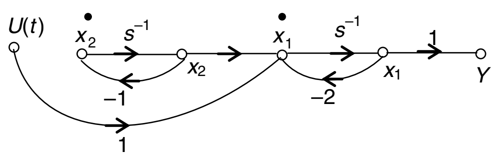
(A) $\begin{bmatrix} \beta & -\gamma & 0 \\ \gamma & \alpha & 0 \\ -\alpha & -\beta & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & \alpha & \gamma \\ 0 & -\alpha & -\gamma \\ 0 & \beta & -\beta \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -\alpha & \beta & 0 \\ -\beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} -\alpha & 0 & \beta \\ \gamma & 0 & \alpha \\ -\beta & 0 & -\alpha \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

3.



In the state diagram of a system shown in the figure, which variables are controllable?

(A) $x_1(t)$

(B) $x_2(t)$

(C) Both $x_1(t)$ and $x_2(t)$

(D) Neither $x_1(t)$ nor $x_2(t)$

4. The state equations of a linear time-invariant system are represented by $\frac{dx(t)}{dt} = AX(t) + BU(t)$

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The state transition matrix ϕ(t) is

(A) $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-t/3} & 0 \\ 0 & e^{t/3} \end{bmatrix}$ (D) $\begin{bmatrix} e^{t/3} & 0 \\ 0 & e^{-t/3} \end{bmatrix}$

5. A particular control system is described by the following state equations:

$$X = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} \frac{x_1}{x_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{U and}$$

$$Y = [1 \quad 0] \begin{bmatrix} \frac{x_1}{x_2} \end{bmatrix}$$

The transfer function of this system is

(A) $\frac{Y(s)}{U(s)} = \frac{2}{s^2+2s+5}$ (B) $\frac{Y(s)}{U(s)} = \frac{4}{s^2+2s+5}$

(C) $\frac{Y(s)}{U(s)} = \frac{1}{s^2+2s+5}$ (D) $\frac{Y(s)}{U(s)} = \frac{3}{s^2+2s+5}$

Direction for questions 6 and 7:

A system is characterized by the following state space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U, (t > 0)$$

$$Y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

6. The transfer function of the system is

(A) $\frac{s}{(s+2)(s+1)}$ (B) $\frac{1}{s(s+2)(s+1)}$

(C) $\frac{s}{(s-2)(s+1)}$ (D) $\frac{1}{(s+2)(s+1)}$

7. The state transition matrix of the system is

(A) $\begin{bmatrix} e^{-t}+2e^{-2t} & e^{-t}+e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}+e^{-2t} \end{bmatrix}$

(B) $\begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-t}-2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}-2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$

(D) $\begin{bmatrix} e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}-2e^{-2t} & 2e^{-t}+e^{-2t} \end{bmatrix}$

8. Match List–I (Matrix) with List–II (dimensions) for the state equations:

$\dot{X}(t) = AX(t) + BU(t)$ and $Y(t) = CX(t) + DU(t)$ and select the correct answer using the codes given in the lists:

List-I	List-II
A	(1) $n \times p$
B	(2) $q \times n$
C	(3) $n \times n$
D	(4) $q \times p$

Codes:

	A	B	C	D
(A)	1	3	4	2
(B)	1	3	2	4
(C)	3	1	4	2
(D)	3	1	2	4

9. Consider the single input, single output system with its state variable representation:

$$\dot{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} U,\ Y = [1 \quad 0 \quad 2]\, X$$

The system is
(A) neither controllable nor observable.
(B) controllable but not observable.
(C) uncontrollable but observable.
(D) both controllable and observable.

10. Consider the state transition matrix: $\phi(t) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$.

The eigen values of the matrix when t = 0 are
(A) 1, 2 (B) 2, 1
(C) 1, 1 (D) 1, 3

11. A linear system is described by the following state equation - $\dot{X} = AX(t) + BU(t)$.

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The state transition matrix of the system is

(A) $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ (B) $\begin{bmatrix} -\cos t & \sin t \\ -\sin t & -\cos t \end{bmatrix}$

(C) $\begin{bmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{bmatrix}$ (D) $\begin{bmatrix} \cos t & -\sin t \\ \cos t & \sin t \end{bmatrix}$

12. Consider the system $\dfrac{dx}{dt} = AX + BU$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix}$ where p and q are arbitrary real numbers. Which of the following statements about the controllability of the system are true?
(A) The system is completely controllable for any non-zero values of p and q.
(B) Only $p = 0$ and $q = 0$ result in controllability.
(C) The system is uncontrollable for all values of p and q.
(D) We cannot conclude about controllability from the given data.

13. The state transition matrix of the system whose state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is

(A) $\begin{bmatrix} \cos\sqrt{2}t & \frac{1}{\sqrt{2}}\cos\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \sin\sqrt{2}t \end{bmatrix}$

(B) $\begin{bmatrix} \cos\sqrt{2}t & \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix}$

(C) $\begin{bmatrix} \sin t & -\cos t \\ +\cos t & \sin t \end{bmatrix}$

(D) $\begin{bmatrix} \sin\sqrt{2}t & \cos\sqrt{2}t \\ -\cos\sqrt{2}t & \sin\sqrt{2}t \end{bmatrix}$

14. If the eigen values of a 3×3 matrix A are 1, – 3, and 4. The eigen values of $P^{-1}AP$ ($\therefore$ P is a linear transformation) are
(A) 1, $\frac{-1}{3}$, and $\frac{1}{4}$ (B) 1, – 3, and 4
(C) 1, 9, and 16 (D) –1, 3, and – 4

15. The state equations of a system are $x = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$. The closed-loop poles of the system are
(A) ±2 (B) – 1, – 1
(C) + 1, – 2 (D) + 2, – 1

Practice Problems 2

Direction for questions 1 to 15: Select the correct alternative from the given choices.

1. The state equation of a linear system is given by $X = AX + BU$, where $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The state transition matrix of the system is

(A) $\begin{bmatrix} e^{-t} & e^{-t} + e^{-2t} \\ 0 & e^{2t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} 0 & e^{-2t} \\ e^{-t} & e^{-t} - e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} 0 & e^{-t} - e^{-2t} \\ e^{-t} & e^{-2t} \end{bmatrix}$

2. The eigen value and eigen vector pairs (λ_I, V_i) for the system are

(A) $\left[-1, \begin{bmatrix}1\\-1\end{bmatrix}\right]$, and $\left[-2, \begin{bmatrix}1\\-2\end{bmatrix}\right]$

(B) $\left[-2, \begin{bmatrix}1\\-1\end{bmatrix}\right]$, and $\left[-1, \begin{bmatrix}1\\-2\end{bmatrix}\right]$

(C) $\left[-1, \begin{bmatrix}1\\-1\end{bmatrix}\right]$, and $\left[2, \begin{bmatrix}1\\-2\end{bmatrix}\right]$

(D) $\left[2, \begin{bmatrix}1\\-1\end{bmatrix}\right]$, and $\left[1, \begin{bmatrix}1\\-2\end{bmatrix}\right]$

3. The system matrix A is

(A) $\begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix}$ (B) $\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$

(C) $\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$ (D) $\begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}$

4. Let $X = \dot{X} = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}X + \begin{bmatrix}0\\1\end{bmatrix}U$, $Y = [b \quad 0]X$

where b is an unknown constant. This system is

(A) observable for all values of b.

(B) unobservable for all values of b.

(C) observable for all non-zero values of b.

(D) unobservable for all non-zero values of b.

5. A state variable representation of a system is given by the expression

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}U(t),$$

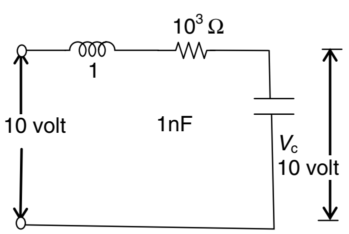
$$Y = [1 \quad 1]\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

The transfer function of the system is

(A) $\dfrac{2}{s+1}$ (B) $\dfrac{2s}{(s-1)(s+1)}$

(C) $\dfrac{2}{(s-1)(s+1)}$ (D) $\dfrac{1}{(s+1)}$

6. The state space model for an electrical network is shown in the figure where the current I and voltage across the capacitor V_c are the state variables.



(A) $\begin{bmatrix}X_1\\\dot{X}_2\end{bmatrix} = \begin{bmatrix}10^6 & 10^3\\10^9 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\10^3\end{bmatrix} \times 10$

(B) $\begin{bmatrix}X_1\\\dot{X}_2\end{bmatrix} = \begin{bmatrix}10^9 & 0\\10^6 & 10^3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\10^3\end{bmatrix} \times 10$

(C) $\begin{bmatrix}X_1\\\dot{X}_2\end{bmatrix} = \begin{bmatrix}-10^6 & -10^3\\10^9 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}10^3\\0\end{bmatrix} \times 10$

(D) None of these

7. The poles and zero of the following system

$$X = \begin{bmatrix}0 & 1\\-20 & -9\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u$$

$y = [-17 - 5]x + [1]u$ are

(A) poles: – 4, – 5, and zeros: – 1, – 3

(B) poles: – 3, – 4, and zeros: – 3, – 2

(C) poles: – 2, – 3, and zeros: – 2, – 4

(D) None of these

8. The second-order system $\dot{x} = Ax$ has $A = \begin{bmatrix}-1 & -2\\+1 & -1\end{bmatrix}$.

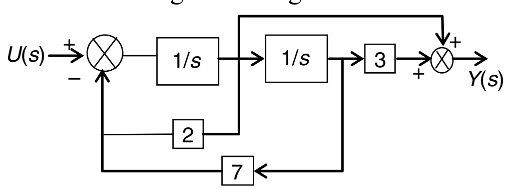
The values of its damping factor and natural frequency ω_n are, respectively

(A) 1.732, 0.577 (B) 1.414, 0.6

(C) 0.577, 1.732 (D) 0.6, 1.414

Direction for questions 9 and 10:

Consider the following block diagram



9. In the above figure, the state space representation in the vector matrix from is

(A) $x = \begin{bmatrix}0 & 1\\-7 & 2\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u$

$Y = [3 \ 1]\,x$

(B) $x = \begin{bmatrix}0 & 1\\-7 & -2\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u$

$Y = [3 \ 1]x$

(C) $x = \begin{bmatrix}1 & 0\\-2 & -7\end{bmatrix}x + \begin{bmatrix}1\\0\end{bmatrix}u$

$Y = [1 \ 3]\,x$

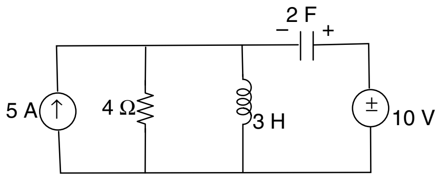
(D) None of these

10. From question 9, the transfer function of the system is

(A) $\dfrac{s}{s^2+2s+7}$ (B) $\dfrac{s+3}{s^2+2s+7}$

(C) $\dfrac{3}{s^2+2s+7}$ (D) $\dfrac{2s+3}{s^2+2s+7}$

Direction for questions 11 and 12:
The circuit diagram is shown in the figure.



11. The state equations of the above circuit are

(A) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & -1/3 \\ -1/2 & -1/8 \end{bmatrix}\begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 1/3 & 0 \\ 1/8 & -1/2 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(B) $\begin{bmatrix} \dot{i}_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} 0 & -1/3 \\ 1/2 & -1/8 \end{bmatrix}\begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 1/3 & 0 \\ 1/8 & -1/2 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(C) $\begin{bmatrix} i_L \\ \dot{v}_c \end{bmatrix} = \begin{bmatrix} -1/3 & 0 \\ 1/8 & -1/2 \end{bmatrix}\begin{bmatrix} i_L \\ v_c \end{bmatrix} + \begin{bmatrix} 0 & 1/3 \\ 1/8 & -1/2 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix}$

(D) None of these

12. From question 11, the eigen values are
(A) $-0.06 \pm j0.06$
(B) $-0.06 \pm j0.403$
(C) $-0.403 \pm j0.06$
(D) $-0.12 \pm j0.12$

13. A continuous time, linear time-invariant system is described by $\frac{d^2y}{dt} + 4\frac{dy}{dt} + 3y(t) = 2\frac{dx}{dt} + 4x$
Assuming zero initial conditions, the response $y(t)$ of the above system for the input $x(t) = e^{-2t}\,u(t)$ is given by
(A) $(e^{-t} - e^{-3t})\,u(t)$
(B) $(e^{-t} + e^{-3t})\,u(t)$
(C) $(e^{t} - e^{+3t})\,u(t)$
(D) $(e^{t} + e^{3t})\,u(t)$

14. The system with the state equation

$$x = Ax + Bu \text{ and } A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix};\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(A) System is controllable
(B) System is uncontrollable
(C) System is controllable and observable
(D) None of these

15. Consider the state transition matrix:

$$\phi(t) = L^{-1}[\phi(s)] = L^{-1}\begin{bmatrix} \frac{s+1}{s^2+3s+2} & \frac{-1}{s^2+3s+2} \\ 0 & \frac{s+2}{s^2+3s+2} \end{bmatrix}$$

The eigen values of the system are
(A) 0 and – 2
(B) –1 and 2
(C) 1 and 2
(D) –1 and – 2

Previous Years' Questions

1. If $A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$, then sin A$t$ is **[2004]**

(A) $\frac{1}{3}\begin{bmatrix} \sin(-4t)+2\sin(-t) & -2\sin(-4t)+2\sin(-t) \\ -\sin(-4t)+\sin(-t) & 2\sin(-4t)+\sin(-t) \end{bmatrix}$

(B) $\begin{bmatrix} \sin(-2t) & \sin(2t) \\ \sin(t) & \sin(-3t) \end{bmatrix}$

(C) $\frac{1}{3}\begin{bmatrix} \sin(4t)+2\sin(t) & 2\sin(-4t)+2\sin(-t) \\ -\sin(-4t)+\sin(t) & 2\sin(4t)+\sin(t) \end{bmatrix}$

(D) $\frac{1}{3}\begin{bmatrix} \cos(t)2\cos(t) & -2\cos(-4t)+2\sin(-t) \\ -\cos(-4t)+\sin(-t) & -2\cos(-4t)+\cos(-t) \end{bmatrix}$

2. The state variable equations of system are

$$\Rightarrow \dot{X}1 = -3x_1 - x_2 + u$$
$$\Rightarrow \dot{X}2 = 2x_1$$
$$y = x_1 + u$$

The system is **[2004]**
(A) controllable but not observable
(B) observable but not controllable
(C) neither controllable nor observable
(D) controllable and observable

3. Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the state transition matrix e^{At} is given by **[2004]**

(A) $\begin{bmatrix} 0 & e^{-t} \\ e^{-t} & 0 \end{bmatrix}$
(B) $\begin{bmatrix} e^{t} & 0 \\ 0 & e^{t} \end{bmatrix}$
(C) $\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}$
(D) $\begin{bmatrix} 0 & e^{t} \\ e^{t} & 0 \end{bmatrix}$

4. A linear system is described by the following state equation:

$$X(t) = AX(t) + BU(t),\ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The state -transition matrix of the system is **[2006]**

(A) $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$
(B) $\begin{bmatrix} -\cos t & \sin t \\ -\sin t & -\cos t \end{bmatrix}$
(C) $\begin{bmatrix} -\cos t & -\sin t \\ -\sin t & \cos t \end{bmatrix}$
(D) $\begin{bmatrix} \cos t & -\sin t \\ \cos t & \sin t \end{bmatrix}$

5. The state space representation of a separately excited DC servo motor dynamics is given as follows:

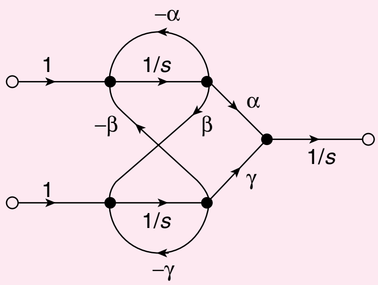
$$\begin{bmatrix} \frac{d\omega}{dt} \\ \frac{di_a}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -10 \end{bmatrix}\begin{bmatrix} \omega \\ i_a \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} U$$

where ω is the speed of the motor, i_a is the armature current and U is the armature voltage. The transfer function $\frac{\omega(s)}{U(s)}$ of the motor is **[2007]**

(A) $\frac{10}{s^2+11s+11}$ (B) $\frac{1}{s^2+11s+11}$

(C) $\frac{10s+10}{s^2+11s+11}$ (D) $\frac{1}{s^2+s+1}$

6. A single flow graph of a system is given as follows:



The set of equations that correspond to this signal flow graph is **[2008]**

(A) $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} \beta & -\gamma & 0 \\ \gamma & \alpha & 0 \\ -\alpha & -\beta & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

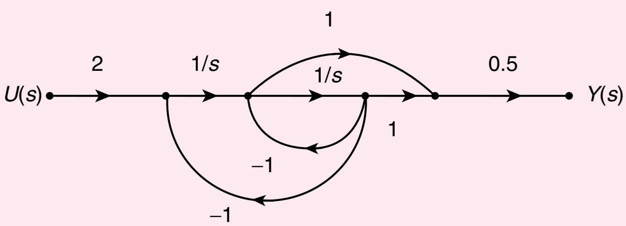
(B) $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 & \alpha & \gamma \\ 0 & -\alpha & -\gamma \\ 0 & \beta & -\beta \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

(C) $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -\alpha & -\beta & 0 \\ -\beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

(D) $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -\gamma & 0 & \beta \\ \gamma & 0 & \alpha \\ -\beta & 0 & -\alpha \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

Direction for questions 7 and 8:

The signal flow graph of a system is shown below.



7. The state variable representation of the system x can be **[2010]**

(A) $X = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$

$y = \begin{bmatrix} 0 & 0.5 \end{bmatrix} x$

(B) $X = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$

$y = \begin{bmatrix} 0 & 0.5 \end{bmatrix} x$

(C) $X = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$

$y = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} x$

(D) $X = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$

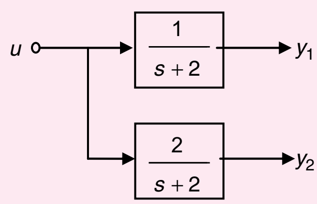
$y = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} x$

8. The transfer function of the system is **[2010]**

(A) $\frac{s+1}{s^2+1}$ (B) $\frac{s-1}{s^2+1}$

(C) $\frac{s+1}{s^2+s+1}$ (D) $\frac{s-1}{s^2+s+1}$

9. The block diagram of a system with one input u and two outputs y_1 and y_2 is given below.



A state space model of the above system in terms of the state vector $\underline{x}$ and the output vector $y = [y_1 \quad y_2]^T$ is **[2011]**

(A) $\dot{X} = [2]x + [1]u; \; y = [1 \quad 2]\, x$

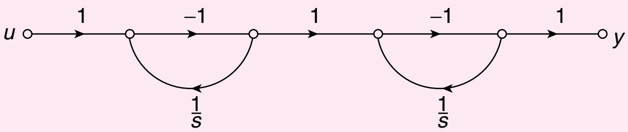
(B) $\dot{X} = [-2]x + [1]u; \; \underline{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x$

(C) $\dot{X} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u; \; y = [1 \quad 2]$

(D) $\dot{X} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u; \; y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x$

Direction for questions 10 and 11:
The state diagram of a system is shown below, A system is described by the state-variable equations

$$\dot{x} = AX + Bu,\ y = CX + Du$$



10. The state-variable equations of the system shown in the figure above are **[2013]**

(A) $\dot{X} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} X + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u\ ;\ Y = [1\ -1]X + U$

(B) $\dot{X} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} X + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u\ ;\ Y = [-1\ -1]X + U$

(C) $\dot{X} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} X + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$
$y = [-1\ \ -1]X - u$

(D) $\dot{X} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$
$y = [1\ \ -1]X - u$

11. The state transition matrix e^{At} of the system shown figure above is **[2013]**

(A) $\begin{bmatrix} e^{-t} & 0 \\ te^{-1} & e^{-t} \end{bmatrix}$ (B) $\begin{bmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{bmatrix}$

(C) $\begin{bmatrix} e^{-t} & 0 \\ e^{-t} & e^{-t} \end{bmatrix}$ (D) $\begin{bmatrix} e^{-t} & -te^{-1} \\ 0 & e^{-t} \end{bmatrix}$

12. Consider the state space model of a system as given below. **[2014]**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} u$$

$$y = [1\ \ 1\ \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The system is
(A) controllable and observable
(B) uncontrollable and observable
(C) uncontrollable and unobservable
(D) controllable and unobservable

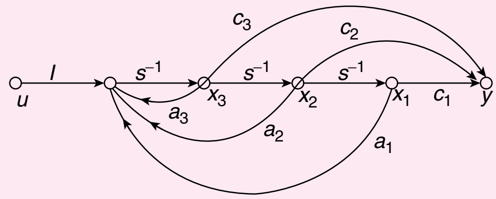
13. An unforced linear time-invariant system is represented by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If the initial conditions are $x_1(0) = 1$ and $x_2(0) = -1$, the solution of the state equation is **[2014]**

(A) $x_1(t) = -1, x_2(t) = 2$
(B) $x_1(t) = -e^{-t}, x_2(t) = 2e^{-t}$
(C) $x_1(t) = e^{-t}, x_2(t) = -e^{-2t}$
(D) $x_1(t) = -e^{-t}, x_2(t) = -2e^{-t}$

14. Consider the state space system expressed by the signal flow diagram shown in the figure. **[2014]**



The corresponding system is
(A) always controllable (B) always observable
(C) always stable (D) always unstable

15. The state equation of a second-order linear system is given by

$$\dot{x}(t) = Ax(t), x(0) = x_0$$

For $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$

and for $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x(t) = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix}$

when $x_0 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, x(t)$ is **[2014]**

(A) $\begin{bmatrix} -8e^{-t} + 11e^{-2t} \\ 8e^{-t} - 22e^{-2t} \end{bmatrix}$ (B) $\begin{bmatrix} 11e^{-t} - 8e^{-2t} \\ -11e^{-t} + 16e^{-2t} \end{bmatrix}$

(C) $\begin{bmatrix} 3e^{-t} - 5e^{-2t} \\ -3e^{-t} + 10e^{-2t} \end{bmatrix}$ (D) $\begin{bmatrix} 5e^{-t} - 3e^{-2t} \\ -5e^{-t} + 6e^{-2t} \end{bmatrix}$

16. The state transition matrix $\Phi(t)$ of a system $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is **[2014]**

(A) $\begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}$ (D) $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

Direction for questions 17 and 18:

Consider a linear system whose state space representation is $\dot{X}(t) = AX(t)$. If the initial state vector of the system is $X(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, then the system response is $X(t) = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}$.

If the initial state vector of the system changes to $X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then the system response becomes $X(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$

17. The eigen value and eigen vector pairs (λ_i, v_i) for the system are **[2014]**

(A) $\left(-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $\left(-2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

(B) $\left(-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $\left(2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

(C) $\left(-2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $\left(-2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

(D) $\left(-2, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $\left(1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

18. The system matrix A is **[2014]**

(A) $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

19. The network is described by the model is

$$\dot{x}_1 = 2x_1 - x_2 + 3u$$

$$\dot{x}_2 = -4x_2 - u$$

$$y = 3x_1 - 2x_2$$

The transfer function $H(s)\left(= \dfrac{Y(s)}{U(s)}\right)$ is **[2015]**

(A) $\dfrac{11s+35}{(s-2)(s+4)}$ (B) $\dfrac{11s-35}{(s-2)(s+4)}$

(C) $\dfrac{11s+38}{(s-2)(s+4)}$ (D) $\dfrac{11s-38}{(s-2)(s+4)}$

20. A second order linear time invariant system is described by the following state equations

$$\frac{d}{dt}x_1(t) + 2x_1(t) = 3u(t)$$

$$\frac{d}{dt}x_2(t) + x_2(t) = u(t)$$

Where $x_1(t)$ and $x_2(t)$ are the two state variables and $u(t)$ denotes the input. If the output $c(t) = x_1(t)$, then the system is **[2016]**

(A) Controllable but not observable
(B) Observable but not controllable
(C) Both controllable and observable
(D) Neither controllable nor observable

Answer Keys

Exercises

Practice Problems 1

1. D	**2.** C	**3.** A	**4.** B	**5.** C	**6.** D	**7.** B	**8.** C	**9.** A	**10.** C
11. A	**12.** C	**13.** B	**14.** B	**15.** B					

Practice Problems 2

1. B	**2.** A	**3.** D	**4.** C	**5.** D	**6.** C	**7.** A	**8.** C	**9.** B	**10.** B
11. B	**12.** B	**13.** A	**14.** B	**15.** D					

Previous Years' Questions

1. A	**2.** D	**3.** B	**4.** A	**5.** A	**6.** D	**7.** D	**8.** C	**9.** B	**10.** A
11. A	**12.** B	**13.** C	**14.** A	**15.** B	**16.** D	**17.** A	**18.** D	**19.** A	**20.** A

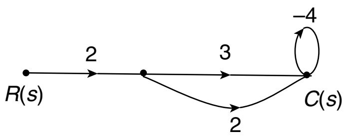
Test

Control Systems

Time: 60 Minutes

Direction for questions 1 to 30: Select the correct alternative from the given choices.

1. The transfer function gain between $C(S)$ and $R(S)$ in the following figure is



(A) 5 (B) 2.5 (C) 10 (D) 2

2. The Laplace transform of a transportation lag of 8 s is

(A) exp $(8s)$ (B) exp $(-8s)$

(C) $\frac{1}{3-8}$ (D) exp $(-s/8)$

3. The transfer function of ZOH (zero-order hold) is

(A) $1 - e^{-Ts}$ (B) $1 - e^{Ts}$

(C) $\frac{1-e^{Ts}}{s}$ (D) $\frac{1-e^{-Ts}}{s}$

4. The main drawback of a feedback system is

(A) inaccuracy (B) inefficiency

(C) unstability (D) insensitivity

5. The transfer function of linear control system is defined as the

(A) Fourier transform of impulse response

(B) Laplace transform of unit step response

(C) Laplace transform of impulse response

(D) None of these

6. Transfer function is defined for

(A) linear and time-variant system

(B) linear and time-invariant system

(C) non-linear and time-variant system

(D) non-linear and time-invariant system

7. The minimum phase transfer function is one having

(A) poles and zeros in RHS of s-plane

(B) poles and zeros in LHS s-plane

(C) poles in LHS and zeros in RHS s-plane

(D) poles in RHS and zeros in RHS s-plane

8. The unit impulse response of a unit feedback control system is given by $c(t) = -e^{-t} + 3e^{+t}$ $(t \geq 0)$. The transfer function is equal to

(A) $\frac{(s+2)}{s^2-1}$ (B) $\frac{s^2-1}{s(s+1)}$

(C) $\frac{2(s-2)}{s^2+1}$ (D) $\frac{2(s+2)}{s^2-1}$

9. The unit step response of a unit feedback control system is given by $c(t) = -e^{-t} + 2.\ e^{-3t}$ $(t \geq 0)$. The impulse response is

(A) $-6.\ e^{-3t} + e^{-t}$ (B) $e^{-t} + 2/3\ e^{-3t}$

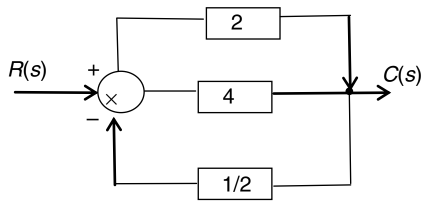
(C) $6.\ e^{3t} - e^{-t}$ (D) None of these

10. The impulse response of an initially relaxed system is $e^{-4t}\,u(t)$. To produce a response of $t.e^{-4t}.u(t)$, the input must be equal to

(A) $e^{+4t}\,u(-t)$ (B) $e^{-4t}\,u(t)$

(C) $t.e^{-4t}$ (D) $e^{-4t}.\ u(t)$

11. The closed loop gain of the system shown in the following figure is



(A) 2 (B) 3 (C) 6/4 (D) −4/6

12. The response $c(t)$ of a system to an input $r(t)$ is given by the following differential equation

$$\frac{d^2c(t)}{dt^2} + 3.\frac{dc(t)}{dt} + c(t) = 2.r(t)$$

The transfer function of the system is given by

(A) $\frac{3}{s^2+2s+1}$ (B) $\frac{-2}{s^2+3s+2}$

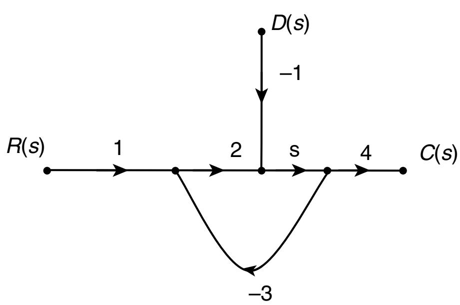
(C) $\frac{2}{s^2+3s+3}$ (D) $\frac{2}{s^2+3s+1}$

13. Given open loop transfer function $G(s) = \frac{1-s}{s(s+1)}$. The system with the transfer function is operated in a closed loop with unity feedback. The closed loop system is

(A) unstable (B) stable

(C) marginally stable (D) conditionally stable

14. The signal flow graph of the system is shown in the given figure. The transfer function $\frac{C(s)}{D(s)}$ of the system is



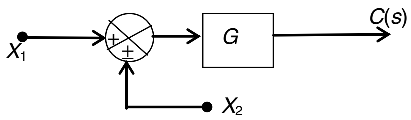
(A) $\frac{8s}{1+6s}$ (B) $\frac{4s}{1+6s}$

(C) $\frac{-4s}{1-6s}$ (D) $\frac{-4s}{1+6s}$

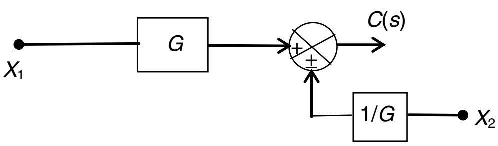
15. If a system has an open-loop transfer function $\frac{1-s}{1+s}$, then the gain of the system at frequency of 1 rad/s will be

(A) 1 (B) 0 (C) –1 (D) 1/2

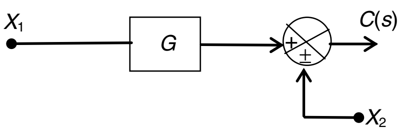
16. The following block diagram is equivalent to



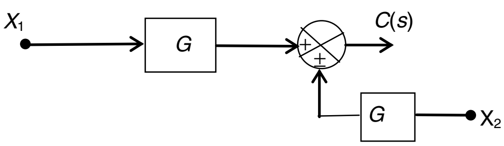
(A)



(B)



(C)



(D) B and C

17. For what value of K, are the two block diagrams, as shown in the following figure, equivalent?

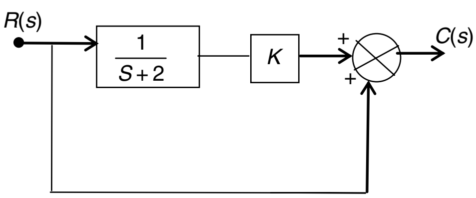


Figure 1

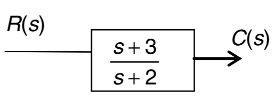
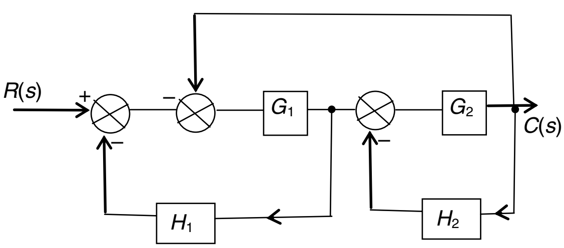


Figure 2

(A) 1 (B) –1 (C) 3 + 1 (D) 3 + 2

18.



The transfer function $\frac{C(s)}{R(s)}$ is

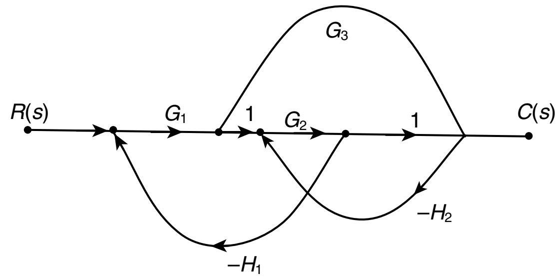
(A) $\frac{G_1 G_2}{1+G_1H_1+G_2H_2}$

(B) $\frac{G_1 G_2}{1+G_1G_2+G_2H_2+G_1H_1+G_1G_2\,H_1H_2}$

(C) $\frac{G_1G_2+1}{1+G_1H_1+G_2H_2+G_1G_2\,H_1H_2}$

(D) None of these

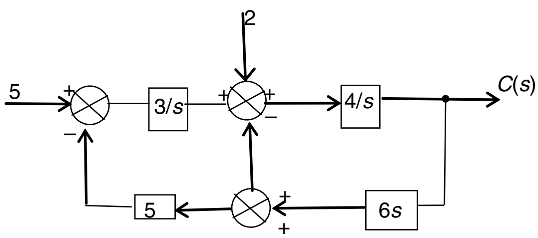
19.



The number of forward paths and individual loops are, respectively,

(A) 2, 2 (B) 2, 3 (C) 3, 2 (D) 2, 4

20.



The output $C(s)$ is

(A) $\frac{4(2s+15)}{5s(5s+52)}$ (B) $\frac{(2s+15)}{s(5s+52)}$

(C) $\frac{4(2s-15)}{5s(s+52)}$ (D) None of these

21. The closed-loop transfer function of a control system is given by $\frac{C(s)}{R(s)} = \frac{2(s+1)}{(s+2)(s-3)}$ for a unit step input the output is

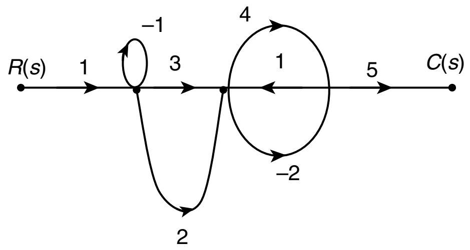
(A) $\left[-\frac{1}{3}+\frac{1}{5}e^{-2t}+\frac{8}{15}e^{3t}\right]u(t)$

(B) 0

(C) $\left[-1+e^{-2t}+e^{3t}\right]u(t)$

(D) $\left[-\frac{1}{5}+\frac{1}{3}e^{-2t}+\frac{8}{15}e^{-3t}\right]u(t)$

22.



In the signal flow graph, the ratio between $\frac{C(s)}{R(s)}$ is equal to

(A) –15 (B) –10 (C) 20 (D) –25

23. A system has a single pole at origin. Its impulse response will be

(A) constant (B) ramp
(C) decaying exponential (D) oscillating

24. An open loop system has a transfer function $\frac{1}{s(s-2)}$.

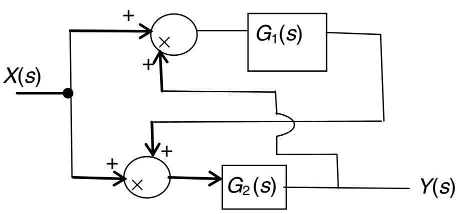
It is converted into a closed loop system by providing a negative feedback having transfer function $(2s + 1)$. The open loop and closed loop systems are, respectively,

(A) stable and unstable
(B) unstable and stable
(C) unstable and unstable
(D) unstable and marginally stable

25. The unit ramp response of a system is $1 - e^{-t}(1 + t)$. Which is this system?

(A) unstable (B) stable
(C) marginally stable (D) None of these

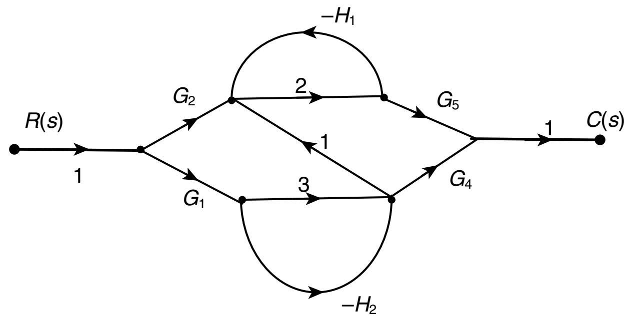
26. The transfer function $\frac{Y(s)}{X(s)}$ of the linear time-invariant system shown in the following figure



(A) $\frac{G_1(s)\,[G_2(s)+1]}{1-G_1(s).G_2(s)}$ (B) $\frac{G_2(s)\,[G_1(s)+1]}{1-G_1(s).G_2(s)}$

(C) $\frac{G_1(s)\,[G_2(s)+1]}{1-G_1(s)+G_2(s)}$ (D) $\frac{G_2(s)\,[G_1(s)+1]}{1+G_1(s).G_2(s)}$

Direction for questions 27 and 28:
For the SFG shown in the following figure:



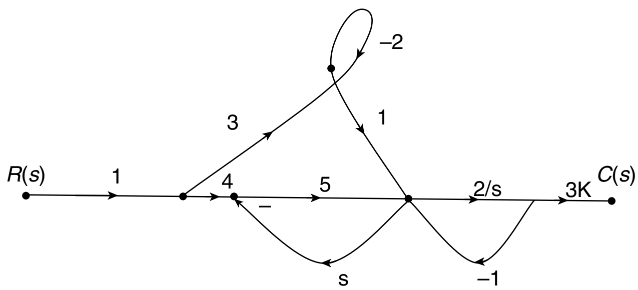
27. The number of forward path gains are

(A) 3 (B) 4 (C) 5 (D) 6

28. The number of individual loop gains are

(A) 1 (B) 2
(C) 3 (D) None of these

Direction for questions 29 and 30:



29. The transfer function $\frac{C(s)}{R(s)}$ is

(A) $\frac{378K}{15s^2+3s+6}$ (B) $\frac{360K}{5s^2+12s+360K}$

(C) $\frac{378K}{5s^2+18s+6}$ (D) None of these

30. The sensitivity of the transfer function with parameter 'K' is

(A) $\frac{378}{5s^2+13s+6}$ (B) $\frac{-378}{(s^2+13s+6)^2}$

(C) 1 (D) –1

Answer Keys

1. D	**2.** B	**3.** D	**4.** C	**5.** C	**6.** B	**7.** B	**8.** D	**9.** A	**10.** B
11. C	**12.** D	**13.** C	**14.** D	**15.** A	**16.** C	**17.** A	**18.** B	**19.** A	**20.** A
21. A	**22.** D	**23.** A	**24.** D	**25.** B	**26.** B	**27.** C	**28.** A	**29.** A	**30.** C