Exercise 11.3

Answer 1E.

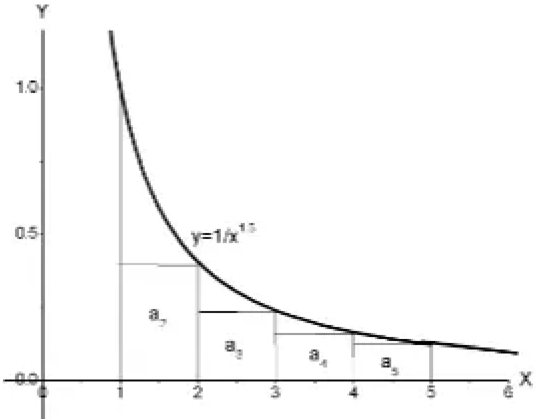


Fig.1

From figure we have

$$a_2 = \frac{1}{2^{13}} < \int_1^2 \frac{1}{x^{13}} dx$$

$$a_3 = \frac{1}{3^{13}} < \int_2^3 \frac{1}{x^{13}} dx$$

$$a_4 = \frac{1}{4^{13}} < \int_3^4 \frac{1}{x^{13}} dx$$

And so on.

Then $\sum_{n=2}^{\infty} \frac{1}{n^{13}} < \int_1^{\infty} \frac{1}{x^{13}} dx$

Since this integral converges with $p = 1.3 > 1$ so this series also converges.

Answer 2E.

Consider,

f is continuous positive decreasing function for $x \ge 1$ and $a_n = f(n)$ and

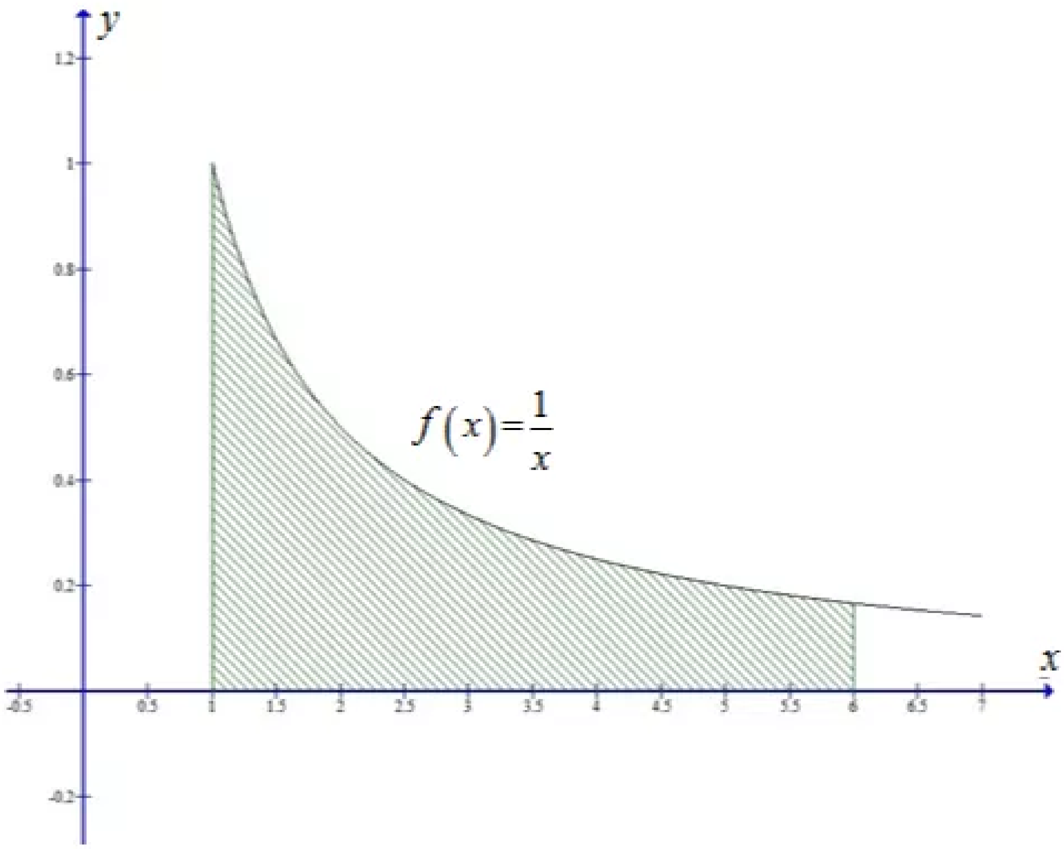
$\int_1^6 f(x)$, $\sum_{i=1}^{5} a_i, \sum_{i=2}^{6} a_i$.

Let f be a continuous positive decreasing function for $x \ge 1$. Define a sequence by $a_n = f(n)$. Let us use a picture to examine the following values:

$\int_1^6 f(x)dx$, $\sum_{i=1}^{5} a_i$, and $\sum_{i=2}^{6} a_i$.

Use a simple function like $f(x) = \frac{1}{x}$. It will be apparent from the pictures that the result will be the same for any function that meets the conditions discussed above.

Draw the picture of $\int_1^6 f(x)dx$ as follows:

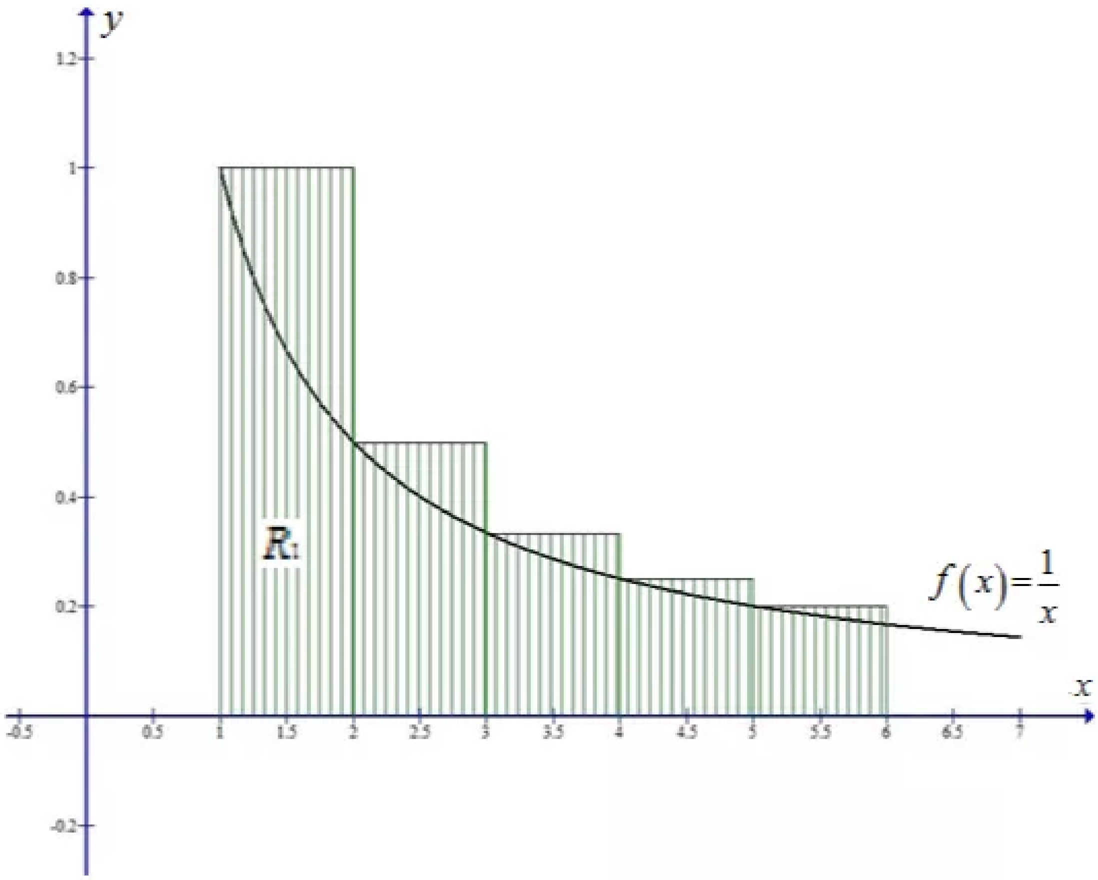


From the graph, note that,

Depict $f(x)=\frac{1}{x}$ for values $x\in[1,7]$ and shaded the region between the function and the axis for $x\in[1,6]$.

Therefore, the shaded region represents a geometric interpretation of the integral $\int_1^6 f(x)dx$.

Again the area covered by the graph of $\sum_{i=1}^{5} a_i$ is as follows:



The shading this picture represents the sum $\sum_{i=1}^{5} a_i$. Look at the rectangle labeled R_1. Its base stretches from 1 to 2. The left hand side of the rectangle stretches all the way up to the point $(1, f(1))$, so its height is $f(1)=1$.

Similarly, all the subsequent rectangles have base 1, and the heights are $f(2)$, $f(3)$, $f(4)$, $f(5)$, because, the bases are always 1.

The areas of the rectangles are $f(1), f(2), f(3), f(4), f(5)$. Since $a_n = f(n)$, the sum of the areas of the rectangles is $\sum_{i=1}^{5} a_i$.

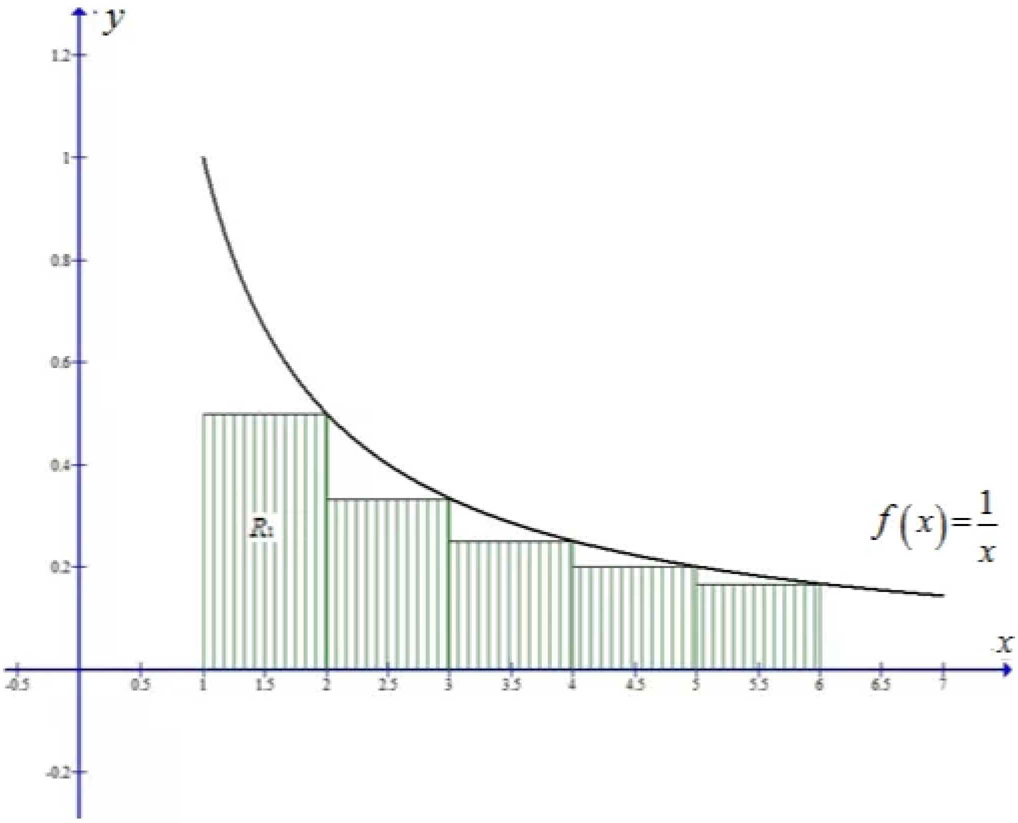
Clearly, the amount under the rectangles is bigger than the area under the curve.

The rectangles all go up and over the curve. This tells us that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$.

f is always decreasing, so in particular it is decreasing on the interval $[1,2]$. Then that means for every point z in $[1,2]$ besides 1, $f(z) < f(1)$. $f(1)$ is the height of the first rectangle, so this means that the function is always shorter than the height of the first rectangle on the first interval. But this is true of all of the intervals and all of the rectangles. So f is always shorter than the rectangles, and this is why the integral of f is smaller than the sum of the rectangles, giving another explanation of $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$.

picture and we'll do the same for $\sum_{i=2}^{6} a_i$.

Similarly the area covered by the graph of $\sum_{i=2}^{6} a_i$ is as follows:



The shading this picture represents the sum $\sum_{i=2}^{6} a_i$. The rectangle labeled R_1. Its base stretches from 1 to 2. But its height is $f(2)=\frac{1}{2}$. Similarly, all the subsequent rectangles have base 1, and the heights are $f(3), f(4), f(5), f(6)$. Since the bases are always 1, the areas of the rectangles are therefore, $f(2), f(3), f(4), f(5), f(6)$ as well.

Because $a_n = f(n)$, the sum of the areas of the rectangles is $\sum_{i=2}^{6} a_i$

The amount under these rectangles is clearly smaller than the amount under the curve, giving us that $\int_1^6 f(x)dx > \sum_{i=2}^{6} a_i$. And there is a similar reason based on the decreasing property of f in this case as well. Because in every interval making up the base of a rectangle, the value of the function decreases down to the value of the right endpoint, the function is always taller than the height of the rectangle, so the area under the function is greater than the sum of the rectangles.

Putting everything together

Therefore, $\boxed{\sum_{i=2}^{6} a_i < \int_1^6 f(x)dx < \sum_{i=1}^{5} a_i}$.

Answer 3E.

Consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}.$$

Determine whether the given series is convergent or divergent by using the Integral Test.

Let $f(x) = \frac{1}{x^{\frac{1}{5}}}$.

The graph of the function $f(x) = \frac{1}{x^{\frac{1}{5}}}$ and its derivative function $f'(x)$ is shown in the figure 1:

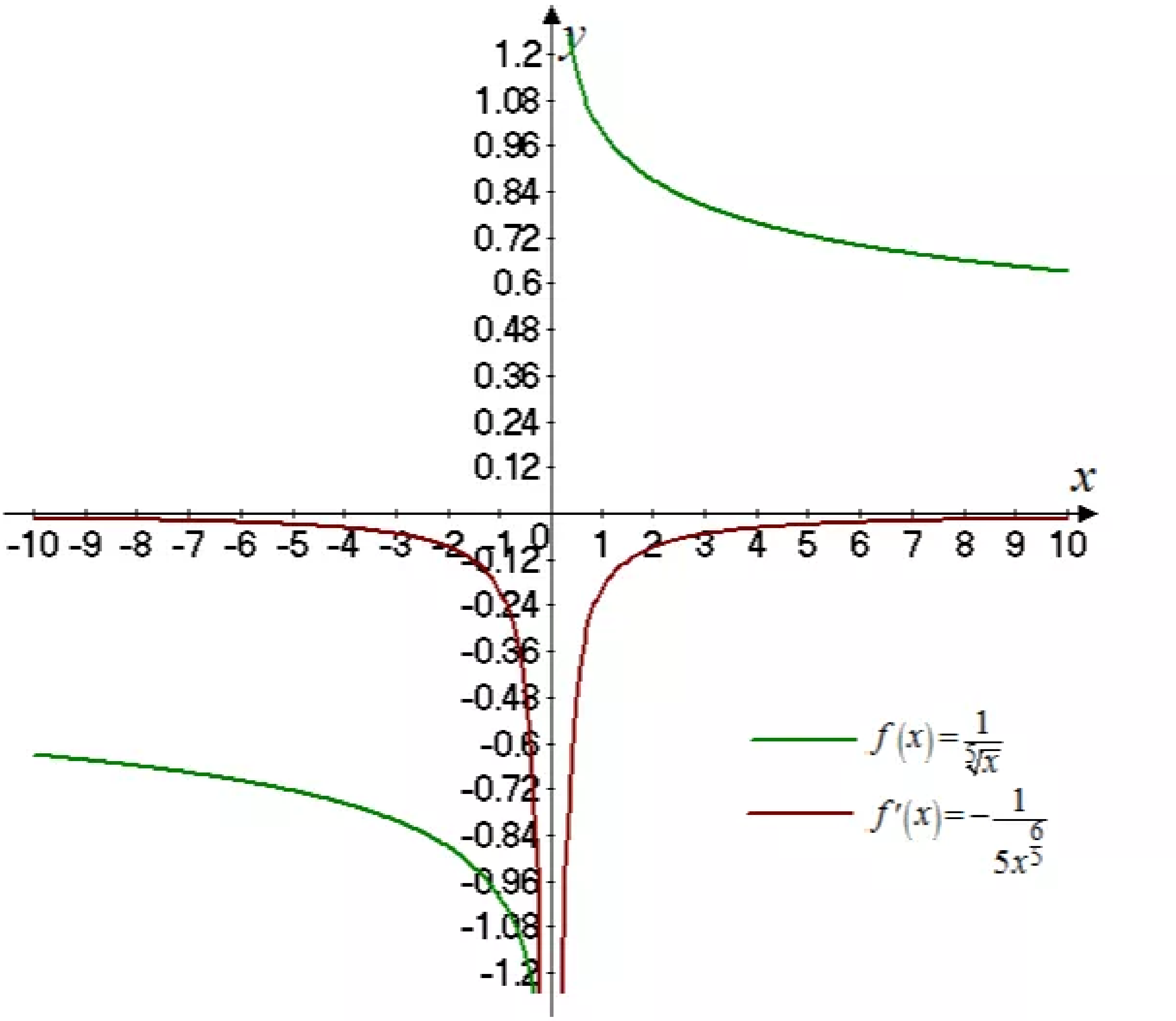


Figure 1

Verify the conditions of the Integral test:

From the graph observe that the function $f(x)=\frac{1}{x^{\frac{1}{5}}}$ is positive and continuous for $x \geq 1$ and also it is decreasing when $x>0$ since $f'(x)<0$ on the interval $(0,\infty)$.

The function $f(x)=\frac{1}{x^{\frac{1}{5}}}$ is continuous, positive, and decreasing on $[1,\infty)$ so use the Integral Test:

$$\int_1^\infty f(x)\,dx=\int_1^\infty \frac{1}{x^{\frac{1}{5}}}\,dx$$

$$=\int_1^\infty x^{-\frac{1}{5}}\,dx$$

$$=\left[\frac{x^{-\frac{1}{5}+1}}{-\frac{1}{5}+1}\right]_1^\infty \qquad \left[\text{Since } \int x^n\,dx=\frac{x^{n+1}}{n+1}+C\right]$$

$$=\left[\frac{5x^{\frac{4}{5}}}{4}\right]_1^\infty$$

$$=\frac{5(\infty)^{\frac{4}{5}}-5(1)^{\frac{4}{5}}}{4}$$

$$=\infty$$

Since the integral $\int_1^\infty f(x)\,dx$ is divergent, the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n}}$ is also divergent by the Integral Test.

Answer 4E.

1679-11.3-4E RID: 1411| 21/02/2016

Consider the series,

$$\sum_{n=1}^{\infty}\frac{1}{n^5}.$$

Use integral test to determine whether the series converges or diverges.

Integral test: suppose f is a continuous, positive, decreasing function on $[1,\infty)$.

If $\int_1^{\infty} f(x)dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

If $\int_1^{\infty} f(x)dx$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Let $f(x)=\frac{1}{x^5}$ is clearly continuous and positive for $x \geq 1$.

The derivative of the function is,

$$f(x)=\frac{-5}{x^6}$$

So f is decreasing on $[1,\infty)$.

Use integral test, the integral value of the function is,

$$\int_1^{\infty}\frac{1}{x^5}dx=\lim_{t\to\infty}\int_1^t x^{-5}dx$$

$$=\lim_{t\to\infty}\left(\frac{x^{-4}}{-4}\right)_1^t$$

$$=\frac{-1}{4}\lim_{t\to\infty}\left(t^{-4}-1\right)$$

$$=\frac{-1}{4}\left(\frac{1}{\infty}-1\right)$$

$$=\frac{1}{4}$$

So, the integral $\int_1^{\infty}\frac{1}{x^5}dx$ converges.

Hence, the series $\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges by using integral test.

Answer 5E.

Consider the series,

$$\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}$$

Use Integral test, to decide the convergence of this series.

Integral test:

If the function f is continuous, positive and decreasing on $[1,\infty)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent, where $a_n = f(n)$

Suppose, $\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3} = \sum_{n=1}^{\infty} a_n$

Then we get,

$$a_n = \frac{1}{(2n+1)^3}$$

Therefore, $f(n) = \frac{1}{(2n+1)^3}$

The function $f(n) = \frac{1}{(2n+1)^3}$ is defined for any n, so it is continuous on $[1,\infty)$

And, $f(n) = \frac{1}{(2n+1)^3} > 0 \;\forall n \in [1,\infty)$

Also, we have

$$(2(n+1)+1)^3 > (2n+1)^3$$
$$\frac{1}{(2(n+1)+1)^3} < \frac{1}{(2n+1)^3}$$
$$f(n+1) < f(n) \;\forall n < n+1$$

So the function $f(n) = \frac{1}{(2n+1)^3}$ is decreasing on $[1,\infty)$

Now consider the integral,

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{(2x+1)^3}\,dx \quad \text{Use } f(x) = \frac{1}{(2x+1)^3}$$

$$= \lim_{t\to\infty} \int_1^{t} (2x+1)^{-3}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{1}{2} \cdot \frac{(2x+1)^{-2}}{-2} \right)_1^t \qquad \text{Use } \int (ax+b)^n\,dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1}, n \neq -1$$

$$= -\frac{1}{4} \lim_{t\to\infty} \left(\frac{1}{(2t+1)^2} - \frac{1}{(2+1)^2} \right)$$

$$= -\frac{1}{4} \left(\frac{1}{\infty} - \frac{1}{9} \right)$$

$$= -\frac{1}{4} \left(0 - \frac{1}{9} \right)$$

$$= \frac{1}{36}$$

Thus $\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx$ is a convergent integral, and hence by Integral test, the series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$ is also convergent.

Answer 6E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

Use Integral test, to decide the convergence of this series.

Integral test:

If the function f is continuous, positive and decreasing on $[1, \infty)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent, where $a_n = f(n)$

Suppose, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}} = \sum_{n=1}^{\infty} a_n$

Then we get,

$$a_n = \frac{1}{\sqrt{n+4}}$$

Therefore, $f(n) = \frac{1}{\sqrt{n+4}}$

The function $f(n)=\frac{1}{\sqrt{n+4}}$ is defined for any n, so it is continuous on $[1,\infty)$

And, $f(n)=\frac{1}{\sqrt{n+4}}>0\ \forall n$

Also, we have

$$\frac{1}{\sqrt{(n+1)+4}}<\frac{1}{\sqrt{n+4}}$$
$$f(n+1)<f(n)\ \forall n<n+1$$

So the function $f(n)=\frac{1}{\sqrt{n+4}}$ is decreasing on $[1,\infty)$

Now consider the integral,

$$\int_1^\infty f(x)dx=\int_1^\infty \frac{1}{\sqrt{x+4}}dx \text{ Use } f(x)=\frac{1}{\sqrt{x+4}}$$

$$=\lim_{t\to\infty}\int_1^t (x+4)^{-\frac{1}{2}}dx$$

$$=\lim_{t\to\infty}\left(\frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right)_1^t \qquad \text{Use } \int x^n dx=\frac{x^{n+1}}{n+1}$$

$$=2\lim_{t\to\infty}\left((t+4)^{\frac{1}{2}}-(1+4)^{\frac{1}{2}}\right)$$

$$=2\lim_{t\to\infty}\left((t+4)^{\frac{1}{2}}-5^{\frac{1}{2}}\right)$$

$$=2\left(\infty-5^{\frac{1}{2}}\right)$$

$$=\infty$$

Thus $\int_1^\infty \frac{1}{\sqrt{x+4}}dx$ is a divergent integral, and hence by Integral test, the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+4}}$ is also divergent.

Answer 7E.

Given series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

The function $f(x) = \frac{x}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned}\int_1^{\infty} \frac{x}{x^2+1} dx &= \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1} dx \\ &= \lim_{t\to\infty} \frac{1}{2}\left[\log(x^2+1)\right]_1^t \\ &= \lim_{t\to\infty} \frac{1}{2}\left[\log(t^2+1) - \log 2\right] \\ &= \infty\end{aligned}$$

Thus $\int_1^{\infty} \frac{x}{x^2+1} dx$ is a divergent integral and so by the Integral Test, the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent.

Answer 8E.

Given series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

The function $f(x) = x^2 e^{-x^3}$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\begin{aligned}\int_1^{\infty} x^2 e^{-x^3} dx &= \lim_{t\to\infty} \int_1^t x^2 e^{-x^3} dx \\ &= \frac{1}{3} \lim_{t\to\infty} \int_1^t e^{-u} du \\ &\qquad (\because x^3 = u \Rightarrow 3x^2 dx = du) \\ &= -\frac{1}{3} \lim_{t\to\infty} \left[e^{-u}\right]_1^t \\ &= \frac{1}{3e}\end{aligned}$$

Thus $\int_1^{\infty} x^2 e^{-x^3} dx$ is a convergent integral and so by the Integral Test, the series $\sum_{n=1}^{\infty} n^2 e^{-x^3}$ is convergent.

Answer 9E.

Given series $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

We know the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$. since in the given series $p = \sqrt{2} > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ is convergent.

Answer 10E.

Given series $\sum_{n=1}^{\infty} n^{-0.9999}$

We know that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$. Since in the given series $p = 0.9999 < 1$, the series $\sum_{n=1}^{\infty} n^{-0.9999}$ is divergent.

Answer 11E.

We have the series $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots\ldots\ldots\ldots$

$$= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots\ldots\ldots\ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3}$$

This is a p- series, with p =3 > 1, Thus the given series converges.

Answer 12E.

We have the series $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \ldots\ldots\ldots$

$$= 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \frac{1}{4^{3/2}} + \frac{1}{5^{3/2}} + \ldots\ldots\ldots\ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This series is a p-series with $p = \frac{3}{2} > 1$, therefore the series converges.

Answer 13E.

Consider the infinite series,

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\ldots$$

Rewrite as $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\ldots=\sum_{n=1}^{\infty}\frac{1}{2n-1}$

Let us take $a_n=\frac{1}{2n-1}$

$=f(n)$.

Recollect **The Integral Test** states that let *f* be a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n=f(n)$. Then the series $\sum_{n=1}^{\infty}a_n$ is convergent if and only if the improper integral $\int_1^{\infty}f(x)dx$ is convergent. That is, if $\int_1^{\infty}f(x)dx$ is convergent, then $\sum_{n=1}^{\infty}a_n$ is convergent and If $\int_1^{\infty}f(x)dx$ is divergent, then $\sum_{n=1}^{\infty}a_n$ is divergent.

Since $\lim_{x\to 1}f(x)=\lim_{x\to 1}\frac{1}{2x-1}$

$$=\frac{1}{2\cdot 1-1}$$
$$=1$$
$$=f(1)$$

So the function $f(x)=\frac{1}{2x-1}$ is continuous on $[1,\infty)$.

Since $x\geq 1$

$$2x\geq 2$$
$$2x-1\geq 1$$
$$\frac{1}{2x-1}\leq 1$$
$$0<f(x)\leq 1$$

So the function $f(x)=\frac{1}{2x-1}$ is positive on $[1,\infty)$.

Since $x\geq 1\Rightarrow f(x)-f(1)=\frac{1}{2x-1}-1$

$$=-\left(\frac{2x-2}{2x-1}\right)$$
$$<0$$

So the function $f(x)=\frac{1}{2x-1}$ is decreasing on $[1,\infty)$.

Since the function $f(x)=\frac{1}{2x-1}$ is continuous, positive, decreasing on $[1,\infty)$ so use the integral test.

$$\int_1^\infty f(x)\,dx=\int_1^\infty \frac{1}{2x-1}\,dx$$

$$=\frac{1}{2}\lim_{t\to\infty}\int_1^t \frac{2}{2x-1}\,dx$$

$$=\frac{1}{2}\lim_{t\to\infty}\left[\ln|2x-1|\right]_1^t \qquad \text{Use } \int\frac{f'(x)}{f(x)}dx=\ln|f(x)|.$$

$$=\frac{1}{2}\lim_{t\to\infty}\left[\ln(2t-1)-\ln(1)\right]$$

$$=\frac{1}{2}\lim_{t\to\infty}\left[\ln(2t-1)-0\right]$$

$$=\frac{1}{2}\lim_{t\to\infty}\ln(2t-1)$$

$$=\frac{1}{2}\times\infty \qquad \lim_{x\to\infty}\ln x=\infty.$$

$$=\infty$$

Since $\int_1^\infty f(x)\,dx$ is divergent, by the integral test,

Therefore the series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\ldots$ is $\boxed{\text{divergent}}$.

Answer 14E.

Consider the infinite series,

$$\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}+\ldots$$

Rewrite as $\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}+\ldots=\sum_{n=1}^{\infty}\frac{1}{3n+2}$

Let us take $a_n=\frac{1}{3n+2}$

$=f(n)$.

Recollect **The Integral Test** states that let *f* be a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n=f(n)$. Then the series $\sum_{n=1}^{\infty}a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. That is, if $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty}a_n$ is convergent and If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty}a_n$ is divergent.

Since $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{3x+2}$

$= \frac{1}{3 \cdot 1 + 2}$

$= \frac{1}{5}$

$= f(1)$

So the function $f(x) = \frac{1}{3x+2}$ is continuous on $[1, \infty)$.

Since $x \geq 1$

$3x \geq 3$

$3x + 2 \geq 5$

$\frac{1}{3x+2} \leq \frac{1}{5} < 1$

$0 < f(x) < 1$

So the function $f(x) = \frac{1}{3x+2}$ is positive on $[1, \infty)$.

Since $x \geq 1 \Rightarrow f(x) - f(1) = \frac{1}{3x+2} - \frac{1}{5}$

$= -\left(\frac{3x-3}{15x+10}\right)$

< 0

So the function $f(x) = \frac{1}{3x+2}$ is decreasing on $[1, \infty)$.

The function $f(x)=\frac{1}{3x+2}$ is continuous, positive, decreasing on $[1,\infty)$ so use the integral test.

$$\int_1^\infty f(x)dx=\int_1^\infty \frac{1}{3x+2}dx$$
$$=\frac{1}{3}\lim_{t\to\infty}\int_1^t \frac{3}{3x+2}dx$$
$$=\frac{1}{3}\lim_{t\to\infty}\left[\ln|3x+2|\right]_1^t \qquad \text{Use } \int\frac{f'(x)}{f(x)}dx=\ln|f(x)|.$$
$$=\frac{1}{3}\lim_{t\to\infty}\left[\ln(3t+2)-\ln(5)\right]$$
$$=\frac{1}{3}\lim_{t\to\infty}\ln\left(\frac{3t+2}{5}\right) \qquad \ln a-\ln b=\ln\left(\frac{a}{b}\right).$$
$$=\frac{1}{3}\times\infty \qquad \lim_{x\to\infty}\ln x=\infty.$$
$$=\infty$$

Since $\int_1^\infty f(x)dx$ is divergent, by the integral test,

Therefore the series $\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}+\ldots$ is $\boxed{\text{divergent}}$.

Answer 15E.

Given series $\sum_{n=1}^{\infty}\frac{\sqrt{n}+4}{n^2}$

The function $f(x)=\frac{\sqrt{n}+4}{n^2}$ is continuous, positive and decreasing on $[1, \alpha]$ so we use the Integral Test:

$$\int_1^\alpha \frac{\sqrt{x}+4}{x^2}dx=\lim_{t\to\alpha}\int_1^t \frac{\sqrt{x}+4}{x^2}dx$$
$$=\lim_{t\to\alpha}\left[\int_1^t \frac{\sqrt{x}}{x^2}dx+\int_1^t \frac{4}{x^2}dx\right]$$
$$=\lim_{t\to\alpha}\left[\int_1^t \frac{1}{x^{3/2}}dx+\int_1^t \frac{4}{x^2}dx\right]$$
$$=\lim_{t\to\alpha}\left[\left[\frac{-2}{\sqrt{x}}\right]_1^t+\left[\frac{-4}{x}\right]_1^t\right]$$

$$= \lim_{t\to\alpha}\left[\left(\frac{-2}{\sqrt{t}}+2\right)+\left(\frac{-4}{t}+4\right)\right]$$

$$= \lim_{t\to\alpha}\left[\left(\frac{-2}{\sqrt{t}}+2\right)+\left(\frac{-4}{t}+4\right)\right]$$

$$= 6$$

Thus $\int_1^{\alpha}\frac{\sqrt{x}+4}{x^2}dx$ is a convergent integral and so, by the Integral Test, the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}+4}{n^2}$ is convergent.

Answer 16E.

Consider the infinite series,

$$\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$$

Let us take $a_n = \frac{n^2}{n^3+1}$

$= f(n)$.

Recollect **The Integral Test** states that let *f* be a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. That is, if $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent and If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Since $\lim_{x\to1} f(x) = \lim_{x\to1}\frac{x^2}{x^3+1}$

$= \frac{1}{1+1}$

$= \frac{1}{2}$

$= f(1)$

So the function $f(x) = \frac{x^2}{x^3+1}$ is continuous on $[1,\infty)$.

Since $x \ge 1$

$x^3 \ge 1$

$x^3+1 \ge 2$

$\frac{x^2}{x^3+1} \le \frac{x^2}{2}$

$0 < f(x)$

So the function $f(x) = \frac{x^2}{x^3+1}$ is positive on $[1,\infty)$.

Since $x \ge 1 \Rightarrow f(x) - f(1) = \frac{x^2}{x^3+1} - \frac{1}{2}$

$= -\left(\frac{x^3-2x^2+1}{x^3+1}\right)$

< 0

So the function $f(x) = \frac{x^2}{x^3+1}$ is decreasing on $[1,\infty)$.

The function $f(x)=\frac{x^2}{x^3+1}$ is continuous, positive, decreasing on $[1,\infty)$ so use the integral test.

$$\int_1^\infty f(x)\,dx=\int_1^\infty \frac{x^2}{x^3+1}\,dx$$
$$=\frac{1}{3}\lim_{t\to\infty}\int_1^t \frac{3x^2}{x^3+1}\,dx$$
$$=\frac{1}{3}\lim_{t\to\infty}\left[\ln\left(x^3+1\right)\right]_1^t \qquad \text{Use } \int\frac{f'(x)}{f(x)}dx=\ln|f(x)|.$$
$$=\frac{1}{3}\lim_{t\to\infty}\left[\ln\left(t^3+1\right)-\ln(2)\right]$$
$$=\frac{1}{3}\lim_{t\to\infty}\ln\left(\frac{t^3+1}{2}\right) \qquad \ln a-\ln b=\ln\left(\frac{a}{b}\right).$$
$$=\frac{1}{3}\times\infty \qquad \lim_{x\to\infty}\ln x=\infty.$$
$$=\infty$$

Since $\int_1^\infty f(x)\,dx$ is divergent, by the integral test,

Therefore the series $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$ is $\boxed{\text{divergent}}$.

Answer 17E.

Cauchy's integral test:

Suppose $f(k)=a_k$, where f is continuous, positive, decreasing function for $x\geq n$ then $\sum_1^\infty a_n$ is convergent if the improper integral $\int_1^\infty f(x)\,dx$ is convergent and $\sum_1^\infty a_n$ is divergent if the improper integral $\int_1^\infty f(x)\,dx$ is divergent.

We have the series $\sum_{n=1}^{\infty}\frac{1}{n^2+4}$

The function $f(x)=\frac{1}{x^2+4}$ is continuous, positive and decreasing on $[1,\infty)$

So we use the integral test:

$$\int_1^{\infty}\frac{1}{x^2+4}dx=\lim_{t\to\infty}\int_1^{t}\frac{1}{x^2+4}dx$$

$$=\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^t \qquad \left(\text{since } \int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$=\frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{2}-\tan^{-1}\frac{1}{2}\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{2}\right)\right]$$

Since the improper integral is convergent, by using Cauchy's integral test

The given series $\sum\frac{1}{n^2+4}$ is also $\boxed{\text{convergent}}$.

Answer 18E.

Given series $\sum_{n=3}^{\infty}\frac{3n-4}{n^2-2n}$

The function $f(x)=\frac{3x-4}{x^2-2x}$ is continuous, positive and decreasing an $[3,\infty]$ so we use the Integral Test:

$$\int_3^{\infty}\frac{3x-4}{x^2-2x}dx=\lim_{t\to\infty}\int_3^{t}\frac{3x-4}{x^2-2x}dx$$

$$=\lim_{t\to\infty}\int_3^{t}\frac{3x-4}{x(x-2)}dx$$

$$=\lim_{t\to\infty}\int_3^{t}\frac{2}{x}+\frac{1}{x-2}dx$$

$$=\lim_{t\to\infty}\left[2\log x+1og(x-2)\right]_3^t$$

$$=\lim_{t\to\infty}\left[\log(x^2(x-2)\right]_3^t$$

$$=\lim_{t\to\infty}\left[\log(t^2(t-2)-\log 9\right]$$

$$=\infty$$

Thus $\int_3^{\infty}\frac{3x-4}{x(x-2)}dx$ is a divergent integral and so, by the Integral Test, the series $\sum_{n=3}^{\infty}\frac{3n-4}{n^2-2n}$ is divergent.

Answer 19E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}.$$

The object is to determine whether the series converge or diverge.

Use integral series test, if there exist a $N \geq k$ so that for all $N \geq n$. $f(n) = a_n$ is positive, continuous and decreasing. Then, $\sum_{n=k}^{\infty} a_n$ and $\int_k^{\infty} f(x)\,dx$ either converge or diverge.

Let $f(n) = \frac{\ln(n)}{n^3}$.

Since $f(n)$ is positive, continuous and decreasing from $n = 2$.

Compute the indefinite integral $\int \frac{\ln(n)}{n^3} dn$ as follows:

Apply integration by parts: $\int uv' = uv - \int u'v$

$$u = \ln(n), u' = \frac{1}{n}, v' = \frac{1}{n^3}, v = \frac{-1}{2n^2}.$$

So the integral can be evaluated as,

$$\begin{aligned}\int \frac{\ln(n)}{n^3} dn &= \ln(n)\left(\frac{-1}{2n^2}\right) - \int \frac{1}{n}\left(\frac{-1}{2n^2}\right) dn \\ &= \frac{-\ln(n)}{2n^2} + \frac{1}{2}\int \frac{1}{n^3} dn \\ &= \frac{-\ln(n)}{2n^2} + \frac{1}{2}\left(\frac{-1}{2n^2}\right) \\ &= \frac{-\ln(n)}{2n^2} - \frac{1}{4n^2}\end{aligned}$$

The definite integral is calculated as,

$$\begin{aligned}\int_2^{\infty} \frac{\ln(n)}{n^3} dn &= \left[\frac{-\ln(n)}{2n^2} - \frac{1}{4n^2}\right]_2^{\infty} \\ &= 0 - \left(\frac{-\ln(2)}{2(2)^2} - \frac{1}{4(2)^2}\right) \\ &= \frac{1}{16} + \frac{1}{8}\ln(2) \\ &= \frac{1}{16}(1 + 2\ln(2))\end{aligned}$$

$$= \frac{1}{16}(1 + \ln(4))$$

Since the integral is exist.

Hence, the series converges by series integral test.

Answer 20E.

To determine whether the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$$

is convergent or divergent use integral test.

The Integral Test:

Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$.

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral

$\int_1^{\infty} f(x)\,dx$ is convergent.

Let $f(x)=\frac{1}{x^2+6x+13}$

The function

$$f(x)=\frac{1}{x^2+6x+13}$$

is continuous, positive and decreasing on $[1,\infty)$ so we use the Integral Test.

Consider the improper integral

$$\begin{aligned}\int_1^{\infty}\frac{1}{x^2+6x+13}dx &= \lim_{t\to\infty}\int_1^{t}\frac{1}{x^2+6x+13}dx\\ &= \lim_{t\to\infty}\int_1^{t}\frac{1}{x^2+2\cdot3\cdot x+9+4}dx\\ &= \lim_{t\to\infty}\int_1^{t}\frac{1}{x^2+2\cdot3\cdot x+3^2+2^2}dx\\ &= \lim_{t\to\infty}\int_1^{t}\frac{1}{(x+3)^2+2^2}dx\end{aligned}$$

Continuation to the above

$$\begin{aligned}\int_1^{\infty}\frac{1}{x^2+6x+13}dx &= \lim_{t\to\infty}\int_1^{t}\frac{1}{(x+3)^2+2^2}dx\\ &= \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)\right]_1^t \quad \text{since } \int\frac{1}{(x+a)^2+b^2}dx=\frac{1}{b}\tan^{-1}\left(\frac{x+a}{b}\right)\\ &= \frac{1}{2}\left\{\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t+3}{2}\right)-\tan^{-1}\left(\frac{1+3}{2}\right)\right]\right\}\end{aligned}$$

Continuation to the above

$$\int_1^\infty \frac{1}{x^2+6x+13}dx = \frac{1}{2}\left\{\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t+3}{2}\right)-\tan^{-1}\left(\frac{1+3}{2}\right)\right]\right\}$$
$$= \frac{1}{2}\left\{\lim_{t\to\infty}\tan^{-1}\left(\frac{t+3}{2}\right)-\lim_{t\to\infty}\tan^{-1}\left(\frac{4}{2}\right)\right\}$$
$$= \frac{1}{2}\left\{\tan^{-1}(\infty)-\tan^{-1}(2)\right\}$$
$$= \frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}(2)\right)$$

Thus $\int_1^\infty \frac{1}{x^2+6x+13}dx$ is a convergent integral and so, by the **Integral Test**the series $\sum_{n=1}^\infty \frac{1}{n^2+6n+13}$ is $\boxed{\text{convergent}}$

Answer 21E.

We have the series $\sum_{n=2}^\infty \frac{1}{n\ln n}$

The function $f(x)=\frac{1}{x\ln x}$ is positive and continuous for $x \ge 2$

$$f'(x)=\frac{(x\ln x).(0)-1\left[x\frac{1}{x}+\ln x\right]}{(x\ln x)^2}$$
$$= \frac{-(1+\ln x)}{(x\ln x)^2}<0 \qquad [\text{ since } \ln x>0 \text{ for } x\ge 2]$$

So f(x) is an decreasing function

So we can use integral test:

$$\int_2^\infty \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}dx$$

Substitute $\ln x = u \Rightarrow \frac{1}{x}dx = du$

Therefore $\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln|u| = \ln|\ln x|$

And so $\int_2^\infty \frac{1}{x\ln x}dx = \lim_{t\to\infty}\left[\ln|\ln x|\right]_2^t$
$$= \lim_{t\to\infty}\left[\ln(\ln t)-\ln(\ln 2)\right]$$
$$= \infty$$

Since the improper integral diverges, the series $\sum_{n=2}^\infty \frac{1}{n\ln n}$ also $\boxed{\text{diverges}}$.

Answer 22E.

Consider the following series:

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}.$$

Test the convergence of the given series.

Let, $a_n = \frac{1}{n(\ln n)^2}$.

Verify the conditions of the Integral test:

Condition (i):

Check whether the function, $f(x)$ is decreasing or not.

Let, $f(x) = \frac{1}{x(\ln x)^2}$.

The first derivative of the function $f(x)$ is, $f'(x) = \frac{-2-\ln(x)}{x^2(\ln x)^3}$.

Clearly, $f'(x) < 0$ on the intervals $\left(0, \frac{1}{e^2}\right)$ and $(1, \infty)$.

Thus, the function $f(x)$ is decreasing on the intervals $\left(0, \frac{1}{e^2}\right)$ and $(1, \infty)$.

This means that, $f(n+1) < f(n)$.

Condition (ii):

Check the continuity of the function, $f(x)$.

Clearly, the function $f(x)$ is a continuous positive function, because the logarithm function is continuous.

The function $f(x)=\frac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $(1,\infty)$, so use the Integral Test:

$$\int_2^\infty f(x)dx=\int_2^\infty \frac{1}{x(\ln x)^2}dx$$
$$=\left[-\frac{1}{\ln x}\right]_2^\infty$$
$$=-\frac{1}{\ln\infty}+\frac{1}{\ln 2}$$
$$=-\frac{1}{\infty}+\frac{1}{\ln 2}$$
$$=-0+\frac{1}{\ln 2}$$
$$=\frac{1}{\ln 2}$$

Since the integral $\int_2^\infty f(x)dx$ is convergent, the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ is also convergent by the Integral Test.

Answer 23E.

Consider the series,

$$\sum_{n=1}^\infty \frac{e^{\frac{1}{n}}}{n^2}.$$

The object is to determine whether the series converge or diverge.

Use integral series test, if there exist a $N\ge k$ so that for all $N\ge n$, $f(n)=a_n$ is positive, continuous and decreasing. Then, $\sum_{n=k}^\infty a_n$ and $\int_k^\infty f(x)dx$ either converge or diverge.

Let $f(n)=\frac{e^{\frac{1}{n}}}{n^2}$.

Since $f(n)$ is positive, continuous and decreasing from $n=1$.

Compute the indefinite integral is as follows:

$$\int \frac{e^{\frac{1}{n}}}{n^2}dn$$

Use substitution method: Let $\frac{1}{n}=t$ then $\frac{-1}{n^2}dn=dt$

So the integral can be calculated as,

$$\int \frac{e^{\frac{1}{n}}}{n^2}dn=-\int e^t dt$$
$$=-e^t+C$$
$$=-e^{\frac{1}{n}}+C$$

Now the definite integral can be evaluated as,

$$\int_1^\infty \frac{e^{\frac{1}{n}}}{n^2}dn=\left(-e^{\frac{1}{n}}\right)_1^\infty$$
$$=-\left(e^{\frac{1}{\infty}}-e^1\right)$$
$$=e-1$$

Since the integral is exist.

Hence, the series converges by series integral test.

Answer 24E.

Consider the following series:

$$\sum_{n=3}^{\infty}\frac{n^2}{e^n}$$

Determine whether the series is convergent or divergent:

Recollect that **The Integral Test** states that let *f* be a continuous, positive, decreasing function on $[a,\infty)$ and let $a_n=f(n)$. Then the series $\sum_{n=1}^{\infty}a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)dx$ is convergent.

If $\int_1^\infty f(x)dx$ is convergent, then $\sum_{n=1}^{\infty}a_n$ is convergent and If $\int_1^\infty f(x)dx$ is divergent,

Then $\sum_{n=1}^{\infty}a_n$ is divergent

Take $a_n = \frac{n^2}{e^n}$

$= f(n)$.

Since $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2}{e^x}$

$$= \frac{3^2}{e^3}$$
$$= \frac{9}{e^3}$$
$$= f(3)$$

So, the function $f(x) = \frac{x^2}{e^x}$ is continuous on $[3, \infty)$.

Take $x \geq 3$

$$x^2 \geq 9$$
$$\frac{x^2}{e^x} \leq \frac{9}{e^x}$$
$$0 < f(x)$$

So, the function $f(x) = \frac{x^2}{e^x}$ is positive on $[3, \infty)$

Take $x \geq 3 \Rightarrow f(x) - f(3) = \frac{x^2}{e^x} - \frac{9}{e^3}$

< 0

So, the function $f(x) = \frac{x^2}{e^x}$ is decreasing on $[3, \infty)$

The function $f(x)=\frac{x^2}{e^x}$ is continuous, positive, decreasing on $[3,\infty)$

So, use the integral test and solve as follows:

$$\int_3^\infty f(x)\,dx=\int_3^\infty \frac{x^2}{e^x}dx$$

$$=\lim_{t\to\infty}\int_3^t \frac{x^2}{e^x}dx$$

$$\int_3^\infty x^2e^{-x}dx=\lim_{t\to\infty}\int_3^t x^2e^{-x}dx$$

$$=\lim_{t\to\infty}\left[x^2\frac{e^{-x}}{-1}\right]_3^t-\lim_{t\to\infty}\int_3^t 2x\left(\frac{e^{-x}}{-1}\right)dx$$

Since $\int(u(x)\cdot v(x))dx=u(x)\int v(x)dx-\int\frac{d}{dx}u(x)\cdot v(x)dx+C$

$$=\lim_{t\to\infty}\left[-x^2e^{-x}\right]_3^t+2\lim_{t\to\infty}\int_3^t xe^{-x}dx$$

$$=\lim_{t\to\infty}\left[-x^2e^{-x}\right]_3^t+2\lim_{t\to\infty}\left[\left[x\frac{e^{-x}}{-1}\right]_3^t-\lim_{t\to\infty}\int_3^t 1\cdot\left(\frac{e^{-x}}{-1}\right)dx\right]$$

Since $\int(u(x)\cdot v(x))dx=u(x)\int v(x)dx-\int\frac{d}{dx}u(x)\cdot v(x)dx+C$

$$=\lim_{t\to\infty}\left[-x^2e^{-x}\right]_3^t+2\lim_{t\to\infty}\left[\left[-xe^{-x}\right]_3^t+\lim_{t\to\infty}\int_3^t e^{-x}dx\right]$$

$$=\lim_{t\to\infty}\left[-x^2e^{-x}\right]_3^t+2\lim_{t\to\infty}\left[\left[-xe^{-x}\right]_3^t-\lim_{t\to\infty}\left(e^{-x}\right)_3^t\right]$$

$$=\lim_{t\to\infty}\left[-x^2e^{-x}\right]_3^t+2\lim_{t\to\infty}\left[-xe^{-x}\right]_3^t-2\lim_{t\to\infty}\left[e^{-x}\right]_3^t$$

$$=\lim_{t\to\infty}\left[-t^2e^{-t}+3^2e^{-3}\right]+2\lim_{t\to\infty}\left[-te^{-t}+3e^{-3}\right]-2\lim_{t\to\infty}\left[e^{-t}-e^{-3}\right]$$

$$=\frac{9}{e^3}+2\cdot\frac{3}{e^3}-2\cdot\left(-\frac{1}{e^3}\right)$$

$$=\frac{9}{e^3}+\frac{6}{e^3}+\frac{2}{e^3}$$

$$=\frac{17}{e^3}$$

Since the improper integral converges.

Thus, $\int_3^\infty f(x)\,dx$ is convergent, by the following integral test.

Therefore, the series $\sum_{n=3}^\infty\frac{n^2}{e^n}$ is $\boxed{\text{convergent}}$.

Answer 25E.

Consider the series,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n^3}.$$

Need to check whether the series converge or diverge.

Use comparison test: Let $\sum a_n, \sum b_n$ be two positive sequences such that for all n, $a_n \le b_n$

If $\sum b_n$ converges, so does $\sum a_n$. If $\sum a_n$ diverges, so does $\sum b_n$.

Rewrite the series as,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n^3} \le \sum_{n=1}^{\infty}\frac{1}{n^3}$$

Recall that, if the series is of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$, where $p>0$.

If $p>1$, then the p-series converges. If $0<p\le 1$, then the p-series diverges.

Compare $\sum_{n=1}^{\infty}\frac{1}{n^3}$ with $\sum_{n=1}^{\infty}\frac{1}{n^p}$, then

$$p=3>1$$

By the p-series test the series converges.

Since the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ converges then by the comparison test $\sum_{n=1}^{\infty}\frac{1}{n^2+n^3}$ also converges.

Hence, the series $\sum_{n=1}^{\infty}\frac{1}{n^2+n^3}$ converges by the comparison test.

Answer 26E.

We have the series $\sum_{n=1}^{\infty}\frac{n}{n^4+1}$

The function $f(x)=\frac{x}{x^4+1}$ is positive and continuous on $[1,\infty)$

And $$f'(x)=\frac{(x^4+1).1-x(4x^3)}{(x^4+1)^2}$$

$$=\frac{x^4+1-4x^4}{(x^4+1)^2}$$

$$=\frac{1-3x^4}{(x^4+1)^2} < 0 \qquad \text{whenever } x>1$$

Thus f is decreasing function. So we can use the integral test:

We have $\int_1^{\infty} \frac{x}{x^4+1} dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1} dx$

Substitute $x^2 = u \Rightarrow 2xdx = du$

Then $\int \frac{x}{x^4+1} dx = \frac{1}{2} \int \frac{du}{u^2+1}$

$= \frac{1}{2} \tan^{-1} u + C$

$= \frac{1}{2} \tan^{-1} x^2 + C$

Thus $\int_1^{\infty} \frac{x}{x^4+1} dx = \lim_{t\to\infty} \left[\frac{1}{2} \tan^{-1} x^2\right]_1^t$

$= \frac{1}{2} \lim_{t\to\infty} \left[\tan^{-1} t^2 - \tan^{-1} 1\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$

$= \frac{1}{2}\left[\frac{\pi}{4}\right] = \frac{\pi}{8}$

Since improper integral converges. Then series $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ also $\boxed{\text{converges}}$.

Answer 27E.

Integral Test cannot be used to determine the convergence of the seriers $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$,

because $f(n) = \frac{\cos \pi n}{\sqrt{n}}$ is not positive on $[1, \infty]$ as $f(1) = \cos \pi = -1 < 0$.

Answer 28E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$$

The comparison test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

2. If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

test to determine the convergence or divergence of the following series,

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$$

The numerator is $\cos^2(n)$, which is always bounded in absolute value by 1 because $-1 \le \cos(n) \le 1$. This gives us the following estimate for the size of the series

$$\left|\frac{\cos^2(n)}{n^2+1}\right| = \left|\cos(n)\cdot\cos(n)\cdot\frac{1}{n^2+1}\right| \qquad \text{Rearrenging the components.}$$

$$\le |\cos(n)|\cdot|\cos(n)|\cdot\left|\frac{1}{n^2+1}\right| \qquad \text{Use } |ab| \le |a||b|.$$

$$\le \left|\frac{1}{n^2+1}\right| \qquad \text{Because } -1 \le \cos(n) \le 1.$$

$$= \frac{1}{n^2+1}$$

So, the terms of our series are smaller in magnitude than $\frac{1}{n^2+1}$.

This series, $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ looks similar to the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. In fact, it is smaller:

$n^2+1 > n^2$

$\frac{1}{n^2+1} < \frac{1}{n^2}$

$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ are series with positive terms.

$\frac{1}{n^2+1} < \frac{1}{n^2}$ for all n, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent series because it is a p-series with $p = 2 > 1$.

So, by the Comparison Test, $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ is a convergent series.

Apply the comparison test.

$\left|\frac{\cos^2(n)}{n^2+1}\right| \le \frac{1}{n^2+1}$ and both $\sum_{n=1}^{\infty}\left|\frac{\cos^2(n)}{n^2+1}\right|$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ are series of positive terms, the fact that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ tells us that the series of absolute values $\sum_{n=1}^{\infty}\left|\frac{\cos^2(n)}{n^2+1}\right|$ converges by the Comparison Test.

Consider,

$$-\left|\frac{\cos^2(n)}{n^2+1}\right| \le \frac{\cos^2(n)}{n^2+1} \le \left|\frac{\cos^2(n)}{n^2+1}\right|$$

$$0 \le \frac{\cos^2(n)}{n^2+1} + \left|\frac{\cos^2(n)}{n^2+1}\right| \le 2\left|\frac{\cos^2(n)}{n^2+1}\right|$$

So, the series $\sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2+1} + \left|\frac{\cos^2(n)}{n^2+1}\right|$ is a series of nonnegative terms.

Further, its terms are all bounded by the terms of the convergent series of nonnegative terms $\sum_{n=1}^{\infty}2\left|\frac{\cos^2(n)}{n^2+1}\right| = 2\sum_{n=1}^{\infty}\left|\frac{\cos^2(n)}{n^2+1}\right|$ by the above inequality.

By the Comparison Test, $\sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2+1} + \left|\frac{\cos^2(n)}{n^2+1}\right|$ converges.

The series $\sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2+1} + \left|\frac{\cos^2(n)}{n^2+1}\right|$ and $\sum_{n=1}^{\infty}\left|\frac{\cos^2(n)}{n^2+1}\right|$ are both convergent series, their difference $\sum_{n=1}^{\infty}\left(\frac{\cos^2(n)}{n^2+1} + \left|\frac{\cos^2(n)}{n^2+1}\right|\right) - \left|\frac{\cos^2(n)}{n^2+1}\right|$ is convergent.

The difference is,

$$\sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2+1} = \sum_{n=1}^{\infty}\left(\frac{\cos^2(n)}{n^2+1} + \left|\frac{\cos^2(n)}{n^2+1}\right|\right) - \left|\frac{\cos^2(n)}{n^2+1}\right|.$$

Therefore, the series $\boxed{\sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2+1} \text{ is convergent.}}$

Answer 29E.

We have the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

The function $f(x) = \frac{1}{x(\ln x)^p}$ *is* positive and continuous . This function is decreasing also.

Since $f'(x) = \left(\frac{1}{x}\right)(-p)(\ln x)^{-p-1}\left(\frac{1}{x}\right) + (-1)x^{-2}(\ln x)^{-p}$

$$= -x^{-2}p(\ln x)^{-p-1} - x^{-2}(\ln x)^{-p}$$

$$= -\frac{p+\ln x}{x^2(\ln x)^{p+1}} < 0 \qquad \text{for } x \geq 2$$

So we can use the Integral test,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^p} dx$$

Now substitute $\ln x = u \Rightarrow \frac{1}{x} dx = du$

$$\int \frac{1}{x(\ln x)^p} dx = \int \frac{1}{u^p} du \quad \Rightarrow \frac{u^{-p+1}}{-p+1} = \frac{(\ln x)^{-p+1}}{-p+1}$$

Thus $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \lim_{t\to\infty} \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^t$

$$= \frac{1}{1-p} \lim_{t\to\infty} \left[(\ln t)^{1-p} - (\ln 2)^{1-p}\right]$$

This improper integral converges when (1-p) <0

By the Integral test, the given series also converges when 1-p<0 or p>1

Answer 30E.

We have to find p such that the series $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^p}$ is convergent.

For p =1, $f(x) = \frac{1}{x \ln x [\ln(\ln x)]}$

Which is continuous and positive on $[3, \infty)$ and since ln x is a increasing function on $[3, \infty)$ then x ln x $[\ln(\ln x) a]$ is also increasing ,therefore f (x) is decreasing function

So we can use integral test

$$\int_3^\infty \frac{1}{x\ln x\ln(\ln x)}dx = \lim_{t\to\infty}\int_3^t \frac{1}{x\ln x\ln(\ln x)}dx$$

Let $\ln(\ln x) = y$ then $\frac{1}{x\ln x}dx = dy$

Then $\int \frac{1}{x\ln x\ln(\ln x)}dx = \int \frac{1}{y}dy$

$$= \ln|y| + c$$

$$= \ln(\ln(\ln x)) + c$$

Then $\lim_{t\to\infty}\int_3^t \frac{1}{x\ln x\ln(\ln x)}dx = \lim_{t\to\infty}\left[\ln(\ln(\ln x))\right]_3^t$

$= \infty$ since $\ln x \to \infty$ as $x \to \infty$

So for $p = 1$, given series is divergent

Now we take $p \neq 1$

Then $f(x) = \frac{1}{x\ln x[\ln(\ln x)]^p}$ is continuous, positive and decreasing

So we can use integral test

$$\int_3^\infty \frac{1}{x\ln x[\ln(\ln x)]^p} = \lim_{t\to\infty}\int_3^t \frac{1}{x\ln x\left[\ln(\ln x)^p\right]}dx$$

Then $\int \frac{1}{x\ln x[\ln(\ln x)]^p}dx = \int \frac{1}{y^p}dy = \frac{y^{-p+1}}{-p+1} + c$

$$= \frac{[\ln(\ln x)]^{1-p}}{1-p} + c$$

Then $\lim_{t\to\infty}\int_3^t \frac{1}{x\ln x[\ln(\ln x)]^p}dx = \lim_{t\to\infty}\left[\frac{[\ln(\ln x)]^{1-p}}{1-p}\right]_3^t$

$$= \lim_{t\to\infty}\left[\frac{[\ln(\ln t)]^{1-p}}{1-p} - \frac{[\ln(\ln 3)]^{1-p}}{1-p}\right]$$

This limit exists whenever $1-p < 0 \Leftrightarrow p > 1$

So the series converges for $\boxed{p > 1}$

Answer 31E.

We have to find p so that $\sum_{n=1}^\infty n(1+n^2)^p$ converges

We see that this cannot converge if $p \geq -1/2$

Since $\lim_{n\to\infty} n(1+n^2)^p \neq 0$

So we have to take $p < -1/2$

Then $f(x) = x(1+x^2)^p$ is continuous and positive

$$f'(x) = px(1+x^2)^{p-1}(2x) + (1+x^2)^p$$

$$\Rightarrow f^I(x) = (1+x^2)^{p-1}(2px^2 + 1 + x^2)$$

$$= \frac{(1+(1+2p)x^2)}{(1+x^2)^{1-p}} \quad \text{this is negative for } x > 1$$

So f (x) is decreasing on $[1, \infty)$

We can use integral test.

So $$\int_1^\infty x(1+x^2)^p\, dx = \lim_{t\to\infty} \int_1^t x(1+x^2)^p\, dx$$

Let $(1+x^2) = y \quad \Rightarrow \quad 2xdx = dy$

$$\Rightarrow xdx = \frac{1}{2}dy$$

Then $$\int x(1+x^2)^p\, dx = \frac{1}{2}\int y^p dy$$

$$= \frac{1}{2}\frac{y^{p+1}}{(p+1)} + c$$

$$= \frac{1}{2}\frac{(1+x^2)^{p+1}}{(p+1)} + c$$

Then we have

$$\lim_{t\to\infty} \int_1^t x(1+x^2)^p\, dx = \frac{1}{2}\lim_{t\to\infty}\left[\frac{(1+x^2)^{p+1}}{(p+1)}\right]_1^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\frac{(1+t^2)^{p+1}}{(p+1)} - \frac{1}{2}\frac{2^{p+1}}{(p+1)}$$

$$= \frac{1}{2}\lim_{t\to\infty}\frac{(1+t^2)^{p+1}}{(p+1)} - \frac{2^p}{(p+1)}$$

This limit is finite and exists when $(p+1) < 0$

$$\Rightarrow \boxed{p < -1}$$

Thus the given series is convergent when $\boxed{p < -1}$

Answer 32E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}.$$

According to the comparison test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

For $n \ge 3$, $\ln n > 1$. So for $n \ge 3$ $\frac{\ln n}{n^p} > \frac{1}{n^p}$.

Apply the Comparison Test, since only the eventual behavior of the series determines convergence rather than the first several terms only.

The $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$and divergent if $p \le 1.$

Consider the case where $p \le 1$. The series $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ are series of positive terms.

As the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a p series with $p \le 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$is convergent.

The terms of $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ are bigger than the terms of $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $\frac{\ln n}{n^p} > \frac{1}{n^p}$ for $n \ge 3$, $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ must also diverge.

So, the series diverges for $p \le 1$.

Consider the case where $p > 1$.

If $f(x) = \frac{\ln x}{x^p}$, $\sum_{n=1}^{\infty} \frac{\ln n}{n^p} = f(n)$ Also, f is eventually decreasing, so use the integral test. We demonstrate that f is eventually decreasing by taking its derivative with the quotient rule:

$$\begin{aligned} f'(x) &= \frac{x^p \frac{1}{x} - \ln x \cdot px^{p-1}}{x^{2p}} \\ &= \frac{x^p \cdot x^{-1} - \ln x \cdot px^{p-1}}{x^{2p}} \qquad \text{Apply } a^{-m} = \frac{1}{a^m}. \\ &= \frac{x^{p-1} - \ln x \cdot px^{p-1}}{x^{2p}} \\ &= \frac{x^{p-1}(1 - p\ln x)}{x^{2p}} \end{aligned}$$

The factors x^{p-1} and x^{2p} are positive for positive values of x. Also, $p > 1$, and we again note that for $x \geq 3$, $\ln n > 1$. So for $x \geq 3$, $p \ln x > 1$, giving that $0 < 1 - p \ln x$. Then for $x \geq 3$, $\frac{x^{p-1}(1 - p\ln x)}{x^{2p}}$ is the product of one negative and two positive factors, therefore negative.

The integral test still applies because it is the eventual behavior of the integral and series that matters for convergence, not its value from $1 \leq x < 3$. So f has a decreasing property that allows us to use the integral test.

According to the integral test:

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

1. If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2. If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Compute the integral $\int_1^{\infty} f(x)dx$. First compute the indefinite integral:

$$\int f(x)dx = \int \frac{\ln x}{x^p} dx$$

Use integration by parts to find this integral.

Let $u = \ln x$, $dv = x^{-p}dx$

Then, $du = \frac{1}{x}dx$, $v = \frac{1}{-p+1}x^{-p+1}$

Note that it's important that $p>1$, so $-p+1\neq 0$. By the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{\ln x}{x^p}dx = \frac{\ln x}{(-p+1)}x^{-p+1} - \int \frac{1}{-p+1}x^{-p+1}\frac{1}{x}dx$$

$$= \frac{\ln x}{(-p+1)}x^{-p+1} - \frac{1}{-p+1}\int x^{-p}dx$$

$$= \frac{\ln x}{(-p+1)}x^{-p+1} - \frac{1}{(-p+1)^2}x^{-p+1}$$

$$= \frac{\ln x - \frac{1}{-p+1}}{(-p+1)x^{p-1}}$$

Take the improper integral.

$$\int_1^\infty \frac{\ln x}{x^p}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^p}dx$$

$$= \lim_{t\to\infty}\left[\frac{\ln x - \frac{1}{-p+1}}{(-p+1)x^{p-1}}\right]_1^t$$

$$= \lim_{t\to\infty}\frac{\ln t - \frac{1}{-p+1}}{(-p+1)t^{p-1}} + \frac{1}{(-p+1)^2}$$

The term, $\ln t \to \infty$ and since $p>1$, $t^{p-1}\to\infty$ as well

So the limit $\lim_{t\to\infty}\frac{\ln t - \frac{1}{-p+1}}{(-p+1)t^{p-1}}$ has the indefinite form $\frac{\infty}{-\infty}$, because the $-\frac{1}{-p+1}$ in the numerator and $(-p+1)$ are constants.

Apply L'Hopital's rule, to calculate:

$$\lim_{t\to\infty}\frac{\ln t-\frac{1}{-p+1}}{(-p+1)t^{p-1}}=\lim_{t\to\infty}\frac{(1/t)}{\frac{-p+1}{p}t^{p}}$$

$$=\lim_{t\to\infty}\frac{(1/t)}{\frac{-p+1}{p}t^{p+1}}$$

$$=0 \qquad \text{Because } p+1>2.$$

So, the above limit converges to zero, and thus the main limit convergent.

$$\lim_{t\to\infty}\frac{\ln t-\frac{1}{-p+1}}{(-p+1)t^{p-1}}+\frac{1}{(-p+1)^2}=\frac{1}{(-p+1)^2}$$

The integral $\int_1^\infty \frac{\ln x}{x^p}dx$ converges.

By the integral test, $\int_1^\infty \frac{\ln x}{x^p}dx$ converges if and only if the series $\sum_{n=1}^{\infty}\frac{\ln n}{n^p}$ converges. thus, the series $\sum_{n=1}^{\infty}\frac{\ln n}{n^p}$ convergent

Therefore, $\boxed{\sum_{n=1}^{\infty}\frac{\ln n}{n^p} \text{ is convergent for } p>1 \text{ and divergent for } p\le 1.}$

Answer 33E.

When have $\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}$

Comparing with p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$, we have p = x.

Since p-series converges when $p>1$ and diverges when $p\le 1$

So $\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}$ defined when $x>1$

Then domain of $\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}$ is $\boxed{(1,\infty)}$

Answer 34E.

Given $\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

(a)

$$\sum_{n=2}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-1$$

$$=\frac{\pi^2}{6}-1$$

(b)

$$\sum_{n=2}^{\infty}\frac{1}{(n+1)^2}=\sum_{n=4}^{\infty}\frac{1}{n^2}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}-1-\frac{1}{2^2}-\frac{1}{3^2}$$

$$=\frac{\pi^2}{6}-\frac{49}{36}$$

(c)

$$\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$=\frac{1}{4}\frac{\pi^2}{6}$$

$$=\frac{\pi^2}{24}$$

Answer 35E.

Given $\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

(a)

$$\sum_{n=1}^{\infty}\frac{3^4}{n^4}=81\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$=(81)\frac{\pi^4}{90}$$

$$=\frac{9\pi^4}{10}$$

(b)

$$\sum_{k=5}^{\infty}\frac{1}{(k-2)^4}=\sum_{k=3}^{\infty}\frac{1}{k^4}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^4}-1-\frac{1}{2^4}$$

$$=\frac{\pi^4}{90}-\frac{17}{16}$$

Answer 36E.

(a)

We know that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \dots$. Then, $s_{10} = \sum_{n=1}^{10} \frac{1}{n^4}$.

$$\begin{aligned} s_{10} &= \frac{1}{1^4} + \frac{1}{2^4} + \dots + \frac{1}{10^4} \\ &= \frac{1}{1} + \frac{1}{16} + \dots + \frac{1}{10000} \\ &\approx 1.08 \end{aligned}$$

Thus, we get $s_{10} \approx 1.082036584$.

We note that the estimation is not precise.

(b) We know that $s_n + \int_{n+1}^{\infty} f(x)dx \le s \le s_n + \int_{n}^{\infty} f(x)dx$.

Let $a_n = \frac{1}{n^4}$. Then, $f(x) = \frac{1}{x^4}$. Since $x \ge 1$, $f(x)$ is continuous, positive, and decreasing. Also, $\int_1^{\infty} \frac{1}{x^4}dx = \left[-\frac{x^{-3}}{3}\right]_1^{\infty}$ or $\int_1^{\infty} \frac{1}{x^4}dx = \frac{1}{3}$.

We have $n = 10$. Then, $\int_{11}^{\infty} \frac{1}{x^4}dx = 0.000250438$.

So, $1.082036584 + 0.000250438 \le 1.082036584 + 0.000333333$. Thus, the improved estimate of the sum is the midpoint of the interval, which is given by 1.08232847.

(c)From part (b), we know that $\sum \frac{1}{n^4} \approx 1.08232847$. Also, we know from exercise 35,. Therefore, the error is 0.00000523629.

(d)We have $R_n \le \int_n^{\infty} \frac{1}{x^4}dx$ or $R_n \le \frac{1}{3n^3}$. We want $\frac{1}{3n^3} < 0.00001$. This means that

$n^3 > \frac{1}{0.00003}$ or $n > \left(\frac{1}{0.00003}\right)^{\frac{1}{3}}$. That is $n > 32.18$.

Therefore, we need 33 terms to ensure the accuracy within 0.00001.

Answer 37E.

(a)We know that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \dots$. Then, $s_{10} = \sum_{n=1}^{10} \frac{1}{n^2}$.

$$s_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{10^2}$$
$$= \frac{1}{1} + \frac{1}{4} + \dots + \frac{1}{100}$$
$$\approx 1.549767731$$

Thus, we get $\boxed{s_{10} \approx 1.549767731}$.
error ≤ 0.1

We note that the estimation is not precise.

(b)

We know that $s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx$. Then, $\int_{n}^{\infty} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_n^{\infty}$ or $\int_{n}^{\infty} \frac{1}{x^2}dx = \frac{1}{n}$. Also, $\int_{10}^{\infty} \frac{1}{x^2}dx = \frac{1}{10}$ and

$$\int_{11}^{\infty} \frac{1}{x^2}dx = \frac{1}{11} \int_{11}^{\infty} \frac{1}{x^2}dx = 0.09090909.$$

So, $1.549767731 + 0.09090909 \leq 1.54976773 + 0.1$.
Thus, the improved estimate of the sum is the midpoint of the interval, which is given by 1.645222276; error ≤ 0.005.

(c) From part **(b)**, we know that $\sum \frac{1}{n^2} \approx 1.64522276$. Also, we know from exercise 34, $\sum \frac{1}{n^2} \approx 1.644934067$. Therefore, the error is 0.00028820915.

(d) We have $R_n \leq \int_{n}^{\infty} \frac{1}{x^2}dx$ or $R_n \leq \frac{1}{n}$. We want $\frac{1}{n} < 0.001$. This means that $n > \frac{1}{0.001}$. That is $n > 1000$.

Therefore, we need 1001 terms to ensure the accuracy within 0.001.

Answer 38E.

Since $f(x)=\frac{1}{x^5}$ is positive and continuous

$\Rightarrow f'(x)=\frac{-5}{x^6}$ negative for $x>1$ so the integral test applies

Now we find $\int_n^\infty \frac{1}{x^5}dx$

$$\Rightarrow \lim_{t\to\infty}\int_n^t \frac{1}{x^5}dx = \lim_{t\to\infty}\left[\frac{x^{-4}}{-4}\right]_n^t$$

$$\Rightarrow \lim_{t\to\infty}\left[\frac{-1}{4t^4}+\frac{1}{4n^4}\right]=\frac{1}{4n^4}$$

We have

$$s_n+\int_{n+1}^\infty f(x)\,dx \le s \le s_n+\int_n^\infty f(x)\,dx$$

Error is $\quad = \frac{1}{2}\left[\int_n^\infty \frac{1}{x^5}dx-\int_{n+1}^\infty \frac{1}{x^5}dx\right]$

Since we have to find the sum correct up to 3 decimal place, so error < 0.0001

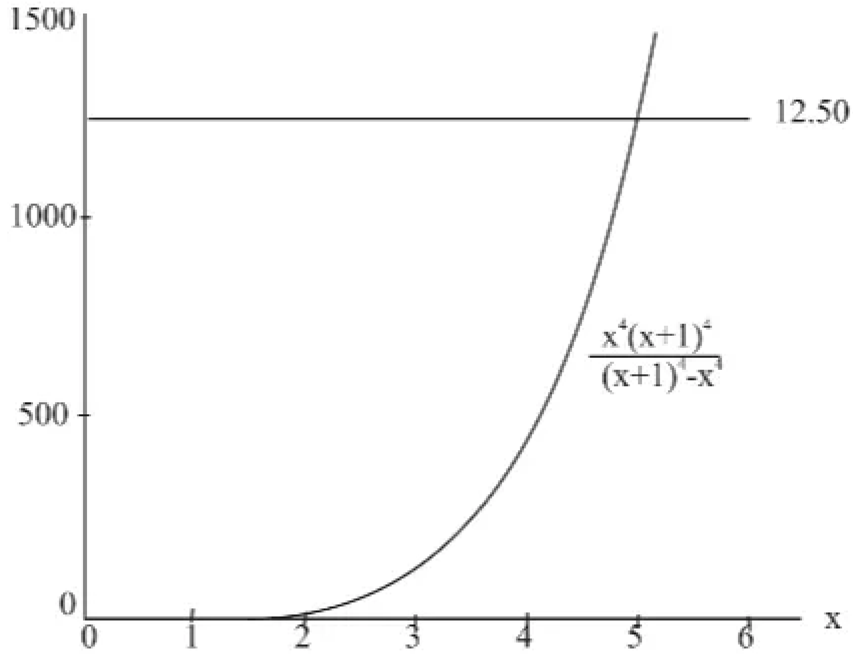
$$\Rightarrow \frac{1}{2}\left[\int_n^\infty \frac{1}{x^5}dx-\int_{n+1}^\infty \frac{1}{x^5}dx\right]<0.0001$$

$$\Rightarrow \frac{1}{4n^4}-\frac{1}{4(n+1)^4}<0.0002$$

$$\Rightarrow \frac{1}{n^4}-\frac{1}{(n+1)^4}<0.0008$$

$$\Rightarrow \frac{(n+1)^4-(n^4)}{n^4(n+1)^4}<0.0008 \qquad \Rightarrow \frac{n^4(n+1)^4}{(n+1)^4-(n^4)}>1250$$

For solving this equation we use graphical method.



$\Rightarrow \quad n > 5.05$

So we take $\quad n = 6$

Now we approximate s by the midpoint of he interval

$$\left(s_n + \int_{n+1}^{\infty} f(x)\,dx \quad , s_n + \int_n^{\infty} f(x)\,dx\right)$$

Mid point $\quad = \frac{1}{2}\left[s_n + \int_{n+1}^{\infty} f(x)\,dx + s_n + \int_n^{\infty} f(x)\,dx\right]$

$$s \approx s_n + \frac{1}{2}\left(\int_{n+1}^{\infty} f(x)\,dx + \int_n^{\infty} f(x)\,dx\right)$$

$$s_6 = \frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \frac{1}{6^5} \approx 1.3679$$

Then $\quad s \approx 1.03679 + \frac{1}{2}\left(\frac{1}{4(7)^4} + \frac{1}{4(6)^4}\right) = 1.03679 + 0.000149$

$$\Rightarrow \boxed{s \approx 1.036939}$$

Answer 39E.

Consider:

$$\sum_{n=1}^{\infty}(2n+1)^{-6}.$$

Estimate to five decimal places.

The function $f(x) = \frac{1}{(2x+1)^6}$ is continuous, positive, and decreasing on $[1, \infty]$.

Now,

$$\sum_{n=1}^{\infty}(2n+1)^{-6} = \int_{1}^{\infty}(2x+1)^{-6}\,dx$$

$$= \lim_{t\to\infty}\int_{1}^{t}(2x+1)^{-6}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(2x+1)^{-6+1}}{-6+1}\right]_{1}^{t}$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(2x+1)^{-6+1}}{-6+1}\right]_{1}^{t}$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(2x+1)^{-5}}{-5}\right]_{1}^{t}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{10(2x+1)^{5}}\right]_{1}^{t}$$

Substitute the limits.

$$= \left[0-\left(-\frac{1}{10(2(1)+1)^{5}}\right)\right] \qquad t\to\infty, -\frac{1}{10(2t+1)^{5}}\to 0$$

$$= \frac{1}{10(2+1)^{5}}$$

$$= \frac{1}{10\cdot 3^{5}}$$

$$= 0.0004115$$

$$\approx 0.00041$$

Thus, $\sum_{n=1}^{\infty}(2n+1)^{-6} = \boxed{0.00041}$.

Answer 40E.

We want the sum of the series $\sum_{n=2}^{\infty} 1/\left[n(\ln n)^2\right]$ within 0.01

Means that we have to find n such that $R_n \le 0.01$

Since $R_n \le \int_n^{\infty} \frac{1}{\left[x(\ln x)^2\right]} dx$

We find $\int_n^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_n^t \frac{1}{x(\ln x)^2} dx$

$$= \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_n^t$$

$$= \lim_{t\to\infty} \left[\frac{-1}{\ln t} + \frac{1}{\ln n}\right] = 1/\ln n$$

So we want $\frac{1}{\ln n} < 0.01$

Or $\ln n > 100$

Or $n > e^{100}$

$\Rightarrow n > 2.6881\times 10^{43} = 2688\times 10^{40}$

So we need $\boxed{2689\times 10^{40}}$ terms to ensure accuracy with in 0.01.

Answer 41E.

The error in the sum is required to be less than 0.000000005. That means we have to find n such that $R_n \le 0.000000005$

i.e $R_n \le 5\times 10^{-9}$

Since $R_n \le \int_n^{\infty} x^{-1.001} dx$

We find that

$$R_n \le \lim_{t\to\infty} \int_n^t x^{-1.001} dx$$

$$\le \lim_{t\to\infty} \left[\frac{x^{-0.001}}{-0.001}\right]_n^t$$

$$\le -1000 \lim_{t\to\infty} \left[\frac{1}{t^{0.001}} - \frac{1}{n^{0.001}}\right]$$

$$\le \frac{1000}{n^{0.001}}$$

Hence we must have

$$\frac{1000}{n^{0.001}} < 5\times10^{-9}$$

$$\Rightarrow n^{0.001} > \frac{1}{5}\times10^{12}$$

$$\Rightarrow n^{0.001} > 0.2\times10^{12} = 2\times10^{11}$$

$$\Rightarrow n > \left(2\times10^{11}\right)^{1000}$$

$$\Rightarrow n > 2^{1000}\times10^{11000}$$

$$\Rightarrow n > 1.07\times10^{301}\times10^{11000} \approx 1.07\times10^{11301}$$

Thus more that $\boxed{10^{11301}}$ terms are required to ensure that the sum error in less than 5×10^{-9}

Answer 42E.

Consider the series $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^2}$

a)

The objective is to prove that the series $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^2}$ is convergent.

Recall the Integral Test:

Suppose *f* is a continuous, positive, decreasing function on $[1,\infty)$.

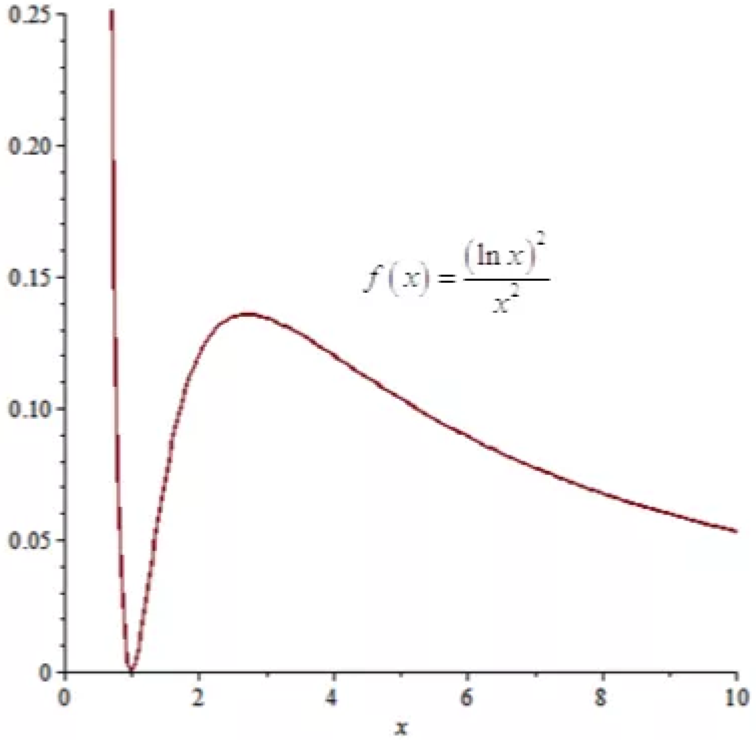
If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Here $f(x)=\frac{(\ln x)^2}{x^2}$

First check whether $f(x)$ is continuous or not:

Sketch the graph of $f(x)=\frac{(\ln x)^2}{x^2}$ is as follows:



From the graph, $f(x)=\frac{(\ln x)^2}{x^2}$ is continuous and increasing for $x>1$ and decreasing for $x>e$

The function $f(x)=\frac{(\ln x)^2}{x^2}$ is continuous, so we can apply Integral Test.

Use CAS to solve the integral

The input command:

int((ln(x))^2/x^2.x=1..infinity,numeric)

The output command

$$> int\left(\frac{(\ln(x))^2}{x^2}, x=1\ ..infinity,\ numeric\right);$$

2.

Thus, $\int_1^\infty \frac{(\ln x)^2}{x^2} = \boxed{2}$

Therefore, $\int_1^\infty \frac{(\ln x)^2}{x^2}$ is convergent.

By Integral Test, the series $\sum_{n=1}^\infty \frac{(\ln n)^2}{n^2}$ is convergent.

b)

The objective is to find an upper bound for the error in the approximation $s \approx s_n$:

The error in $s \approx s_n$ is given by the following representation:

$$R_n \le \int_n^\infty \frac{(\ln x)^2}{x^2} dx$$

Now use CAS to solve the integral.

The input command

int((lnx)^2/x^2,x=n..infinity);

The output command is given as follows:

$$> int\left(\frac{(\ln(x))^2}{x^2}, x=n\ ..infinity\right);$$

$$\frac{\ln(n)^2+2\ln(n)+2}{n}$$

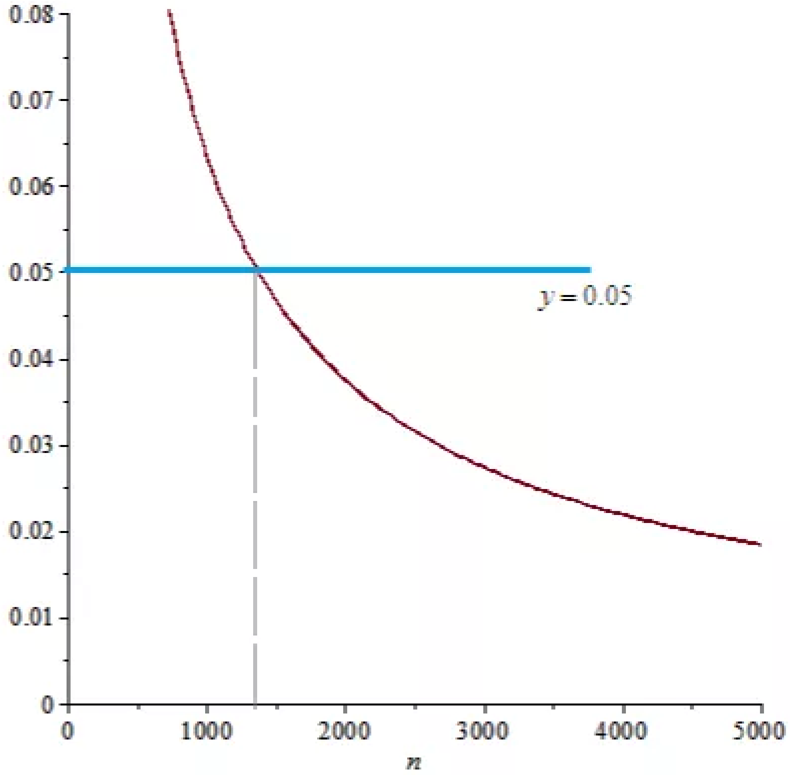
Therefore, $R_n \le \int_n^\infty \frac{(\ln x)^2}{x^2} dx = \boxed{\frac{\ln(n)^2+2\ln n+2}{n}}$

c)

Find the smallest value of *n* such that this upper bound is less than 0.05:

Now solve the equation $\frac{\ln(n)^2+2\ln n+2}{n}=0.05$

Sketch the graph of $\frac{\ln(n)^2+2\ln n+2}{n}$ and $y=0.05$ as follows:



From the graph the intersecting point is $n=\boxed{1373}$

Therefore, the smallest value of n is $\boxed{1373}$

d)

Find S_n for the value of n is 1373:

Use CAS to find S_n:

The **input command**:

evalf(add(((ln(n))^2/n^2),n=1..1373));

The output

>

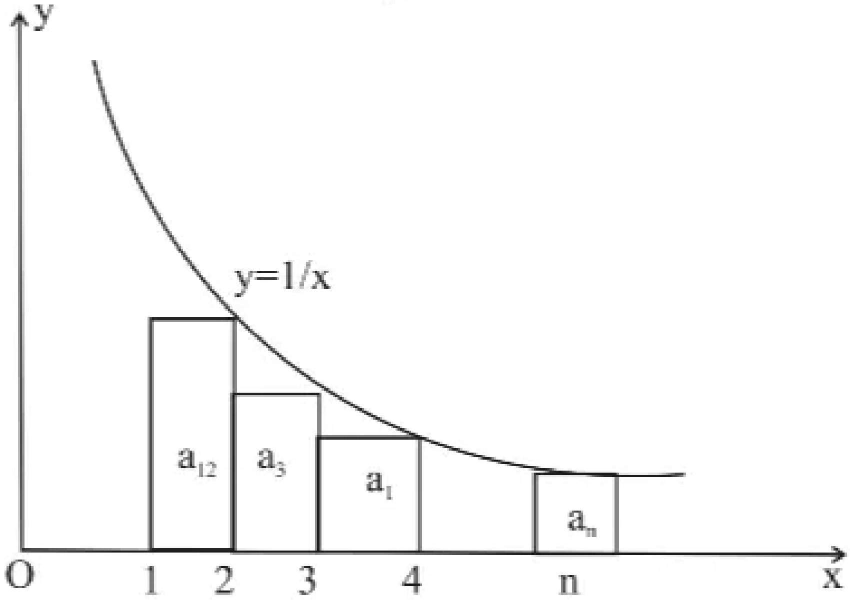
$$evalf\left(add\left(\left(\frac{(\ln(n))^2}{n^2}\right), n=1..1373\right)\right);$$

1.939296577

Therefore, $S_{1373}=\boxed{1.9392}$

Answer 43E.

(A) We take $f(x)=\frac{1}{x}$ so $\sum_{n=1}^{\infty}\frac{1}{n}$ is a harmonic series



From figure

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{-----}+\frac{1}{n}\le\int_1^n\frac{1}{x}dx=\ln n$$

So

$$s_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{-----}+\frac{1}{n}\le 1+\ln n$$

$$\Rightarrow \boxed{s_n\le 1+\ln n}$$

(B) Since $10^6=1\text{ million}$ $10^9=1\text{ billion}$

If number of terms $n=10^6$

Then sum $s_{10^6}\le 1+\ln 10^6\approx 14.82<15$

so $\boxed{s_{10^6}<15}$

If number of terms $n=10^9$

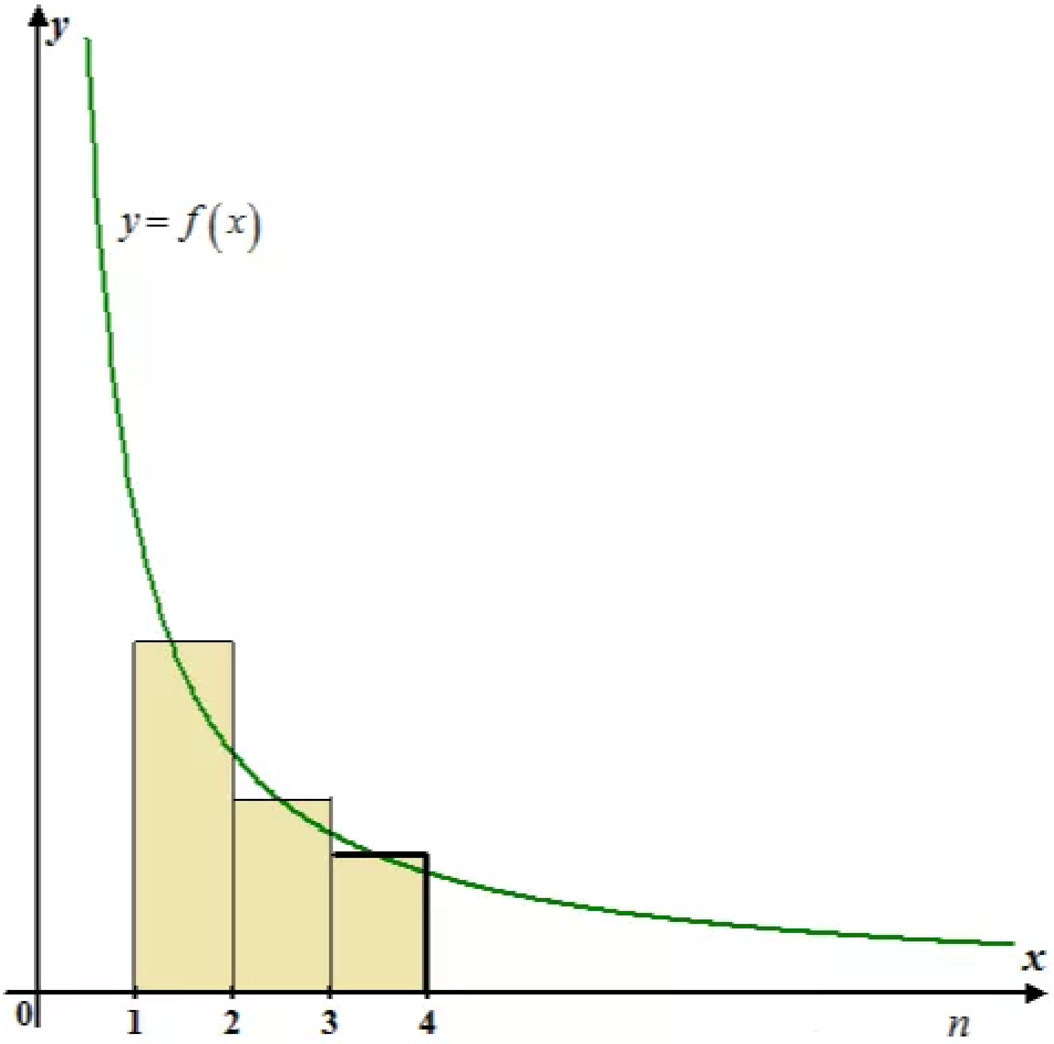
The sum $s_{10^9}\le 1+\ln 10^9\approx 21.72<22$

so $\boxed{s_{10^9}<22}$

Answer 44E.

(a)

The objective is to draw a picture with $y = f(x) = \frac{1}{x}$ and interpret t_n as an area to show that $t_n > 0$ for all n.



The total area of the rectangles formed in the graph of the following function is calculated as follows:

$$y = f(x)$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Also, integrate the function and apply the relevant limits.

$$\int_1^{1+n} \frac{1}{x} dx = [\ln x]_1^{1+n}$$
$$= \ln(n+1) - \ln(1)$$
$$= \ln(n+1) \qquad \text{Since } \ln(1) = 0$$

$\int_1^{1+n} \frac{1}{x} dx$ represents the area that lies below the curve and above the $x-\text{axis}$.

So, the sum of areas of the rectangles will be greater than this area, which is represented as follows:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{1+n} \frac{1}{x} dx$$
$$> \ln(n+1) \qquad \text{Since } \int_1^{1+n} \frac{1}{x} dx = \ln(n+1)$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1) > 0$$
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n) > 0 \qquad \text{Since } \ln(n+1) > \ln(n)$$

So $t_n > 0$ from given value of t_n.

(b)

The area between $x = n$ and $x = n+1$ and the graph is represented as follows:

$$\int_1^{1+n} \frac{1}{x} dx = [\ln x]_1^{1+n}$$
$$= \ln(n+1) - \ln(1)$$

Since $\ln(n+1) > \ln(1)$ so area between $x = n$ and $x = n+1$ and which lies below the graph is greater than rectangle formed.

So, obtain the following result:

$$t_n - t_{n+1} = \ln(n+1) - \ln(n) - \frac{1}{n+1}$$
$$> 0$$
$$t_n > t_{n+1}$$

This implies that $\{t_n\}$ is a decreasing sequence.

(c)

The objective to show that t_n converges to a limit γ, where γ is called as the Euler's constant.

First show that $\{t_n\}$ is bounded

Consider the following sequence:

$\{t_n\} = \{S_n - \ln n\}$ Since $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

Now, proceed as follows:

$\ln(n+1) \le S_n \le 1 + \ln n$

Subtracting $\ln n$ from both sides gives

$\ln(n+1) - \ln n \le S_n - \ln n \le 1$

Use the fact that $\ln x$ is an increasing function.

So $\ln(n+1) - \ln n > 0$, for $n \ge 1$

Therefore, solve the following inequality:

$0 < \ln(n+1) - \ln n \le S_n - \ln n \le 1$

$0 \le S_n - \ln n \le 1$

$0 \le a_n \le 1$

Thus, the sequence $\{t_n\}$ is bounded.

Since $\{t_n\}$ is bounded and monotonically decreasing sequence.

So, it must be convergent.

So there exists a limit for $\{t_n\}$. Let it be γ

Thus, t_n converges to a limit γ.

Answer 45E.

We have the series $\sum_{n=1}^{\infty} b^{\ln n}$

Since $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b}$

$= n^{\ln b}$

$= \frac{1}{n^{-\ln b}}$

So series is $\sum_{n=1}^{\infty} b^{\ln n} = \sum_{n=1}^{\infty} \frac{1}{n^{-\ln b}}$

Which is a p-series with p = -ln b

So this series converges when p > 1

$$\Rightarrow -\ln b > 1$$
$$\Rightarrow \ln b < -1$$
$$\Rightarrow b < e^{-1}$$
$$\Rightarrow \boxed{b < 1/e}$$

So for $\boxed{b < 1/e}$ the given series converges.

Answer 46E.

Consider the following:

$$\sum_{n=1}^{\infty}\left(\frac{c}{n}-\frac{1}{n+1}\right).$$

The objective is to find for what values of *c*, the series is convergent.

Rewrite the series as follows:

$$\begin{aligned}
s_n &= \left(\frac{c}{1}-\frac{1}{1+1}\right)+\left(\frac{c}{2}-\frac{1}{2+1}\right)+\left(\frac{c}{3}-\frac{1}{3+1}\right)+\ldots+\left(\frac{c}{n}-\frac{1}{n+1}\right)\\
&= \left(\frac{c}{1}-\frac{1}{2}\right)+\left(\frac{c}{2}-\frac{1}{3}\right)+\left(\frac{c}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{c}{n}-\frac{1}{n+1}\right)\\
&= c+\left(\frac{c}{2}-\frac{1}{2}\right)+\left(\frac{c}{3}-\frac{1}{3}\right)+\left(\frac{c}{4}-\frac{1}{4}\right)+\ldots+\left(\frac{c}{n}-\frac{1}{n}\right)-\frac{1}{n+1}\\
&= c+\frac{1}{2}(c-1)+\frac{1}{3}(c-1)+\ldots+\frac{1}{n}(c-1)-\frac{1}{n+1}
\end{aligned}$$

Factor out $(c-1)$ to get the following result:

$$\begin{aligned}
&= c+(c-1)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}\right)-\frac{1}{n+1}\\
&= c+(c-1)\sum_{k=2}^{n}\frac{1}{k}-\frac{1}{n+1}
\end{aligned}$$

Now, let the sum when $n\to\infty$ be S.

Then, obtain the following results:

$$\begin{aligned}
S &= \lim_{n\to\infty} s_n\\
&= \lim_{n\to\infty}\left(c+(c-1)\sum_{k=2}^{n}\frac{1}{k}-\frac{1}{n+1}\right)
\end{aligned}$$

The sum exists only when the limit exists.

The limit exists for $c-1=0$.

That is, $\boxed{c=1}$.