Chapter 5

Conic Section

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Apollonius

The synthetic approach to the subject of geometry as given by Euclid and in Sulbasutras etc. was continued for some 1300 yrs. In the 200 B.C. Apollonius wrote a book called 'The conic' which was all about conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

Many important discoveries, both in mathematics and science, have been linked to the conic-sections.

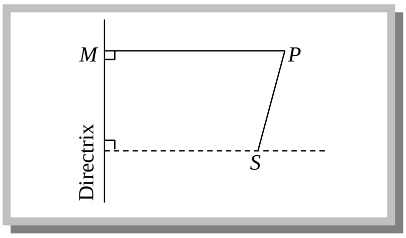
5.0 Conic Section : General

5.0.1. Introduction

The curves obtained by intersection of a plane and a double cone in different orientation are called conic section.

In other words "Graph of a quadratic equation (in two variables) is a "Conic section".

A conic section or conic is the locus of a point P, which moves in such a way that its distance from a fixed point S always bears a constant ratio to its distance from a fixed straight line, all being in the same plane.



$$\frac{SP}{PM} = \text{constant} = e \text{ (eccentricity)}$$

or $\quad SP = e.\, PM$

5.0.2. Definitions of Various important Terms

(1) **Focus :** The fixed point is called the focus of the conic-section.

(2) **Directrix :** The fixed straight line is called the directrix of the conic section.

In general, every central conic has four foci, two of which are real and the other two are imaginary. Due to two real foci, every conic has two directrices corresponding to each real focus.

(3) **Eccentricity :** The constant ratio is called the eccentricity of the conic section and is denoted by e.

If $e = 1$, the conic is called **Parabola**.

If $e < 1$, the conic is called **Ellipse**.

If $e > 1$, the conic is called **Hyperbola**.

If $e = 0$, the conic is called **Circle**.

If $e = \infty$, the conic is called **Pair of the straight lines**.

(4) **Axis:** The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section. A conic is always symmetric about its axis.

(5) **Vertex:** The points of intersection of the conic section and the axis are called vertices of conic section.

(6) **Centre:** The point which bisects every chord of the conic passing through it, is called the centre of conic.

(7) **Latus-rectum:** The latus-rectum of a conic is the chord passing through the focus and perpendicular to the axis.

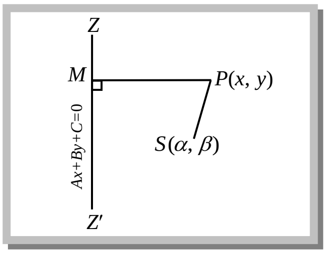
(8) **Double ordinate:** The double ordinate of a conic is a chord perpendicular to the axis.

(9) **Focal chord:** A chord passing through the focus of the conic is called a focal chord.

(10) **Focal distance:** The distance of any point on the conic from the focus is called the focal distance of the point.

5.0.3. General equation of a Conic section when its Focus, Directrix and Eccentricity are given

Let $S(\alpha,\beta)$ be the focus, $Ax + By + C = 0$ be the directrix and e be the eccentricity of a conic. Let $P(h,k)$ be any point on the conic. Let PM be the perpendicular from P, on the directrix. Then by definition



$$SP = ePM \Rightarrow SP^2 = e^2 PM^2$$

$$\Rightarrow (h-\alpha)^2 + (k-\beta)^2 = e^2\left(\frac{Ah + Bk + C}{\sqrt{A^2+B^2}}\right)^2$$

Thus the locus of (h,k) is $(x-\alpha)^2 + (y-\beta)^2 = e^2 \dfrac{(Ax+By+C)^2}{(A^2+B^2)}$ this is the cartesian equation of the conic section which, when simplified, can be written in the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which is general equation of second degree.

5.0.4. Recognisation of Conics

The equation of conics is represented by the general equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{......(i)}$$

and discriminant of above equation is represented by Δ, where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

Case I: When $\Delta = 0$

In this case equation (i) represents the degenerate conic whose nature is given in the following table.

S. No.	Condition	Nature of conic
1.	$\Delta = 0$ and $ab - h^2 = 0$`	A pair of coincident straight lines
2.	$\Delta = 0$ and $ab - h^2 < 0$	A pair of intersecting straight lines
3.	$\Delta = 0$ and $ab - h^2 > 0$	A point

Case II: When $\Delta \neq 0$

In this case equation (i) represents the non-degenerate conic whose nature is given in the following table.

S. No.	Condition	Nature of conic
1.	$\Delta \neq 0,\ h = 0,\ a = b$	A circle
2.	$\Delta \neq 0,\ ab - h^2 = 0$	A parabola
3.	$\Delta \neq 0,\ ab - h^2 > 0$	An ellipse
4.	$\Delta \neq 0,\ ab - h^2 < 0$	A hyperbola
5.	$\Delta \neq 0,\ ab - h^2 < 0$ and $a + b = 0$	A rectangular hyperbola

5.0.5. Method to find centre of a Conic

Let $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ be the given conic. Find $\frac{\partial S}{\partial x}; \frac{\partial S}{\partial y}$

Solve $\frac{\partial S}{\partial x} = 0, \frac{\partial S}{\partial y} = 0$ for x, y we shall get the required centre (x, y)

$$(x, y) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

Example: 1 The equation $x^2 - 2xy + y^2 + 3x + 2 = 0$ represents **[UPSEAT 2001]**

(a) A parabola (b) An ellipse (c) A hyperbola (d) A circle

Solution: (a) Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Here, $a = 1, b = 1, h = -1, g = \frac{3}{2}, f = 0, c = 2$

Now $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$\Rightarrow \Delta = (1)(1)(2) + 2\left(\frac{3}{2}\right)(0)(-1) - (1)(0)^2 - 1\left(\frac{3}{2}\right)^2 - 2(-1)^2 \Rightarrow \Delta = \frac{-9}{4}$ *i.e.,* $\Delta \neq 0$ and $h^2 - ab = 1 - 1 = 0$ *i.e.,* $h^2 = ab$

So given equation represents a parabola.

Example: 2 The centre of $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ is **[BIT Ranchi1986]**

(a) (2, 3) (b) (2, –3) (c) (–2, 3) (d) (–2, –3)

Solution: (a) Centre of conic is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

Here, $a = 14$, $h = -2$, $b = 11$, $g = -22$, $f = -29$, $c = 71$

Centre $\equiv \left(\frac{(-2)(-29) - (11)(-22)}{(14)(11) - (-2)^2}, \frac{(-22)(-2) - (14)(-29)}{(14)(11) - (-2)^2}\right)$

Centre $\equiv (2, 3)$.

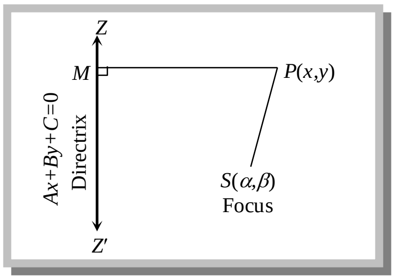
***

5.1 Parabola

5.1.1 Definition

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (*i.e.*, focus) in the plane is always equal to its distance from a fixed straight line (*i.e.*, directrix) in the same plane.

General equation of a parabola : Let S be the focus, ZZ' be the directrix and let P be any point on the parabola. Then by definition,



$$SP = PM \qquad (\because\ e = 1)$$

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = \frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

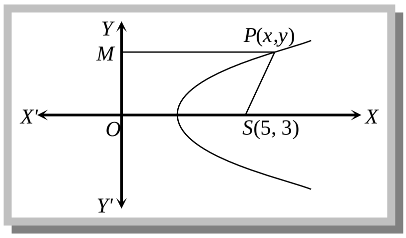
Or $\quad (A^2 + B^2)\{(x-\alpha)^2 + (y-\beta)^2\} = (Ax + By + C)^2$

Example: 1 The equation of parabola whose focus is (5, 3) and directrix is $3x - 4y + 1 = 0$, is **[MP PET 2002]**

(a) $(4x + 3y)^2 - 256x - 142y + 849 = 0$ (b) $(4x - 3y)^2 - 256x - 142y + 849 = 0$

(c) $(3x + 4y)^2 - 142x - 256y + 849 = 0$ (d) $(3x - 4y)^2 - 256x - 142y + 849 = 0$

Solution: (a) $PM^2 = PS^2 \Rightarrow (x-5)^2 + (y-3)^2 = \left(\dfrac{3x - 4y + 1}{\sqrt{9 + 16}}\right)^2$



$\Rightarrow 25(x^2 + 25 - 10x + y^2 + 9 - 6y)$

$= 9x^2 + 16y^2 + 1 - 12xy + 6x - 8y - 12xy$

$\Rightarrow 16x^2 + 9y^2 - 256x - 142y + 24xy + 849 = 0$

$\Rightarrow (4x + 3y)^2 - 256x - 142y + 849 = 0$

5.1.2 Standard equation of the Parabola

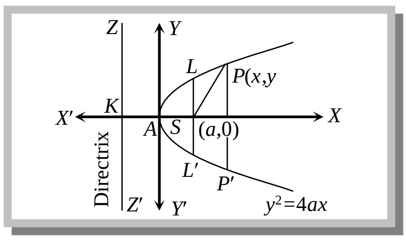
Let S be the focus ZZ' be the directrix of the parabola and (x, y) be any point on parabola.

Let $AS = AK = a (> 0)$ then coordinate of S is $(a, 0)$ and the equation of KZ is $x = -a$ or $x + a = 0$

Now $SP = PM \Rightarrow (SP)^2 = (PM)^2$

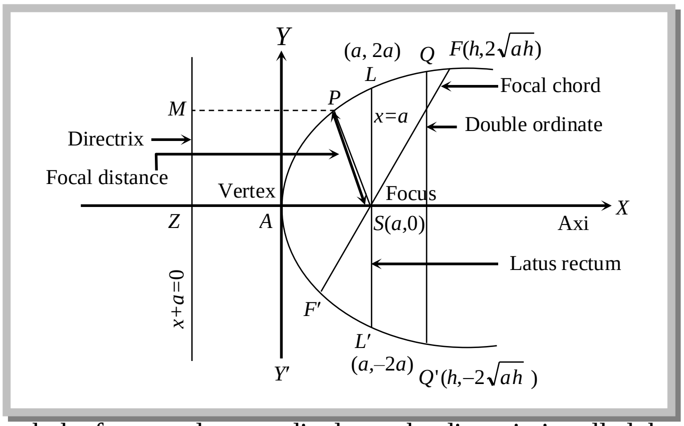
$\Rightarrow (x - a)^2 + (y - 0)^2 = (a + x)^2$

$\therefore \boxed{y^2 = 4ax}$



which is the equation of the parabola in its standard form.

Some terms related to parabola



For the parabola $y^2 = 4ax$**,**

(1) **Axis :** A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola.

For the parabola $y^2 = 4ax$, x-axis is the axis. Here all powers of y are even in $y^2 = 4ax$. Hence parabola $y^2 = 4ax$ is symmetrical about x-axis.

(2) **Vertex :** The point of intersection of a parabola and its axis is called the vertex of the parabola. The vertex is the middle point of the focus and the point of intersection of axis and the directrix.

For the parabola $y^2 = 4ax$, $A(0,0)$ i.e., the origin is the vertex.

(3) **Double-ordinate :** The chord which is perpendicular to the axis of parabola or parallel to directrix is called double ordinate of the parabola.

Let QQ' be the double-ordinate. If abscissa of Q is h then ordinate of Q, $y^2 = 4ah$ or $y = 2\sqrt{ah}$ (for I^{st} Quadrant) and ordinate of Q' is $y = -2\sqrt{ah}$ (for IVth Quadrant). Hence coordinates of Q and Q' are $(h, 2\sqrt{ah})$ and $(h, -2\sqrt{ah})$ respectively.

(4) **Latus-rectum :** If the double-ordinate passes through the focus of the parabola, then it is called latus-rectum of the parabola.

Coordinates of the extremeties of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$ respectively.

Since $LS = L'S = 2a$ $\therefore$ Length of latus rectum $LL' = 2(LS) = 2(L'S) = 4a$.

(5) **Focal Chord :** A chord of a parabola which is passing through the focus is called a focal chord of the parabola. Here PP' and LL' are the focal chords.

(6) **Focal distance (Focal length) :** The focal distance of any point P on the parabola is its distance from the focus S *i.e.*, SP.

Here, Focal distance $SP = PM = x + a$

Note : ❑ If length of any double ordinate of parabola $y^2 = 4ax$ is $2l$, then coordinates of end points of this double ordinate are $\left(\frac{l^2}{4a}, l\right)$ and $\left(\frac{l^2}{4a}, -l\right)$.

Important Tips

☞ *The area of the triangle inscribed in the parabola* $y^2 = 4ax$ *is* $\frac{1}{8a}(y_1 \sim y_2)(y_2 \sim y_3)(y_3 \sim y_1)$, *where* $y_1, y_2 y_3$ *are the ordinate of the vertices*

☞ *The length of the side of an equilateral triangle inscribed in the parabola* $y^2 = 4ax$ *is* $8a\sqrt{3}$ *(one angular point is at the vertex).*

Example: 2 The point on the parabola $y^2 = 18x$, for which the ordinate is three times the abscissa, is **[MP PET 2003]**

(a) (6, 2) (b) (–2, –6) (c) (3, 18) (d) (2, 6)

Solution: (d) Given $y = 3x$, then $(3x)^2 = 18x \Rightarrow 9x^2 = 18x \Rightarrow x = 2$ and $y = 6$.

Example: 3 The equation of the directrix of parabola $5y^2 = 4x$ is **[UPSEAT 1998]**

(a) $4x - 1 = 0$ (b) $4x + 1 = 0$ (c) $5x + 1 = 0$ (d) $5x - 1 = 0$

Solution: (c) The given parabola is $y^2 = \frac{4}{5}x$. Here $a = \frac{1}{5}$. Directrix is $x = -a = \frac{-1}{5} \Rightarrow 5x + 1 = 0$

Example: 4 The point on the parabola $y^2 = 8x$. Whose distance from the focus is 8, has *x*-coordinate as

(a) 0 (b) 2 (c) 4 (d) 6

Solution: (d) If $P(x_1, y_1)$ is a point on the parabola $y^2 = 4ax$ and S is its focus, then $SP = x_1 + a$

Here $4a = 8 \;\Rightarrow\; a = 2;\; SP = 8$

$\therefore\; 8 = x_1 + 2 \;\Rightarrow\; x_1 = 6$

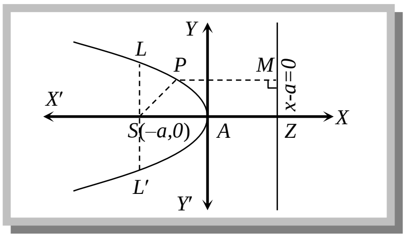
Example: 5 If the parabola $y^2 = 4ax$ passes through (–3, 2), then length of its latus rectum is **[Rajasthan PET 1986, 95]**

(a) 2/3 (b) 1/3 (c) 4/3 (d) 4

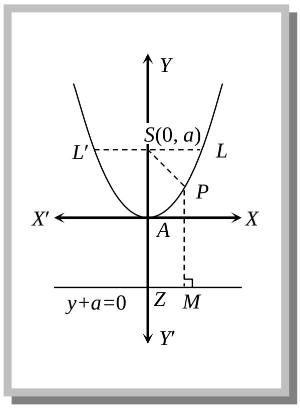
Solution: (c) The point $(-3, 2)$ will satisfy the equation $y^2 = 4ax \Rightarrow 4 = -12a \Rightarrow$ Latus rectum $= 4|a| = 4 \times \left|-\frac{1}{3}\right| = \frac{4}{3}$

5.1.3 Some other standard forms of Parabola

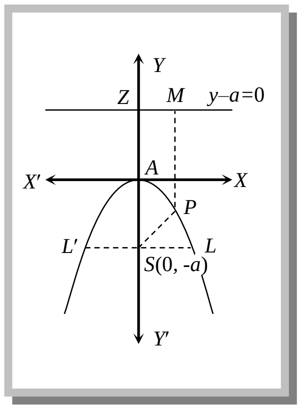
(1) Parabola opening to left (*i.e.* $y^2 = -4ax$); $(a > 0)$



(2) Parabola opening upwards (*i.e.* $x^2 = 4ay$); $(a>0)$



(3) Parabola opening down wards (*i.e.* $x^2 = -4ay$); $(a > 0)$



Important terms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Equation of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the latusrectum	$4a$	$4a$	$4a$	$4a$
Focal distance of a point $P(x, y)$	$x + a$	$a - x$	$y + a$	$a - y$

Example: 6 Focus and directrix of the parabola $x^2 = -8ay$ are **[Rajasthan PET 2001]**

(a) $(0, -2a)$ and $y = 2a$ (b) $(0, 2a)$ and $y = -2a$ (c) $(2a, 0)$ and $x = -2a$ (d) $(-2a, 0)$ and $x = 2a$

Solution: (a) Given equation is $x^2 = -8ay$

Comparing the given equation with $x^2 = -4AY$, $A = 2a$

Focus of parabola (0,–A) i.e. (0, –2a)

Directrix $y = A$, *i.e.* $y = 2a$

Example: 7 The equation of the parabola with its vertex at the origin, axis on the y-axis and passing through the point (6, –3) is **[MP PET 2001]**

(a) $y^2 = 12x + 6$ (b) $x^2 = 12y$ (c) $x^2 = -12y$ (d) $y^2 = -12x + 6$

Solution: (c) Since the axis of parabola is y-axis with its vertex at origin.

$\therefore$ Equation of parabola $x^2 = 4ay$. Since it passes through (6, –3) ; $\therefore 36 = -12a \Rightarrow a = -3$

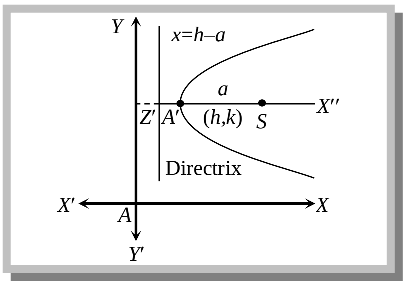
$\therefore$ Equation of parabola is $x^2 = -12y$.

5.1.4 Special form of Parabola $(y-k)^2 = 4a(x-h)$

The equation of a parabola with its vertex at (h, k) and axis as parallel to x-axis is $(y-k)^2 = 4a(x-h)$

If the vertex of the parabola is (p, q) and its axis is parallel to y-axis, then the equation of the parabola is $(x-p)^2 = 4b(y-q)$

When origin is shifted at $A'(h, k)$ without changing the direction of axes, its equation becomes $(y-k)^2 = 4a(x-h)$ or $(x-p)^2 = 4b(y-q)$



Equation of Parabola	Vertex	Axis	Focus	Directrix	Equation of L.R.	Length of L.R.
$(y-K)^2 = 4a(x-h)$	(h,k)	$y = k$	$(h+a, k)$	$x + a - h = 0$	$x = a + h$	$4a$
$(x-p)^2 = 4b(y-q)$	(p,q)	$x = p$	$(p, b+q)$	$y + b - q = 0$	$y = b + q$	$4b$

Important Tips

☞ $y^2 = 4a(x+a)$ *is the equation of the parabola whose focus is the origin and the axis is x-axis.*

☞ $y^2 = 4a(x-a)$ *is the equation of parabola whose axis is x-axis and y-axis is directrix.*

☞ $x^2 = 4a(y+a)$ *is the equation of parabola whose focus is the origin and the axis is y-axis.*

☞ $x^2 = 4a(y-a)$ *is the equation of parabola whose axis is y-axis and the directrix is x-axis.*

☞ *The equation to the parabola whose vertex and focus are on x-axis at a distance a and a' respectively from the origin is* $y^2 = 4(a'-a)(x-a)$.

☞ *The equation of parabola whose axis is parallel to x-axis is* $x = Ay^2 + By + C$ *and* $y = Ax^2 + Bx + C$ *is a parabola with its axis parallel to y-axis.*

Example: 8 Vertex of the parabola $x^2 + 4x + 2y - 7 = 0$ is **[MP PET 1990]**

(a) (–2,11 / 2) (b) (–2,2) (c) (–2,11) (d) (2,11)

Solution: (a) Equation of the parabola is $(x+2)^2 = -2y + 7 + 4 \Rightarrow (x+2)^2 = -2\left(y - \frac{11}{2}\right)$. Hence vertex is $\left(-2, \frac{11}{2}\right)$.

Example: 9 The focus of the parabola $4y^2 - 6x - 4y = 5$ is **[Rajasthan PET 1997]**

(a) $(-8/5, 2)$ (b) $(-5/8, 1/2)$ (c) $(1/2, 5/8)$ (d) $(6/8, -1/2)$

Solution: (b) Given equation of parabola when written in standard form, we get

$$4\left(y-\frac{1}{2}\right)^2 = 6(x+1) \Rightarrow \left(y-\frac{1}{2}\right)^2 = \frac{3}{2}(x+1) \Rightarrow Y^2 = \frac{3}{2}X \text{ where, } Y = y-\frac{1}{2}, X = x+1$$

$$\therefore y = Y+\frac{1}{2}, x = X-1 \qquad ....(i)$$

Focus $\Rightarrow X = a, \quad Y = 0$; $\therefore 4a = \frac{3}{2} \Rightarrow a = \frac{3}{8} \Rightarrow x = \frac{3}{8}-1 = -\frac{5}{8}$; $y = 0+\frac{1}{2} = \frac{1}{2} \Rightarrow$ Focus $= \left(-\frac{5}{8}, \frac{1}{2}\right)$

Example: 10 The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is **[IIT Screening 2001]**

(a) $x = -1$ (b) $x = 1$ (c) $x = \frac{-3}{2}$ (d) $x = \frac{3}{2}$

Solution: (d) Here, $y^2 + 4y + 4 + 4x - 2 = 0$ or $(y+2)^2 = -4\left(x - \frac{1}{2}\right)$

Let $y + 2 = Y$, $\frac{1}{2} - x = X$. Then the parabola is $Y^2 = 4X$. $\therefore$ The directrix is $X + 1 = 0$ or $\frac{1}{2} - x + 1 = 0$, $\therefore x = \frac{3}{2}$

Example: 11 The line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$. Then one of the values of k is **[IIT Screening 2000]**

(a) $\frac{1}{8}$ (b) 8 (c) 4 (d) $\frac{1}{4}$

Solution: (c) The parabola is $y^2 = 4\frac{k}{4}\left(x - \frac{8}{k}\right)$. Putting $y = Y$, $x - \frac{8}{k} = X$. The equation is $Y^2 = 4.\frac{k}{4}.X$

$\therefore$ The directrix is $X + \frac{k}{4} = 0$ *i.e.,* $x - \frac{8}{k} + \frac{k}{4} = 0$. But $x - 1 = 0$ is the directrix. So $\frac{8}{k} - \frac{k}{4} = 1 \Rightarrow k = -8, 4$.

Example: 12 Equation of the parabola with its vertex at (1, 1)and focus (3, 1)is **[Karnataka CET 2001, 2002]**

(a) $(x-1)^2 = 8(y-1)$ (b) $(y-1)^2 = 8(x-3)$ (c) $(y-1)^2 = 8(x-1)$ (d) $(x-3)^2 = 8(y-1)$

Solution: (c) Given vertex of parabola $(h, k) \equiv (1,1)$ and its focus $(a+h, k) \equiv (3,1)$ or $a + h = 3$ or $a = 2$. We know that as the y-coordinates of vertex and focus are same, therefore axis of parabola is parallel to x-axis. Thus equation of the parabola is $(y-k)^2 = 4a(x-h)$ or $(y-1)^2 = 4 \times 2(x-1)$ or $(y-1)^2 = 8(x-1)$.

5.1.5 Parametric equations of a Parabola

The simplest and the best form of representing the coordinates of a point on the parabola $y^2 = 4ax$ is $(at^2 2at)$ because these coordinates satisfy the equation $y^2 = 4ax$ for all values of t. The equations $x = at^2, y = 2at$ taken together are called the parametric equations of the parabola $y^2 = 4ax$, t being the parameter.

The following table gives the parametric coordinates of a point on four standard forms of the parabola and their parametric equation.

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Parametric Coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric Equations	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = at^2$	$x = 2at$, $y = -at^2$

Note : ❑ The parametric equation of parabola $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$

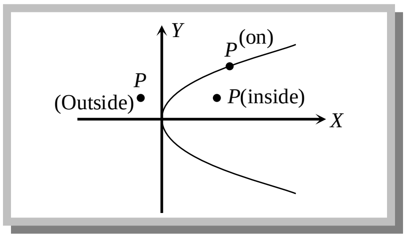
Example: 13 $x - 2 = t^2, y = 2t$ are the parametric equations of the parabola

(a) $y^2 = 4x$ (b) $y^2 = -4x$ (c) $x^2 = -4y$ (d) $y^2 = 4(x-2)$

Solution: (d) Here $\frac{y}{2} = t$ and $x - 2 = t^2 \Rightarrow (x-2) = \left(\frac{y}{2}\right)^2 \Rightarrow y^2 = 4(x-2)$

5.1.6 Position of a point and a Line with respect to a Parabola

(1) **Position of a point with respect to a parabola**: The point $P(x_1, y_1)$ lies outside on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =, \text{or} < 0$



(2) **Intersection of a line and a parabola:** Let the parabola be $y^2 = 4ax$ .....(i)

And the given line be $y = mx + c$ .....(ii)

Eliminating y from (i) and (ii) then $(mx + c)^2 = 4ax$ or $m^2x^2 + 2x(mc - 2a) + c^2 = 0$ .....(iii)

This equation being quadratic in x, gives two values of x. It shows that every straight line will cut the parabola in two points, may be real, coincident or imaginary, according as discriminate of (iii) >, = or < 0

$\therefore$ The line $y = mx + c$ does not intersect, touches or intersect a parabola $y^2 = 4ax$, according as $c >, =, < \frac{a}{m}$

Condition of tangency : The line $y = mx + c$ touches the parabola, if $c = \frac{a}{m}$

Example: 14 The equation of a parabola is $y^2 = 4x$. $P(1,3)$ and $Q(1,1)$ are two points in the xy-plane. Then, for the parabola

(a) P and Q are exterior points
(b) P is an interior point while Q is an exterior point
(c) P and Q are interior points
(d) P is an exterior point while Q is an interior point

Solution: (d) Here, $S \equiv y^2 - 4x = 0$; $S(1,3) = 3^2 - 4.1 > 0 \Rightarrow P(1,3)$ is an exterior point.

$S(1,1) = 1^2 - 4.1 < 0 \Rightarrow Q(1,1)$ is an interior point.

Example: 15 The ends of a line segment are $P(1,3)$ and $Q(1,1)$. R is a point on the line segment PQ such that $PQ : QR = 1 : \lambda$. If R is an interior point of the parabola $y^2 = 4x$, then

(a) $\lambda \in (0, 1)$
(b) $\lambda \in \left(-\frac{3}{5}, 1\right)$
(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$
(d) None of these

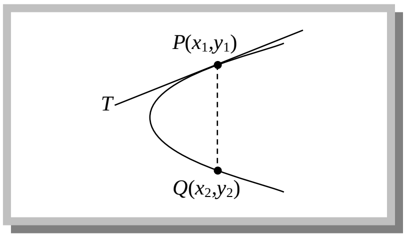
Solution: (a) $R = \left(1, \frac{1+3\lambda}{1+\lambda}\right)$ It is an interior point of $y^2 - 4x = 0$ *iff* $\left(\frac{1+3\lambda}{1+\lambda}\right)^2 - 4 < 0$

$\Rightarrow \left(\frac{1+3\lambda}{1+\lambda} - 2\right)\left(\frac{1+3\lambda}{1+\lambda} + 2\right) < 0 \Rightarrow \left(\frac{\lambda - 1}{1+\lambda}\right)\left(\frac{5\lambda + 3}{1+\lambda}\right) < 0 \Rightarrow (\lambda - 1)\left(\lambda + \frac{3}{5}\right) < 0$

Therefore, $-\frac{3}{5} < \lambda < 1$. But $\lambda > 0$ $\therefore 0 < \lambda < 1 \Rightarrow \lambda \in (0, 1)$.

5.1.7 Equation of Tangent in Different forms

(1) **Point Form:** The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $yy_1 = 2a(x + x_1)$



Equation of tangent of all other standard parabolas at (x_1, y_1)	
Equation of parabolas	**Tangent at (x_1, y_1)**
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$

$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

Note : ❑ The equation of tangent at (x_1, y_1) to a curve can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$ and xy by $\frac{xy_1 + x_1 y}{2}$ provided the equation of curve is a polynomial of second degree in x and y.

(2) **Parametric form :** The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $ty = x + at^2$

Equations of tangent of all other standard parabolas at '*t*'		
Equations of parabolas	**Parametric co-ordinates '*t*'**	**Tangent at '*t*'**
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

(3) **Slope Form:** The equation of a tangent of slope m to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is $y = mx + \frac{a}{m}$

Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)	Condition of Tangency
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$	$c = am^2$

Important Tips

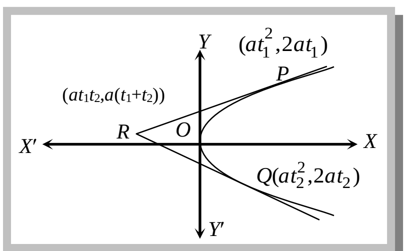
☞ *If the straight line* $lx + my + n = 0$ *touches the parabola* $y^2 = 4ax$ *then* $ln = am^2$.

☞ *If the line* $x\cos\alpha + y\sin\alpha = p$ *touches the parabola* $y^2 = 4ax$, *then* $P\cos\alpha + a\sin^2\alpha = 0$ *and point of contact is* $(a\tan^2\alpha, -2a\tan\alpha)$

☞ *If the line* $\frac{x}{l} + \frac{y}{m} = 1$ *touches the parabola* $y^2 = 4a(x + b)$, *then* $m^2(l + b) + al^2 = 0$

5.1.8 Point of intersection of Tangents at any two points on the Parabola

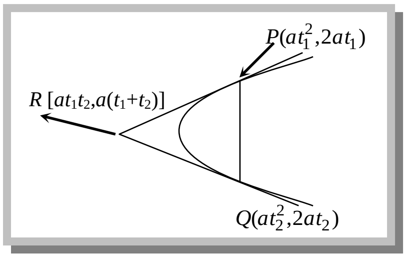
The point of intersection of tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $(at_1t_2, a(t_1 + t_2))$.



The locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle α is $(x+a)^2 \tan^2 \alpha = y^2 - 4ax$.

Director circle: The locus of the point of intersection of perpendicular tangents to a conic is known as its director circle. The director circle of a parabola is its directrix.

Note : ❑ Clearly, x-coordinates of the point of intersection of tangents at P and Q on the parabola is the G.M of the x-coordinate of P and Q and y-coordinate is the A.M. of y-coordinate of P and Q.



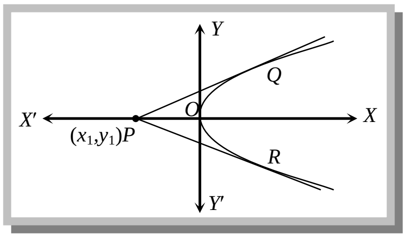
❑ The equation of the common tangents to the parabola $y^2 = 4ax$ and $x^2 = 4by$ is $a^{\frac{1}{3}}x + b^{\frac{1}{3}}y + a^{\frac{2}{3}}b^{\frac{2}{3}} = 0$

❑ The tangents to the parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at R. Then the area of triangle PQR is $\frac{1}{2}a^2(t_1 - t_2)^3$

5.1.9 Equation of Pair of Tangents from a point to a Parabola

If $y_1^2 - 4ax_1 > 0$, then any point $P(x_1, y_1)$ lies out side the parabola and a pair of tangents PQ, PR can be drawn to it from P

The combined equation of the pair of the tangents drawn from a point to a parabola is $SS' = T^2$ where $S = y^2 - 4ax; S' = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$



Note : ❑ The two tangents can be drawn from a point to a parabola. The two tangent are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

Important Tips

☞ *Tangents at the extremities of any focal chord of a parabola meet at right angles on the directrix.*

☞ *Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.*

☞ *If the tangents at the points P and Q on a parabola meet in T, then ST is the geometric mean between SP and SQ, i.e.* $ST^2 = SP.SQ$

☞ *Tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.*

☞ *The angle of intersection of two parabolas* $y^2 = 4ax$ *and* $x^2 = 4by$ *is given by* $\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}$

Example: 16 The straight line $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$, if **[MP PET 1993; MNR 1977]**

(a) $\lambda < \frac{1}{4}$ (b) $\lambda > \frac{1}{4}$ (c) $\lambda = 4$ (d) $\lambda = 1$

Solution: (b) $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$, If $\lambda > \frac{a}{m} = \frac{1}{2.2} = \frac{1}{4} \Rightarrow \lambda > \frac{1}{4}$

Example: 17 If the parabola $y^2 = 4ax$ passes through the point $(1, -2)$, then the tangent at this point is **[MP PET 1998]**

(a) $x + y - 1 = 0$ (b) $x - y - 1 = 0$ (c) $x + y + 1 = 0$ (d) $x - y + 1 = 0$

Solution: (c) $\because$ Parabola passes through the point (1, –2), then $4 = 4a \Rightarrow a = 1$. From $yy_1 = 2a(x + x_1) \Rightarrow -2y = 2(x+1)$

$\therefore$ Required tangent is $x+y+1=0$

Example: 18 The equation of the tangent to the parabola $y^2=16x$, which is perpendicular to the line $y=3x+7$ is **[MP PET 1998]**

(a) $y-3x+4=0$ (b) $3y-x+36=0$ (c) $3y+x-36=0$ (d) $3y+x+36=0$

Solution: (a) A line perpendicular to the given line is $3y+x=\lambda \Rightarrow y=-\frac{1}{3}x+\frac{\lambda}{3}$

Here $m=-\frac{1}{3}$, $c=\frac{\lambda}{3}$. If we compare $y^2=16x$ with $y^2=4ax$, then $a=4$

Condition for tangency is $c=\frac{a}{m} \Rightarrow \frac{\lambda}{3}=\frac{4}{(-1/3)} \Rightarrow \lambda=-36$. $\therefore$ Required equation is $x+3y+36=0$.

Example: 19 If the tangent to the parabola $y^2=ax$ makes an angle of 45^o with x-axis, then the point of contact is **[Rajasthan PET 1985, 90, 2003]**

(a) $\left(\frac{a}{2},\frac{a}{2}\right)$ (b) $\left(\frac{a}{4},\frac{a}{4}\right)$ (c) $\left(\frac{a}{2},\frac{a}{4}\right)$ (d) $\left(\frac{a}{4},\frac{a}{2}\right)$

Solution: (d) Parabola is $y^2=ax$ *i.e.* $y^2=4\left(\frac{a}{4}\right)x$ .....(i)

Let point of contact is (x_1, y_1). $\therefore$ Equation of tangent is $y-y_1=\frac{2(a/4)}{y_1}(x-x_1) \Rightarrow y=\frac{a}{2y_1}(x)-\frac{ax_1}{2y_1}+y_1$

Here, $m=\frac{a}{2y_1}=\tan 45^o \Rightarrow \frac{a}{2y_1}=1 \Rightarrow y_1=\frac{a}{2}$. From (i), $x_1=\frac{a}{4}$, So point is $\left(\frac{a}{4},\frac{a}{2}\right)$.

Example: 20 The line $x-y+2=0$ touches the parabola $y^2=8x$ at the point **[Roorkee 1998]**

(a) $(2,-4)$ (b) $(1,2\sqrt{2})$ (c) $(4,-4\sqrt{2})$ (d) $(2, 4)$

Solution: (d) The line $x-y+2=0$ i.e. $x=y-2$ meets parabola $y^2=8x$, if

$\Rightarrow y^2=8(y-2)=8y-16 \Rightarrow y^2-8y+16=0 \Rightarrow (y-4)^2=0 \Rightarrow y=4,4$

$\because$ Roots are equal, $\therefore$ Given line touches the given parabola.

$\therefore$ $x=4-2=2$, Thus the required point is (2, 4).

Example: 21 The equation of the tangent to the parabola at point $(a/t^2, 2a/t)$ is **[Rajasthan PET 1996]**

(a) $ty=xt^2+a$ (b) $ty=x+at^2$ (c) $y=tx+at^2$ (d) $y=tx+(a/t^2)$

Solution: (a) Equation of the tangent to the parabola, $y^2=4ax$ is $yy_1=2a(x+x_1)$

$\Rightarrow y.\frac{2a}{t}=2a\left(x+\frac{a}{t^2}\right) \Rightarrow \frac{y}{t}=\left(x+\frac{a}{t^2}\right) \Rightarrow \frac{y}{t}=\frac{t^2x+a}{t^2} \Rightarrow ty=t^2x+a$

Example: 22 Two tangents are drawn from the point $(-2,-1)$ to the parabola $y^2=4x$. If α is the angle between these tangents, then $\tan\alpha=$

(a) 3 (b) 1/3 (c) 2 (d) 1/2

Solution: (a) Equation of pair of tangent from $(-2,-1)$ to the parabola is given by $SS_1=T^2$ *i.e.* $(y^2-4x)(1+8)=[y(-1)-2(x-2)]^2$

$\Rightarrow 9y^2-36x=[-y-2x+4]^2 \Rightarrow 9y^2-36x=y^2+4x^2+16+4xy-16x-8y \Rightarrow 4x^2-8y^2+4xy+20x-8y+16=0$

$\therefore \tan\alpha=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|=\left|\frac{2\sqrt{4-4(-8)}}{4-8}\right|=\left|\frac{12}{-4}\right|=3$

Example: 23 If $\left(\frac{a}{b}\right)^{1/3}+\left(\frac{b}{a}\right)^{1/3}=\frac{\sqrt{3}}{2}$, then the angle of intersection of the parabola $y^2=4ax$ and $x^2=4by$ at a point other than the origin is

(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) None of these

Solution: (b) Given parabolas are $y^2=4ax$ ....(i) and $x^2=4by$ .....(ii)

These meet at the points (0, 0), $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

Tangent to (i) at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ is $y.4a^{2/3}b^{1/3} = 2a(x + 4a^{2/3}b^{1/3})$

Slope of the tangent $(m_1) = \frac{2a}{4a^{2/3}b^{1/3}} = \frac{a^{1/3}}{2b^{1/3}}$

Tangent to (ii) at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ is $x.4a^{1/3}b^{2/3} = 2b(y + 4a^{2/3}b^{1/3})$

Slope of the tangent $(m_2) = \frac{2a^{1/3}}{b^{1/3}}$

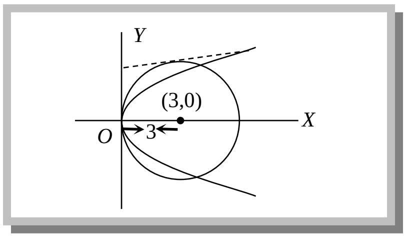
If θ is the angle between the two tangents, then $\Rightarrow$ $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{a^{1/3}}{2b^{1/3}} - \frac{2a^{1/3}}{b^{1/3}}}{1 + \frac{a^{1/3}}{2b^{1/3}} \cdot \frac{2a^{1/3}}{b^{1/3}}}\right|$

$$= \frac{3}{2} \cdot \frac{1}{\left(\frac{a}{b}\right)^{1/3} + \left(\frac{b}{a}\right)^{1/3}} = \frac{3}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} = \sqrt{3}\ ; \ \therefore \theta = 60^o = \frac{\pi}{3}$$

Example: 24 The equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis , is **[IIT Screening 2001]**

(a) $\sqrt{3}y = 3x + 1$ (b) $\sqrt{3}y = -(x + 3)$ (c) $\sqrt{3}y = x + 3$ (d) $\sqrt{3}y = -(3x + 1)$

Solution: (c) Any tangent to $y^2 = 4x$ is $y = mx + \frac{1}{m}$. It touches the circle if $3 = \left|\frac{3m + \frac{1}{m}}{\sqrt{1 + m^2}}\right|$



or $9(1 + m^2) = \left(3m + \frac{1}{m}\right)^2$ or $\frac{1}{m^2} = 3$, $\therefore m = \pm\frac{1}{\sqrt{3}}$

For the common tangent to be above the x-axis, $m = \frac{1}{\sqrt{3}}$

$\therefore$ Common tangent is $y = \frac{1}{\sqrt{3}}x + \sqrt{3} \Rightarrow \sqrt{3}y = x + 3$

Example: 25 If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ then

(a) $d^2 + (3b - 2c)^2 = 0$ (b) $d^2 + (3b + 2c)^2 = 0$ (c) $d^2 + (2b - 3c)^2 = 0$ (d) $d^2 + (2b + 3c)^2 = 0$

Solution: (d) Given parabolas are $y^2 = 4ax$ .....(i) and $x^2 = 4ay$ .....(ii)

from (i) and (ii) $\left(\frac{x^2}{4a}\right)^2 = 4ax \Rightarrow x^4 - 64a^3x = 0 \Rightarrow x = 0\ ,\ 4a \quad \therefore y = 0,\ 4a$

So points of intersection are $(0,0)$ and $(4a, 4a)$

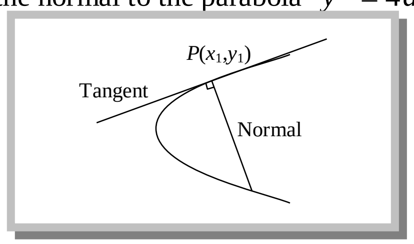
Given, the line $2bx + 3cy + 4d = 0$ passes through (0,0) and $(4a, 4a)$

$\therefore d = 0 \Rightarrow d^2 = 0$ and $(2b + 3c)^2 = 0$ $(\because a \neq 0)$

Therefore $d^2 + (2b + 3c)^2 = 0$

5.1.10 Equations of Normal in Different forms

(1) **Point form :** The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$



Equation of normals of all other standard parabolas at (x_1, y_1)	
Equation of parabolas	**Normal at (x_1, y_1)**
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = -\frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

(2) **Parametric form**: The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y + tx = 2at + at^3$

Equations of normal of all other standard parabola at 't'		
Equations of parabolas	**Parametric co-ordinates**	**Normals at 't'**
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

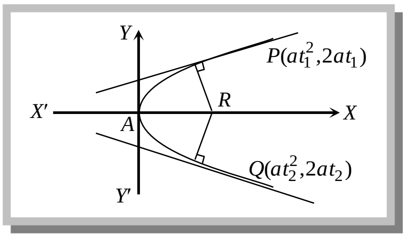
(3) **Slope form:** The equation of normal of slope m to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ at the point $(am^2, -2am)$.

Equations of normal, point of contact, and condition of normality in terms of slope (m)			
Equations of parabola	**Point of contact in terms of slope (m)**	**Equations of normal in terms of slope (m)**	**Condition of normality**
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

Note : ❑ The line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$ if $al(l^2 + 2m^2) + m^2 n = 0$

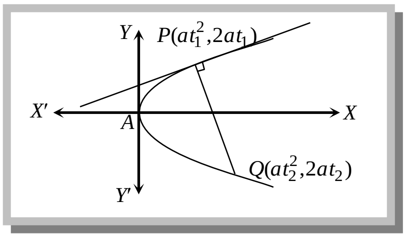
5.1.11 Point of intersection of normals at any two points on the Parabola

If R is the point of intersection then point of intersection of normals at any two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R[2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2(t_1 + t_2)]$



5.1.12 Relation between 't_1' and 't_2' if Normal at 't_1' meets the Parabola again at 't_2'

If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$



Important Tips

- *If the normals at points $(at_1^2, 2at)$ and $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet on the parabola then $t_1 t_2 = 2$*
- *If the normal at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola then $t^2 = 2$.*
- *If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis, then it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.*
- *The normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.*
- *If the normal at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve. Then the product of the ordinate of P and Q is $8a^2$.*

Example: 26 If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then k is **[IIT Screening 2000]**

(a) 3 (b) 9 (c) –9 (d) –3

Solution: (b) Any normal to the parabola $y^2 = 12x$ is $y + tx = 6t + 3t^3$. It is identical with $x + y = k$ if $\frac{t}{1} = \frac{1}{1} = \frac{6t + 3t^3}{k}$

$\therefore t = 1$ and $1 = \frac{6+3}{k} \Rightarrow k = 9$

Example: 27 The equation of normal at the point $\left(\frac{a}{4}, a\right)$ to the parabola $y^2 = 4ax$, is **[Rajasthan PET 1984]**

(a) $4x + 8y + 9a = 0$ (b) $4x + 8y - 9a = 0$ (c) $4x + y - a = 0$ (d) $4x - y + a = 0$

Solution: (b) From $y - y_1 = -\frac{-y_1}{2a}(x - x_1)$

$\Rightarrow y - a = \frac{-a}{2a}\left(x - \frac{a}{4}\right) \Rightarrow 2y + x = 2a + \frac{a}{4} = \frac{9a}{4} \Rightarrow 2y + x - \frac{9a}{4} = 0 \Rightarrow 4x + 8y - 9a = 0$

Example: 28 The point on the parabola $y^2 = 8x$ at which the normal is parallel to the line $x - 2y + 5 = 0$ is

(a) $(-1/2, 2)$ (b) $(1/2, -2)$ (c) $(2, -1/2)$ (d) $(-2, 1/2)$

Solution: (b) Let point be (h, k). Normal is $y - k = \frac{-k}{4}(x - h)$ or $-kx - 4y + kh + 4k = 0$

Gradient $= \frac{-K}{4} = \frac{1}{2} \Rightarrow k = -2$. Substituting (h, k) and $k = -2$ in $y^2 = 8x$, we get $h = \frac{1}{2}$. Hence point is $\left(\frac{1}{2}, -2\right)$

Trick: Here only point $\left(\frac{1}{2}, -2\right)$ will satisfy the parabola $y^2 = 8x$.

Example: 29 The equations of the normal at the ends of the latus rectum of the parabola $y^2 = 4ax$ are given by

(a) $x^2 - y^2 - 6ax + 9a^2 = 0$ (b) $x^2 - y^2 - 6ax - 6ay + 9a^2 = 0$

(c) $x^2 - y^2 - 6ay + 9a^2 = 0$ (d) None of these

Solution: (a) The coordinates of the ends of the latus rectum of the parabola $y^2 = 4ax$ are $(a,\ 2a)$ and $(a, -2a)$ respectively.

The equation of the normal at $(a,\ 2a)$ to $y^2 = 4ax$ is $y - 2a = \frac{-2a}{2a}(x - a)$ $\left\{\text{using } y - y_1 = \frac{-y_1}{2a}(x - x_1)\right\}$

Or $x + y - 3a = 0$ .....(i)

Similarly the equation of the normal at $(a, -2a)$ is $x - y - 3a = 0$ .....(ii)

The combined equation of (i) and (ii) is $x^2 - y^2 - 6ax + 9a^2 = 0$.

Example: 30 The locus of the point of intersection of two normals to the parabola $x^2 = 8y$, which are at right angles to each other, is

[Roorkee 1997]

(a) $x^2 = 2(y - 6)$ (b) $x^2 = 2(y + 6)$ (c) $x^2 = -2(y - 6)$ (d) None of these

Solution: (a) Given parabola is $x^2 = 8y$ .....(i)

Let $(4t_1\ 2t_1^2)$ and $Q(4t_2, 2t_2^2)$ be two points on the parabola (i)

Normal at P, Q are $y - 2t_1^2 = -\frac{1}{t_1}(x - 4t_1)$ ......(ii) and $y - 2t_2^2 = -\frac{1}{t_2}(x - 4t_2)$ ......(iii)

(ii)–(iii) gives $2\left(t_2^2 - t_1^2\right) = x\left(\frac{1}{t_2} - \frac{1}{t_1}\right) = x\,\frac{t_1 - t_2}{t_1 t_2}$, $\therefore x = -2t_1 t_2(t_2 + t_1)$ .....(iv)

From (ii), $y = 2t_1^2 - \frac{1}{t_1}(-2t_1 t_2(t_2 + t_1) - 4t_1) = 2t_1^2 + 2t_2(t_1 + t_2) + 4 \Rightarrow y = 2t_1^2 + 2t_1 t_2 + 2t_2^2 + 4$ .....(v)

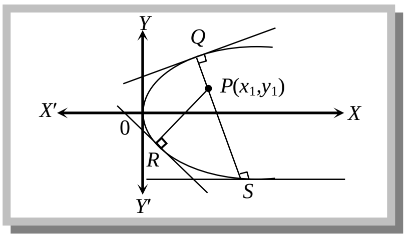
Since normals (ii) and (iii) are at right angles, $\therefore$ $\left(-\frac{1}{t_1}\right)\left(-\frac{1}{t_2}\right) = -1 \Rightarrow t_1 t_2 = -1$

$\therefore$ From (iv), $x = 2(t_1 + t_2)$ and from (v) $y = 2t_1^2 - 2 + 2t_2^2 + 4 \Rightarrow y = 2[t_1^2 + t_2^2 + 1] = 2[(t_1 + t_2)^2 - 2t_1 t_2 + 1]$

$\Rightarrow$ $y = 2[(t_1 + t_2)^2 + 2 + 1] = 2[(t_1 + t_2)^2 + 3 \Rightarrow y = 2\left[\frac{x^2}{4} + 3\right] = \frac{x^2}{2} + 6 \Rightarrow x^2 = 2(y - 6)$, which is the required locus.

5.1.13 Co-normal Points

The points on the curve at which the normals pass through a common point are called co-normal points.



Q, R, S are co-normal points. The co- normal points are also called the feet of the normals.

If the normal passes through point $P(x_1, y_1)$ which is not on parabola, then $y_1 = mx_1 - 2am - am^3 \Rightarrow am^3 + (2a - x_1)m + y_1 = 0$ ......(i)

Which gives three values of m**.** Let three values of m are m_1, m_2 and m_3, which are the slopes of the normals at Q, R and S respectively, then the coordinates of Q, R and S are $(am_1^2, -2am_1), (am_2^2, -2am_2)$ and $(am_{3,}^2\ - 2am_3)$ respectively. These three points are called the feet of the normals.

Now $m_1 + m_2 + m_3 = 0$, $m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{(2a - x_1)}{a}$ and $m_1 m_2 m_3 = \frac{-y_1}{a}$

In general, three normals can be drawn from a point to a parabola.

(1) The algebraic sum of the slopes of three concurrent normals is zero.

(2) The sum of the ordinates of the co-normal points is zero.

(3) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola.

(4) The centroid of a triangle formed by joining the foots of the normal of the parabola lies on its axis and is given by $\left(\frac{am_1^2+am_2^2+am_3^2}{3}, \frac{2am_1+2am_2+2am_3}{3}\right)=\left(\frac{am_1^2+am_2^2+am_3^2}{3}, 0\right)$

(5) If three normals drawn to any parabola $y^2=4ax$ from a given point (h, k) be real, then $h>2a$ for $a=1$, normals drawn to the parabola $y^2=4x$ from any point (h, k) are real, if $h>2$.

(6) Out of these three at least one is real, as imaginary normals will always occur in pairs.

5.1.14 Circle through Co-normal points

Equation of the circle passing through the three (co-normal) points on the parabola $y^2=4ax$, normal at which pass through a given point (α, β); is $x^2+y^2-(2a+\alpha)x-\frac{\beta}{2}y=0$

(1) The algebraic sum of the ordinates of the four points of intersection of a circle and a parabola is zero.

(2) The common chords of a circle and a parabola are in pairs, equally inclined to the axis of parabola.

(3) The circle through co-normal points passes through the vertex of the parabola.

(4) The centroid of four points; in which a circle intersects a parabola, lies on the axis of the parabola.

Example: 31 The normals at three points P, Q, R of the parabola $y^2=4ax$ meet in (h, k), the centroid of triangle PQR lies on **[MP PET 1999]**

(a) $x=0$ (b) $y=0$ (c) $x=-a$ (d) $y=a$

Solution: (b) Since the centroid of the triangle formed by the co-normal points lies on the axis of the parabola.

Example: 32 If two of the three feet of normals drawn from a point to the parabola $y^2=4x$ be (1, 2) and (1, – 2)then the third foot is

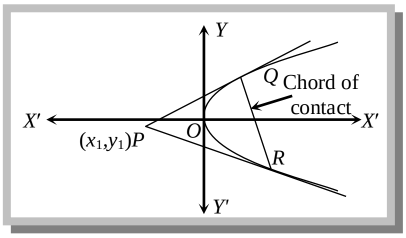
(a) $(2, 2\sqrt{2})$ (b) $(2, -2\sqrt{2})$ (c) $(0, 0)$ (d) None of these

Solution: (c) The sum of the ordinates of the foot $= y_1+y_2+y_3=0$

$\therefore 2+(-2)+y_3=0 \Rightarrow y_3=0$

5.1.15 Equation of the Chord of contact of Tangents to a Parabola

Let PQ and PR be tangents to the parabola $y^2=4ax$ drawn from any external point $P(x_1, y_1)$ then QR is called the 'Chord of contact' of the parabola $y^2=4ax$.

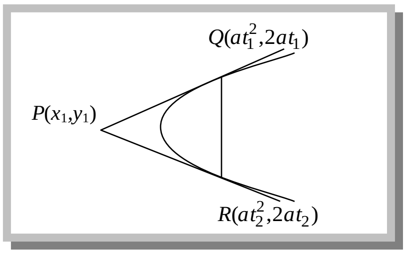


The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$

The equation is same as equation of the tangents at the point (x_1, y_1).

Note : ❑ The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.

❑ If tangents are drawn from the point (x_1, y_1) to the parabola $y^2=4ax$, then the length of their chord of contact is

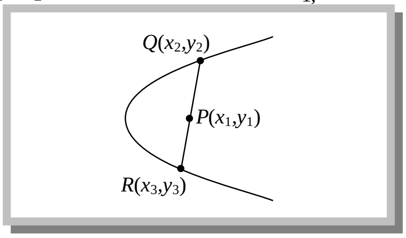


$$\frac{1}{|a|}\sqrt{(y_1^2-4ax_1)(y_1^2+4a^2)}$$

- The area of the triangle formed by the tangents drawn from (x_1,y_1) to $y^2=4ax$ and their chord of contact is $\dfrac{(y_1^2-4ax_1)^{3/2}}{2a}$.

5.1.16 Equation of the Chord of the Parabola which is bisected at a given point

The equation of the chord at the parabola $y^2=4ax$ bisected at the point (x_1,y_1) is given by $T=S_1$, where $T=yy_1-2a(x+x_1)$ and $S_1=y_1^2-4ax_1$. *i.e.*, $yy_1-2a(x+x_1)=y_1^2-4ax_1$



5.1.17 Equation of the Chord joining any two points on the Parabola

Let $P(at_1^2,2at_1), Q(at_2^2,2at_2)$ be any two points on the parabola $y^2=4ax$. Then, the equation of the chord joining these points is, $y-2at_1=\dfrac{2at_2-2at_1}{at_2^2-at_1^2}(x-at_1^2)$ or $y-2at_1=\dfrac{2}{t_1+t_2}(x-at_1^2)$ or $y(t_1+t_2)=2x+2at_1t_2$

(1) **Condition for the chord joining points having parameters t_1 and t_2 to be a focal chord:** If the chord joining points $(at_1^2,2at_1)$ and $(at_2^2,2at_2)$ on the parabola passes through its focus, then $(a,0)$ satisfies the equation $y(t_1+t_2)=2x+2at_1t_2 \Rightarrow 0=2a+2at_1t_2 \Rightarrow t_1t_2=-1$ or $t_2=-\dfrac{1}{t_1}$

(2) **Length of the focal chord:** The length of a focal chord having parameters t_1 and t_2 for its end points is $a(t_2-t_1)^2$.

Note:
- If one extremity of a focal chord is $(at_1^2,2at_1)$, then the other extremity $(at_2^2,2at_2)$ becomes $\left(\dfrac{a}{t_1^2},\dfrac{-2a}{t_1}\right)$ by virtue of relation $t_1t_2=-1$.
- If one end of the focal chord of parabola is $(at^2,2at)$, then other end will be $\left(\dfrac{a}{t^2},-2at\right)$ and length of chord $=a\left(t+\dfrac{1}{t}\right)^2$.
- The length of the chord joining two point 't_1' and 't_2' on the parabola $y^2=4ax$ is $a(t_1-t_2)\sqrt{(t_1+t_2)^2+4}$
- The length of intercept made by line $y=mx+c$ between the parabola $y^2=4ax$ is $\dfrac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$.

Important Tips

☞ *The focal chord of parabola $y^2=4ax$ making an angle α with the x-axis is of length $4a\operatorname{cosec}^2\alpha$.*

☞ *The length of a focal chord of a parabola varies inversely as the square of its distance from the vertex.*

☞ *If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latus-rectum is* $\dfrac{4l_1l_2}{l_1+l_2}$

☞ *The semi latus rectum of the parabola $y^2=4ax$ is the harmonic mean between the segments of any focal chord of the parabola.*

Example: 33 If the points $(au^2, 2au)$ and $(av^2, 2av)$ are the extremities of a focal chord of the parabola $y^2=4ax$, then **[MP PET 1998, 93]**

(a) $uv-1=0$ (b) $uv+1=0$ (c) $u+v=0$ (d) $u-v=0$

Solution: (b) Equation of focal chord for the parabola $y^2=4ax$ passes through the point $(au^2, 2au)$ and $(av^2, 2av)$

$$\Rightarrow \quad y-2au=\frac{2av-2au}{av^2-au^2}(x-au^2)\Rightarrow y-2au=\frac{2a(v-u)}{a(v-u)(v+u)}(x-au^2)\Rightarrow y-2au=\frac{2}{v+u}(x-au^2)$$

It this is focal chord, so it would passes through focus $(a, 0)$

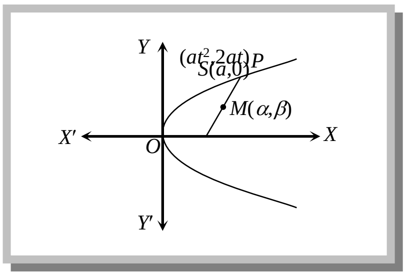
$$\Rightarrow \quad 0-2au=\frac{2}{v+u}(a-au^2)\Rightarrow -uv-u^2=1-u^2,\ \therefore\ uv+1=0$$

Trick : Given points $(au^2,\ 2au)$ and $(av^2,\ 2av)$, then $t_1=u$ and $t_2=v$, we know that $t_1t_2=-1$. Hence $uv+1=0$.

Example: 34 The locus of the midpoint of the line segment joining the focus to a moving point on the parabola $y^2=4ax$ is another parabola with the directrix **[IIT Screening 2002]**

(a) $x=-a$ (b) $x=-\dfrac{a}{2}$ (c) $x=0$ (d) $x=\dfrac{a}{2}$

Solution: (c) Let $M(\alpha,\ \beta)$ be the mid point of PS.



$$\alpha=\frac{at^2+a}{2},\ \beta=\frac{2at+0}{2}\ \Rightarrow\ 2\alpha=at^2+a,\ \ at=\beta$$

$$\therefore\ \ 2\alpha=a.\frac{\beta^2}{a^2}+a \text{ or } 2a\alpha=\beta^2+a^2$$

$$\therefore\ \text{The locus is } y^2=\frac{4a}{2}\left(x-\frac{a}{2}\right)=4b(x-b),\left\{b=\frac{a}{2}\right\}$$

$\therefore$ Directrix is $(x-b)+b=0$ or $x=0$.

Example: 35 The length of chord of contact of the tangents drawn from the point (2, 5) to the parabola $y^2=8x$, is **[MNR 1976]**

(a) $\dfrac{1}{2}\sqrt{41}$ (b) $\sqrt{41}$ (c) $\dfrac{3}{2}\sqrt{41}$ (d) $2\sqrt{41}$

Solution: (c) Equation of chord of contact of tangents drawn from a point (x_1, y_1) to parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$. So that $5y=2\times 2(x+2)\Rightarrow 5y=4x+8$.

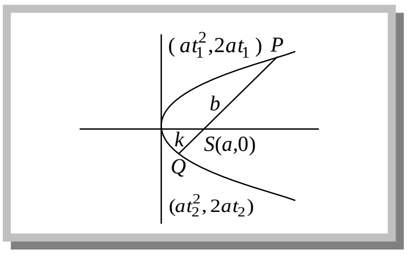
Point of intersection of chord of contact with parabola $y^2=8x$ are $\left(\dfrac{1}{2},\ 2\right)$, (8, 8), So the length of chord is $\dfrac{3}{2}\sqrt{41}$.

Example: 36 If b, k are the intercept of a focal chord of the parabola $y^2=4ax$, then K is equal to **[Rajasthan PET 1999]**

(a) $\dfrac{ab}{b-a}$ (b) $\dfrac{b}{b-a}$ (c) $\dfrac{a}{b-a}$ (d) $\dfrac{ab}{a-b}$

Solution: (a) Let 't_1' 't_2' be the ends of focal chords

$\therefore$. $t_1t_2=-1$. If S is the focus and P, Q are the ends of the focal chord, then



$$SP=\sqrt{(at_1^2-a)^2+(2at_1-0)^2}=a\left(t_1^2+1\right)=b \quad \text{(Given).... (i)}$$

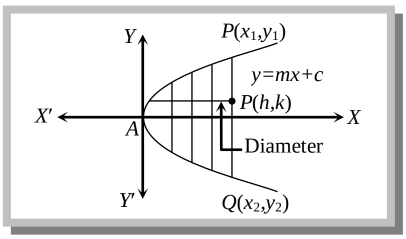
$$\therefore\ SQ=a\left(t_2^2+1\right)=a\left(\frac{1}{t_1^2}+1\right)\ \text{(Given)}\ \left[\because t_2=-\frac{1}{t_1}\Rightarrow t_2^2=\frac{1}{t_1^2}\right]$$

$$=\frac{a(t_1^2+1)}{t_1^2}=k \quad \text{....(ii)},\ \therefore\ \frac{b}{k}=t_1^2 \quad \text{[Divide (i) by (ii)]}$$

Putting in (1), we get $a\left(\dfrac{b}{k}+1\right)=b\Rightarrow \dfrac{ab}{k}+a=b\Rightarrow k=\dfrac{ab}{b-a}$

5.1.18 Diameter of a Parabola

The locus of the middle points of a system of parallel chords is called a diameter and in case of a parabola this diameter is shown to be a straight line which is parallel to the axis of the parabola.



The equation of the diameter bisecting chords of the parabola $y^2 = 4ax$ of slope m is $y = \dfrac{2a}{m}$

Note : ❑ Every diameter of a parabola is parallel to its axis.

❑ The tangent at the end point of a diameter is parallel to corresponding system of parallel chords.

❑ The tangents at the ends of any chord meet on the diameter which bisects the chord.

Example: 37 Equation of diameter of parabola $y^2 = x$ corresponding to the chord $x - y + 1 = 0$ is **[Rajasthan PET 2003]**

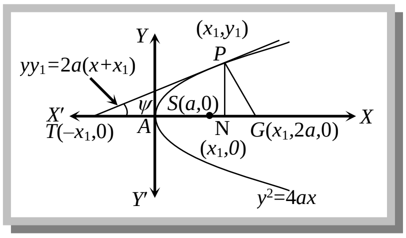
(a) $2y = 3$ (b) $2y = 1$ (c) $2y = 5$ (d) $y = 1$

Solution: (b) Equation of diameter of parabola is $y = \dfrac{2a}{m}$, Here $a = \dfrac{1}{4}, m = 1 \Rightarrow y = \dfrac{2.\frac{1}{4}}{1} \Rightarrow 2y = 1$

5.1.19 Length of Tangent, Subtangent ,Normal and Subnormal

Let the parabola $y^2 = 4ax$. Let the tangent and normal at $P(x_1, y_1)$ meet the axis of parabola at T and G respectively, and tangent at $P(x_1, y_1)$ makes angle ψ with the positive direction of x-axis.

$A(0,0)$ is the vertex of the parabola and $PN = y$. Then,



(1) Length of tangent $= PT = PN \operatorname{cosec} \psi = y_1 \operatorname{cosec} \psi$

(2) Length of normal $= PG = PN \operatorname{cosec}(90^o - \psi) = y_1 \sec \psi$

(3) Length of subtangent $= TN = PN \cot \psi = y_1 \cot \psi$

(4) Length of subnormal $= NG = PN \cot(90^o - \psi) = y_1 \tan \psi$

where , $\tan \psi = \dfrac{2a}{y_1} = m$, [slope of tangent at $P(x, y)$]

Length of tangent, subtangent, normal and subnormal to $y^2 = 4ax$ at $(at^2, 2at)$

(1) Length of tangent at $(at^2, 2at) = 2at \operatorname{cosec} \psi \; = 2at\sqrt{(1 + \cot^2 \psi)} \; = 2at\sqrt{1 + t^2}$

(2) Length of normal at $(at^2, 2at) = 2at \sec \psi = 2at\sqrt{(1 + \tan^2 \psi)} = 2a\sqrt{t^2 + t^2 \tan^2 \psi} \; = 2a\sqrt{(t^2 + 1)}$

(3) Length of subtangent at $(at^2, 2at) = 2at \cot \psi = 2at^2$

(4) Length of subnormal at $(at^2, 2at) = 2at \tan \psi = 2a$

Example: 38 The length of the subtangent to the parabola $y^2 = 16x$ at the point whose abscissa is 4, is

(a) 2 (b) 4 (c) 8 (d) None of these

Solution: (c) Since the length of the subtangent at a point to the parabola is twice the abscissa of the point. Therefore, the required length is 8.

Example: 39 If P is a point on the parabola $y^2 = 4ax$ such that the subtangent and subnormal at P are equal, then the coordinates of P are

(a) $(a, 2a)$ or $(a, -2a)$ (b) $(2a, 2\sqrt{2}a)$ or $(2a, -2\sqrt{2}a)$

(c) $(4a, -4a)$ or $(4a, 4a)$ (d) None of these

Solution: (a) Since the length of the subtangent at a point on the parabola is twice the abscissa of the point and the length of the subnormal is equal to semi-latus-rectum. Therefore if $P(x, y)$ is the required point, then $2x = 2a \Rightarrow x = a$

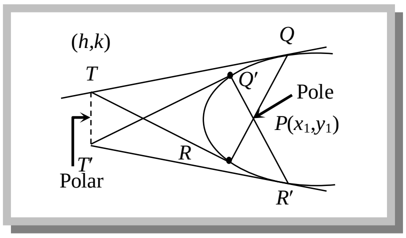
Now (x, y) lies on the parabola $y^2 = 4ax \Rightarrow 4a^2 = y^2 \Rightarrow y = \pm 2a$

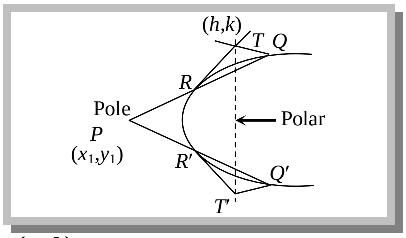
Thus the required points are $(a, 2a)$ and $(a, -2a)$.

5.1.20 Pole and Polar

The locus of the point of intersection of the tangents to the parabola at the ends of a chord drawn from a fixed point P is called the polar of point P and the point P is called the pole of the polar.

Equation of polar: Equation of polar of the point (x_1, y_1) with respect to parabola $y^2 = 4ax$ is same as chord of contact and is given by $yy_1 = 2a(x + x_1)$

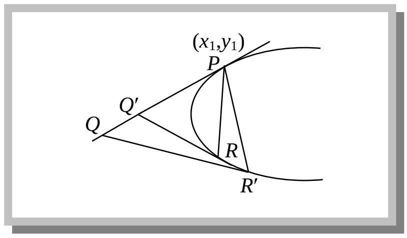




(1) **Polar of the focus is directrix:** Since the focus is $(a, 0)$

$\therefore$ Equation of polar of $y^2 = 4ax$ is $y.0 = 2a(x + a) \Rightarrow x + a = 0$, which is the directrix of the parabola $y^2 = 4ax$.

(2) **Any tangent is the polar of its point of contact:** If the point $P(x_1 y_1)$ be on the parabola. Its polar and tangent at P are identical. Hence the tangent is the polar of its own point of contact.



Coordinates of pole: The pole of the line $lx + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

(i) Pole of the chord joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{y_1 y_2}{4a}, \frac{y_1 + y_2}{2}\right)$ which is the same as the point of intersection of tangents at (x_1, y_1) and (x_2, y_2).

(ii) The point of intersection of the polar of two points Q and R is the pole of QR.

5.1.21 Characterstics of Pole and Polar

(1) **Conjugate points:** If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the parabola $y^2 = 4ax$, if $y_1 y_2 = 2a(x_1 + x_2)$.

(2) **Conjugate lines:** If the pole of a line $ax + by + c = 0$ lies on the another line $a_1 x + b_1 y + c_1 = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

Two lines $l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$ are conjugate lines with respect to parabola $y^2 = 4ax$, if $(l_1 n_2 + l_2 n_1) = 2am_1 m_2$

Note : ❑ The chord of contact and polar of any point on the directrix always passes through focus.

❑ The pole of a focal chord lies on directrix and locus of poles of focal chord is the directrix.

❑ The polars of all points on directrix always pass through a fixed point and this fixed point is focus.

Example: 40 The pole of the line $2x = y$ with respect to the parabola $y^2 = 2x$ is

(a) $\left(0, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 0\right)$ (c) $\left(0, -\frac{1}{2}\right)$ (d) None of these

Solution: (a) Let (x_1, y_1) be the pole of line $2x = y$ *w.r.t.* parabola $y^2 = 2x$ its polar is $yy_1 = x + x_1$

Also polar is $y = 2x$, $\therefore \frac{y_1}{1} = \frac{1}{2} = \frac{x_1}{0}$, $\therefore x_1 = 0, y_1 = \frac{1}{2}$. So Pole is $\left(0, \frac{1}{2}\right)$

Example: 41 If the polar of a point with respect to the circle $x^2 + y^2 = r^2$ touches the parabola $y^2 = 4ax$, the locus of the pole is **[EAMCET 1995]**

(a) $y^2 = -\frac{r^2}{a}x$ (b) $x^2 = \frac{-r^2}{a}y$ (c) $y^2 = \frac{r^2}{a}x$ (d) $x^2 = \frac{r^2}{a}y$

Solution: (a) Polar of a point (x_1, y_1) *w.r.t.* $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$ *i.e.* $yy_1 = -xx_1 + r^2$

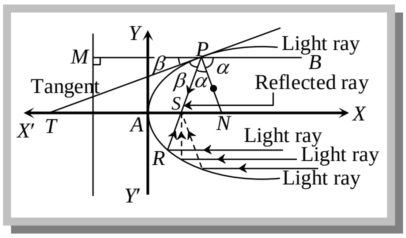
$\Rightarrow y = -\frac{x_1}{y_1}x + \frac{r^2}{y_1} \Rightarrow y = mx + c$, where $m = -\frac{x_1}{y_1}; c = \frac{r^2}{y_1}$

This touches the parabola $y^2 = 4ax$, If $c = \frac{a}{m} \Rightarrow \frac{r^2}{y_1} = \frac{a}{-x_1/y_1} = -\frac{ay_1}{x_1}$

$\therefore$ Required locus of pole (x_1, y_1) is $\frac{r^2}{y} = \frac{-ay}{x}$ *i.e.*, $y^2 = \frac{-r^2}{a}x$

5.1.22 Reflection property of a Parabola

The tangent (PT) and normal (PN) of the parabola $y^2 = 4ax$ at. P are the internal and external bisectors of $\angle SPM$ and BP is parallel to the axis of the parabola and $\angle BPN = \angle SPN$



Note : ❑ When the incident ray is parallel to the axis of the parabola, the reflected ray will always pass through the focus.

Example: 42 A ray of light moving parallel to the *x*-axis gets reflected from a parabolic mirror whose equation is $(y-2)^2 = 4(x+1)$. After reflection, the ray must pass through the point

(a) (0, 2) (b) (2, 0) (c) (0, –2) (d) (–1, 2)

Solution: (a) The equation of the axis of the parabola is $y - 2 = 0$, which is parallel to the *x*-axis. So, a ray parallel to *x*-axis is parallel to the axis of the parabola. We know that any ray parallel to the axis of a parabola passes through the focus after reflection. Here (0, 2) is the focus.

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Assignment

Conic Section: General

Basic Level

1. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents **[IIT Screening 1994]**

(a) No locus, if $k > 0$ (b) An ellipse, if $k < 0$ (c) A point, if $k = 0$ (d) A hyperbola, if $k > 0$

2. The equation $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ represents **[BIT 1986]**

(a) A circle (b) An ellipse (c) A hyperbola (d) A rectangular hyperbola

3. Eccentricity of the parabola $x^2 - 4x - 4y + 4 = 0$ is **[Rajasthan PET 1996]**

(a) $e = 0$ (b) $e = 1$ (c) $e > 4$ (d) $e = 4$

4. $x^2 - 4y^2 - 2x + 16y - 40 = 0$ represents **[DCE 1999]**

(a) A pair of straight lines (b) An ellipse (c) A hyperbola (d) A parabola

5. The centre of the conic represented by the equation $2x^2 - 72xy + 23y^2 - 4x - 28y - 48 = 0$ is

(a) $\left(\frac{11}{15}, \frac{2}{25}\right)$ (b) $\left(\frac{2}{25}, \frac{11}{25}\right)$ (c) $\left(\frac{11}{25}, -\frac{2}{25}\right)$ (d) $\left(-\frac{11}{25}, -\frac{2}{25}\right)$

Definition, Standard Equation of Parabola and Terms related to Parabola

Basic Level

6. The equation of the parabola with focus (a, b) and directrix $\frac{x}{a} + \frac{y}{b} = 1$ is given by **[MP PET 1997]**

(a) $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0$ (b) $(ax + by)^2 - 2a^3x - 2b^3y - a^4 + a^2b^2 - b^4 = 0$

(c) $(ax - by)^2 + a^4 + b^4 - 2a^3x = 0$ (d) $(ax - by)^2 - 2a^3x = 0$

7. The equation of the parabola with focus $(3, 0)$ and the directrix $x + 3 = 0$ is **[EAMCET 2002]**

(a) $y^2 = 3x$ (b) $y^2 = 2x$ (c) $y^2 = 12x$ (d) $y^2 = 6x$

8. The parabola $y^2 = x$ is symmetric about **[Kerala (Engg.) 2002]**

(a) *x*-axis (b) *y*-axis (c) Both *x*-axis and *y*-axis (d) The line $y = x$

9. The focal distance of a point on the parabola $y^2 = 16x$ whose ordinate is twice the abscissa, is

(a) 6 (b) 8 (c) 10 (d) 12

10. The points on the parabola $y^2 = 12x$, whose focal distance is 4, are

(a) $(2, \sqrt{3}), (2, -\sqrt{3})$ (b) $(1, 2\sqrt{3}), (1, -2\sqrt{3})$ (c) $(1, 2)$ (d) None of these

11. The coordinates of the extremities of the latus rectum of the parabola $5y^2 = 4x$ are

(a) $(1/5, 2/5); (-1/5, 2/5)$ (b) $(1/5, 2/5); (1/5, -2/5)$ (c) $(1/5, 4/5); (1/5, -4/5)$ (d) None of these

12. If the vertex of a parabola be at origin and directrix be $x + 5 = 0$, then its latus rectum is **[Rajasthan PET 1991]**

(a) 5 (b) 10 (c) 20 (d) 40

13. The equation of the lines joining the vertex of the parabola $y^2 = 6x$ to the points on it whose abscissa is 24, is

(a) $y \pm 2x = 0$ (b) $2y \pm x = 0$ (c) $x \pm 2y = 0$ (d) $2x \pm y = 0$

14. PQ is a double ordinate of the parabola $y^2 = 4ax$. The locus of the points of trisection of PQ is

(a) $9y^2 = 4ax$ (b) $9x^2 = 4ay$ (c) $9y^2 + 4ax = 0$ (d) $9x^2 + 4ay = 0$

15. The equation of a parabola is $25\{(x-2)^2 + (y+5)^2\} = (3x + 4y - 1)^2$. For this parabola

(a) Vertex $= (2,-5)$ (b) Focus $= (2,-5)$

(c) Directrix has the equation $3x + 4y - 1 = 0$ (d) Axis has the equation $3x + 4y - 1 = 0$

16. The co-ordinates of a point on the parabola $y^2 = 8x$, whose focal distance is 4, is

(a) $(2, 4)$ (b) $(4, 2)$ (c) $(2, -4)$ (d) $(4, -2)$

17. The equation of the parabola with $(-3, 0)$ as focus and $x + 5 = 0$ as directrix, is **[Rajasthan PET 1985, 86, 89; MP PET 1991]**

(a) $x^2 = 4(y+4)$ (b) $x^2 = 4(y-4)$ (c) $y^2 = 4(x+4)$ (d) $y^2 = 4(x-4)$

Advance Level

18. A double ordinate of the parabola $y^2 = 8px$ is of length $16p$. The angle subtended by it at the vertex of the parabola is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) None of these

19. If $(2,-8)$ is at an end of a focal chord of the parabola $y^2 = 32x$; then the other end of the chord is

(a) $(32, 32)$ (b) $(32,-32)$ (c) $(-2, 8)$ (d) None of these

20. A square has one vertex at the vertex of the parabola $y^2 = 4ax$ and the diagonal through the vertex lies along the axis of the parabola. If the ends of the other diagonal lie on the parabola, the co-ordinates of the vertices of the square are

(a) $(4a, 4a)$ (b) $(4a, -4a)$ (c) $(0, 0)$ (d) $(8a, 0)$

Other standard forms of Parabola

Basic Level

21. A parabola passing through the point (–4,–2) has its vertex at the origin and y-axis as its axis. The latus rectum of the parabola is

(a) 6 (b) 8 (c) 10 (d) 12

22. The focus of the parabola $x^2 = -16y$ is **[Rajasthan PET 1987; MP PET 1988, 1992]**

(a) (4, 0) (b) (0, 4) (c) (– 4, 0) (d) (0, – 4)

23. The end points of latus rectum of the parabola $x^2 = 4ay$ are **[Rajasthan PET 1997]**

(a) $(a, 2a), (2a, -a)$ (b) $(-a, 2a), (2a, a)$ (c) $(a, -2a), (2a, a)$ (d) $(-2a, a), (2a, a)$

24. The ends of latus rectum of parabola $x^2 + 8y = 0$ are **[MP PET 1995]**

(a) (–4, –2) and (4, 2) (b) (4, –2) and (–4, 2) (c) (–4, –2) and (4, –2) (d) (4, 2) and (–4, 2)

25. Given the two ends of the latus rectum, the maximum number of parabolas that can be drawn is

(a) 1 (b) 2 (c) 0 (d) Infinite

26. The length of the latus rectum of the parabola $9x^2 - 6x + 36y + 19 = 0$ is **[MP PET 1994]**

(a) 36 (b) 9 (c) 6 (d) 4

Special form of Parabola

Basic Level

27. Vertex of the parabola $y^2 + 2y + x = 0$ lies in the quadrant **[MP PET 1989]**

(a) First (b) Second (c) Third (d) Fourth

28. The vertex of the parabola $3x - 2y^2 - 4y + 7 = 0$ is **[Rajasthan PET 1996]**

(a) (3, 1) (b) (–3, –1) (c) (–3, 1) (d) None of these

29. The vertex of parabola $(y-2)^2 = 16(x-1)$ is **[Karnataka CET 2001]**

(a) (2, 1) (b) (1, –2) (c) (–1, 2) (d) (1, 2)

30. The vertex of the parabola $x^2 + 8x + 12y + 4 = 0$ is **[DCE 1999]**

(a) (–4, 1) (b) (4, –1) (c) (–4, –1) (d) (4, 1)

31. The axis of the parabola $9y^2 - 16x - 12y - 57 = 0$ is **[MNR 1995]**

(a) $3y = 2$ (b) $x + 3y = 3$ (c) $2x = 3$ (d) $y = 3$

32. The directrix of the parabola $x^2 - 4x - 8y + 12 = 0$ is **[Karnataka CET 2003]**

(a) $x = 1$ (b) $y = 0$ (c) $x = -1$ (d) $y = -1$

33. The length of the latus rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is **[MP PET 2000]**

(a) 4 (b) 6 (c) 8 (d) 10

34. The latus rectum of the parabola $y^2 = 5x + 4y + 1$ is **[MP PET 1996]**

(a) $\frac{5}{4}$ (b) 10 (c) 5 (d) $\frac{5}{2}$

35. If (2, 0) is the vertex and y-axis the directrix of a parabola then its focus is **[MNR 1981]**

(a) (2, 0) (b) (–2, 0) (c) (4, 0) (d) (–4, 0)

36. The length of latus rectum of the parabola $4y^2 + 2x - 20y + 17 = 0$ is **[MP PET 1999]**

(a) 3 (b) 6 (c) $\frac{1}{2}$ (d) 9

37. The focus of the parabola $y^2 = 4y - 4x$ is **[MP PET 1991]**

(a) (0, 2) (b) (1, 2) (c) (2, 0) (d) (2, 1)

38. Focus of the parabola $(y-2)^2 = 20(x+3)$ is **[Karnataka CET 1999]**

(a) (3, –2) (b) (2, –3) (c) (2, 2) (d) (3, 3)

39. The focus of the parabola $y^2 - x - 2y + 2 = 0$ is **[UPSEAT 2000]**

(a) (1/4, 0) (b) (1, 2) (c) (3/4, 1) (d) (5/4, 1)

40. The focus of the parabola $y = 2x^2 + x$ is **[MP PET 2000]**

(a) (0, 0) (b) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (c) $\left(-\frac{1}{4}, 0\right)$ (d) $\left(-\frac{1}{4}, \frac{1}{8}\right)$

41. The vertex of a parabola is the point (a, b) and latus rectum is of length l. If the axis of the parabola is along the positive direction of y-axis, then its equation is

(a) $(x+a)^2 = \frac{l}{2}(2y - 2b)$ (b) $(x-a)^2 = \frac{l}{2}(2y - 2b)$ (c) $(x+a)^2 = \frac{l}{4}(2y - 2b)$ (d) $(x-a)^2 = \frac{l}{8}(2y - 2b)$

42. $y^2 - 2x - 2y + 5 = 0$ represents **[Roorkee 1986, 95]**

(a) A circle whose centre is (1, 1) (b) A parabola whose focus is (1, 2)

(c) A parabola whose directrix is $x = \frac{3}{2}$ (d) A parabola whose directrix is $x = -\frac{1}{2}$

43. The length of the latus rectum of the parabola whose focus is (3, 3) and directrix is $3x - 4y - 2 = 0$ is **[UPSEAT 2001]**

(a) 2 (b) 1 (c) 4 (d) None of these

44. The equation of the parabola whose vertex is at (2, –1) and focus at (2, – 3)is **[Kerala (Engg.) 2002]**

(a) $x^2+4x-8y-12=0$ (b) $x^2-4x+8y+12=0$ (c) $x^2+8y=12$ (d) $x^2-4x+12=0$

45. The equation of the parabola with focus (0, 0) and directrix $x+y=4$ is **[EAMCET 2003]**

(a) $x^2+y^2-2xy+8x+8y-16=0$ (b) $x^2+y^2-2xy+8x+8y=0$

(c) $x^2+y^2+8x+8y-16=0$ (d) $x^2-y^2+8x+8y-16=0$

46. The equation of the parabola whose vertex and focus lies on the *x*-axis at distance a and a' from the origin, is **[Rajasthan PET 2000]**

(a) $y^2=4(a'-a)(x-a)$ (b) $y^2=4(a'-a)(x+a)$ (c) $y^2=4(a'+a)(x-a)$ (d) $y^2=4(a'+a)(x+a)$

47. The equation of parabola whose vertex and focus are (0, 4)and (0, 2) respectively, is **[Rajasthan PET 1987, 1989, 1990, 1991]**

(a) $y^2-8x=32$ (b) $y^2+8x=32$ (c) $x^2+8y=32$ (d) $x^2-8y=32$

48. The equation of the parabola, whose vertex is (–1, –2) axis is vertical and which passes through the point (3, 6)is

(a) $x^2+2x-2y-3=0$ (b) $2x^2=3y$ (c) $x^2-2x-y+3=0$ (d) None of these

49. The length of the latus rectum of the parabola whose focus is $\left(\frac{u^2}{2g}\sin 2\alpha, -\frac{u^2}{2g}\cos 2\alpha\right)$ and directrix is $y=\frac{u^2}{2g}$, is

(a) $\frac{u^2}{g}\cos^2\alpha$ (b) $\frac{u^2}{g}\cos 2\alpha$ (c) $\frac{2u^2}{g}\cos 2\alpha$ (d) $\frac{2u^2}{g}\cos^2\alpha$

50. The equation of the parabola whose axis is vertical and passes through the points (0, 0), (3, 0) and (–1, 4), is

(a) $x^2-3x-y=0$ (b) $x^2+3x+y=0$ (c) $x^2-4x+2y=0$ (d) $x^2-4x-2y=0$

51. If the vertex and the focus of a parabola are (–1, 1) and (2, 3) respectively, then the equation of the directrix is

(a) $3x+2y+14=0$ (b) $3x+2y-25=0$ (c) $2x-3y+10=0$ (d) None of these

52. If the focus of a parabola is (–2, 1) and the directrix has the equation $x+y=3$, then the vertex is

(a) (0, 3) (b) (–1, 1/2) (c) (–1, 2) (d) (2, –1)

53. The vertex of a parabola is (a, 0) and the directrix is $x+y=3a$. The equation of the parabola is

(a) $x^2+2xy+y^2+6ax+10ay+7a^2=0$ (b) $x^2-2xy+y^2+6ax+10ay=7a^2$

(c) $x^2-2xy+y^2-6ax+10ay=7a^2$ (d) None of these

54. The equation of a locus is $y^2+2ax+2by+c=0$, then

(a) It is an ellipse (b) It is a parabola (c) Its latus rectum $=a$ (d) Its latus rectum$=2a$

55. If the vertex of the parabola $y=x^2-8x+c$ lies on *x*-axis, then the value of c is

(a) –16 (b) –4 (c) 4 (d) 16

56. If the vertex of a parabola is the point (–3, 0) and the directrix is the line $x+5=0$ then its equation is

(a) $y^2=8(x+3)$ (b) $x^2=8(y+3)$ (c) $y^2=-8(x+3)$ (d) $y^2=8(x+5)$

57. If the parabola $y^2=4ax$ passes through (3, 2), then the length of its latusrectum is

(a) 2/3 (b) 4/3 (c) 1/3 (d) 4

58. The extremities of latus rectum of the parabola $(y-1)^2=2(x+2)$ are

(a) $\left(-\frac{3}{2}, 2\right)$ (b) $(-2, 1)$ (c) $\left(-\frac{3}{2}, 0\right)$ (d) $\left(-\frac{3}{2}, 1\right)$

59. The equation of parabola is given by $y^2+8x-12y+20=0$. Tick the correct options given below

(a) Vertex (2, 6) (b) Focus (0, 6) (c) Latus rectum $=4$ (d) axis $y=6$

Advance Level

60. The length of the latus rectum of the parabola $169\{(x-1)^2+(y-3)^2\}=(5x-12y+17)^2$ is

(a) $\frac{14}{13}$ (b) $\frac{28}{13}$ (c) $\frac{12}{13}$ (d) None of these

61. The length of the latus rectum of the parabola $x=ay^2+by+c$ is

(a) $\frac{a}{4}$ (b) $\frac{a}{3}$ (c) $\frac{1}{a}$ (d) $\frac{1}{4a}$

62. If the vertex = (2, 0) and the extremities of the latus rectum are (3, 2) and (3, –2), then the equation of the parabola is

(a) $y^2=2x-4$ (b) $x^2=4y-8$ (c) $y^2=4x-8$ (d) None of these

63. Let there be two parabolas with the same axis, focus of each being exterior to the other and the latus recta being $4a$ and $4b$. The locus of the middle points of the intercepts between the parabolas made on the lines parallel to the common axis is a

(a) Straight line if $a=b$ (b) Parabola if $a\neq b$ (c) Parabola for all a, b (d) None of these

64. A line L passing through the focus of the parabola $y^2=4(x-1)$ intersects the parabola in two distinct points. If 'm' be the slope of the line L, then

(a) $-1<m<1$ (b) $m<-1$ or $m>1$ (c) $m\in R$ (d) None of these

Parametric equations of Parabola

Basic Level

65. Which of the following points lie on the parabola $x^2=4ay$ **[Rajasthan PET 2002]**

(a) $x=at^2, y=2at$ (b) $x=2at, y=at$ (c) $x=2at^2, y=at$ (d) $x=2at, y=at^2$

66. The parametric equation of a parabola is $x=t^2+1, y=2t+1$. The cartesian equation of its directrix is

(a) $x=0$ (b) $x+1=0$ (c) $y=0$ (d) None of these

67. The parametric representation $(2+t^2, 2t+1)$ represents

(a) A parabola with focus at (2, 1) (b) A parabola with vertex at (2, 1)
(c) An ellipse with centre at (2, 1) (d) None of these

68. The graph represented by the equations $x=\sin^2 t$, $y=2\cos t$ is **[EAMCET 1993]**

(a) A portion of a parabola (b) A parabola (c) A part of a sine graph (d) A Part of a hyperbola

69. The curve described parametrically by $x=t^2+t+1, y=t^2-t+1$ represents **[IIT 1999]**

(a) A pair of straight lines (b) An ellipse (c) A parabola (d) A hyperbola

Position of a Point, Intersection of Line and Parabola, Tangents and Pair of Tangents

Basic Level

70. The equation of the tangent at a point $P(t)$ where 't' is any parameter to the parabola $y^2=4ax$, is **[MNR 1983]**

(a) $yt=x+at^2$ (b) $y=xt+at^2$ (c) $y=xt+\frac{a}{t}$ (d) $y=tx$

71. The condition for which the straight line $y=mx+c$ touches the parabola $y^2=4ax$ is **[MP PET 1997, 2001]**

(a) $a=c$ (b) $\frac{a}{c}=m$ (c) $m=a^2c$ (d) $m=ac^2$

72. The line $y=mx+c$ touches the parabola $x^2=4ay$, if **[MNR 1973; MP PET 1994, 1999]**

(a) $c=-am$ (b) $c=-a/m$ (c) $c=-am^2$ (d) $c=a/m^2$

73. The line $y=2x+c$ is tangent to the parabola $y^2=16x$, if c equals **[MNR 1988]**

(a) –2 (b) –1 (c) 0 (d) 2

74. The line $y = 2x + c$ is tangent to the parabola $y^2 = 4x$, then $c =$ **[MP PET 1996]**

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) 4

75. If line $x = my + k$ touches the parabola $x^2 = 4ay$, then $k =$ **[MP PET 1995]**

(a) $\frac{a}{m}$ (b) am (c) am^2 (d) $-am^2$

76. The line $y = mx + 1$ is a tangent to the parabola $y^2 = 4x$, if **[MNR 1990; Kurukshetra CEE 1998; DCE 2000]**

(a) $m = 1$ (b) $m = 2$ (c) $m = 4$ (d) $m = 3$

77. The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if **[Rajasthan PET 1988; MNR 1977; MP PET 2003]**

(a) $mn = al^2$ (b) $lm = an^2$ (c) $ln = am^2$ (d) $mn = al$

78. The equation of the tangent to the parabola $y^2 = 4x + 5$ parallel to the line $y = 2x + 7$ is **[MNR 1979]**

(a) $2x - y - 3 = 0$ (b) $2x - y + 3 = 0$ (c) $2x + y + 3 = 0$ (d) None of these

79. If $lx + my + n = 0$ is tangent to the parabola $x^2 = y$, then condition of tangency is **[Rajasthan PET 1999]**

(a) $l^2 = 2mn$ (b) $l = 4m^2n^2$ (c) $m^2 = 4ln$ (d) $l^2 = 4mn$

80. The point at which the line $y = mx + c$ touches the parabola $y^2 = 4ax$ is **[Rajasthan PET 2001]**

(a) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ (b) $\left(\frac{a}{m^2}, \frac{-2a}{m}\right)$ (c) $\left(-\frac{a}{m^2}, \frac{2a}{m}\right)$ (d) $\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$

81. The locus of a foot of perpendicular drawn to the tangent of parabola $y^2 = 4ax$ from focus, is **[Rajasthan PET 1989]**

(a) $x = 0$ (b) $y = 0$ (c) $y^2 = 2a(x + a)$ (d) $x^2 + y^2(x + a) = 0$

82. The equation of tangent at the point (1, 2) to the parabola $y^2 = 4x$, is **[Rajasthan PET 1984, 85, 86]**

(a) $x - y + 1 = 0$ (b) $x + y + 1 = 0$ (c) $x + y - 1 = 0$ (d) $x - y - 1 = 0$

83. The tangent to the parabola $y^2 = 4ax$ at the point $(a, 2a)$ makes with x-axis an angle equal to **[SCRA 1996]**

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

84. A tangents to the parabola $y^2 = 8x$ makes an angle of 45^o with the straight line $y = 3x + 5$; then the equation of tangent is

(a) $2x + y - 1 = 0$ (b) $x + 2y - 1 = 0$ (c) $2x + y + 1 = 0$ (d) None of these

85. The equation of the tangent to the parabola $y^2 = 9x$ which goes through the point $(4, 10)$ is **[MP PET 2000]**

(a) $x + 4y + 1 = 0$ (b) $9x + 4y + 4 = 0$ (c) $x - 4y + 36 = 0$ (d) $9x - 4y + 4 = 0$

86. The angle of intersection between the curves $y^2 = 4x$ and $x^2 = 32y$ at point $(16, 8)$ is **[Rajasthan PET 1987, 96]**

(a) $\tan^{-1}\left(\frac{3}{5}\right)$ (b) $\tan^{-1}\left(\frac{4}{5}\right)$ (c) π (d) $\frac{\pi}{2}$

87. The equation of the tangent to the parabola $y = x^2 - x$ at the point where $x = 1$, is **[MP PET 1992]**

(a) $y = -x - 1$ (b) $y = -x + 1$ (c) $y = x + 1$ (d) $y = x - 1$

88. The point of intersection of the tangents to the parabola $y^2 = 4ax$ at the points t_1 and t_2 is **[Rajasthan PET 2002]**

(a) $(at_1t_2, a(t_1 + t_2))$ (b) $(2at_1t_2, a(t_1 + t_2))$ (c) $(2at_1t_2, 2a(t_1 + t_2))$ (d) None of these

89. The tangents drawn from the ends of latus rectum of $y^2 = 12x$ meets at **[Rajasthan PET 2000]**

(a) Directrix (b) Vertex (c) Focus (d) None of these

90. Two perpendicular tangents to $y^2 = 4ax$ always intersect on the line **[Karnataka CET 2000]**

(a) $x = a$ (b) $x + a = 0$ (c) $x + 2a = 0$ (d) $x + 4a = 0$

91. The locus of the point of intersection of the perpendicular tangents to the parabola $x^2 = 4ay$ is **[MP PET 1994]**

(a) Axis of the parabola (b) Directrix of the parabola

(c) Focal chord of the parabola (d) Tangent at vertex to the parabola

92. The angle between the tangents drawn from the origin to the parabola $y^2 = 4a(x - a)$ is **[MNR 1994; UPSEAT 1999, 2000]**

(a) 90^o (b) 30^o (c) $\tan^{-1}\frac{1}{2}$ (d) 45^o

93. The angle between tangents to the parabola $y^2 = 4ax$ at the points where it intersects with the line $x - y - a = 0$, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

94. The equation of latus rectum of a parabola is $x + y = 8$ and the equation of the tangent at the vertex is $x + y = 12$, then length of the latus rectum is **[MP PET 2002]**

(a) $4\sqrt{2}$ (b) $2\sqrt{2}$ (c) 8 (d) $8\sqrt{2}$

95. If the segment intercepted by the parabola $y^2 = 4ax$ with the line $lx + my + n = 0$ subtends a right angle at the vertex, then

(a) $4al + n = 0$ (b) $4al + 4am + n = 0$ (c) $4am + n = 0$ (d) $al + n = 0$

96. Tangents at the extremities of any focal chord of a parabola intersect

(a) At right angles (b) On the directrix (c) On the tangent at vertex (d) None of these

97. Angle between two curves $y^2 = 4(x + 1)$ and $x^2 = 4(y + 1)$ is **[UPSEAT 2002]**

(a) 0^o (b) 90^o (c) 60^o (d) 30^o

98. The angle of intersection between the curves $x^2 = 4(y + 1)$ and $x^2 = -4(y + 1)$ is **[UPSEAT 2002]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) 0 (d) $\frac{\pi}{2}$

99. If the tangents drawn from the point (0, 2) to the parabola $y^2 = 4ax$ are inclined at an angle $\frac{3\pi}{4}$, then the value of a is

(a) 2 (b) –2 (c) 1 (d) None of these

100. The point of intersection of the tangents to the parabola $y^2 = 4x$ at the points, where the parameter 't' has the value 1 and 2, is

(a) (3, 8) (b) (1, 5) (c) (2, 3) (d) (4, 6)

101. The tangents from the origin to the parabola $y^2 + 4 = 4x$ are inclined at

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

102. The number of distinct real tangents that can be drawn from (0, –2) to the parabola $y^2 = 4x$ is

(a) One (b) Two (c) Zero (d) None of these

103. If two tangents drawn from the point (α, β) to the parabola $y^2 = 4x$ be such that the slope of one tangent is double of the other, then

(a) $\beta = \frac{2}{9}\alpha^2$ (b) $\alpha = \frac{2}{9}\beta^2$ (c) $2\alpha = 9\beta^2$ (d) None of these

104. If $y + b = m_1(x + a)$ and $y + b = m_2(x + a)$ are two tangents to the parabola $y^2 = 4ax$, then

(a) $m_1 + m_2 = 0$ (b) $m_1 m_2 = 1$ (c) $m_1 m_2 = -1$ (d) None of these

105. If $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$, then

(a) $c = \frac{a}{m}$ (b) $c = am + \frac{a}{m}$ (c) $c = a + \frac{a}{m}$ (d) None of these

106. The angle between the tangents drawn from a point $(-a, 2a)$ to $y^2 = 4ax$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

107. The tangents to the parabola $y^2 = 4ax$ at $(at_1^2, 2at_1)$; (at_2^2, at_2) intersect on its axis, then **[EAMCET 1995]**

(a) $t_2 = t_2$ (b) $t_1 = -t_2$ (c) $t_1 t_2 = 2$ (d) $t_1 t_2 = -1$

108. If perpendiculars are drawn on any tangent to a parabola $y^2 = 4ax$ from the points $(a \pm k, 0)$ on the axis. The difference of their squares is

(a) 4 (b) $4a$ (c) $4k$ (d) $4ak$

109. The straight line $kx + y = 4$ touches the parabola $y = x - x^2$, if

(a) $k = -5$ (b) $k = 0$ (c) $k = 3$ (d) k takes any real value

110. If a tangent to the parabola $y^2 = ax$ makes an angle 45^o with x-axis, its points of contact will be

(a) $(a/2, a/4)$ (b) $(-a/2, a/4)$ (c) $(a/4, a/2)$ (d) $(-a/4, a/2)$

111. The equations of common tangent to the parabola $y^2 = 4ax$ and $x^2 = 4by$ is

(a) $xa^{1/3} + yb^{1/3} + (ab)^{2/3} = 0$ (b) $\frac{x}{a^{1/3}} + \frac{y}{b^{1/3}} + \frac{1}{(ab)^{2/3}} = 0$

(c) $xb^{\frac{1}{3}} + ya^{\frac{1}{3}} - (ab)^{\frac{2}{3}} = 0$ (d) $\frac{x}{b^{1/3}} + \frac{y}{a^{1/3}} - \frac{1}{(ab)^{2/3}} = 0$

112. The range of values of λ for which the point $(\lambda, -1)$ is exterior to both the parabolas $y^2 = |x|$ is

(a) (0, 1) (b) (–1, 1) (c) (–1, 0) (d) None of these

Advance Level

113. The line $x\cos\alpha + y\sin\alpha = p$ will touch the parabola $y^2 = 4a(x+a)$, if

(a) $p\cos\alpha + a = 0$ (b) $p\cos\alpha - a = 0$ (c) $a\cos\alpha + p = 0$ (d) $a\cos\alpha - p = 0$

114. If the straight line $x + y = 1$ touches the parabola $y^2 - y + x = 0$, then the coordinates of the point of contact are **[Rajasthan PET 1991]**

(a) (1, 1) (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (c) (0, 1) (d) (1, 0)

115. The equation of common tangent to the circle $x^2 + y^2 = 2$ and parabola $y^2 = 8x$ is **[Rajasthan PET 1997]**

(a) $y = x + 1$ (b) $y = x + 2$ (c) $y = x - 2$ (d) $y = -x + 2$

116. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is **[IIT Screening 2002]**

(a) $3y = 9x + 2$ (b) $y = 2x + 1$ (c) $2y = x + 8$ (d) $y = x + 2$

117. Two common tangents to the circle $x^2 + y^2 = 2a^2$ and parabola $y^2 = 8ax$ are **[AIEEE 2002]**

(a) $x = \pm(y + 2a)$ (b) $y = \pm(x + 2a)$ (c) $x = \pm(y + a)$ (d) $y = \pm(x + a)$

118. If the line $lx + my + n = 0$ is a tangent to the parabola $y^2 = 4ax$, then locus of its point of contact is **[Rajasthan PET 1997]**

(a) A straight line (b) A circle (c) A parabola (d) Two straight lines

119. The tangent drawn at any point P to the parabola $y^2 = 4ax$ meets the directrix at the point K, then the angle which KP subtends at its focus is **[Rajasthan PET 1996, 2002]**

(a) 30^o (b) 45^o (c) 60^o (d) 90^o

120. The point of intersection of tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is **[IIT 1994; Kurukshetra CEE 1998]**

(a) (1, 0) (b) (–1, 0) (c) (0, 1) (d) (0, –1)

121. If y_1, y_2 are the ordinates of two points *P* and *Q* on the parabola and y_3 is the ordinate of the point of intersection of tangents at *P* and *Q*, then

(a) y_1, y_2, y_3 are in A. P. (b) y_1, y_3, y_2 are in A. P. (c) y_1, y_2, y_3 are in G.P. (d) y_1, y_3, y_2 are in G. P.

122. If the tangents at *P* and *Q* on a parabola meet in *T*, then *SP*,*ST* and *SQ* are in

(a) A. P. (b) G. P. (c) H. P. (d) None of these

123. The equation of the parabola whose focus is the point (0, 0) and the tangent at the vertex is $x - y + 1 = 0$ is **[Orissa JEE 2002]**

(a) $x^2 + y^2 - 2xy - 4x + 4y - 4 = 0$ (b) $x^2 + y^2 - 2xy + 4x - 4y - 4 = 0$

(c) $x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$ (d) $x^2 + y^2 + 2xy - 4x - 4y + 4 = 0$

124. The two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at a point *P*, whose abscissae is not zero, such that

(a) They both touch each other at *P*

(b) They cut at right angles at *P*

(c) The tangents to each curve at *P* make complementary angles with the *x*-axis

(d) None of these

125. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then , a point of intersection of the circle and the parabola is **[IIT 1995]**

(a) $\left(\frac{p}{2}, p\right)$ (b) $\left(\frac{p}{2}, -p\right)$ (c) $\left(\frac{-p}{2}, p\right)$ (d) $\left(\frac{-p}{2}, -p\right)$

126. The angle of intersection of the curves $y^2 = 2x/\pi$ and $y = \sin x$, is **[Roorkee Qualifying 1998]**

(a) $\cot^{-1}(-1/\pi)$ (b) $\cot^{-1}\pi$ (c) $\cot^{-1}(-\pi)$ (d) $\cot^{-1}(1/\pi)$

127. *P* is a point. Two tangents are drawn from it to the parabola $y^2 = 4x$ such that the slope of one tangent is three times the slope of the other. The locus of *P* is

(a) A straight line (b) A circle (c) A parabola (d) An ellipse

128. The parabola $y^2 = kx$ makes an intercept of length 4 on the line $x - 2y = 1$ .Then *k* is

(a) $\frac{\sqrt{105} - 5}{10}$ (b) $\frac{5 - \sqrt{105}}{10}$ (c) $\frac{5 + \sqrt{105}}{10}$ (d) None of these

129. The triangle formed by the tangents to a parabola $y^2 = 4ax$ at the ends of the latus rectum and the double ordinates through the focus is

(a) Equilateral (b) Isosceles

(c) Right-angled isosceles (d) Dependent on the value of *a* for its classification

130. The equation of the tangent at the vertex of the parabola $x^2 + 4x + 2y = 0$ is

(a) $x = -2$ (b) $x = 2$ (c) $y = 2$ (d) $y = -2$

131. The locus of the point of intersection of the perpendicular tangents to the parabola $x^2 - 8x + 2y + 2 = 0$ is

(a) $2y - 15 = 0$ (b) $2y + 15 = 0$ (c) $2x + 9 = 0$ (d) None of these

132. If *P*,*Q*,*R* are three points on a parabola $y^2 = 4ax$, whose ordinates are in geometrical progression, then the tangents at *P* and *R* meet on

(a) The line through *Q* parallel to *x*-axis (b) The line through *Q* parallel to *y*-axis

(c) The line joining *Q* to the vertex (d) The line joining *Q* to the focus

133. The tangents at three points A, *B*, *C* on the parabola $y^2 = 4x$; taken in pairs intersect at the points *P*, *Q* and *R*. If Δ, Δ' be the areas of the triangles *ABC* and *PQR* respectively, then

(a) $\Delta = 2\Delta'$ (b) $\Delta' = 2\Delta$ (c) $\Delta = \Delta'$ (d) None of these

134. If the line $y = mx + a$ meets the parabola $y^2 = 4ax$ in two points whose abscissa are x_1 and x_2 , then $x_1 + x_2$ is equal to zero if

(a) $m = -1$ (b) $m = 1$ (c) $m = 2$ (d) $m = -1/2$

135. Two tangents of the parabola $y^2 = 8x$, meet the tangent at its vertex in the points P and Q. If $PQ = 4$, locus of the point of intersection of the two tangents is

(a) $y^2 = 8(x+2)$ (b) $y^2 = 8(x-2)$ (c) $x^2 = 8(y-2)$ (d) $x^2 = 8(y+2)$

136. If perpendicular be drawn from any two fixed points on the axis of a parabola at a distance d from the focus on any tangent to it, then the difference of their squares is

(a) $a^2 - d^2$ (b) $a^2 + d^2$ (c) $4ad$ (d) $2ad$

137. Two straight lines are perpendicular to each other. One of them touches the parabola $y^2 = 4a(x+a)$ and the other touches $y^2 = 4b(x+b)$. Their point of intersection lies on the line

(a) $x - a + b = 0$ (b) $x + a - b = 0$ (c) $x + a + b = 0$ (d) $x - a - b = 0$

138. The point $(a,\ 2a)$ is an interior point of the region bounded by the parabola $y^2 = 16x$ and the double ordinate through the focus. Then a belongs to the open interval

(a) $a < 4$ (b) $0 < a < 4$ (c) $0 < a < 2$ (d) $a > 4$

139. The number of points with integral coordinates that lie in the interior of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 4x$ is

(a) 8 (b) 10 (c) 16 (d) None of these

Normals in different forms, Intersection of Normals

Basic Level

140. The maximum number of normal that can be drawn from a point to a parabola is **[MP PET 1990]**

(a) 0 (b) 1 (c) 2 (d) 3

141. The centroid of the triangle formed by joining the feet of the normals drawn from any point to the parabola $y^2 = 4ax$, lies on **[MP PET 1999]**

(a) Axis (b) Directrix (c) Latus rectum (d) Tangent at vertex

142. If the line $2x + y + k = 0$ is normal to the parabola $y^2 = -8x$, then the value of k will be **[Rajasthan PET 1986, 1997]**

(a) –16 (b) –8 (c) –24 (d) 24

143. The point on the parabola $y^2 = 8x$ at which the normal is inclined at 60^o to the x-axis has the coordinates **[MP PET 1993]**

(a) $(6, -4\sqrt{3})$ (b) $(6, 4\sqrt{3})$ (c) $(-6, -4\sqrt{3})$ (d) $(-6,\ 4\sqrt{3})$

144. If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of ordinates of P and Q is

(a) $4a^2$ (b) $2a^2$ (c) $-4a^2$ (d) $8a^2$

145. The equation of normal to the parabola at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, is **[Rajasthan PET 1987]**

(a) $y = m^2x - 2mx - am^3$ (b) $m^3y = m^2x - 2am^2 - a$ (c) $m^3y = 2am^2 - m^2x + a$ (d) None of these

146. At what point on the parabola $y^2 = 4x$, the normal makes equal angles with the coordinate axes **[Rajasthan PET 1994]**

(a) $(4, 4)$ (b) $(9, 6)$ (c) $(4, -4)$ (d) $(1, -2)$

147. The slope of the normal at the point $(at^2, 2at)$ of the parabola $y^2 = 4ax$, is **[MNR 1991; UPSEAT 2000]**

(a) $\frac{1}{t}$ (b) t (c) $-t$ (d) $-\frac{1}{t}$

148. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then

[MNR 1986; Rajasthan PET 2003; AIEEE 2003]

(a) $t_2 = -t_1 - \frac{2}{t_1}$ (b) $t_2 = -t_1 + \frac{2}{t_1}$ (c) $t_2 = t_1 - \frac{2}{t_1}$ (d) $t_2 = t_1 + \frac{2}{t_1}$

149. The normal to the parabola $y^2 = 8x$ at the point (2, 4) meets the parabola again at the point **[Orissa JEE 2003]**

(a) $(-18, -12)$ (b) $(-18, 12)$ (c) $(18, 12)$ (d) $(18, -12)$

150. If a normal drawn to the parabola $y^2 = 4ax$ at the point $(a, 2a)$ meets parabola again on $(at^2, 2at)$, then the value of t will be

[Rajasthan PET 1990]

(a) 1 (b) 3 (c) –1 (d) –3

151. The arithmetic mean of the ordinates of the feet of the normals from (3, 5) to the parabola $y^2 = 8x$ is

(a) 4 (b) 0 (c) 8 (d) None of these

152. If the normal to $y^2 = 12x$ at (3, 6) meets the parabola again in (27, –18) and the circle on the normal chord as diameter is

[Kurukshetra CEE 1998]

(a) $x^2 + y^2 + 30x + 12y - 27 = 0$ (b) $x^2 + y^2 + 30x + 12y + 27 = 0$

(c) $x^2 + y^2 - 30x - 12y - 27 = 0$ (d) $x^2 + y^2 - 30x + 12y - 27 = 0$

153. The number of distinct normal that can be drawn from $\left(\frac{11}{4}, \frac{1}{4}\right)$ to the parabola $y^2 = 4x$ is

(a) 3 (b) 2 (c) 1 (d) 4

154. The normal chord of a parabola $y^2 = 4ax$ at (x_1, x_1) subtends a right angle at the

(a) Focus (b) Vertex (c) End of the latus-rectum (d) None of these

155. The normal at $(ap^2, 2ap)$ on $y^2 = 4ax$, meets the curve again at $(aq^2, 2aq)$ then

(a) $p^2 + pq + 2 = 0$ (b) $p^2 - pq + 2 = 0$ (c) $q^2 + pq + 2 = 0$ (d) $p^2 + pq + 1 = 0$

156. The angle between the normals to the parabola $y^2 = 24x$ at points (6, 12) and (6, –12) is

(a) 30^o (b) 45^o (c) 60^o (d) 90^o

Advance Level

157. The centre of a circle passing through the point (0,1) and touching the curve $y = x^2$ at (2, 4) is **[IIT 1983]**

(a) $\left(\frac{-16}{5}, \frac{27}{10}\right)$ (b) $\left(\frac{-16}{7}, \frac{5}{10}\right)$ (c) $\left(\frac{-16}{5}, \frac{53}{10}\right)$ (d) None of these

158. The length of the normal chord to the parabola $y^2 = 4x$, which subtends right angle at the vertex is **[Rajasthan PET 1999]**

(a) $6\sqrt{3}$ (b) $3\sqrt{3}$ (c) 2 (d) 1

159. Three normals to the parabola $y^2 = x$ are drawn through a point $(C, 0)$ then **[IIT 1991]**

(a) $C = \frac{1}{4}$ (b) $C = \frac{1}{2}$ (c) $C > \frac{1}{2}$ (d) None of these

160. If the tangent and normal at any point P of a parabola meet the axes in T and G respectively, then **[Rajasthan PET 2001]**

(a) $ST \neq SG = SP$ (b) $ST - SG \neq SP$ (c) $ST = SG = SP$ (d) $ST = SG \, . \, SP$

161. The number of distinct normals that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$ is

(a) 1 (b) 2 (c) 3 (d) 0

162. The set of points on the axis of the parabola $y^2 = 4x + 8$ from which the 3 normals to the parabola are all real and different is

(a) $\{(k,0)|\ k \le -2\}$ (b) $\{(k,0)|\ k > -2\}$ (c) $\{(0,k)|\ k > -2\}$ (d) None of these

163. The area of the triangle formed by the tangent and the normal to the parabola $y^2 = 4ax$; both drawn at the same end of the latus rectum, and the axis of the parabola is

(a) $2\sqrt{2}\,a^2$ (b) $2a^2$ (c) $4a^2$ (d) None of these

164. If a chord which is normal to the parabola $y^2 = 4ax$ at one end subtends a right angle at the vertex, then its slope is

(a) 1 (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) 2

165. If the normals from any point to the parabola $x^2 = 4y$ cuts the line $y = 2$ in points whose abscissae are in A.P., then the slopes of the tangents at the three co-normal points are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

166. If $x = my + c$ is a normal to the parabola $x^2 = 4ay$, then the value of c is

(a) $-2am - am^3$ (b) $2am + am^3$ (c) $-\frac{2a}{m} - \frac{a}{m^3}$ (d) $\frac{2a}{m} + \frac{a}{m^3}$

167. The normal at the point $P(ap^2, 2ap)$ meets the parabola $y^2 = 4ax$ again at $Q(aq^2, 2aq)$ such that the lines joining the origin to P and Q are at right angle. Then

(a) $p^2 = 2$ (b) $q^2 = 2$ (c) $p = 2q$ (d) $q = 2p$

168. If $y = 2x + 3$ is a tangent to the parabola $y^2 = 24x$, then its distance from the parallel normal is

(a) $5\sqrt{5}$ (b) $10\sqrt{5}$ (c) $15\sqrt{5}$ (d) None of these

169. If $P(-3, 2)$ is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$, then the slope of the normal at Q is

(a) $\frac{-1}{2}$ (b) 2 (c) $\frac{1}{2}$ (d) –2

170. The distance between a tangent to the parabola $y^2 = 4ax$ which is inclined to axis at an angle α and a parallel normal is

(a) $\frac{a\cos\alpha}{\sin^2\alpha}$ (b) $\frac{a\sin\alpha}{\cos^2\alpha}$ (c) $\frac{a}{\sin\alpha\cos^2\alpha}$ (d) $\frac{a}{\cos\alpha\sin^2\alpha}$

171. If the normal to the parabola $y^2 = 4ax$ at the point $P(at^2, 2at)$ cuts the parabola again at $Q(aT^2, 2aT)$, then

(a) $-2 \le \dot{T} \le 2$ (b) $T \in (-\infty, -8) \cup (8, \infty)$ (c) $T^2 < 8$ (d) $T^2 \ge 8$

Chords

Basic Level

172. The locus of the middle points of the chords of the parabola $y^2 = 4ax$ which passes through the origin is

[Rajasthan PET 1997; UPSEAT 1999]

(a) $y^2 = ax$ (b) $y^2 = 2ax$ (c) $y^2 = 4ax$ (d) $x^2 = 4ay$

173. In the parabola $y^2 = 6x$, the equation of the chord through vertex and negative end of latus rectum, is

(a) $y = 2x$ (b) $y + 2x = 0$ (c) $x = 2y$ (d) $x + 2y = 0$

174. From the point $(-1, 2)$ tangent lines are drawn to the parabola $y^2 = 4x$, then the equation of chord of contact is **[Roorkee 1994]**

(a) $y = x + 1$ (b) $y = x - 1$ (c) $y + x = 1$ (d) None of these

175. A set of parallel chords of the parabola $y^2 = 4ax$ have their mid points on

(a) Any straight line through the vertex
(b) Any straight line through the focus
(c) A straight line parallel to the axis
(d) Another parabola

176. The length of the chord of the parabola $y^2 = 4ax$ which passes through the vertex and makes an angle θ with the axis of the parabola, is

(a) $4a\cos\theta\,\text{cosec}^2\theta$ (b) $4a\cos^2\theta\,\text{cosec}\,\theta$ (c) $a\cos\theta\,\text{cosec}^2\theta$ (d) $a\cos^2\theta\,\text{cosec}\,\theta$

177. If PSQ is the focal chord of the parabola $y^2 = 8x$ such that $SP = 6$. Then the length SQ is

(a) 6 (b) 4 (c) 3 (d) None of these

178. The locus of the middle points of parallel chords of a parabola $x^2 = 4ay$ is a

(a) Straight line parallel to the axis
(b) Straight line parallel to the y-axis
(c) Circle
(d) Straight line parallel to a bisector of the angles between the axes

179. The locus of the middle points of chords of the parabola $y^2 = 8x$ drawn through the vertex is a parabola whose

(a) focus is (2, 0) (b) Latus rectum =8 (c) Focus is (0, 2) (d) Latus rectum =4

180. 't_1' and 't_2' are two points on the parabola $y^2 = 4x$. If the chord joining them is a normal to the parabola at 't_1', then

(a) $t_1 + t_2 = 0$ (b) $t_1(t_1 + t_2) = 0$ (c) $t_1(t_1 + t_2) + 2 = 0$ (d) $t_1t_2 + 1 = 0$

181. The locus of the middle points of chords of a parabola which subtend a right angle at the vertex of the parabola is

(a) A circle (b) An ellipse (c) A parabola (d) None of these

182. AB is a chord of the parabola $y^2 = 4ax$. If its equation is $y = mx + c$ and it subtends a right angle at the vertex of the parabola then

(a) $c = 4am$ (b) $a = 4mc$ (c) $c = -4am$ (d) $a + 4mc = 0$

183. The length of a focal chord of parabola $y^2 = 4ax$ making an angle θ with the axis of the parabola is

(a) $4a\,\text{cosec}^2\theta$ (b) $4a\sec^2\theta$ (c) $a\,\text{cosec}^2\theta$ (d) None of these

184. If (a, b) is the mid point of a chord passing through the vertex of the parabola $y^2 = 4x$, then

(a) $a = 2b$ (b) $2a = b$ (c) $a^2 = 2b$ (d) $2a = b^2$

185. The mid-point of the chord $2x + y - 4 = 0$ of the parabola $y^2 = 4x$ is

(a) $\left(\frac{5}{2}, -1\right)$ (b) $\left(-1, \frac{5}{2}\right)$ (c) $\left(\frac{3}{2}, -1\right)$ (d) None of these

186. If $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are two variable points on the curve $y^2 = 4ax$ and PQ subtends a right angle at the vertex, then t_1t_2 is equal to

(a) –1 (b) – 2 (c) –3 (d) –4

187. If $(at^2, 2at)$ are the coordinates of one end of a focal chord of the parabola $y^2 = 4ax$, then the coordinate of the other end are

(a) $(at^2, -2at)$ (b) $(-at^2, -2at)$ (c) $\left(\frac{a}{t^2}, \frac{2a}{t}\right)$ (d) $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

188. If b and c are the lengths of the segments of any focal chord of a parabola $y^2 = 4ax$, then the length of the semi- latusrectum is

(a) $\frac{b+c}{2}$ (b) $\frac{bc}{b+c}$ (c) $\frac{2bc}{b+c}$ (d) $\sqrt{bc}$

189. The ratio in which the line segment joining the points $(4, -6)$ and $(3, 1)$ is divided by the parabola $y^2 = 4x$ is

(a) $\frac{-20 \pm \sqrt{155}}{11} : 1$ (b) $\frac{-2 \pm 2\sqrt{155}}{11} : 1$ (c) $-20 \pm 2\sqrt{155} : 11$ (d) $-2 \pm \sqrt{155} : 11$

190. If the lengths of the two segments of focal chord of the parabola $y^2 = 4ax$ are 3 and 5, then the value of a will be

(a) $\frac{15}{8}$ (b) $\frac{15}{4}$ (c) $\frac{15}{2}$ (d) 15

Advance Level

191. If $'a'$ and $'c'$ are the segments of a focal chord of a parabola and b the semi-latus rectum, then **[MP PET 1995]**

(a) a, b, c are in A. P. (b) a, b, c are in G. P. (c) a, b, c are in H. P. (d) None of these

192. The locus of mid point of that chord of parabola which subtends right angle on the vertex will be **[UPSEAT 1999]**

(a) $y^2 - 2ax + 8a^2 = 0$ (b) $y^2 = a(x - 4a)$ (c) $y^2 = 4a(x - 4a)$ (d) $y^2 + 3ax + 4a^2 = 0$

193. The HM of the segments of a focal chord of the parabola $y^2 = 4ax$ is

(a) $4a$ (b) $2a$ (c) a (d) a^2

194. The length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c. Then

(a) $2a^2 = bc$ (b) $a^3 = b^2c$ (c) $ac = b^2$ (d) $b^2c = 4a^3$

195. A chord PP' of a parabola cuts the axis of the parabola at O. The feet of the perpendiculars from P and P' on the axis are M and M' respectively. If V is the vertex then VM, VO, VM' are in

(a) *A.P.* (b) *G.P.* (c) *H.P.* (d) None of these

196. The chord AB of the parabola $y^2 = 4ax$ cuts the axis of the parabola at C. If $A = (at_1^2, 2at_2)$; $B = (at_2^2, 2at_2)$ and $AC : AB = 1 : 3$, then

(a) $t_2 = 2t_1$ (b) $t_2 + 2t_1 = 0$ (c) $t_1 + 2t_2 = 0$ (d) None of these

197. The locus of the middle points of the focal chord of the parabola $y^2 = 4ax$ is

(a) $y^2 = a(x - a)$ (b) $y^2 = 2a(x - a)$ (c) $y^2 = 4a(x - a)$ (d) None of these

198. If $(4, -2)$ is one end of a focal chord of the parabola $y^2 = x$, then the slope of the tangent drawn at its other end will be

(a) $-\frac{1}{4}$ (b) -4 (c) 4 (d) $\frac{1}{4}$

199. If (a_1, b_1) and (a_2, b_2) are extremities of a focal chord of the parabola $y^2 = 4ax$, then $a_1a_2 =$

(a) $4a^2$ (b) $-4a^2$ (c) a^2 (d) $-a^2$

200. The length of the chord of the parabola $y^2 = 4ax$ whose equation is $y - x\sqrt{2} + 4a\sqrt{2} = 0$ is

(a) $2\sqrt{11}a$ (b) $4\sqrt{2}a$ (c) $8\sqrt{2}a$ (d) $6\sqrt{3}a$

201. If the line $y = x\sqrt{3} - 3$ cuts the parabola $y^2 = x + 2$ at P and Q and if A be the point $(\sqrt{3}, 0)$, then $AP.\ AQ$ is

(a) $\frac{2}{3}(\sqrt{3} + 2)$ (b) $\frac{4}{3}(\sqrt{3} + 2)$ (c) $\frac{4}{3}(2 - \sqrt{3})$ (d) $2\sqrt{3}$

202. A triangle ABC of area Δ is inscribed in the parabola $y^2 = 4ax$ such that the vertex A lies at the vertex of the parabola and BC is a focal chord. The difference of the distances of B and C from the axis of the parabola is

(a) $\frac{2\Delta}{a}$ (b) $\frac{2\Delta}{a^2}$ (c) $\frac{a}{2\Delta}$ (d) None of these

Diameter of Parabola, Length of tangent, Normal and Subnormal, Pole and Polar

Basic Level

203. The length of the subnormal to the parabola $y^2 = 4ax$ at any point is equal to **[UPSEAT 2000]**

(a) $\sqrt{2}a$ (b) $2\sqrt{2}$ (c) $a/\sqrt{2}$ (d) $2a$

204. The polar of focus of a parabola is **[Rajasthan PET 1999]**

(a) x-axis (b) y-axis (c) Directrix (d) Latus rectum

205. Locus of the poles of focal chords of a parabola is .....of parabola **[EAMCET 2002]**

(a) The tangent at the vertex (b) The axis (c) A focal chord (d) The directrix

206. The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in **[EAMCET 1993]**

(a) *A.P.* (b) *G.P.* (c) *H.P.* (d) None of these

Miscellaneous Problems

Basic Level

207. The equation of a circle passing through the vertex and the extremities of the latus rectum of the parabola $y^2 = 8x$ is **[MP PET 1998]**

(a) $x^2 + y^2 + 10x = 0$ (b) $x^2 + y^2 + 10y = 0$ (c) $x^2 + y^2 - 10x = 0$ (d) $x^2 + y^2 - 5x = 0$

208. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, whose vertices are at the parabola, then the length of its side is equal to

(a) $8a$ (b) $8a\sqrt{3}$ (c) $a\sqrt{2}$ (d) None of these

209. The area of triangle formed inside the parabola $y^2 = 4x$ and whose ordinates of vertices are 1, 2 and 4 will be **[Rajasthan PET 1990]**

(a) $\frac{7}{2}$ (b) $\frac{5}{2}$ (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

210. The area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum is

(a) 12 sq. units (b) 16 sq. units (c) 18 sq. units (d) 24 sq. units

211. The vertex of the parabola $y^2 = 8x$ is at the centre of a circle and the parabola cuts the circle at the ends of its latus rectum. Then the equation of the circle is

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 20$ (c) $x^2 + y^2 = 80$ (d) None of these

212. The circle $x^2 + y^2 + 2\lambda x = 0, \lambda \in R,$ touches the parabola $y^2 = 4x$ externally. Then

(a) $\lambda > 0$ (b) $\lambda < 0$ (c) $\lambda > 1$ (d) None of these

213. The length of the common chord of the parabola $2y^2 = 3(x+1)$ and the circle $x^2 + y^2 + 2x = 0$ is

(a) $\sqrt{3}$ (b) $2\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) None of these

214. The circles on focal radii of a parabola as diameter touch

(a) The tangent at the vertex (b) The axis (c) The directrix (d) None of these

Advance Level

215. The ordinates of the triangle inscribed in parabola $y^2 = 4ax$ are y_1, y_2, y_3 , then the area of triangle is

(a) $\frac{1}{8a}(y_1 + y_2)(y_2 + y_3)(y_3 + y_1)$ (b) $\frac{1}{4a}(y_1 + y_2)(y_2 + y_3)(y_3 + y_1)$

(c) $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ (d) $\frac{1}{4a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$

216. Which one of the following curves cuts the parabola $y^2 = 4ax$ at right angles **[IIT Screening 1994]**

(a) $x^2 + y^2 = a^2$ (b) $y = e^{-x/2a}$ (c) $y = ax$ (d) $x^2 = 4ay$

217. On the parabola $y = x^2$, the point least distant from the straight line $y = 2x - 4$ is **[AMU 2001]**

(a) (1, 1) (b) (1, 0) (c) (1, –1) (d) (0, 0)

218. Let the equations of a circle and a parabola be $x^2 + y^2 - 4x - 6 = 0$ and $y^2 = 9x$ respectively. Then

(a) (1, –1) is a point on the common chord of contact (b) The equation of the common chord is $y+1=0$

(c) The length of the common chord is 6 (d) None of these

219. *P* is a point which moves in the *x*-*y* plane such that the point *P* is nearer to the centre of square than any of the sides. The four vertices of the square are $(\pm a, \pm a)$. The region in which *P* will move is bounded by parts of parabola of which one has the equation

(a) $y^2 = a^2 + 2ax$ (b) $x^2 = a^2 + 2ay$ (c) $y^2 + 2ax = a^2$ (d) None of these

220. The focal chord to $y^2 = 16x$ is tangent to $(x-6)^2 + y^2 = 2$, then the possible values of the slope of this chord, are **[IIT Screening 2003]**

(a) {–1, 1} (b) {–2, 2} (c) {–2, 1/2} (d) {2, –1/2}

221. Let *PQ* be a chord of the parabola $y^2 = 4x$. A circle drawn with *PQ* as a diameter passes through the vertex *V* of the parabola. If $ar(\Delta PVQ) = 20$ unit ^{2}, then the coordinates of *P* are

(a) (16, 8) (b) (16, –8) (c) (–16, 8) (d) (–16, –8)

222. A normal to the parabola $y^2 = 4ax$ with slope *m* touches the rectangular hyperbola $x^2 - y^2 = a^2$, if

(a) $m^6 + 4m^4 - 3m^2 + 1 = 0$ (b) $m^6 - 4m^4 + 3m^2 - 1 = 0$ (c) $m^6 + 4m^4 + 3m^2 + 1 = 0$ (d) $m^6 - 4m^4 - 3m^2 + 1 = 0$

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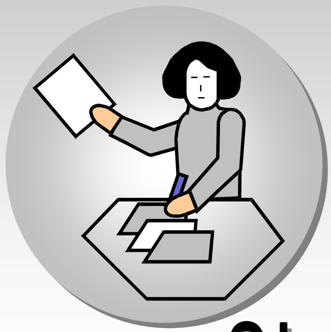

Answer Sheet

Conic Section : Parabola *Assignment (Basic and Advance Level)*

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
c	b	b	c	d	a	c	a	b	b	a	c	c	a	b,c	a,c	c	b	a	a,b,c,d
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
b	d	d	c	b	d	d	b	d	a	a	d	c	c	c	c	a	c	d	c
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
b	c	a	b	a	a	c	a	d	a	a	c	b	b,d	a	a	b	a,c	a,b,d	b
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**
c	c	a,b	d	d	a	b	b	c	a	b	c	d	b	a	a	c	b	d	a
81	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**
a	a	b	c	c,d	a	d	a	a	b	b	a	d	d	a	a,b	b	c	a,b	c
101	**102**	**103**	**104**	**105**	**106**	**107**	**108**	**109**	**110**	**111**	**112**	**113**	**114**	**115**	**116**	**117**	**118**	**119**	**120**
d	b	b	c	b	b	b	d	a,c	c	a	b	a	c	b	d	b	c	d	b
121	**122**	**123**	**124**	**125**	**126**	**127**	**128**	**129**	**130**	**131**	**132**	**133**	**134**	**135**	**136**	**137**	**138**	**139**	**140**
b	b	c	c	a,b	b	c	a	c	c	a	b	a	c	a	c	c	b	a	d
141	**142**	**143**	**144**	**145**	**146**	**147**	**148**	**149**	**150**	**151**	**152**	**153**	**154**	**155**	**156**	**157**	**158**	**159**	**160**
a	d	a	d	c	d	c	a	d	d	b	d	a	a	a	d	c	a	c	c
161	**162**	**163**	**164**	**165**	**166**	**167**	**168**	**169**	**170**	**171**	**172**	**173**	**174**	**175**	**176**	**177**	**178**	**179**	**180**
a	d	c	c	b	a	a	c	a	c	d	b	b	b	c	a	c	b	d	c
181	**182**	**183**	**184**	**185**	**186**	**187**	**188**	**189**	**190**	**191**	**192**	**193**	**194**	**195**	**196**	**197**	**198**	**199**	**200**
c	c	a	d	a	d	d	c	c	a	c	a	b	d	b	b	b	c	c	d
201	**202**	**203**	**204**	**205**	**206**	**207**	**208**	**209**	**210**	**211**	**212**	**213**	**214**	**215**	**216**	**217**	**218**	**219**	**220**
b	a	d	c	d	b	c	b	d	c	b	a	a	a	c	b	a	a,c	a,b,c	a
221	**222**																		

a,b	c

Chapter

5.2

Ellipse

Contents

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5.2.2	Standard equation of the Ellipse
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5.2.4	Parametric form of the Ellipse
5.2.5	Special form of an Ellipse
5.2.6	Position of a point with respect to an Ellipse
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5.2.12	Properties of Eccentric angles of the co-normal points
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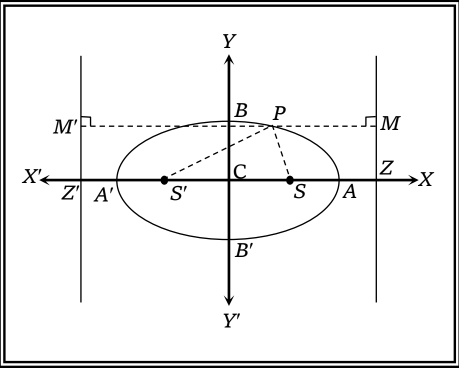
5.2.19	Concyclic points
5.2.20	Reflection property of an Ellipse
Assignment (Basic and Advance Level)	
Answer Sheet of Assignment	



The Greeks particularly Archimedes (287-212 B.C.) and Apollonius (200 B.C.) studied conic sections for their own beauty. These curves are important tools for present day exploration of outer space and also for research into behaviour of atomic particles.

Kepler was first to declare that the planets of our solar system travel around the sun in elliptic path. The ellipse is also used in many art forms and in the construction of bridges. Due to our knowledge of the ellipse, it is now possible to predict accurately the time and place of solar and lunar eclipses.

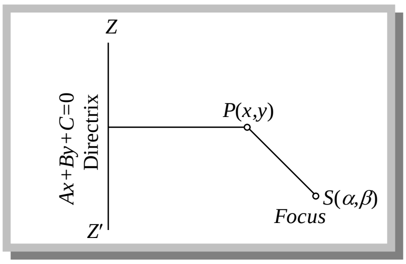
5.2 Ellipse

5.2.1 Definition

An ellipse is the locus of a point which moves in such a way that its distance from a fixed point is in constant ratio (<1) to its distance from a fixed line. The fixed point is called the **focus** and fixed line is called the **directrix** and the constant ratio is called the **eccentricity** of the ellipse, denoted by (e).

In other words, we can say an ellipse is the locus of a point which moves in a plane so that the sum of its distances from two fixed points is constant and is more than the distance between the two fixed points.

Let $S(\alpha, \beta)$ is the focus, ZZ' is the directrix and P is any point on the ellipse. Then by definition,



$$\frac{SP}{PM} = e \Rightarrow SP = e.PM$$

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e\,\frac{Ax + By + C}{\sqrt{A^2 + B^2}}$$

Squaring both sides, $(A^2 + B^2)[(x-\alpha)^2 + (y-\beta)^2] = e^2(Ax + By + C)^2$

Note : ❑ The condition for second degree equation in x and y to represent an ellipse is that $h^2 - ab < 0$ and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

Example: 1 The equation of an ellipse whose focus is (–1, 1), whose directrix is $x - y + 3 = 0$ and whose eccentricity is $\frac{1}{2}$, is given by **[MP PET 1993]**

(a) $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$ (b) $7x^2 - 2xy + 7y^2 - 10x + 10y + 7 = 0$

(c) $7x^2 - 2xy + 7y^2 - 10x - 10y - 7 = 0$ (d) $7x^2 - 2xy + 7y^2 + 10x + 10y - 7 = 0$

Solution: (a) Let any point on it be (x, y) then by definition,

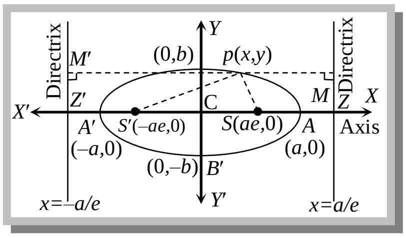
$$\sqrt{(x+1)^2 + (y-1)^2} = \frac{1}{2}\left|\frac{x - y + 3}{\sqrt{1^2 + 1^2}}\right|$$

Squaring and simplifying, we get

$7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$, which is the required ellipse.

5.2.2 Standard equation of the Ellipse

Let S be the focus, ZM be the directrix of the ellipse and $P(x, y)$ is any point on the ellipse, then by definition



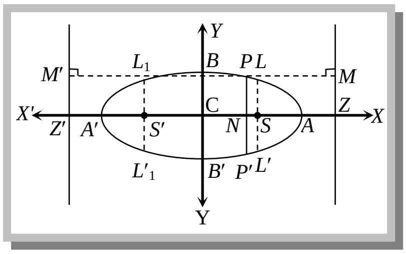
$$\frac{SP}{PM} = e \Rightarrow (SP)^2 = e^2(PM)^2$$

$$(x - ae)^2 + (y-0)^2 = e^2\left(\frac{a}{e} - x\right)^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1-e^2)$

Since $e<1$, therefore $a^2(1-e^2)<a^2 \Rightarrow b^2<a^2$. Some terms related to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$:

(1) **Centre:** The point which bisects each chord of the ellipse passing through it, is called centre $(0, 0)$ denoted by C.



(2) **Major and minor axes:** The diameter through the foci, is called the major axis and the diameter bisecting it at right angles is called the minor axis. The major and minor axes are together called principal axes.

Length of the major axis $AA'=2a$, Length of the minor axis $BB'=2b$

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is symmetrical about both the axes.

(3) **Vertices:** The extremities of the major axis of an ellipse are called vertices.
The coordinates of vertices A and A' are $(a, 0)$ and $(-a, 0)$ respectively.

(4) **Foci:** S and S' are two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively. Distance between foci $SS'=2ae$.

(5) **Directrices:** ZM and $Z'M'$ are two directrices of the ellipse and their equations are $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ respectively. Distance between directrices $ZZ'=\frac{2a}{e}$.

(6) **Eccentricity of the ellipse:** For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

we have $b^2=a^2(1-e)^2 \Rightarrow e^2=1-\frac{b^2}{a^2}=1-\frac{4b^2}{4a^2}=1-\left(\frac{2b}{2a}\right)^2$; $e=\sqrt{1-\left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}$

This formula gives the eccentricity of the ellipse.

(7) **Ordinate and double ordinate:** Let P be a point on the ellipse and let PN be perpendicular to the major axis AA' such that PN produced meets the ellipse at P'. Then PN is called the ordinate of P and PNP' the double ordinate of P.

If abscissa of P is h, then ordinate of P, $\frac{y^2}{b^2}=1-\frac{h^2}{a^2} \Rightarrow y=\frac{b}{a}\sqrt{(a^2-h^2)}$ (For first quadrant)

And ordinate of P' is $y=\frac{-b}{a}\sqrt{(a^2-h^2)}$ (For fourth quadrant)

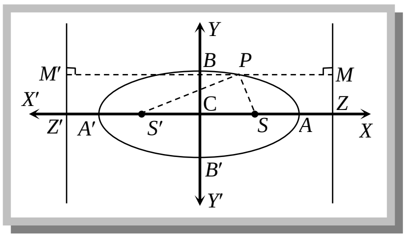
Hence coordinates of P and P' are $\left(h, \frac{b}{a}\sqrt{(a^2-h^2)}\right)$ and $\left(h, \frac{-b}{a}\sqrt{(a^2-h^2)}\right)$ respectively.

(8) **Latus-rectum:** Chord through the focus and perpendicular to the major axis is called its latus rectum.
The double ordinates LL' and L_1L_1' are latus rectum of the ellipse.

Length of latus rectum $LL'=L_1L_1'=\frac{2b^2}{a}$ and end points of latus-rectum are $L=\left(ae, \frac{b^2}{a}\right), L'=\left(ae, \frac{-b^2}{a}\right)$ and $L_1=\left(-ae, \frac{b^2}{a}\right)$; $L_1'=\left(-ae, \frac{-b^2}{a}\right)$

(9) **Focal chord:** A chord of the ellipse passing through its focus is called a focal chord.

(10) **Focal distances of a point:** The distance of a point from the focus is its focal distance. The sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis of the ellipse.



Let $P(x_1, y_1)$ be any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$SP = ePM = e\left(\dfrac{a}{e} - x_1\right) = a - ex_1$ and $S'P = ePM' = e\left(\dfrac{a}{e} + x_1\right) = a + ex_1$

$\therefore \quad SP + S'P = (a - ex_1) + (a + ex_1) = 2a = AA' =$ major axis.

Example: 2 The length of the latus-rectum of the ellipse $5x^2 + 9y^2 = 45$ is **[MNR 1978, 80, 81; Kurukshetra CEE 1999]**

(a) $\sqrt{5}/4$ (b) $\sqrt{5}/2$ (c) $5/3$ (d) $10/3$

Solution: (d) Here the ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

Here $a^2 = 9$ and $b^2 = 5$. So, latus-rectum $= \dfrac{2b^2}{a} = \dfrac{2(5)}{3} = \dfrac{10}{3}$.

Example: 3 In an ellipse the distance between its foci is 6 and its minor axis is 8. Then its eccentricity is **[EAMCET 1994]**

(a) $\dfrac{4}{5}$ (b) $\dfrac{1}{\sqrt{52}}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{1}{2}$

Solution: (c) Distance between foci $= 6 \Rightarrow 2ae = 6 \Rightarrow ae = 3$, Minor axis $= 8 \Rightarrow 2b = 8 \Rightarrow b = 4 \Rightarrow b^2 = 16$

From $b^2 = a^2(1 - e^2)$, $\Rightarrow 16 = a^2 - a^2e^2 \Rightarrow a^2 - 9 = 16 \Rightarrow a = 5$

Hence, $ae = 3 \Rightarrow e = \dfrac{3}{5}$

Example: 4 What is the equation of the ellipse with foci $(\pm 2, 0)$ and eccentricity $\dfrac{1}{2}$ **[DCE 1999]**

(a) $3x^2 + 4y^2 = 48$ (b) $4x^2 + 3y^2 = 48$ (c) $3x^2 + 4y^2 = 0$ (d) $4x^2 + 3y^2 = 0$

Solution: (a) Here $ae = \pm 2$, $\because e = \dfrac{1}{2}$, $\therefore a = \pm 4$

Form $b^2 = a^2(1 - e^2) \Rightarrow b^2 = 16\left(1 - \dfrac{1}{4}\right) \Rightarrow b^2 = 12$

Hence, the equation of ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$ or $3x^2 + 4y^2 = 48$

Example: 5 If $P(x, y), F_1 = (3, 0)$, $F_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals **[IIT 1998]**

(a) 8 (b) 6 (c) 10 (d) 12

Solution: (c) We have $16x^2 + 25y^2 = 400 \Rightarrow \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 16$

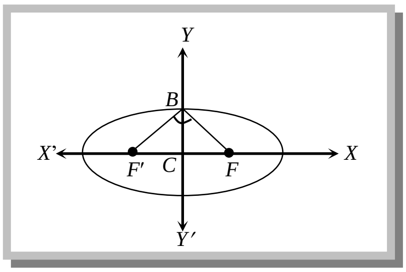
This equation represents an ellipse with eccentricity given by $e^2 = 1 - \dfrac{b^2}{a^2} = 1 - \dfrac{16}{25} = \dfrac{9}{25} \Rightarrow e = 3/5$

So, the coordinates of the foci are $(\pm ae, 0)$ i.e. $(3, 0)$ and $(-3, 0)$, Thus, F_1 and F_2 are the foci of the ellipse.

Since, the sum of the focal distance of a point on an ellipse is equal to its major axis, $\therefore PF_1 + PF_2 = 2a = 10$

Example: 6 An ellipse has OB as semi minor axis. F and F' are its foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is **[Pb. CET 2001, IIT 1997, DCE 2002]**

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{1}{3}$



Solution: (b) Since $\angle FBF' = \frac{\pi}{2}$

$\therefore \quad \angle FBC = \angle F'BC = \frac{\pi}{4}$

$\therefore \quad CB = CF \Rightarrow b = ae$

$\Rightarrow b^2 = a^2e^2 \Rightarrow a^2(1-e^2) = a^2e^2$

$\Rightarrow 1 - e^2 = e^2 \Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$.

Example: 7 Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then the maximum value of A is **[IIT 1994]**

(a) $2abe$ (b) abe (c) $\frac{1}{2}abe$ (d) None of these

Solution: (b) Let $P(a\cos\theta, b\sin\theta)$ and $F_1(-ae, 0)$, $F_2(ae, 0)$

$$A = \text{Area of } \Delta PF_1F_2 = \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix} = \frac{1}{2}|\,2aeb\sin\theta| = aeb\,|\sin\theta|$$

$\therefore$ A is maximum, when $|\sin\theta| = 1$.

Hence, maximum value of $A = abe$

Example: 8 The eccentricity of an ellipse, with its centre at the origin is $\frac{1}{2}$. If one of the directrices is $x = 4$, then the equation of the ellipse is **[AIEEE 2004]**

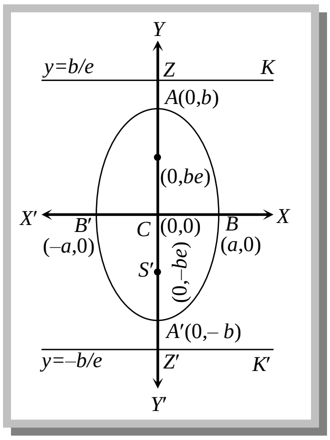
(a) $4x^2 + 3y^2 = 1$ (b) $3x^2 + 4y^2 = 12$ (c) $4x^2 + 3y^2 = 12$ (d) $3x^2 + 4y^2 = 1$

Solution: (b) Given $e = \frac{1}{2}, \frac{a}{e} = 4$. So, $a = 2 \Rightarrow a^2 = 4$

From $b^2 = a^2(1-e^2) \Rightarrow b^2 = 4\left(1 - \frac{1}{4}\right) = 4 \times \frac{3}{4} = 3$

Hence the equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$, *i.e.* $3x^2 + 4y^2 = 12$

5.2.3 Equation of Ellipse in other form



In the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a > b$ or $a^2 > b^2$ (denominator of x^2 is greater than that of y^2), then the major and minor axis lie along x-axis and y-axis respectively. But if $a < b$ or $a^2 < b^2$ (denominator of x^2 is less than that of y^2), then the major axis of the ellipse lies along the y-axis and is of length $2b$ and the minor axis along the x-axis and is of length $2a$.

The coordinates of foci S and S' are (0, be) and (0, – be) respectively.

The equation of the directrices ZK and $Z'K'$ are $y = \pm b/e$ and eccentricity e is given by the formula $a^2 = b^2(1-e^2)$ or $e = \sqrt{1 - \frac{a^2}{b^2}}$

Difference between both ellipse will be clear from the following table.

Ellipse / **Basic fundamentals**	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For $a > b$	**For $b > a$**
Centre	$(0, 0)$	$(0, 0)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus-rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations	$(a\cos\phi, b\sin\phi)$	$(a\cos\phi, b\sin\phi)$ $(0 \le \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ and $S'P = a + ex_1$	$SP = b - ey_1$ and $S'P = b + ey_1$
Sum of focal radii $SP + S'P =$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrices	$2a/e$	$2b/e$
Tangents at the vertices	$x = -a, x = a$	$y = b, y = -b$

Example: 9 The equation of a directrix of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is

(a) $y = \frac{25}{3}$ (b) $x = 3$ (c) $x = -3$ (d) $x = \frac{3}{25}$

Solution: (a) From the given equation of ellipse $a^2 = 16, b^2 = 25$ (since $b > a$)

So, $a^2 = b^2(1-e^2)$, $\therefore$ $16 = 25(1-e^2) \Rightarrow 1-e^2 = \frac{16}{25} \Rightarrow e^2 = \frac{9}{25} \Rightarrow e = \frac{3}{5}$

$\therefore$ One directrix is $y = \frac{b}{e} = \frac{5}{3/5} = \frac{25}{3}$

Example: 10 The distances from the foci of $P(x_1, y_1)$ on the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$ are

(a) $4 \pm \frac{5}{4}y_1$ (b) $5 \pm \frac{4}{5}x_1$ (c) $5 \pm \frac{4}{5}y_1$ (d) None of these

Solution: (c) For the given ellipse $b > a$, so the two foci lie on y-axis and their coordinates are $(0, \pm be)$,

Where $b = 5, a = 3$. So $e = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

The focal distances of a point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Where $b^2 > a^2$ are given by $b \pm ey_1$. So, Required distances are $b \pm ey_1 = 5 \pm \frac{4}{5}y_1$.

5.2.4 Parametric form of the Ellipse

Let the equation of ellipse in standard form will be given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then the equation of ellipse in the parametric form will be given by $x=a\cos\phi, y=b\sin\phi$, where ϕ is the eccentric angle whose value vary from $0\le\phi<2\pi$. Therefore coordinate of any point P on the ellipse will be given by $(a\cos\phi, b\sin\phi)$

Example: 11 The curve represented by $x=3(\cos t+\sin t)$, $y=4(\cos t-\sin t)$ is **[EAMCET 1988; DCE 2000]**

(a) Ellipse (b) Parabola (c) Hyperbola (d) Circle

Solution: (a) Given, $x=3(\cos t+\sin t), y=4(\cos t-\sin t)\Rightarrow\frac{x}{3}=(\cos t+\sin t), \frac{y}{4}=(\cos t-\sin t)$

Squaring and adding, we get $\frac{x^2}{9}+\frac{y^2}{16}=(1+\sin 2t)+(1-\sin 2t)\Rightarrow\frac{x^2}{9}+\frac{y^2}{16}=2$, which represents ellipse.

Example: 12 The distance of the point 'θ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from a focus is

(a) $a(e+\cos\theta)$ (b) $a(e-\cos\theta)$ (c) $a(1+e\cos\theta)$ (d) $a(1+2e\cos\theta)$

Solution: (c) Focal distance of any point $P(x,y)$ on the ellipse is equal to $SP=a+ex$. Here $x=\cos\theta$.

Hence, $SP=a+ae\cos\theta=a(1+e\cos\theta)$

5.2.5 Special forms of an Ellipse

(1) If the centre of the ellipse is at point (h,k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

If we shift the origin at (h, k) without rotating the coordinate axes, then $x=X+h$ and $y=Y+k$

(2) If the equation of the curve is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$ where $lx+my+n=0$ and $mx-ly+P=0$ are perpendicular lines, then we substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X, \frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

Example: 13 The foci of the ellipse $25(x+1)^2+9(y+2)^2=225$ are **[MP PET 1998, UPSEAT 1991, 2000]**

(a) $(-1,2),(6,1)$ (b) $(-1,-2),(1,6)$ (c) $(1,-2),(1,-6)$ (d) $(-1,2),(-1,-6)$

Solution: (d) Given ellipse is $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{25}=1$ *i.e.* $\frac{X^2}{9}+\frac{Y^2}{25}=1$, where $X=x+1$ and $Y=y+2$

Here $a^2=25, b^2=9$ [Type : $\frac{X^2}{b^2}+\frac{Y^2}{a^2}=1$]

Eccentricity is given by $e^2=\frac{a^2-b^2}{a^2}=\frac{25-9}{25}=\frac{16}{25}$, $\therefore$ $e=\frac{4}{5}$

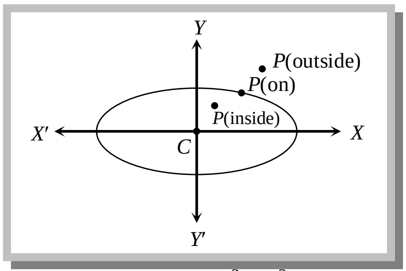
Foci are given by $Y=\pm ae=\pm 5\left(\frac{4}{5}\right)=\pm 4$

$X=0\Rightarrow y+2=\pm 4\Rightarrow y=-2\pm 4=-6$ or 2

$x+1=0\Rightarrow x=-1$. Hence foci are (–1, –6) or (–1, 2).

5.2.6 Position of a point with respect to an Ellipse

Let $P(x_1, y_1)$ be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse. The point lies outside, on or inside the ellipse as if $S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0, =0, <0$



Example: 14 Let E be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and C be the circle $x^2+y^2=9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then

[IIT 1994]

(a) Q lies inside C but outside E (b) Q lies outside both C and E
(c) P lies inside both C and E (d) P lies inside C but outside E

Solution: (d) The given ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$. The value of the expression $\frac{x^2}{9}+\frac{y^2}{4}-1$ is positive for $x=1, y=2$ and negative for $x=2, y=1$. Therefore P lies outside E and Q lies inside E. The value of the expression x^2+y^2-9 is negative for both the points P and Q. Therefore P and Q both lie inside C. Hence P lies inside C but outside E.

5.2.7 Intersection of a Line and an Ellipse

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ......(i) and the given line be $y=mx+c$ ......(ii)

Eliminating y from equation (i) and (ii), then $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

i.e., $(a^2m^2+b^2)x^2+2mca^2x+a^2(c^2-b^2)=0$

The above equation being a quadratic in x, its discriminant $=4m^2c^2a^4-4a^2(a^2m^2+b^2)(c^2-b^2)$

$$=b^2\{(a^2m^2+b^2)-c^2\}$$

Hence the line intersects the ellipse in two distinct points if $a^2m^2+b^2>c^2$ in one point if $c^2=a^2m^2+b^2$ and does not intersect if $a^2m^2+b^2<c^2$.

5.2.8 Equations of Tangent in Different forms

(1) **Point form:** The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

(2) **Slope form:** If the line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2=a^2m^2+b^2$. Hence, the straight line $y=mx\pm\sqrt{a^2m^2+b^2}$ always represents the tangents to the ellipse.

Points of contact: Line $y=mx\pm\sqrt{a^2m^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at

$$\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$$

(3) **Parametric form:** The equation of tangent at any point $\phi(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$

Note : ❑ The straight line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a^2l^2 + b^2m^2 = n^2$.

❑ The line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ and that point of contact is $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$.

❑ Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

❑ The tangents at the extremities of latus-rectum of an ellipse intersect on the corresponding directrix.

Important Tips

☞ *A circle of radius r is concentric with the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *then the common tangent is inclined to the major axis at an angle* $\tan^{-1}\sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$.

☞ *The locus of the foot of the perpendicular drawn from centre upon any tangent to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ *or* $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$ *(in polar coordinates)*

☞ *The locus of the mid points of the portion of the tangents to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *intercepted between the axes is* $a^2y^2 + b^2x^2 = 4x^2y^2$.

☞ *The product of the perpendiculars from the foci to any tangent of an ellipse is equal to the square of the semi minor axis, and the feet of these perpendiculars lie on the auxiliary circle.*

Example: 15 The number of values of 'c' such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$ is **[IIT 1998]**

(a) 0 (b) 1 (c) 2 (d) Infinite

Solution: (c) We know that the line $y = mx + c$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *iff* $c^2 = a^2m^2 + b^2$

Here, $a^2 = 4, b^2 = 1, m = 4 \quad \therefore c^2 = 64 + 1 \Rightarrow c = \pm\sqrt{65}$

Example: 16 On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are **[IIT 1999]**

(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(\frac{-2}{5}, \frac{1}{5}\right)$ (c) $\left(\frac{-2}{5}, \frac{-1}{5}\right)$ (d) $\left(\frac{2}{5}, \frac{-1}{5}\right)$

Solution: (b,d) Ellipse is $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1 \Rightarrow a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$. The equation of its tangent is $4xx' + 9yy' = 1$

$\therefore\ m = -\frac{4x'}{9y'} = \frac{8}{9} \Rightarrow x' = -2y'$ and $4x'^2 + 9y'^2 = 1 \Rightarrow 4x'^2 + 9\frac{x'^2}{4} = 1 \Rightarrow x' = \pm\frac{2}{5}$

When $x' = \frac{2}{5}$, then $y' = \frac{-1}{5}$ and when $x' = \frac{-2}{5}$, then $y' = \frac{1}{5}$.

Hence points are $\left(\frac{2}{5}, \frac{-1}{5}\right), \left(\frac{-2}{5}, \frac{1}{5}\right)$

Example: 17 If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepts equal lengths l on the axes, then l=

(a) a^2+b^2 (b) $\sqrt{a^2+b^2}$ (c) $(a^2+b^2)^2$ (d) None of these

Solution: (b) The equation of any tangent to the given ellipse is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

This line meets the coordinate axes at $P\left(\frac{a}{\cos\theta},0\right)$ and $Q\left(0,\frac{b}{\sin\theta}\right)$

$\therefore \frac{a}{\cos\theta}=l=\frac{b}{\sin\theta}\Rightarrow \cos\theta=\frac{a}{l}$ and $\sin\theta=\frac{b}{l}\Rightarrow \cos^2\theta+\sin^2\theta=\frac{a^2}{l^2}+\frac{b^2}{l^2}\Rightarrow l^2=a^2+b^2\Rightarrow l=\sqrt{a^2+b^2}$.

Example: 18 The area of the quadrilateral formed by the tangents at the end points of latus- rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is **IIT Screening 2003]**

(a) 27/4 *sq.* units (b) 9 *sq.* units (c) 27/2 *sq.* units (d) 27*sq.* units

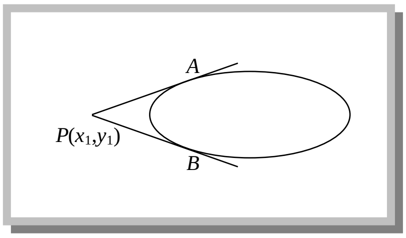
Solution: (d) By symmetry the quadrilateral is a rhombus. So area is four times the area of the right angled triangle formed by the tangents and axes in the 1st quadrant.

Now $ae=\sqrt{a^2-b^2}\Rightarrow ae=2\Rightarrow$ Tangent (in the first quadrant) at one end of latus rectum $\left(2,\frac{5}{3}\right)$ is $\frac{2}{9}x+\frac{5}{3}.\frac{y}{5}=1$

i.e. $\frac{x}{9/2}+\frac{y}{3}=1$. Therefore area $=4.\frac{1}{2}.\frac{9}{2}.3=27\,sq.$ units.

5.2.9 Equation of Pair of Tangents $SS_1=T^2$

Pair of tangents: Let $P(x_1,y_1)$ be any point lying outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let a pair of tangents *PA*, *PB* can be drawn to it from *P*.



Then the equation of pair of tangents *PA* and *PB* is $SS_1=T^2$

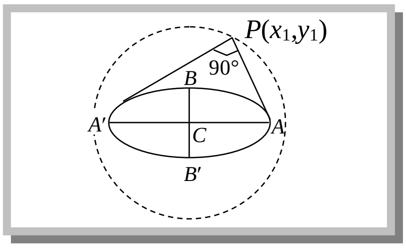
where $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

$S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0$

$T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$

Director circle: The director circle is the locus of points from which perpendicular tangents are drawn to the ellipse.

Let $P(x_1,y_1)$ be any point on the locus. Equation of tangents through $P(x_1,y_1)$ is given by $SS_1=T^2$



i.e., $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left[\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right]^2$

They are perpendicular, So coeff. of x^2 + coeff. of $y^2=0$

$\therefore\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right)=0$ or $x_1^2+y_1^2=a^2+b^2$

Hence locus of $P(x_1,y_1)$ *i.e.*, equation of director circle is $x^2+y^2=a^2+b^2$

Example: 19 The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2+2y^2=5$ is **[UPSEAT 2001]**

(a) $\tan^{-1}(12/5)$ (b) $\tan^{-1}(6/\sqrt{5})$ (c) $\tan^{-1}(12/\sqrt{5})$ (d) $\tan^{-1}(6/5)$

Solution: (c) The combined equation of the pair of tangents drawn from (1,2) to the ellipse $3x^2+2y^2=5$ is $(3x^2+2y^2-5)(3+8-5)=(3x+4y-5)^2$ **[**using $SS_1=T^2$ **]**

$\Rightarrow\ 9x^2-24xy-4y^2+.......=0$

The angle between the lines given by this equation is $\tan\theta=\dfrac{2\sqrt{h^2-ab}}{a+b}$

Where $a=9$, $h=-12$, $b=-4\ \Rightarrow\ \tan\theta=12/\sqrt{5}\Rightarrow\theta=\tan^{-1}(12/\sqrt{5})$

Example: 20 The locus of the point of intersection of the perpendicular tangents to the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ is **[Karnataka CET 2003]**

(a) $x^2+y^2=9$ (b) $x^2+y^2=4$ (c) $x^2+y^2=13$ (d) $x^2+y^2=5$

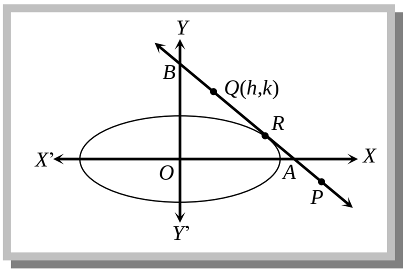
Solution: (c) The locus of point of intersection of two perpendicular tangents drawn on the ellipse is $x^2+y^2=a^2+b^2$, which is called "director circle".

Given ellipse is $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$. $\therefore$ Locus is $x^2+y^2=9+4$, *i.e.* $x^2+y^2=13$.

Example: 21 The locus of the middle point of the intercept of the tangents drawn from an external point to the ellipse $x^2+2y^2=2$ between the coordinate axes, is **[IIT Screening 2004]**

(a) $\dfrac{1}{x^2}+\dfrac{1}{2y^2}=1$ (b) $\dfrac{1}{4x^2}+\dfrac{1}{2y^2}=1$ (c) $\dfrac{1}{2x^2}+\dfrac{1}{4y^2}=1$ (d) $\dfrac{1}{2x^2}+\dfrac{1}{y^2}=1$

Solution: (c) Let the point of contact be $R\equiv(\sqrt{2}\cos\theta,\sin\theta)$



Equation of tangent AB is $\dfrac{x}{\sqrt{2}}\cos\theta+y\sin\theta=1$

$\Rightarrow\ A\equiv(\sqrt{2}\sec\theta,0)$; $B\equiv(0,\text{coses}\,\theta)$

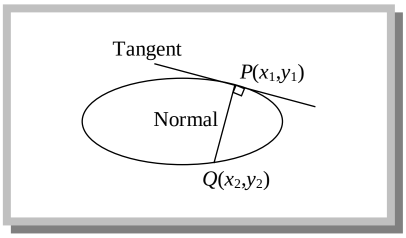
Let the middle point Q of AB be (h,k).

$\Rightarrow\ h=\dfrac{\sec\theta}{\sqrt{2}},k=\dfrac{\text{cosec}\,\theta}{2}\Rightarrow\cos\theta=\dfrac{1}{h\sqrt{2}},\sin\theta=\dfrac{1}{2k}\Rightarrow\dfrac{1}{2h^2}+\dfrac{1}{4k^2}=1$

Thus required locus is $\dfrac{1}{2x^2}+\dfrac{1}{4y^2}=1$

5.2.10 Equations of Normal in Different forms

(1) **Point form:** The equation of the normal at (x_1,y_1) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{a^2x}{x_1}-\dfrac{b^2y}{y_1}=a^2-b^2$.



(2) **Parametric form:** The equation of the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at $(a\cos\phi,b\sin\phi)$ is $ax\sec\phi-by\,\text{cosec}\phi=a^2-b^2$.

(3) **Slope form:** If m is the slope of the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then the equation of normal is

$$y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2 m^2}}$$

The coordinates of the point of contact are $\left(\frac{\pm a^2}{\sqrt{a^2 + b^2 m^2}}, \frac{\pm mb^2}{\sqrt{a^2 + b^2 m^2}}\right)$

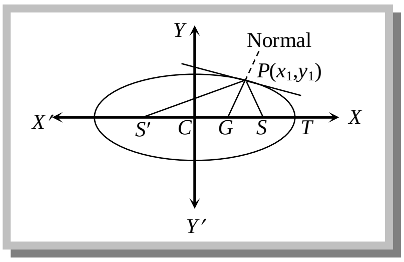
Note : ❑ If $y = mx + c$ is the normal of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then condition of normality is $c^2 = \frac{m^2(a^2 - b^2)^2}{(a^2 + b^2 m^2)}$.

❑ The straight line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \left(\frac{a^2 - b^2}{n^2}\right)^2$.

❑ Four normals can be drawn from a point to an ellipse.

Important Tips

☞ *If S be the focus and G be the point where the normal at P meets the axis of an ellipse, then* $SG = e.SP$ *, and the tangent and normal at P bisect the external and internal angles between the focal distances of* P.



☞ *Any point P of an ellipse is joined to the extremities of the major axis then the portion of a directrix intercepted by them subtends a right angle at the corresponding focus.*

☞ *With a given point and line as focus and directrix, a series of ellipse can be described. The locus of the extermities of their minor axis is a parabola.*

☞ *The equations to the normals at the end of the latera recta and that each passes through an end of the minor axis, if* $e^4 + e^2 + 1 = 0$

☞ *If two concentric ellipse be such that the foci of one be on the other and if e and e' be their eccentricities. Then the angle between their axes is* $\cos^{-1}\sqrt{\frac{e^2 + e'^2 - 1}{ee'}}$.

Example: 22 The equation of normal at the point (0, 3) of the ellipse $9x^2 + 5y^2 = 45$ is **[MP PET 1998]**

(a) $y - 3 = 0$ (b) $y + 3 = 0$ (c) x-axis (d) y-axis

Solution: (d) For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, equation of normal at point (x_1, y_1), is $\frac{(x - x_1)a^2}{x_1} = \frac{(y - y_1)b^2}{y_1}$

Here, $(x_1, y_1) = (0, 3)$ and $a^2 = 5$, $b^2 = 9$, Therefore $\frac{(x-0)}{0}.5 = \frac{(y-3)}{3}.9$ or $x = 0$ *i.e.*, y-axis.

Example: 23 If the normal at any point P on the ellipse cuts the major and minor axes in G and g respectively and C be the centre of the ellipse, then **[Kurukshetra CEE 1998]**

(a) $a^2(CG)^2 + b^2(Cg)^2 = (a^2 - b^2)^2$ (b) $a^2(CG)^2 - b^2(Cg)^2 = (a^2 - b^2)^2$

(c) $a^2(CG)^2 - b^2(Cg)^2 = (a^2 + b^2)^2$ (d) None of these

Solution: (a) Let at point (x_1, y_1) normal will be $\frac{(x - x_1)}{x_1}a^2 = \frac{(y - y_1)b^2}{y_1}$

At G, $y=0 \Rightarrow x = CG = \frac{x_1(a^2-b^2)}{a^2}$ and at g, $x=0 \Rightarrow y = Cg = \frac{y_1(b^2-a^2)}{b^2}$

$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \Rightarrow a^2(CG)^2+b(Cg)^2=(a^2-b^2)^2$.

Example: 24 The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the positive end of the latus-rectum is

(a) $x+ey+e^3a=0$ (b) $x-ey-e^3a=0$ (c) $x-ey-e^2a=0$ (d) None of these

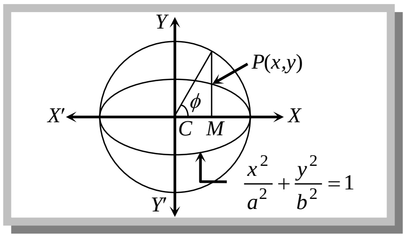
Solution: (b) The equation of the normal at (x_1, y_1) to the given ellipse is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$. Here, $x_1=ae$ and $y_1=\frac{b^2}{a}$

So, the equation of the normal at positive end of the latus- rectum is

$$\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2e^2 \quad [\because\ b^2=a^2(1-e^2)] \Rightarrow \frac{ax}{e}-ay=a^2e^2 \Rightarrow x-ey-e^3a=0$$

5.2.11 Auxiliary Circle

The circle described on the major axis of an ellipse as diameter is called an auxiliary circle of the ellipse.



If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse, then its auxiliary circle is $x^2+y^2=a^2$

Eccentric angle of a point: Let P be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to meet the auxiliary circle in Q. Join CQ. The angle $\angle XCQ=\phi$ is called the eccentric angle of the point P on the ellipse.

Note that the angle $\angle XCP$ is not the eccentric angle of point P.

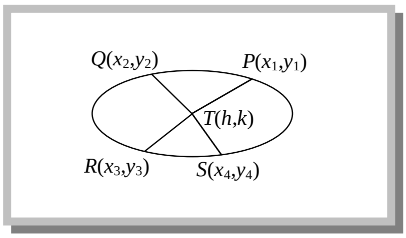
5.2.12 Properties of Eccentric angles of the Co-normal points

(1) The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to odd multiple of π.

(2) If α, β, γ are the eccentric angles of three points on the ellipse, the normals at which are concurrent, then $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$.

(3)**Co-normal points lie on a fixed curve:** Let $P(x_1,y_1)$, $Q(x_2,y_2)$, $R(x_3,y_3)$ and $S(x_4,y_4)$ be co-normal points, then $PQRS$ lie on the curve $(a^2-b^2)xy+b^2kx-a^2hy=0$

This curve is called Apollonian rectangular hyperbola.



Note : ❑ The feet of the normals from any fixed point to the ellipse lie at the intersections of the apollonian rectangular hyperbola with the ellipse.

Important Tips

☞ *The area of the triangle formed by the three points, on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *whose eccentric angles are* θ, ϕ *and* ψ *is* $2ab\sin\left(\frac{\phi-\psi}{2}\right)\sin\left(\frac{\psi-\theta}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$.

☞ *The eccentricity of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is given by* $2\cot w=\frac{e^2\sin 2\theta}{\sqrt{(1-e^2)}}$, *where w is one of the angles between the normals at the points whose eccentric angles are* θ *and* $\frac{\pi}{2}+\theta$.

Example: 25 The eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$, whose distance from the centre of the ellipse is 2, is

[WB JEE 1990]

(a) $\pi/4$ (b) $3\pi/2$ (c) $5\pi/3$ (d) $7\pi/6$

Solution: (a) Let θ be the eccentric angle of the point P. Then the coordinates of P are $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$

The centre of the ellipse is at the origin, It is given that $OP=2$

$\Rightarrow \sqrt{6\cos^2\theta+2\sin^2\theta}=2 \Rightarrow 6\cos^2\theta+2\sin^2\theta=4 \Rightarrow 3\cos^2\theta+\sin^2\theta=2 \Rightarrow 2\sin^2\theta=1$

$\Rightarrow \sin^2\theta=\frac{1}{2} \Rightarrow \sin\theta=\pm\frac{1}{\sqrt{2}} \Rightarrow \theta=\pm\pi/4$

Example: 26 The area of the rectangle formed by the perpendiculars from the centre of the ellipse to the tangent and normal at the point-whose eccentric angle is $\pi/4$, is

(a) $\left(\frac{a^2-b^2}{a^2+b^2}\right)ab$ (b) $\left(\frac{a^2+b^2}{a^2-b^2}\right)ab$ (c) $\frac{1}{ab}\left(\frac{a^2-b^2}{a^2+b^2}\right)ab$ (d) $\frac{1}{ab}\left(\frac{a^2+b^2}{a^2-b^2}\right)ab$

Solution: (a) The given point is $(a\cos\pi/4, b\sin\pi/4)$ *i.e.* $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

So, the equation of the tangent at this point is $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ ......(i)

$\therefore\ p_1 =$ length of the perpendicular form (0, 0) on (i) $=\left|\frac{\frac{0}{a}+\frac{0}{b}-\sqrt{2}}{\sqrt{1/a^2+1/b^2}}\right|=\frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$

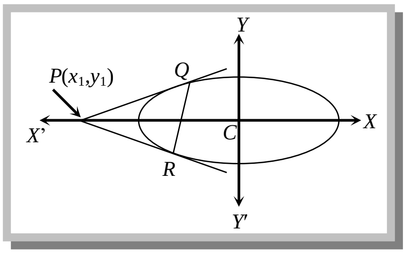
Equation of the normal at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ is $\frac{a^2x}{a/\sqrt{2}}-\frac{b^2y}{b/\sqrt{2}}=a^2-b^2 \Rightarrow \sqrt{2}ax-\sqrt{2}by=a^2-b^2$ .....(ii)

Therefore, $p_2 =$ length of the perpendicular form (0, 0) on (ii) $=\frac{a^2b^2}{\sqrt{(\sqrt{2a})^2+(-\sqrt{2b})^2}}=\frac{a^2-b^2}{\sqrt{2(a^2+b^2)}}$

So, area of the rectangle $=p_1p_2=\frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}\times\frac{a^2-b^2}{\sqrt{2(a^2+b^2)}}=\left(\frac{a^2-b^2}{a^2+b^2}\right)ab$

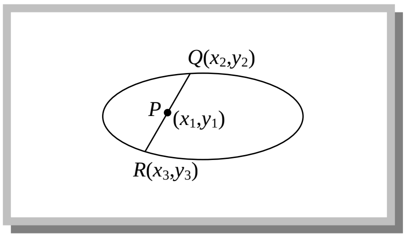
5.2.13 Chord of Contact

If PQ and PR be the tangents through point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact QR is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or $T=0$ at (x_1, y_1)



5.2.14 Equation of Chord with Mid point (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid point be (x_1,y_1) is $T=S_1$, where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$, $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0$



5.2.15 Equation of the Chord joining two points on an Ellipse

Let $P(a\cos\theta, b\sin\theta)$; $Q(a\cos\phi, b\sin\phi)$ be any two points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then ,the equation of the chord joining these two points is $y-b\sin\theta=\frac{b\sin\phi-b\sin\theta}{a\cos\phi-a\cos\theta}(x-a\cos\theta)$

Thus, the equation of the chord joining two points having eccentric angles θ and ϕ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$

Note :

- ❑ If the chord joining two points whose eccentric angles are α and β cut the major axis of an ellipse at a distance 'c' from the centre, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}$.
- ❑ If α and β be the eccentric angles of the extremities of a focal chord of an ellipse of eccentricity e, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\frac{1\mp e}{1\pm e}=0$.

Example: 27 What will be the equation of that chord of ellipse $\frac{x^2}{36}+\frac{y^2}{9}=1$ which passes from the point $(2,1)$ and bisected on the point

[UPSEAT 1999]

(a) $x+y=2$ (b) $x+y=3$ (c) $x+2y=1$ (d) $x+2y=4$

Solution: (d) Let required chord meets to ellipse on the points P and Q whose coordinates are (x_1,y_1) and (x_2,y_2) respectively

$\because$ Point (2,1) is mid point of chord PQ

$\therefore$ $2=\frac{1}{2}(x_1+x_2)$ or $x_1+x_2=4$ and $1=\frac{1}{2}(y_1+y_2)$ or $y_1+y_2=2$

Again points (x_1,y_1) and (x_2,y_2) are situated on ellipse; $\therefore \frac{x_1^2}{36}+\frac{y_1^2}{9}=1$ and $\frac{x_2^2}{36}+\frac{y_2^2}{9}=1$

On subtracting $\frac{x_2^2-x_1^2}{36}+\frac{y_2^2-y_1^2}{9}=0$ or $\frac{y_2-y_1}{x_2-x_1}=-\frac{(x_2+x_1)}{4(y_2+y_1)}=\frac{-4}{4\times 2}=\frac{-1}{2}$

$\therefore$ Gradient of chord $PQ=\frac{y_2-y_1}{x_2-x_1}=\frac{-1}{2}$

Therefore, required equation of chord PQ is as follows, $y-1=-\frac{1}{2}(x-2)$ or $x+2y=4$

Alternative: $S_1=T$ (If mid point of chord is known)

$\therefore \frac{2^2}{36}+\frac{1^2}{9}-1=\frac{2x}{36}+\frac{1y}{9}-1 \Rightarrow x+2y=4$

Example: 28 What will be the equation of the chord of contact of tangents drawn from (3, 2) to the ellipse $x^2 + 4y^2 = 9$

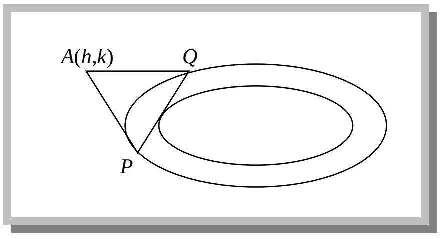
(a) $3x + 8y = 9$ (b) $3x + 8y = 25$ (c) $3x + 4y = 9$ (d) $3x + 8y + 9 = 0$

Solution: (a) The required equation is $T = 0$ *i.e.* , $3x + 4(2y) - 9 = 0$ or $3x + 8y = 9$.

Example: 29 A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. The angle between the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ is **[IIT 1997]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Solution: (a) The given ellipse $x^2 + 4y^2 = 4$ can be written as $\frac{x^2}{4} + \frac{y^2}{1} = 1$ .....(i)



Any tangent to ellipse (i) is $\frac{x}{2}\cos\theta + y\sin\theta = 1$ .....(ii)

Second ellipse is $x^2 + 2y^2 = 6$, *i.e.* $\frac{x^2}{6} + \frac{y^2}{3} = 1$ .....(iii)

Let the tangents at P, Q meet at (h, k) .

$\therefore$ Equation of *PQ*, *i.e.* chord of contact is $\frac{hx}{6} + \frac{ky}{3} = 1$ .....(iv)

Since (ii) and (iv) represent the same line, $\therefore \frac{h/6}{(\cos\theta)/2} = \frac{k/3}{\sin\theta} = \frac{1}{1} \Rightarrow h = 3\cos\theta$ and $k = 3\sin\theta$

So, $h^2 + k^2 = 9$ or $x^2 + y^2 = 9$ is the locus of (h, k) which is the director circle of the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$

$\therefore$ The angle between the tangents at P and Q will be $\pi/2$.

Example: 30 The locus of mid-points of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is **[EAMCET 1995]**

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$ (c) $x^2 + y^2 = a^2 + b^2$ (d) None of these

Solution: (a) Let (h, k) be the mid point of a focal chord. Then its equation is $S_1 = T$ or $\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. This passes through $(ae, 0)$,

$\therefore \frac{hae}{a^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. So, locus of (h, k) is $\frac{xe}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Example: 31 If α and β are the eccentric angles of the extremities of a focal chord of an ellipse, then the eccentricity of the ellipse is

(a) $\frac{\cos\alpha + \cos\beta}{\cos(\alpha - \beta)}$ (b) $\frac{\sin\alpha - \sin\beta}{\sin(\alpha - \beta)}$ (c) $\frac{\cos\alpha - \cos\beta}{\cos(\alpha - \beta)}$ (d) $\frac{\sin\alpha + \sin\beta}{\sin(\alpha + \beta)}$

Solution: (d) The equation of a chord joining points having eccentric angles α and β is given by

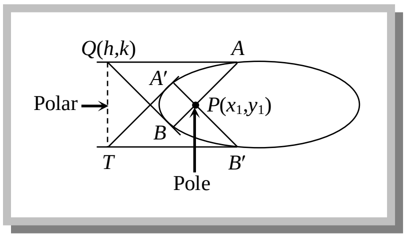
$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

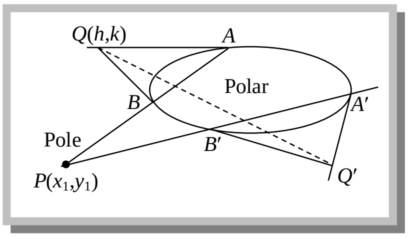
If it passes through $(ae, 0)$ then $e\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

$$\Rightarrow e = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow e = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow e = \frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$$

5.2.16 Pole and Polar

Let $P(x_1, y_1)$ be any point inside or outside the ellipse. A chord through P intersects the ellipse at A and B respectively. If tangents to the ellipse at A and B meet at $Q(h,k)$ then locus of Q is called polar of P with respect to ellipse and point P is called pole.



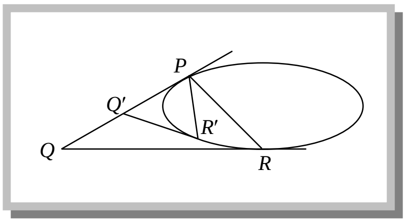


Equation of polar: Equation of polar of the point (x_1, y_1) with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad (i.e.\ T=0)$$

Coordinates of pole: The pole of the line $lx+my+n=0$ with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$P\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$$



Note : ❑ The polar of any point on the directrix, passes through the focus.

❑ Any tangent is the polar of its own point of contact.

Properties of pole and polar

(1) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

(2) If the pole of a line $l_1x+m_1y+n_1=0$ lies on the another line $l_2x+m_2y+n_2=0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

(3) Pole of a given line is same as point of intersection of tangents at its extremities.

Example: 32 The pole of the straight line $x+4y=4$ with respect to ellipse $x^2+4y^2=4$ is **[EAMCET 2002]**

(a) (1, 4) (b) (1, 1) (c) (4, 1) (d) (4, 4)

Solution: (b) Equation of polar of (x_1, y_1) *w.r.t* the ellipse is $xx_1+4yy_1=4$ .....(i)

Comparing with $x+4y=4$ .....(ii)

$\frac{x_1}{1}=\frac{4y_1}{4}=1 \Rightarrow x_1=1, y_1=1$. $\therefore$ Coordinates of pole $(x_1, y_1)=(1, 1)$

Example: 33 If the polar with respect to $y^2=4ax$ touches the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, the locus of its pole is **[EAMCET 1995]**

(a) $\frac{x^2}{\alpha^2}-\frac{y^2}{(4a^2\alpha^2/\beta^2)}=1$ (b) $\frac{x^2}{\alpha^2}+\frac{\beta^2y^2}{4a^2}=1$

(c) $\alpha^2x^2+\beta^2y^2=1$ (d) None of these

Solution: (a) Let $P(h,k)$ be the pole. Then the equation of the polar is $ky = 2a(x+h)$ or $y = \frac{2a}{k}x + \frac{2ah}{k}$.

This touches $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, So $\left(\frac{2ah}{k}\right)^2 = \alpha^2\left(\frac{2a}{k}\right)^2 + \beta^2$, (using $c^2 = a^2m^2 + b^2$)

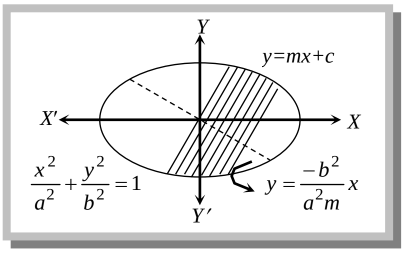
$\Rightarrow 4a^2h^2 = 4a^2\alpha^2 + k^2\beta^2$. So, locus of (h,k) is $4a^2x^2 = 4a^2\alpha^2 + \beta^2y^2$ or $\frac{x^2}{\alpha^2} - \frac{y^2}{\left(\frac{4a^2\alpha^2}{\beta^2}\right)} = 1$

5.2.17 Diameter of the Ellipse

Definition : The locus of the mid- point of a system of parallel chords of an ellipse is called a diameter and the chords are called its double ordinates *i.e.* A line through the centre of an ellipse is called a diameter of the ellipse.

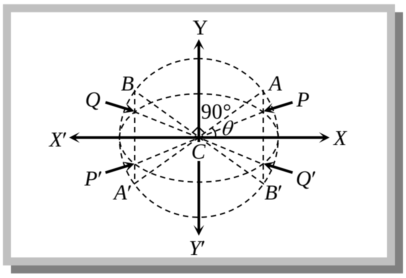
The point where the diameter intersects the ellipse is called the vertex of the diameter.

Equation of a diameter to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: Let $y = mx + c$ be a system of parallel chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where m is a constant and c is a variable.



The equation of the diameter bisecting the chords of slope m of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $y = -\frac{b^2}{a^2m}x$, which is passing through (0, 0).

Conjugate diameter: Two diameters of an ellipse are said to be conjugate diameter if each bisects all chords parallel to the other.



Conjugate diameter of circle *i.e.* AA' and BB' are perpendicular to each other. Hence, conjugate diameter of ellipse are PP' and QQ'. Hence, angle between conjugate diameters of ellipse $> 90^o$.

Now the coordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi); P'(-a\cos\phi, -b\sin\phi); Q(-a\sin\phi, b\cos\phi); Q'(a\sin\phi, -b\cos\phi)$

If $y = m_1x$ and $y = m_2x$ be two conjugate diameters of an ellipse, then $m_1m_2 = \frac{-b^2}{a^2}$

(1) Properties of diameters

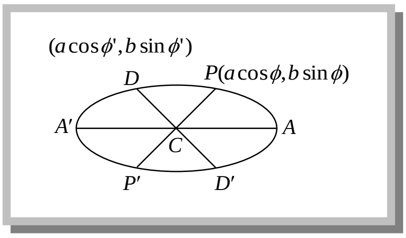
(i) The tangent at the extremity of any diameter is parallel to the chords it bisects or parallel to the conjugate diameter.

(ii) The tangent at the ends of any chord meet on the diameter which bisects the chord.

(2) Properties of conjugate diameters

(i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle,

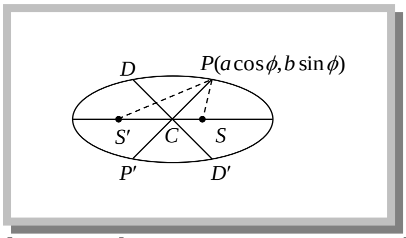
i.e. $\phi - \phi' = \frac{\pi}{2}$



(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi axes of the ellipse, *i.e.* $CP^2 + CD^2 = a^2 + b^2$

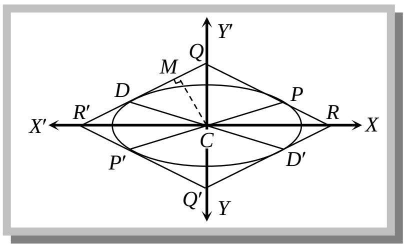
(iii) The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point,

i.e., $SP.S'P = CD^2$



(iv) The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to product of the axes, *i.e.*

Area of parallelogram $=(2a)(2b)=$ Area of rectangle contained under major and minor axes.



(v) The polar of any point with respect to ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid point is (x_1, y_1), *i.e.* chord is $T = S_1$.

(3) **Equi-conjugate diameters:** Two conjugate diameters are called equi-conjugate, if their lengths are equal *i.e.* $(CP)^2 = (CD)^2$

$$\therefore\ a^2\cos^2\phi + b^2\sin^2\phi = a^2\sin^2\phi + b^2\cos^2\phi$$

$$\Rightarrow\ a^2(\cos^2\phi - \sin^2\phi) - b^2(\cos^2\phi - \sin^2\phi) = 0 \Rightarrow (a^2 - b^2)(\cos^2\phi - \sin^2\phi) = 0$$

$$\because\ (a^2 - b^2) \neq 0,\ \therefore \cos 2\phi = 0.\ \text{So, } \phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\therefore\quad (CP) = (CD) = \sqrt{\frac{(a^2 + b^2)}{2}} \text{ for equi-conjugate diameters.}$$

Important Tips

- *If the point of intersection of the ellipses* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and* $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ *be at the extremities of the conjugate diameters of the former, then* $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 2$
- *The sum of the squares of the reciprocal of two perpendicular diameters of an ellipse is constant.*
- *In an ellipse, the major axis bisects all chords parallel to the minor axis and vice-versa, therefore major and minor axes of an ellipse are conjugate diameters of the ellipse but they do not satisfy the condition* $m_1.m_2 = -b^2/a^2$ *and are the only perpendicular conjugate diameters.*

Example: 34 If one end of a diameter of the ellipse $4x^2 + y^2 = 16$ is $(\sqrt{3}, 2)$, then the other end is

(a) $(-\sqrt{3}, 2)$ (b) $(\sqrt{3}, -2)$ (c) $(-\sqrt{3}, -2)$ (d) $(0, 0)$

Solution: (c) Since every diameter of an ellipse passes through the centre and is bisected by it, therefore the coordinates of the other end are $(-\sqrt{3}, -2)$.

Example: 35 If θ and ϕ are eccentric angles of the ends of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\theta - \phi$ is equal to

(a) $\pm\frac{\pi}{2}$ (b) $\pm\pi$ (c) 0 (d) None of these

Solution: (a) Let $y = m_1 x$ and $y = m_2 x$ be a pair of conjugate diameter of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be ends of these two diameters. Then $m_1 m_2 = \frac{-b^2}{a^2}$

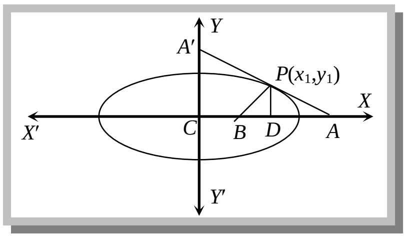
$\Rightarrow \frac{b\sin\theta - 0}{a\cos\theta - 0} \times \frac{b\sin\phi - 0}{a\cos\phi - 0} = \frac{-b^2}{a^2} \Rightarrow \sin\theta\sin\phi = -\cos\theta\cos\phi \Rightarrow \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = \pm\pi/2$.

5.2.18 Subtangent and Subnormal

Let the tangent and normal at $P(x_1, y_1)$ meet the x-axis at A and B respectively.

Length of subtangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $DA = CA - CD = \frac{a^2}{x_1} - x_1$

Length of sub-normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is



$$BD = CD - CB = x_1 - \left(x_1 - \frac{b^2}{a^2}x_1\right) = \frac{b^2}{a^2}x_1 = (1 - e^2)x_1 .$$

Note : ❑ The tangent and normal to any point of an ellipse bisects respectively the internal and external angles between the focal radii of that point.

Example: 36 Length of subtangent and subnormal at the point $\left(\frac{-5\sqrt{3}}{2}, 2\right)$ of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ are

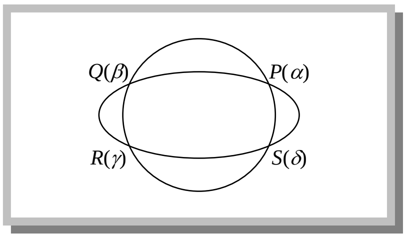
(a) $\left(\frac{5\sqrt{3}}{2} - \frac{10}{\sqrt{3}}\right), \frac{8\sqrt{3}}{5}$ (b) $\left(\frac{5\sqrt{3}}{2} + \frac{10}{\sqrt{3}}\right), \frac{8\sqrt{3}}{10}$ (c) $\left(\frac{5\sqrt{3}}{2} + \frac{12}{\sqrt{3}}\right), \frac{16\sqrt{3}}{5}$ (d) None of thee

Solution: (a) Here $a^2 = 25, b^2 = 16, x_1 = \frac{-5\sqrt{3}}{2}$. Length of subtangent $= \left|\frac{a^2}{x_1} - x_1\right| = \left|\frac{25}{-5\sqrt{3}/2} + \frac{5\sqrt{3}}{2}\right| = \left|\frac{5\sqrt{3}}{2} - \frac{10}{\sqrt{3}}\right|$.

Length of subnormal $= \left|\frac{b^2}{a^2}x_1\right| = \left|\frac{16}{25}\left(\frac{-5\sqrt{3}}{2}\right)\right| = \left|\frac{8\sqrt{3}}{5}\right|$

5.2.19 Concyclic points

Any circle intersects an ellipse in two or four points. They are called concyclic points and the sum of their eccentric angles is an even multiple of π .



If $\alpha, \beta.\gamma, \delta$ be the eccentric angles of the four concyclic points on an ellipse, then $\alpha + \beta + \gamma + \delta = 2n\pi$, where n is any integer.

Note : ❑ The common chords of a circle and an ellipse are equally inclined to the axes of the ellipse.

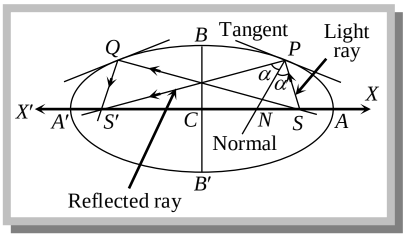
Important Tips

☞ *The centre of a circle* $x^2+y^2+2gx+2fy+c=0$ *passing through the three points, on an ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *(whose eccentric angles are* a,β,γ *) is* $-g=\left(\frac{a^2-b^2}{4a}\right)\{\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)\}$ *and* $-f=\left(\frac{b^2-a^2}{4a}\right)\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}$

☞ $P'CP$ *and* $D'CD$ *are conjugate diameters of an ellipse and* α *is the eccentric angles of P. Then the eccentric angles of the point where the circle through* P, P', D *again cuts the ellipse is* $\pi/2-3\alpha$.

5.2.20 Reflection property of an Ellipse

Let S and S' be the foci and PN the normal at the point P of the ellipse, then $\angle SPS'=\angle SQS'$. Hence if an incoming light ray aimed towards one focus strike the concave side of the mirror in the shape of an ellipse then it will be reflected towards the other focus.



Example: 37 A ray emanating from the point $(-3,0)$ is incident on the ellipse $16x^2+25y^2=400$ at the point P with ordinate 4. Then the equation of the reflected ray after first reflection is

(a) $4x+3y=12$ (b) $3x+4y=12$ (c) $4x-3y=12$ (d) $3x-4y=12$

Solution: (a) For point P y-coordinate =4

Given ellipse is $16x^2+25y^2=400$

$16x^2+25(4)^2=400$, $\therefore x=0$

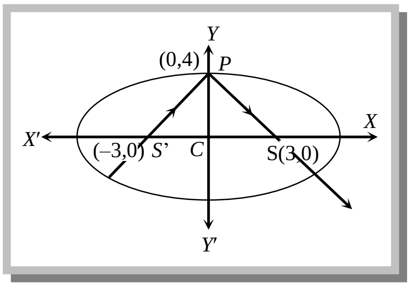
$\therefore$ co-ordinate of P is (0, 4)

$e^2=1-\frac{16}{25}=\frac{9}{25}$

$\therefore\ e=\frac{3}{5}$

$\therefore$ Foci $(\pm ae,0)$, *i.e.* $(\pm 3,0)$

$\therefore$ Equation of reflected ray $(i.e. PS)$ is $\frac{x}{3}+\frac{y}{4}=1$ or $4x+3y=12$.



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Assignment

Definition of the Ellipse

Basic Level

1. If a bar of given length moves with its extremities on two fixed straight lines at right angles, then the locus of any point on bar marked on the bar describes a/an **[Orissa JEE 2003]**

(a) Circle (b) Parabola (c) Ellipse (d) Hyperbola

2. If the eccentricity of an ellipse becomes zero, then it takes the form of

(a) A circle (b) A parabola (c) A straight line (d) None of these

3. The locus of a variable point whose distance from $(-2,0)$ is $\frac{2}{3}$ times its distance from the line $x=-\frac{9}{2}$, is **[IIT Screening 1994]**

(a) Ellipse (b) Parabola (c) Hyperbola (d) None of these

4. If *A* and *B* are two fixed points and *P* is a variable point such that $PA+PB=4$, where $AB<4$, then the locus of *P* is

(a) A parabola (b) An ellipse (c) A hyperbola (d) None of these

5. Equation of the ellipse whose focus is $(6,7)$ directrix is $x+y+2=0$ and $e=1/\sqrt{3}$ is

(a) $5x^2+2xy+5y^2-76x-88y+506=0$ (b) $5x^2-2xy+5y^2-76x-88y+506=0$

(c) $5x^2-2xy+5y^2+76x+88y-506=0$ (d) None of these

6. The locus of the centre of the circle $x^2+y^2+4x\cos\theta-2y\sin\theta-10=0$ is

(a) An ellipse (b) A circle (c) A hyperbola (d) A parabola

Standard and other forms of an Ellipse, Terms related to an Ellipse

Basic Level

7. The equation $2x^2+3y^2=30$ represents **[MP PET 1988]**

(a) A circle (b) An ellipse (c) A hyperbola (d) A parabola

8. The equation $\frac{x^2}{2-r}+\frac{y^2}{r-5}+1=0$ represents an ellipse, if **[MP PET 1995]**

(a) $r>2$ (b) $2<r<5$ (c) $r>5$ (d) None of these

9. Equation of the ellipse with eccentricity $\frac{1}{2}$ and foci at $(\pm 1,0)$ is **[MP PET 2002]**

(a) $\frac{x^2}{3}+\frac{y^2}{4}=1$ (b) $\frac{x^2}{4}+\frac{y^2}{3}=1$ (c) $\frac{x^2}{4}+\frac{y^2}{3}=\frac{4}{3}$ (d) None of these

10. The equation of the ellipse whose foci are $(\pm 5,0)$ and one of its directrix is $5x=36$, is

(a) $\frac{x^2}{36}+\frac{y^2}{11}=1$ (b) $\frac{x^2}{6}+\frac{y^2}{\sqrt{11}}=1$ (c) $\frac{x^2}{6}+\frac{y^2}{11}=1$ (d) None of these

11. The equation of ellipse whose distance between the foci is equal to 8 and distance between the directrix is 18, is

(a) $5x^2 - 9y^2 = 180$ (b) $9x^2 + 5y^2 = 180$ (c) $x^2 + 9y^2 = 180$ (d) $5x^2 + 9y^2 = 180$

12. The equation of the ellipse whose one of the vertices is $(0, 7)$ and the corresponding directrix is $y = 12$, is

(a) $95x^2 + 144y^2 = 4655$ (b) $144x^2 + 95y^2 = 4655$ (c) $95x^2 + 144y^2 = 13680$ (d) None of these

13. The equation of the ellipse whose centre is at origin and which passes through the points $(-3, 1)$ and $(2, -2)$ is

(a) $5x^2 + 3y^2 = 32$ (b) $3x^2 + 5y^2 = 32$ (c) $5x^2 - 3y^2 = 32$ (d) $3x^2 + 5y^2 + 32 = 0$

14. An ellipse passes through the point $(-3, 1)$ and its eccentricity is $\sqrt{\frac{2}{5}}$. The equation of the ellipse is

(a) $3x^2 + 5y^2 = 32$ (b) $3x^2 + 5y^2 = 25$ (c) $3x^2 + y^2 = 4$ (d) $3x^2 + y^2 = 9$

15. If the centre, one of the foci and semi- major axis of an ellipse be (0, 0), (0, 3) and 5 then its equation is **[AMU 1981]**

(a) $\frac{x^2}{16} + \frac{y^2}{25} = 1$ (b) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (c) $\frac{x^2}{9} + \frac{y^2}{25} = 1$ (d) None of these

16. The equation of the ellipse whose latus rectum is 8 and whose eccentricity is $\frac{1}{\sqrt{2}}$, referred to the principal axes of coordinates, is **[MP PET 1993]**

(a) $\frac{x^2}{18} + \frac{y^2}{32} = 1$ (b) $\frac{x^2}{8} + \frac{y^2}{9} = 1$ (c) $\frac{x^2}{64} + \frac{y^2}{32} = 1$ (d) $\frac{x^2}{16} + \frac{y^2}{24} = 1$

17. The lengths of major and minor axes of an ellipse are 10 and 8 respectively and its major axis is along the *y*-axis. The equation of the ellipse referred to its centre as origin is

(a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (b) $\frac{x^2}{16} + \frac{y^2}{25} = 1$ (c) $\frac{x^2}{100} + \frac{y^2}{64} = 1$ (d) $\frac{x^2}{64} + \frac{y^2}{100} = 1$

18. The equation of the ellipse whose vertices are $(\pm 5, 0)$ and foci are $(\pm 4, 0)$ is

(a) $9x^2 + 25y^2 = 225$ (b) $25x^2 + 9y^2 = 225$ (c) $3x^2 + 4y^2 = 192$ (d) None of these

19. The latus rectum of an ellipse is 10 and the minor axis is equal to the distance between the foci. The equation of the ellipse is

(a) $x^2 + 2y^2 = 100$ (b) $x^2 + \sqrt{2}y^2 = 10$ (c) $x^2 - 2y^2 = 100$ (d) None of these

20. The eccentricity of the ellipse $4x^2 + 9y^2 = 36$, is **[MP PET 2000]**

(a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{5}}{3}$ (d) $\frac{\sqrt{5}}{6}$

21. Eccentricity of the conic $16x^2 + 7y^2 = 112$ is **[MNR 1981]**

(a) $\frac{3}{\sqrt{7}}$ (b) $\frac{7}{16}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

22. Eccentricity of the ellipse $9x^2 + 25y^2 = 225$ is **[Kerala (Engg.) 2002]**

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{9}{25}$ (d) $\frac{\sqrt{34}}{5}$

23. The eccentricity of the ellipse $25x^2 + 16y^2 = 400$ is **[MP PET 2001]**

(a) $\frac{3}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

24. For the ellipse $\frac{x^2}{64} + \frac{y^2}{28} = 1$, the eccentricity is **[MNR 1974]**

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{2}{\sqrt{7}}$ (d) $\frac{1}{3}$

25. If the latus rectum of an ellipse be equal to half of its minor axis, then its eccentricity is **[MP PET 1991, 1997; Karnataka CET 2000]**

(a) $\frac{3}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2}{3}$ (d) $\frac{\sqrt{2}}{3}$

26. If the length of the major axis of an ellipse is three times the length of its minor axis, then its eccentricity is **[EAMCET 1990]**

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{2\sqrt{2}}{3}$

27. The length of the latus rectum of an ellipse is $\frac{1}{3}$ of the major axis. Its eccentricity is **[EMACET 1991]**

(a) $\frac{2}{3}$ (b) $\sqrt{\frac{2}{3}}$ (c) $\frac{5\times 4\times 3}{7^3}$ (d) $\left(\frac{3}{4}\right)^4$

28. Eccentricity of the ellipse whose latus rectum is equal to the distance between two focus points, is

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{\sqrt{5}-1}{2}$ (c) $\frac{\sqrt{5}}{2}$ (d) $\frac{\sqrt{3}}{2}$

29. If the distance between the foci of an ellipse be equal to its minor axis, then its eccentricity is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{3}}$

30. The length of the latus rectum of the ellipse $\frac{x^2}{36}+\frac{y^2}{49}=1$ is **[Karnataka CET 1993]**

(a) $\frac{98}{6}$ (b) $\frac{72}{7}$ (c) $\frac{72}{14}$ (d) $\frac{98}{12}$

31. For the ellipse $3x^2+4y^2=12$, the length of latus rectum is **[MNR 1973]**

(a) $\frac{3}{2}$ (b) 3 (c) $\frac{8}{3}$ (d) $\sqrt{\frac{3}{2}}$

32. The length of the latus rectum of the ellipse $9x^2+4y^2=1$, is **[MP PET 1999]**

(a) $\frac{3}{2}$ (b) $\frac{8}{3}$ (c) $\frac{4}{9}$ (d) $\frac{8}{9}$

33. In an ellipse, minor axis is 8 and eccentricity is $\frac{\sqrt{5}}{3}$. Then major axis is **[Karnataka CET 2002]**

(a) 6 (b) 12 (c) 10 (d) 16

34. The distance between the foci of an ellipse is 16 and eccentricity is $\frac{1}{2}$. Length of the major axis of the ellipse is **[Karnataka CET 2001]**

(a) 8 (b) 64 (c) 16 (d) 32

35. If the eccentricity of an ellipse be $1/\sqrt{2}$, then its latus rectum is equal to its

(a) Minor axis (b) Semi-minor axis (c) Major axis (d) Semi-major axis

36. If the distance between a focus and corresponding directrix of an ellipse be 8 and the eccentricity be $1/2$, then the length of the minor axis is

(a) 3 (b) $4\sqrt{2}$ (c) 6 (d) None of these

37. The sum of focal distances of any point on the ellipse with major and minor axes as $2a$ and $2b$ respectively, is equal to **[MP PET 2003]**

(a) $2a$ (b) $2\frac{a}{b}$ (c) $2\frac{b}{a}$ (d) $\frac{b^2}{a}$

38. P is any point on the ellipse $9x^2+36y^2=324$ whose foci are S and S'. Then $SP+S'P$ equals **[DCE 1999]**

(a) 3 (b) 12 (c) 36 (d) 324

39. The foci of $16x^2 + 25y^2 = 400$ are **[BIT Ranchi 1996]**

(a) $(\pm 3, 0)$ (b) $(0, \pm 3)$ (c) $(3, -3)$ (d) $(-3, 3)$

40. In an ellipse $9x^2 + 5y^2 = 45$, the distance between the foci is **[Karnataka CET 2002]**

(a) $4\sqrt{5}$ (b) $3\sqrt{5}$ (c) 3 (d) 4

41. The distance between the directrices of the ellipse $\frac{x^2}{36} + \frac{y^2}{20} = 1$ is

(a) 8 (b) 12 (c) 18 (d) 24

42. If the eccentricity of the two ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are equal, then the value of a/b is **[UPSEAT 2001]**

(a) $\frac{5}{13}$ (b) $\frac{6}{13}$ (c) $\frac{13}{5}$ (d) $\frac{13}{6}$

43. The equation of the ellipse whose one focus is at (4, 0) and whose eccentricity is 4/5, is **[Karnataka CET 1993]**

(a) $\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$ (b) $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ (c) $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ (d) $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$

44. *S* and *T* are the foci of an ellipse and *B* is an end of the minor axis. If *STB* is an equilateral triangle, the eccentricity of the ellipse is **[EMACET 1992; DCE 1995]**

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

45. If *C* is the centre of the ellipse $9x^2 + 16y^2 = 144$ and *S* is one focus, the ratio of *CS* to semi-major axis, is

(a) $\sqrt{7} : 16$ (b) $\sqrt{7} : 4$ (c) $\sqrt{5} : \sqrt{7}$ (d) None of these

46. If $L.R. = 10$, distance between foci = length of minor axis, then equation of ellipse is

(a) $\frac{x^2}{50} + \frac{y^2}{100} = 1$ (b) $\frac{x^2}{100} + \frac{y^2}{50} = 1$ (c) $\frac{x^2}{50} + \frac{y^2}{20} = 1$ (d) None of these

47. Line joining foci subtends an angle of 90° at an extremity of minor axis, then eccentricity is

(a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) None of these

48. If foci are points $(0, 1), (0, -1)$ and minor axis is of length 1, then equation of ellipse is

(a) $\frac{x^2}{1/4} + \frac{y^2}{5/4} = 1$ (b) $\frac{x^2}{5/4} + \frac{y^2}{1/4} = 1$ (c) $\frac{x^2}{3/4} + \frac{y^2}{1/4} = 1$ (d) $\frac{x^2}{1/4} + \frac{y^2}{3/4} = 1$

49. The eccentricity of the ellipse $5x^2 + 9y^2 = 1$ is **[EMACET 2000]**

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{4}{5}$ (d) $\frac{1}{2}$

50. For the ellipse $x^2 + 4y^2 = 9$ **[Roorkee 1999]**

(a) The eccentricity is $\frac{1}{2}$ (b) The latus rectum is $\frac{2}{3}$ (c) A focus is $(3\sqrt{3}, 0)$ (d) A directrix is $x = 2\sqrt{3}$

51. The sum of the distances of any point on the ellipse $3x^2 + 4y^2 = 24$ from its foci is **[Kerala (Engg.) 2001]**

(a) $8\sqrt{2}$ (b) $4\sqrt{2}$ (c) $16\sqrt{2}$ (d) None of these

52. The sum of the focal distances from any point on the ellipse $9x^2 + 16y^2 = 144$ is **[Roorkee 1997; Pb.CET 2002]**

(a) 32 (b) 18 (c) 16 (d) 8

53. The distance of a focus of the ellipse $9x^2 + 16y^2 = 144$ from an end of the minor axis is

(a) $\frac{3}{2}$ (b) 3 (c) 4 (d) None of these

54. The equation of ellipse in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, given the eccentricity to be $\frac{2}{3}$ and latus rectum $\frac{2}{3}$ is **[BIT Ranchi 1998]**

(a) $25x^2 + 45y^2 = 9$ (b) $25x^2 - 4y^2 = 9$ (c) $25x^2 - 45y^2 = 9$ (d) $25x^2 + 4y^2 = 1$

55. The equation of the ellipse with axes along the x-axis and the y-axis, which passes through the points P (4, 3) and Q (6, 2) is

(a) $\frac{x^2}{50} + \frac{y^2}{13} = 1$ (b) $\frac{x^2}{52} + \frac{y^2}{13} = 1$ (c) $\frac{x^2}{13} + \frac{y^2}{52} = 1$ (d) $\frac{x^2}{52} + \frac{y^2}{17} = 1$

56. P is a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with AA' as the major axis. Then the maximum value of the area of the triangle APA' is

(a) ab (b) $2ab$ (c) $\frac{ab}{2}$ (d) None of these

57. The latus rectum of the ellipse $x^2 \tan^2 \alpha + y^2 \sec^2 \alpha = 1$ is 1/2 then $\alpha (0 < \alpha < \pi)$ is equal to

(a) $\pi / 12$ (b) $\pi / 6$ (c) $5\pi / 12$ (d) None of these

Advance Level

58. An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 *cm* and 4 *cm*, the necessary length of the string and the distance between the pins respectively in *cm*, are **[MNR 1989]**

(a) $6, 2\sqrt{5}$ (b) $6, \sqrt{5}$ (c) $4, 2\sqrt{5}$ (d) None of these

59. A man running round a race-course notes that the sum of the distances of two flag-posts from him is always 10 *meters* and the distance between the flag-posts is 8 *meters*. The area of the path he encloses in square metres is **[MNR 1991; UPSEAT 2000]**

(a) 15π (b) 12π (c) 18π (d) 8π

60. The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, $r > 1$ represents **[IIT 1981]**

(a) An ellipse (b) A hyperbola (c) A circle (d) An imaginary ellipse

61. The radius of the circle having its centre at (0,3) and passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, is **[IIT 1995]**

(a) 3 (b) 3.5 (c) 4 (d) $\sqrt{12}$

62. The centre of an ellipse is C and PN is any ordinate and A, A' are the end points of major axis, then the value of $\frac{PN^2}{AN.A'N}$ is

(a) $\frac{b^2}{a^2}$ (b) $\frac{a^2}{b^2}$ (c) $a^2 + b^2$ (d) 1

63. Let P be a variable point on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ with foci at S and S'. If A be the area of triangle PSS', then the maximum value of A is

(a) 24 sq. units (b) 12 sq. units (c) 36 sq. units (d) None of these

64. The eccentricity of the ellipse which meets the straight line $\frac{x}{7} + \frac{y}{2} = 1$ on the axis of x and the straight line $\frac{x}{3} - \frac{y}{5} = 1$ on the axis of y and whose axes lie along the axes of coordinates, is

(a) $\frac{3\sqrt{2}}{7}$ (b) $\frac{2\sqrt{6}}{7}$ (c) $\frac{\sqrt{3}}{7}$ (d) None of these

65. If the focal distance of an end of the minor axis of an ellipse (referred to its axes as the axes of x and y respectively) is k and the distance between its foci is $2h$, then its equation is

(a) $\frac{x^2}{k^2}+\frac{y^2}{h^2}=1$ (b) $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$ (c) $\frac{x^2}{k^2}+\frac{y^2}{h^2-k^2}=1$ (d) $\frac{x^2}{k^2}+\frac{y^2}{k^2+h^2}=1$

66. If (5, 12) and (24, 7) are the foci of a conic passing through the origin, then the eccentricity of conic is

(a) $\frac{\sqrt{386}}{38}$ (b) $\frac{\sqrt{386}}{12}$ (c) $\frac{\sqrt{386}}{13}$ (d) $\frac{\sqrt{386}}{25}$

67. The maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the vertex at one end of the major axis is

[Roorkee 1994, Him. CET 2002]

(a) $\sqrt{3}ab$ (b) $\frac{3\sqrt{3}}{4}ab$ (c) $\frac{5\sqrt{3}}{4}ab$ (d) None of these

68. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having its centre (0, 3) is **[IIT 1995]**

(a) 4 (b) 3 (c) $\sqrt{12}$ (d) $\frac{7}{2}$

69. The locus of extremities of the latus rectum of the family of ellipse $b^2x^2+y^2=a^2b^2$ is

(a) $x^2-ay=a^2$ (b) $x^2-ay=b^2$ (c) $x^2+ay=a^2$ (d) $x^2+ay=b^2$

Special form of an Ellipse, Parametric equation of an Ellipse

Basic Level

70. The equation of the ellipse whose centre is (2,–3), one of the foci is (3, –3) and the corresponding vertex is (4, –3) is

(a) $\frac{(x-2)^2}{3}+\frac{(y+3)^2}{4}=1$ (b) $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{3}=1$ (c) $\frac{x^2}{3}+\frac{y^2}{4}=1$ (d) None of these

71. The equation of an ellipse, whose vertices are (2, –2), (2, 4) and eccentricity $\frac{1}{3}$, is **[Karnataka CET 1999]**

(a) $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{8}=1$ (b) $\frac{(x-2)^2}{8}+\frac{(y-1)^2}{9}=1$ (c) $\frac{(x+2)^2}{8}+\frac{(y+1)^2}{9}=1$ (d) $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{8}=1$

72. The equation of an ellipse whose eccentricity 1/2 is and the vertices are (4, 0) and (10, 0) is

(a) $3x^2+4y^2-42x+120=0$ (b) $3x^2+4y^2+42x+120=0$

(c) $3x^2+4y^2+42x-120=0$ (d) $3x^2+4y^2-42x-120=0$

73. For the ellipse $3x^2+4y^2-6x+8y-5=0$ **[BTT Ranchi 2000]**

(a) Centre is (2, –1) (b) Eccentricity is $\frac{1}{3}$

(c) Foci are(3, 1) and (–1, 1) (d) Centre is (1, –1), $e=\frac{1}{2}$, foci are (3, –1) and (–1, –1)

74. The eccentricity of the ellipse $9x^2+5y^2-18x-2y-16=0$ **[EAMCET 2003]**

(a) 1/2 (b) 2/3 (c) 1/3 (d) 3/4

75. The eccentricity of the ellipse $\frac{(x-1)^2}{9}+\frac{(y+1)^2}{25}=1$ is **[AMU 1999]**

(a) 4/5 (b) 3/5 (c) 5/4 (d) Imaginary

76. The eccentricity of the ellipse $9x^2+5y^2-30y=0$, is **[MNR 1993]**

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) None of these

77. The eccentricity of the ellipse $4x^2+9y^2+8x+36y+4=0$ is **[MP PET 1996]**

(a) $\frac{5}{6}$ (b) $\frac{3}{5}$ (c) $\frac{\sqrt{2}}{3}$ (d) $\frac{\sqrt{5}}{3}$

78. The eccentricity of the curve represented by the equation $x^2+2y^2-2x+3y+2=0$ is **[Roorkee 1998]**

(a) 0 (b) 1/2 (c) $1/\sqrt{2}$ (d) $\sqrt{2}$

79. The centre of the ellipse $\frac{(x+y-2)^2}{9}+\frac{(x-y)^2}{16}=1$, is **[EAMCET 1994]**

(a) (0, 0) (b) (1, 1) (c) (1, 0) (d) (0, 1)

80. The centre of the ellipse $4x^2+9y^2-16x-54y+61=0$ is **[MP PET 1992]**

(a) (1, 3) (b) (2, 3) (c) (3, 2) (d) (3, 1)

81. Latus rectum of ellipse $4x^2+9y^2-8x-36y+4=0$ is **[MP PET 1989]**

(a) 8 / 3 (b) 4 / 3 (c) $\frac{\sqrt{5}}{3}$ (d) 16 / 3

82. The length of the axes of the conic $9x^2+4y^2-6x+4y+1=0$, are **[Orissa JEE 2002]**

(a) $\frac{1}{2},9$ (b) $3,\frac{2}{5}$ (c) $1,\frac{2}{3}$ (d) 3, 2

83. Equations $x=a\cos\theta,\ y=b\sin\theta\,(a>b)$ represent a conic section whose eccentricity e is given by

(a) $e^2=\frac{a^2+b^2}{a^2}$ (b) $e^2=\frac{a^2+b^2}{b^2}$ (c) $e^2=\frac{a^2-b^2}{a^2}$ (d) $e^2=\frac{a^2-b^2}{b^2}$

84. The curve with parametric equations $x=1+4\cos\theta,\ y=2+3\ \sin\theta$ is

(a) An ellipse (b) A parabola (c) A hyperbola (d) A circle

85. The equations $x=a\cos\theta, y=b\ \sin\theta, 0\le\theta<2\pi, a\ne b$, represent

(a) An ellipse (b) A parabola (c) A circle (d) A hyperbola

86. The curve represented by $x=2(\cos t+\sin t), y=5(\cos t-\sin t)$ is **[EAMCET 2000]**

(a) A circle (b) A parabola (c) An ellipse (d) A hyperbola

87. The equations $x=a\left(\frac{1-t^2}{1+t^2}\right),\ y=\frac{2bt}{1+t^2}; t\in R$ represent

(a) A circle (b) An ellipse (c) A parabola (d) A hyperbola

88. The eccentricity of the ellipse represented by $25x^2+16y^2-150x-175=0$ is **[JMIEE 2000]**

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{4}{5}$ (d) None of these

89. The set of values of a for which $(13x-1)^2+(13y-2)^2=a(5x+12y-1)^2$ represents an ellipse is

(a) $1<a<2$ (b) $0<a<1$ (c) $2<a<3$ (d) None of these

Advance Level

90. The parametric representation of a point on the ellipse whose foci are (–1, 0) and (7, 0) and eccentricity 1/2 is

(a) $(3+8\cos\theta, 4\sqrt{3}\sin\theta)$ (b) $(8\cos\theta, 4\sqrt{3}\sin\theta)$ (c) $(3+4\sqrt{3}\cos\theta, 8\sin\theta)$ (d) None of these

91. If $P(\theta)$ and $Q\left(\frac{\pi}{2}+\theta\right)$ are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then locus of the mid-point of PQ is

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$ (b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$ (c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$ (d) None of these

Position of a point, Tangents, Pair of tangents, and Director circle of an Ellipse

Basic Level

92. The line $lx+my-n=0$ will be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if

(a) $a^2l^2+b^2m^2=n^2$ (b) $al^2+bm^2=n^2$ (c) $a^2l+b^2m=n$ (d) None of these

93. The line $x\cos\alpha+y\sin\alpha=p$ will be a tangent to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if **[Roorkee 1978]**

(a) $p^2=a^2\sin^2\alpha+b^2\cos^2\alpha$ (b) $p^2=a^2+b^2$

(c) $p^2=b^2\sin^2\alpha+a^2\cos^2\alpha$ (d) None of these

94. The equations of the tangents of the ellipse $9x^2+16y^2=144$, which passes through the point (2, 3) is **[MP PET 1996]**

(a) $y=3, x+y=5$ (b) $y=-3, x-y=5$ (c) $y=4, x+y=3$ (d) $y=-4, x-y=3$

95. The equation of the tangent to the conic $x^2-y^2-8x+2y+11=0$ at (2, 1) is **[Karnataka CET 1993]**

(a) $x+2=0$ (b) $2x+1=0$ (c) $x-2=0$ (d) $x+y+1=0$

96. The position of the point (1, 3) with respect to the ellipse $4x^2+9y^2-16x-54y+61=0$ is **[MP PET 1991]**

(a) Outside the ellipse (b) On the ellipse (c) On the major axis (d) On the minor axis

97. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the straight line $y=mx+c$ intersect in real points only if **[MNR 1984, 1995]**

(a) $a^2m^2<c^2-b^2$ (b) $a^2m^2>c^2-b^2$ (c) $a^2m^2\geq c^2-b^2$ (d) $c\geq b$

98. If the line $y=mx+c$ touches the ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, then c= **[MNR 1975; MP PET 1994,95,99]**

(a) $\pm\sqrt{b^2m^2+a^2}$ (b) $\pm\sqrt{a^2m^2+b^2}$ (c) $\pm\sqrt{b^2m^2-a^2}$ (d) $\pm\sqrt{a^2m^2-b^2}$

99. If the line $y=2x+c$ be a tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$, then c= **[MNR 1979; DCE 2000]**

(a) ±4 (b) ±6 (c) ±1 (d) ±8

100. The equation of the tangent to the ellipse $x^2+16y^2=16$ making an angle of 60^o with x-axis

(a) $\sqrt{3}x-y+7=0$ (b) $\sqrt{3}x-y-7=0$ (c) $\sqrt{3}x-y\pm 7=0$ (d) None of these

101. The position of the point (4, –3) with respect to the ellipse $2x^2+5y^2=20$ is

(a) Outside the ellipse (b) On the ellipse (c) On the major axis (d) None of these

102. The angle between the pair of tangents drawn to the ellipse $3x^2+2y^2=5$ from the point (1, 2) is **[MNR 1984]**

(a) $\tan^{-1}\left(\frac{12}{5}\right)$ (b) $\tan^{-1}(6\sqrt{5})$ (c) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$ (d) $\tan^{-1}(12\sqrt{5})$

103. If any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts off intercepts of length h and k on the axes, then $\frac{a^2}{h^2}+\frac{b^2}{k^2}=$

(a) 0 (b) 1 (c) –1 (d) None of these

104. The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2+5y^2-30y=0$, are **[MP PET 1999]**

(a) $y=\pm 3$ (b) $x=\pm\sqrt{5}$ (c) $y=0, y=6$ (d) None of these

105. The locus of the point of intersection of mutually perpendicular tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is **[MP PET 1995]**

(a) A straight line (b) A parabola (c) A circle (d) None of these

106. Two perpendicular tangents drawn to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ intersect on the curve

(a) $x=\frac{a}{e}$ (b) $x^2+y^2=41$ (c) $x^2+y^2=9$ (d) $x^2-y^2=41$

107. The product of the perpendiculars drawn from the two foci of an ellipse to the tangent at any point of the ellipse is **[EAMCAT 2000]**

(a) a^2 (b) b^2 (c) $4a^2$ (d) $4b^2$

108. The equations of the tangents to the ellipse $4x^2+3y^2=5$, which are inclined at 60^o to the axis of x are

(a) $y=\sqrt{3}x\pm\sqrt{\frac{65}{12}}$ (b) $y=\sqrt{3}x\pm\sqrt{\frac{12}{65}}$ (c) $y=\frac{x}{\sqrt{3}}\pm\sqrt{\frac{65}{12}}$ (d) None of these

109. If the straight line $y=4x+c$ is a tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$, then c will be equal to

(a) ± 4 (b) ± 6 (c) ± 1 (d) $\pm\sqrt{(132)}$

110. Tangents are drawn to the ellipse $3x^2+5y^2=32$ and $25x^2+9y^2=450$ passing through the point (3, 5). The number of such tangents are

(a) 2 (b) 3 (c) 4 (d) 0

111. If $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then its eccentric angle θ is equal to **[EAMCET 1995]**

(a) 0^o (b) 90^o (c) 45^o (d) 60^o

112. Locus of point of intersection of tangents at $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$ for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is **[IIIT Allahabad 2001]**

(a) A circle (b) A straight line (c) An ellipse (d) A parabola

113. The equation of the tangent at the point $(1/4, 1/4)$ of the ellipse $\frac{x^2}{4}+\frac{y^2}{12}=1$ is

(a) $3x+y=48$ (b) $3x+y=3$ (c) $3x+y=16$ (d) None of these

114. If F_1 and F_2 be the feet of the perpendiculars from the foci S_1 and S_2 of an ellipse $\frac{x^2}{5}+\frac{y^2}{3}=1$ on the tangent at any point P on the ellipse, then $(S_1F_1)(S_2F_2)$ is equal to

(a) 2 (b) 3 (c) 4 (d) 5

115. Equations of tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, which cut off equal intercepts on the axes is

(a) $y=x+\sqrt{13}$ (b) $y=-x+\sqrt{13}$ (c) $y=x-\sqrt{13}$ (d) $y=-x-\sqrt{13}$

116. The line $x = at^2$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the real points , if

(a) $|t|< 2$ (b) $|t| \le 1$ (c) $|t| > 1$ (d) None of these

Advance Level

117. The locus of mid points of parts in between axes and tangents of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will be **[UPSEAT 1999]**

(a) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ (b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=2$ (c) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=3$ (d) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$

118. The angle of intersection of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle $x^2+y^2=ab$, is

(a) $\tan^{-1}\left(\frac{a-b}{ab}\right)$ (b) $\tan^{-1}\left(\frac{a+b}{ab}\right)$ (c) $\tan^{-1}\left(\frac{a+b}{\sqrt{ab}}\right)$ (d) $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$

119. Locus of the foot of the perpendicular drawn from the centre upon any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is

(a) $(x^2+y^2)^2=b^2x^2+a^2y^2$ (b) $(x^2+y^2)^2=b^2x^2-a^2y^2$

(c) $(x^2+y^2)^2=a^2x^2-b^2y^2$ (d) $(x^2+y^2)^2=a^2x^2+b^2y^2$

120. If a tangent having slope of $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ intersects the major and minor axes in points A and B respectively, then the area of ΔOAB is equal to (O is centre of the ellipse)

(a) 12 sq. units (b) 48 sq. units (c) 64 sq. units (d) 24 sq. units

121. Tangent is drawn to ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$(where $\theta \in \left(0, \frac{\pi}{2}\right)$). Then the value of θ such that sum of intercepts on axes made by this tangent is minimum, is **[IIT Screening 2003]**

(a) $\pi/3$ (b) $\pi/6$ (c) $\pi/8$ (d) $\pi/4$

122. If the tangent at the point $\left(4\cos\phi, \frac{16}{\sqrt{11}}\sin\phi\right)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x=15$, then the value of ϕ is

(a) $\pm\frac{\pi}{2}$ (b) $\pm\frac{\pi}{4}$ (c) $\pm\frac{\pi}{3}$ (d) $\pm\frac{\pi}{6}$

123. An ellipse passes through the point (4, –1) and its axes are along the axes of co-ordinates. If the line $x+4y-10=0$ is a tangent to it, then its equation is

(a) $\frac{x^2}{100}+\frac{y^2}{5}=1$ (b) $\frac{x^2}{80}+\frac{y^2}{5/4}=1$ (c) $\frac{x^2}{20}+\frac{y^2}{5}=1$ (d) None of these

124. The sum of the squares of the perpendiculars on any tangent to the ellipse $x^2/a^2+y^2/b^2=1$ from two points on the minor axis each distance $\sqrt{a^2-b^2}$ from the centre is

(a) a^2 (b) b^2 (c) $2a^2$ (d) $2b^2$

125. The tangent at a point $P(a\cos\theta, b\sin\theta)$ of an ellipse $x^2/a^2+y^2/b^2=1$, meets its auxiliary circle in two points, the chord joining which subtends a right angle at the centre, then the eccentricity of the ellipse is

(a) $(1+\sin^2\theta)^{-1}$ (b) $(1+\sin^2\theta)^{-1/2}$ (c) $(1+\sin^2\theta)^{-3/2}$ (d) $(1+\sin^2\theta)^{-2}$

126. The locus of the point of intersection of tangents to an ellipse at two points, sum of whose eccentric angles is constant is

(a) A parabola (b) A circle (c) An ellipse (d) A straight line

127. The sum of the squares of the perpendiculars on any tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from two points on the minor axis each at a distance *ae* from the centre is

(a) $2a^2$ (b) $2b^2$ (c) a^2+b^2 (d) a^2-b^2

128. The equation of the circle passing through the points of intersection of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ is

(a) $x^2+y^2=a^2$ (b) $x^2+y^2=b^2$ (c) $x^2+y^2=\frac{a^2b^2}{a^2+b^2}$ (d) $x^2+y^2=\frac{2a^2b^2}{a^2+b^2}$

129. The slope of a common tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a concentric circle of radius *r* is

(a) $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$ (b) $\sqrt{\frac{r^2-b^2}{a^2-r^2}}$ (c) $\left(\frac{r^2-b^2}{a^2-r^2}\right)$ (d) $\sqrt{\frac{a^2-r^2}{r^2-b^2}}$

130. The tangents from which of the following points to the ellipse $5x^2+4y^2=20$ are perpendicular

(a) $(1, 2\sqrt{2})$ (b) $(2\sqrt{2}, 1)$ (c) $(2, \sqrt{5})$ (d) $(\sqrt{5}, 2)$

Normals , Eccentric angles of the Co-normal points

Basic Level

131. The line $y=mx+c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $c=$

(a) $-(2am+bm^2)$ (b) $\frac{(a^2+b^2)m}{\sqrt{a^2+b^2m^2}}$ (c) $-\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$ (d) $\frac{(a^2-b^2)m}{\sqrt{a^2+b^2}}$

132. The line $lx+my+n=0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if **[DCE 2000]**

(a) $\frac{a^2}{m^2}+\frac{b^2}{l^2}=\frac{(a^2-b^2)}{n^2}$ (b) $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$ (c) $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$ (d) None of these

133. If the line $x\cos\alpha+y\sin\alpha=p$ be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then **[MP PET 2001]**

(a) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=a^2-b^2$ (b) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=(a^2-b^2)^2$

(c) $p^2(a^2\sec^2\alpha+b^2\text{cosec}^2\alpha)=a^2-b^2$ (d) $p^2(a^2\sec^2\alpha+b^2\text{cosec}^2\alpha)=(a^2-b^2)^2$

134. The equation of the normal at the point (2, 3) on the ellipse $9x^2+16y^2=180$, is **[MP PET 2000]**

(a) $3y=8x-10$ (b) $3y-8x+7=0$ (c) $8y+3x+7=0$ (d) $3x+2y+7=0$

135. The eccentric angles of the extremities of latus-rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by

(a) $\tan^{-1}\left(\pm\frac{ae}{b}\right)$ (b) $\tan^{-1}\left(\pm\frac{be}{a}\right)$ (c) $\tan^{-1}\left(\pm\frac{b}{ae}\right)$ (d) $\tan^{-1}\left(\pm\frac{a}{be}\right)$

136. The number of normals that can be drawn from a point to a given ellipse is

(a) 2 (b) 3 (c) 4 (d) 1

137. The eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$, whose distances from the centre of the ellipse is 2, is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{3}$ (d) $\frac{7\pi}{6}$

Advance Level

138. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

139. If the normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the coordinates axes in G and g respectively, then $PG : Pg$=

(a) $a:b$ (b) $a^2:b^2$ (c) $b^2:a^2$ (d) $b:a$

140. If α and β are eccentric angles of the ends of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ is equal to

(a) $\frac{1-e}{1+e}$ (b) $\frac{e-1}{e+1}$ (c) $\frac{e+1}{e-1}$ (d) None of these

141. If the normal at one end of the latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the one end of the minor axis, then

(a) $e^4-e^2+1=0$ (b) $e^2-e+1=0$ (c) $e^2+e+1=0$ (d) $e^4+e^2-1=0$

142. The line $2x+y=3$ cuts the ellipse $4x^2+y^2=5$ at P and Q. If θ be the angle between the normals at these points, then $\tan\theta=$ **[DCE 1995]**

(a) 1/2 (b) 3/4 (c) 3/5 (d) 5

143. The eccentric angles of extremities of a chord of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are θ_1 and θ_2. If this chord passes through the focus, then

(a) $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}+\frac{1-e}{1+e}=0$ (b) $\cos\frac{\theta_1-\theta_2}{2}=e.\cos\frac{\theta_1+\theta_2}{2}$

(c) $e=\frac{\sin\theta_1+\sin\theta_2}{\sin(\theta_1+\theta_2)}$ (d) $\cot\frac{\theta_1}{2}.\cot\frac{\theta_2}{2}=\frac{e+1}{e-1}$

144. Let F_1, F_2 be two foci of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P then

(a) PN bisects $\angle F_1PF_2$ (b) PT bisects $\angle F_1PF_2$

(c) PT bisects angle $(180^o-\angle F_1PF_2)$ (d) None of these

145. If CF is the perpendicular from the centre C of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the tangent at any point P and G is the point when the normal at P meets the major axis, then $CF.PG$=

(a) a^2 (b) ab (c) b^2 (d) b^3

Chord of contact, Equation of the chord joining two points of an Ellipse

Basic Level

146. The equation of the chord of the ellipse $2x^2 + 5y^2 = 20$ which is bisected at the point (2, 1)is

(a) $4x + 5y + 13 = 0$ (b) $4x + 5y = 13$ (c) $5x + 4y + 13 = 0$ (d) None of these

147. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1 x_2}{y_1 y_2}$ is equal to

(a) $\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$ (c) $-\frac{a^4}{b^4}$ (d) $-\frac{b^4}{a^4}$

148. Chords of an ellipse are drawn through the positive end of the minor axis. Then their mid-point lies on

(a) A circle (b) A parabola (c) An ellipse (d) A hyperbola

149. The length of the common chord of the ellipse $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1$ and the circle $(x-1)^2 + (y-2)^2 = 1$ is

(a) Zero (b) One (c) Three (d) Eight

Advance Level

150. If $\tan\theta_1 \tan\theta_2 = -\frac{a^2}{b^2}$, then the chord joining two points θ_1 and θ_2 on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will subtend a right angle at

(a) Focus (b) Centre (c) End of the major axis (d) End of the minor axis

151. If θ and ϕ are the eccentric angles of the ends of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

(a) $\cos\frac{\theta-\phi}{2} = e\cos\frac{\theta+\phi}{2}$ (b) $\cos\frac{\theta-\phi}{2} + e\cos\frac{\theta+\phi}{2} = 0$ (c) $\cos\frac{\theta+\phi}{2} = e\cos\frac{\theta-\phi}{2}$ (d) None of these

Diameter of an ellipse, Pole and Polar and Conjugate diameters

Basic Level

152. With respect to the ellipse $3x^2 + 2y^2 = 1$, the pole of the line $9x + 2y = 1$ is

(a) $(-1, -3)$ (b) $(-1, 3)$ (c) $(3, -1)$ (d) $(3, 1)$

153. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of diameter conjugate to the diameter $y = \frac{b}{a}x$, is

(a) $y = -\frac{b}{a}x$ (b) $y = -\frac{a}{b}x$ (c) $x = -\frac{b}{a}y$ (d) None of these

154. If CP and CD are semi conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $CP^2 + CD^2 =$

(a) $a + b$ (b) $a^2 + b^2$ (c) $a^2 - b^2$ (d) $\sqrt{a^2 + b^2}$

155. The eccentricity of an ellipse whose pair of a conjugate diameter are $y = x$ and $3y = -2x$ is

(a) 2/3 (b) 1/3 (c) $1/\sqrt{3}$ (d) None of these

156. If eccentric angle of one diameter is $\frac{5\pi}{6}$, then eccentric angle of conjugate diameter is

(a) $\frac{2\pi}{3}$ (b) $\frac{4\pi}{3}$ (c) $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ (d) None of these

157. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of the diameter conjugate to $ax-by=0$ is

(a) $bx+ay=0$ (b) $bx-ay=0$ (c) $a^3y+b^3x=0$ (d) $a^3y-b^3x=0$

158. Equation of equi-conjugate diameter for an ellipse $\frac{x^2}{25}+\frac{y^2}{16}$ is

(a) $x=\pm\frac{5}{4}y$ (b) $y=\pm\frac{5}{4}x$ (c) $x=\pm\frac{25}{16}y$ (d) None of these

Advance Level

159. The locus of the point of intersection of tangents at the ends of semi-conjugate diameter of ellipse is

(a) Parabola (b) Hyperbola (c) Circle (d) Ellipse

160. AB is a diameter of $x^2+9y^2=25$. The eccentric angle of A is $\pi/6$. Then the eccentric angle of B is

(a) $5\pi/6$ (b) $-5\pi/6$ (c) $-2\pi/3$ (d) None of these

161. If the points of intersection of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$ be the extremities of the conjugate diameter of first ellipse, then

(a) $\frac{x^2}{p^2}+\frac{y^2}{q^2}=2$ (b) $\frac{a^2}{p^2}+\frac{b^2}{q^2}=1$ (c) $\frac{a}{p}+\frac{b}{q}=1$ (d) $\frac{a^2}{p^2}+\frac{b^2}{q^2}=2$

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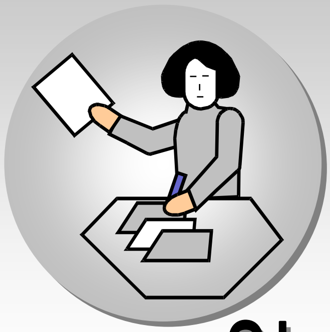

Answer Sheet

Conic Section : Ellipse *Assignment (Basic and Advance level)*

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
c	a	a	b	b	a	b	b	b	a	d	b	b	a	a	c	b	a	a	c
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
c	b	a	a	b	d	b	b	b	b	b	c	b	d	d	d	a	b	a	d
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
c	c	b	c	b	b	c	a	a	d	b	d	c	a	b	a	a,c	d	a	d
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**
c	a	b	b	b	a,b	b	a	a,c	b	b	a	d	b	a	b	d	c	b	b
81	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**
a	c	c	a	a	c	b	b	b	a	a	a	c	a	c	c	c	a	b	c
101	**102**	**103**	**104**	**105**	**106**	**107**	**108**	**109**	**110**	**111**	**112**	**113**	**114**	**115**	**116**	**117**	**118**	**119**	**120**
a	c	b	c	c	b	b	a	d	b	c	c	d	b	a,b,c,d	b	d	d	d	d
121	**122**	**123**	**124**	**125**	**126**	**127**	**128**	**129**	**130**	**131**	**132**	**133**	**134**	**135**	**136**	**137**	**138**	**139**	**140**
b	c	b,c	c	b	d	a	d	b	a,b,c,d	c	b	d	b	c	c	a	b	c	b
141	**142**	**143**	**144**	**145**	**146**	**147**	**148**	**149**	**150**	**151**	**152**	**153**	**154**	**155**	**156**	**157**	**158**	**159**	**160**
d	c	a,b,c,d	a,c	c	b	c	c	a	b	a	d	a	b	c	c	c	a	d	b
161																			
d																			

Chapter

5.3

Hyperbola

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5.3.6	Position of a point with respect to a hyperbola
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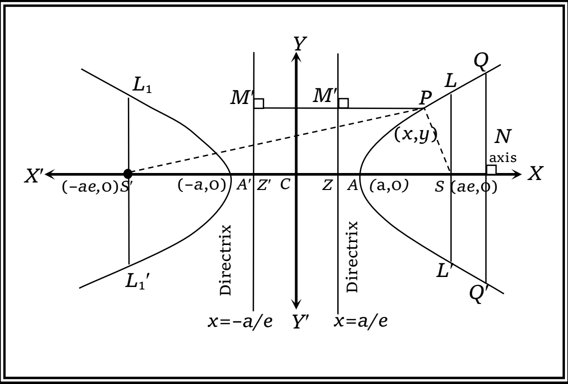
5.3.19	Rectangular or equilateral hyperbola
5.3.20	Intersection of a circle and rectangular hyperbola
Assignment (Basic and Advance Level)	
Answer Sheet of Assignment	



Apollonius writes Conics in which he introduces the terms "parabola", " ellipse" and "hyperbola".

De Beaune writes Notes brieves which contains the many results on "Cartesian geometry", in particular giving the now familiar equations for hyperbolas, parabolas and ellipses.

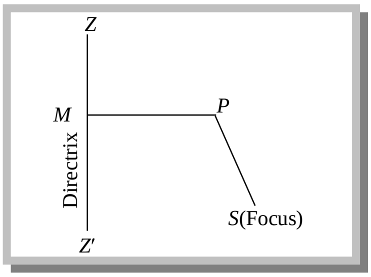
The hyperbola is also useful for describing the path of an alpha particle in the electric field of the nucleus of an atom.

Hyperbola has its application in the field of Ballistics. Suppose a gun is fired. If the sound reaches two listening posts, situated at two foci of the hyperbola at different times, from the time difference, the distance between the two listening posts (two foci) can be calculated.

5.3 Hyperbola

5.3.1 Definition

A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is always greater than unity.



Fixed point is called focus, fixed straight line is called directrix and the constant ratio is called eccentricity of the hyperbola. Eccentricity is denoted by e and $e > 1$.

A hyperbola is the particular case of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

When, $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ *i.e.*, $\Delta \neq 0$ and $h^2 > ab$.

Let $S(h,k)$ is the focus, directrix is the line $ax + by + c = 0$ and the eccentricity is e. Let $P(x_1, y_1)$ be a point which moves such that $SP = e.PM$

$$\Rightarrow \quad \sqrt{(x_1 - h)^2 + (y_1 - k)^2} = e.\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad (a^2 + b^2)[(x_1 - h)^2 + (y_1 - k)^2] = e^2(ax_1 + by_1 + c)^2$$

Hence, locus of (x_1, y_1) is given by $(a^2 + b^2)[(x - h)^2 + (y - k)^2] = e^2(ax + by + c)^2$

Which is a second degree equation to represent a hyperbola ($e > 1$).

Example: 1 The equation of the conic with focus at (1, – 1), directrix along $x - y + 1 = 0$ and with eccentricity $\sqrt{2}$ is

[EAMCET 1994; DCE 1998]

(a) $x^2 - y^2 = 1$ (b) $xy = 1$ (c) $2xy - 4x + 4y + 1 = 0$ (d) $2xy + 4x - 4y - 1 = 0$

Solution: (c) Here, focus (S) = (1, –1), eccentricity (e)= $\sqrt{2}$

From definition, $SP = e\,PM$

$$\sqrt{(x - 1)^2 + (y + 1)^2} = \frac{\sqrt{2}.(x - y + 1)}{\sqrt{1^2 + 1^2}}$$

$\Rightarrow (x - 1)^2 + (y + 1)^2 = (x - y + 1)^2 \Rightarrow 2xy - 4x + 4y + 1 = 0$, which is the required equation of conic (Rectangular hyperbola)

Example: 2 The centre of the hyperbola $9x^2 - 36x - 16y^2 + 96y - 252 = 0$ is **[Karnataka CET 1993]**

(a) (2, 3) (b) (– 2, – 3) (c) (–2, 3) (d) (2, – 3)

Solution: (a) Here $a = 9$, $b = -16$, $h = 0$, $g = -18$, $f = 48$, $c = -252$

Centre of hyperbola = $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right) = \left(\frac{(0)(48) - (-16)(-18)}{(9)(-16) - 0}, \frac{(-18)(0) - (9)(48)}{(9)(-16) - 0}\right)$ = (2, 3)

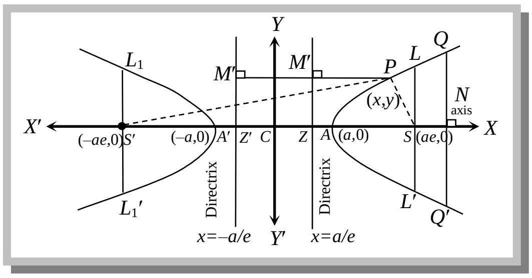
5.3.2 Standard equation of the Hyperbola

Let S be the focus, ZM be the directrix and e be the eccentricity of the hyperbola, then by definition,

$$\Rightarrow \frac{SP}{PM} = e \Rightarrow (SP)^2 = e^2(PM)^2$$

$$\Rightarrow (x - a.e)^2 + (y-0)^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(e^2-1)$$



This is the standard equation of the hyperbola.

Some terms related to hyperbola : Let the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(1) **Centre :** All chords passing through C are bisected at C. Here $C(0,0)$

(2) **Vertex:** The point A and A' where the curve meets the line joining the foci S and S' are called vertices of hyperbola. The co-ordinates of A and A' are $(a, 0)$ and $(-a, 0)$ respectively.

(3) **Transverse and conjugate axes :** The straight line joining the vertices A and A′ is called transverse axis of the hyperbola. The straight line perpendicular to the transverse axis and passing through the centre is called conjugate axis.

Here, transverse axis = $AA' = 2a$

Conjugate axis = $BB' = 2b$

(4) **Eccentricity :** For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have $b^2 = a^2(e^2-1)$, $e = \sqrt{1 + \left(\frac{2b}{2a}\right)^2} = \sqrt{1 + \left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2}$

(5) **Double ordinates :** If Q be a point on the hyperbola, QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then QQ' is called a double ordinate at Q.

If abscissa of Q is h, then co-ordinates of Q and Q' are $\left(h, \frac{b}{a}\sqrt{h^2 - a^2}\right)$ and $\left(h, -\frac{b}{a}\sqrt{h^2-a^2}\right)$ respectively.

(6) **Latus-rectum :** The chord of the hyperbola which passes through the focus and is perpendicular to its transverse axis is called latus-rectum.

Length of latus-rectum $LL' = L_1L_1' = \frac{2b^2}{a} = 2a(e^2-1)$ and end points of latus-rectum $L\left(ae, \frac{b^2}{a}\right)$; $L'\left(ae, \frac{-b^2}{a}\right)$; $L_1\left(-ae, \frac{b^2}{a}\right)$; $L_1'\left(-ae, -\frac{b^2}{a}\right)$ respectively.

(7) **Foci and directrices:** The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the hyperbola and ZM and $Z'M'$ are two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.

Distance between foci $SS' = 2ae$ and distance between directrices $ZZ' = 2a/e$.

(8) **Focal chord :** A chord of the hyperbola passing through its focus is called a focal chord.

(9) **Focal distance :** The difference of any point on the hyperbola from the focus is called the focal distance of the point.

From the figure, $SP = ePM = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$, $S'P = ePM' = e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$

The difference of the focal distance of a point on the hyperbola is constant and is equal to the length of transverse axis.

$|S'P - SP| = 2a = AA' =$ Transverse axis

Example: 3 The eccentricity of the hyperbola which passes through (3, 0) and $(3\sqrt{2}, 2)$ is **[UPSEAT 2000]**

(a) $\sqrt{(13)}$ (b) $\frac{\sqrt{13}}{3}$ (c) $\sqrt{\frac{13}{4}}$ (d) None of these

Solution: (b) Let equation of hyperbola is $x^2/a^2 - y^2/b^2 = 1$. Point (3, 0) lies on hyperbola

So, $\frac{(3)^2}{a^2} - \frac{0}{b^2} = 1$ or $\frac{9}{a^2} = 1$ or $a^2 = 9$ and point $(3\sqrt{2}, 2)$ also lies on hyperbola. So, $\frac{3(\sqrt{2})^2}{a^2} - \frac{(2)^2}{b^2} = 1$

Put $a^2 = 9$ we get, $\frac{18}{9} - \frac{4}{b^2} = 1$ or $2 - \frac{4}{b^2} = 1$ or $-\frac{4}{b^2} = 1 - 2$ or $\frac{4}{b^2} = 1$ or $b^2 = 4$

We know that $b^2 = a^2(e^2 - 1)$. Putting values of a^2 and b^2

$4 = 9(e^2 - 1)$ or $e^2 - 1 = \frac{4}{9}$ or $e^2 = 1 + \frac{4}{9}$ or $e = \sqrt{(1 + 4/9)}$ or $e = \sqrt{(13)/9} = \frac{\sqrt{13}}{3}$.

Example: 4 The foci of the hyperbola $9x^2 - 16y^2 = 144$ are **[MP PET 2001]**

(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$ (c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

Solution: (c) The equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Now, $b^2 = a^2(e^2 - 1) \Rightarrow 9 = 16(e^2 - 1) \Rightarrow e = \frac{5}{4}$. Hence foci are $(\pm ae, 0) = \left(\pm 4 . \frac{5}{4}, 0\right)$ *i.e.*, $(\pm 5, 0)$

Example: 5 If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

[MNR 1992; UPSEAT 2001; AIEEE 2003]

(a) 1 (b) 5 (c) 7 (d) 9

Solution: (c) For hyperbola, $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

$$A = \sqrt{\frac{144}{25}}, B = \sqrt{\frac{81}{25}}, e_1 = \sqrt{1 + \frac{B^2}{A^2}} = \sqrt{1 + \frac{81}{144}} = \sqrt{\frac{225}{144}} = \frac{5}{4}$$

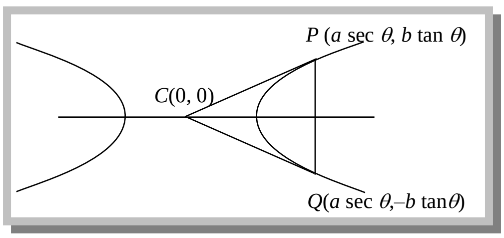
Therefore foci $= (\pm ae_1, 0) = \left(\pm \frac{12}{5} . \frac{5}{4}, 0\right) = (\pm 3, 0)$. Therefore foci of ellipse *i.e.*, $(\pm 4e, 0) = (\pm 3, 0)$ (For ellipse $a = 4$)

$\Rightarrow e = \frac{3}{4}$, Hence $b^2 = 16\left(1 - \frac{9}{16}\right) = 7$.

Example: 6 If PQ is a double ordinate of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that CPQ is an equilateral triangle, C being the centre of the hyperbola. Then the eccentricity e of the hyperbola satisfies **[EAMCET 1999]**

(a) $1 < e < 2/\sqrt{3}$ (b) $e = 2/\sqrt{3}$ (c) $e = \sqrt{3}/2$ (d) $e > 2/\sqrt{3}$

Solution: (d) Let $P(a\sec\theta, b\tan\theta)$; $Q(a\sec\theta, -b\tan\theta)$ be end points of double ordinates and $C(0,0)$ is the centre of the hyperbola



Now $PQ = 2b\tan\theta$; $CQ = CP = \sqrt{a^2\sec^2\theta + b^2\tan^2\theta}$

Since $CQ = CP = PQ$, $\therefore$ $4b^2\tan^2\theta = a^2\sec^2\theta + b^2\tan^2\theta$

$\Rightarrow 3b^2\tan^2\theta = a^2\sec^2\theta \Rightarrow 3b^2\sin^2\theta = a^2$

$\Rightarrow 3a^2(e^2 - 1)\sin^2\theta = a^2 \Rightarrow 3(e^2 - 1)\sin^2\theta = 1$

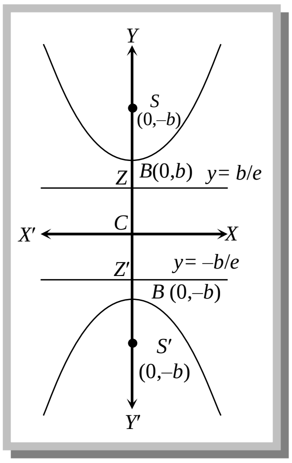
$\Rightarrow \frac{1}{3(e^2 - 1)} = \sin^2\theta < 1$ $(\because \sin^2\theta < 1)$

$\Rightarrow \frac{1}{e^2 - 1} < 3 \Rightarrow e^2 - 1 > \frac{1}{3} \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

5.3.3 Conjugate Hyperbola

The hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola is called conjugate hyperbola of the given hyperbola.

Fundamentals \ Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **or** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x=\pm a/e$	$y=\pm b/e$
Eccentricity	$e=\sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e=\sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric co-ordinates	$(a\sec\phi, b\tan\phi)$, $0\le\phi<2\pi$	$(b\sec\phi, a\tan\phi)$, $0\le\phi<2\pi$
Focal radii	$SP=ex_1-a$ & $S'P=ex_1+a$	$SP=ey_1-b$ & $S'P=ey_1+b$
Difference of focal radii $(S'P-SP)$	$2a$	$2b$
Tangents at the vertices	$x=-a, x=a$	$y=-b, y=b$
Equation of the transverse axis	$y=0$	$x=0$
Equation of the conjugate axis	$x=0$	$y=0$



Note : ❑ If e and e' are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2}+\frac{1}{e'^2}=1$.

❑ The foci of a hyperbola and its conjugate are concyclic.

Example: 7 The eccentricity of the conjugate hyperbola of the hyperbola $x^2-3y^2=1$, is **[MP PET 1999]**

(a) 2 (b) $\frac{2}{\sqrt{3}}$ (c) 4 (d) $\frac{4}{3}$

Solution: (a) The given hyperbola is $\frac{x^2}{1}-\frac{y^2}{1/3}=1$. Here $a^2=1$ and $b^2=\frac{1}{3}$

Since $b^2=a^2(e^2-1)\Rightarrow\frac{1}{3}=1(e^2-1)\Rightarrow e^2=\frac{4}{3}\Rightarrow e=\frac{2}{\sqrt{3}}$

If e' is the eccentricity of the conjugate hyperbola, then $\frac{1}{e^2}+\frac{1}{e'^2}=1\Rightarrow\frac{1}{e'^2}=1-\frac{1}{e^2}=1-\frac{3}{4}=\frac{1}{4}\Rightarrow e'=2$.

5.3.4 Special form of Hyperbola

If the centre of hyperbola is (h, k) and axes are parallel to the co-ordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. By shifting the origin at (h, k) without rotating the co-ordinate axes, the above equation reduces to $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$, where $x=X+h, y=Y+k$.

Example: 8 The equation of the hyperbola whose foci are (6, 4) and (– 4, 4) and eccentricity 2 is given by **[MP PET 1993]**

(a) $12x^2-4y^2-24x+32y-127=0$ (b) $12x^2+4y^2+24x-32y-127=0$

(c) $12x^2 - 4y^2 - 24x - 32y + 127 = 0$ (d) $12x^2 - 4y^2 + 24x + 32y + 127 = 0$

Solution: (a) Foci are (6, 4) and (– 4, 4) and $e = 2$.

$\therefore$ Centre is $\left(\frac{6-4}{2}, \frac{4+4}{2}\right) = (1, 4)$

So, $ae + 1 = 6 \Rightarrow ae = 5 \Rightarrow a = \frac{5}{2}$ and $b = \frac{5}{2}\sqrt{3}$

Hence, the required equation is $\frac{(x-1)^2}{25/4} - \frac{(y-4)^2}{(75/4)} = 1$ or $12x^2 - 4y^2 - 24x + 32y - 127 = 0$

Example: 9 The equations of the directrices of the conic $x^2 + 2x - y^2 + 5 = 0$ are

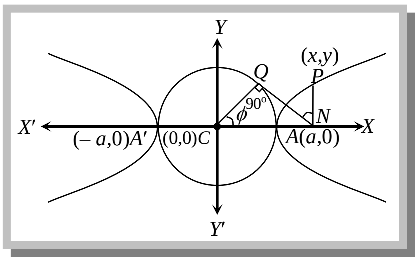
(a) $x = \pm 1$ (b) $y = \pm 2$ (c) $y = \pm\sqrt{2}$ (d) $x = \pm\sqrt{3}$

Solution: (c) $(x+1)^2 - y^2 - 1 + 5 = 0 \Rightarrow -\frac{(x+1)^2}{4} + \frac{y^2}{4} = 1$

Equation of directrices of $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are $y = \pm\frac{b}{e}$

Here $b = 2$, $e = \sqrt{1+1} = \sqrt{2}$. Hence, $y = \pm\frac{2}{\sqrt{2}} \Rightarrow y = \pm\sqrt{2}$.

5.3.5 Auxiliary circle of Hyperbola



Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ Equation of the auxiliary circle is $x^2 + y^2 = a^2$

Let $\angle QCN = \phi$

Here P and Q are the corresponding points on the hyperbola and the auxiliary circle $(0 \le \phi < 2\pi)$

(1) **Parametric equations of hyperbola :** The equations $x = a\sec\phi$ and $y = b\tan\phi$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This $(a\sec\phi, b\tan\phi)$ lies on the hyperbola for all values of ϕ.

Position of points Q on auxiliary circle and the corresponding point P which describes the hyperbola and $0 \le \phi < 2\pi$		
ϕ varies from	**$Q(a\cos\varphi, a\sin\varphi)$**	**$P(a\sec\varphi, b\tan\varphi)$**
0 to $\frac{\pi}{2}$	I	I
$\frac{\pi}{2}$ to π	II	III
π to $\frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}$ to 2π	IV	IV

Note : ❑ The equations $x = a\cosh\theta$ and $y = b\sinh\theta$ are also known as the parametric equations of the hyperbola and the co-ordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are expressible as $(a\cosh\theta,\, b\sinh\theta)$, where $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$ and $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$.

Example: 10 The distance between the directrices of the hyperbola $x = 8\sec\theta$, $y = 8\tan\theta$ is **[Karnataka CET 2003]**

(a) $16\sqrt{2}$ (b) $\sqrt{2}$ (c) $8\sqrt{2}$ (d) $4, \sqrt{2}$

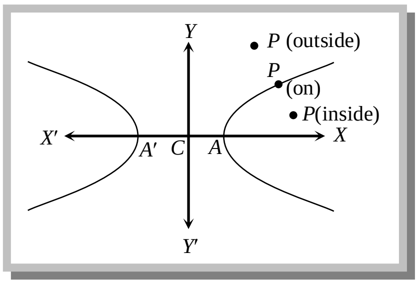
Solution: (c) Equation of hyperbola is $x = 8\sec\theta, y = 8\tan\theta \Rightarrow \frac{x}{8} = \sec\theta$, $\frac{y}{8} = \tan\theta$

$\therefore \sec^2\theta - \tan^2\theta = 1 \Rightarrow \frac{x^2}{8^2} - \frac{y^2}{8^2} = 1$

Here $a = 8, b = 8$. Now $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{8^2}{8^2}} = \sqrt{2}$

$\therefore$ Distance between directrices $= \frac{2a}{e} = \frac{2 \times 8}{\sqrt{2}} = 8\sqrt{2}$.

5.3.6 Position of a point with respect to a Hyperbola



Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then $P(x_1, y_1)$ will lie inside, on or outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative.

Example: 11 The number of tangents to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ through (4, 1) is **[AMU 1998]**

(a) 1 (b) 2 (c) 0 (d) 3

Solution: (c) Since the point (4, 1) lies inside the hyperbola $\left[\because \frac{16}{4} - \frac{1}{3} - 1 > 0\right]$; $\therefore$ Number of tangents through (4, 1) is 0.

5.3.7 Intersection of a Line and a Hyperbola

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in two points may be real, coincident or imaginary according as $c^2 >, =, < a^2m^2 - b^2$.

Condition of tangency : If straight line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$.

5.3.8 Equations of Tangent in Different forms

(1) **Point form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(2) **Parametric form :** The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\phi, b\tan\phi)$ is

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$$

(3) **Slope form :** The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=mx\pm\sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

Note : ❑ If the straight line $lx+my+n=0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2l^2-b^2m^2=n^2$.

❑ If the straight line $x\cos\alpha+y\sin\alpha=p$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$

❑ Two tangents can be drawn from an outside point to a hyperbola.

Important Tips

☞ *For hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *and* $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, *the equation of common tangent is* $y=\pm x\pm\sqrt{a^2-b^2}$, *points of contacts are* $\left(\pm\frac{a^2}{\sqrt{a^2-b^2}};\pm\frac{b^2}{\sqrt{a^2-b^2}}\right)$ *and length of common tangent is* $\sqrt{2}.\frac{(a^2+b^2)}{\sqrt{a^2-b^2}}$.

☞ *If the line* $y=mx\pm\sqrt{a^2m^2-b^2}$ *touches the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *at the point* $(a\sec\theta, b\tan\theta)$, *then* $\theta=\sin^{-1}\left(\frac{b}{am}\right)$.

Example: 12 The value of m for which $y=mx+6$ is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$, is **[Karnataka CET 1993]**

(a) $\sqrt{\frac{17}{20}}$ (b) $\sqrt{\frac{20}{17}}$ (c) $\sqrt{\frac{3}{20}}$ (d) $\sqrt{\frac{20}{3}}$

Solution: (a) For condition of tangency, $c^2=a^2m^2-b^2$. Here $c=6$, $a=10, b=7$

Then, $(6)^2=(10)^2.m^2-(7)^2$

$36=100m^2-49 \Rightarrow 100m^2=85 \Rightarrow m^2=\frac{17}{20} \Rightarrow m=\sqrt{\frac{17}{20}}$

Example: 13 If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ which pass through the point (6, 2), then

(a) $m_1+m_2=\frac{24}{11}$ (b) $m_1m_2=\frac{20}{11}$ (c) $m_1+m_2=\frac{48}{11}$ (d) $m_1m_2=\frac{11}{20}$

Solution: (a, b) The line through (6, 2) is $y-2=m(x-6) \Rightarrow y=mx+2-6m$

Now, from condition of tangency $(2-6m)^2=25m^2-16$

$\Rightarrow 36m^2+4-24m-25m^2+16=0 \Rightarrow 11m^2-24m+20=0$

Obviously, its roots are m_1 and m_2, therefore $m_1+m_2=\frac{24}{11}$ and $m_1m_2=\frac{20}{11}$

Example: 14 The points of contact of the line $y=x-1$ with $3x^2-4y^2=12$ is **[BIT Ranchi 1996]**

(a) (4, 3) (b) (3, 4) (c) (4, – 3) (d) None of these

Solution: (a) The equation of line and hyperbola are $y=x-1$ .....(i) and $3x^2-4y^2=12$ .....(ii)

From (i) and (ii), we get $3x^2-4(x-1)^2=12$

$\Rightarrow 3x^2-4(x^2-2x+1)=12$ or $x^2-8x+16=0 \Rightarrow x=4$

From (i), $y=3$ so points of contact is (4, 3)

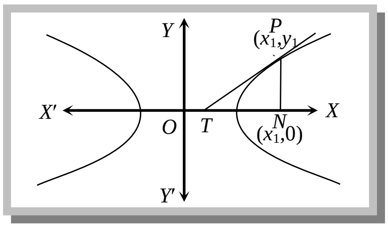
Trick : Points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

Here $a^2=4$, $b^2=3$ and $m=1$. So the required points of contact is (4, 3).

Example: 15 P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, N$ is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT.ON$ is equal to

(a) e^2 (b) a^2 (c) b^2 (d) $\frac{b^2}{a^2}$

Solution: (b) Let $P(x_1, y_1)$ be a point on the hyperbola. Then the co-ordinates of N are $(x_1, 0)$.



The equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

This meets x-axis at $T\left(\frac{a^2}{x_1}, 0\right)$; $\therefore$ $OT.ON = \frac{a^2}{x_1}\times x_1 = a^2$

Example: 16 If the tangent at the point $(2\sec\phi, 3\tan\phi)$ on the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to $3x-y+4=0$, then the value of ϕ is

(a) 45^o (b) 60^o (c) 30^o (d) 75^o

Solution: (c) Here $x=2\sec\phi$ and $y=3\tan\phi$

Differentiating *w.r.t.* ϕ

$\frac{dx}{d\phi}=2\sec\phi\tan\phi$ and $\frac{dy}{d\phi}=3\sec^2\phi$

$\therefore$ Gradient of tangent $\frac{dy}{dx}=\frac{dy/d\phi}{dx/d\phi}=\frac{3\sec^2\phi}{2\sec\phi\tan\phi}$; $\therefore$ $\frac{dy}{dx}=\frac{3}{2}\text{cosec}\,\phi$ .....(i)

But tangent is parallel to $3x-y+4=0$; $\therefore$ Gradient $m=3$ ......(ii)

From (i) and (ii), $\frac{3}{2}\text{cosec}\,\phi=3 \Rightarrow \text{cosec}\,\phi=2$, $\therefore$ $\phi=30^o$

Example: 17 The slopes of the common tangents to the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ and $\frac{y^2}{9}-\frac{x^2}{16}=1$ are **[Roorkee 1997]**

(a) – 2, 2 (b) – 1, 1 (c) 1, 2 (d) 2, 1

Solution: (b) Given hyperbola are $\frac{x^2}{9}-\frac{y^2}{16}=1$ .....(i) and $\frac{y^2}{9}-\frac{x^2}{16}=1$ .....(ii)

Any tangent to (i) having slope m is $y=mx\pm\sqrt{9m^2-16}$ .....(iii)

Putting in (ii), we get, $16[mx\pm\sqrt{9m^2-16}]^2-9x^2=144$

$\Rightarrow (16m^2-9)x^2\pm 32m(\sqrt{9m^2-16})x+144m^2-256-144=0$

$\Rightarrow (16m^2-9)x^2\pm 32m(\sqrt{9m^2-16})x+(144m^2-400)=0$ .....(iv)

If (iii) is a tangent to (ii), then the roots of (iv) are real and equal.

$\therefore$ Discriminant = 0; $32\times 32m^2(9m^2-16) = 4(16m^2-9)(144m^2-400) = 64(16m^2-9)(9m^2-25)$

$\Rightarrow 16m^2(9m^2-16)=(16m^2-9)(9m^2-25)\Rightarrow 144m^4-256m^2=144m^4-481m^2+225$

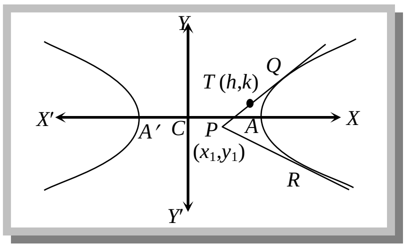
$\Rightarrow 225m^2=225 \Rightarrow m^2=1 \Rightarrow m=\pm 1$

5.3.9 Equation of Pair of Tangents

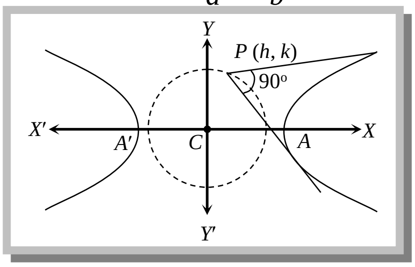
If $P(x_1, y_1)$ be any point outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then a pair of tangents PQ, PR can be drawn to it from P.

The equation of pair of tangents PQ and PR is $SS_1=T^2$

where, $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$



Director circle : The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola. The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$



Example: 18 The locus of the point of intersection of tangents to the hyperbola $4x^2-9y^2=36$ which meet at a constant angle $\pi/4$, is

(a) $(x^2+y^2-5)^2=4(9y^2-4x^2+36)$ (b) $(x^2+y^2-5)=4(9y^2-4x^2+36)$

(c) $4(x^2+y^2-5)^2=(9y^2-4x^2+36)$ (d) None of these

Solution: (a) Let the point of intersection of tangents be $P(x_1,y_1)$. Then the equation of pair of tangents from $P(x_1,y_1)$ to the given hyperbola is $(4x^2-9y^2-36)(4x_1^2-9y_1^2-36)=[4x_1x-9y_1y-36]^2$ ......(i)

From $SS_1=T^2$ or $x^2(y_1^2+4)+2x_1y_1xy+y^2(x_1^2-9)+.....=0$ .....(ii)

Since angle between the tangents is $\pi/4$.

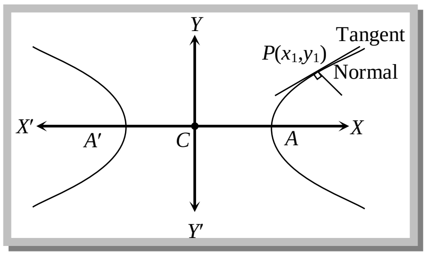
$\therefore\ \tan(\pi/4)=\frac{2\sqrt{[x_1^2y_1^2-(y_1^2+4)(x_1^2-9)]}}{y_1^2+4+x_1^2-9}$. Hence locus of $P(x_1,y_1)$ is $(x^2+y^2-5)^2=4(9y^2-4x^2+36)$.

5.3.10 Equations of Normal in Different forms

(1) **Point form :** The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1,y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

(2) **Parametric form:** The equation of normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$ax\cos\theta+by\cot\theta=a^2+b^2$$



(3) **Slope form:** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope m of the normal is $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$

(4) **Condition for normality :** If $y=mx+c$ is the normal of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

then $c=\mp\frac{m(a^2+b^2)}{\sqrt{a^2-m^2b^2}}$ or $c^2=\frac{m^2(a^2+b^2)^2}{(a^2-m^2b^2)}$, which is condition of normality.

(5) **Points of contact :** Co-ordinates of points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}},\mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$

Note : ❑ If the line $lx+my+n=0$ will be normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$.

Important Tip

☞ *In general, four normals can be drawn to a hyperbola from any point and if $\alpha,\beta,\gamma,\delta$ be the eccentric angles of these four co-normal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.*

☞ *If α,β,γ are the eccentric angles of three points on the hyperbola. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the normals at which are concurrent, then,* $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$

☞ *If the normal at P meets the transverse axis in G, then $SG=e.SP$. Also the tangent and normal bisect the angle between the focal distances of P.*

☞ *The feet of the normals to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from (h,k) lie on $a^2y(x-h)+b^2x(y-k)=0$.*

Example: 19 The equation of the normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point $(8,3\sqrt{3})$ is **[MP PET 1996]**

(a) $\sqrt{3}x+2y=25$ (b) $x+y=25$ (c) $y+2x=25$ (d) $2x+\sqrt{3}y=25$

Solution: (d) From $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

Here $a^2=16$, $b^2=9$ and $(x_1,y_1)=(8,3\sqrt{3})$

$\Rightarrow \frac{16x}{8}+\frac{9y}{3\sqrt{3}}=16+9$ *i.e.*, $2x+\sqrt{3}y=25$.

Example: 20 If the normal at 'ϕ' on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets transverse axis at *G*, then $AG.A'G=$

(Where *A* and A' are the vertices of the hyperbola)

(a) $a^2(e^4\sec^2\phi-1)$ (b) $(a^2e^4\sec^2\phi-1)$ (c) $a^2(1-e^4\sec^2\phi)$ (d) None of these

Solution: (a) The equation of normal at $(a\sec\phi, b\tan\phi)$ to the given hyperbola is $ax\cos\phi+by\cot\phi=(a^2+b^2)$

This meets the transverse axis *i.e.*, *x*-axis at *G*. So the co-ordinates of *G* are $\left(\left(\frac{a^2+b^2}{a}\right)\sec\phi, 0\right)$ and the co-ordinates of the vertices *A* and A' are $A(a,0)$ and $A'(-a,0)$ respectively.

$\therefore \quad AG.A'G=\left(-a+\left(\frac{a^2+b^2}{a}\right)\sec\phi\right)\left(a+\left(\frac{a^2+b^2}{a}\right)\sec\phi\right)=\left(\frac{a^2+b^2}{a}\right)^2\sec^2\phi-a^2=(ae^2)^2\sec^2\phi-a^2=a^2(e^4\sec^2\phi-1)$

Example: 21 The normal at *P* to a hyperbola of eccentricity *e*, intersects its transverse and conjugate axis at *L* and *M* respectively, then the locus of the middle point of *LM* is a hyperbola whose eccentricity is

(a) $\frac{e}{\sqrt{e^2-1}}$ (b) $\frac{e}{\sqrt{e^4-1}}$ (c) $\frac{e}{\sqrt{a^2e^2-1}}$ (d) None of these

Solution: (a) The equation of the normal at $P(a\sec\phi, b\tan\phi)$ to the hyperbola is $ax\cos\phi+by\cot\phi=a^2+b^2=a^2e^2$

It meets the transverse and conjugate axes at *L* and *M*, then $L(ae^2\sec\phi,0)$; $M\left(0,\frac{a^2e^2\tan\phi}{b}\right)$

Let the middle point of *LM* is (α,β); then $\alpha=\frac{ae^2\sec\phi}{2} \Rightarrow \sec\phi=\frac{2\alpha}{ae^2}$ .....(i)

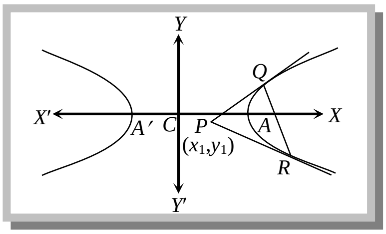
and $\beta=\frac{a^2e^2\tan\phi}{2b} \Rightarrow \tan\phi=\frac{2b\beta}{a^2e^2}$ ......(ii)

$\because\quad 1=\sec^2\phi-\tan^2\phi$; $1=\dfrac{4\alpha^2}{a^2e^4}-\dfrac{4b^2\beta^2}{a^4e^4}$, $\therefore$ Locus of (α,β) is $\dfrac{x^2}{\left(\frac{a^2e^4}{4}\right)}-\dfrac{y^2}{\left(\frac{a^4e^4}{4b^2}\right)}=1$

It is a hyperbola, let its eccentricity $e_1=\dfrac{\sqrt{\left(\frac{a^2e^4}{4}+\frac{a^4e^4}{4b^2}\right)}}{\left(\frac{a^2e^4}{4}\right)}=\sqrt{1+\dfrac{a^2}{b^2}}=\sqrt{\dfrac{a^2+b^2}{b^2}}=\sqrt{\dfrac{a^2e^2}{a^2(e^2-1)}}$; $\therefore\quad e_1=\dfrac{e}{\sqrt{e^2-1}}$.

5.3.11 Equation of Chord of Contact of Tangents drawn from a Point to a Hyperbola

Let PQ and PR be tangents to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ drawn from any external point $P(x_1,y_1)$.



Then equation of chord of contact QR is

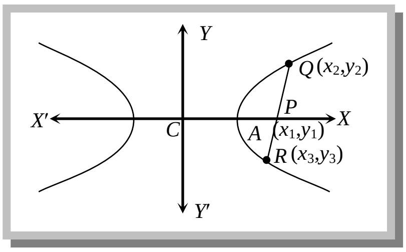
or $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$

or $T=0$ (At x_1,y_1)

5.3.12 Equation of the Chord of the Hyperbola whose Mid point (x_1, y_1) is given

Equation of the chord of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, bisected at the given point (x_1,y_1) is $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}-1=\dfrac{x_1^2}{a^2}-\dfrac{y_1^2}{b^2}-1$

i.e., $T=S_1$



Note : ❑ The length of chord cut off by hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ from the line $y=mx+c$ is

$$\frac{2ab\sqrt{[c^2-(a^2m^2-b^2)](1+m^2)}}{(b^2-a^2m^2)}$$

5.3.13 Equation of the Chord joining Two points on the Hyperbola

The equation of the chord joining the points $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$y-b\tan\phi_1=\frac{b\tan\phi_2-b\tan\phi_1}{a\sec\phi_2-a\sec\phi_1}(x-a\sec\phi_1)$$

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

Note : ❑ If the chord joining two points $(a\sec\theta_1, b\tan\theta_1)$ and $(a\sec\theta_2, b\tan\theta_2)$ passes through the focus of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, then $\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}=\dfrac{1-e}{1+e}$.

Example: 22 The equation of the chord of contact of tangents drawn from a point (2, –1) to the hyperbola $16x^2-9y^2=144$ is

(a) $32x+9y=144$ (b) $32x+9y=55$ (c) $32x+9y+144=0$ (d) $32x+9y+55=0$

Solution: (a) From $T=0$ *i.e.*, $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$. Here, $16x^2-9y^2=144$ *i.e.*, $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$

So, the equation of chord of contact of tangents drawn from a point (2, –1) to the hyperbola is $\frac{2x}{9} - \frac{(-1)y}{16} = 1$

i.e., $32x + 9y = 144$

Example: 23 The point of intersection of tangents drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the points where it is intersected by the line $lx + my + n = 0$ is

(a) $\left(\frac{-a^2l}{n}, \frac{b^2m}{n}\right)$ (b) $\left(\frac{a^2l}{n}, \frac{-b^2m}{n}\right)$ (c) $\left(-\frac{a^2n}{l}, \frac{b^2n}{m}\right)$ (d) $\left(\frac{a^2n}{l}, \frac{-b^2n}{m}\right)$

Solution: (a) Let (x_1, y_1) be the required point. Then the equation of the chord of contact of tangents drawn from (x_1, y_1) to the given hyperbola is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ ......(i)

The given line is $lx + my + n = 0$ .....(ii)

Equation (i) and (ii) represent the same line

$\therefore \frac{x_1}{a^2l} = -\frac{y_1}{b^2m} = \frac{1}{-h} \Rightarrow x_1 = \frac{-a^2l}{n}, y_1 = \frac{b^2m}{n}$; Hence the required point is $\left(-\frac{a^2l}{n}, \frac{b^2m}{n}\right)$.

Example: 24 What will be equation of that chord of hyperbola $25x^2 - 16y^2 = 400$, whose mid point is (5, 3) **[UPSEAT 1999]**

(a) $115x - 117y = 17$ (b) $125x - 48y = 481$ (c) $127x + 33y = 341$ (d) $15x + 121y = 105$

Solution: (b) According to question, $S = 25x^2 - 16y^2 - 400 = 0$

Equation of required chord is $S_1 = T$ ......(i)

Here $S_1 = 25(5)^2 - 16(3)^2 - 400 = 625 - 144 - 400 = 81$ and $T = 25xx_1 - 16yy_1 - 400$, where $x_1 = 5$, $y_1 = 3$

$\Rightarrow 25x(5) - 16y(3) - 400 = 125x - 48y - 400$

So, from (i) required chord is $125x - 48y - 400 = 81 \Rightarrow 125x - 48y = 481$.

Example: 25 The locus of the mid-points of the chords of the circle $x^2 + y^2 = 16$ which are tangent to the hyperbola $9x^2 - 16y^2 = 144$ is

(a) $(x^2 + y^2)^2 = 16x^2 - 9y^2$ (b) $(x^2 + y^2)^2 = 9x^2 - 16y^2$

(c) $(x^2 - y^2)^2 = 16x^2 - 9y^2$ (d) None of these

Solution: (a) The given hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$ ……(i)

Any tangent to (i) is $y = mx + \sqrt{16m^2 - 9}$ ……(ii)

Let (x_1, y_1) be the mid point of the chord of the circle $x^2 + y^2 = 16$

Then equation of the chord is $T = S_1$ *i.e.,* $xx_1 + yy_1 - (x_1^2 + y_1^2) = 0$ ……(iii)

Since (ii) and (iii) represent the same line.

$\therefore \frac{m}{x_1} = \frac{-1}{y_1} = \frac{\sqrt{16m^2 - 9}}{-(x_1^2 + y_1^2)}$

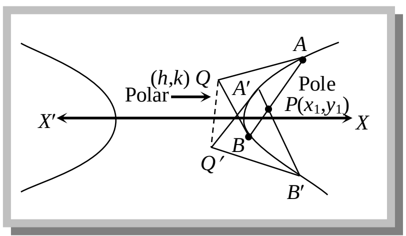
$\Rightarrow m = -\frac{x_1}{y_1}$ and $(x_1^2 + y_1^2)^2 = y_1^2(16m^2 - 9) \Rightarrow (x_1^2 + y_1^2)^2 = 16.\frac{x_1^2}{y_1^2}y_1^2 - 9y_1^2 = 16x_1^2 - 9y_1^2$

$\therefore$ Locus of (x_1, y_1) is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.

5.3.14 Pole and Polar

Let P be any point inside or outside the hyperbola. If any straight line drawn through P interesects the hyperbola at A and B. Then the locus of the point of intersection of the tangents to the hyperbola at A and B is called the polar of the given point P with respect to the hyperbola and the point P is called the pole of the polar.

The equation of the required polar with (x_1, y_1) as its pole is



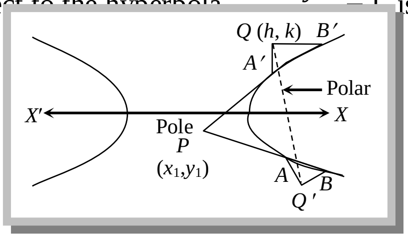
$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Note : ❑ Polar of the focus is the directrix.

❑ Any tangent is the polar of its point of contact.

(1) **Pole of a given line :** The pole of a given line $lx + my + n = 0$ with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$(x_1, y_1) = \left(-\frac{a^2 l}{n}, \frac{b^2 m}{n}\right)$$



(2) **Properties of pole and polar**

(i) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points.

(ii) If the pole of a line $lx + my + n = 0$ lies on the another line $l'x + m'y + n' = 0$ then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

(iii) Pole of a given line is same as point of intersection of tangents as its extremities.

Important Tips

☞ *If the polars of* (x_1, y_1) *and* (x_2, y_2) *with respect to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *are at right angles, then* $\frac{x_1 x_2}{y_1 y_2} + \frac{a^4}{b^4} = 0$

Example: 26 If the polar of a point *w.r.t.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the locus of the point is **[Pb. CET 1999]**

(a) Given hyperbola (b) Ellipse

(c) Circle (d) None of these

Solution: (a) Let (x_1, y_1) be the given point.

Its polar *w.r.t.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ *i.e.*, $y = \frac{b^2}{y_1}\left(1 - \frac{xx_1}{a^2}\right) = -\frac{b^2 x_1}{a^2 y_1} x + \frac{b^2}{y_1}$

This touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\left(\frac{b^2}{y_1}\right)^2 = a^2 . \left(\frac{b^2 x_1}{a^2 y_1}\right) - b^2 \Rightarrow \frac{b^4}{y_1^2} = \frac{a^2 b^4 x_1^2}{a^4 y_1^2} - b^2 \Rightarrow \frac{b^2}{y_1^2} = \frac{b^2 x_1^2}{a^2 y_1^2} - 1 \Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$\therefore$ Locus of (x_1, y_1) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Which is the same hyperbola.

Example: 27 The locus of the poles of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which subtend a right angle at the centre is

(a) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$ (c) $\frac{x^2}{a^4} - \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$ (d) $\frac{x^2}{a^4} - \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$

Solution: (a) Let (x_1, y_1) be the pole w.r.t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ......(i)

Then equation of polar is $\frac{hx}{a^2} - \frac{ky}{b^2} = 1$ .....(ii)

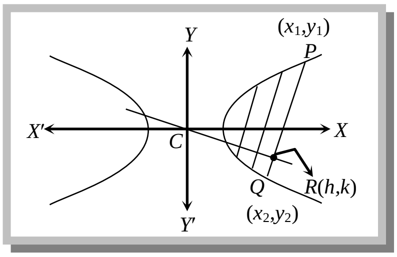
The equation of lines joining the origin to the points of intersection of (i) and (ii) is obtained by making homogeneous (i) with the help of (ii), then $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\left(\frac{hx}{a^2}-\frac{ky}{b^2}\right)^2 \Rightarrow x^2\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)-y^2\left(\frac{1}{b^2}+\frac{k^2}{b^4}\right)+\frac{2hk}{a^2b^2}xy=0$

Since the lines are perpendicular, then coefficient of x^2 + coefficient of $y^2=0$

$\frac{1}{a^2}-\frac{h^2}{a^4}-\frac{1}{b^2}-\frac{k^2}{b^4}=0$ or $\frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}-\frac{1}{b^2}$. Hence required locus is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}-\frac{1}{b^2}$

5.3.15 Diameter of the Hyperbola

The locus of the middle points of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.



Let $y=mx+c$ a system of parallel chords to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for different chords then the equation of diameter of the hyperbola is $y=\frac{b^2x}{a^2m}$, which is passing through (0, 0)

Conjugate diameter : Two diameters are said to be conjugate when each bisects all chords parallel to the others.

If $y=m_1x$, $y=m_2x$ be conjugate diameters, then $m_1m_2=\frac{b^2}{a^2}$.

Note :
- ❑ If a pair of diameters be conjugate with respect to a hyperbola, they are conjugate with respect to its conjugate hyperbola also.
- ❑ In a pair of conjugate diameters of a hyperbola. Only one meets the curve in real points.
- ❑ The condition for the lines $AX^2+2HXY+BY^2=0$ to be conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $a^2A=b^2B$.

Important Tips

☞ *If CD is the conjugate diameter of* a *diameter CP of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *where P is* $(a\sec\phi, b\tan\phi)$ *then coordinates of D is* $(a\tan\phi, b\sec\phi)$, *where C is (0, 0).*

Example: 28 If a pair of conjugate diameters meet the hyperbola and its conjugate in *P* and *D* respectively, then $CP^2-CD^2=$

(a) a^2+b^2 (b) a^2-b^2 (c) $\frac{a^2}{b^2}$ (d) None of these

Solution: (b) Coordinates of *P* and *D* are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

Then $(CP)^2-(CD)^2 = a^2\sec^2\phi+b^2\tan^2\phi-a^2\tan^2\phi-b^2\sec^2\phi$

$= a^2(\sec^2\phi-\tan^2\phi)-b^2(\sec^2\phi-\tan^2\phi) = a^2(1)-b^2(1) = a^2-b^2$.

Example: 29 If the line $lx+my+n=0$ passes through the extremities of a pair of conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then

(a) $a^2l^2-b^2m^2=0$ (b) $a^2l^2+b^2m^2=0$ (c) $a^2l^2+b^2m^2=n^2$ (d) None of these

Solution: (a) The extremities of a pair of conjugate diameters of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.

According to the question, since extremities of a pair of conjugate diameters lie on $lx+my+n=0$

$\therefore \quad l(a \sec \phi) + m(b \tan \phi) + n = 0 \Rightarrow l(a \tan \phi) + m(b \sec \phi) + n = 0$ ……(i)

Then from (i), $al \sec \phi + bm \tan \phi = -n$ or $a^2 l^2 \sec^2 \phi + b^2 m^2 \tan^2 \phi + 2ablm \sec \phi \tan \phi = n^2$ ……(ii)

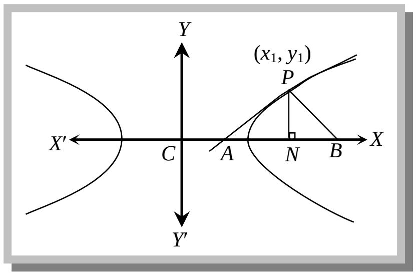
And from (ii), $al \tan \phi + bm \sec \phi = -n$ or $a^2 l^2 \tan^2 \phi + b^2 m^2 \sec^2 \phi + 2ablm \sec \phi \tan \phi = n^2$ ……(iii)

Then subtracting (ii) from (iii)

$\therefore \quad a^2 l^2 (\sec^2 \phi - \tan^2 \phi) + b^2 m^2 (\tan^2 \phi - \sec^2 \phi) = 0$ or $a^2 l^2 - b^2 m^2 = 0$.

5.3.16 Subtangent and Subnormal of the Hyperbola

Let the tangent and normal at $P(x_1, y_1)$ meet the x-axis at A and B respectively.



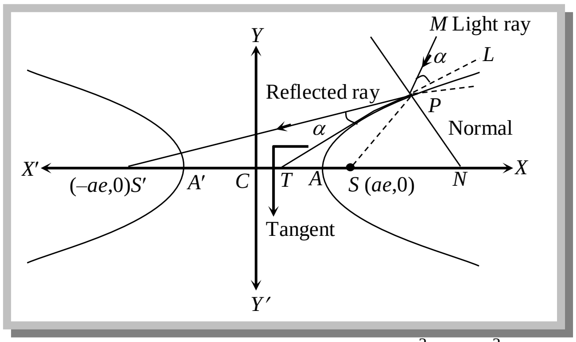
Length of subtangent $AN = CN - CA = x_1 - \dfrac{a^2}{x_1}$

Length of subnormal $BN = CB - CN = \dfrac{(a^2 + b^2)}{a^2} x_1 - x_1 = \dfrac{b^2}{a^2} x_1 = (e^2 - 1) x_1$

5.3.17 Reflection property of the Hyperbola

If an incoming light ray passing through one focus (S) strike convex side of the hyperbola then it will get reflected towards other focus (S')

$\angle TPS' = \angle LPM = \alpha$



Example: 30 A ray emanating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8; then the equation of reflected ray after first reflection is (Point P lies in first quadrant)

(a) $3\sqrt{3}x - 13y + 15\sqrt{3} = 0$ (b) $3x - 13y + 15 = 0$ (c) $3\sqrt{3}x + 13y - 15\sqrt{3} = 0$ (d) None of these

Solution: (a) Given hyperbola is $9x^2 - 26y^2 = 144$. This equation can be rewritten as $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ .....(i)

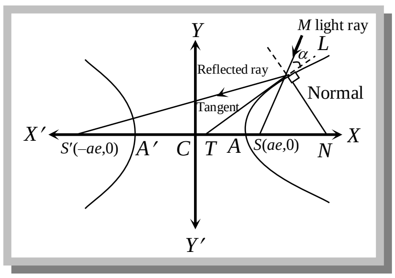
Since x coordinate of P is 8. Let y-coordinate of P is α

$\therefore \quad (8, \alpha)$ lies on (i)

$\therefore \quad \dfrac{64}{16} - \dfrac{\alpha^2}{9} = 1$; $\therefore \quad \alpha = 27$ ($\because$ P lies in first quadrant)

$\alpha = 3\sqrt{3}$

Hence coordinate of point P is $(8, 3\sqrt{3})$



$\because$ Equation of reflected ray passing through $P(8, 3\sqrt{3})$ and $S'(-5, 0)$; $\therefore$ Its equation is $y - 3\sqrt{3} = \dfrac{0 - 3\sqrt{3}}{-5 - 8}(x - 8)$

or $13y - 39\sqrt{3} = 3\sqrt{3}x - 24\sqrt{3}$ or $3\sqrt{3}\,x - 13y + 15\sqrt{3} = 0$

5.3.18 Asymptotes of a Hyperbola

An asymptote to a curve is a straight line, at a finite distance from the origin, to which the tangent to a curve tends as the point of contact goes to infinity.

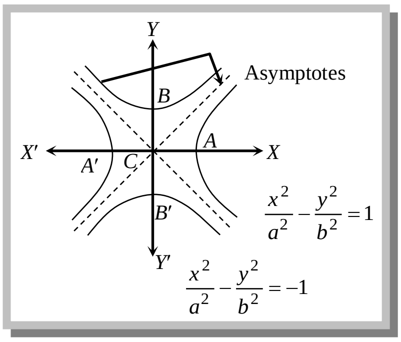
The equations of two asymptotes of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $y = \pm \dfrac{b}{a} x$ or $\dfrac{x}{a} \pm \dfrac{y}{b} = 0$.

Note : ❑ The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

❑ When $b=a$ *i.e.* the asymptotes of rectangular hyperbola $x^2-y^2=a^2$ are $y=\pm x$, which are at right angles.

❑ A hyperbola and its conjugate hyperbola have the same asymptotes.

❑ The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only *i.e.* Hyperbola – Asymptotes = Asymptotes – Conjugated hyperbola or, $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)-\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}+1\right)$.



❑ The asymptotes pass through the centre of the hyperbola.

❑ The bisectors of the angles between the asymptotes are the coordinate axes.

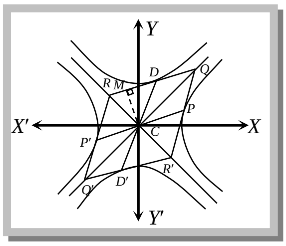
❑ The angle between the asymptotes of the hyperbola $S=0$ *i.e.*, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\frac{b}{a}$ or $2\sec^{-1}e$.

❑ Asymptotes are equally inclined to the axes of the hyperbola.

Important Tips

☞ *The parallelogram formed by the tangents at the extremities of conjugate diameters of a hyperbola has its vertices lying on the asymptotes and is of constant area.*

Area of parallelogram $QRQ'R'$ = *4(Area of parallelogram QDCP)* = $4ab$ = *Constant*



☞ *The product of length of perpendiculars drawn from any point on the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *to the asymptotes is* $\frac{a^2b^2}{a^2+b^2}$.

Example: 31 From any point on the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$. The area cut-off by the chord of contact on the asymptotes is equal to

(a) $\frac{ab}{2}$ (b) ab (c) $2ab$ (d) $4ab$

Solution: (d) Let $P(x_1,y_1)$ be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$

The chord of contact of tangent from P to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=2$ .....(i)

The equation of asymptotes are $\frac{x}{a}-\frac{y}{b}=0$ ......(ii)

And $\frac{x}{a}+\frac{y}{b}=0$ ......(iii)

The point of intersection of the asymptotes and chord are $\left(\frac{2a}{x_1/a-y_1/b},\frac{2b}{x_1/a-y_1/b}\right);\left(\frac{2a}{x_1/a+y_1/b},\frac{-2b}{x_1/a+y_1/b}\right)$, (0, 0)

$\therefore$ Area of triangle $= \frac{1}{2}|(x_1y_2 - x_2y_1)| = \frac{1}{2}\left|\left(\frac{-8ab}{x_1^2/a^2 - y_1^2/b^2}\right)\right| = 4ab$.

Example: 32 The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ **[Karnataka CET 2002]**

(a) $2x^2 + 5xy + 2y^2 = 0$

(b) $2x^2 + 5xy + 2y^2 - 4x + 5y + 2 = 0 = 0$

(c) $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$

(d) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

Solution: (d) Given, equation of hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ and equation of asymptotes

$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ ......(i) which is the equation of a pair of straight lines. We know that the standard equation of a pair of straight lines is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Comparing equation (i) with standard equation, we get $a = 2, b = 2$, $h = \frac{5}{2}, g = 2, f = \frac{5}{2}$ and $c = \lambda$.

We also know that the condition for a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

Therefore, $4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$ or $\frac{-9\lambda}{4} + \frac{9}{2} = 0$ or $\lambda = 2$

Substituting value of λ in equation (i), we get $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$.

5.3.19 Rectangular or Equilateral Hyperbola

(1) **Definition :** A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$.

The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0$, $h^2 > ab$ and coefficient of x^2 + coefficient of $y^2 = 0$

The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $y = \pm\frac{b}{a}x$.

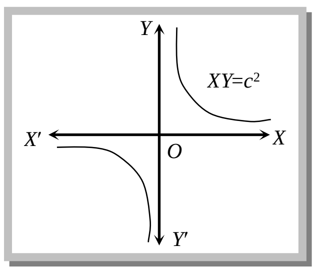
The angle between these two asymptotes is given by $\tan\theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \frac{b}{a}\left(\frac{-b}{a}\right)} = \frac{2b/a}{1 - b^2/a^2} = \frac{2ab}{a^2 - b^2}$.

If the asymptotes are at right angles, then $\theta = \pi/2 \Rightarrow \tan\theta = \tan\frac{\pi}{2} \Rightarrow \frac{2ab}{a^2 - b^2} = \tan\frac{\pi}{2} \Rightarrow a^2 - b^2 = 0$ $\Rightarrow a = b \Rightarrow 2a = 2b$. Thus the transverse and conjugate axis of a rectangular hyperbola are equal and the equation is $x^2 - y^2 = a^2$. The equations of the asymptotes of the rectangular hyperbola are $y = \pm x$ *i.e.*, $y = x$ and $y = -x$. Clearly, each of these two asymptotes is inclined at 45° to the transverse axis.

(2) **Equation of the rectangular hyperbola referred to its asymptotes as the axes of coordinates :** Referred to the transverse and conjugate axis as the axes of coordinates, the equation of the rectangular hyperbola is

$$x^2 - y^2 = a^2 \qquad .....(i)$$

The asymptotes of (i) are $y = x$ and $y = -x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis, So, if we rotate the coordinate axes through an angle of $-\pi/4$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola and $x = X\cos(-\pi/4) - Y\sin(-\pi/4) = \frac{X+Y}{\sqrt{2}}$ and $y = X\sin(-\pi/4) + Y\cos(-\pi/4) = \frac{Y-X}{\sqrt{2}}$.



Substituting the values of x and y in (i), We obtain the $\left(\frac{X+Y}{\sqrt{2}}\right)^2 - \left(\frac{Y-X}{\sqrt{2}}\right)^2 = a^2 \Rightarrow XY = \frac{a^2}{2} \Rightarrow XY = c^2$

where $c^2 = \frac{a^2}{2}$.

This is transformed equation of the rectangular hyperbola (i).

(3) **Parametric co-ordinates of a point on the hyperbola $XY = c^2$:** If t is non–zero variable, the coordinates of any point on the rectangular hyperbola $xy = c^2$ can be written as $(ct, c/t)$. The point $(ct, c/t)$ on the hyperbola $xy = c^2$ is generally referred as the point 't'.

For rectangular hyperbola the coordinates of foci are $(\pm a\sqrt{2}, 0)$ and directrices are $x = \pm a\sqrt{2}$.

For rectangular hyperbola $xy = c^2$, the coordinates of foci are $(\pm c\sqrt{2}, \pm c\sqrt{2})$ and directrices are $x + y = \pm c\sqrt{2}$.

(4) **Equation of the chord joining points t_1 and t_2 :** The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$ is $y - \frac{c}{t_1} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}(x - ct_1) \Rightarrow x + y\,t_1 t_2 = c(t_1 + t_2)$.

(5) **Equation of tangent in different forms**

(i) **Point form :** The equation of tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$

(ii) **Parametric form :** The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$ .On replacing x_1 by ct and y_1 by $\frac{c}{t}$ on the equation of the tangent at (x_1, y_1) i.e. $xy_1 + yx_1 = 2c^2$ we get $\frac{x}{t} + yt = 2c$.

Note : ❑ Point of intersection of tangents at $'t_1'$ and $'t_2'$ is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$

(6) **Equation of the normal in different forms :** (i) **Point form :** The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$. As discussed in the equation of the tangent, we have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1}{x_1}$

So, the equation of the normal at (x_1, y_1) is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1) \Rightarrow y - y_1 = \frac{x_1}{y_1}(x - x_1)$

$\Rightarrow yy_1 - y_1^2 = xx_1 - x_1^2 \quad \Rightarrow xx_1 - yy_1 = x_1^2 - y_1^2$

This is the required equation of the normal at (x_1, y_1).

(ii) **Parametric form:** The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

On replacing x_1 by ct and y_1 by c/t in the equation.

We obtain $xx_1 - yy_1 = x_1^2 - y_1^2, \quad xct - \frac{yc}{t} = c^2t^2 - \frac{c^2}{t^2} \Rightarrow xt^3 - yt - ct^4 + c = 0$

Note : ❑ The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree in t. So, in general, four normals can be drawn from a point to the hyperbola $xy = c^2$

- If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in $'t''$ then; $t' = \frac{-1}{t^3}$.
- Point of intersection of normals at $'t_1'$ and $'t_2'$ is $\left(\frac{c\{t_1t_2(t_1^2 + t_1t_2 + t_2^2) - 1\}}{t_1t_2(t_1 + t_2)}, \frac{c\{t_1^3t_2^3 + (t_1^2 + t_1t_2 + t_2^2)\}}{t_1t_2(t_1 + t_2)}\right)$

Important Tips

- *A triangle has its vertices on a rectangular hyperbola; then the orthocentre of the triangle also lies on the same hyperbola.*
- *All conics passing through the intersection of two rectangular hyperbolas are themselves rectangular hyperbolas.*
- *An infinite number of triangles can be inscribed in the rectangular hyperbola $xy = c^2$ whose all sides touch the parabola $y^2 = 4ax$.*

Example: 33 If $5x^2 + \lambda y^2 = 20$ represents a rectangular hyperbola, then λ equals

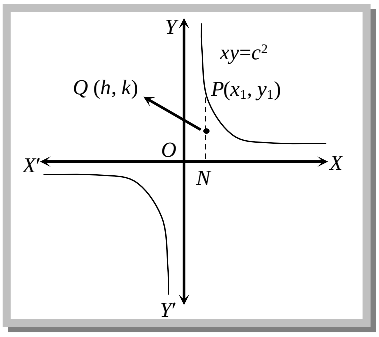
(a) 5 (b) 4 (c) – 5 (d) None of these

Solution: (c) Since the general equation of second degree represents a rectagular hyperbola if $\Delta \neq 0, h^2 > ab$ and coefficient of x^2 + coefficient of $y^2 = 0$. Therefore the given equation represents a rectangular hyperbola if $\lambda + 5 = 0$ *i.e.,* $\lambda = -5$

Example: 34 If *PN* is the perpendicular from a point on a rectangular hyperbola to its asymptotes, the locus, the mid-point of *PN* is

(a) Circle (b) Parabola (c) Ellipse (d) Hyperbola

Solution: (d) Let $xy = c^2$ be the rectangular hyperbola, and let $P(x_1, y_1)$ be a point on it. Let $Q(h, k)$ be the mid-point of *PN*. Then the coordinates of *Q* are $\left(x_1, \frac{y_1}{2}\right)$.



$\therefore$ $x_1 = h$ and $\frac{y_1}{2} = k \Rightarrow x_1 = h$ and $y_1 = 2k$

But (x_1, y_1) lies on $xy = c^2$.

$\therefore$ $h.(2k) = c^2 \Rightarrow hk \Rightarrow c^2/2$

Hence, the locus of (h, k) is $xy = c^2/2$, which is a hyperbola.

Example: 35 If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in t', then

(a) $t' = -\frac{1}{t^3}$ (b) $t' = -\frac{1}{t}$ (c) $t' = \frac{1}{t^2}$ (d) $t'^2 = -\frac{1}{t^2}$

Solution: (a) The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ is $ty = t^3x - ct^4 + c$

If it passes through $\left(ct', \frac{c}{t'}\right)$ then

$\Rightarrow \frac{tc}{t'} = t^3ct' - ct^4 + c \Rightarrow t = t^3t'^2 - t^4t' + t' \Rightarrow t - t' = t^3t'(t' - t) \Rightarrow t' = -\frac{1}{t^3}$

Example: 36 If the tangent and normal to a rectangular hyperbola cut off intercepts a_1 and a_2 on one axis and b_1 and b_2 on the other axis, then

(a) $a_1b_1 + a_2b_2 = 0$ (b) $a_1b_2 + b_2a_1 = 0$ (c) $a_1a_2 + b_1b_2 = 0$ (d) None of these

Solution: (c) Let the hyperbola be $xy = c^2$. Tangent at any point *t* is $x + yt^2 - 2ct = 0$

Putting $y = 0$ and then $x = 0$ intercepts on the axes are $a_1 = 2ct$ and $b_1 = \frac{2c}{t}$

Normal is $xt^3 - yt - ct^4 + c = 0$.

Intercepts as above are $a_2 = \frac{c(t^4 - 1)}{t^3}$, $b^2 = \frac{-c(t^4 - 1)}{t}$

$\therefore \quad a_1a_2 + b_1b_2 = 2ct \times \frac{c(t^4-1)}{t^3} + \frac{2c}{t} \times \frac{-c(t^4-1)}{t} = \frac{2c^2}{t^2}(t^4-1) - \frac{2c^2}{t^2}(t^4-1) = 0$; $\therefore \quad a_1a_2 + b_1b_2 = 0$.

Example: 37 A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. The locus of the point which divides the line segment between these two points in the ratio 1 : 2 is **[IIT 1997]**

(a) $16x^2 + 10xy + y^2 = 2$ (b) $16x^2 - 10xy + y^2 = 2$ (c) $16x^2 + 10xy + y^2 = 4$ (d) None of these

Solution: (a) Let $P(h,k)$ be any point on the locus. Equation of the line through P and having slope 4 is $y - k = 4(x - h)$ .....(i)

Suppose this meets $xy = 1$ ......(ii) in $A(x_1, y_1)$ and $B(x_2, y_2)$

Eliminating y between (i) and (ii), we get $\frac{1}{x} - k = 4(x - h)$

$\Rightarrow \quad 1 - xk = 4x^2 - 4hx \quad \Rightarrow \quad 4x^2 - (4h - k)x - 1 = 0$ ......(iii)

This has two roots say x_1, x_2; $x_1 + x_2 = \frac{4h-k}{4}$ ......(iv) and $x_1x_2 = -\frac{1}{4}$ ......(v)

Also, $\frac{2x_1 + x_2}{3} = h$ [$\because$ P divides AB in the ratio 1 : 2]

i.e., $2x_1 + x_2 = 3h$ ......(vi)

(vi) – (iv) gives, $x_1 = 3h - \frac{4h-k}{4} = \frac{8h+k}{4}$ and $x_2 = 3h - 2.\frac{8h+k}{4} = -\frac{2h+k}{2}$

Putting in (v), we get $\frac{8h+k}{4}\left(-\frac{2h+k}{2}\right) = -\frac{1}{4}$

$\Rightarrow \quad (8h+k)(2h+k) = 2 \quad \Rightarrow \quad 16h^2 + 10hk + k^2 = 2$

$\therefore$ Required locus of $P(h,k)$ is $16x^2 + 10xy + y^2 = 2$.

Example: 38 PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then the product of the slopes of CP, CQ, CR and CS is equal to

(a) – 1 (b) 1 (c) 0 (d) None of these

Solution: (b) Let t_1, t_2, t_3, t_4 be the parameters of the points P, Q, R and S respectively. Then, the coordinates of P, Q, R and S are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$, $\left(ct_3, \frac{c}{t_3}\right)$ and $\left(ct_4, \frac{c}{t_4}\right)$ respectively.

Now, $PQ \perp RS \Rightarrow \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = -1 \Rightarrow -\frac{1}{t_1t_2} \times -\frac{1}{t_3t_4} = -1 \Rightarrow t_1t_2t_3t_4 = -1$ .....(i)

$\therefore$ Product of the slopes of CP, CQ, CR and CS

$\frac{1}{t_1^2} \times \frac{1}{t_2^2} \times \frac{1}{t_3^2} \times \frac{1}{t_4^2} = \frac{1}{t_1^2t_2^2t_3^2t_4^2} = 1$ [Using (i)]

5.3.20 Intersection of a Circle and a Rectangular Hyperbola

If a circle $x^2 + y^2 + 2gx + 2fy + k = 0$ cuts a rectangular hyperbola $xy = c^2$ in A, B, C and D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively; then

(1) (i) $\Sigma t_1 = -\frac{2g}{c}$ (ii) $\Sigma t_1t_2 = \frac{k}{c^2}$

(iii) $\Sigma t_1t_2t_3 = \frac{-2f}{c}$ (iv) $t_1t_2t_3t_4 = 1$ (v) $\Sigma \frac{1}{t_1} = -\frac{2f}{c}$

(2) Orthocentre of ΔABC is $H\left(-ct_4, \frac{-c}{t_4}\right)$ but D is $\left(ct_4, \frac{c}{t_4}\right)$

Hence H and D are the extremities of a diagonal of rectangular hyperbola.

(3) Centre of mean position of four points is $\left\{\frac{c}{4}\Sigma t_1, \frac{c}{4}\Sigma\left(\frac{1}{t_1}\right)\right\}$ *i.e.,* $\left(-\frac{g}{2}, -\frac{f}{2}\right)$

$\because$ Centres of the circles and rectangular hyperbola are $(-g, -f)$ and (0, 0); mid point of centres of circle and hyperbola is $\left(-\frac{g}{2}, -\frac{f}{2}\right)$. Hence the centre of the mean position of the four points bisects the distance between the centres of the two curves (circle and rectangular hyperbola)

(4) If the circle passing through ABC meet the hyperbola in fourth points D; then centre of circle is $(-g, -f)$

i.e., $\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}\right); \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 t_2 t_3\right)\right\}$

Example: 39 If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively. Then **[Kurukshetra CEE 1998]**

(a) $t_1 t_2 = t_3 t_4$ (b) $t_1 t_2 t_3 t_4 = 1$ (c) $t_1 = t_2$ (d) $t_3 = t_4$

Solution: (b) Let the equation of circle be $x^2 + y^2 = a^2$ ......(i)

Parametric equation of rectangular hyperbola is $x = c\,t, y = \frac{c}{t}$

Put the values of x and y in equation (i) we get $c^2t^2 + \frac{c^2}{t^2} = a^2 \Rightarrow c^2t^4 - a^2t^2 + c^2 = 0$

Hence product of roots $t_1 t_2 t_3 t_4 = \frac{c^2}{c^2} = 1$

Example: 40 If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$, $S(x_4, y_4)$ then **[IIT 1998]**

(a) $x_1 + x_2 + x_3 + x_4 = 0$ (b) $y_1 + y_2 + y_3 + y_4 = 0$ (c) $x_1 x_2 x_3 x_4 = c^4$ (d) $y_1 y_2 y_3 y_4 = c^4$

Solution: (a,b,c,d) Given, circle is $x^2 + y^2 = a^2$ ....(i) and hyperbola be $xy = c^2$ .....(ii)

from (ii) $y = \frac{c^2}{x}$. Putting in (i), we get $x^2 + \frac{c^4}{x^2} = a^2 \Rightarrow x^4 - a^2x^2 + c^4 = 0$

$\therefore$ $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 x_2 x_3 x_4 = c^4$

Since both the curves are symmetric in x and y, $\therefore$ $y_1 + y_2 + y_3 + y_4 = 0$; $y_1 y_2 y_3 y_4 = c^4$.

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Assignment

Definition, Standard form of hyperbola, Conjugate hyperbola

Basic Level

1. The locus of the centre of a circle, which touches externally the given two circle, is**[Karnataka CET 1999; Kurukshetra CE**

(a) Circle (b) Parabola (c) Hyperbola (d) Ellipse

2. The locus of a point which moves such that the difference of its distances from two fixed points is always a constant is

[UPSEAT 1995; Kerala (Engg.) 1998; Karnataka CET 2003]

(a) A straight line (b) A circle (c) An ellipse (d) A hyperbola

3. The one which does not represent a hyperbola is **[MP PET 1992]**

(a) $xy = 1$ (b) $x^2 - y^2 = 5$ (c) $(x-1)(y-3) = 3$ (d) $x^2 - y^2 = 0$

4. The equation of the hyperbola whose directrix is $x + 2y = 1$, focus (2, 1) and eccentricity 2 will be **[MP PET 1988, 1989]**

(a) $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$ (b) $3x^2 + 16xy + 15y^2 - 4x - 14y - 1 = 0$

(c) $x^2 + 16xy + 11y^2 - 12x - 6y + 21 = 0$ (d) None of these

5. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different value of k is

(a) Circle (b) Parabola (c) Hyperbola (d) Ellipse

6. Locus of the point of intersection of straight line $\frac{x}{a} - \frac{y}{b} = m$ and $\frac{x}{a} + \frac{y}{b} = \frac{1}{m}$ is **[MP PET 1991, 2003]**

(a) An ellipse (b) A circle (c) A hyperbola (d) A parabola

7. The eccentricity of the hyperbola $2x^2 - y^2 = 6$ is **[MP PET 1992]**

(a) $\sqrt{2}$ (b) 2 (c) 3 (d) $\sqrt{3}$

8. Centre of hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ is

(a) (1, –1) (b) (–1, 1) (c) (–1, –1) (d) (1, 1)

9. The eccentricity of the conic $x^2 - 4y^2 = 1$, is **[MP PET 1999; Kurukshetra CEE 1998]**

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{2}$

10. The eccentricity of a hyperbola passing through the point (3, 0), $(3\sqrt{2}, 2)$ will be **[MNR 1985]**

(a) $\sqrt{13}$ (b) $\frac{\sqrt{13}}{3}$ (c) $\frac{\sqrt{13}}{4}$ (d) $\frac{\sqrt{13}}{2}$

11. If (4, 0) and (–4, 0)be the vertices and (6, 0) and (–6, 0) be the foci of a hyperbola, then its eccentricity is

(a) 5/2 (b) 2 (c) 3/2 (d) $\sqrt{2}$

12. If e and e' are eccentricities of hyperbola and its conjugate respectively, then

[UPSEAT 1999; EAMCET 1994, 95; MNR 1984; MP PET 1995; DCE 2000]

(a) $\left(\frac{1}{e}\right)^2+\left(\frac{1}{e'}\right)^2=1$ (b) $\frac{1}{e}+\frac{1}{e'}=1$ (c) $\left(\frac{1}{e}\right)^2+\left(\frac{1}{e'}\right)^2=0$ (d) $\frac{1}{e}+\frac{1}{e'}=2$

13. If e and e' are the eccentricities of the ellipse $5x^2+9y^2=45$ and the hyperbola $5x^2-4y^2=45$ respectively, then $ee'=$ **[EAMCET 2002]**

(a) 9 (b) 4 (c) 5 (d) 1

14. The directrix of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{4}=1$ **[UPSEAT 2003]**

(a) $x=9/\sqrt{13}$ (b) $y=9/\sqrt{13}$ (c) $x=6/\sqrt{13}$ (d) $y=6/\sqrt{13}$

15. The latus rectum of the hyperbola $16x^2-9y^2=144$, is **[MP PET 2000]**

(a) $\frac{16}{3}$ (b) $\frac{32}{3}$ (c) $\frac{8}{3}$ (d) $\frac{4}{3}$

16. The foci of the hyperbola $2x^2-3y^2=5$, is **[MP PET 2000]**

(a) $\left(\pm\frac{5}{\sqrt{6}},0\right)$ (b) $\left(\pm\frac{5}{6},0\right)$ (c) $\left(\pm\frac{\sqrt{5}}{6},0\right)$ (d) None of these

17. The distance between the directrices of a rectangular hyperbola is 10 units, then distance between its foci is **[MP PET**

(a) $10\sqrt{2}$ (b) 5 (c) $5\sqrt{2}$ (d) 20

18. The difference of the focal distances of any point on the hyperbola $9x^2-16y^2=144$, is **[MP PET 1995]**

(a) 8 (b) 7 (c) 6 (d) 4

19. If the length of the transverse and conjugate axes of a hyperbola be 8 and 6 respectively, then the difference of focal distances of any point of the hyperbola will be

(a) 8 (b) 6 (c) 14 (d) 2

20. The length of transverse axis of the hyperbola $3x^2-4y^2=32$ is **[Karnataka CET 2001]**

(a) $\frac{8\sqrt{2}}{\sqrt{3}}$ (b) $\frac{16\sqrt{2}}{\sqrt{3}}$ (c) $\frac{3}{32}$ (d) $\frac{64}{3}$

21. A hyperbola passes through the points (3, 2) and (–17, 12) and has its centre at origin and transverse axis is along x-axis. The length of its transverse axis is

(a) 2 (b) 4 (c) 6 (d) None of these

22. The equation of the hyperbola whose foci are the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the eccentricity is 2, is

(a) $\frac{x^2}{4}+\frac{y^2}{12}=1$ (b) $\frac{x^2}{4}-\frac{y^2}{12}=1$ (c) $\frac{x^2}{12}+\frac{y^2}{4}=1$ (d) $\frac{x^2}{12}-\frac{y^2}{4}=1$

23. The distance between the foci of a hyperbola is double the distance between its vertices and the length of its conjugate axis is 6. The equation of the hyperbola referred to its axes as axes of coordinates is

(a) $3x^2-y^2=3$ (b) $x^2-3y^2=3$ (c) $3x^2-y^2=9$ (d) $x^2-3y^2=9$

24. If $(0,\pm4)$ and $(0,\pm2)$ be the foci and vertices of a hyperbola then its equation is

(a) $\frac{x^2}{4}-\frac{y^2}{12}=1$ (b) $\frac{x^2}{12}-\frac{y^2}{4}=1$ (c) $\frac{y^2}{4}-\frac{x^2}{12}=1$ (d) $\frac{y^2}{12}-\frac{x^2}{4}=1$

25. The length of the transverse axis of a hyperbola is 7 and it passes through the point (5, –2), the equation of the hyperbola is

(a) $\frac{4}{49}x^2-\frac{196}{51}y^2=1$ (b) $\frac{49}{4}x^2-\frac{51}{196}y^2=1$ (c) $\frac{4}{49}x^2-\frac{51}{196}y^2=1$ (d) None of these

26. If the centre, vertex and focus of a hyperbola be (0, 0),(4, 0) and (6, 0) respectively, then the equation of the hyperbola is

(a) $4x^2 - 5y^2 = 8$ (b) $4x^2 - 5y^2 = 80$ (c) $5x^2 - 4y^2 = 80$ (d) $5x^2 - 4y^2 = 8$

27. The equation of a hyperbola, whose foci are (5, 0) and (–5, 0) and the length of whose conjugate axis is 8, is

(a) $9x^2 - 16y^2 = 144$ (b) $16x^2 - 9y^2 = 144$ (c) $9x^2 - 16y^2 = 12$ (d) $16x^2 - 9y^2 = 12$

28. If the latus rectum of an hyperbola be 8 and eccentricity be $3/\sqrt{5}$, then the equation of the hyperbola is

(a) $4x^2 - 5y^2 = 100$ (b) $5x^2 - 4y^2 = 100$ (c) $4x^2 + 5y^2 = 100$ (d) $5x^2 + 4y^2 = 100$

29. The equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13, is

(a) $25x^2 - 144y^2 = 900$ (b) $144x^2 - 25y^2 = 900$ (c) $144x^2 + 25y^2 = 900$ (d) $25x^2 + 144y^2 = 900$

30. For hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ which of the following remains constant with change in 'α' **[IIT Screening 2003]**

(a) Abscissae of vertices (b) Abscissae of foci (c) Eccentricity (d) Directrix

31. The hyperbola is the conic with eccentricity **[BIT Ranchi 1998, UPSEAT 1998]**

(a) $e > 1$ (b) $e < 1$ (c) $e = 1$ (d) $e = 0$

32. The eccentricity of the conic $9x^2 - 16y^2 = 144$ is **[DCE 1994]**

(a) $\frac{4}{5}$ (b) $\frac{5}{4}$ (c) $\frac{4}{3}$ (d) $\sqrt{7}$

33. If e, e' be the eccentricities of two conics S and S' and if $e^2 + e'^2 = 3$, then both S and S' can be **[Kerala (Engg.) 2001]**

(a) Ellipses (b) Parabolas (c) Hyperbolas (d) None of these

34. If e_1, e_2 be respectively the eccentricities of ellipse $9x^2 + 4y^2 = 36$ and hyperbola $9x^2 - 4y^2 = 36$, then

(a) $e_1^2 + e_2^2 > 3$ (b) $e_1^2 + e_2^2 = 2$ (c) $e_1^2 + e_2^2 > 4$ (d) $e_1^2 + e_2^2 < 4$

35. The length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is

(a) $\frac{2a^2}{b}$ (b) $\frac{2b^2}{a}$ (c) $\frac{b^2}{a}$ (d) $\frac{a^2}{b}$

36. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then the equation of hyperbola is **[DCE 1998; MNR 1984; UPSEAT 2000]**

(a) $x^2 + y^2 = 32$ (b) $x^2 - y^2 = 16$ (c) $x^2 + y^2 = 16$ (d) $x^2 - y^2 = 32$

37. The equation of the hyperbola with vertices (3, 0) and (–3, 0) and semi-latus-rectum 4, is given by

(a) $4x^2 - 3y^2 + 36 = 0$ (b) $4x^2 - 3y^2 + 12 = 0$ (c) $4x^2 - 3y^2 - 36 = 0$ (d) None of these

38. Equation of the hyperbola with eccentricity 3/2 and foci at $(\pm 2, 0)$ is

(a) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$ (b) $\frac{x^2}{9} - \frac{y^2}{9} = \frac{4}{9}$ (c) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (d) None of these

39. The eccentricity of the hyperbola with latus rectum 12 and semi-conjugate axis $2\sqrt{3}$, is

(a) 2 (b) 3 (c) $\frac{\sqrt{3}}{2}$ (d) $2\sqrt{3}$

40. The eccentricity of the hyperbola $3x^2 - 4y^2 = -12$ is

(a) $\sqrt{\frac{7}{3}}$ (b) $\frac{\sqrt{7}}{2}$ (c) $-\sqrt{\frac{7}{3}}$ (d) $-\frac{\sqrt{7}}{2}$

41. The equation $\frac{x^2}{12-k}+\frac{y^2}{8-k}=1$ represents

(a) A hyperbola if $k<8$ (b) An ellipse if $k>8$

(c) A hyperbola if $8 < k < 12$ (d) None of these

Parametric equations of Hyperbola, Special form of Hyperbola

Basic Level

42. The auxiliary equation of circle of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is

(a) $x^2+y^2=a^2$ (b) $x^2+y^2=b^2$ (c) $x^2+y^2=a^2+b^2$ (d) $x^2+y^2=a^2-b^2$

43. A point on the curve $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ is **[Karnataka CET 1993; MP PET 1988]**

(a) $(A\cos\theta, B\sin\theta)$ (b) $(A\sec\theta, B\tan\theta)$ (c) $(A\cos^2\theta, B\sin^2\theta)$ (d) None of these

44. The locus of the point of intersection of the lines $ax\sec\theta+by\tan\theta=a$ and $ax\tan\theta+by\sec\theta=b$, where θ is the parameter, is

(a) A straight line (b) A circle (c) An ellipse (d) A hyperbola

45. The eccentricity of the conic represented by $x^2-y^2-4x+4y+16=0$ is

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 1/2

46. The latus rectum of the hyperbola $9x^2-16y^2-18x-32y-151=0$ is **[MP PET 1996]**

(a) $\frac{9}{4}$ (b) 9 (c) $\frac{3}{2}$ (d) $\frac{9}{2}$

47. The vertices of a hyperbola are at $(0,0)$ and $(10,0)$ and one of its foci is at $(18,0)$. The equation of the hyperbola is

(a) $\frac{x^2}{25}-\frac{y^2}{144}=1$ (b) $\frac{(x-5)^2}{25}-\frac{y^2}{144}=1$ (c) $\frac{x^2}{25}-\frac{(y-5)^2}{144}=1$ (d) $\frac{(x-5)^2}{25}-\frac{(y-5)^2}{144}=1$

48. The equations of the transverse and conjugate axis of the hyperbola $16x^2-y^2+64x+4y+44=0$ are

(a) $x=2, y+2=0$ (b) $x=2, y=2$ (c) $y=2, x+2=0$ (d) None of these

49. Foci of the hyperbola $\frac{x^2}{16}-\frac{(y-2)^2}{9}=1$ are

(a) $(5,2),(-5,2)$ (b) $(5,2),(5,-2)$ (c) $(5,2),(-5-2)$ (d) None of these

50. The eccentricity of the conic $x^2-2x-4y^2=0$ is

(a) $\frac{1}{4}$ (b) $\frac{3}{2}$ (c) $\frac{\sqrt{5}}{2}$ (d) $\frac{\sqrt{5}}{4}$

51. The equation $16x^2-3y^2-32x+12y-44=0$ represents a hyperbola

(a) The length of whose transverse axis is $4\sqrt{3}$ (b) The length of whose conjugate axis is 4

(c) Whose centre is (–1, 2) (d) Whose eccentricity is $\sqrt{\frac{19}{3}}$

52. The equation of the hyperbola whose foci are $(6,5),(-4,5)$ and eccentricity $\frac{5}{4}$ is

(a) $\frac{(x-1)^2}{16}-\frac{(y-5)^2}{9}=1$ (b) $\frac{x^2}{16}-\frac{y^2}{9}=1$ (c) $\frac{(x-1)^2}{16}-\frac{(y-5)^2}{9}=-1$ (d) None of these

53. The equation $x=\frac{e^t+e^{-t}}{2}; y=\frac{e^t-e^{-t}}{2}; t\in R$ represents **[Kerala (Engg.) 2001]**

(a) An ellipse (b) A parabola (c) A hyperbola (d) A circle

54. The vertices of the hyperbola $9x^2-16y^2-36x+96y-252=0$ are

(a) (6, 3) and (–6, 3) (b) (6, 3) and (–2, 3) (c) (–6, 3) and (–6, –3) (d) None of these

55. The curve represented by $x=a(\cos h\theta+\sin h\theta), y=b(\cos h\theta-\sin h\theta)$ is **[EAMCET 1994]**

(a) A hyperbola (b) An ellipse (c) A parabola (d) A circle

56. The foci of the hyperbola $9x^2-16y^2+18x+32y-151=0$ are

(a) (2, 3), (5, 7) (b) (4, 1), (–6, 1) (c) (0, 0), (5, 3) (d) None of these

Advance Level

57. The equations of the transverse and conjugate axes of a hyperbola respectively are $x+2y-3=0$, $2x-y+4=0$ and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. The equation of the hyperbola is

(a) $\frac{2}{5}(x+2y-3)^2-\frac{3}{5}(2x-y+4)^2=1$ (b) $\frac{2}{5}(2x-y+4)^2-\frac{3}{5}(x+2y-3)^2=1$

(c) $2(2x-y+4)^2-3(x+2y-3)^2=1$ (d) $2(x+2y-3)^2-3(2x-y+4)^2=1$

58. The points of intersection of the curves whose parametric equations are $x=t^2+1, y=2t$ and $x=2s, y=2/s$ is given by

(a) (1, –3) (b) (2, 2) (c) (–2, 4) (d) (1, 2)

59. Equation $\frac{1}{r}=\frac{1}{8}+\frac{3}{8}\cos\theta$ represents **[EAMCET 2002]**

(a) A rectangular hyperbola (b) A hyperbola (c) An ellipse (d)

Position of a Point, Intersection of a line and Hyperbola, Tangents, Director circle, Pair of

Basic Level

60. The line $y=mx+c$ touches the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if **[Kerala (Engg.) 2002]**

(a) $c^2=a^2m^2+b^2$ (b) $c^2=a^2m^2-b^2$ (c) $c^2=b^2m^2-a^2$ (d) $a^2=b^2m^2+c^2$

61. The line $lx+my+n=0$ will be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if **[MP PET 2001]**

(a) $a^2l^2+b^2m^2=n^2$ (b) $a^2l^2-b^2m^2=n^2$ (c) $a^2m^2-b^2n^2=a^2l^2$ (d) None of these

62. If the straight line $x\cos\alpha+y\sin\alpha=p$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then **[Karnataka CET 1999]**

(a) $a^2\cos^2\alpha+b^2\sin^2\alpha=p^2$ (b) $a^2\cos^2\alpha-b^2\sin^2\alpha=p^2$

(c) $a^2\sin^2\alpha+b^2\cos^2\alpha=p^2$ (d) $a^2\sin^2\alpha-b^2\cos^2\alpha=p^2$

63. The equation of the tangent at the point $(a\sec\theta, b\tan\theta)$ of the conic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is

(a) $x\sec^2\theta - y\tan^2\theta = 1$ (b) $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

(c) $\frac{x+a\sec\theta}{a^2}-\frac{y+b\tan\theta}{b^2}=1$ (d) None of these

64. If the line $y=2x+\lambda$ be a tangent to the hyperbola $36x^2-25y^2=3600$, then $\lambda=$

(a) 16 (b) –16 (c) ±16 (d) None of these

65. The equation of the tangent to the hyperbola $4y^2=x^2-1$ at the point (1, 0) is **[Karnataka CET 1994]**

(a) $x=1$ (b) $y=1$ (c) $y=4$ (d) $x=4$

66. The straight line $x+y=\sqrt{2}p$ will touch the hyperbola $4x^2-9y^2=36$, is **[Orissa JEE 2003]**

(a) $p^2=2$ (b) $p^2=5$ (c) $5p^2=2$ (d) $2p^2=5$

67. The equation of the tangent to the hyperbola $2x^2-3y^2=6$ which is parallel to the line $y=3x+4$, is **[UPSEAT 1993, 99, 2**

(a) $y=3x+5$ (b) $y=3x-5$ (c) $y=3x+5$ and $y=3x-5$ (d) None of these

68. The equation of tangents to the hyperbola $3x^2-4y^2=12$ which cuts equal intercepts from the axes, are

(a) $y+x=\pm1$ (b) $y-x=\pm1$ (c) $3x+4y=\pm1$ (d) $3x-4y=\pm1$

69. The line $3x-4y=5$ is a tangent to the hyperbola $x^2-4y^2=5$. The point of contact is

(a) (3, 1) (b) (2, 1/4) (c) (1, 3) (d) None of these

70. The equation of a common tangent to the conics $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, is

(a) $x+y=a^2-b^2$ (b) $x+y=\sqrt{a^2-b^2}$ (c) $x-y=\sqrt{a^2-b^2}$ (d) $x+y=\sqrt{b^2-a^2}$

71. The equation of common tangents to the parabola $y^2=8x$ and hyperbola $3x^2-y^2=3$, is

(a) $2x\pm y+1=0$ (b) $2x\pm y-1=0$ (c) $x\pm 2y+1=0$ (d) $x\pm 2y-1=0$

72. The radius of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is **[MP PET 1999]**

(a) $a-b$ (b) $\sqrt{a-b}$ (c) $\sqrt{a^2-b^2}$ (d) $\sqrt{a^2+b^2}$

73. The tangents to the hyperbola $x^2-y^2=3$ are parallel to the straight line $2x+y+8=0$ at the following points. **[Roorkee**

(a) (2, 1) or (1, 2) (b) (2, –1) or (–2, 1) (c) (–1, –2) (d) (–2, –1)

74. The line $y=4x+c$ touches the hyperbola $x^2-y^2=1$ iff **[Kurukshetra CEE 2001]**

(a) $c=0$ (b) $c=\pm\sqrt{2}$ (c) $c=\pm\sqrt{15}$ (d) $c=\pm\sqrt{17}$

75. The line $5x+12y=9$ touches the hyperbola $x^2-9y^2=9$ at the point

(a) $\left(-5,\frac{4}{3}\right)$ (b) $\left(5,-\frac{4}{3}\right)$ (c) $\left(3,-\frac{1}{2}\right)$ (d) None of these

76. The number of tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from an external point is

(a) 2 (b) 4 (c) 6 (d) 5

77. The slope of the tangent to the hyperbola $2x^2-3y^2=6$ at (3, 2) is **[SCRA 1999]**

(a) –1 (b) 1 (c) 0 (d) 2

78. A common tangent to $9x^2-16y^2=144$ and $x^2+y^2=9$ is

(a) $y=\frac{3}{\sqrt{7}}x+\frac{\pi}{\sqrt{7}}$ (b) $y=3\sqrt{\frac{2}{7}}x+\frac{15}{\sqrt{7}}$ (c) $y=2\sqrt{\frac{3}{7}}x+15\sqrt{7}$ (d) None of these

79. The product of the perpendiculars from two foci on any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(a) a^2 (b) $-a^2$ (c) b^2 (d) $-b^2$

80. If the two intersecting lines intersect the hyperbola and neither of them is a tangent to it, then number of intersecting points are

[IIIT Allahabad 2001]

(a) 1 (b) 2 (c) 2, 3 or 4 (d) 2 or 3

81. The equation of a tangent parallel to $y=x$ drawn to $\frac{x^2}{3}-\frac{y^2}{2}=1$ is

(a) $x-y+1=0$ (b) $x+y+2=0$ (c) $x+y-1=0$ (d) $x-y+2=0$

82. The equation of the tangent to the conic $x^2-y^2-8x+2y+11=0$ at (2, 1) is **[Karnataka CET 1993]**

(a) $x+2=0$ (b) $2x+1=0$ (c) $x-2=0$ (d) $x+y+1=0$

83. The equation of tangents to the hyperbola $x^2-4y^2=36$ which are perpendicular to the line $x-y+4=0$

(a) $y=-x+3\sqrt{3}$ (b) $y=-x-3\sqrt{3}$ (c) $y=-x\pm 2$ (d) None of these

84. The position of point (5, – 4) relative to the hyperbola $9x^2-y^2=1$

(a) Outside the hyperbola (b) Inside the hyperbola (c) On the conjugate axis(d)

Advance Level

85. If the two tangents drawn on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in such a way that the product of their gradients is c^2, then they intersects on the curve

(a) $y^2+b^2=c^2(x^2-a^2)$ (b) $y^2+b^2=c^2(x^2+a^2)$ (c) $ax^2+by^2=c^2$ (d) None of these

86. C the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. The tangent at any point P on this hyperbola meets the straight lines $bx-ay=0$ and $bx+ay=0$ in the points Q and R respectively. Then $CQ.\,CR=$

(a) a^2+b^2 (b) a^2-b^2 (c) $\frac{1}{a^2}+\frac{1}{b^2}$ (d) $\frac{1}{a^2}-\frac{1}{b^2}$

87. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h,k) is the point of intersection of the normals at P and Q, then k is equal to **[IIT 1999; MP PET 2002]**

(a) $\frac{a^2+b^2}{a}$ (b) $-\left(\frac{a^2+b^2}{a}\right)$ (c) $\frac{a^2+b^2}{b}$ (d) $-\left(\frac{a^2+b^2}{b}\right)$

88. Let P be a point on the hyperbola $x^2-y^2=a^2$ where a is a parameter such that P is nearest to the line $y=2x$. The locus of P is

(a) $x-2y=0$ (b) $2y-x=0$ (c) $x+2y=0$ (d) $2y+x=0$

89. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2},1\right)$. Its one directrix is the common tangent nearer to the point P, to the circle $x^2+y^2=1$ and the hyperbola $x^2-y^2=1$. The equation of the ellipse in the standard form, is **[IIT 1996]**

(a) $\frac{(x-1/3)^2}{1/9}+\frac{(y-1)^2}{1/12}=1$ (b) $\frac{(x-1/3)^2}{1/9}+\frac{(y+1)^2}{1/12}=1$

(c) $\frac{(x-1/3)^2}{1/9}-\frac{(y-1)^2}{1/12}=1$ (d) $\frac{(x-1/3)^2}{1/9}-\frac{(y+1)^2}{1/12}=1$

Normals, Co-normal points

Basic Level

90. The condition that the straight line $lx+my=n$ may be a normal to the hyperbola $b^2x^2-a^2y^2=a^2b^2$ is given by **[MP PET**

(a) $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$ (b) $\frac{l^2}{a^2}-\frac{m^2}{b^2}=\frac{(a^2+b^2)^2}{n^2}$ (c) $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$ (d) $\frac{l^2}{a^2}+\frac{m^2}{b^2}=\frac{(a^2-b^2)^2}{n^2}$

91. The equation of the normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at (–4, 0) is **[UPSEAT 2002]**

(a) $y=0$ (b) $y=x$ (c) $x=0$ (d) $x=-y$

92. The equation of the normal at the point $(a\sec\theta, b\tan\theta)$ of the curve $b^2x^2-a^2y^2=a^2b^2$ is **[Karnataka CET 1999]**

(a) $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2+b^2$ (b) $\frac{ax}{\tan\theta}+\frac{by}{\sec\theta}=a^2+b^2$ (c) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$ (d) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2-b^2$

93. The number of normals to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from an external point is **[EAMCET 1995]**

(a) 2 (b) 4 (c) 6 (d) 5

Chord of Contact, Equation of the Chord whose Mid point is given and Equation of Chord joining two points

Basic Level

94. The locus of the middle points of the chords of hyperbola $3x^2-2y^2+4x-6y=0$ parallel to $y=2x$ is **[EAMCET 1989]**

(a) $3x-4y=4$ (b) $3y-4x+4=0$ (c) $4x-4y=3$ (d) $3x-4y=2$

95. The equation of the chord of the hyperbola $x^2-y^2=9$ which is bisected at $(5,-3)$ is

(a) $5x+3y=9$ (b) $5x-3y=16$ (c) $5x+3y=16$ (d) $5x-3y=9$

96. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are at right angles, then $\frac{x_1x_2}{y_1y_2}$ is equal to

(a) $-\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$ (c) $-\frac{b^4}{a^4}$ (d) $-\frac{a^4}{b^4}$

97. Equation of the chord of the hyperbola $25x^2-16y^2=400$ which is bisected at the point (6, 2) is

(a) $16x-75y=418$ (b) $75x-16y=418$ (c) $25x-4y=400$ (d) None of these

Advance Level

98. If $x=9$ is the chord of contact of the hyperbola $x^2-y^2=9$, then the equation of the corresponding pair of tangent is **[IIT 1999]**

(a) $9x^2-8y^2+18x-9=0$ (b) $9x^2-8y^2-18x+9=0$ (c) $9x^2-8y^2-18x-9=0$ (d) $9x^2-8y^2+18x+9=0$

99. If $(a\sec\theta, b\tan\theta)$ and $(a\sec\phi, b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equals to

(a) $\frac{e-1}{e+1}$ (b) $\frac{1-e}{1+e}$ (c) $\frac{1+e}{1-e}$ (d) $\frac{e+1}{e-1}$

100. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ and $x^2-y^2=c^2$ cut at right angles, then

(a) $a^2+b^2=2c^2$ (b) $b^2-a^2=2c^2$ (c) $a^2-b^2=2c^2$ (d) $a^2b^2=2c^2$

101. The locus of the middle points of the chords of contact of tangents to the hyperbola $x^2-y^2=a^2$ from points on the auxiliary circle, is

(a) $a^2(x^2+y^2)=(x^2-y^2)$ (b) $a^2(x^2+y^2)=(x^2-y^2)^2$ (c) $a^2(x^2+y^2)=(x-y)^2$ (d) None of these

102. The locus of the mid points of the chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which subtend a right angle at the origin

(a) $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{x^2}{a^4}+\frac{y^2}{b^4}$ (b) $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

(c) $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ (d) None of these

Pole and Polar, Diameter and Conjugate diameter

Basic Level

103. The diameter of $16x^2-9y^2=144$ which is conjugate to $x=2y$ is

(a) $y=\frac{16}{9}x$ (b) $y=\frac{32}{9}x$ (c) $x=\frac{16}{9}y$ (d) $x=\frac{32}{9}y$

104. The lines $2x+3y+4=0$ and $3x-2y+5=0$ may be conjugate *w.r.t* the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if

(a) $a^2+b^2=\frac{10}{3}$ (b) $a^2-b^2=\frac{10}{3}$ (c) $b^2-a^2=\frac{10}{3}$ (d) None of these

105. The polars of (x_1, y_1) and (x_2, y_2) *w.r.t* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are perpendicular to each other if **[AMU 1998]**

(a) $\frac{x_1x_2}{y_1y_2}=-\frac{b^2}{a^4}$ (b) $\frac{x_1x_2}{y_1y_2}=-\frac{a^4}{b^4}$ (c) $x_1x_2+y_1y_2=\frac{a^2}{b^2}$ (d) $x_1x_2-y_1y_2=\frac{a^2}{b^2}$

Advance Level

106. The locus of the pole of normal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

(a) $a^6/x^2-b^6/y^2=(a^2+b^2)^2$ (b) $x^2/a^2-y^2/b^2=(a^2+b^2)^2$

(c) $a^2/x^2-b^2/y^2=(a^2+b^2)^2$ (d) None of these

107. The locus of the pole with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ of any tangent to the circle, whose diameter is the line joining the foci is the

(a) Ellipse (b) Hyperbola (c) Parabola (d) None of these

Asymptotes of Hyperbola

Basic Level

108. The product of the lengths of perpendicular drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes is **[EAMCET 2003]**

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 2

109. The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is equal to **[BIT Ranchi 1999]**

(a) $2\tan^{-1}\left(\frac{b}{a}\right)$ (b) $2\tan^{-1}\frac{a}{b}$ (c) $\tan^{-1}\frac{a}{b}$ (d) $\tan^{-1}\frac{b}{a}$

Advance Level

110. The product of perpendicular drawn from any point on a hyperbola to its asymptotes is **[Karnataka CET 2000]**

(a) $\frac{a^2b^2}{a^2+b^2}$ (b) $\frac{a^2+b^2}{a^2b^2}$ (c) $\frac{ab}{\sqrt{a}+\sqrt{b}}$ (d) $\frac{ab}{a^2+b^2}$

111. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. The area cut-off by the chord of contact on the asymptotes is equal to

(a) $\frac{ab}{2}$ (b) ab (c) $2ab$ (d) $4ab$

112. The equation of the hyperbola whose asymptotes are the straight lines $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ and which passes through origin is

(a) $(3x - 4y + 7)(4x + 3y + 1) = 0$ (b) $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$

(c) $12x^2 - 7xy + 2y^2 = 0$ (d) None of these

113. The equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$ are

(a) $2x^2 + 5xy + 2y^2 - 11x - 7y - 5 = 0$ (b) $2x^2 + 4xy + 2y^2 - 7x - 11y + 5 = 0$

(c) $2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$ (d) None of these

Rectangular Hyperbola

Basic Level

114. Eccentricity of the curve $x^2 - y^2 = a^2$ is **[UPSEAT 2002]**

(a) 2 (b) $\sqrt{2}$ (c) 4 (d) None of these

115. The eccentricity of curve $x^2 - y^2 = 1$ is **[MP PET 1995]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 2 (d) $\sqrt{2}$

116. The eccentricity of the hyperbola $x^2 - y^2 = 25$ is **[MP PET 1987]**

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 2 (d) $1+\sqrt{2}$

117. If transverse and conjugate axes of a hyperbola are equal, then its eccentricity is **[MP PET 2003]**

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 2

118. The eccentricity of the hyperbola $\frac{\sqrt{1999}}{3}(x^2-y^2)=1$ is **[Karnataka CET 1999]**

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$

119. Eccentricity of the rectangular hyperbola $\int_0^1 e^x\left(\frac{1}{x}-\frac{1}{x^3}\right)dx$ is **[UPSEAT 2002]**

(a) 2 (b) $\sqrt{2}$ (c) 1 (d) $\frac{1}{\sqrt{2}}$

120. The reciprocal of the eccentricity of rectangular hyperbola, is **[MP PET 1994]**

(a) 2 (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

121. The locus of the point of intersection of the lines $(x+y)t=a$ and $x-y=at$, where t is the parameter, is

(a) A circle (b) An ellipse (c) A rectangular hyperbola (d) None of these

122. Curve $xy=c^2$ is said to be

(a) Parabola (b) Rectangular hyperbola (c) Hyperbola (d) Ellipse

123. What is the slope of the tangent line drawn to the hyperbola $xy=a(a\neq 0)$ at the point $(a,1)$ **[AMU 2000]**

(a) $\frac{1}{a}$ (b) $\frac{-1}{a}$ (c) a (d) $-a$

124. The coordinates of the foci of the rectangular hyperbola $xy=c^2$ are

(a) $(\pm c,+c)$ (b) $(\pm c\sqrt{2},\pm c\sqrt{2})$ (c) $\left(\pm\frac{c}{\sqrt{2}},\pm\frac{c}{\sqrt{2}}\right)$ (d) None of these

125. A tangent to a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intercepts a length of unity from each of the coordinate axes, then the point (a, b) lies on the rectangular hyperbola

(a) $x^2-y^2=2$ (b) $x^2-y^2=1$ (c) $x^2-y^2=-1$ (d) None of these

126. A rectangular hyperbola is one in which

(a) The two axes are rectangular (b) The two axes are equal

(c) The asymptotes are perpendicular (d) The two branches are perpendicular

127. If e and e_1 are the eccentricities of the hyperbolas $xy=c^2$ and $x^2-y^2=c^2$, then $e^2+e_1^2$ is equal to **[EAMCET 1995; UPSE**

(a) 1 (b) 4 (c) 6 (d) 8

128. If the line $ax+by+c=0$ is a normal to the curve $xy = 1$, then

(a) $a>0, b>0$ (b) $a>0, b<0$ or $a<0, b>0$ (c) $a<0, b<0$ (d) None of these

129. The number of normals that can be drawn from any point to the rectangular hyperbola $xy=c^2$ is

(a) 1 (b) 2 (c) 3 (d) 4

130. The equation of the chord joining two points (x_1,y_1) and (x_2,y_2) on the rectangular hyperbola $xy=c^2$ is

(a) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$ (b) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$ (c) $\frac{x}{y_1+y_2}+\frac{y}{x_1+x_2}=1$ (d) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

131. If a triangle is inscribed in a rectangular hyperbola, its orthocentre lies

(a) Inside the curve (b) Outside the curve (c) On the curve (d) None of these

Advance Level

132. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is **[IIT Screening 2002]**

(a) $3y = 9x + 2$ (b) $y = 2x + 1$ (c) $2y = x + 8$ (d) $y = x + 2$

133. A rectangular hyperbola whose centre is C is cut by any circle of radius r in four points P, Q, R and S, then $CP^2 + CQ^2 + CR^2 + CS^2 =$

(a) r^2 (b) $2r^2$ (c) $3r^2$ (d) $4r^2$

134. If $P(x_1, y_1), Q(x_2, y_2) R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, the coordinates of orthocentre of the ΔPQR are

(a) $(x_4, -y_4)$ (b) (x_4, y_4) (c) $(-x_4, -y_4)$ (d) $(-x_4, y_4)$

135. If a circle cuts the rectangular hyperbola $xy = 1$ in the points (x_r, y_r) where $r = 1,2,3,4$ then

(a) $x_1 x_2 x_3 x_4 = 2$ (b) $x_1 x_2 x_3 x_4 = 1$ (c) $x_1 + x_2 + x_3 + x_4 = 0$ (d) $y_1 + y_2 + y_3 + y_4 = 0$

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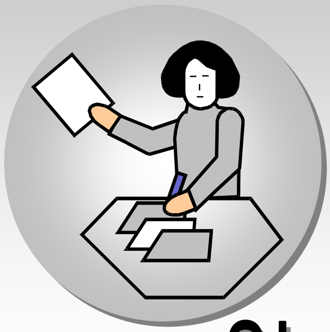

Answer Sheet

Conic Section : Hyperbola *Assignment (Basic and Advance level)*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	d	d	a	c	c	d	b	d	b	c	a	d	a	b	a	d	a	a	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
a	b	c	c	c	c	b	a	a	b	a	b	c	a,d	a	d	c	a	a	a
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
c	a	b	d	b	d	b	c	a	c	d	a	c	b	a	b	b	b	b	b
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
b	b	b	c	a	d	c	b	a	b	a	c	b	c	b	a	b	b	c	c
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
a	c	a,b	a	a	a	d	a,b	a	a	a	c	b	a	c	d	b	b	b	c
101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
b	a	b	a	b	a	a	b	a	a	d	b	c	b	d	a	b	b	b	d
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135					
c	b	b	b	b	a,b,c	b	b	d	a	c	d	d	d	b					