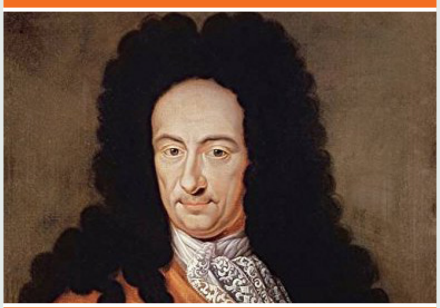
4 Differential Equations



Gottfried Wilhelm Leibniz

(1^{st} July 1646 - 14^{th} November 1716)

Introduction

Differential equations, began with Gottfried Wilhelm Leibniz. He was a Germen Philosopher, mathematicians, and logician.

In lower class, we have studied algebraic equations like $5x-3(x-6)=4x$, $x^2-7x+12=0$, $|18x-5|=3$. The goal here was to solve the equation, which meant to find the value (or values) of the variable that makes the equation true.

For example, $x=9$ is the solution to the first equation because only where 9 is substituted for x both sides of the equation are identical.

In general each type of algebraic equation had its own particular method of solution; quadratic equations were solved by one method equations involving absolute values by another, and so on. In each case, an equation was presented, and a certain method was employed to arrive at a solution, a method appropriate or the particular equation at hand.

These same general ideas carry over to differential equations, which are equations involving derivatives. There are different types of differential equations, and each type requires its own particular solution method.

Many problems related to economics, commerce and engineering, when formulated in mathematical forms, lead to differential equations. Many of these problems are complex in nature and very difficult to understand. But when they are described by differential equations, it is easy to analyse them.

Learning Objectives

After studying this chapter , the students will be able to understand

- order of the differential equations
- degree of the differential equations
- general and Particular solution of Differential equations
- formation of Differential equations
- differential equations with variable separable
- homogeneous differential equations
- linear differential equations
- second order linear differential equations with constant coefficients.

A differential equation is an equation with a function and one or more of its derivatives. (i.e) an equation with the function $y = f(x)$ and its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ … is called differential equation.

For example,

(i) $\frac{dy}{dx} = x + 5$ (ii) $\frac{dy}{dx} = \frac{\sqrt{1-x^2}}{\sqrt{2-y^2}}$

(iii) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$ (iv) $\frac{d^2x}{dt^2} + m^2x = 0$

Differential equations are of two types. One is ordinary differential equations and other one partial differential equations. Here we study only ordinary differential equations.

4.1 Formation of ordinary differential equations

4.1.1 Definition of ordinary differential equation

An ordinary differential equation is an equation that involves some ordinary derivatives $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right)$ of a function $y = f(x)$. Here we have one independent variable.

4.1.2 Order and degree of a differential equation

The highest order derivative present in the differential equation is the order of the differential equation.

Degree is the highest power of the highest order derivative in the differential equation, after the equation has been cleared from fractions and the radicals as for as the derivatives are concerned.

For example, consider the differential equation

$$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + y = 7$$

Here the highest order derivatives is $\frac{d^3y}{dx^3}$ (i.e 3rd order derivative). So the order of the differential equation is 3.

Now the power of highest order derivative $\frac{d^3y}{dx^3}$ is 1.

$\therefore$ The degree of the differential equation is 1.

Example 4.1

Find the order and degree of the following differential equations.

(i) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 + 4y = 0$

(ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$

(iii) $\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)^6 + 2y = x^2$

(iv) $\left[1 + \frac{d^2y}{dx^2}\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$

(v) $y' + (y'')^2 = (x + y'')^2$

(vi) $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$

(vii) $y = 2\left(\frac{dy}{dx}\right)^2 + 4x\frac{dx}{dy}$

Solution

(i) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 + 4y = 0$

Highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ order = 2

Power of the highest order derivative $\frac{d^2y}{dx^2}$ is 1.

$\therefore$ Degree =1

(ii) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = 0$

Highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ order =2

Power of the highest order derivative $\frac{d^2y}{dx^2}$ is 1.

$\therefore$ Degree = 1

(iii) $\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)^6 + 2y = x^2$

$\therefore$ order =3, Degree =1

(iv) $\left[1 + \frac{d^2y}{dx^2}\right]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$

Here we eliminate the radical sign.

Squaring both sides, we get

$$\left[1 + \frac{d^2y}{dx^2}\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

$\therefore$ Order=2, degree =3

(v) $y' + (y'')^2 = (x + y'')^2$

$$y' + (y'')^2 = x^2 + 2xy'' + (y'')^2$$

$$y' = x^2 + 2xy'' \Rightarrow \frac{dy}{dx} = x^2 + 2x\frac{d^2y}{dx^2}$$

$\therefore$ Order=2, degree=1

(vi) $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^{\frac{1}{2}} = 0$

Here we eliminate the radical sign.

For this write the equation as

$$\frac{d^3y}{dx^3} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$$

Squaring both sides, we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{dy}{dx}$$

$\therefore$ Order=3, degree=2

(vii) $y = 2\left(\frac{dy}{dx}\right)^2 + 4x\frac{dx}{dy}$

$$y = 2\left(\frac{dy}{dx}\right)^2 + 4x\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$y\frac{dy}{dx} = 2\left(\frac{dy}{dx}\right)^3 + 4x$$

$\therefore$ order=1, degree=3

DO YOU KNOW?

Order and degree (if defined) of a differential equation are always positive integers.

Family of Curves

Sometimes a family of curves can be represented by a single equation with one or more arbitrary constants. By assigning different values for constants, we get a family of curves. The arbitrary constants are called the parameters of the family.

For example,

(i) $y^2 = 4ax$ represents the equation of a family of parabolas having the origin as vertex where 'a' is the parameter.

(ii) $x^2 + y^2 = a^2$ represents the equation of family of circles having the origin as centre, where 'a' is the parameter.

(iii) $y = mx + c$ represents the equation of a family of straight lines in a plane, where m and c are parameters.

4.1.3 Formation of ordinary differential equation:

Consider the equation $f\left(x, y, c_1\right) = 0$ --(1) where c_1 is the arbitrary constant. We form the differential equation from this equation. For this, differentiate equation (1) with respect to the independent variable occur in the equation.

Eliminate the arbitrary constant c_1 from (1) and its derivative. Then we get the required differential equation.

Suppose we have $f\left(x, y, c_1, c_2\right) = 0$. Here we have two arbitrary constants c_1 and c_2. So, find the first two successive derivatives. Eliminate c_1 and c_2 from the given function and the successive derivatives. We get the required differential equation.

Note

The order of the differential equation to be formed is equal to the number of arbitrary constants present in the equation of the family of curves.

Example 4.2

Find the differential equation of the family of straight lines $y = mx + c$ when (i) m is the arbitrary constant (ii) c is the arbitrary constant (iii) m and c both are arbitrary constants.

Solution:

(i) m is an arbitrary constant

$y = mx + c$ ...(1)

Differentiating w.r.t. to x,

we get $\dfrac{dy}{dx} = m$ ...(2)

Now we eliminate m from (1) and (2)

For this substitute (2) in (1)

$y = x\dfrac{dy}{dx} + c$

$x\dfrac{dy}{dx} - y + c = 0$ which is the required differential equation of first order.

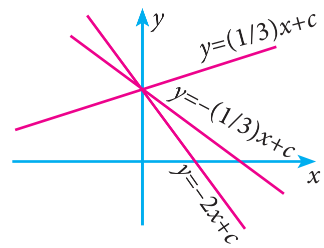


Fig. 4.1

(ii) c is an arbitrary constant

Differentiating (1), we get $\dfrac{dy}{dx} = m$

Here c is eliminated from the given equation

$\therefore \dfrac{dy}{dx} = m$ is the required differential equation.

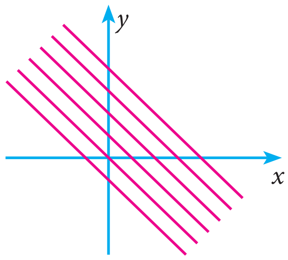


Fig. 4.2

(iii) both m and c are arbitrary constants

Since m and c are two arbitrary constants differentiating (1) twice we get

$$\frac{dy}{dx} = m$$

$$\frac{d^2y}{dx^2} = 0$$

Here m and c are eliminated from the given equation.

$\dfrac{d^2y}{dx^2} = 0$ which is the required differential equation.

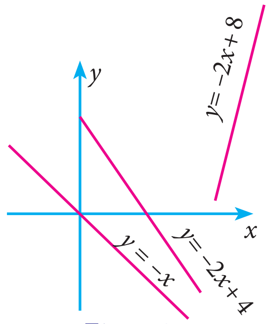


Fig. 4.3

Example 4.3

Find the differential equation of the family of curves $y=\frac{a}{x}+b$ where a and b are arbitrary constants.

Solution:

Given $\quad y=\frac{a}{x}+b$

Differentiating w.r.t x, we get

$$\frac{dy}{dx}=\frac{-a}{x^2}$$

$$x^2\frac{dy}{dx}=-a$$

Again differentiating w.r.t x we get

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0$$

$\Rightarrow x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ which is the required differential equation

Example 4.4

Find the differential equation corresponding to $y=ae^{4x}+be^{-x}$ where a, b are arbitrary constants,

Solution:

Given $y=ae^{4x}+be^{-x}$. $\qquad (1)$

Here a and b are arbitrary constants

From (1), $\quad \frac{dy}{dx} = 4ae^{4x}-be^{-x} \qquad (2)$

and $\quad \frac{d^2y}{dx^2} = 16ae^{4x}+be^{-x} \qquad (3)$

(1)+(2) $\Rightarrow \quad y+\frac{dy}{dx} = 5ae^{4x} \qquad (4)$

(2)+(3) $\Rightarrow \quad \frac{dy}{dx}+\frac{d^2y}{dx^2} = 20ae^{4x}$

$$= 4\left(5ae^{4x}\right)$$

$$= 4\left(y+\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}+\frac{d^2y}{dx^2} = 4y+4\frac{dy}{dx}$$

$\Rightarrow \quad \frac{d^2y}{dx^2}-3\frac{dy}{dx}-4y = 0$, which is the required differential equation.

Example 4.5

Find the differential equation of the family of curves $y=e^x\left(a\cos x+b\sin x\right)$ where a and b are arbitrary constants.

Solution :

$y=e^x\left(a\cos x+b\sin x\right) \qquad (1)$

Differentiating (1) w.r.t x, we get

$$\frac{dy}{dx} = e^x\left(a\cos x+b\sin x\right)+ e^x\left(-a\sin x+b\cos x\right)$$

$$= y+e^x(-a\sin x+b\cos x)$$

(from (1))

$\Rightarrow \frac{dy}{dx}-y = e^x(-a\sin x+b\cos x) \quad (2)$

Again differentiating, we get

$$\frac{d^2y}{dx^2}-\frac{dy}{dx} = e^x(-a\sin x+b\cos x)+e^x(-a\cos x-b\sin x)$$

$$\frac{d^2y}{dx^2}-\frac{dy}{dx} = e^x(-a\sin x+b\cos x)-e^x(a\cos x+b\sin x)$$

$\frac{d^2y}{dx^2}-\frac{dy}{dx} = \left(\frac{dy}{dx}-y\right)-y \quad$ (from (1) and (2))

$\Rightarrow \frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y = 0$, which is the requited differential equation.

Exercise 4.1

1. Find the order and degree of the following differential equations.

 (i) $\frac{dy}{dx}+2y=x^3$

 (ii) $\frac{d^3y}{dx^3}+3\left(\frac{dy}{dx}\right)^3+2\frac{dy}{dx}=0$

(iii) $\dfrac{d^2y}{dx^2} = \sqrt{y - \dfrac{dy}{dx}}$

(iv) $\dfrac{d^3y}{dx^3} = 0$

(v) $\dfrac{d^2y}{dx^2} + y + \left(\dfrac{dy}{dx} - \dfrac{d^3y}{dx^3}\right)^{3/2} = 0$

(vi) $(2 - y'')^2 = y''^2 + 2y'$

(vii) $\left(\dfrac{dy}{dx}\right)^3 + y = x - \dfrac{dx}{dy}$

2. Find the differential equation of the following

   (i) $y = cx + c - c^3$ (ii) $y = c(x-c)^2$

   (iii) $xy = c^2$ (iv) $x^2 + y^2 = a^2$

3. Form the differential equation by eliminating α and β from $(x-\alpha)^2 + (y-\beta)^2 = r^2$.

4. Find the differential equation of the family of all straight lines passing through the origin.

5. Form the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to the x axis.

6. Find the differential equation of all circles passing through the origin and having their centers on the y axis.

7. Find the differential equation of the family of parabola with foci at the origin and axis along the x-axis.

Solution of a Differential Equation:

The relation between the dependent and independent variables not involving derivatives is called the solution of the differential equation.

Solution of the differential equation must contain the same number of arbitrary constants as the order of the equation. Such a solution is called General (complete) solution of the differential equation.

4.2 First order and first degree differential equations

A differential equation of first order and first degree can be written as $f\left(x, y, \dfrac{dy}{dx}\right) = 0$. Here we will discuss the solution of few types of equations.

4.2.1 General solution and particular solution

For any differential equations it is possible to find the general solution and particular solution.

4.2.2 Differential Equation in which variables are separable

If in an equation it is possible to collect all the terms of x and dx on one side and all the terms of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is

$$f(x)dx = g(y)dy \text{ (or) } f(x)dx + g(y)dy = 0$$

By direct integration we get the solution.

Example 4.6

Solve: $(x^2 + x + 1)dx + (y^2 - y + 3)dy = 0$

Solution:

Given $(x^2 + x + 1)dx + (y^2 - y + 3)dy = 0$

It is of the form $f(x)dx + g(y)dy = 0$

Integrating , we get

$$\int (x^2 + x + 1)dx + \int (y^2 - y + 3)\, dy = c$$

$$\left(\frac{x^3}{3} + \frac{x^2}{2} + x\right) + \left(\frac{y^3}{3} - \frac{y^2}{2} + 3y\right) = c$$

Example 4.7

Solve $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y}$

Solution :

Given $\dfrac{dy}{dx} = e^{x-y} + x^2e^{-y} = e^{-y}e^x + e^{-y}x^2$

$= e^{-y}\left(e^x + x^2\right)$

Separating the variables,

we get $e^y dy = \left(e^x + x^2\right)dx$

Integrating, we get $\int e^y dy = \int\left(e^x + x^2\right)dx$

$$e^y = e^x + \frac{x^3}{3} + c$$

Example 4.8

Solve $3e^x \tan y dx + (1+e^x)\sec^2 y dy = 0$

given $y(0) = \frac{\pi}{4}$

Solution:

Given $3e^x \tan y dx + (1+e^x)\sec^2 y \, dy = 0$

$$3e^x \tan y \; dx = -(1+e^x)\sec^2 y \; dy$$

$$\frac{3e^x}{1+e^x}dx = -\frac{\sec^2 y}{\tan y}dy$$

Integrating, we get $3\int\frac{e^x}{1+e^x}dx = -\int\frac{\sec^2 y}{\tan y}dy + c$

$$3\log(1+e^x) = -\log\tan y + \log c$$

$$\left[\because \int\frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$$\log(1+e^x)^3 + \log\tan y = \log c$$

$$\log\left[\left(1+e^x\right)^3 \tan y\right] = \log c$$

$$(1+e^x)^3 \tan y = c \qquad (1)$$

Given $y(0) = \frac{\pi}{4}$ (i.e) $y = \frac{\pi}{4}$ at $x = 0$

$$(1) \Rightarrow (1+e^0)^3 \tan\frac{\pi}{4} = c$$

$$2^3(1) = c$$

$$\Rightarrow c = 8$$

Hence the required solution is $(1+e^x)^3 \tan y = 8$.

Example 4.9

Solve $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

Solution:

Separating the variables, we get

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we get

$$\int\frac{\sec^2 x}{\tan x}dx + \int\frac{\sec^2 y}{\tan y}dy = c$$

$$\log\tan x + \log\tan y = \log c$$

$$\log(\tan x \tan y) = \log c$$

$$\tan x \tan y = c$$

Example 4.10

Solve $ydx - xdy - 3x^2y^2e^{x^3}dx = 0$.

Solution:

Given equation can be written as

$$\frac{ydx - xdy}{y^2} - 3x^2e^{x^3}dx = 0$$

Integrating, $\int\frac{ydx-xdy}{y^2} - \int 3x^2e^{x^3}dx = c$

$$\int d\left(\frac{x}{y}\right) - \int e^t dt = c$$

(where $t = x^3$ and $dt = 3x^2dx$)

$$\frac{x}{y} - e^t = c$$

$$\frac{x}{y} - e^{x^3} = c$$

Example 4.11

Solve : $x - y\frac{dx}{dy} = a\left(x^2 + \frac{dx}{dy}\right)$

Solution:

Given $x - y\frac{dx}{dy} = a\left(x^2 + \frac{dx}{dy}\right)$

$$x - y\frac{dx}{dy} = ax^2 + a\frac{dx}{dy}$$

$$x - ax^2 = a\frac{dx}{dy} + y\frac{dx}{dy}$$

$$x(1-ax) = (a+y)\frac{dx}{dy}$$

By separating the variables

we get, $\frac{dx}{x(1-ax)} = \frac{dy}{a+y}$

$$\left(\frac{a}{1-ax} + \frac{1}{x}\right)dx = \frac{dy}{a+y}$$

Integrating, $\int\left(\frac{a}{1-ax}+\frac{1}{x}\right)dx = \frac{dy}{a+y}$

$$-\log(1-ax)+\log x = \log(a+y)+\log c$$

$$\log\left(\frac{x}{1-ax}\right) = \log\left(c\left(a+y\right)\right)$$

$$\left(\frac{x}{1-ax}\right) = c\ (a+y)$$

$x = (1-ax)(a+y)c$ which is the required solution.

Example 4.12

The marginal cost function of manufacturing x gloves is $6+10x-6x^2$. The total cost of producing a pair of gloves is ₹100. Find the total and average cost function.

Solution:

Given $MC = 6+10x-6x^2$

i.e., $\frac{dc}{dx} = 6+10x-6x^2$

$$dc = (6+10x-6x^2)dx$$

$$\int dc = \int(6+10x-6x^2)dx+k$$

$$c = 6x+10\frac{x^2}{2}-6\frac{x^3}{3}+k$$

$$c = 6x+5x^2-2x^3+k \quad (1)$$

Given $c=100$ when $x=2$

$\therefore$ (1) $\Rightarrow$ $100 = 12+5(4)-2(8)+k$

$\Rightarrow$ $k = 84$

$\therefore$ (1) $\Rightarrow$ $c\ (x) = 6x+5x^2-2x^3+84$

Average Cost $AC = \frac{c}{x} = 6+5x-2x^2+\frac{84}{x}$

Example 4.13

The normal lines to a given curve at each point (x,y) on the curve pass through the point (1,0). The curve passes through the point (1,2). Formulate the differential equation representing the problem and hence find the equation of the curve.

Solution:

Slope of the normal at any point $P(x,y) = -\frac{dx}{dy}$

Let Q be (1,0)

Slope of the normal PQ *is* $\frac{y_2-y_1}{x_2-x_1}$

i.e., $\frac{y-0}{x-1} = \frac{y}{x-1}$

$\therefore\ -\frac{dx}{dy}=\frac{y}{x-1} \Rightarrow \frac{dx}{dy}=\frac{y}{1-x}$, which is the differential equation

i.e., $(1-x)dx = ydy$

$$\int(1-x)dx = \int ydy+c$$

$$x-\frac{x^2}{2} = \frac{y^2}{2}+c \quad (1)$$

Since it passes through (1,2)

$$1-\frac{1}{2} = \frac{4}{2}+c$$

$$c=\frac{1}{2}-2 = -\frac{3}{2}$$

Put $c = \frac{-3}{2}$ in (1)

$$x-\frac{x^2}{2} = \frac{y^2}{2}-\frac{3}{2}$$

$$2x-x^2 = y^2-3$$

$\Rightarrow$ $y^2 = 2x-x^2+3$,

which is the equation of the curve

Example 4.14

The sum of ₹2,000 is compounded continuously, the nominal rate of interest being 5% per annum. In how many years will the amount be double the original principal? $(\log_e 2 = 0.6931)$

Solution:

Let P be the principal at time 't'

$$\frac{dP}{dt} = \frac{5}{100}P = 0.05P$$

$$\Rightarrow \quad \int \frac{dP}{P} = \int 0.05\, dt + c$$

$$\log_e P = 0.05t + c$$

$$P = e^{0.05t} e^{c}$$

$$P = c_1 e^{0.05t} \quad (1)$$

Given $P = 2000$ when $t = 0$

$$\Rightarrow \quad c_1 = 2000$$

$$\therefore (1) \Rightarrow \quad P = 2000e^{0.05t}$$

To find t, when $P = 4000$

$$(2) \Rightarrow \quad 4000 = 2{,}000e^{0.05t}$$

$$2 = e^{0.05t}$$

$$0.05t = \log 2$$

$$t = \frac{0.6931}{0.05} = 14 \text{ years (approximately)}$$

Exercise 4.2

1. Solve: (i) $\frac{dy}{dx} = ae^{y}$ (ii) $\frac{1+x^2}{1+y} = xy\frac{dy}{dx}$

2. Solve : $y(1-x) - x\frac{dy}{dx} = o$

3. Solve: (i) $ydx - xdy = 0$

   (ii) $\frac{dy}{dx} + e^x + ye^x = 0$

4. Solve: $\cos x(1+\cos y)dx - \sin y(1+\sin x)dy = 0$

5. Solve: $(1-x)dy - (1+y)dx = 0$

6. Solve: (i) $\frac{dy}{dx} = y \sin 2x$

   (ii) $\log\left(\frac{dy}{dx}\right) = ax + by$

7. Find the curve whose gradient at any point $P(x,y)$ on it is $\frac{x-a}{y-b}$ and which passes through the origin.

4.2.3 Homogeneous Differential Equations

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is called homogeneous differential equation if $f(x,y)$ and $g(x, y)$ are homogeneous functions of the same degree in x and y (or) Homogeneous differential can be written as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

Method of solving first order Homogeneous differential equation

Check $f(x,y)$ and $g(x,y)$ are homogeneous functions of same degree.

i.e. $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

The given differential equation becomes $v + x\frac{dv}{dx} = F(v)$

Separating the variables, we get

$$x\frac{dv}{dx} = F(v) - v \Rightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$$

By integrating we get the solution in terms of v and x.

Replacing v by $\frac{y}{x}$ we get the solution.

Note

Sometimes it becomes easier by taking the Homogeneous differential equation as $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$ (1)

In this method we have to substitute $x = vy$ and $\frac{dx}{dy} = v + x\frac{dv}{dy}$ then (1) reduces to variable separable type. By integrating, we get the solution in terms of v and y. The solution is deduced by replacing $v = \frac{x}{y}$.

Example 4.15

Solve the differential equation $y^2dx+(xy+x^2)dy=0$

Solution

$$y^2dx+(xy+x^2)\,dy = 0$$

$$(xy+x^2)\,dy = -y^2dx$$

$$\frac{dy}{dx} = \frac{-y^2}{xy+x^2} \qquad (1)$$

It is a homogeneous differential equation, same degree in x and y

Put $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ (1) becomes

$$v+x\frac{dv}{dx} = \frac{-v^2x^2}{x\,vx+x^2}$$

$$= \frac{-v^2}{v+1}$$

$$x\frac{dv}{dx} = \frac{-v^2}{v+1}-v$$

$$= \frac{-v^2-v^2-v}{v+1}$$

$$x\frac{dv}{dx} = \frac{-\left(v+2v^2\right)}{1+v}$$

Now, separating the variables

$$\frac{1+v}{v\left(1+2v\right)}dv = \frac{-dx}{x}$$

$$\frac{\left(1+2v\right)-v}{v\left(1+2v\right)}dv = \frac{-dx}{x}$$

$$\left(\because\ 1+v = 1+2v-v\right)$$

$$\frac{1}{v}-\frac{1}{1+2v}\,dv = \frac{-dx}{x}$$

On Integration we have

$$\int\left(\frac{1}{v}-\frac{1}{1+2v}\right)dv = -\int\frac{dx}{x}$$

$$\log v-\frac{1}{2}\log\left(1+2v\right) = -\log x+\log c$$

$$\log\left(\frac{v}{\sqrt{1+2v}}\right) = \log\left(\frac{c}{x}\right)$$

$$\frac{v}{\sqrt{1+2v}} = \frac{c}{x}$$

Replace $v = \frac{y}{x}$ we get

$$\frac{\frac{y}{x}}{\sqrt{1+\frac{2y}{x}}} = \frac{c}{x}$$

$$\frac{y\sqrt{x}}{\sqrt{x+2y}} = c$$

$$\frac{y^2x}{x+2y} = k$$

where $k = c^2$

Note

$\int\frac{1+v}{v^2+2v}\,dv$ can be done by the method of partial fraction also.

Example 4.16

Solve the differential equation $\frac{dy}{dx}=\frac{x-y}{x+y}$.

Solution:

$$\frac{dy}{dx}=\frac{x-y}{x+y} \qquad (1)$$

This is a homogeneous differential equation.

Now put $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore (1) \Rightarrow \quad v+x\frac{dv}{dx} = \frac{x-vx}{x+vx}$$

$$= \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v}{1+v}-v$$

$$= \frac{1-2v-v^2}{1+v}$$

$$\frac{1+v}{v^2+2v-1}dv = \frac{-dx}{x}$$

Multiply 2 on both sides

$$\frac{2+2v}{v^2+2v-1}\,dv = -2\frac{dx}{x}$$

On Integration

$$\int\frac{2+2v}{v^2+2v-1}\,dv = -2\int\frac{dx}{x}$$

$$\log(v^2+2v-1) = -2\log x + \log c$$

$$v^2+2v-1 = \frac{c}{x^2}$$

$$x^2\,(v^2+2v-1) = c$$

$$\text{Now, Replace } v = \frac{y}{x}$$

$$x^2\left[\frac{y^2}{x^2}+\frac{2y}{x}-1\right] = c$$

$$y^2+2xy-x^2 = c \text{ is the solution.}$$

Example 4.17

Find the particular solution of the differential equation $x^2\ dy + y\left(x+y\right)dx = 0$ given that $x=1,\ y=1$

Solution

$$x^2dy + y\left(x+y\right)dx = 0$$

$$x^2dy = -y\left(x+y\right)dx$$

$$\frac{dy}{dx} = \frac{-(xy+y^2)}{x^2} \quad (1)$$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{-(x\,vx + v^2x^2)}{x^2}$$

$$= -\left(v+v^2\right)$$

$$x\frac{dv}{dx} = -v^2 - v - v$$

$$= -(v^2+2v)$$

On separating the variables

$$\frac{dv}{v^2+2v} = \frac{-dx}{x}$$

$$\frac{dv}{v\left(v+2\right)} = \frac{-dx}{x}$$

$$\frac{1}{2}\left[\frac{\left(v+2\right)-v}{v\left(v+2\right)}\right]dv = \frac{-dx}{x}$$

$$\frac{1}{2}\int\left(\frac{1}{v}-\frac{1}{v+2}\right)dv = -\int\frac{dx}{x}$$

$$\frac{1}{2}\left[\log v - \log\left(v+2\right)\right] = -\log x + \log c$$

$$\frac{1}{2}\log\frac{v}{v+2} = \log\frac{c}{x}$$

We have

$$\frac{v}{v+2} = \frac{c^2}{x^2}$$

Replace $v = \frac{y}{x}$, we get

$$\frac{y}{x\left(\frac{y}{x}+2\right)} = \frac{k}{x^2}$$

where $c^2 = k$

$$\frac{y\,x^2}{y+2x} = \text{k} \qquad (2)$$

When $x=1\,, y=1$

$$\therefore (2) \Rightarrow k = \frac{1}{1+2}$$

$$k = \frac{1}{3}$$

$\therefore$ The solution is $3x^2\ y = 2x + y$

Example 4.18

If the marginal cost of producing x shoes is given by $(3xy+y^2)\ dx + (x^2+xy)\ dy = 0$. and the total cost of producing a pair of shoes is given by ₹12. Then find the total cost function.

Solution:

Given marginal cost function is $(x^2+xy)\ dy + (3xy+y^2)\ dx = 0$

$$\frac{dy}{dx} = \frac{-(3xy+y^2)}{x^2+xy} \qquad (1)$$

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{-(3x\ vx + v^2x^2)}{x^2 + x\ vx}$$

$$= \frac{-(3v+v^2)}{1+v}$$

Now, $$x\frac{dv}{dx} = \frac{-3v-v^2}{1+v} - v$$

$$= \frac{-3v-v^2-v-v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{-4v-2v^2}{1+v}$$

$$\frac{1+v}{4v+2v^2}dv = \frac{-dx}{x}$$

On Integration

$$\int\frac{1+v}{4v+2v^2}dv = -\int\frac{dx}{x}$$

Now, multiply 4 on both sides

$$\int\frac{4+4v}{4v+2v^2}dv = -4\int\frac{dx}{x}$$

$$\log(4v+2v^2) = -4\log x + \log c$$

$$4v+2v^2 = \frac{c}{x^4}$$

$$x^4(4v+2v^2) = \text{c}$$

Replace $\text{v} = \dfrac{y}{x}$

$$x^4\left(4\frac{y}{x}+2\frac{y^2}{x^2}\right) = \text{c}$$

$$x^4\left[\frac{4xy+2y^2}{x^2}\right] = \text{c}$$

$$\text{c} = 2x^2(2xy+y^2) \quad (2)$$

Cost of producing a pair of shoes = ₹12

$(i.e)$ $y = 12$ when $x = 2$

$$\text{c} = 8\left[48+144\right] = 1536$$

∴ The cost function is $x^2(2xy+y^2) = 768$

Example 4.19

The marginal revenue 'y' of output 'q' is given by the equation $\dfrac{dy}{dq} = \dfrac{q^2+3y^2}{2qy}$. Find the total Revenue function when output is 1 unit and Revenue is ₹5.

Solution:

Given that $MR = \dfrac{dy}{dq} = \dfrac{q^2+3y^2}{2qy} \qquad (1)$

Put $y = vq$ and $\dfrac{dy}{dq} = v + q\dfrac{dv}{dq}$ in (1)

Now (1) becomes

$$v + q\frac{dv}{dq} = \frac{q^2+3v^2q^2}{2q\ vq}$$

$$= \frac{1+3v^2}{2v}$$

$$q\frac{dv}{dq} = \frac{1+3v^2}{2v} - v$$

$$= \frac{1+3v^2-2v^2}{2v}$$

$$= \frac{1+v^2}{2v}$$

$$\frac{2v}{1+v^2}dv = \frac{dq}{q}$$

On Integration

$$\int\frac{2v}{1+v^2}dv = \int\frac{dq}{q}$$

$$\log(1+v^2) = \log q + \log c$$

$$1+v^2 = \text{cq}$$

Replace $\text{v} = \dfrac{y}{q}$

$$1+\frac{y^2}{q^2} = cq$$

$$q^2+y^2 = c\,q^3 \qquad (2)$$

Given output is 1 unit and revenue is ₹5

$\therefore$ (2) $\Rightarrow$ $\quad 1+25=c \;\Rightarrow\; c=26$

$\therefore$ The total revenue fonction is

$$q^2+y^2=26q^3$$

Exercise 4.3

Solve the following homogeneous differential equations.

1. $x\dfrac{dy}{dx}=x+y$
2. $(x-y)\dfrac{dy}{dx}=x+3y$
3. $x\dfrac{dy}{dx}-y=\sqrt{x^2+y^2}$
4. $\dfrac{dy}{dx}=\dfrac{3x-2y}{2x-3y}$
5. $(y^2-2xy)dx=(x^2-2xy)dy$
6. The slope of the tangent to a curve at any point (x, y) on it is given by $(y^3-2yx^2)dx+(2xy^2-x^3)\,dy=0$ and the curve passes through $(1, 2)$. Find the equation of the curve.
7. An electric manufacturing company makes small household switches. The company estimates the marginal revenue function for these switches to be $(x^2+y^2)dy=xydx$ where x represents the number of units (in thousands). What is the total revenue function?

4.2.4 Linear differential equations of first order:

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and no product of these occur.

The most general form of a linear equation of the first order is $\dfrac{dy}{dx}+Py=Q$ (1)

P and Q are functions of x alone.

Equation (1) is linear in y. The solution is given by $ye^{\int pdx}=\int Qe^{\int pdx}dx+c$. Here $e^{\int pdx}$ is known as an integrating factor and is denoted by I.F.

Note

For the differential equation $\dfrac{dx}{dy}+Px=Q$ (linear in x) where P and Q are functions of y alone, the solution is $xe^{\int pdy}=\int Qe^{\int pdy}dy+c$

Example 4.20

Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=x^3$

Solution:

Given $\dfrac{dy}{dx}+\dfrac{1}{x}y=x^3$

It is of the form $\dfrac{dy}{dx}+Py=Q$

Here $P=\dfrac{1}{x},\quad Q=x^3$

$$\int Pdx = \int\frac{1}{x}dx=\log x$$

$$\text{I.F} = e^{\int pdx}=e^{\log x}=x$$

The required solution is

$$y(I.F) = \int Q(I.F)dx+c$$

$$yx = \int x^3.x\;dx+c$$

$$= \int x^4dx+c$$

$$= \frac{x^5}{5}+c$$

$$\therefore\; yx = \frac{x^5}{5}+c$$

A linear differential equation is always of first degree but every differential equation of the first degree need not be linear

$$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y^2 = 0$$

is not linear.

Example 4.21

Solve $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

Solution:

The given equation can be written as

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $\quad P = \sec^2 x, Q = \tan x \sec^2 x$

$$\int P dx = \int \sec^2 x \, dx = \tan x$$

$$\text{I.F} = e^{\int pdx} = e^{\tan x}$$

The required solution is

$$y(I.F) = \int Q(I.F)dx + c$$

$$ye^{\tan x} = \int \tan x \sec^2 x \, e^{\tan x} \, dx + c$$

Put $\tan x = t$

Then $\sec^2 x dx = dt$

$$\therefore ye^{\tan x} = \int te^t \; dt + c$$

$$= \int td(e^t) + c$$

$$= te^t - e^t + c$$

$$= \tan x \, e^{\tan x} - e^{\tan x} + c$$

$$ye^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

Example 4.22

Solve $\left(x^2+1\right)\dfrac{dy}{dx} + 2xy = 4x^2$

Solution:

The given equation can be reduced to

$$\frac{dy}{dx} + \frac{2x}{x^2+1} y = \frac{4x^2}{x^2+1}$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $\quad P = \dfrac{2x}{x^2+1}, Q = \dfrac{4x^2}{x^2+1}$

$$\int Pdx = \int \frac{2x}{x^2+1}dx = \log(x^2+1)$$

$$\text{I.F} = e^{\int pdx} = e^{\log\left(x^2+1\right)} = x^2+1$$

The required solution is

$$y(IF) = \int Q(I.F)dx + c$$

$$y(x^2+1) = \int \frac{4x^2}{x^2+1}\left(x^2+1\right)dx + c$$

$$y(x^2+1) = \frac{4x^3}{3} + c$$

Example 4.23

Solve $\dfrac{dy}{dx} - 3y\cot x = \sin 2x$ given that $y = 2$ when $x = \dfrac{\pi}{2}$

Solution:

Given $\dfrac{dy}{dx} - \left(3\cot x\right).y = \sin 2x$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $\quad P = -3\cot x, \; Q = \sin 2x$

$$\int P dx = \int -3\cot x dx = -3\log \sin x$$

$$= -\log \sin^3 x = \log \frac{1}{\sin^3}$$

$$\text{I.F.} = e^{\log \frac{1}{\sin^3 x}} = \frac{1}{\sin^3 x}$$

The required solution is

$$y\,(I.F) = \int Q(I.F)dx + c$$

$$y\frac{1}{\sin^3 x} = \int \sin 2x \frac{1}{\sin^3 x}\, dx + c$$

$$y\frac{1}{\sin^3 x} = \int 2\sin x \cos x \times \frac{1}{\sin^3 x} dx + c$$

$$= 2\int \frac{1}{\sin x} \times \frac{\cos x}{\sin x}\, dx + c$$

$$= 2\int \operatorname{cosec} x \cot x dx + c$$

$$y\frac{1}{\sin^3 x} = -2\operatorname{cosec} x + c \qquad (1)$$

$$\text{Now } y = 2 \text{ when } x = \frac{\pi}{2}$$

$$(1) \Rightarrow 2\left(\frac{1}{1}\right) = -2\times 1 + c \Rightarrow c = 4$$

$$\therefore (1) \Rightarrow y\frac{1}{\sin^3 x} = -2\operatorname{cosec} x + 4$$

Example 4.24

A firm has found that the cost C of producing x tons of certain product by the equation $x\frac{dC}{dx} = \frac{3}{x} - C$ and C = 2 when x = 1.

Find the relationship between C and x.

Solution:

$$x\frac{dC}{dx} = \frac{3}{x} - C$$

$$\frac{dC}{dx} = \frac{3}{x^2} - \frac{C}{x}$$

$$\frac{dC}{dx} + \frac{C}{x} = \frac{3}{x^2}$$

$$\text{i.e.,} \quad \frac{dC}{dx} + \frac{1}{x}C = \frac{3}{x^2}$$

It is of the form $\frac{dC}{dx} + PC = Q$

Here, $P = \frac{1}{x}, Q = \frac{3}{x^2}$

$$\int P dx = \int \frac{1}{x} dx = \log x$$

$$\text{I.F} = e^{\int p dx} = e^{\log x} = x$$

The Solution is

$\text{C(I.F)} = \int Q(I.F)dx + k$ where k is constant

$$\text{Cx} = \int \frac{3}{x^2} x dx + k$$

$$= 3\int \frac{1}{x} dx + k$$

$$\text{Cx} = 3\log x + k \qquad (1)$$

Given C = 2 When $x = 1$

$$(1) \Rightarrow 2\times 1 = k \Rightarrow k = 2$$

$\therefore$ The relationship between C and x is

$$\text{Cx} = 3\log x + 2$$

Exercise 4.4

Solve the following:

1. $\frac{dy}{dx} - \frac{y}{x} = x$
2. $\frac{dy}{dx} + y\cos x = \sin x \cos x$
3. $x\frac{dy}{dx} + 2y = x^4$
4. $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{1+x^2}{1+x^3}$
5. $\frac{dy}{dx} + \frac{y}{x} = xe^x$
6. $\frac{dy}{dx} + y\tan x = \cos^3 x$
7. If $\frac{dy}{dx} + 2y\tan x = \sin x$ and if $y = 0$ when $x = \frac{\pi}{3}$ express y in terms of x

8. $\dfrac{dy}{dx}+\dfrac{y}{x}=xe^{x}$

9. A bank pays interest by continuous compounding, that is by treating the interest rate as the instantaneous rate of change of principal. A man invests ₹1,00,000 in the bank deposit which accures interest, 8% per year compounded continuously. How much will he get after 10 years.

4.3 Second Order first degree differential equations with constant coefficients:

4.3.1 A general second order linear differential equation with constant coefficients is of the form

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)}$$

$aD^2y+bDy+cy=f(x)$, where $\dfrac{d}{dx}=D, \dfrac{d^2}{dx^2}=D^2$

$$\phi(D)y=f(x) \qquad (1)$$

where $\phi(D)=aD^2+bD+c$ (a,b and c are constants)

To solve the equation (1), we first solve the equation $\phi(D)y=0$. The solution so obtained is called complementary function (C.F).

Next we operate on $f(x)$ with $\dfrac{1}{\phi(D)}$, the solution so obtained is called particular integral (P.I)

$$PI=\frac{1}{\phi(D)}f(x)$$

General solution is $y=C.F+P.I$

Type 1 : $f(x)=0$

$$(i.e)\phi(D)y=0$$

To solve this, put $\phi(D)=0$

Replace D by m. This equation is called auxiliary equation $.\phi(m)=0$ *is a* quadratic equation. So we have two roots, say m_1 and m_2.

Now we have the following three cases.

	Nature of roots	Complementary function
1	Real and different $(m_1\neq m_2)$	$Ae^{m_1x}+Be^{m_2x}$
2	Real and equal $m_1=m_2=m(say)$	$(Ax+B)e^{mx}$
3	Complex roots $(\alpha\pm i\beta)$	$e^{\alpha x}(A\cos\beta x+B\sin\beta x)$

Here A and B are arbitrary constants

Example 4.25

Solve $(D^2-3D-4)y=0$

Solution:

Given $\qquad (D^2-3D-4)y = 0$

The auxiliary equations is

$$m^2-3m-4 = 0$$

$$\Rightarrow (m-4)(m+1) = 0$$

$$m = -1, 4$$

Roots are real and different

$\therefore$ The complementary function is $Ae^{-x}+Be^{4x}$

The general solution is $y = Ae^{-x}+Be^{4x}$

Example 4.26

Solve $9y''-12y'+4y=0$

Solution:

Given $(9D^2-12D+4)y = 0$

The auxiliary equation is $(3m-2)^2=0$

$(3m-2)(3m-2)=0 \quad \Rightarrow m=\dfrac{2}{3}, \dfrac{2}{3}$

Roots are real and equal.

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The C.F. is $(Ax+B)e^{\frac{2}{3}x}$

The general solution is $y = (Ax+B)e^{\frac{2}{3}x}$

Example 4.27

Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

Solution:

Given $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

$$(D^2 - 4D + 5)y = 0$$

The auxiliary equation is

$$m^2 - 4m + 5 = 0$$

$$\Rightarrow (m-2)^2 - 4 + 5 = 0$$

$$(m-2)^2 = -1$$

$$m - 2 = \pm\sqrt{-1}$$

m $= 2 \pm i$, it is if the form $\alpha \pm i\beta$

$\therefore$ C.F= $e^{2x}\left[A\cos x + B\sin x\right]$

The general solution is

$$y = e^{2x}\left[A\cos x + B\sin x\right]$$

Example 4.28

Solve $\frac{d^2x}{dt^2} - \frac{3dx}{dt} + 2x = 0$ given that when $t = 0,\ x = 0$ *and* $\frac{dx}{dt} = 1$

Solution:

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$$

Given $(D^2 - 3D + 2)x = 0$ where $D = \frac{d}{dt}$

A.E is $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$\text{C.F} = Ae^t + Be^{2t}$$

The general solution is x = $Ae^t + Be^{2t}$ (1)

Now when $t = 0,\ x = 0$ (given)

(1) $\Rightarrow$ $0 = A + B$ (2)

Differentiating (1) w.r.t 't'

$$\frac{dx}{dt} = Ae^t + 2Be^{2t}$$

When t = 0, $\frac{dx}{dt} = 1$

$A + 2B = 1$ (3)

Thus we have $A + B = 0$ and $A + 2B = 1$

Solving, we get A = –1, B = 1

$\therefore$ (1) $\Rightarrow$ x = $-e^t + e^{2t}$

(i.e.) x = $e^{2t} - e^t$

Type II : $f(x) = e^{ax}$ (i.e) $\phi(D)y = e^{ax}$

$$\text{P.I} = \frac{1}{\phi(D)}e^{ax}$$

Replace D by a, provided $\phi(D) \neq 0$ when D = a

If $\phi(D) = 0$ when D = a, then

$$\text{P.I} = x\frac{1}{\phi'(D)}e^{ax}$$

Replace D by a, provided $\phi'(D) \neq 0$ when D = a

If $\phi'(D) = 0$ when D = a, then

P.I $= x^2\frac{1}{\phi''(D)}e^{ax}$ and so on

Example 4.29

Solve : $(D^2 - 4D - 1)y = e^{-3x}$

Solution:

$$(D^2 - 4D - 1)y = e^{-3x}$$

The auxiliary equation is

$$m^2-4m-1 = 0$$
$$(m-2)^2-4-1 = 0$$
$$(m-2)^2 = 5$$
$$m-2 = \pm\sqrt{5}$$
$$\text{m} = 2\pm\sqrt{5}$$
$$\text{C.F} = Ae^{(2+\sqrt{5})x}+Be^{(2-\sqrt{5})x}$$
$$\text{PI} = \frac{1}{\phi(D)}f(x)$$
$$= \frac{1}{D^2-4D-1}e^{-3x}$$
$$= \frac{1}{(-3)^2-4(-3)-1}e^{-3x}$$
(replace D *by* −3)
$$= \frac{1}{9+12-1}e^{-3x}$$
$$= \frac{e^{-3x}}{20}$$

Hence the general solution is

$$\text{y} = \text{C.F+P.I}$$
$$\Rightarrow \quad \text{y} = Ae^{(2+\sqrt{5})x}+Be^{(2-\sqrt{5})x}+\frac{e^{-3x}}{20}$$

Example 4.30

Solve: $\left(D^2-2D+1\right)y = e^{2x}+e^{x}$

Solution:

$$\left(D^2-2D+1\right)y = e^{2x}+e^{x}$$

The auxiliary equation is

$$m^2-2m+1 = 0$$
$$\Rightarrow (m-1)(m-1) = 0$$
$$\text{m} = 1, 1$$
$$\text{C.F} = (Ax+B)e^{x}$$
$$\text{PI} = \frac{1}{\phi(D)}f(x) = \frac{1}{D^2-2D+1}\left(e^{2x}+e^{x}\right)$$

Now, $$P.I_1 = \frac{1}{D^2-2D+1}e^{2x}$$
$$= \frac{1}{4-4+1}e^{2x}$$
(replace D *by* 2)
$$= e^{2x}$$

and $$P.I_2 = \frac{1}{D^2-2D+1}e^{x}$$
$$= \frac{1}{(D-1)^2}e^{x}$$

Replace D by 1. $(D-1)^2 = 0$ when D = 1

$$\therefore\ P.I_2 = x.\frac{1}{2(D-1)}e^{x}$$

Replace D by 1. $(D-1) = 0$ when D = 1

$$\therefore\ P.I_2 = x^2\frac{1}{2}e^{x}$$

The general solution is

$$\text{y} = \text{C.F+P.I}_1\text{+P.I}_2$$
$$\text{y} = (Ax+B)e^{x}+e^{2x}+\frac{x^2}{2}e^{x}$$

Example 4.31

Solve: $\left(3D^2+D-14\right)y = 4-13e^{\frac{-7}{3}x}$

Solution:

$$\left(3D^2+D-14\right)y = 4-13e^{\frac{-7}{3}x}$$

The auxiliary equation is

$$3m^2+m-14 = 0$$
$$(3m+7)(m-2) = 0$$
$$\text{m} = \frac{-7}{3}, 2$$
$$\text{C.F} = Ae^{\frac{-7}{3}x}+Be^{2x}$$
$$\Rightarrow \text{PI} = \frac{1}{\phi(D)}f(x) = \frac{1}{3D^2+D-14}\left(4-13e^{\frac{-7}{3}x}\right)$$

$$= \frac{1}{3D^2+D-14}(4)+\frac{1}{3D^2+D-14}\left(-13e^{\frac{-7}{3}x}\right)$$

$$= PI_1 + PI_2$$

$$\text{P.I}_1 = \frac{1}{3D^2+D-14}4e^{0x}$$

$$= \frac{1}{0+0-14}4e^{0x}$$

(replace D *by* 0)

$$\text{P.I}_1 = \frac{-4}{14} = \frac{-2}{7}$$

$$P.I_2 = \frac{1}{3D^2+D-14}\times(-13)e^{\frac{-7}{3}x}$$

replace D by $\frac{-7}{3}$

Here $3D^2+D-14=0$, when $D=-\frac{7}{3}$

$$\therefore\ P.I_2 = x.\frac{1}{6D+1}\left(-13e^{\frac{-7}{3}x}\right)$$

replace D by $\frac{-7}{3}$

$$\therefore\ P.I_2 = x\frac{1}{6\left(\frac{-7}{3}\right)+1}\left(-13e^{\frac{-7}{3}x}\right)$$

$$= x\frac{1}{-13}\left(-13e^{\frac{-7}{3}x}\right)$$

$$= xe^{\frac{-7}{3}x}$$

The general solution is $y = C.F.+P.I_1+P.I_2$

$$y = Ae^{\frac{-7}{3}x}+Be^{2x}-\frac{2}{7}+xe^{\frac{-7}{3}x}$$

Example 4.32

Suppose that the quantity demanded $Q_d = 29-2p-5\frac{dp}{dt}+\frac{d^2p}{dt^2}$ and quantity supplied $Q_s = 5+4p$ where p is the price. Find the equilibrium price for market clearance.

Solution:

For market clearance, the required condition is $Q_d = Q_s$

$$\Rightarrow 29-2p-5\frac{dp}{dt}+\frac{d^2p}{dt} = 5+4p$$

$$\Rightarrow 24-6p-5\frac{dp}{dt}+\frac{d^2p}{dt^2} = 0$$

$$\Rightarrow \frac{d^2p}{dt^2}-5\frac{dp}{dt}-6p = -24$$

$$(D^2-5D-6)p = -24$$

The auxiliary equation is

$$m^2-5m-6 = 0$$

$$(m-6)(m+1) = 0$$

$$\Rightarrow m = 6, -1$$

$$\text{C.F} = \text{A}e^{6t}+Be^{-t}$$

$$\text{P.I} = \frac{1}{\phi(D)}f(x)$$

$$= \frac{1}{D^2-5D-6}(-24)e^{0t}$$

$$= \frac{-24}{-6} \quad \text{(Replace D by 0)}$$

$$= 4$$

The general solution is $\text{p} = C.F+P.I$

$$= Ae^{6t}+Be^{-t}+4$$

Exercise 4.5

Solve the following differential equations

(1) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+8y=0$

(2) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$

(3) $\left(D^2+2D+3\right)y=0$

(4) $\dfrac{d^2y}{dx^2}-2k\dfrac{dy}{dx}+k^2y=0$

(5) $\left(D^2-2D-15\right)y=0$ given that $\dfrac{dy}{dx}=0$ and $\dfrac{d^2y}{dx^2}=2$ when $x=0$

(6) $\left(4D^2+4D-3\right)y=e^{2x}$

(7) $\dfrac{d^2y}{dx^2}+16y=0$

(8) $\left(D^2-3D+2\right)y=e^{3x}$ which shall vanish for $x=0$ and for $x=\log 2$

(9) $\left(D^2+D-6\right)y=e^{3x}+e^{-3x}$

(10) $\left(D^2-10D+25\right)y=4e^{5x}+5$

(11) $\left(4D^2+16D+15\right)y=4e^{\frac{-3}{2}x}$

(12) $\left(3D^2+D-14\right)y=13e^{2x}$

(13) Suppose that the quantity demanded $Q_d=13-6p+2\dfrac{dp}{dt}+\dfrac{d^2p}{dt^2}$ and quantity supplied $Q_s=-3+2p$ where p is the price. Find the equilibrium price for market clearance.

Exercise 4.6

Choose the Correct answer

1. The degree of the differential equation $\dfrac{d^4y}{dx^4}-\left(\dfrac{d^2y}{dx^2}\right)^4+\dfrac{dy}{dx}=3$

   (a) 1 (b) 2 (c) 3 (d) 4

2. The order and degree of the differential equation $\sqrt{\dfrac{d^2y}{dx^2}}=\sqrt{\dfrac{dy}{dx}+5}$ are respectively

   (a) 2 and 3 (b) 3 and 2

   (c) 2 and 1 (d) 2 and 2

3. The order and degree of the differential equation $\left(\dfrac{d^2y}{dx^2}\right)^{\frac{3}{2}}-\sqrt{\left(\dfrac{dy}{dx}\right)}-4=0$ are respectively.

   (a) 2 and 6 (b) 3 and 6

   (c) 1 and 4 (d) 2 and 4

4. The differential equation $\left(\dfrac{dx}{dy}\right)^3+2y^{\frac{1}{2}}=x$ is

   (a) of order 2 and degree 1

   (b) of order 1 and degree 3

   (c) of order 1 and degree 6

   (d) of order 1 and degree 2

5. The differential equation formed by eliminating a and b from $y=ae^x+be^{-x}$ is

   (a) $\dfrac{d^2y}{dx^2}-y=0$ (b) $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=0$

   (c) $\dfrac{d^2y}{dx^2}=0$ (d) $\dfrac{d^2y}{dx^2}-x=0$

6. If $y=cx+c-c^3$ then its differential equation is

   (a) $y=x\dfrac{dy}{dx}+\dfrac{dy}{dx}-\left(\dfrac{dy}{dx}\right)^3$

   (b) $y+\left(\dfrac{dy}{dx}\right)^3=x\dfrac{dy}{dx}-\dfrac{dy}{dx}$

   (c) $\dfrac{dy}{dx}+y=\left(\dfrac{dy}{dx}\right)^3-x\dfrac{dy}{dx}$

   (d) $\dfrac{d^3y}{dx^3}=0$

7. The integrating factor of the differential equation $\dfrac{dx}{dy}+Px=Q$ is

   (a) $e^{\int Pdx}$ (b) $\int Pdx$

   (c) $\int Pdy$ (d) $e^{\int Pdy}$

8. The complementary function of $\left(D^2+4\right)y=e^{2x}$ is

(a) $\left(Ax+B\right)e^{2x}$

(b) $\left(Ax+B\right)e^{-2x}$

(c) $A\cos 2x+B\sin 2x$

(d) $Ae^{-2x}+Be^{2x}$

9. The differential equation of $y=mx+c$ is (m and c are arbitrary constants)

(a) $\frac{d^2y}{dx^2}=o$ (b) $y=x\frac{dy}{dx}+c$

(c) $xdy+ydx=0$ (d) $ydx-xdy=0$

10. The particular integral of the differential equation is $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=2e^{4x}$

(a) $\frac{x^2e^{4x}}{2!}$ (b) $\frac{e^{4x}}{2!}$

(c) x^2e^{4x} (d) xe^{4x}

11. Solution of $\frac{dx}{dy}+Px=0$

(a) $x=ce^{py}$ (b) $x=ce^{-py}$

(c) $x=py+c$ (d) $x=cy$

12. If $\sec^2 x$ is an integrating factor of the differential equation $\frac{dy}{dx}+Py=Q$ then $P=$

(a) $2\tan x$ (b) $\sec x$

(c) $\cos^2 x$ (d) $\tan^2 x$

13. The integrating factor of $x\frac{dy}{dx}-y=x^2$ is

(a) $\frac{-1}{x}$ (b) $\frac{1}{x}$

(c) $\log x$ (d) x

14. The solution of the differential equation $\frac{dy}{dx}+Py=Q$ where P and Q are the function of x is

(a) $y=\int Qe^{\int Pdx}dx+c$

(b) $y=\int Qe^{-\int Pdx}dx+c$

(c) $ye^{\int Pdx}=\int Qe^{\int Pdx}dx+c$

(d) $ye^{\int Pdx}=\int Qe^{-\int Pdx}dx+C$

15. The differential equation formed by eliminating A and B from $y=e^{-2x}\left(A\cos x+B\sin x\right)$ is

(a) $y_2-4y_1+5=0$

(b) $y_2+4y-5=0$

(c) $y_2-4y_1-5=0$

(d) $y_2+4y_1+5=0$

16. The particular integral of the differential equation $f\left(D\right)y=e^{ax}$ where $f\left(D\right)=\left(D-a\right)^2$

(a) $\frac{x^2}{2}e^{ax}$ (b) xe^{ax}

(c) $\frac{x}{2}e^{ax}$ (d) x^2e^{ax}

17. The differential equation of $x^2+y^2=a^2$

(a) $xdy+ydx=0$ (b) $ydx-xdy=0$

(c) $xdx-ydx=0$ (d) $xdx+ydy=0$

18. The complementary function of $\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$ is

(a) $A+Be^x$ (b) $\left(A+B\right)e^x$

(c) $\left(Ax+B\right)e^x$ (d) Ae^x+B

19. The P.I of $(3D^2+D-14)y=13e^{2x}$ is

(a) $\frac{x}{2}e^{2x}$ (b) xe^{2x}

(c) $\frac{x^2}{2}e^{2x}$ (d) $13xe^{2x}$

20. The general solution of the differential equation $\frac{dy}{dx}=\cos x$ is

(a) $y=\sin x+1$ (b) $y=\sin x$ - 2

(c) $y=\cos x+c$, c is an arbitrary constant

(d) $y=\sin x+c$, c is an arbitrary constant

21. A homogeneous differential equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ can be solved by making substitution,

(a) $y = v\,x$ (b) $v = y\,x$

(c) $x = v\,y$ (d) $x = v$

22. A homogeneous differential equation of the form $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$ can be solved by making substitution,

(a) $x = v\,y$ (b) $y = v\,x$

(c) $y = v$ (d) $x = v$

23. The variable separable form of $\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)}$ by taking $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ is

(a) $\frac{2v^2}{1+v}dv = \frac{dx}{x}$

(b) $\frac{2v^2}{1+v}dv = -\frac{dx}{x}$

(c) $\frac{2v^2}{1-v}dv = \frac{dx}{x}$

(d) $\frac{1+v}{2v^2}dv = -\frac{dx}{x}$

24. Which of the following is the homogeneous differential equation?

(a) $(3x-5)\,dx = (4y-1)\,dy$

(b) $xy\,dx - (x^3 + y^3)\,dy = 0$

(c) $y^2\,dx + (x^2 - xy - y^2)\,dy = 0$

(d) $(x^2 + y)\,dx = (y^2 + x)\,dy$

25. The solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \frac{f\left(\frac{y}{x}\right)}{f'\left(\frac{y}{x}\right)}$ is

(a) $f\left(\frac{y}{x}\right) = k\,x$ (b) $x\,f\left(\frac{y}{x}\right) = k$

(c) $f\left(\frac{y}{x}\right) = k\,y$ (d) $y\,f\left(\frac{y}{x}\right) = k$

Miscellaneous Problems

1. Suppose that $Q_d = 30 - 5P + 2\frac{dP}{dt} + \frac{d^2P}{dt^2}$ and $Q_s = 6 + 3P$. Find the equilibrium price for market clearance.

2. Form the differential equation having for its general solution $y = ax^2 + bx$

3. Solve $yx^2 dx + e^{-x} dy = 0$

4. Solve $(x^2 + y^2)dx + 2xy\,dy = 0$

5. Solve $x\frac{dy}{dx} + 2y = x^4$

6. A manufacturing company has found that the cost C of operating and maintaining the equipment is related to the length 'm' of intervals between overhauls by the equation $m^2\frac{dC}{dm} + 2mC = 2$ and $c = 4$ and when $m = 2$. Find the relationship between C and m.

7. Solve $(D^2 - 3D + 2)y = e^{4x}$ given $y = 0$ when $x = 0$ and $x = 1$.

8. Solve $\frac{dy}{dx} + y\cos x + x = 2\cos x$

9. Solve $x^2 y dx - (x^3 + y^3)dy = 0$

10. Solve $\frac{dy}{dx} = xy + x + y + 1$

Summary

- A differential equation is an equation with a function and one or more of its derivatives. (i.e) an equation with the function $y = f(x)$ and its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$... is called differential equation.
- Order of the highest order derivative present in the differential equation is the order of the differential equation.
- Degree is the highest power of the highest order derivative in the differential equation, after the equation has been cleared from fractions and the radicals as for as the derivatives are concerned.
- A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.
- To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.
- In an equation it is possible to collect all the terns of x and dx on one side and all the terms of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is $f(x)dx = g(y)dy$ (or) $f(x)dx + g(y)dy = 0$ By direct integration, we get the solution.
- A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$ where $f(x, y)$ and $g(x, y)$ are homogeneous function of degree zero is called a homogeneous differential equation.
- A differential equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are constants or functions of x only is called a first order linear differential equation.
- A general second order linear differential equation with constant coefficients is of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

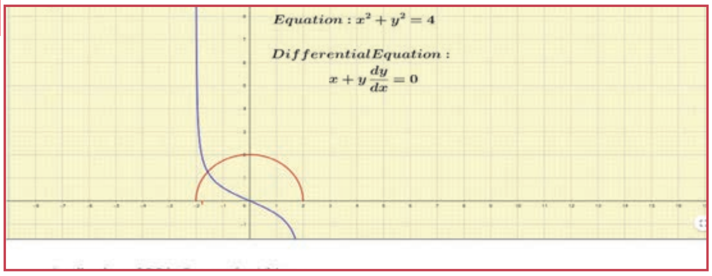
GLOSSARY (கலைச்சொற்கள்)

Abscissa	கிடை அச்சுத் தொலைவு
Arbitrary constant	மாறத்தக்க மாறிலி
Auxiliary equation	துணைச் சமன்பாடு
Complementary function	நிரப்புச் சார்பு
Constant	மாறிலி
Degree	படி
Differential equation	வகைக்கெழுச் சமன்பாடுகள்
Fixed constant	நிலையான மாறிலி

General solution	பொதுத் தீர்வு
Homogeneous equations	சமபடித்தான சமன்பாடுகள்
Linear differential equations	நேரிய வகைக்கெழுச் சமன்பாடு
Order	வரிசை
Ordinary differential equation	சாதாரண வகைக்கெழுச் சமன்பாடுகள்
Ordinate	குத்தாயம்
Partial differential equation	பகுதி வகைக்கெழுச் சமன்பாடுகள்
Particular integra	சிறப்புத் தீர்வு
Second order linear differential equations	இரண்டாம் வரிசை நேரிய வகைக்கெழுச் சமன்பாடுகள்
Variable separable	மாறிகள் பிரிக்கக் கூடியன
Variable	மாறி

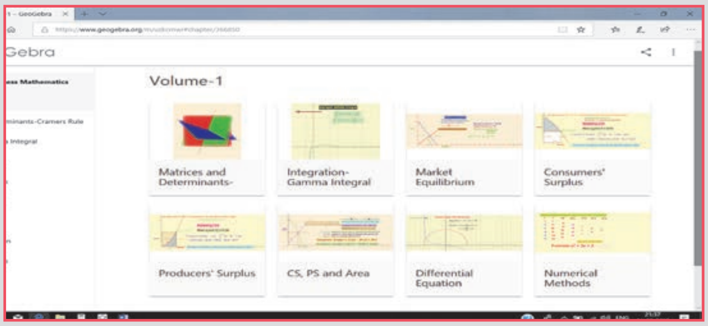
ICT Corner

Expected Result is shown in this picture



Step 1

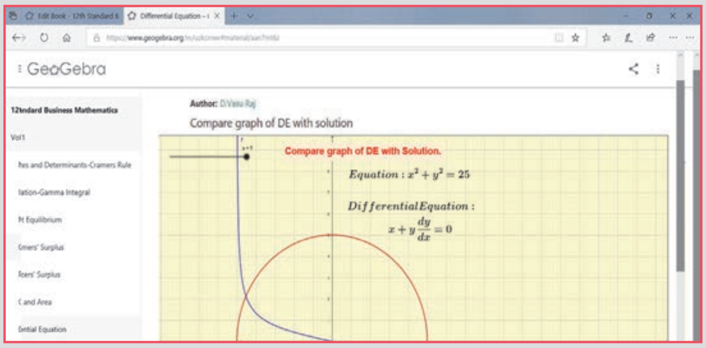
Open the Browser, type the URL Link given below (or) Scan the QR Code. GeoGebra work Book named "12^{th} Standard Business Mathematics and Statistics " will open. In the work book there are two Volumes. Select "Volume-1".



Step 2

Select the worksheet named" Differential Equation"

There is a video explanation and a graph in a single worksheet. Observe and learn the concepts.



Browse in the link

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