Integration

Integration -

The process of finding a function, given it's derivative, is called anti-differentiation (or) integration.

→ If $F'(x) = f(x)$, we say $F(x)$ is an anti-derivative of $f(x)$.

We write

$$\int f(x) = F(x) + C$$

ex:

$$\int x = \frac{x^2}{2} + c$$

Important formulas -

Derivatives	Integrals (Anti-Derivatives)
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(-\cot x) = \cancel{\sec}\ \operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + c$

Derivatives	Integrals
$\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
$\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + c$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
$\frac{d}{dx}(-\cot^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = -\cot^{-1} x + c$
$\frac{d}{dx} \log_e x = \frac{1}{x} + c$	$\int \frac{1}{x} dx = \log_e x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x \quad a>0, a\neq 1$	$\int a^x dx = \frac{a^x}{\log_e a} + c$
$\frac{d}{dx}(\log(\sin x)) = \cot x$	$\int \cot x \, dx = \log\lvert \sin x\rvert + c$
$\frac{d}{dx}(-\log(\cos x)) = \tan x$	$\int \tan x \, dx = -\log\lvert \cos x\rvert + c$
$\frac{d}{dx}(\log(\sec x + \tan x)) = \sec x$	$\int \sec x \, dx = \log\lvert \sec x + \tan x\rvert + c$
$\frac{d}{dx}(\log(\operatorname{cosec} x - \cot x)) = \operatorname{cosec} x$	$\int \operatorname{cosec} x \, dx = \log\lvert \operatorname{cosec} x - \cot x\rvert + c$
$\frac{d}{dx}\left(\sin^{-1} \frac{x}{a}\right) = \frac{1}{x\sqrt{a^2-x^2}}$	$\int \frac{1}{x\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c$

Derivatives	Integrals
$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\frac{x}{a} + c$
$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2+x^2}$	$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Ex-1

$$\int x^6\,dx = \frac{x^7}{7} + c$$

Ex-2

$$\int \frac{x^3-1}{x^2}\,dx$$

$$= \int x\,dx - \int \frac{1}{x^2}\,dx$$

$$= \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1} + c$$

$$= \frac{x^2}{2} + \frac{1}{x} + c$$

Ex-3 $\int \cos 2x\,dx$

$$\Rightarrow \frac{1}{2}\sin 2x + c$$

Ex-4

$$\int \operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\,dx.$$

$$= \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x)\,dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

• Methods of Integration –

1. Integration by Substitution –

$\int f(x)$ can be transformed into another form by changing Independent variable 'x' to 't' by substituting $x = g(t)$

consider,

$$I = \int f(x)\, dx$$

put $x = g(t)$

$$\frac{dx}{dt} = g'(t)$$

$$dx = g'(t)\, dt$$

$$\boxed{\therefore I = \int f(x)\, dx = \int f(g(t))\, g'(t)\, dt}.$$

ex: 5

$\int \tan x \, dx$.

$\Rightarrow \int \frac{\sin x}{\cos x}\, dx$ put, $\cos x = t$

$= \int \frac{-dt}{t}$ $\Rightarrow -\sin x\, dx = dt$

$= -\int \frac{1}{t}\, dt$

$= -\log_e t + c$

$= -\log|\cos x| + c = \log|\sec x| + c$.

(Q6) $\int \sin^3 x \cos^2 x \, dx$

$= \int \sin^2 x \cos^2 x \sin x \, dx$

$= \int (1 - \cos^2 x) \cos^2 x \, \cancel{dx} \sin x \, dx$

Let, $\cos x = t$

$-\sin x \, dx = dt$

$= -\int (1-t^2) t^2 \, dt$

$= -\int (t^2 - t^4) dt$

$= -\left(\frac{t^3}{3} - \frac{t^5}{5}\right)$

$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c$

2. Integration by parts:

This method is used to integrate the product of two function.

If $f(x)$ and $g(x)$ be two integrable functions then;

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int \left(f'(x) \int g(x) dx\right) dx$$

(Q7) $\int x \cdot e^x \, dx$

$\rightarrow x \int e^x dx - \int (1 \cdot \int e^x dx) dx$

$= xe^x - e^x$

$= e^x (x-1) + c$

(Q.8) $\int \frac{\log x}{x^2}\,dx$

$= \int \log x \cdot x^{-2}\,dx$

$= \log x \int x^{-2}\,dx - \int\left(\frac{1}{x}\cdot\int x^{-2}\,dx\right)dx$

$= \log x \cdot \frac{x^{-\cancel{3}}}{-\cancel{3}} - \int\left(\frac{1}{x}\cdot\frac{x^{-\cancel{4}}}{-\cancel{4}}\right)dx$

$\cancel{= -\frac{\log x}{3x^3} - \frac{1}{3}\left(\frac{x^{-5}}{-5}\right)}$

$\cancel{= -\frac{1}{3}\frac{\log x}{x^3} + \frac{1}{15}\cdot\frac{1}{x^5}}$

$= -\frac{\log x}{x} + \int x^{-2}\,dx$

$= -\frac{\log x}{x} + \frac{x^{-1}}{-1} + c$

$= -\frac{\log x}{x} - \frac{1}{x} + c$

$= -\frac{1}{x}(\log x + 1) + c$

(Q.9) $\int x\cos x\,dx$

$= x\int \cos x\,dx - \int\left(\int \cos x\,dx\right)dx$

$= x\cdot\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + c$

$=$

3. Intregration by partial fractions —
~~when~~ when we have the ratio of two polynomials in the form of $\frac{P(x)}{Q(x)}$, where $P(x), Q(x)$ are polynomials in x and $Q(x) \neq 0$

ex- $\int \frac{1}{x(x+1)} dx$.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$A = 1$
$B = -1$

$$= \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \log|x| - \log|x+1| + c.$$

Q- find, $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$.

put, $x^2 = y$

$$= \frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$$

$$= \int \left(\frac{-1/3}{(x^2+1)} + \frac{4/3}{x^2+4} \right) dx$$

$A(y+4) + B(y+1) = y$
$\Rightarrow y(B+A) + 4A + B = y$
$A + B = 1$ / $A = 1 - B$
$4A + B = 0$ $A = 1 + 4A$
$\Rightarrow B = -4A$ $A = -1/3$
$= 4/3$

$$= -\frac{1}{3}\int \frac{1}{1+x^2}\,dx + \frac{4}{3}\int \frac{1}{2^2+x^2}\,dx$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{\cancel{4}2}{3}\tan^{-1}\frac{x}{2} + c$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + c.$$

~~Q/-~~

- **Definite integrals:**

Theorem: Let $f(x)$ be continuous function in $[a,b]$ and $F(x)$ be the anti-derivative of $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ex
$$\int_0^{\pi/2} \cos x\,dx$$

$$= \left[\sin x\right]_0^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1 - 0$$

$$= 1$$

- **properties of Difinite Integrals –**

1. $\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx.$

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \boxed{\forall c \in (a,b)}$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

3. $\int_0^a f(x) = \int_0^a f(a-x)\,dx.$

4. $\int_a^b \frac{f(x)}{f(x)+f(a+b\pm x)} = \left(\frac{b-a}{2}\right).$

5. $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd.} \end{cases}$$

6. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x). \end{cases}$$

7. $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(a+x) = f(x)$

8. $\int_0^a x.f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$ if $(a-x) = f(x)$

*9. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3} \text{ or } \frac{1}{2}\right]K$$

where, $K = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$

10. $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

$$= \frac{[(m-1)(m-3)\cdots 2 \text{ or } 1]\,[(n-1)(n-3)\cdots 2 \text{ or } 1]}{(m+n)(m+n-2)\cdots 2 \text{ or } 1}K$$

where,

$m, n \in z^+$ and $K = \begin{cases} \pi/2 & \text{when } m, n \text{ are even} \\ 1 & \text{others} \end{cases}$

ex-1

$$\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\, dx. \qquad \int_a^b \frac{f(x)}{f(x) + f(a+b-x)} = \frac{b-a}{2}$$

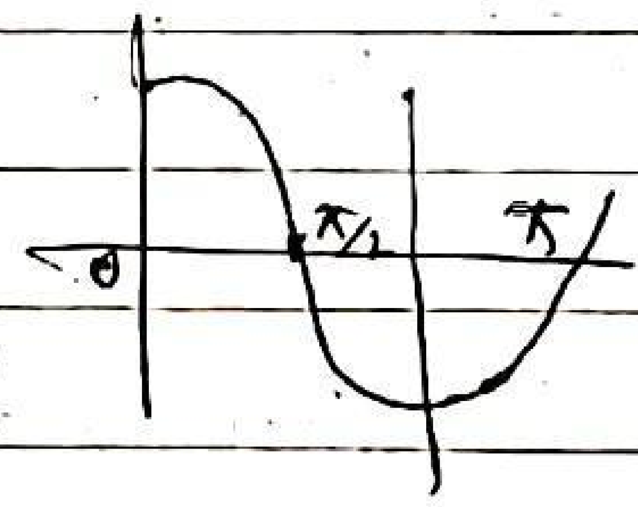
$$= \int_0^{\pi/2} \frac{\tan x}{\tan x + \tan(0 + \frac{\pi}{2} - x)}\, dx$$

$$= \frac{\pi/2 - 0}{2}$$

$$= \frac{\pi}{4}.$$

ex-2

$$\int_0^{\pi} |\cos x|\, dx.$$



$$= \int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx.$$

$$= \int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx.$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= (\sin \pi/2 - 0) - (\sin \pi - \sin \pi/2)$$

$$= 1 - (0 - 1)$$

$$= 2$$

ex-3 $\int_0^n [x]\,dx$

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \dots \int_{n-1}^n (n-1)\,dx.$

$= 0 + [x]_1^2 + 2[x]_2^3 + \dots + (n-1)[x]_{n-1}^n.$

$= (2-1) + 2.(3-2) + \dots + (n-1)(n-n+1).$

$= 0+1+2+3+\dots+(n-1).$

$= \cancel{\frac{n(n+1)}{2}}\ \frac{n(n-1)}{2}$

ex-4 $\int_0^1 x(1-x)^{99}\,dx. = \_\_\_\_.$

$\Rightarrow \int_0^1 x(1-x)^{99}\,dx.$

$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$

$= \int_0^1 (1-x)\,(1-(1-x))^{99}\,dx$

$= \int_0^1 (1-x).x^{99}\,dx.$

$= \int_0^1 (x^{99} - x^{100})\,dx$

$= \left.\frac{x^{100}}{100} - \frac{x^{101}}{101}\right]_0^1$

$= \left(\frac{1}{100} - \frac{1}{101}\right)$

$= \frac{1}{10100}$ Ans.

ex-4 $\int_0^{\pi/2} \log \tan x \, dx = \underline{0}$

$$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$

$$I = \int_0^{\pi/2} \log(\tan x)\, dx \quad \text{---} (1)$$

$$I = \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \log(\cot x)\, dx \quad \text{---} (2)$$

(1) + (2)

$$2I = \int_0^{\pi/2} \log(\tan x)\, dx + \int_0^{\pi/2} \log(\cot x)\, dx$$

$$= \int_0^{\pi/2} \log(\tan x \cdot \cot x)\, dx$$

$$2I = 0$$

$$\Rightarrow I = 0$$

ex-5 $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx.$

$0 \quad \pi/4$

Let, $x = \tan t$

$dx = \sec^2 t\,dt.$

$= \int_0^{\pi/4} \frac{\log(1+\tan t)}{1+\tan^2 t}\, \sec^2 t\,dt.$

$= \int_0^{\pi/4} \frac{\log(1+\tan t)}{\cancel{\sec^2 t}} \cdot \cancel{\sec^2 t}\,dt$

$= \int_0^{\pi/4} \log(1+\tan t)\,dt. = I$ —— (1)

$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt = I$

$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan t}{1+\tan t}\right)dt = I$

$\tan\left(\frac{\pi}{4}-t\right) = \frac{1-\tan t}{1+\tan t}$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan t}\right)dt = I$ —— (2)

(1) + (2) $\Rightarrow 2I = \int_0^{\pi/4} \log\left[(1+\cancel{\tan t}) \cdot \frac{2}{(1+\cancel{\tan t})}\right] dt.$

$= \int_0^{\pi/4} \log(2)\,dt.$

$2I = \log(2)\,[t]_0^{\pi/4}$

$= \log 2 \cdot \pi/4$

$I = \frac{\pi}{8}\log 2$

Q-6 if,

$$I_1 = \int_0^{\pi} f(\cos^2 x)\,dx.$$

$$I_2 = \int_0^{3\pi} f(\cos^2 x)\,dx$$ then which of the following is true?

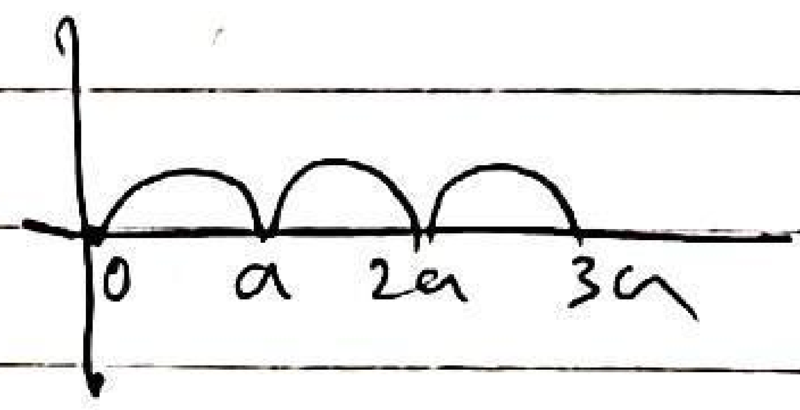
(a) $I_1 = I_2$ (b) $I_1 = 3I_2$

(c) $I_2 = 3I_1$ (d) None.

$\Rightarrow$ $$\boxed{\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx \quad \text{if } f(x+a) = f(a)}$$

$$f(x) = f(x+a)$$

$$\cos^2 x = \cos^2(\pi + 0)$$
$$= (-\cos x)^2$$
$$= \cos^2 x$$



$$I_2 = 3\int_0^{\pi} f(\cos^2 x)\,dx$$

$$\boxed{I_2 = 3I_1}$$

Q-7 $$\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx = \_\_\_\_$$

$$\boxed{\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}}$$

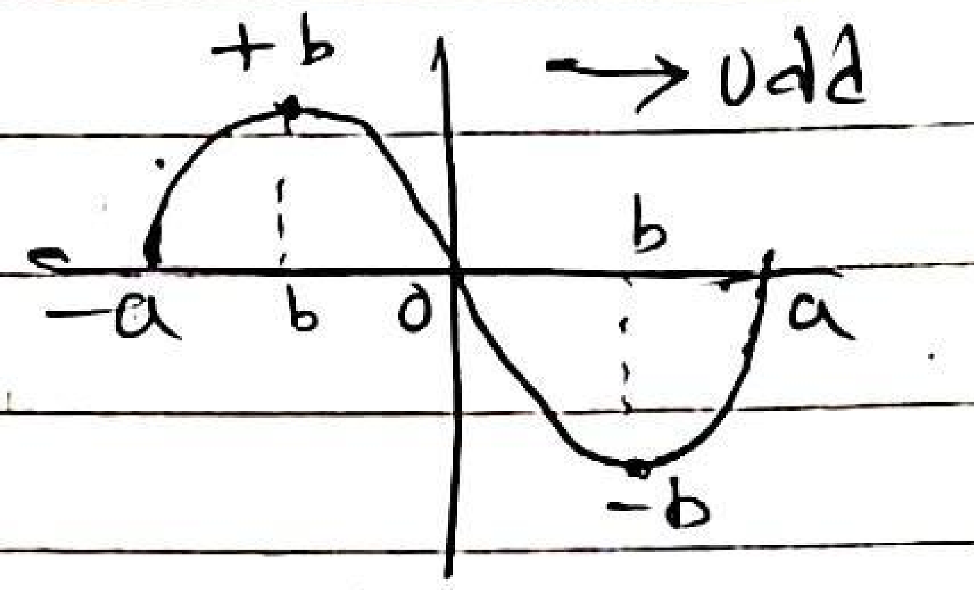
$$f(x) = \cos x \log \frac{1+x}{1-x}$$

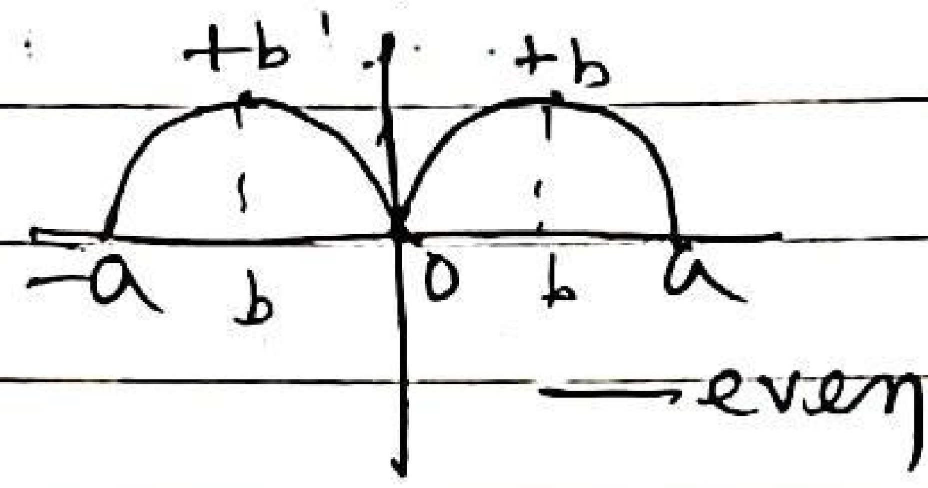
$$f(-x) = \cos(-x) \log \frac{1+(-x)}{1-(-x)}$$

$$= \cos x \log \frac{1-x}{1+x}$$

$$= \cos x \log\left(\frac{1+x}{1-x}\right)^{-1}$$

$$= -\cos x \log\left(\frac{1+x}{1-x}\right)$$





$f(x)$ is odd function.

$$\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx = 0.$$

Q-8 $\displaystyle\int_0^{\pi} \frac{x \cdot \sin x}{1+\cos^2 x}$ — $f(x)$

$$\boxed{\int_0^a x \cdot f(x)\, dx = \frac{a}{2} \int_0^a f(x)\, dx \quad \text{if } f(a-x) = f(x)}$$

$\Rightarrow$ $$f(x) = \frac{\cancel{x} \cdot \sin x}{1+\cos^2 x}$$

$$f(\pi - x) = \frac{\sin(\pi-x)}{1+\cos^2(\pi-x)} = \frac{\sin x}{1+\cos^2 x}$$

$$\boxed{f(x) = f(\pi - x)}$$

$$\Rightarrow \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

let, $\cos x = t$

$-\sin x\, dx = dt$

$$= \frac{\pi}{2} \int_{1}^{-1} -\frac{dt}{1+t^2}$$

$$= -\frac{\pi}{2} \left[\tan^{-1} t\right]_{1}^{-1}$$

$$= -\frac{\pi}{2} \left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= -\frac{\pi}{2} \left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$= -\frac{\pi}{2} \times -\frac{\pi}{2}$$

$$\Rightarrow \left(\frac{\pi}{2}\right)^2$$

**

Q-9

$$\boxed{\begin{aligned} \int_0^{\pi/2} \sin^n x\, dx &= \int_0^{\pi/2} \cos^n x\, dx \\ &= \left[\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \text{ (or) } \frac{1}{2}\right] K \\ \text{where, } K &= \begin{cases} \frac{\pi}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \end{aligned}}$$

Q-1 $\int_0^{\pi/2} \sin^8 x\, dx = \left[\frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2}\right] \times \frac{\pi}{2} = \frac{35\pi}{256}$

Q-2 $\int_0^{\pi/2} \sin^5 x\, dx = \left[\frac{4}{5} \cdot \frac{2}{3}\right] \cancel{\frac{\pi}{2}} \cdot 1$ $K = \text{odd}$

$$= \frac{8}{15}$$

Q-3 $\int_0^{\pi/2} \cos^3 x\, dx = \left[\frac{2}{3}\right] \cdot 1$

$$= \frac{2}{3}$$

Q4)
$$\int_0^{\pi/2} \cos^6 x\,dx = \left[\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\right]\cdot\frac{\pi}{2} = \frac{15}{48}\cdot\frac{\pi}{2}$$
$$= \frac{5\pi}{16\times 2} = \frac{5\pi}{32}$$

Q-10) **

$$\int_0^{\pi/2} \sin^{m} x \cos^{n} x\,dx$$ (m, n) should be positive int.

$$= \frac{\left[(m-1)(m-2)(m-3)\cdots 2(\text{or})1\right]\left[(n-1)\cdot(n-2)\cdots 2(\text{or})1\right]}{(m+n)(m+n-2)\cdots 2(\text{or})1}\cdot k$$

Where, $k = \begin{cases} \pi/2 & \text{when, } m, n \text{ is even.} \\ 1 & \text{otherwise} \end{cases}$

Q1)
$$\int_0^{\pi/2} \sin^8 x \cos^3 x\,dx = \frac{[7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1]\cdot[2\cdot1]}{11\cdot9\cdot7\cdot5\cdot3\cdot1}\cdot 1$$
$$= \frac{2}{99}$$

Q2)
$$\int_0^{\pi} \sin^4 x \cos^3 x\,dx$$
$$= 2\int_0^{\pi/2} \sin^4 x \cos^3 x\,dx \qquad \boxed{f(\pi-x) = -f(x)} \to \text{odd}$$
$$= 0.$$

Q3)
$$\int_{-2\pi}^{2\pi} \underbrace{\sin^4 x \cos^8 x}_{f(x)}\,dx \to \text{even fun.}$$

$\cos(\pi-x)$
$\cos(2\times 90-x)$

$$= 2\int_0^{2\pi} f(x)\,dx$$
$$= 4\int_0^{\pi} f(x)\,dx = 8\int_0^{\pi/2} f(x)\,dx$$
$$= 8\int_0^{\pi/2} \sin^4 x\cdot\cos^8 x\,dx$$
$$= \underline{(3\cdot2\cdot1)\,(7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1)}\cdot\frac{\pi}{2}$$

Test series examples -

1. $\lim_{n\to\infty} \sqrt{n^2-n}-\sqrt{n^2-1} \Rightarrow \lim_{n\to\infty} \frac{(\sqrt{n^2-n})^2-(\sqrt{n^2-1})^2}{\sqrt{n^2-n}+\sqrt{n^2-1}}$

$$\Rightarrow \lim_{n\to\infty} \frac{1-n}{\sqrt{n^2-n}+\sqrt{n^2-1}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\cancel{n}\left(\frac{1}{n}-1\right)}{\cancel{n}\left(\sqrt{1-\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}\right)}$$

$$\Rightarrow -\frac{1}{2}$$