CHAPTER

5

Hyperbola

HYPERBOLA: DEFINITION 1

The hyperbola is the set of all points in a plane, the difference of whose distance from two fixed points in the plane is a constant.

The term "*difference*" that means the distance to the farther point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The midpoint of the line segment joining the foci is called the centre of the hyperbola. The line through the foci is called the transverse axis and the line through the centre and perpendicular to the transverse axis is called the conjugate axis. The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola.

We denote the distance between the two foci by $2c$, the distance between the two vertices (the length of the transverse axis) by $2a$ and we define the quantity b as $b = \sqrt{c^2 - a^2}$.

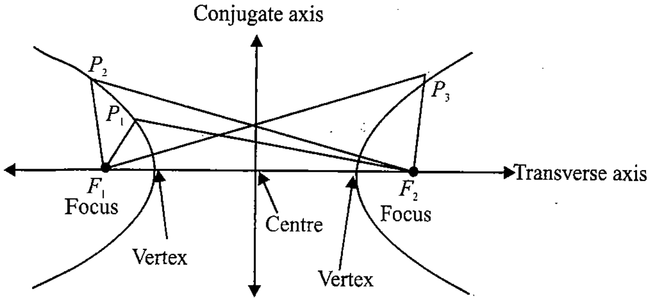


Fig. 5.1

$$P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

Standard Equation of Hyperbola

Let the foci of a hyperbola be $(\pm c, 0)$. Then its centre is $(0, 0)$.

According to the definition of hyperbola,

$$PF_1 - PF_2 = 2a \ (2a < 2c, \text{ i.e. } c > a)$$

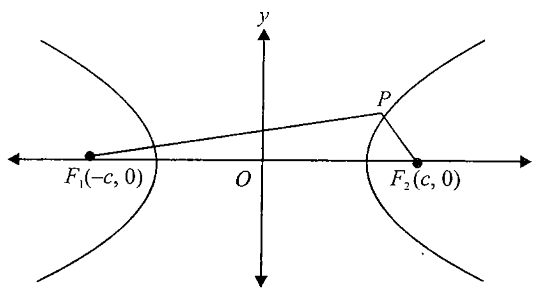


Fig. 5.2

$$\Rightarrow \quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

i.e. $$\sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Squaring both sides, we get

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

On simplifying, we get

$$\frac{cx}{a} - a = \sqrt{(x-c)^2 + y^2}$$

On squaring again and further simplifying, we get

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

i.e. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (\text{since } c^2 - a^2 = b^2)$$

Hence, any point on the hyperbola satisfies the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is same as the standard equation of ellipse except b^2 has been replaced by $-b^2$.

This is a second degree equation in which the powers of x and y are even, hence the curve is symmetric about both the axes. For each $x > a$ or $x < -a$ there are two values of y symmetrically situated on either side of x-axis and for each value of x lying in $(-a, a)$ the curve fails to exist. Hence, the curve denoted by Es. (i) consists of two symmetrical branches, each extending to infinity in two directions.

Eccentricity

$$e = \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

$$= \frac{c}{a}$$

$$e^2 = \frac{c^2}{a^2} = \frac{a^2 + b^2}{a^2} = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow \quad a^2e^2 = a^2 + b^2$$

Therefore, equation of hyperbola in terms of eccentricity can be written as

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

Coordinates of the foci are $(\pm ae, 0)$.

Two hyperbolas are said to be similar if they have the same value of eccentricity.

Rectangular or Equilateral Hyperbola

If $a = b$, hyperbola is said to be equilateral or rectangular and has the equation $x^2 - y^2 = a^2$.

Eccentricity for such a hyperbola is $\sqrt{2}$ as

$$e^2 = 1 + \frac{b^2}{a^2} = 1 + 1$$
$$= 2$$

Directrix

It is possible to define two lines, $x = \pm \frac{a}{e}$, corresponding to each focus, which satisfy the focal directrix property of the hyperbola, i.e. $PF_1 = e\,PM_1$ and $PF_2 = e\,PM_2$.

Hence, for any point P on the hyperbola,

$$\frac{PF_1}{PM_1} = e \text{ (constant)}$$

Focal Distance (Focal Radius)

The difference of the focal radii of any point on the hyperbola is equal to the length of the major axis. We have

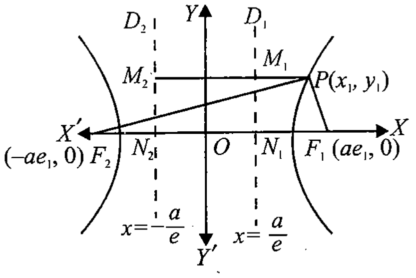


Fig. 5.3

$$PF_1 = ePM_1 = e\left(x_1 - \frac{a}{e}\right)$$
$$= ex_1 - a \quad \text{(i)}$$
$$PF_2 = e\left(x_1 + \frac{a}{e}\right)$$
$$= ex_1 + a \quad \text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get

$$PF_2 - PF_1 = 2a$$

Equation of Hyperbola Whose Axes are Parallel to Coordinate Axes and Centre is (*h*, *k*)

Equation of such hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(x-k)^2}{b^2} = 1 \quad (a > b)$$

Foci: $(h \pm ae, k)$

Directrix: $x = h \pm \frac{a}{e}$

Definition and Basic Terminology

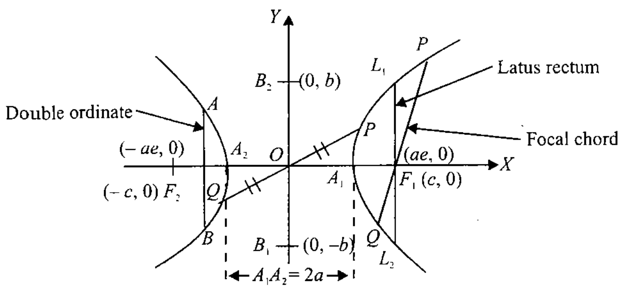


Fig. 5.4

- Line containing the fixed points F_1 and F_2 (called foci) is called transverse axis (TA) or focal axis and the distance between F_1 and F_2 is called focal length.
- The points of intersection A_1 and A_2 of the curve with the transverse axis are called vertices of the hyperbola. The length '$2a$' between the vertices is called the length of transverse axis (TA).
- The points B_1 $(0, b)$ and $B_2(0, -b)$ which have special significance, are known as the extremities of conjugate axis and the length '$2b$' is called the length of conjugate axis. The point of intersection of these two axes is called the centre 'O' of the hyperbola. (Transverse axis and conjugate axis together are called the principal axes).
- Any chord passing through centre is called diameter (PQ) and is bisected by it.
- Any chord passing through focus is called a focal chord and any chord perpendicular to the transverse axis is called a double ordinate (AB).
- A particular double ordinates which passes through focus and perpendicular to focal axis is called the latus rectum (L_1L_2)

Latus-Rectum Length

The two foci are $(\pm ae, 0)$

We have the equation of the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Putting $x = ae$, we get

$$\Rightarrow \quad \frac{y^2}{b^2} = e^2 - 1 = \left(\frac{b^2}{a^2} + 1\right) - 1$$
$$= \frac{b^2}{a^2}$$
$$\Rightarrow \quad y = \pm \frac{b^2}{a}$$

Hence, coordinate of the extremities of LR

$$= \left(\pm ae, \pm b^2/a\right)$$

Length of LR $= \frac{2b^2}{a}$

$$= \frac{4b^2}{2a} = \frac{(\text{minor axis})^2}{\text{major axis}}$$

Also $L_1L_2 = 2a(e^2 - 1)$

$= 2e\left(ae - \frac{a}{e}\right)$

$= 2e(OF_1 - OA_1)$ (as shown in Fig. 5.4)

$= 2e \times$ (distance between focus and corresponding foot of the directrix)

POSITION OF A POINT (h, k) WITH RESPECT TO A HYPERBOLA

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative as the point (x_1, y_1) lies within, upon or outside the curve.

Explanation

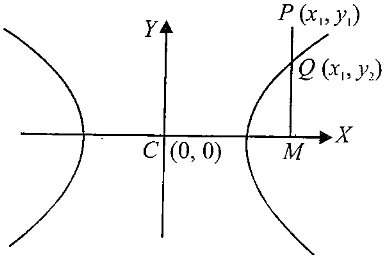


Fig. 5.5

If (x_1, y_1) lies on the hyperbola, then

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$\Rightarrow \quad \left(\frac{x_1^2}{a^2} - 1\right) b^2 = y_1^2$$

Now if P lies outside the curve, then

$$y_1^2 > y_2^2$$

$$\Rightarrow \quad y_1^2 > \left(\frac{x_1^2}{a^2} - 1\right) b^2$$

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$$

CONJUGATE HYPERBOLA

Corresponding to every hyperbola there exists a hyperbola such that the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of the other. Such hyperbolas are known as conjugate to each other.

Hence, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(i)}$$

The conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad \text{(ii)}$$

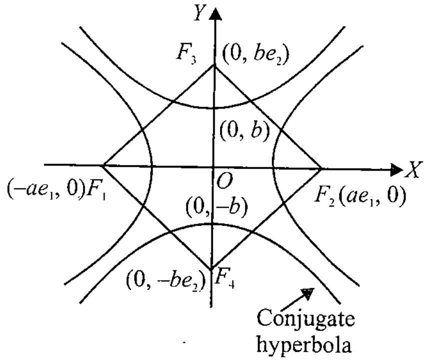


Fig. 5.6

Notes

- ***If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, respectively, then $e_1^{-2} + e_2^{-2} = 1$.***

  Proof:

  For hyperbola, $\quad e_1^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$

  And for conjugate hyperbola,

  $$e_2^2 = 1 + \frac{a^2}{b^2} = \frac{a^2 + b^2}{b^2}$$

  $$\therefore \quad \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

- ***The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.***

  Proof:

  All the four sides of the quadrilateral $F_1F_3F_2F_4$ are obviously equal to their diagonals at right angles.

  Hence it is a rhombus.

  Now to prove that $F_1F_3F_2F_4$ is a square it is sufficient to prove that

  $$ae_1 = be_2$$

  or $\quad a^2e_1^2 = b^2e_2^2 = a^2(e_1^2 - 1)e_2^2$

  or $\quad e_1^2 = e_1^2e_2^2 - e_2^2$

  or $\quad e_1^2 + e_2^2 = e_1^2e_2^2$

  or $\quad \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ *which is true*

  Hence $\quad ae_1 = be_2$

  $\Rightarrow$ ***$F_1F_3F_2F_4$ is a square.***

AUXILIARY CIRCLE AND ECCENTRIC ANGLE

Definition

A circle drawn with centre C and transverse axis as a diameter is called the *auxiliary circle* of the hyperbola. Hence, equation of the auxiliary circle is $x^2 + y^2 = a^2$.

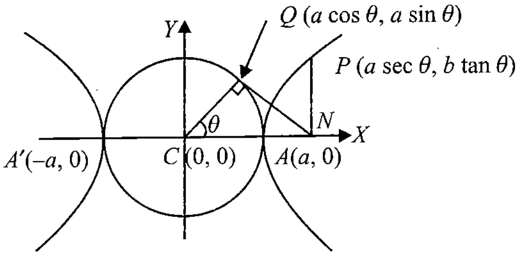


Fig. 5.7

Note: ***From Fig. 5.7, P and Q are called the "corresponding points" on the hyperbola and the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola*** $(0 \le \theta < 2\pi)$.

The equations $x = a\sec\theta$ and $y = b\tan\theta$ together represent the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where θ is a parameter.

θ	**$Q(a\cos\theta, a\sin\theta)$**	**$P(a\sec\theta, b\tan\theta)$**
$\theta \in \left[0, \frac{\pi}{2}\right)$	I	I
$\theta \in \left(\frac{\pi}{2}, \pi\right]$	II	III
$\theta \in \left[\pi, \frac{3\pi}{2}\right)$	III	II
$\theta \in \left(\frac{3\pi}{2}, 2\pi\right]$	IV	IV

The parametric form of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be represented as $x = a\sec\theta$, $y = b\tan\theta$.

For hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, parametric form is

$$x = h + a\sec\theta$$

$$y = k + b\tan\theta$$

COMPARISON OF HYPERBOLA AND ITS CONJUGATE HYPERBOLA

Fundamentals	**Hyperbola**	**Conjugate Hyperbola**
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric coordinates	$(a\sec\theta, b\tan\theta)$,	$(b\sec\theta, a\tan\theta)$,
Focal radii of point $P(x_1, y_1)$	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii $(S'P - SP)$	$2a$	$2b$
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

Example 5.1 **Find the equation of hyperbola**

a. whose axes are coordinate axes and the distances of one of its vertices from the foci are 3 and 1;

b. whose centre is (1, 0), focus is (6, 0) and transverse axis is 6;

c. whose centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2);

d. whose centre is (–3, 2), one vertex is (–3, 4) and eccentricity is $\frac{5}{2}$;

e. whose foci are (4, 2) and (8, 2) and eccentricity is 2.

Sol.

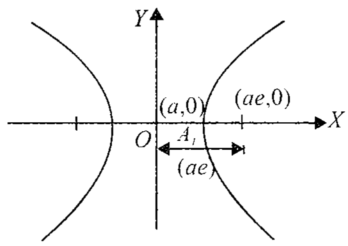


Fig. 5.8

$$ae - a = 1 \text{ and } ae + a = 3$$

$\therefore$ $\frac{e+1}{e-1} = 3$

$\Rightarrow$ $e = 2$ and $a = 1$

Also from $b^2 = a^2(e^2 - 1) = 3$, the equation is

$$x^2 - \frac{y^2}{3} = 1$$

or $$\frac{x^2}{3} - \frac{y^2}{1} = -1$$

b. Equation of hyperbola with centre (1, 0) is

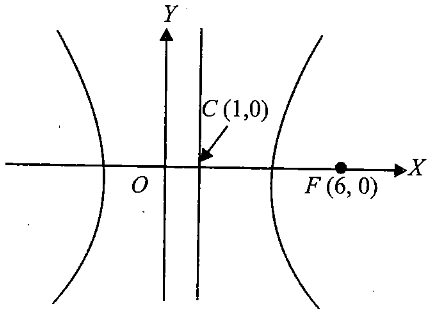


Fig. 5.9

$$\frac{(x-1)^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 \quad \text{(i)}$$

Given $a = 3$ and $ae = 5$; hence $e = \frac{5}{3}$.

Therefore, Eq. (i) becomes

$$\frac{(x-1)^2}{9} - \frac{y^2}{9\left(\frac{25}{9}-1\right)} = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$$

c. Equation of hyperbola with centre (3, 2) is

$$\frac{(x-3)^2}{a^2} - \frac{(y-2)^2}{a^2(e^2-1)} = 1, \text{ with axis parallel to } x\text{-axis}$$

a = distance between centre and vertex = 1

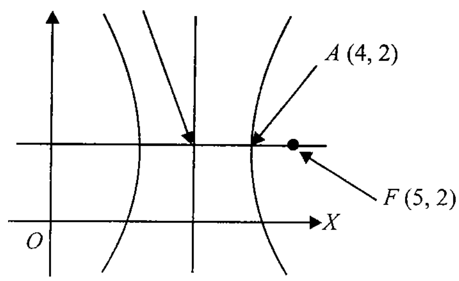


Fig. 5.10

ae = distance between centre and focus.

Hence, $ae = 2 \Rightarrow e = 2$

Hence, the equation is

$$\frac{(x-3)^2}{1} - \frac{(y-2)^2}{(4-1)} = 1$$

$$\Rightarrow \quad (x-3)^2 - \frac{(y-2)^2}{3} = 1$$

d. Equation of the hyperbola is $\frac{(x+3)^2}{a^2} - \frac{(y-2)^2}{b^2} = -1$ (as the line joining centre to the vertex is parallel to y-axis)

Now $b = 2$ (distance between centre and vertex)

Focus = $be = 2 \times \frac{5}{2} = 5$

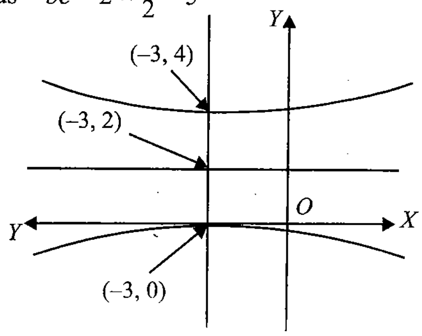


Fig. 5.11

Also, $$a^2 = b^2(e^2 - 1)$$
$$= 4\left(\frac{25}{4} - 1\right)$$
$$= 21$$

Therefore, the required equation is

$$\frac{(x+3)^2}{21} - \frac{(y-2)^2}{4} = -1$$

e. Line joining the foci is parallel to x-axis.

Distance between the two foci = 4 = $2ae$

Hence, $a = 1$ as $e = 2$

$\therefore$ $$b^2 = a^2(e^2 - 1)$$
$$= 3$$

Now centre is midpoint of the line joining the foci which is (6, 2).

Hence, equation of the hyperbola is

$$\frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1$$

Example 5.2 **If base of triangle and ratio of tangents of half of base angles are given, then identify the locus of opposite vertex.**

Sol. In ΔABC, base $BC = a$ is given

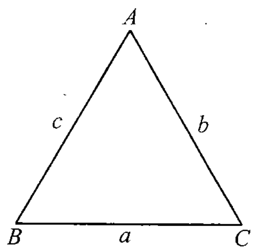


Fig. 5.12

Also, $$\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} = k \text{ (constant)}$$

$$\Rightarrow \quad \frac{\sqrt{\dfrac{s(s-b)}{(s-a)(s-c)}}}{\sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}} = k$$

$$\Rightarrow \quad \frac{s-b}{s-c} = k$$

$$\Rightarrow \quad \frac{(s-b)-(s-c)}{(s-b)+(s-c)} = \frac{k-1}{k+1}$$

$$\Rightarrow \quad \frac{c-b}{2s-(b+c)} = \frac{k-1}{k+1}$$

$$\Rightarrow \quad c-b = a\frac{k-1}{k+1} = \text{constant}$$

$$\Rightarrow \quad AB - AC = \text{constant}$$

So, locus of A is a hyperbola.

Example 5.3 **Two circles are given such that they neither intersect nor touch. Then identify the locus of centre of variable circle which touches both the circles externally.**

Sol.

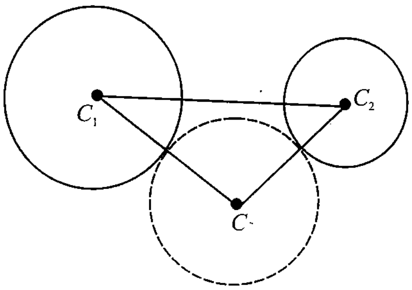


Fig. 5.13

In the figure circles with solid line have centres C_1 and C_2 and radii r_1 and r_2.

Let the circle with dotted line is variable circle which touches given two circles as explained in the question which has centre C and radius r.

Now, $\quad CC_2 = r + r_2$

and $\quad CC_1 = r_1 + r$

Hence, $\quad CC_1 - CC_2 = r_1 - r_2$ (= constant)

Then locus of C is hyperbola whose focii are C_1 and C_2.

Example 5.4 **Two rods are rotating about two fixed points in opposite directions. If they start from their position of co-incidence and one rotates at the rate double that of the other, then find the locus of point of intersection of the two rods.**

Sol. Suppose two rods are co-incident on the x-axis. One rotates about point O and the other about point $A(a, 0)$.

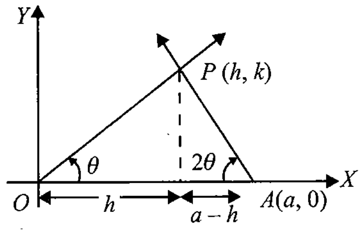


Fig. 5.14

If they rotate according to question, then at some time t, they are in the position as shown in the figure.

From the figure $\tan\theta = \frac{k}{h}$ and $\tan 2\theta = \frac{k}{a-h}$

$$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} = \frac{k}{a-h}$$

$$\Rightarrow \quad \frac{\frac{2k}{h}}{1-\frac{k^2}{h^2}} = \frac{k}{a-h}$$

$$\Rightarrow \quad \frac{2hk}{h^2-k^2} = \frac{k}{a-h}$$

$$\Rightarrow \quad 2h(a-h) = h^2 - k^2$$

$$\Rightarrow \quad 2ah - 2h^2 = h^2 - k^2$$

$$\Rightarrow \quad 3x^2 - y^2 - 2ax = 0$$

$\Rightarrow$ Locus is hyperbola.

Example 5.5 **Find the vertices of the hyperbola $9x^2 - 16y^2 - 36x + 96y - 252 = 0$.**

Sol. The equation can be rewritten as

$$9(x^2 - 4x + 4 - 4) - 16(y^2 - 6y + 9 - 9) = 252$$

$$9(x-2)^2 - 16(y-3)^2 = 252 + 36 - 144 = 144$$

$$\Rightarrow \quad \frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

or $$\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$$

Hence, vertices are $\quad X = \pm A,\ Y = 0$

$$\Rightarrow \quad (x-2) = \pm 4,\ y - 3 = 0$$

$$\Rightarrow \quad x = 6, -2 \text{ and } y = 3$$

$\Rightarrow$ Vertices are (6, 3), (–2, 3)

Example 5.6 **Find the coordinates of foci, the eccentricity and latus – rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.**

Sol. Equation can be rewritten as

$$\frac{(x-4)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1$$

So $\quad a = 4, b = 3$

$$b^2 = a^2(e^2 - 1) \text{ gives } e = \frac{5}{4}.$$

Foci: $x - 4 = \pm ae,$

$y - 3 = 0$

give the foci as (9, 3), (–1, 3)

Centre: $x - 4 = 0, y - 3 = 0,$

i.e. centre is (4, 3)

Directrices: $x - 4 = \pm\frac{a}{e},$

i.e. $x - 4 = \pm\frac{16}{5}$

Hence, directrices are $5x - 36 = 0$; $5x - 4 = 0$.

Latus rectum: $\frac{2b^2}{a} = 2 \times \frac{9}{4} = \frac{9}{2}$

Example 5.7 **If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then find the value of b^2.**

Sol. For hyperbola,

$$e^2 = 1 + \frac{b^2}{a^2}$$

$$= 1 + \frac{81}{144} = \frac{225}{144}$$

$\therefore$ $e = \frac{15}{12} = \frac{5}{4}$

Also, $a^2 = \frac{144}{25}$

Hence, the foci are $(\pm ae, 0) \equiv \left(\pm\frac{12}{5} \times \frac{5}{4}, 0\right) \equiv (\pm 3, 0)$

Now for ellipse $ae = 3$ or $a^2e^2 = 9$

Now $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $b^2 = a^2 - a^2e^2$

$= 16 - 9 = 7$

Example 5.8 **If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then find range of the eccentricity e of the hyperbola.**

Sol.

Let double ordinate PQ be such that $P \equiv (a \sec\theta, b \tan\theta)$, and $Q \equiv (a\sec\theta, -b\tan\theta)$ and O is centre (0, 0).

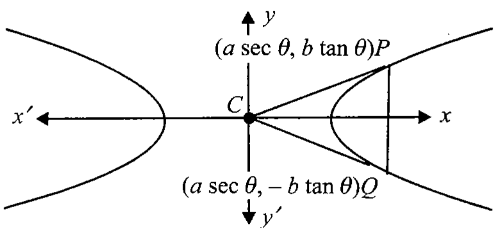


Fig. 5.15

ΔOPQ is equilateral

$\Rightarrow$ $\tan 30° = \frac{b\tan\theta}{a\sec\theta}$

$\Rightarrow$ $3\frac{b^2}{a^2} = \text{cosec}^2\theta$

$\Rightarrow$ $3(e^2 - 1) = \text{cosec}^2\theta$

Now, $\text{cosec}^2\theta \ge 1$

$\Rightarrow$ $3(e^2 - 1) \ge 1$

$\Rightarrow$ $e^2 \ge \frac{4}{3}$

$\Rightarrow$ $e > \frac{2}{\sqrt{3}}$

Example 5.9 **If the latus rectum subtends a right angle at the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then find its eccentricity.**

Sol.

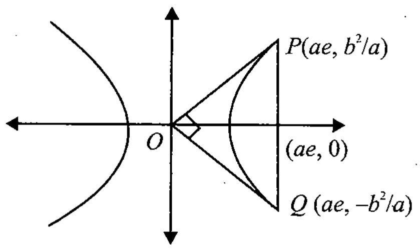


Fig. 5.16

$$m_{OP} \times m_{OQ} = -1$$

$\Rightarrow$ $\left(\frac{\frac{b^2}{a}}{ae}\right) \times -\left(\frac{\frac{b^2}{a}}{ae}\right) = -1$

$\Rightarrow$ $\frac{b^4}{a^4e^2} = 1$

$\Rightarrow$ $(e^2 - 1)^2 = e^2$

$\Rightarrow$ $e^4 - 3e^2 + 1 = 0$

$\Rightarrow$ $e^2 = \frac{3 \pm \sqrt{5}}{2}$

$\Rightarrow$ $e^2 = \frac{3 + \sqrt{5}}{2}$

$\Rightarrow$ $e = \frac{\sqrt{5} + 1}{2}$

Concept Application Exercise 5.1

1. If the latus rectum of a hyperbola forms an equilateral triangle with the vertex at the centre of the hyperbola, then find the eccentricity of the hyperbola.
2. The distance between the two directrix of a rectangular hyperbola is 10 units, then find the distance between its foci.
3. An ellipse and hyperbola are confocal (have the same focus) and the conjugate axis of the hyperbola is

equal to the minor axis of the ellipse. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola, then prove that $\frac{1}{e_1^2}+\frac{1}{e_2^2}=2$.

4. Two straight lines pass through the fixed points $(\pm a, 0)$ and have slopes whose product is $p > 0$. Show that the locus of the points of intersection of the lines is a hyperbola.
5. Show that the locus represented by $x = \frac{1}{2}a\left(t+\frac{1}{t}\right)$, $y=\frac{1}{2}a\left(t-\frac{1}{t}\right)$ is a rectangular hyperbola.
6. Find the lengths of transverse axis and conjugate axis, eccentricity, the coordinates of foci, vertices, lengths of the latus rectum and equations of the directrices of the following hyperbola: $16x^2 - 9y^2 = -144$.
7. If AOB and COD are two straight lines which bisect one another at right angles, show that the locus of a point P which moves so that $PA \times PB = PC \times PD$ is a hyperbola. Find its eccentricity.
8. If S and S' be the foci, C the centre and P be a point on a rectangular hyperbola, show that $SP \times S'P = (CP)^2$.
9. Find the equation of the hyperbola whose foci are (8, 3), (0, 3) and eccentricity $= \frac{4}{3}$.

HYPERBOLA: DEFINITION 2

We can also define hyperbola w.r.t. one fixed point and a fixed line. Hyperbola is the locus of a point which moves in a plane such that ratio of its distances from a fixed point (i.e. focus) and the fixed line (i.e. directrix) is constant and more than 1. This ratio is called eccentricity and is denoted by e. For a hyperbola $e > 1$.

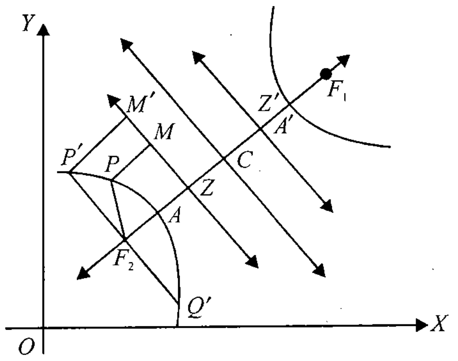


Fig. 5.17

From the diagram, for any point P on the curve, we have by definition,

$$\frac{F_2P}{PM} = e$$

or $F_2P = e\,PM$ (focal length or focal radius of point P)

Also, A and A' divide F_2Z in the ratio e:1 internally and externally, respectively.

If the focus F_2 has coordinates (α, β) and equation of directrix ZM is $lx + my + n = 0$, then equation of hyperbola is

$$\sqrt{(x-\alpha)^2+(y-\beta)^2} = \frac{e\,|lx+my+n|}{\sqrt{l^2+m^2}}$$

which is of the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

and $\quad h^2 > ab.$

From the diagram length of latus rectum

$= P'Q'$

$= 2F_2P'$

$= 2(e \times P'M')$

$= 2(e \times F_2Z)$

$= 2e \times$ (distance of focus from corresponding directrix)

Note:

If $\phi(x, y) = 0$ is the equation of hyperbola and $\frac{\partial\phi}{\partial x}$ denotes the differential coefficient of ϕ w.r.t. x keeping y as constant and likewise $\frac{\partial\phi}{\partial y}$, then centre C is the solution of **$\frac{\partial\phi}{\partial x} = 0$, $\frac{\partial\phi}{\partial y} = 0$. (This is valid for all conic.)**

Example 5.10 **Find the equation of the hyperbola whose directrix is $x + 2y = 1$, focus is (2, 1) and eccentricity is 2.**

Sol. Let $P(x, y)$ be any point on the hyperbola, then by definition

$$SP = e \times PM$$

where S is focus and M is foot of $\perp$ from P on directrix.

$\Rightarrow \quad SP^2 = e^2 PM^2$

$\Rightarrow \quad (x-2)^2 + (y-1)^2 = 4\left(\frac{x+2y-1}{\sqrt{5}}\right)^2$

$\Rightarrow \quad x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$

This is the required equation of the hyperbola.

Example 5.11 **The equation of one of the directrices of a hyperbola is $2x + y = 1$, the corresponding focus is (1, 2) and $e = \sqrt{3}$. Find the equation of the hyperbola and coordinates of the centre and second focus.**

Sol. Let S be the focus and PM be perpendicular distance of a point $P(x, y)$ from the directrix, then

$$PS = e\,PM \text{ gives}$$

$$(x-1)^2+(y-2)^2=3\left[\frac{(2x+y-1)}{\sqrt{5}}\right]^2$$

$$\Rightarrow 5[x^2+y^2-2x-4y+5]$$

$$=3[4x^2+y^2+4xy-4x-2y+1]$$

$$\Rightarrow \quad 7x^2+12xy-2y^2-2x+14y-22=0 \qquad \text{(i)}$$

Equation of the perpendicular from S to directrix, i.e. of SZ is

$$x-2y=-3 \qquad \text{(ii)}$$

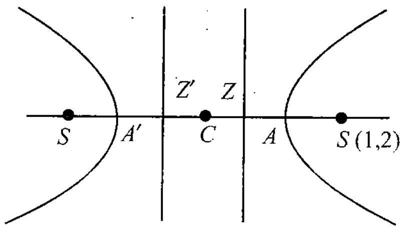


Fig. 5.18

Solving Eq. (ii) and $2x+y-1=0$, we get

$$Z=\left(-\frac{1}{5},\frac{7}{5}\right)$$

Since A and A' divide SZ in the ratio $\sqrt{3}:1$ internally and externally, we get

$$A=\left(\frac{5-\sqrt{3}}{5(\sqrt{3}+1)},\frac{10+7\sqrt{3}}{5(\sqrt{3}+1)}\right)$$

and $$A'=\left(-\frac{5+\sqrt{3}}{5(\sqrt{3}-1)},\frac{7\sqrt{3}-10}{5(\sqrt{3}-1)}\right)$$

Since C is midpoint of AA', we get after simplification,

$$C=\left(-\frac{4}{5},\frac{11}{10}\right)$$

Now, if S' is the second focus, C is midpoint of SS' and $S'=(x_1,y_1)$, then

$$\frac{x_1+1}{2}=-\frac{4}{5}$$

and $$\frac{y_1+2}{2}=\frac{11}{10}.$$

So, $$S'=\left(-\frac{13}{5},\frac{1}{5}\right)$$

Example 5.12 ***OA*, *OB* are fixed straight lines, *P* is any point and *PM*, *PN* are the perpendiculars from *P* on *OA*, *OB*. Find the locus of *P* if the quadrilateral *OMPN* is of constant area.**

Sol.

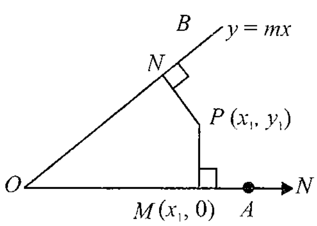


Fig. 5.19

Taking OA as x-axis, O as origin, let the equation of OB be $y=mx$.

Then $$M=(x_1,0)$$

Equation of the perpendicular PN is

$$my+x=my_1+x_1$$

Solving the equations OB and PN, we get

$$N=\left(\frac{my_1+x_1}{1+m^2},\frac{m(my_1+x_1)}{1+m^2}\right)$$

$\therefore$ Area of the quadrilateral $OMPN$ (by stair method)

$$=\frac{1}{2}\left|\begin{array}{cc} 0 & 0 \\ x_1 & 0 \\ x_1 & y_1 \\ \dfrac{my_1+x_1}{1+m^2} & \dfrac{m(my_1+x_1)}{1+m^2} \\ 0 & 0 \end{array}\right|$$

$$=\frac{1}{2}x_1y_1+x_1\frac{m(my_1+x_1)}{1+m^2}$$

$$-y_1\frac{my_1+x_1}{1+m^2}=\pm k \text{ (say)}$$

Therefore, locus of P is

$$mx^2+2m^2xy-my^2\pm 2(1+m^2)k=0$$

Here $h=m^2$,

$a=m$,

$b=-m$,

$f=g=0$

and $c=\pm 2(1+m^2)k$

$\therefore$ $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$=\pm 2m^2(1+m^2)\,k\pm 2m^4(1+m^2)k^2\neq 0$

and $h^2>ab$

So the locus is a hyperbola.

Equation of a Hyperbola Referred to Two Perpendicular Lines

Consider the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as shown in figure.

Let $P(x,y)$ be any point on the hyperbola. Then, $PM=y$ and $PN=x$.

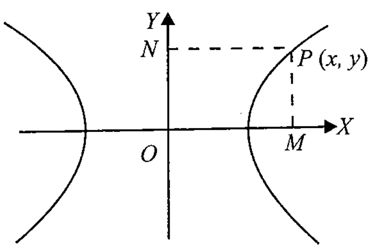


Fig. 5.20

$$\therefore \quad \frac{PN^2}{a^2} - \frac{PM^2}{b^2} = 1$$

It follows from this that if perpendicular distance p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv b_1x - a_1y + c_2 = 0$, respectively, satisfy the equation

$$\frac{p_1^2}{a^2} - \frac{p_2^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} - \frac{\left(\frac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1$$

then the locus of point P denotes an hyperbola in the plane of the given lines such that

i. the centre of the hyperbola is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$;

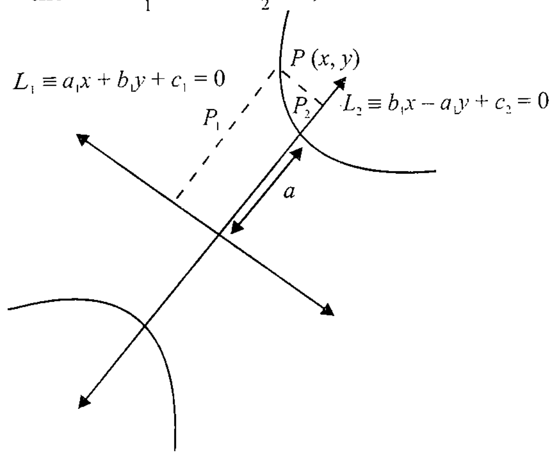


Fig. 5.21

ii. the transverse axis lies along $L_2 = 0$ and the conjugate axis lies along $L_1 = 0$;

iii. the length of the transverse and conjugate axes are $2a$ and $2b$, if $a > b$.

Example 5.13 **Find the coordinates of the foci and the centre of the hyperbola**

$$\frac{(3x - 4y - 12)^2}{100} - \frac{(4x + 3y - 12)^2}{225} = 1$$

Sol. Let $3x - 4y - 12 = X$ and $4x + 3y - 12 = Y$.

$\frac{X^2}{10^2} - \frac{Y^2}{15^2} = 1$ (note that X and Y are two perpendicular lines)

Centre is the point of intersection of $X = 0$ and $Y = 0$.

Hence, $\quad 3x - 4y = 12 \quad$ (i)

$\quad 4x + 3y = 12 \quad$ (ii)

Solving (i) and (ii), we get $x = \frac{84}{25}, y = -\frac{12}{25}$

Now, $\quad e^2 = 1 + \frac{225}{100} = \frac{325}{100}$

$\Rightarrow \quad e = \frac{\sqrt{13}}{2}$

Also, $\quad a = 10; b = 15.$

Focus is $(ae, 0)$.

Hence, $\quad X = ae$

and $\quad Y = 0$

$\Rightarrow \quad 3x - 4y - 12 = ae$

and $\quad 4x + 3y - 12 = 0$

Solving, we get $\quad x = \frac{84 + 3ae}{25} = \frac{84 + 15\sqrt{13}}{25}$

and $\quad y = \frac{-12 + 15\sqrt{13}}{25}$

Hence, focus is $\left(\frac{84 + 15\sqrt{3}}{25}, \frac{-12 + 15\sqrt{13}}{25}\right)$.

Example 5.14 **Find the eccentricity of the conic $4(2y - x - 3)^2 - 9(2x + y - 1)^2 = 80$.**

Sol. Here $2y - x - 3 = 0$ and $2x + y - 1 = 0$ are perpendicular to each other.

Therefore, the equation of the conic can be written as

$$4 \times 5\left[\frac{2y - x - 3}{\sqrt{2^2 + 1^2}}\right]^2 - 9 \times 5\left[\frac{2x + y - 1}{\sqrt{2^2 + 1^2}}\right]^2 = 80$$

$$\Rightarrow \quad 4\left[\frac{2y - x - 3}{\sqrt{5}}\right]^2 - 9\left[\frac{2x + y - 1}{\sqrt{5}}\right]^2 = 16$$

On putting $\frac{2y - x - 3}{\sqrt{5}} = X$ and $\frac{2x + y - 1}{\sqrt{5}} = Y$, the given equation can be written as

$$4X^2 - 9Y^2 = 16$$

$$\Rightarrow \quad \frac{X^2}{4} - \frac{Y^2}{(4/3)^2} = 1$$

The eccentricity is given by

$$e = \sqrt{1 + \frac{4/3}{4}}$$

$$= \frac{2}{\sqrt{3}}$$

Intersection of a Line and Hyperbola

$$y = mx + c \quad \text{(i)}$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(ii)}$$

Solving (i) and (ii), we get

$$b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$$\Rightarrow (b^2 - a^2m^2)x^2 - 2a^2 mc\, x - a^2(b^2 + c^2) = 0 \quad \text{(iii)}$$

$D > 0$ Secant | $D = 0$ Tangent $\Downarrow$ $c^2 = a^2m^2 - b^2$ | $D < 0$ Neither tangent nor secant

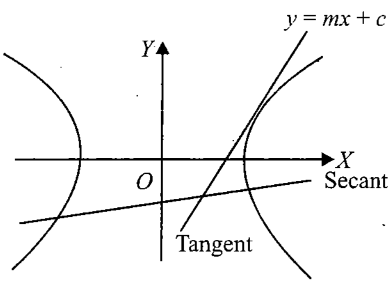


Fig. 5.22

Hence, $y = mx \pm \sqrt{a^2m^2 - b^2}$ is a tangent to the standard hyperbola.

In the above equation $a^2m^2 - b^2 \geq 0 \Rightarrow m^2 \geq \frac{b^2}{a^2}$

$$\Rightarrow m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right).$$

Hence, for given m, there can be two parallel tangents to the hyperbola.

If tangents passes through (h, k) then

$$(k - mh)^2 = a^2m^2 - b^2$$

$$(h^2 - a^2)m^2 - 2kmh + k^2 + b^2 = 0 \quad \text{(iv)}$$

Hence, passing through a given point (h, k) there can be a maximum of two tangents.

Now, $$m_1 + m_2 = \frac{2kh}{h^2 - a^2} \quad \text{(v)}$$

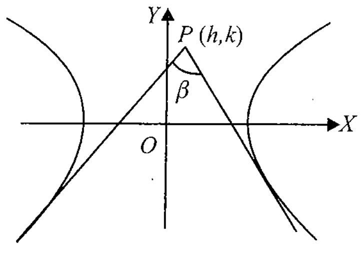


Fig. 5.23

$$m_1m_2 = \frac{k^2 + b^2}{h^2 - a^2} \quad \text{(vi)}$$

β is the angle enclosed by the tangents as shown in the figure.

Now, $$\tan^2\beta = \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$$

(substituting the value of $m_1 + m_2$ and m_1m_2 to get the locus of P)

If $\beta = 90°$ then $m_1m_2 = -1$

hence from (vi), we get

$$k^2 + b^2 = a^2 - h^2$$

$x^2 + y^2 = a^2 - b^2$ which is the equation to the *director circle* of the hyperbola.

If $a > b$, director circle is real with finite radius.

If $a = b$, director circle is a point circle which is origin.

If $a < b$, no real circle or no such point on the plane.

Note:

For hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, ***equation of tangent at point*** $P(x_1, y_1)$ **is**

$$y - y_1 = m(x - x_1) \pm \sqrt{a^2m^2 - b^2}$$

Equation of Tangent to the Hyperbola at Point (x_1, y_1)

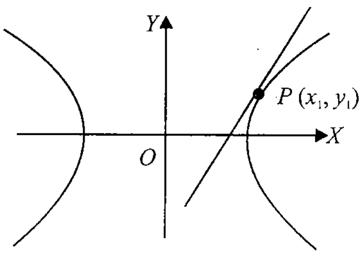


Fig. 5.24

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t. x, we have

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b^2x_1}{a^2y_1}$$

Hence, equation of tangent is $y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1)$

or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

But (x_1, y_1) lies on the hyperbola. So,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

Hence, equation of tangent is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \text{(i)}$$

or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$ or $T = 0$

where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Note:

Equation of tangent at point P (x_1, y_1) to the hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)}{b^2} = 1$ is $\frac{(x-h)(x_1-h)}{a^2} - \frac{(y-k)(y_1-k)}{b^2} = 1$

Equation of Tangent at Point ($a \sec\theta$, $b \tan\theta$)

Putting $x_1 = a\sec\theta$ and $y_1 = b\tan\theta$ in (i), we have

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \quad \text{(ii)}$$

Point of Contact When Line $y = mx + c$ Touches the Hyperbola

Line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, when $c = \pm\sqrt{a^2m^2 - b^2}$.

Comparing lines $y = mx \pm \sqrt{a^2m^2 - b^2}$ with $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$ we have

$$\frac{\frac{x_1}{a^2}}{m} = \frac{\frac{-y_1}{b^2}}{-1} = \frac{1}{\mp\sqrt{a^2m^2 - b^2}}$$

$$\Rightarrow \quad (x_1, y_1) \equiv \left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

or $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$, where $c = \sqrt{a^2m^2 - b^2}$

Point of Intersection of Tangent at Point $P(\alpha)$ and $Q(\beta)$

If point of intersection is $R(x_1, y_1)$, then

$$x_1 = a\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$$

and
$$y_1 = b\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$$

Example 5.15 **For all real values of m the straight line $y = mx + \sqrt{9m^2 - 4}$ is a tangent to which of the following certain hyperbolas:**

a. $9x^2 + 4y^2 = 36$ **b.** $4x^2 + 9y^2 = 36$
c. $9x^2 - 4y^2 = 36$ **d.** $4x^2 - 9y^2 = 36$

Sol. Comparing $y = mx + \sqrt{9m^2 - 4}$ with $y = mx + \sqrt{a^2m^2 - b^2}$, we have

$$a^2 = 9 \text{ and } b^2 = 4$$

Therefore, the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

or $4x^2 - 9y^2 = 36$

Example 5.16 **Find the equations of tangents to the curve $4x^2 - 9y^2 = 1$ which is parallel to $4y = 5x + 7$.**

Sol. Let m be the slope of the tangent to $4x^2 - 9y^2 = 1$.

Then, m = (slope of the line $4y = 5x + 7$)

$$= \frac{5}{4}$$

We have, $\frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$

where $a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$

The equations of the tangents are $y = mx \pm \sqrt{a^2m^2 - b^2}$

or $y = \frac{5}{4}x \pm \sqrt{\frac{25}{64} - \frac{1}{9}}$.

or $30x - 24y \pm \sqrt{161} = 0$

or $24y - 30x = \pm\sqrt{161}$

Example 5.17 **If the line $5x + 12y = 9$ touches the hyperbola $x^2 - 9y^2 = 9$, then find its point of contact.**

Sol. Solving line $5x + 12y = 9$ or $y = \frac{9 - 5x}{12}$ and $x^2 - 9y^2 = 9$, we have

$$x^2 - 9\left(\frac{9-5x}{12}\right)^2 = 9$$

$\Rightarrow \quad x^2 - \frac{1}{16}(9-5x)^2 = 9$

$\Rightarrow 16x^2 - (25x^2 - 90x + 81) = 144$

$\Rightarrow \quad 9x^2 - 90x + 225 = 0$

$\Rightarrow \quad x^2 - 10x + 25 = 0$

$\Rightarrow \quad x = 5$

$\Rightarrow \quad y = \frac{9-25}{12} = -\frac{4}{3}$

Example 5.18 **Find the equation of the tangent to the conic $x^2 - y^2 - 8x + 2y + 11 = 0$ at (2, 1).**

Sol. The equation of the tangent to $x^2 - y^2 - 8x + 2y + 11 = 0$ at (2, 1) is

$$2x - y - 4(x + 2) + (y + 1) + 11 = 0$$

$\Rightarrow \quad x = 2$

Example 5.19 **Find the value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$.**

Sol. If $y = mx + c$ touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$.

Here $\quad c = 6, a^2 = 100, b^2 = 49$

$\therefore \quad 36 = 100m^2 - 49$

$\Rightarrow \quad 100m^2 = 85$

$\Rightarrow \quad m = \sqrt{\frac{17}{20}}$

Example 5.20 ***P* is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *N* is the foot of the perpendicular from *P* on the transverse axis. The tangent to the hyperbola at *P* meets the transverse axis at *T*. If *O* is the centre of the hyperbola, then find the value of $OT \times ON$.**

Sol. Let $P(x_1, y_1)$ be a point on the hyperbola.

Then the coordinates of N are $(x_1, 0)$.

The equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

This meets x-axis at $T\left(\frac{a^2}{x_1}, 0\right)$

$\therefore \quad OT \,.\, ON = \frac{a^2}{x_1} x_1 = a^2$

Example 5.21 **On which curve does the perpendicular tangents drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ intersect ?**

Sol. The locus of the point of intersection of prependicular tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the director circle given by

$$x^2 + y^2 = a^2 - b^2.$$

Hence, the perpendicular tangents drawn to $\frac{x^2}{25} - \frac{y^2}{16} = 1$ intersect on the curve $x^2 + y^2 = 25 - 16$,

i.e. $\quad x^2 + y^2 = 9.$

Example 5.22 **Tangents drawn from the point (c, d) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ make angles α and β with the x-axis. If $\tan\alpha \tan\beta = 1$, then find the value of $c^2 - d^2$.**

Sol. Any tangent to hyperbola is $y = mx + \sqrt{a^2m^2 - b^2}$. It passes through (c, d), so

$$d = mc + \sqrt{a^2m^2 - b^2}$$

$\Rightarrow \quad (d - mc)^2 = a^2m^2 - b^2$

$\Rightarrow (c^2 - a^2)m^2 - 2cdm + d^2 + b^2 = 0$

$\Rightarrow \quad$ product of roots $= m_1m_2 = \frac{d^2 + b^2}{c^2 - a^2}$

$\Rightarrow \quad \tan\alpha \tan\beta = \frac{d^2 + b^2}{c^2 - a^2} = 1$

$\Rightarrow \quad d^2 + b^2 = c^2 - a^2$

$\Rightarrow \quad c^2 - d^2 = a^2 + b^2$

Example 5.23 **Tangents are drawn from the points on a tangent of the hyperbola $x^2 - y^2 = a^2$ to the parabola $y^2 = 4ax$. If all the chords of contact pass through a fixed point Q, prove that the locus of the point Q for different tangents on the hyperbola is an ellipse.**

Sol. Tangent at a point $(a \sec\theta, a \tan\theta)$ on the hyperbola $x^2 - y^2 = a^2$ is

$$x \sec\theta - y \tan\theta = a \qquad \text{(i)}$$

Any point on (i) will be of the form $\left(t, \frac{t \sec\theta - a}{\tan\theta}\right)$

Equation of chord of contact of the point w.r.t. parabola $y^2 = 4ax$ will be

$$y\left(\frac{t\sec\theta - a}{\tan\theta}\right) - 2a(x + t) = 0$$

$$\Rightarrow \left(\frac{-ay}{\tan\theta} - 2ax\right) + t\left(\frac{y\sec\theta}{\tan\theta} - 2a\right) = 0 \qquad \text{(ii)}$$

(ii) represents a family of straight lines each member of which passes through the point of intersection of straight lines

$-\frac{ay}{\tan\theta} - 2ax = 0$ and $y\frac{\sec\theta}{\tan\theta} - 2a = 0$

$\Rightarrow \quad y = 2a\sin\theta, x = -a\cos\theta$

So the point Q is $(-a\cos\theta, 2a\sin\theta)$

Let $\quad \alpha = -a\cos\theta, \beta = 2a\sin\theta$

$\Rightarrow \frac{\alpha^2}{a^2} + \frac{\beta^2}{4a^2} = 1$. So locus of Q is $\frac{x^2}{a^2} + \frac{y^2}{4a^2} = 1$, which is an ellipse.

Example 5.24 **Find the equations to the common tangents to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.**

Sol. Tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y=m_1x\pm\sqrt{a^2m_1^2-b^2}$$

The other hyperbola is

$$\frac{x^2}{(-b^2)}-\frac{y^2}{(-a^2)}=1$$

Any tangent to this hyperbola is given by

$$y=m_2x\pm\sqrt{(-b^2)m_2^2-(-a^2)} \quad \text{(ii)}$$

If (i) and (ii) are same, then $m_1=m_2$ and

$$\Rightarrow \quad a^2m_1^2-b^2=a^2-b^2m_1^2$$

$$\Rightarrow \quad (a^2+b^2)m_1^2=a^2+b^2$$

$$\Rightarrow \quad m_1^2=1$$

$$\Rightarrow \quad m_1=\pm1$$

Equation of Pair of Tangents from Point (x_1, y_1)

Combined equation for pair of tangents PQ and PR is given by $T^2=SS_1$

where $$T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1,$$

$$S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1$$

and $$S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

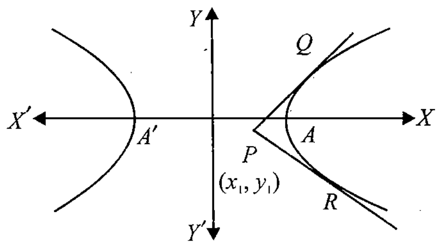


Fig. 5.25

Example 5.25 **How many real tangents can be drawn from the point (4, 3) to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$? Find the equation of these tangents and angle between them.**

Sol. Given point $P\equiv(4, 3)$

$$\text{Hyperbola } S\equiv\frac{x^2}{16}-\frac{y^2}{9}-1$$

$$\because \quad S_1\equiv\frac{16}{16}-\frac{9}{9}-1=-1<0$$

$\Rightarrow$ Point $P\equiv(4, 3)$ lies outside the hyperbola

Hence, two tangents can be drawn from the point P (4, 3).

Equation of pair of tangents is

$$SS_1=T^2$$

$$\Rightarrow\left(\frac{x^2}{16}-\frac{y^2}{9}-1\right)(-1)=\left(\frac{4x}{16}-\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \quad -\frac{x^2}{16}+\frac{y^2}{9}+1=\frac{x^2}{16}+\frac{y^2}{9}+1-\frac{xy}{6}-\frac{x}{2}+\frac{2y}{3}$$

$$\Rightarrow \quad \frac{x^2}{8}-\frac{xy}{6}-\frac{x}{2}+\frac{2y}{3}=0$$

$$\Rightarrow 3x^2-4xy-12x+16y=0$$

$$\theta=\tan^{-1}\left(\frac{4}{3}\right)$$

Concept Application Exercise 5.2

1. From the centre C of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ perpendicular CN is drawn on any tangent to it at the point P ($a\sec\theta$, $b\tan\theta$) in the first quadrant. Find the value of θ so that area of ΔCPN is maximum.
2. Find the common tangent to $9x^2-16y^2=144$ and $x^2+y^2=9$.
3. Find the equation of the tangent to the curve $4x^2-9y^2=1$ which is parallel to $4y=5x+7$.
4. Find the locus of a point P (α, β) moving under the condition that the line $y=\alpha x+\beta$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
5. A point P moves such that the chord of contact of the pair of tangents from P on the parabola $y^2=4ax$ touches the rectangular hyperbola $x^2-y^2=c^2$. Show that the locus of P is the ellipse $\frac{x^2}{c^2}+\frac{y^2}{(2a)^2}=1$.
6. PN is the ordinate of any point P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and AA' is its transverse axis. If Q divides AP in the ratio $a^2:b^2$, then prove that NQ is perpendicular to $A'P$.
7. C is the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

   The tangents at any point P on this hyperbola meet the straight lines $bx-ay=0$ and $bx+ay=0$ at points Q are R, respectively, Then prove that $CQ\cdot CR=a^2+b^2$.

Equation of Normal to the Hyperbola at Point (x_1, y_1)

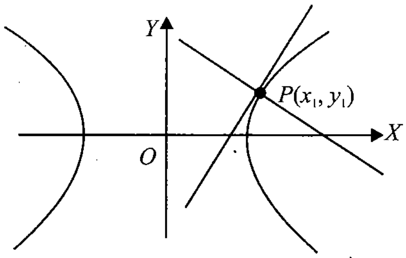


Fig. 5.26

Slope of normal at point (x_1, y_1) is $-\dfrac{a^2y_1}{b^2x_1}$.

Hence, equation of normal is

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

or $$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \quad \text{(i)}$$

Normal at Point P (a sec θ, b tan θ)

Putting $x_1 = a\sec\theta$ and $y_1 = b\tan\theta$ in (i), we get

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \quad \text{(ii)}$$

Note:

- ***Normal other than transverse axis never passes through the focus.***
- ***Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle, i.e. $x^2 + y^2 = a^2$.***
- ***The product of the feet of these perpendiculars is b^2 (semi-conjugate axis)^2.***
- ***The portion of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.***
- ***The tangent and normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as "an incoming light ray" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. If follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common points.***

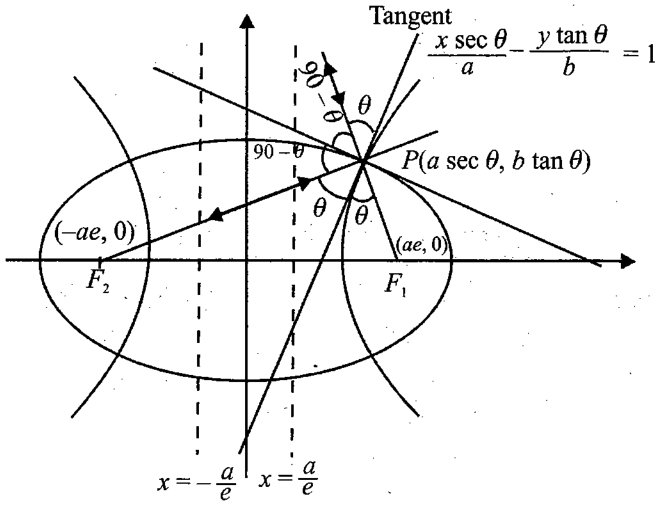


Fig. 5.27

Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ ($a > k > b > 0$) are confocal and therefore orthogonal.

- ***The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.***

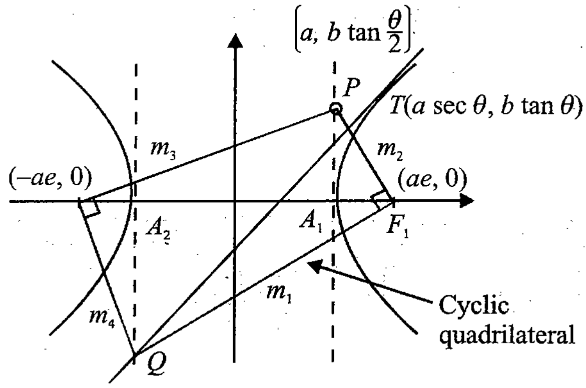


Fig. 5.28

Example 5.26 **If the normal at P (θ) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{2a^2} = 1$ meets the transverse axis at G, then prove that $AG \times A'G = a^2(e^4\sec^2\theta - 1)$ (where A and A' are the vertices of the hyperbola).**

Sol. 4. The equation of the normal at P ($a\sec\theta$, $b\tan\theta$) to the given hyperbola is $ax\cos\theta + by\cot\theta = (a^2 + b^2)$.

This meets the transverse axis, i.e. x-axis at G.

So, the coordinates of G are $\left(\left(\frac{a^2+b^2}{a}\right)\sec\theta, 0\right)$

The coordinates of the vertices A and A' are $(a, 0)$ and $(-a, 0)$, respectively.

$$\therefore \quad AG \cdot A'G = \left(-a + \frac{a^2+b^2}{a}\sec\theta\right)\left(a + \frac{a^2+b^2}{a}\sec\theta\right)$$

$$= (-a + ae^2\sec\theta)(a + ae^2\sec\theta) = a^2(e^4\sec^2\theta - 1)$$

Example 5.27 **Normals are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at points θ_1 and θ_2 meeting the conjugate axis at G_1 and G_2 respectively. If $\theta_1 + \theta_2 = \frac{\pi}{2}$, prove that $CG_1 \cdot CG_2 = \frac{a^2e^4}{e^2-1}$ where C is the centre of the hyperbola and e is its eccentricity.**

Sol. Normal at point $P(a\sec\theta_1, b\tan\theta_1)$ is

$ax\cos\theta_1 + by\cot\theta_1 = (a^2+b^2)$.

It meets the conjugate axis at $G_1\left(0, \frac{a^2+b^2}{b}\tan\theta_1\right)$.

Normal at point $Q(a\sec\theta_2, b\tan\theta_2)$ is

$ax\cos\theta_2 + by\cot\theta_2 = (a^2+b^2)$.

It meets the conjugate axis at $G_2\left(0, \frac{a^2+b^2}{b}\tan\theta_2\right)$

$$\Rightarrow \quad CG_1 \cdot CG_2 = \frac{(a^2+b^2)^2}{b^2}\tan\theta_1\tan\theta_2$$

$$= \frac{(a^2+b^2)^2}{b^2} \quad \left(\because \theta_1 + \theta_2 = \frac{\pi}{2}\right)$$

$$= \frac{a^4\left(1+\frac{b^2}{a^2}\right)^2}{b^2}$$

$$= \frac{a^2e^4}{e^2-1}$$

Example 5.28 **Normal is drawn at one of the extremities of the latus rectum of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which meets the axis at points A and B. Then find the area of triangle OAB (O being the origin).**

Sol. Normal at point $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

It meets the axes at $A\left(\frac{(a^2+b^2)x_1}{a^2}, 0\right)$ and $B\left(0, \frac{(a^2+b^2)y_1}{b^2}\right)$

$$\text{Area of } \Delta OAB = \frac{1}{2}\left[\frac{(a^2+b^2)x_1}{a^2}\right]\left[\frac{(a^2+b^2)y_1}{b^2}\right]$$

$$= \frac{1}{2}\left[\frac{(a^2+b^2)^2x_1y_1}{a^2b^2}\right]$$

Now normal is drawn at the extremity of latus rectum.

$$\text{Hence,} \quad (x_1, y_1) \equiv \left(ae, \frac{b^2}{a}\right)$$

$$\Rightarrow \quad \text{Area} = \frac{1}{2}\left[\frac{(a^2+b^2)^2b^2e}{a^2b^2}\right]$$

$$= \frac{1}{2}\left[\frac{a^4\left(1+\frac{b^2}{a^2}\right)^2e}{a^2}\right]$$

$$= \frac{1}{2}a^2e^5$$

Example 5.29 **A ray emanating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection if point P lies in the first quadrant.**

Sol. Given hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \text{(i)}$$

Now, x coordinate of point P is 8. Let y coordinate of P is α.

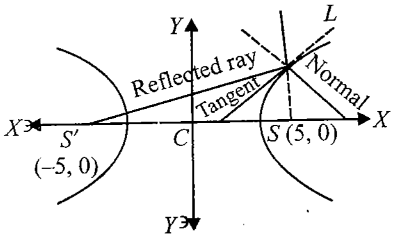


Fig. 5.29

As $(8, \alpha)$ lies on (i),

$$\therefore \quad \frac{64}{16} - \frac{\alpha^2}{9} = 1$$

$$\Rightarrow \quad \alpha^2 = 27$$

$$\Rightarrow \quad \alpha = 3\sqrt{3} \quad (\because P \text{ lies in first quadrant})$$

Hence, coordinates of point P are $(8, 3\sqrt{3})$.

The equation of reflected ray passes through $P(8, 3\sqrt{3})$ and $S'(-5, 0)$.

Therefore, its equation is

$$y - 0 = \frac{0 - 3\sqrt{3}}{-5-8}(x+5)$$

or $\quad 3\sqrt{3}x - 13y + 15\sqrt{3} = 0$

Equation of Chord Joining Points $P(\alpha)$ and $Q(\beta)$

Equation of chord passing through the points P (a sec α, b tan α) and Q (a sec β, b tan β) is given by

$$\begin{vmatrix} x & y & 1 \\ a\sec\alpha & b\tan\alpha & 1 \\ a\sec\beta & b\tan\beta & 1 \end{vmatrix} = 0.$$

$\Rightarrow$ $bx(\tan\alpha - \tan\beta) - ay(\sec\alpha - \sec\beta) + ab(\sec\alpha\tan\beta - \sec\beta\tan\alpha) = 0$

$\Rightarrow$ $bx\sin(\alpha-\beta) - ay(\cos\beta - \cos\alpha) + ab(\sin\beta - \sin\alpha) = 0$

$$\Rightarrow bx\cos\left(\frac{\alpha-\beta}{2}\right) - ay\sin\left(\frac{\alpha+\beta}{2}\right) - ab\cos\left(\frac{\alpha+\beta}{2}\right) = 0$$

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

Example 5.30 **If (a sec θ, b tan θ) and (a sec ϕ, b tan ϕ) are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that $\tan\frac{x}{a}\tan\frac{\phi}{2} = \frac{1-e}{1+e}$.**

Sol. The equation of the chord joining (a sec θ, b tan θ) and (a sec ϕ, b tan ϕ) is

$$\frac{x}{a}\cos\frac{\theta-\phi}{2} - \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta+\phi}{2}$$

This passes through (ae, 0)

$$\Rightarrow e\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow e = \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\Rightarrow \frac{e-1}{e+1} = \frac{\cos\left(\frac{\theta+\phi}{2}\right) - \cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right) + \cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\Rightarrow \frac{e-1}{e+1} = -\tan\frac{\theta}{2}\tan\frac{\phi}{2}$$

$$\Rightarrow \tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{1-e}{1+e}$$

Chord of Contact

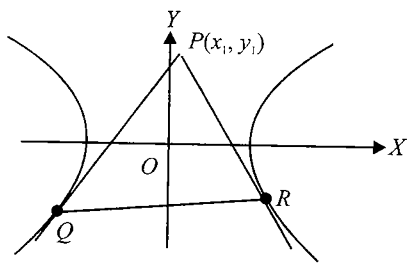


Fig. 5.30

In the diagram from point P tangents PQ and PR are drawn. Line QR is called chord of contact.

Its equation is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

or $$T = 0$$

where $$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

Example 5.31 **If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.**

Sol. 4 Let $P \equiv (h, k)$ be the point of intersection of tangents at A and B. Therefore, the equation of chord of contact AB of hyperbola is

$$\frac{xh}{a^2} - \frac{yk}{b^2} = 1$$

or $y = \frac{xb^2h}{ka^2} - \frac{b^2}{k}$ which touches the parabola $y^2 = 4ax$

Then $$-\frac{b^2}{k} = \frac{a}{\left(\frac{b^2h}{a^2k}\right)}$$

$$\Rightarrow -\frac{b^2}{k} = \frac{ka^3}{b^2h}$$

$$\Rightarrow y^2 = -\frac{b^4}{a^3}x$$

Equation of the Chord of the Hyperbola Whose Midpoint is (x_1, y_1)

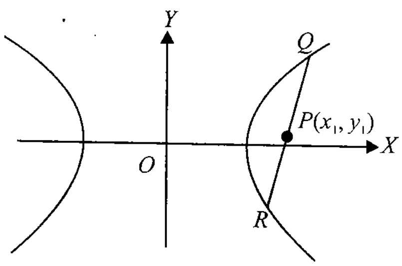


Fig. 5.31

Here chord QR is bisected at point P.

Its equation is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

or $T = S_1$, where

$$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

and

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

Example 5.32 **Find the locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.**

Sol. By $T = S_1$ the equation of chord whose midpoint is (h, k) is

$$3xh - 2yk + 2(x + h) - 3(y + k) = 3h^2 - 2k^2 + 4h - 6k$$

$$\Rightarrow x(3h + 2) - y(2k + 3) + \cdots = 0$$

Its slope is $\frac{3h+2}{2k+3} = 2$ (as it is parallel to $y = 2x$)

$$\Rightarrow \quad 3h - 4k = 4$$

$$\Rightarrow \quad 3x - 4y = 4$$

Example 5.33 **Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line $x + y = b$.**

Sol. Let the line $x + y = b$ bisect the chord at $P(\alpha, b - \alpha)$

Therefore, the equation of chord whose midpoint is $P(\alpha, b - \alpha)$ is given by

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b), therefore

$$\frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

or $\alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0$

or $\alpha = 0, \alpha = \frac{1}{\frac{1}{a} + \frac{1}{b}}$

Hence, the required condition is $a \neq -b$.

Concept Application Exercise 5.3

1. If any line perpendicular to the transverse axis cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at points P and Q, then prove that normals at P and Q meet on the x-axis.
2. A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.
3. Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$, which is bisected at the point (5, 3).
4. Find the equation to the locus of the middle points of the chords of the hyperbola $2x^2 - 3y^2 = 1$, each of which makes an angle of 45° with the x-axis.
5. Prove that the locus of the point of intersection of the tangents at the ends of the normal chords of the hyperbola $x^2 - y^2 = a^2$ is $a^2(y^2 - x^2) = 4x^2y^2$.
6. If $\alpha + \beta = 3\pi$, then the chord joining the points α and β for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through which of the following points:

   a. focus
   b. centre
   c. one of the end points of the transverse axis
   d. one of the end points of the conjugates axis

Asymptotes of Hyperbola : Definition

If the length of the perpendicular from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the asymptote of the hyperbola.

The asymptote of the hyperbola can be found as follows,

Let $y = mx + c$ be the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two Eq. we get the quadratic as

$$(b^2 - a^2m^2)x^2 - 2a^2 mcx - a^2(b^2 + c^2) = 0 \quad \text{(i)}$$

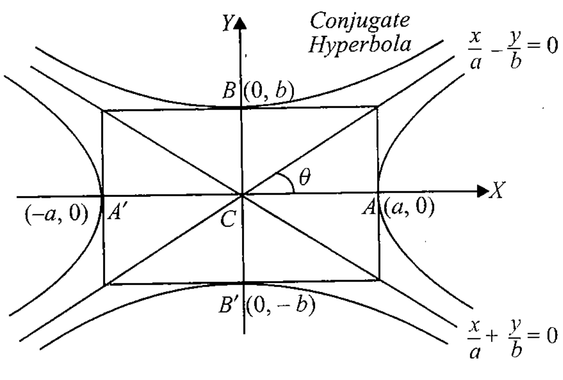


Fig. 5.32

In order that $y = mx + c$ be an asymptote, both roots of Eq. (i) must approach infinity, the conditions for which are as follows:

coefficient of $x^2 = 0$ and coefficient of $x = 0$

$\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ and $a^2\,mc = 0 \Rightarrow c = 0$

Therefore, equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and

$$\frac{x}{a} - \frac{y}{b} = 0$$

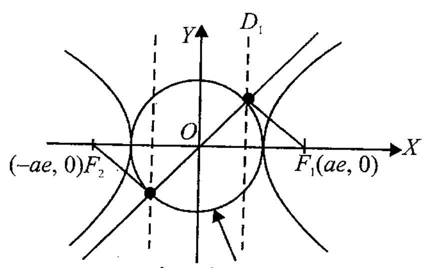
The combined equation to the asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Important Points

1. If the angle between the asymptotes of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ, then $e = \sec\theta$.

   Also acute angle between the asymptotes is $\theta = \tan^{-1}\left|\frac{2ab}{a^2 - b^2}\right|$

2. A hyperbola and its conjugate have the same asymptote.
3. The asymptotes pass through the centre of the hyperbola and the bisectors of the angles between the asymptotes are the axes of the hyperbola.
4. The equation of the pair of asymptotes differs from the equation of the hyperbola and the conjugate hyperbola by some constant only.
5. The asymptotes of a hyperbola are the diagonals of the rectangle formed by the line drawn through the extremities of each axis parallel to the other axis.
6. For rectangular hyperbola we have $b = a$. Then the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angle.
7. If from any point on the asymptotes a straight line is drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi-conjugate axis.
8. Perpendicular from the foci on either asymptote meets it at the same point as the corresponding directrix and the common points of intersection lie on the auxiliary circle.



$$\frac{a^2}{e^2} + \frac{b^2}{e^2} = \frac{a^2 + b^2}{a^2} = a^2$$

Fig. 5.33

Proof:

$$\frac{x}{a} - \frac{y}{b} = 0$$

Asymptote: $y = \frac{b}{a}x$ (i)

Line through the focus and perpendicular to asymptote:

$$y - 0 = -\frac{a}{b}(x - ae)$$

or $by + ax = a^2e$ (ii)

Solving (i) and (ii) for x, we have

$$\left(\frac{b^2}{a} + a\right)x = a^2e$$

$\Rightarrow$ $(b^2 + a^2)\,x = aa^2e$

$\Rightarrow$ $(a^2e^2)x = a^2ae$

$\Rightarrow$ $x = \frac{a}{e}$, hence $y = \frac{b}{a}\frac{a}{e} = \frac{b}{e}$

Now $\left(\frac{a}{e}, \frac{b}{e}\right)$ lies on the auxiliary circle.

9. The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C meets the asymptotes at Q and R and cuts off ΔCQR of constant area equal to ab from the asymptotes and the portion of the tangent intercepted between the asymptotes is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing ΔCQR in case of a rectangular hyperbola is the hyperbola itself.

Example 5.34 **Find the eccentricity of the hyperbola with asymptotes $3x + 4y = 2$ and $4x - 3y = 2$.**

Sol. Since the asymptotes are perpendicular, hyperbola is rectangular and hence eccentricity is $\sqrt{2}$.

Example 5.35 **Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ as its asymptotes and which passes through the origin.**

Sol. Combined equation of the asymptotes is

$$(3x - 4y + 7)\,(4x + 3y + 1) = 0$$

or $12x^2 - 7xy - 12y^2 + 31x + 17y + 7 = 0$ (i)

Since equation of hyperbola and combined equation of its asymptotes differ by a constant, therefore the equation of hyperbola may be taken as

$$12x^2 - 7xy - 12y^2 + 31x + 17y + k = 0 \quad \text{(ii)}$$

As (ii) passes through origin (0, 0), we have $k = 0$.

Hence, equation to the required hyperbola is

$$12x^2 - 7xy - 12y^2 + 31x + 17y = 0$$

Example 5.36 **Find the equations of the asymptotes of the hyperbola $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$.**

Sol. Since equation of hyperbola and combined equation of its asymptotes differ by a constant, equations of asymptotes should be

$$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0 \quad \text{(i)}$$

λ is to be chosen so that (i) represents a pair of straight lines.

Comparing (1) with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{(ii)}$$

we have

$$a = 3, b = 8, h = 5, g = 7,$$
$$f = 11, c = \lambda$$

We know that (ii) represents pair of straight lines if $abc + 2hgf - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow$ $3 \cdot 8 \cdot \lambda + 2 \cdot 7 \cdot 11 \cdot 5 - 3 \cdot 121 - 8 \cdot 49 - \lambda \cdot 25 = 0$

$\Rightarrow$ $\lambda = 15$

Hence, combined equation of the asymptotes is $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$.

Example 5.37 **If a hyperbola passing through the origin has $3x - 4y - 1 = 0$ and $4x - 3y - 6 = 0$ as its asymptotes, then find the equations of its transverse and conjugate axes.**

Sol. Axes of hyperbola are bisectors of pair of asymptotes.

Transverse axis is the bisector which contains the origin and is given by

$$\frac{3x - 4y - 1}{5} = + \frac{4x - 3y - 6}{5}$$

or $x + y - 5 = 0$

Conjugate axis is

$$\frac{3x - 4y - 1}{5} = - \frac{4x - 3y - 6}{5}$$

or $x - y - 1 = 0$

Example 5.38 **Find the product of the lengths of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2 = 0$ to its asymptotes**

Sol.

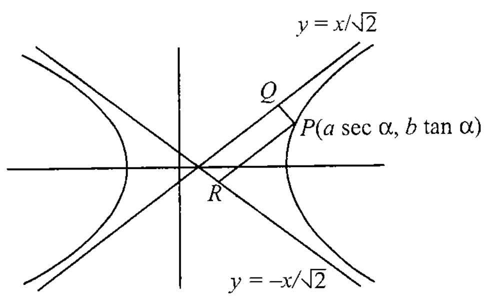


Fig. 5.34

Given hyperbola is $x^2 - 2y^2 - 2 = 0$

or $\frac{x^2}{2} - \frac{y^2}{1} = 1$

$\Rightarrow$ $PQ \cdot PR = \frac{|a \sec \alpha - \sqrt{2} b \tan \alpha|}{\sqrt{3}} \frac{|a \sec \alpha + \sqrt{2} b \tan \alpha|}{\sqrt{3}}$

$= \frac{a^2 \sec^2 \alpha - 2b^2 \tan^2 \alpha}{3}$

$= \frac{2(\sec^2 \alpha - \tan^2 \alpha)}{3}$ (as $a = \sqrt{2}$ and $b = 1$)

$= \frac{2}{3}$

Concept Application Exercise 5.4

1. Find the angle between the asymptotes of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
2. Find the area of the triangle formed by any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with its asymptotes.
3. Find the asymptotes of the curve $xy - 3y - 2x = 0$.
4. Show that the acute angle between the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a^2 > b^2)$ is $2 \cos^{-1}\left(\frac{1}{e}\right)$, where e is the eccentricity of the hyperbola.

Rectangular Hyperbola Referred to Its Asymptotes as the Axes of Coordinates

Referred to the transverse and conjugate axes along the axes of coordinates, the equation of the rectangular hyperbola is

$$x^2 - y^2 = a^2 \quad \text{(i)}$$

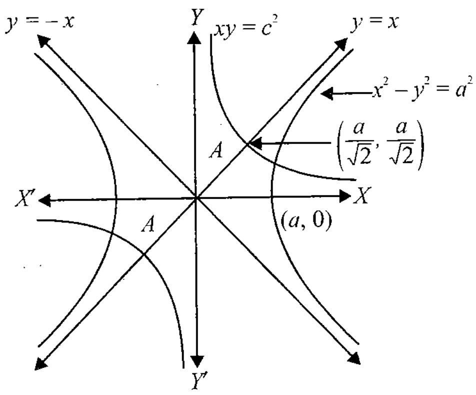


Fig. 5.35

The asymptotes of (i) are $y = x$ and $y = -x$. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis. So, if we rotate the coordinate axes through an angle of $\frac{\pi}{4}$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola.

Now for new hyperbola equation of asymptotes is $xy = 0$.

Then equation of hyperbola is $xy = k$ (constant)

The hyperbola passes through the point $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

$$\therefore \quad k = \frac{a^2}{2}$$

Then equation of hyperbola is $xy = \frac{a^2}{2}$ or $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$, then its equation is $(x - \alpha)(y - \beta) = c^2$.

Important Points

For hyperbola $xy = c^2$

- Asymptotes: $x = 0; y = 0$
- Transverse axis: $y = x$; conjugate axis: $y = -x$
  - Vertex: $A(c, c)$ and $A'(-c, -c)$
  - Foci: $(c\sqrt{2}, c\sqrt{2})$ and $(-c\sqrt{2}, -c\sqrt{2})$
  - Length of latus rectum = length of $AA' = 2\sqrt{2}c$
  - Equation of auxiliary circle: $x^2 + y^2 = 2c^2$
  - Equation of director circle: $x^2 + y^2 = 0$
  - $x^2 - y^2 = 1$ and $xy = c^2$ intersect at right angle

Properties of Rectangular Hyperbola $xy = c^2$

- Eccentricity of rectangular hyperbola is $\sqrt{2}$.
- Parametric form of rectangular hyperbola $xy = c^2$ is $P(ct, \frac{c}{t})$ where $t \in R - \{0\}$.
- Slope of chord joining the point $P(ct_1, \frac{c}{t_1})$ and $Q(ct_2, \frac{c}{t_2})$ is $-\frac{1}{t_1 t_2}$.
- Slope of tangent at point $(ct, \frac{c}{t})$ is $-\frac{1}{t^2}$.
- Equation of tangent at point whose parameter is 't' is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

or $\quad x + yt^2 - 2ct = 0$

- Equation of normal at point whose parameter is 't' is

$$y - \frac{c}{t} = t^2(x - ct)$$

or $\quad xt^3 - yt - ct^4 + c = 0$

- Equation of tangent at (x_1, y_1) is

$$xy_1 + x_1 y = 2c^2$$

or $\quad T = 0$

where

$$T = xy_1 + x_1 y - 2c^2$$

- Equation of normal at (x_1, y_1) is

$$xx_1 - yy_1 = x_1^2 - y_1^2$$

- Chord with a given middle point as (h, k) is

$$kx + hy = 2hk$$

Example 5.39 **A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.**

Sol. Let $A(t_1)$, $B(t_2)$, $C(t_3)$ be the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ Coordinates of A, B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$, $\left(ct_3, \frac{c}{t_3}\right)$, respectively.

Now slope of BC is $\dfrac{ct_3 - ct_2}{\frac{c}{t_3} - \frac{c}{t_2}} = -\frac{1}{t_2 t_3}$

Hence, slope of AD is $t_2 t_3$.

Equation of altitude AD is

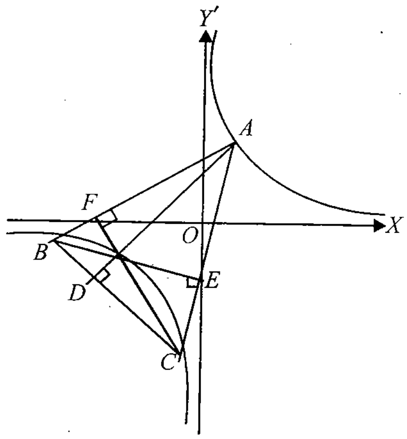


Fig. 5.36

$$y - \frac{c}{t_1} = t_2 t_3 (x - ct_1)$$

or $\quad t_1 y - c = x t_1 t_2 t_3 - c t_1^2 t_2 t_3 \quad$ (i)

Similarly equation of altitude BE is

$$t_2 y - c = x t_1 t_2 t_3 - c t_1 t_2^2 t_3 \quad \text{(ii)}$$

Solving Eqns. (i) and (ii) we get the orthocentre $\left(-\frac{c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$ which lies on $xy = c^2$.

Example 5.40 **If A, B and C be three points on the hyperbola $xy = c^2$ such that AB subtends a right angle at**

C. then prove that AB is parallel to normal to hyperbola at point C.

Sol.

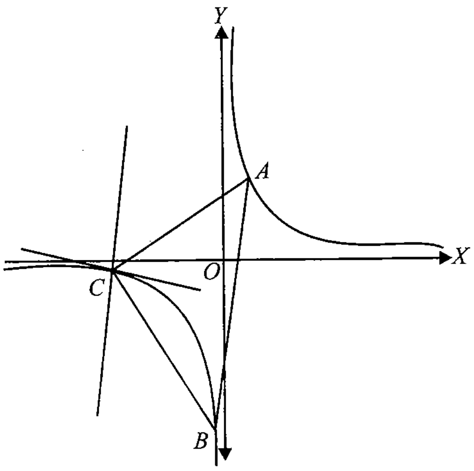


Fig. 5.37

Let coordinates of A, B and C be $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$

$$\text{Slope of } CA = -\frac{1}{t_1t_3}$$

$$\text{Slope of } CB = -\frac{1}{t_2t_3}$$

Given that $CA \perp CB$

$$\Rightarrow \quad \left(-\frac{1}{t_1t_3}\right) \times \left(-\frac{1}{t_2t_3}\right) = -1$$

$$\Rightarrow \quad \left(-\frac{1}{t_3^2}\right) \times \left(-\frac{1}{t_1t_2}\right) = -1$$

$\Rightarrow$ slope of tangent at point C × slope of $AB = -1$

$\Rightarrow$ Tangent at $C \perp AB$

$\Rightarrow$ Normal at C is parallel to AB

Example 5.41 **If PN is the perpendicular from a point on a rectangular hyperbola $xy = c^2$ to its asymptotes, then find the locus of the midpoint of PN.**

Sol. Let $xy = c^2$ be the rectangular hyperbola, and let $P(x_1, y_1)$ be the point on it.

Let $Q(h, k)$ be the midpoint of PN. Then the coordinates of Q are $\left(x_1, \frac{y_1}{2}\right)$.

$$\therefore \quad x_1 = h \text{ and } \frac{y_1}{2} = k$$

$$\Rightarrow \quad x_1 = h \text{ and } y_1 = 2k$$

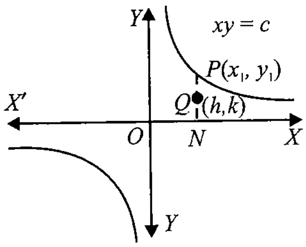


Fig. 5.38

But (x_1, y_1) lies on $\quad xy = c^2$

$$\therefore \quad h(2k) = c^2$$

$$\Rightarrow \quad hk = \frac{c^2}{2}$$

Hence, the locus of (h, k) is $xy = \frac{c^2}{2}$, which is a rectangular hyperbola.

Example 5.42 **PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then find the value of product of the slopes of CP, CQ, CR and CS.**

Sol. Let coordinates of P, Q, R, S be $\left(ct_i, \frac{c}{t_i}\right)$, respectively (where $i = 1, 2, 3, 4$)

Now, $PQ \perp RS$

$$\Rightarrow \quad \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = -1$$

$$\Rightarrow \quad t_1t_2t_3t_4 = -1$$

$$\text{Now slope of } CP = \frac{\frac{c}{t_1}}{ct_1} = \frac{1}{t_1^2}$$

Hence, product of slopes of CP, CQ, CR and CS is

$$\frac{1}{t_1^2t_2^2t_3^2t_4^2} = 1$$

Concyclic Points on the Hyperbola $xy = c^2$

If a circle and the rectangular hyperbola $xy = c^2$ meet at the four points t_1, t_2, t_3 and t_4, then

a. $t_1t_2t_3t_4 = 1$

b. the centre of the mean position of the four points bisects the distance between the centres of the two curves

Proof:

a. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + d = 0 \qquad \text{(i)}$$

Solving Eq. (i) and the equation of hyperbola, we have

$$x^2 + \frac{c^4}{x^2} + 2gx + 2f\frac{c^2}{x} + d = 0$$

$$\Rightarrow \quad x^4 + 2gx^3 + dx^2 + 2fc^2x + c^4 = 0 \quad \text{(ii)}$$

From Eq. (i),

$$x_1x_2x_3x_4 = c^4$$

$$c^4[t_1t_2t_3t_4] = c^4$$

$$\Rightarrow \quad t_1t_2t_3t_4 = 1$$

b. Again, centre of the mean position of the four points of intersection is $\left(\frac{\Sigma x_i}{4}, \frac{\Sigma y_i}{4}\right)$

Now from Eq. (i),

$$x_1 + x_2 + x_3 + x_4 = -2g \quad \text{(iii)}$$

$$\Rightarrow \quad \frac{\Sigma x_i}{4} = -\frac{g}{2}$$

using $xy = c^2$, we have

$$y_1 + y_2 + y_3 + y_4 = c^2\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right]$$

$$= \frac{c^2}{x_1x_2x_3x_4}\Sigma x_1x_2x_3$$

$$= \frac{c^2}{c^4}(-2fc^2) = -2f$$

$$\Rightarrow \quad \frac{\Sigma y_i}{4} = -\frac{f}{2}$$

Hence, $\left(\frac{\Sigma x_i}{4}, \frac{\Sigma y_i}{4}\right) \equiv \left(-\frac{g}{2}, -\frac{f}{2}\right)$

EXERCISES

Subjective Type

Solutions on page 5.36

1. A variable line $y = mx - 1$, cuts the lines $x = 2y$ and $y = -2x$ at points A and B. Prove that locus of the centroid of the triangle OAB (O being origin) is a hyperbola passing through origin.
2. Two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slopes m_1 and m_2 cut the axes in four concyclic points. Find the value of m_1m_2.
3. Let P be a point on the hyperbola $x^2 - y^2 = a^2$ where 'a' is a parameter, such that P is nearest to the line $y = 2x$. Find the locus of P.
4. Find the range of parameter a for which a unique circle will pass through the points of intersection of the rectangular hyperbola $x^2 - y^2 = a^2$ and the parabola $y = x^2$. Find also the equation of the circle.
5. Show that the midpoints of focal chords of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on another similar hyperbola.
6. A tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at P and Q. Show that the locus of the mid-point of PQ is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.
7. Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendicular drawn from the foci on the normal at the same point.
8. If one axis of varying central conic (hyperbola) be fixed in magnitude and position, prove that the locus of the point of contact of a tangent drawn to it from a fixed point on the other axis is a parabola.
9. A transverse axis cuts the same branch of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P, P' and the asymptotes in Q. Q'. Prove that (i) $PQ = P'Q'$ and (ii) $PQ' = P'Q$.
10. A normal is drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P which meets the transverse axis (TA) at G. If perpendicular from G on the asymptote meets it at L, show that LP is parallel to conjugate axis.
11. Find the angle between the rectangular hyperbolas $(y - mx)(my + x) = a^2$ and $(m^2 - 1)(y^2 - x^2) + 4mxy = b^2$.

Objective Type

Solutions on page 5.39

Each question has four choices a, b, c, d, out of which *only one* answer is correct. Find the correct answer.

1. If the distance between two parallel tangents drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{49} = 1$ is 2, then their slope is equal to

   a. $\pm\frac{5}{2}$ **b.** $\pm\frac{4}{5}$
   c. $\pm\frac{7}{2}$ **d.** none of these

2. If the distance between the foci and the distance between the two directries of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are in the ratio 3:2, then $b{:}a$ is

   a. $1{:}\sqrt{2}$ **b.** $\sqrt{3}{:}\sqrt{2}$ **c.** 1:2 **d.** 2:1

3. A tangent drawn to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ square-units, with coordinate axes, then the square of its eccentricity is

a. 15 **b.** 24 **c.** 17 **d.** 14

4. The length of the transverse axis of the rectangular hyperbola $xy = 18$ is

a. 6 **b.** 12 **c.** 18 **d.** 9

5. A straight line has its extremities on two fixed straight lines and cuts off from them a triangle of constant area c^2. Then the locus of the middle point of the line is

a. $2xy = c^2$ **b.** $xy + c^2 = 0$
c. $4x^2y^2 = c$ **d.** none of these

6. If a variable line has its intercepts on the coordinate axes e, e', where $\frac{e}{2}, \frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where $r =$

a. 1 **b.** 2
c. 3 **d.** cannot be decided

7. The equation of the transverse axis of the hyperbola $(x - 3)^2 + (y + 1)^2 = (4x + 3y)^2$ is

a. $x + 3y = 0$ **b.** $4x + 3y = 9$
c. $3x - 4y = 13$ **d.** $4x + 3y = 0$

8. The family of curves for which the length of the normal at any point is equal to the radius vector of that point is

a. hyperbola **b.** straight line
c. parabola **d.** ellipse

9. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is

a. 1 **b.** $\sqrt{2}$ **c.** 2 **d.** $\frac{1}{2}$

10. The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ represents a hyperbola

a. the length of whose transverse axis is $4\sqrt{3}$
b. the length of whose conjugate axis is 4
c. whose centre is $(-1, 2)$
d. whose eccentricity is $\sqrt{\frac{19}{3}}$

11. If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then ratio of square of its conjugate axis to the square of its transverse axis is

a. 2 **b.** 4 **c.** 6 **d.** 3

12. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is (axes are coordinate axes)

a. $\frac{4}{3}$ **b.** $\frac{4}{\sqrt{3}}$
c. $\frac{2}{\sqrt{3}}$ **d.** none of these

13. Let LL' be the latus rectum through the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and A' be the farther vertex. If $\Delta A'LL'$ is equilateral, then the eccentricity of the hyperbola is (axes are coordinate axes)

a. $\sqrt{3}$ **b.** $\sqrt{3} + 1$
c. $\frac{\sqrt{3} + 1}{\sqrt{2}}$ **d.** $\frac{\sqrt{3} + 1}{\sqrt{3}}$

14. The latus rectum of the hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$ is

a. $\frac{9}{4}$ **b.** 9 **c.** $\frac{3}{2}$ **d.** $\frac{9}{2}$

15. The eccentricity of the conjugate hyperbola of the hyperbola $x^2 - 3y^2 = 1$ is

a. 2 **b.** $\frac{2}{\sqrt{3}}$ **c.** 4 **d.** $\frac{4}{5}$

16. The equations of the transverse and conjugate axes of a hyperbola are respectively $x + 2y - 3 = 0$, $2x - y + 4 = 0$, and their respective lengths are $\sqrt{2}$ and $\frac{2}{\sqrt{3}}$. The equation of the hyperbola is

a. $\frac{2}{5}(x + 2y - 3)^2 - \frac{3}{5}(2x - y + 4)^2 = 1$
b. $\frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$
c. $2(2x - y + 4)^2 - 3(x + 2y - 3)^2 = 1$
d. $2(x + 2y - 3)^2 - 3(2x - y + 4)^2 = 1$

17. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}t = 0$ and $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is

a. $\sqrt{3}$ **b.** 2 **c.** $\frac{2}{\sqrt{3}}$ **d.** $\frac{4}{3}$

18. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then a value of α is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

19. With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre, a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

a. less than 2 **b.** 2
c. 1/3 **d.** none of these

20. If $ax + by = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $a^2 - b^2$ equals to

a. $\frac{1}{a^2e^2}$ **b.** a^2e^2
c. b^2e^2 **d.** none of these

21. Locus of a point whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is a/an

a. ellipse b. circle
c. hyperbola d. parabola

22. The sides AC and AB of a ΔABC touch the conjugate hyperbola of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the vertex A lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the side BC must touch

a. parabola b. circle
c. hyperbola d. ellipse

23. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point $(0, -b)$ and the normal at P passes through the point $(2a\sqrt{2}, 0)$ then eccentricity of the hyperbola is

a. 2 b. $\sqrt{2}$ c. 3 d. $\sqrt{3}$

24. If values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are roots of the equation $x^2 - (a + b)x - 4 = 0$, then value of $(a + b)$ is equal to

a. 2 b. 4 c. zero d. none of these

25. Portion of asymptote of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (between centre and the tangent at vertex) in the first quadrant is cut by the line $y + \lambda(x - a) = 0$ (λ is a parameter) then

a. $\lambda \in R$ b. $\lambda \in (0, \infty)$
c. $\lambda \in (-\infty, 0)$ d. none of these

26. If angle between asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 120° and product of perpendiculars drawn from foci upon its any tangent is 9, then locus of point of intersection of perpendicular tangents of the hyperbola can be

a. $x^2 + y^2 = 6$ b. $x^2 + y^2 = 9$
c. $x^2 + y^2 = 3$ d. $x^2 + y^2 = 18$

27. The co-ordinates of a point on the hyperbola, $\frac{x^2}{24} - \frac{y^2}{18} = 1$, which is nearest to the line $3x + 2y + 1 = 0$ are

a. (6, 3) b. (– 6, – 3) c. (6, – 3) d. (– 6, 3)

28. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is

a. zero b. 1 c. 2 d. 4

29. Locus of the feet of the perpendiculars drawn from either focus on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

a. $x^2 + y^2 = 9$ b. $x^2 + y^2 = \frac{1}{9}$
c. $x^2 + y^2 = \frac{7}{144}$ d. $x^2 + y^2 = \frac{1}{16}$

30. The locus of the foot of the perpendicular form the centre of the hyperbola $xy = 1$ on a variable tangent is

a. $(x^2 - y^2)^2 = 4xy$ b. $(x^2 + y^2)^2 = 2xy$
c. $(x^2 + y^2) = 4xy$ d. $(x^2 + y^2)^2 = 4xy$

31. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, the OT. ON is equal to

a. e^2 b. a^2 c. b^2 d. $\frac{b^2}{a^2}$

32. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the directrix in F. If PF subtends an angle θ at the corresponding focus, then θ equals

a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$ c. $\frac{3\pi}{4}$ d. π

33. The locus of a point, from where tangents to the rectangular hyperbola $x^2 - y^2 = a^2$ contain an angle of 45°, is

a. $(x^2 + y^2)^2 + a^2(x^2 - y^2) = 4a^2$
b. $2(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^2$
c. $(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^2$
d. $(x^2 + y^2)^2 + a^2(x^2 - y^2) = a^4$

34. If tangents PQ and PR are drawn from variable point P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a > b)$ so that the fourth vertex S of parallelogram $PQSR$ lies on circumcircle of triangle PQR, then locus of P is

a. $x^2 + y^2 = b^2$ b. $x^2 + y^2 = a^2$
c. $x^2 + y^2 = (a^2 - b^2)$ d. none of these

35. Number of points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 3$, from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$, is /are

a. 0 b. 2 c. 3 d. 4

36. A normal to the hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$ has equal intercepts on positive x- and y-axes. If this normal touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $a^2 + b^2$ is equal to

a. 5 b. 25
c. 16 d. none of these

37. If the normal to the given hyperbola at the point $(ct, \frac{c}{t})$ meets the curve again at $(ct', \frac{c}{t'})$, then

a. $t^3t' = 1$ b. $t^3t' = -1$
c. $tt' = 1$ d. $tt' = -1$

38. If the sum of the slopes of the normal from a point P to the hyperbola $xy = c^2$ is equal to λ ($\lambda \in R^+$), then locus of point P is

a. $x^2 = \lambda c^2$ **b.** $y^2 = \lambda c^2$

c. $xy = \lambda c^2$ **d.** none of these

39. If a ray of light incident along the line $3x + (5 - 4\sqrt{2})y = 15$ gets reflected from the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then its reflected ray goes along the line

a. $x\sqrt{2} - y + 5 = 0$ **b.** $\sqrt{2}y - x + 5 = 0$

c. $\sqrt{2}\,y - x - 5 = 0$ **d.** none of these

40. Let any double ordinate PNP' of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ be produced on both sides to meet the asymptotes in Q and Q', then $PQ \cdot P'Q$ is equal to

a. 25 **b.** 16

c. 41 **d.** none of these

41. For a hyperbola whose centre is at (1, 2) and asymptotes are parallel to lines $2x + 3y = 0$ and $x + 2y = 1$, then equation of hyperbola passing through (2, 4) is

a. $(2x + 3y - 5)(x + 2y - 8) = 40$

b. $(2x + 3y - 8)(x + 2y - 5) = 40$

c. $(2x + 3y - 8)(x + 2y - 5) = 30$

d. none of these

42. The chords of contact of a point 'P' w.r.t. a hyperbola and its auxiliary circle are at right angle, then the point P lies on

a. conjugate hyperbola

b. one of the directirx

c. one of the asymptotes

d. none of these

43. Asymptotes of the hyperbola $\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$ and $\frac{x^2}{a_2^2} - \frac{y^2}{b_2^2} = 1$ are perpendicular to each other, then

a. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ **b.** $a_1 a_2 = b_1 b_2$

c. $a_1 a_2 + b_1 b_2 = 0$ **d.** $a_1 - a_2 = b_1 - b_2$

44. If $S = 0$ be the equation of the hyperbola $x^2 + 4xy + 3y^2 - 4x + 2y + 1 = 0$, then the value of k for which $S + K = 0$ represents its asymptotes is

a. 20 **b.** -16 **c.** -22 **d.** 18

45. If two distinct tangents can be drawn from the point (α, 2) on different branches of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then

a. $|\alpha| < \frac{3}{2}$ **b.** $|\alpha| > \frac{2}{3}$

c. $|\alpha| > 3$ **c.** none of these

46. A hyperbola passes through (2, 3) and has asymptotes $3x - 4y + 5 = 0$ and $12x + 5y - 40 = 0$, then the equation of its transverse axis is

a. $77x - 21y - 265 = 0$

b. $21x - 77y + 265 = 0$

c. $21x - 77y - 265 = 0$

d. $21x + 77y - 265 = 0$

47. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. Then area cut-off by the chord of contact on the asymptotes is equal to

a. $\frac{a}{2}$ **b.** ab **c.** $2ab$ **d.** $4ab$

48. From a point P (1, 2) two tangents are drawn to a hyperbola 'H' in which one tangent is drawn to each arm of the hyperbola. If the equations of asymptotes of hyperbola H are $\sqrt{3}x - y + 5 = 0$ and $\sqrt{3}x + y - 1 = 0$, then eccentricity of 'H' is

a. 2 **b.** $\frac{2}{\sqrt{3}}$ **c.** $\sqrt{2}$ **d.** $\sqrt{3}$

49. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is

a. $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

b. $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$

c. $2x^2 + 5xy + 2y^2 = 0$

d. none of these

50. The asymptotes of the hyperbola $xy = hx + ky$ are

a. $x - k = 0$ and $y - h = 0$

b. $x + h = 0$ and $y + k = 0$

c. $x - k = 0$ and $y + h = 0$

d. $x + k = 0$ and $y - h = 0$

51. The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2 \tan \lambda$ in magnitude then its eccentricity is

a. $\sec \lambda$ **b.** $\operatorname{cosec} \lambda$ **c.** $\sec^2 \lambda$ **d.** $\operatorname{cosec}^2 \lambda$

52. The centre of a rectangular hyperbola lies on the line $y = 2x$. If one of the asymptotes is $x + y + c = 0$, then the other asymptote is

a. $x - y - 3c = 0$ **b.** $2x - y + c = 0$

c. $x - y - c = 0$ **d.** none of these

Hence, equation of other asymptote is $x + y - 3c = 0$.

53. Equation of a rectangular hyperbola whose asymptotes are $x = 3$ and $y = 5$ and passing through (7, 8) is

a. $xy - 3y + 5x + 3 = 0$

b. $xy + 3y + 4x + 3 = 0$

c. $xy - 3y + 5x - 3 = 0$

d. $xy - 3y - 5x + 3 = 0$

54. If foci of hyperbola lie on $y = x$ and one of the asymptote is $y = 2x$, then equation of the hyperbola, given that it passes through (3, 4) is

a. $x^2 - y^2 - \frac{5}{2}xy + 5 = 0$

b. $2x^2 - 2y^2 + 5xy + 5 = 0$

c. $2x^2 + 2y^2 - 5xy + 10 = 0$

d. none of these

55. $(x - 1)(y - 2) = 5$ and $(x - 1)^2 + (y + 2)^2 = r^2$ intersect at four points A, B, C, D and if centroid of ΔABC lies on line $y = 3x - 4$, then locus of D is

a. $y = 3x$ **b.** $x^2 + y^2 + 3x + 1 = 0$

c. $3y = x + 1$ **d.** $y = 3x + 1$

56. If tangents OQ and OR are dawn to variable circles having radius r and the centre lying on the rectangular hyperbola $xy = 1$, then locus of circumcentre of triangle OQR is (O being the origin)

a. $xy = 4$ **b.** $xy = \frac{1}{4}$

c. $xy = 1$ **d.** none of these

57. Four points are such that the line joining any two points is perpendicular to the line joining other two points. If three points out of these lie on a rectangular hyperbola then the fourth point will lie on

a. the same hyperbola

b. conjugate hyperbola

c. one of the directrix

d. one of the asymptotes

58. Equation of conjugate axis of hyperbola $xy - 3y - 4x + 7 = 0$ is

a. $y + x = 3$ **b.** $y + x = 7$

c. $y - x = 3$ **d.** none of these

59. If S_1 and S_2 are the foci of the hyperbola whose transverse axis length is 4 and conjugate axis length is 6, S_3 and S_4 are the foci of the conjugate hyperbola, then the area of the quadrilateral $S_1S_3S_2S_4$ is

a. 24 **b.** 26

c. 22 **d.** None of these

60. Suppose the circle having equation $x^2 + y^2 = 3$ intersects the rectangular hyperbola $xy = 1$ at the points A, B, C and D. The equation $x^2 + y^2 - 3 + \lambda(xy - 1) = 0$, $\lambda \in R$, represents

a. a pair of lines through origin for $\lambda = -3$

b. an ellipse through A, B, C and D for $\lambda = -3$

c. a parabola through A, B, C and D for $\lambda = -3$

d. a circle for any $\lambda \in R$

61. The family of the curves which intersects the family of rectangular hyperbola $xy = c^2$ orthogonally is

a. family of parabola

b. family of ellipse

c. family of circle

d. family of rectangular hyperbola

62. If two points P and Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose centre C be such that CP is perpendicular to CQ, $a < b$, then the value of $\frac{1}{CP^2} + \frac{1}{CQ^2}$ is

a. $\frac{b^2 - a^2}{2ab}$ **b.** $\frac{1}{a^2} + \frac{1}{b^2}$ **c.** $\frac{2ab}{b^2 - a^2}$ **d.** $\frac{1}{a^2} - \frac{1}{b^2}$

63. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

a. $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$

b. $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

c. $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$

d. $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

64. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, then coordinates of the orthocentre of the triangle PQR is

a. $(x_4, -y_4)$ **b.** (x_4, y_4) **c.** $(-x_4, -y_4)$ **d.** $(-x_4, y_4)$

65. The chord PQ of the rectangular hyperbola $xy = a^2$ meets the axis of x at A; C is the midpoint of PQ and 'O' is the origin. Then the ΔACO is

a. equilateral **b.** isosceles

c. right angled **d.** right isosceles

66. The curve $xy = c$ $(c > 0)$ and the circle $x^2 + y^2 = 1$ touch at two points, then distance between the points of contacts is

a. 1 **b.** 2 **c.** $2\sqrt{2}$ **d.** none of these

67 Let 'C' be a curve which is locus of the point of the intersection of lines $x = 2 + m$ and $my = 4 - m$. A circle $s \equiv (x - 2)^2 + (y + 1)^2 = 25$ intersects the curve C at four points P, Q, R and S. If O is centre of the curve 'C', then $OP^2 + OQ^2 + OR^2 + OS^2$ is

a. 50 **b.** 100 **c.** 25 **d.** $\frac{25}{2}$

68. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $a^2x^2 - y^2 = 4$ intersect at right angles then the equation of the circle through the points of intersection of two conic is

a. $x^2 + y^2 = 5$

b. $\sqrt{5}(x^2 + y^2) - 3x - 4y = 0$

c. $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$

d. $x^2 + y^2 = 25$

69. The locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ forms a triangle of constant area with the coordinate axe is

a. a straight line b. a hyperbola
c. an ellipse d. a circle

70. The angle between lines joining the origin to the points of intersection of the line $\sqrt{3}x+y=2$ and the curve $y^2-x^2=4$ is

a. $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ b. $\frac{\pi}{6}$
c. $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ d. $\frac{\pi}{2}$

71. A variable chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $(b>a)$ subtends a right angle at the centre of the hyperbola, if this chord touches

a. a fixed circle concentric with the hyperbola
b. a fixed ellipse concentric with the hyperbola
c. a fixed hyperbola concentric with the hyperbola
d. a fixed parabola having vertex at (0, 0)

72. The exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2+\alpha^2y^2=\alpha^2$ such that the portion of the tangent intercepted by the hyperbola $\alpha^2x^2-y^2=1$ subtends a right angle at the centre of the curves is

a. $\left[\frac{\sqrt{5}+1}{2}, 2\right]$ b. $(1, 2]$
c. $\left[\frac{\sqrt{5}-1}{2}, 1\right)$ d. $\left[\frac{\sqrt{5}-1}{2}, 1\right)\cup\left(1, \frac{\sqrt{5}+1}{2}\right]$

Multiple Correct Answers Type

Solutions on page 5.49

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. The equation $\left|\sqrt{x^2+(y-1)^2}-\sqrt{x^2+(y+1)^2}\right|=K$ will represent a hyperbola for

a. $K\in(0, 2)$ b. $K\in(-2, 1)$
c. $K\in(1, \infty)$ d. $K\in(0, \infty)$

2. If $x, y\in R$ then the equation $3x^4-2(19y+8)x^2+(361y^2+2(100+y^4)+64)=2(190y+2y^2)$ represents in rectangular Cartesian system

a. parabola b. hyperbola
c. circle d. ellipse

3. If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin, then

a. $e=\frac{\sqrt{386}}{12}$ b. $e=\frac{\sqrt{386}}{13}$
c. $LR=\frac{121}{6}$ d. $LR=\frac{121}{3}$

4. If (5, 12) and (24, 7) are the foci of a conic passing through the origin then the eccentricity of conic is

a. $\frac{\sqrt{386}}{12}$ b. $\frac{\sqrt{386}}{13}$
c. $\frac{\sqrt{386}}{25}$ d. $\frac{\sqrt{386}}{38}$

5. For which of the hyperbolas, we can have more than one pair of perpendicular tangents?

a. $\frac{x^2}{4}-\frac{y^2}{9}=1$ b. $\frac{x^2}{4}-\frac{y^2}{9}=-1$
c. $x^2-y^2=4$ d. $xy=44$

6. For the hyperbola $9x^2-16y^2-18x+32y-151=0$

a. one of the directrix is $x=\frac{21}{5}$
b. length of latus rectum $=\frac{9}{2}$
c. focii are (6, 1) and (−4, 1)
d. eccentricity is $\frac{5}{4}$

7. If foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ coincide with the foci of $\frac{x^2}{25}+\frac{y^2}{9}=1$ and eccentricity of the hyperbola is 2, then

a. $a^2+b^2=16$
b. there is no director circle to the hyperbola
c. centre of the director circle is (0, 0)
d. length of latus rectum of the hyperbola = 12

8. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$ then

a. $x_1+x_2+x_3+x_4=0$
b. $y_1+y_2+y_3+y_4=0$
c. $x_1x_2x_3x_4=c^4$
d. $y_1y_2y_3y_4=c^4$

9. The differential equation $\frac{dy}{dx}=\frac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity

a. $\sqrt{\frac{3}{5}}$ b. $\sqrt{\frac{5}{3}}$
c. $\sqrt{\frac{2}{5}}$ d. $\sqrt{\frac{5}{2}}$

10. Circles are drawn on chords of the rectangular hyperbola $xy = 4$ parallel to the line $y = x$ as diameters. All such circles pass through two fixed points whose coordinates are

   a. (2, 2) b. (2, –2) c. (–2, 2) d. (–2, –2)

11. The equation $(x - \alpha)^2 + (y - \beta)^2 = k(lx + my + n)^2$ represents
   a. a parabola for $k < (l^2 + m^2)^{-1}$
   b. an ellipse for $0 < k < (l^2 + m^2)^{-1}$
   c. a hyperbola for $k > (1^2 + m^2)^{-1}$
   d. a point circle for $k = 0$

12. If P is a point on a hyperbola, then
   a. locus of excentre of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci) is tangents at vertex
   b. locus of excentre of the circle described opposite to $\angle S'$ is hyperbola
   c. locus of excentre of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci), is hyperbola
   d. locus of excentre of the circle described opposite to $\angle S'$, is tangent at vertex

13. The lines parallel to normal to the curve $xy = 1$ is/are
   a. $3x + 4y + 5 = 0$ b. $3x - 4y + 5 = 0$
   c. $4x + 3y + 5 = 0$ d. $3y - 4x + 5 = 0$

14. From point (2, 2) tangents are drawn to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ then point of contact lies in
   a. I quadrant b. II quadrant
   c. quadrant d. IV quadrant

15. If the two intersecting lines intersect the hyperbola and neither of them is a tangent to it, then number of intersecting points are
   a. 1 b. 2 c. 3 d. 4

16. For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn.
   a. if $n = 1$, then $e = \sqrt{2}$
   b. if $n > 1$, then $0 < e < \sqrt{2}$
   c. if $n = 0$, then $e > \sqrt{2}$
   d. none of these

17. If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then
   a. $PG = PC$ b. $Pg = PC$
   c. $PG = Pg$ d. $Gg = 2PC$

Reasoning Type

Solutions on page 5.52

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

   a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.
   b. Both the statements are True but Statement 2 is not the correct explanation of Statement 1.
   c. Statement 1 is True and Statement 2 is False.
   d. Statement 1 is False and Statement 2 is True.

1. **Statement 1:** Asymptotes of hyperbola $3x + 4y = 2$ and $4x - 3y = 5$ are bisectors of transverse and conjugate axes of hyperbola.

   Statement 2: Transverse and conjugate axes of hyperbola are bisectors of the asymptotes.

2. **Statement 1:** Every line which cuts the hyperbola in two distinct points has slope lies in $(-2, 2)$.

   Statement 2: Slope of tangents of hyperbola lies in $(-\infty, -2) \cup (2, \infty)$.

3. **Statement 1:** A bullet is fired and it hits a target. An observer in the same plane heard two sounds, the crack of the rifle and the thud of the bullet striking the target at the same instant, then locus of the observer is hyperbola where velocity of sound is smaller than velocity of bullet.

   Statement 2: If difference of distances of a point 'P' from the two fixed points is constant and less than the distance between the fixed points then locus of 'P' is a hyperbola.

4. **Statement 1:** Ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $12x^2 - 4y^2 = 27$ intersect each other at right angle.

   Statement 2: Given ellipse and hyperbola have same foci.

5. **Statement 1:** If a circle S = 0 intersects a hyperbola $xy = 4$ at four points. Three of them are (2, 2), (4, 1) and (6, 2/3), then coordinates of the fourth point are (1/4, 16).

   Statement 2: If a circle S = 0 intersects a hyperbola $xy = c^2$ at t_1, t_2, t_3, t_4, then $t_1 - t_2 - t_3 - t_4 = 1$.

6. **Statement 1:** If a point (x_1, y_1) lies in the shaded region $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, shown in the figure, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 0$.

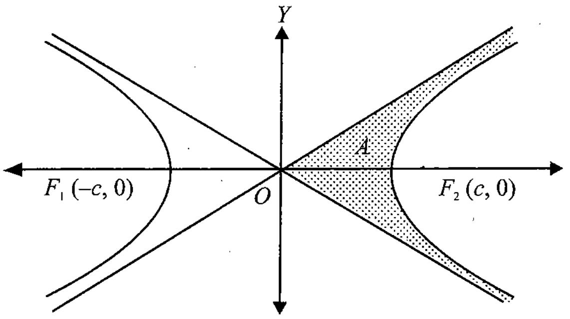


Fig. 5.39

Statement 2: If $P(x_1, y_1)$ lies outside the a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$.

7. Statement 1: Equations of tangents to the hyperbola $2x^2 - 3y^2 = 6$ which is parallel to the line $y = 3x + 4$ is $y = 3x - 5$ and $y = 3x + 5$.

Statement 2: For given slope two parallel tangents can be drawn to the hyperbola.

8. Statement 1: There are infinite points from which two mutually perpendicular tangents can be drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

Statement 2: The locus of point of intersection of perpendicular tangents lies on the circle.

9. Statement 1: If from any point $P(x_1, y_1)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, tangents are draws to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then corresponding chord of contact lies on another branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Statement 2: From any point outside the hyperbola two tangents can be drawn to the hyperbola.

10. Statement 1: If (3, 4) is a point on a hyperbola having focus (3, 0) and (λ, 0) and length of the transverse axis being 1 unit then λ can take the value 0 or 3.

Statement 2: $|S'P - SP| = 2a$, where S and S' are the two foci, $2a$ = length of the transverse axis and P be any point on the hyperbola.

11. Statement 1: Given the base BC of the triangle and the ratio radius of the ex-circles opposite to the angles B and C. Then locus of the vertex A is hyperbola.

Statement 2: $|S'P - SP| = 2a$, where S and S' are the two foci, $2a$ = length of the transverse axis and P be any point on the hyperbola.

Linked Comprehension Type

Solutions on page 5.53

Based upon each paragraph, three multiple choice questions have to be answered. Each question has 4 choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

A conic passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the coordinate axes is bisected at the point of tangency.

1. The eccentricity of the conic is

a. 2 **b.** $\sqrt{2}$ **c.** $\sqrt{3}$ **d.** $\sqrt{\frac{3}{2}}$

2. The foci of the conic are

a. $(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$

b. $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$

c. (4, 4) and (–4, –4)

d. $(4\sqrt{2}, 4\sqrt{2})$ and $(-4\sqrt{2}, -4\sqrt{2})$

3. The equations of directrix are

a. $x + y = \pm 8$ **b.** $x + y = \pm 4$

c. $x + y = \pm 4\sqrt{2}$ **d.** none of these

For Problems 4–6

The locus of foot of perpendicular from any focus of a hyperbola upon any tangent to the hyperbola is the auxiliary circle of the hyperbola. Consider the foci of a hyperbola as (–3, –2) and (5, 6) and the foot of perpendicular from the focus (5, 6) upon a tangent to the hyperbola as (2, 5).

4. The conjugate axis of the hyperbola is

a. $4\sqrt{11}$ **b.** $2\sqrt{11}$ **c.** $4\sqrt{22}$ **d.** $2\sqrt{22}$

5. The directrix of the hyperbola corresponding to the focus (5, 6) is

a. $2x + 2y - 1 = 0$ **b.** $2x + 2y - 11 = 0$

c. $2x + 2y - 7 = 0$ **d.** $2x + 2y - 9 = 0$

6. The point of contact of the tangent with the hyperbola is

a. $\left(\frac{2}{9}, \frac{31}{3}\right)$ **b.** $\left(\frac{7}{4}, \frac{23}{4}\right)$ **c.** $\left(\frac{2}{3}, 9\right)$ **d.** $\left(\frac{7}{9}, 7\right)$

For Problems 7–9

Let $P(x, y)$ is a variable point such that

$\left|\sqrt{(x-1)^2 + (y-2)^2} - \sqrt{(x-5) + (y-5)^2}\right| = 3$ which represents hyperbola.

7. The eccentricity e' of the corresponding conjugate hyperbola is

a. $\frac{5}{3}$ **b.** $\frac{4}{3}$ **c.** $\frac{5}{4}$ **d.** $\frac{3}{\sqrt{7}}$

8. Locus of intersection of two perpendicular tangents to the hyperbola is

a. $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{55}{4}$

b. $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{25}{4}$

c. $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{7}{4}$

d. none of these

9. If origin is shifted to point $\left(3,\frac{7}{2}\right)$ and the axes are rotated through an angle θ is clockwise sense so that equation of given hyperbola changes to the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then θ is

a. $\tan^{-1}\left(\frac{4}{3}\right)$ b. $\tan^{-1}\left(\frac{3}{4}\right)$

c. $\tan^{-1}\left(\frac{5}{3}\right)$ d. $\tan^{-1}\left(\frac{3}{5}\right)$

For Problems 10–12

In hyperbola portion of tangent intercepted between asymptotes is bisected at the point of contact.

Consider a hyperbola whose centre is at origin. A line $x+y=2$ touches this hyperbola at $P(1, 1)$ and interests the asymptotes at A and B such that $AB=6\sqrt{2}$ units.

10. Equation of asymptotes are

a. $5xy+2x^2+2y^2=0$

b. $3x^2+4y^2+6xy=0$

c. $2x^2+2y^2-5xy=0$

d. none of these

11. Angle subtended by AB at centre of the hyperbola is

a. $\sin^{-1}\frac{4}{5}$ b. $\sin^{-1}\frac{2}{5}$ c. $\sin^{-1}\frac{3}{5}$ d. none of these

12. Equation of the tangent to the hyperbola at $\left(-1,\frac{7}{2}\right)$ is

a. $5x+2y=2$ b. $3x+2y=4$

c. $3x+4y=11$ d. none of these

For Problems 13–15

A point P moves such that sum of the slopes of the normals drawn from it to the hyperbola $xy=16$ is equal to the sum of ordinates of feet of normals. The locus of P is a curve C.

13. The equation of the curve C is

a. $x^2=4y$ b. $x^2=16y$ c. $x^2=12y$ d. $y^2=8x$

14. If the tangent to the curve C cuts the co-ordinate axes at A and B, then the locus of the middle point of AB is

a. $x^2=4y$ b. $x^2=2y$ c. $x^2+2y=0$ d. $x^2+4y=0$

15. Area of the equilateral triangle, inscribed in the curve C, having one vertex as the vertex of curve C is

a. $772\sqrt{3}$ sq. units b. $776\sqrt{3}$ sq. units

c. $760\sqrt{3}$ sq. units d. $768\sqrt{3}$ sq. units

For Problems 16–18

The vertices of ΔABC lie on a rectangular hyperbola such that the orthocentre of the triangle is (3, 2) and the asymptotes of the rectangular hyperbola are parallel to the coordinate axes. The two perpendicular tangents of the hyperbola intersect at the point (1, 1).

16. The equation of the asymptotes is

a. $xy-1=x-y$ b. $xy+1=x+y$

c. $2xy=x+y$ d. none of these

17. Equation of the rectangular hyperbola is

a. $xy=2x+y-2$ b. $2xy=x+2y+5$

c. $xy=x+y+1$ d. none of these

18. Number of real tangents that can be drawn from the point (1, 1) to the rectangular hyperbola is

a. 4 b. 0 c. 3 d. 2

Matrix-Match Type

Solutions on page 5.56

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a → p, a → s, b → q, b → r, c → p, c → q, and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:

a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. Let the foci of the hyperbola $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ be the vertices of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the foci of the ellipse be the vertices of the hyperbola. Let the eccentricities of the ellipse and hyperbola be e_E and e_H, respectively, then match the following.

Column I	Column II
a. $\frac{b}{B}$ is equal to	p. 1
b. e_H+e_E is always greater than	q. 2
c. if angle between the asymptotes of hyperbola is $\frac{2\pi}{3}$, then $4e_E$ is equal to	r. 3
d. If $e_E^2=\frac{1}{2}$ and (x, y) is point of intersection of ellipse and the hyperbola then $\frac{9x^2}{2y^2}$ is	s. 4

2.

Column I	Column II
a. The points common to the hyperbola $x^2 - y^2 = 9$ and circle $x^2 + y^2 = 41$ are	**p.** $(-5, -4)$
b. Tangents are drawn from point $\left(0, -\frac{9}{4}\right)$ to the hyperbola $x^2 - y^2 = 9$, then the point of tangency may have coordinate(s)	**q.** $(5, 4)$
c. The point which is diametrically opposite of point $(5, 4)$ with respect to the hyperbola $x^2 - y^2 = 9$ is	**r.** $(-5, 4)$
d. If P and Q lie on the hyperbola $x^2 - y^2 = 9$ such that area of the isosceles triangle PQR where $PR = QR$ is 10 sq. units, where $R \equiv (0, -6)$, then P can have the co-ordinate(s)	**s.** $(5, -4)$

3. $A(-2, 0)$ and $B(2, 0)$ are the two fixed points and P is a point such that $PA - PB = 2$. Let S be the circle $x^2 + y^2 = r^2$, then match the following.

Column I	Column II
a. If $r = 2$, then the number of points P satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	**p.** 2
b. If $r = 1$, then he number of points satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	**q.** 4
c. For $r = 2$ the number of common tangents is	**r.** 0
d. For $r = 1/2$, the number of common tangents is	**s.** 1

4.

Column I	Column II
a. If z is a complex number such that $\text{Im}(z^2) = 3$, then eccentricity of the locus is	**p.** $\sqrt{3}$
b. If the latus rectum of a hyperbola through one focus subtends 60° angle at the other focus, then its eccentricity is	**q.** 2
c. If $A(3, 0)$ and $B(-3, 0)$ and $PA - PB = 4$, then eccentricity of conjugate hyperbola is	**r.** $\sqrt{2}$
d. If the angle between the asymptotes of hyperbola is $\pi/3$, then the eccentricity of its conjugate hyperbola is	**s.** $\frac{3}{\sqrt{5}}$

5. If e_1 and e_2 are the roots of the equation $x^2 - ax + 2 = 0$, then match the following.

Column I	Column II
a. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola, respectively then the values of a are	**p.** 6
b. If both e_1 and e_2 are the eccentricities of the hyperbolas, then values of a are	**q.** $\frac{5}{2}$
c. If e_1 and e_2 are eccentricities of hyperbola and conjugate hyperbola, then values of a are	**r.** $2\sqrt{2}$
d. If e_1 is the eccentricity of the hyperbola for which there exists infinite points from which perpendicular tangents can be drawn and e_2 is the eccentricity of the hyperbola in which no such points exist then the values of a are	**s.** 5

Integer Type

Solutions on page 5.57

1. Eccentricity of the hyperbola $\left|\sqrt{(x-3)^2 + (y-2)^2} - \sqrt{(x+1)^2 + (y+1)^2}\right| = 1$ is
2. If $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having eccentricity 5, then the least positive integral value of m is
3. Consider the graphs of $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$, where A is a positive constant and $x, y \in R$. Number of points in which the two graphs intersect is
4. Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x-axis. If $\tan\theta . \tan\phi = 2$, then the value of $2\alpha^2 - \beta^2$ is
5. If tangents drawn from the point $(a, 2)$ to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are perpendicular, then the value of a^2 is
6. If hyperbola $x^2 - y^2 = 4$ is rotated by 45° in anticlockwise direction about its center keeping the axis intact then equation of hyperbola is $xy = a^2$, where a^2 is equal to
7. The area of triangle formed by the tangents from point (3, 2) to hyperbola $x^2 - 9y^2 = 9$ and the chord of contact w.r.t. point (3, 2) is

8. If a variable line has its intercepts on the co-ordinates axes e, e', where $\frac{e}{2}, \frac{e'}{2}$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where $r =$
9. If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then ratio of square of its conjugate axis to the square of its transverse axis is
10. If distance between two parallel tangents having slope m drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{49} = 1$ is 2, then the value of $2|m|$ is
11. If L is the length of latus rectum of hyperbola for which $x = 3$ and $y = 2$ are the equations of asymptotes and which passes through the point (4, 6), then the value of $L/\sqrt{2}$ is
12. If the chord $x \cos\alpha + y \sin\alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre, and the diameter of the circle, concentric with the hyperbola, to which the given chord is a tangent is d then the value of $d/4$ is
13. A tangent drawn to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P\left(\frac{\pi}{6}\right)$ forms a triangle of area $3a^2$ square units, with coordinate axes. If the eccentricity of hyperbola is e, then the value of $e^2 - 9$ is
14. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\theta = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$, then smallest positive value of θ is $\frac{\pi}{p}$, value of 'p' is
15. If locus of a point, whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is $xy = c^2$, then value of c^2 is

Archives

Solutions on page 5.59

Subjective Type

1. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid-point of the chord of contact. **(IIT-JEE, 2005)**

Objective Type

Fill in the blanks

1. An ellipse has eccentricity $\frac{1}{2}$ and one focus at $S\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent (nearer to S) to the circle $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$. The equation of the ellipse in standard form is_________. **(IIT-JEE, 1996)**

Multiple choice questions with one correct answer

1. The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1, r > 1$ represents
   a. an ellipse **b.** a hyperbola
   c. a circle **d.** none of these
   which is not possible for any values of x and y.
   (IIT-JEE, 1981)
2. Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any two points (x_1, y_1) and (x_2, y_2) in the region, the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is also in the region. The inequality defining this region is
   a. $x^2 + 2y^2 \le 1$
   b. $\max\{|x|, |y|\} \le 1$
   c. $x^2 - y^2 \le 1$
   d. $y^2 - x \le 0$ **(IIT-JEE, 1981)**
3. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents
   a. no locus if $k > 0$ **b.** an ellipse if $k > 0$
   c. a point if $k = 0$ **d.** a hyperbola if $k > 0$
   (IIT-JEE, 1994)
4. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents
   a. no locus if $k > 0$
   b. an ellipse if $k < 0$
   c. a point if $k = 0$
   d. a hyperbola if $k > 0$
   (IIT-JEE, 1994)
5. Let P ($a \sec\theta$, $b \tan\theta$) and Q ($a \sec\phi$, $b \tan\phi$), where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to
   a. $\frac{a^2 + b^2}{a}$ **b.** $-\left(\frac{a^2 + b^2}{a}\right)$
   c. $\frac{a^2 + b^2}{b}$ **d.** $-\left(\frac{a^2 + b^2}{b}\right)$ **(IIT-JEE, 1999)**
6. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is
   a. $9x^2 - 8y^2 + 18x - 9 = 0$
   b. $9x^2 - 8y^2 - 18x + 9 = 0$

c. $9x^2 - 8y^2 - 18x - 9 = 0$

d. $9x^2 - 8y^2 + 18x + 9 = 0$ **(IIT-JEE, 1999)**

7. Which of the following is independent of α in the hyperbola $(0 < \alpha < \frac{\pi}{2})\ \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$?

a. eccentricity **b.** abscissa of foci

c. directrix **d.** vertex

8. If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

a. $(-2, \sqrt{6})$ **b.** $(-5, 2\sqrt{6})$

c. $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ **d.** $(4, -\sqrt{6})$ **(IIT-JEE, 2004)**

9. A hyperbola, having the transverse axis of length $2\sin\theta$ is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is

a. $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$

b. $x^2 \sec^2 - y^2 \operatorname{cosec}^2\theta = 1$

c. $x^2 \sin^2\theta - y^2\cos^2\theta = 1$

d. $x^2\cos^2\theta - y^2\sin^2\theta = 1$ **(IIT-JEE, 2007)**

10. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

a. $1 - \sqrt{\frac{2}{3}}$ **b.** $\sqrt{\frac{3}{2}} - 1$

c. $1 + \sqrt{\frac{2}{3}}$ **d.** $\sqrt{\frac{3}{2}} + 1$ **(IIT-JEE, 2008)**

11. Let a and b be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

a. four straight lines, when $c = 0$ and a, b are of the same sign

b. two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a

c. two straight lines and hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

d. a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

(IIT-JEE, 2009)

12. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x–axis at $(9, 0)$, then the eccentricity of the hyperbola is

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$

(C) $\sqrt{2}$ (D) $\sqrt{3}$ **(IIT-JEE, 2011)**

Multiple choice questions with one or more than one correct answer

1. Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of given ellipse and hyperbola is 1, then

a. the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

b. the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

c. focus of hyperbola is $(5, 0)$

d. vertex of hyperbola is $(5\sqrt{3}, 0)$ **(IIT-JEE, 2006)**

2. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then **(IIT-JEE, 2009)**

a. equation of ellipse is $x^2 + 2y^2 = 2$

b. the foci of ellipse are $(\pm 1, 0)$

c. equation of ellipse is $x^2 + 2y^2 = 4$

d. the foci of ellipse are $(\pm\sqrt{2}, 0)$

3. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

a. equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

b. a focus of hyperbola is $(2, 0)$

c. eccentricity of hyperbola is $\frac{2}{\sqrt{3}}$

d. equation of hyperbola is $x^2 - 3y^2 = 3$

(IIT-JEE, 2011)

Comprehension Type

For Problems 1–2

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

1. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is
   a. $2x - \sqrt{5}y - 20 = 0$ **b.** $2x - \sqrt{5}y + 4 = 0$
   c. $3x - 4y + 8 = 0$ **d.** $4x - 3y + 4 = 0$
   (IIT-JEE, 2010)
2. Equation of the circle with AB as its diameter is
   a. $x^2 + y^2 - 12x + 24 = 0$ **b.** $x^2 + y^2 + 12x + 24 = 0$
   c. $x^2 + y^2 + 24x - 12 = 0$ **d.** $x^2 + y^2 - 24x - 12 = 0$
   (IIT-JEE, 2010)

Match the following

1. Match the conic in Column I with the statements/ expressions in Column II. **(IIT-JEE, 2009)**

Column I	Column II
a. Circle	**p.** The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
b. Parabola	**q.** Point z in the complex plane satisfying $\|z + 2\| - \|z - 2\| = \pm 3$
c. Ellipse	**r.** Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1 - t^2}{1 + t^2}\right), y = \frac{2t}{1 + t^2}$
d. Hyperbola	**s.** The eccentricity of the conic lies in the interval $1 \le x < \infty$
	t. Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \|z\|^2 + 1$

Integer type

1. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **(IIT-JEE, 2010)**

ANSWERS AND SOLUTIONS

Subjective Type

1.

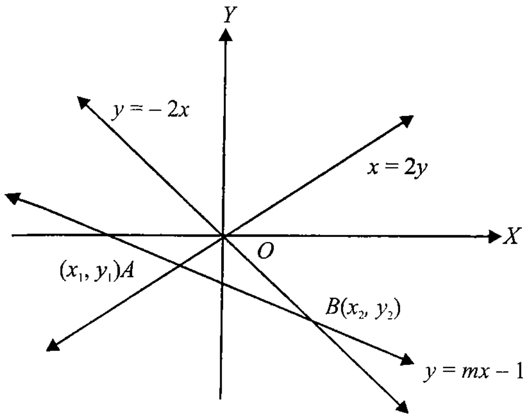


Fig. 5.40

Solving the variable line $y = mx - 1$ with $x = 2y$, we get

$$x_1 = \frac{2}{2m - 1} \quad \text{(i)}$$

Solving with $y = -2x$, we get

$$x_2 = \frac{1}{m + 2}$$

Now, $y_1 + y_2 = m(x_1 + x_2) - 2$

Let centroid of the triangle OAB be (h, k)

So, $h = \frac{x_1 + x_2}{3}$

and $k = \frac{y_1 + y_2}{3} = \frac{m(x_1 + x_2) - 2}{3}$

$\Rightarrow m = \frac{3k + 2}{3h}$

So, $3h = x_1 + x_2 = \frac{2}{2\left(\frac{3k + 2}{3h}\right) - 1} + \frac{1}{\left(\frac{3k + 2}{3h}\right) + 2}$ [using (i)]

$$\Rightarrow \frac{2}{6k - 3h + 4} + \frac{1}{6h + 3k + 2} = 1$$

Simplifying we get the final locus as $6x^2 - 9xy - 6y^2 - 3x - 4y = 0$ which is a hyperbola passing through origin, as $h^2 > ab$ and $\Delta \neq 0$.

2.

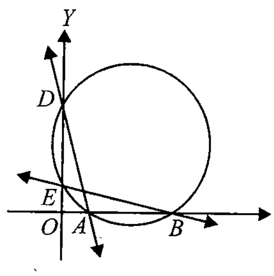


Fig. 5.41

Let the tangent be $y = m_1 x + \sqrt{a^2 m_1^2 - b}$

$y = m_2 x + \sqrt{a^2 m_2^2 - b^2}$

Points of intersection of these tangents with axes are

$A\left(\frac{-\sqrt{a^2m_1^2 - b^2}}{m_1}, 0\right), C\left(0, \sqrt{a^2m_1^2 - b^2}\right),$

$B\left(-\frac{\sqrt{a^2m_2^2 - b^2}}{m_2}, 0\right) D\left(0, \sqrt{a^2 m_2^2 - b^2}\right).$

Now as the four points are concyclic

$$OA \cdot OB = OC \cdot OD$$

$$\Rightarrow \left(\frac{-\sqrt{a^2m_1^2 - b^2}}{m_1}\right)\left(\frac{-\sqrt{a^2m_2^2 - b^2}}{m_2}\right) = \sqrt{a^2m_1^2 - b^2}\sqrt{a^2m_2^2 - b^2}$$

$$\Rightarrow \quad m_1 m_2 = 1$$

3. Consider any point P ($a \sec \theta$, $a \tan \theta$) on $x^2 - y^2 = a^2$. This point will be nearest to $y = 2x$, if tangent at this point is parallel to $y = 2x$.

Differentiating $x^2 - y^2 = a^2$ w.r.t. x, we get $\frac{dy}{dx} = \frac{x}{y}$

$\therefore \left[\frac{dy}{dx}\right]_{(a \sec \theta,\, a \tan \theta)} = \operatorname{cosec} \theta$

Slope of $y = 2x$ is 2,

$\Rightarrow \operatorname{cosec} \theta = 2$ or $\theta = \pi/6$

Thus $P \equiv (a \sec \pi/6, a \tan \pi/6)$

$$\equiv \left(\frac{2a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) \equiv (h, k)$$

$$h = \frac{2a}{\sqrt{3}} \text{ and } k = \frac{a}{\sqrt{3}} \Rightarrow k = \frac{h}{2}$$

So required locus is $2y - x = 0$.

4. Equation of family of curves, passing through the points of intersection of $x^2 - y^2 = a^2$ and $y = x^2$ is

$x^2 - y^2 - a^2 + \lambda(x^2 - y) = 0$

$\Rightarrow x^2(1 + \lambda) - y^2 - a^2 - \lambda y = 0$

It will be a circle if $\lambda = -2$

$\Rightarrow -x^2 - y^2 - a^2 + 2y = 0$

$\Rightarrow x^2 + y^2 - 2y = -a^2$

$\Rightarrow x^2 + (y - 1)^2 = 1 - a^2$

$\Rightarrow 1 - a^2 > 0 \Rightarrow a^2 < 1$

$\Rightarrow a \in (-1, 1)$ (i)

Also both the curves will intersect at real points if $y^2 - y + a^2 = 0$ for some real y,

if $-\frac{1}{2} < a < \frac{1}{2}$

(i) and (ii), $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

5.

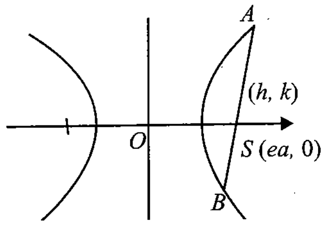


Fig. 5.42

Equation of chord AB with $T = S_1$ is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

If passes through (ae, 0)

$$\Rightarrow \frac{he}{a} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

Locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$

$$\Rightarrow \frac{1}{a^2}[x^2 - aex] - \frac{y^2}{b^2} = 0$$

$$\Rightarrow \frac{\left(x - \frac{ae}{2}\right)^2 - \frac{a^2e^2}{4}}{a^2} - \frac{y^2}{b^2} = 0$$

$$\Rightarrow \frac{\left(x - \frac{ae}{2}\right)^2}{a^2} - \frac{y^2}{b^2} = \frac{e^2}{4}$$

which is also a hyperbola with eccentricity e.

6.

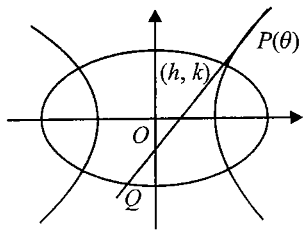


Fig. 5.43

Tangent to hyperbola at $P(\theta)$ is

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1 \quad \text{(i)}$$

Also the chord of the ellipse with middle point (h, k) is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} = \lambda \text{ (say)} \quad \text{(ii)}$$

Comparing (i) and (ii), we get

$$\frac{\sec\theta}{a}\frac{a^2}{h} = -\frac{\tan\theta}{b}\frac{b^2}{k} = \frac{1}{\lambda}$$

$\Rightarrow$ $$\sec\theta = \frac{h}{\lambda a} \text{ and } \tan\theta = -\frac{k}{\lambda b}$$

$\therefore$ $$\frac{h^2}{\lambda^2 a^2} - \frac{k^2}{\lambda^2 b^2} = 1 \text{ (eliminating } \theta)$$

$\Rightarrow$ $$\frac{h^2}{a^2} - \frac{k^2}{b^2} = \lambda^2 = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 = 1$$

Hence, locus is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

7. Tangent at point P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$;

Normal at point P is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2e^2$

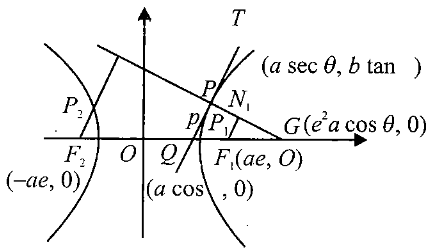


Fig. 5.44

From the diagram $$\frac{p}{p_1} = \frac{QG}{F_1G}$$

$$= \frac{e^2a\sec\theta - a\cos\theta}{e^2a\sec\theta - ae}$$

$$= \frac{e^2 - \cos^2\theta}{e^2 - e\cos\theta}$$

$$= \frac{e + \cos\theta}{e}$$

$\Rightarrow$ $$\frac{p}{p_1} = 1 + \frac{\cos\theta}{e}$$

Similarly, $$\frac{p}{p_2} = 1 - \frac{\cos\theta}{e}$$

$\Rightarrow$ $$\frac{p}{p_1} + \frac{p}{p_2} = 2$$

$\Rightarrow$ $$\frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

8. Equation of tangent to hyperbola at point P ($a\sec\theta$, $b\tan\theta$) is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \qquad \text{(i)}$$

Since (i) passes through the point $(0, c)$, therefore

$$-c\tan\theta = b \qquad \text{(ii)}$$

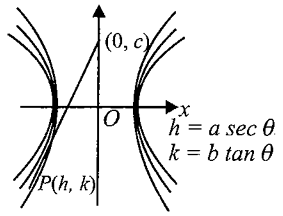


Fig. 5.45

Now substituting the value of b in (ii), we get

$$\frac{k}{c} = -\tan^2\theta \text{ and } \frac{h^2}{a^2} = \sec^2\theta$$

Adding, we have

$$\frac{k}{c} + \frac{h^2}{a^2} = 1$$

$\Rightarrow$ $$\frac{h^2}{a^2} = 1 - \frac{k}{c}$$

$\Rightarrow$ $$h^2 = \frac{a^2}{c}(c - k)$$

$$x^2 = -\frac{a^2}{c}(y - c)$$ which is a parabola

9. Hyperbola is $b^2x^2 - a^2y^2 = a^2b^2$ (i)

Let the transversal be $y = mx + c$ (ii)

Solving (i) and (ii), we get

$$b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$$\Rightarrow (b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$$

$\Rightarrow$ $$\frac{x_1 + x_2}{2} = \frac{2a^2mc}{b^2 - a^2m^2} \qquad \text{(iii)}$$

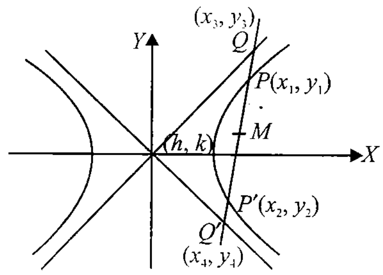


Fig. 5.46

Solving $y = mx + c$ with pair of asymptotes $b^2x^2 - a^2y^2 = 0$, we have

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2c^2 = 0 \qquad \text{(iv)}$$

$\Rightarrow$ $$\frac{x_3 + x_4}{2} = \frac{2a^2mc}{b^2 - a^2m^2} \qquad \text{(v)}$$

$\Rightarrow$ $$MQ = MQ' \text{ and } MP = MP'$$

$\Rightarrow$ $$PQ = P'Q'$$

10. Equation of GL with slope $-a/b$ and passing through $(e^2x_1, 0)$ is

$$y - 0 = -\frac{a}{b}(x - e^2x_1)$$

$$ax + by = ae^2x_1 \quad \text{(i)}$$

Put $$y = \frac{b}{a}x$$

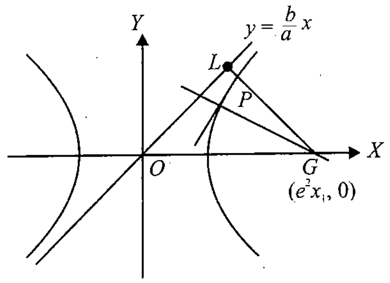


Fig. 5.47

$$ax + b\frac{b}{a}x = ae^2x_1$$

$$x\left[\frac{a^2+b^2}{a}\right] = ae^2x_1$$

$$x = x_1$$

Thus abscissa of point L is x_1 which is same as that of point P.

Hence, LP is parallel to conjugate axis.

11. For first hyperbola,

$$(y - mx)\left(m\frac{dy}{dx} + 1\right) + (my + x)\left(\frac{dy}{dx} - m\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y + m^2y + 2mx}{2my + x - m^2x} = m_1$$

For second hyperbola,

$$(m^2 - 1)\left(2y\frac{dy}{dx} - 2x\right) + 4m\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2my + m^2x - x}{m^2y - y + 2mx} = m_2$$

$\therefore$ $m_1m_2 = -1$

$\Rightarrow$ Angle between the hyperbolas $= \frac{\pi}{2}$

Objective Type

1. a. According to the question $\frac{2\sqrt{9m^2 - 49}}{\sqrt{1 + m^2}} = 2$

$$\Rightarrow \quad 9m^2 - 49 = 1 + m^2$$

$$\Rightarrow \quad 8m^2 = 50$$

$$\Rightarrow \quad m = \pm\frac{5}{2}$$

2. a. Given that

$$\frac{\text{Distance between foci}}{\text{Distance between two directrix}} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{2ae}{2\frac{a}{e}} = \frac{3}{2}$$

$$\Rightarrow \quad e^2 = \frac{3}{2}$$

$$\Rightarrow \quad 1 + \frac{b^2}{a^2} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{1}{\sqrt{2}}$$

3. c. $P\left(a\sec\frac{\pi}{2}, b\tan\frac{\pi}{6}\right) \equiv P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$

Therefore, equation of tangent at P is $\dfrac{x}{\frac{\sqrt{3}a}{2}} - \dfrac{y}{\sqrt{3}b} = 1$

$\therefore$ Area of the triangle $= \frac{1}{2} \times \frac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2$

$$\therefore \quad \frac{b}{a} = 4$$

$$\therefore \quad e^2 = 1 + \frac{b^2}{a^2} = 17$$

4. b. Transverse axis is along the line $y = x$.

Solving $y = x$ and $xy = 18$, we have $x^2 = 18$ or $x = \pm3\sqrt{2}$.

Then two vertices of the hyperbola are $(\pm3\sqrt{2}, \pm3\sqrt{2})$.

Distance between them $= \sqrt{72 + 72} = 12$.

5. a. Let the given straight line be axis of coordinates and let the equation of the variable line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

This line cuts the coordinate axis at the point A $(a, 0)$ and B $(0, b)$.

Therefore, the area of ΔAOB is

$$\frac{1}{2}ab = c^2$$

$$\Rightarrow \quad ab = 2c^2 \quad \text{(i)}$$

If (h, k) be the coordinates of the middle point of AB, then

$$h = \frac{a}{2} \text{ and } k = \frac{b}{2}$$

On eliminating a and b from Eqns. (i) and (ii), we get

$$2hk = c^2$$

Hence, the locus of (h, k) is $2xy = c^2$.

6. b. Since $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate hyperbola, therefore

$$\frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\Rightarrow \quad 4 = \frac{e^2e'^2}{e'^2 + e'^2}$$

The line passing through the points $(e, 0)$ and $(0, e')$ is

$$e'x + ey - ee' = 0$$

It is tangent to the circle $x^2 + y^2 = r^2$.

Hence, $$\frac{ee'}{\sqrt{e^2 + e'^2}} = r$$

$\Rightarrow \quad r^2 = \dfrac{e^2e'^2}{e^2+e'^2} = 4$

$\Rightarrow \quad r = 2$

7. **c.** $(x-3)^2 + (y+1)^2 = (4x+3y)^2$

$\Rightarrow \quad (x-3)^2 + (y+1)^2 = 25\left(\dfrac{4x+3y}{5}\right)^2$

$\Rightarrow \quad PS = 5PM$

$\Rightarrow$ directrix is $4x + 3y = 0$ and focus $(3, -1)$

So equation of transverse axis is $y + 1 = \frac{3}{4}(x-3)$

$\Rightarrow \quad 3x - 4y = 13$

8. **a.** Given that

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{x^2+y^2}$$

$\Rightarrow \quad y^2\left(\dfrac{dy}{dx}\right) = x^2$

$\Rightarrow \quad ydy = \pm xdx$

$\Rightarrow \quad y^2 \pm x^2 = k^2$

$\Rightarrow$ family of curves may be a circle or rectangular hyperbola

9. **b.** We have

$x^2 - y^2 - 4x + 4y + 16 = 0$

$\Rightarrow \quad (x-2)^2 - (y-2)^2 = -16$

$\Rightarrow \quad \dfrac{(x-2)^2}{4^2} - \dfrac{(y-2)^2}{4^2} = -1$

This is a rectangular hyperbola, whose eccentricity is always $\sqrt{2}$.

10. **d.** We have

$16(x^2 - 2x) - 3(y^2 - 4y) = 44$

$\Rightarrow \quad 16(x-1)^2 - 3(y-2)^2 = 48$

$\Rightarrow \quad \dfrac{(x-1)^2}{3} - \dfrac{(y-2)^2}{16} = 1$

This equation represents a hyperbola with eccentricity

$$e = \sqrt{1+\frac{16}{3}} = \sqrt{\frac{19}{3}}$$

11. **c.** Let the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

then $\quad 2a = ae$, i.e., $e = 2$

$\therefore \quad \dfrac{b^2}{a^2} = e^2 - 1 = 3$

$\therefore \quad \dfrac{(2b)^2}{(2a)^2} = 3$

12. **c.** We have

$\dfrac{2b^2}{a} = 8$

and $\quad 2b = \frac{1}{2}(2ae)$

$\therefore \quad \frac{2}{a}\left(\frac{ae}{2}\right)^2 = 8$

$\Rightarrow \quad ae^2 = 16 \quad$ (i)

Also, $\quad 2\frac{b^2}{a} = 8$

$\Rightarrow \quad b^2 = 4a$

$\Rightarrow \quad a^2(e^2 - 1) = 4a$

$\Rightarrow \quad ae^2 - a = 4 \quad$ (ii)

From (i) and (ii), we have

$16 - \dfrac{16}{e^2} = 4$

$\Rightarrow \quad \dfrac{16}{e^2} = 12$

$\Rightarrow \quad e = \dfrac{2}{\sqrt{3}}$

13. **d.**

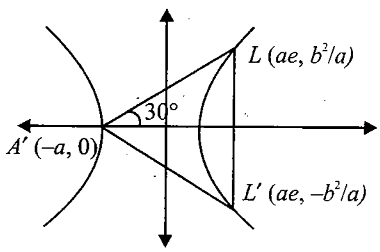


Fig. 5.48

$\tan 30° = \dfrac{\frac{b^2}{a}}{a+ae}$

$\Rightarrow \quad \dfrac{1+e}{\sqrt{3}} = e^2 - 1$

$\Rightarrow \quad e - 1 = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad e = \dfrac{\sqrt{3}+1}{\sqrt{3}}$

14. **d.** Hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$ can be written as

$9(x^2 - 2x) - 16(y^2 + 2y) = 151$

$\Rightarrow \quad 9(x-1)^2 - 16(y+1)^2 = 151 + 9 - 16 = 144$

$\Rightarrow \quad \dfrac{(x-1)^2}{16} - \dfrac{(y+1)^2}{9} = 1$

or $\quad \dfrac{X^2}{16} - \dfrac{Y^2}{9} = 1$

[where $X = x - 1$, $Y = y + 1$]

Here $\quad a^2 = 16, b^2 = 9$

Latus rectum $= 2\dfrac{b^2}{a} = \dfrac{2(9)}{4} = \dfrac{9}{2}$

15. **a.** Given hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{\frac{1}{3}} = 1$$

Its eccentricity 'e' is given by

$$\frac{1}{3} = 1\,(e^2 - 1)$$

Hence, eccentricity e' of the conjugate hyperbola is given by

$$1 = \frac{1}{3}(e'^2 - 1)$$

$\Rightarrow \quad e'^2 = 4$

$\Rightarrow \quad e' = 2$

16. b. The equation of the hyperbola is

$$\frac{\left(\frac{2x - y + 4}{\sqrt{5}}\right)^2}{\frac{1}{2}} - \frac{\left(\frac{x + 2y - 3}{\sqrt{5}}\right)^2}{\frac{1}{3}} = 1$$

or $\frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$

17. b. Eliminating t from the given two equation, we have

$$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

whose eccentricity is

$$e = \sqrt{1 + \frac{48}{16}} = 2$$

18. b. For hyperbola

$$\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$$

We have

$$e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{5\cos^2\alpha}{5}$$

$$= 1 + \cos^2\alpha$$

For ellipse

$$\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$$

We have

$$e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = \sin^2\alpha$$

Given that

$$e_1 = \sqrt{3}e_2$$

$\Rightarrow \quad e_1^2 = 3e_2^2$

$\Rightarrow \quad 1 + \cos^2\alpha = 3\sin^2\alpha$

$\Rightarrow \quad 2 = 4\sin^2\alpha$

$\Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{2}}$

19. b. Given hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \qquad \text{(i)}$$

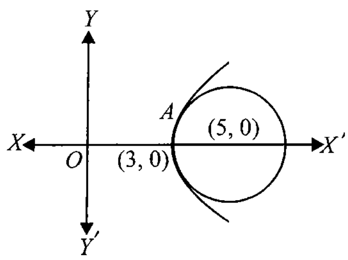


Fig. 5.49

$\Rightarrow \quad e^2 = 1 + \frac{16}{9} = \frac{25}{9}$

$\Rightarrow \quad e = \frac{5}{3}$

Hence, its foci are (± 5, 0).

The equation of the circle with (5, 0) as centre is

$$(x - 5)^2 + y^2 = r^2 \qquad \text{(ii)}$$

Solving (i) and (ii), we have

$$16x^2 - 9[r^2 - (x - 5)^2] = 144$$

or $\quad 25x^2 - 90x - 9r^2 + 81 = 0$

Since the circle touches the hyperbola, above equation must have equal roots. Hence,

$$90^2 - 4(25)(81 - 9r^2) = 0$$

$\Rightarrow \quad 9 - (9 - r^2) = 0$

$\Rightarrow \quad r = 0$ which is not possible.

Hence, the circle cannot touch at two points.

It can only be tangent at the vertex. Hence, $r = 5 - 3 = 2$.

20. a. Any tangent to hyperbola is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \qquad \text{(i)}$$

Given tangent is

$$ax + by = 1 \qquad \text{(ii)}$$

Comparing Eqs. (i) and (ii), we have

$$\sec\theta = a^2 \text{ and } \tan\theta = -b^2$$

Eliminating θ, we have

$$a^4 - b^4 = 1$$

$\Rightarrow \quad (a^2 - b^2)(a^2 + b^2) = 1$

Also $\quad a^2 + b^2 = a^2e^2$

$\Rightarrow \quad a^2 - b^2 = \frac{1}{a^2e^2}$

21. c. Let the point be (h, k).

Then equation of the chord of contact is $hx + ky = 4$.

Since $hx + ky = 4$ is tangent to $xy = 1$

$\therefore \quad x\left(\frac{4 - hx}{k}\right) = 1$ has two equal roots

$\Rightarrow \quad hx^2 - 4x + k = 0$

$\Rightarrow \quad hk = 4$

$\Rightarrow \quad$ locus of (h, k) is $xy = 4$

$\Rightarrow \quad c^2 = 4$

22. d. Let the vertex A be $(a\cos\theta, b\sin\theta)$

Since AC and AB touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

BC is the chord of contact. Its equation is

$$\frac{x\cos\theta}{a} - \frac{y\sin\theta}{b} = -1$$

or $\quad -\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

or $\frac{x\cos(\pi-\theta)}{a}+\frac{y\sin(\pi-\theta)}{b}=1$

which is the equation of the tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the point $(a\cos(\pi-\theta), b\sin(\pi-\theta))$.

Hence, BC touches the given ellipse.

23. b. Equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

It passing through $(0, -b)$. So,

$$0+\frac{y_1}{b}=1 \Rightarrow y_1=b$$

Equation of normal is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2e^2$$

which passes through $(2a\sqrt{2}, 0)$. Hence,

$$\frac{a^2 2a\sqrt{2}}{x_1}=a^2e^2$$

$$\Rightarrow \quad x_1=\frac{2a\sqrt{2}}{e^2}$$

Now (x_1, y_1) lies on the hyperbola

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

$$\Rightarrow \quad \frac{8}{e^4}-1=1$$

$$\Rightarrow \quad e^2=2$$

24. c. Equation of hyperbola is

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

Equation of tangent is

$$y=mx+\sqrt{9m^2-16}$$

$$\Rightarrow \quad \sqrt{9m^2-16}=2\sqrt{5}$$

$$\Rightarrow \quad m=\pm 2$$

$$\Rightarrow \quad a+b=\text{sum of roots}=0$$

25. b.

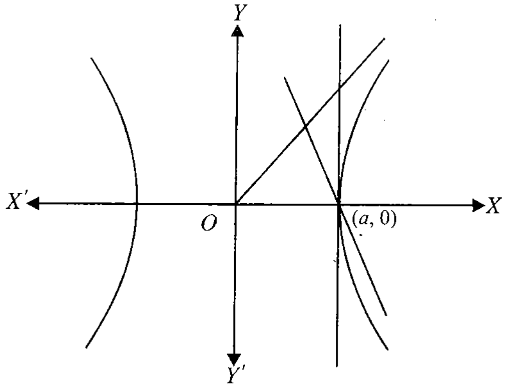


Fig. 5.50

The line $y+\lambda(x-a)=0$ will intersect the portion of the asymptote in the first quadrant only if its slope is negative. Hence,

$$-\lambda<0$$

$$\Rightarrow \quad \lambda>0$$

$$\therefore \quad \lambda\in(0,\infty)$$

26. d. Product of perpendiculars drawn from foci upon any of its tangents = 9

$$\Rightarrow \quad b^2=9$$

Also, $\frac{b}{a}=\tan 30°=\frac{1}{\sqrt{3}}$

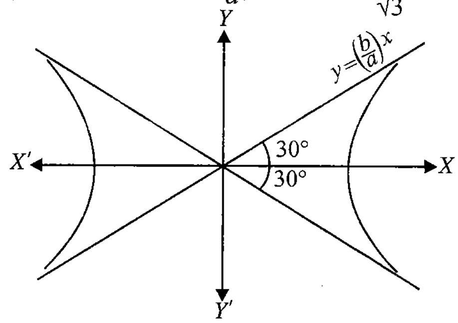


Fig. 5.51

$$\therefore \quad a^2=3b^2=27$$

Therefore, required locus is the director circle of the hyperbola which is given by $x^2+y^2=27-9$

or $\quad x^2+y^2=18$

if $\quad \frac{b}{a}=\tan 60°$, then

$$a^2=\frac{b^2}{3}=\frac{9}{3}=3$$

Hence, the required locus is $x^2+y^2=3-9=-6$ which is not possible.

27. c.

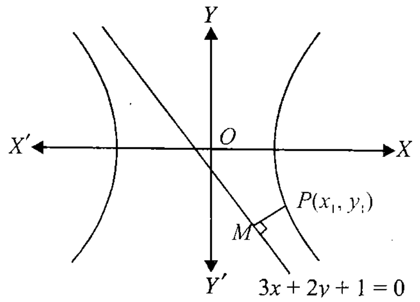


Fig. 5.52

Point P is nearest to the given line if tangent at P is parallel to the given line.

Now slope of tangent at $P(x_1, y_1)$ is

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{18y_1}{24x_1}=\frac{3}{4}\frac{y_1}{x_1}$ which must be equal to $-\frac{3}{2}$

$$\Rightarrow \quad \frac{3}{4}\frac{y_1}{x_1}=-\frac{3}{2}$$

$\Rightarrow \quad y_1 = -2x_1$ (i)

Also (x_1, y_1) lies on the curve. Hence,

$$\frac{x_1^2}{24} - \frac{y_1^2}{18} = 1 \quad \text{(ii)}$$

Solving (i) and (ii) we get two points (6, −3) and (−6, 3) of which (6, −3) is nearest.

28. a. Tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ at $P(3 \sec\theta, 2\tan\theta)$ is

$$\frac{x}{3}\sec\theta - \frac{y}{2}\tan\theta = 1$$

This is perpendicular to

$$5x + 2y - 10 = 0$$

$$\Rightarrow \frac{2\sec\theta}{3\tan\theta} = \frac{2}{5}$$

$$\Rightarrow \sin\theta = \frac{5}{3} \text{ which is not possible.}$$

Hence, there is no such tangent.

29. d. $\frac{y^2}{\frac{1}{16}} - \frac{x^2}{\frac{1}{9}} = 1$

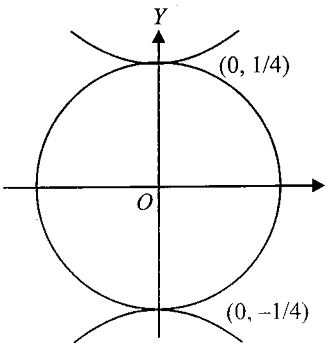


Fig. 5.53

Locus will be the auxiliary circle

$$x^2 + y^2 = \frac{1}{16}$$

30. d. Let the foot of perpendicular from $O(0, 0)$ to tangent to hyperbola is $P(h, k)$. Slope of $OP = \frac{k}{h}$

Then equation of tangent to hyperbola is

$$y - k = -\frac{h}{k}(x - h)$$

or $\quad hx + ky = h^2 + k^2$

Solving it with $\quad xy = 1$, we have

$$hx + \frac{k}{x} = h^2 + k^2$$

or $\quad hx^2 - (h^2 + k^2)x + k = 0$

This equation must have real and equal roots. Hence,

$$D = 0$$

$$\Rightarrow (h^2 + k^2)^2 - 4hk = 0$$

$$\Rightarrow (x^2 + y^2)^2 = 4xy$$

31. b.

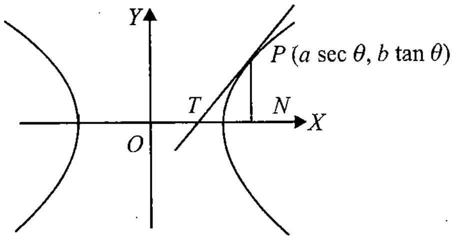


Fig. 5.54

Tangent at point P is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

It meets the x-axis at point $T(a\cos\theta, 0)$ and foot of perpendicular from P to x-axis is $N(a\sec\theta, 0)$

From the diagram,

$$OT = a\cos\theta \text{ and } ON = a\sec\theta$$

$$\Rightarrow OT \cdot ON = a^2$$

32. b. Let directrix be $x = \frac{a}{e}$ and focus be $S(ae, 0)$. Let $P(a\sec\theta, b\tan\theta)$ be any point on the curve.

Equation of tangent at P is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

Let F be the intersection point of tangent of directrix.

Then $\quad F = \left(\frac{a}{e}, \frac{b(\sec\theta - e)}{e\tan\theta}\right)$

$$\Rightarrow m_{SF} = \frac{b(\sec\theta - e)}{-e\tan\theta(a^2 - 1)},$$

$$m_{PS} = \frac{b\tan\theta}{a(\sec\theta - e)}$$

$$\Rightarrow m_{SF} \cdot m_{PS} = -1$$

33. c. Let $y = mx \pm \sqrt{m^2a^2 - a^2}$ be two tangents and passes through (h, k). Then

$$(k - mh)^2 = m^2a^2 - a^2$$

$$\Rightarrow m^2(h^2 - a^2) - 2khm + k^2 + a^2 = 0$$

$$\Rightarrow m_1 + m_2 = \frac{2kh}{h^2 - a^2}$$

and $\quad m_1m_2 = \frac{k^2 + a^2}{h^2 - a^2},$

Now, $\quad \tan 45° = \frac{m_1 - m_2}{1 + m_1m_2}$

$$\Rightarrow 1 = \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$$

$$\Rightarrow \quad \left(1+\frac{k^2+a^2}{h^2-a^2}\right)^2 = \left(\frac{2kh}{h^2-a^2}\right)^2 - 4\left(\frac{k^2+a^2}{h^2-a^2}\right)$$

$$\Rightarrow \quad (h^2+k^2)^2 = 4h^2k^2 - 4(k^2+a^2)(h^2-a^2)$$

$$\Rightarrow \quad (x^2+y^2)^2 = 4(a^2y^2 - a^2x^2 + a^4)$$

$$\Rightarrow (x^2+y^2)^2 + 4a^2(x^2-y^2) = 4a^2$$

34. c. Fourth vertex of parallelogram lies on circumcircle

$\Rightarrow$ parallelogram is cyclic

$\Rightarrow$ parallelogram is a rectangle

$\Rightarrow$ tangents are perpendicular

$\Rightarrow$ locus of P is the director circle

35. a. Director circle of circle $x^2+y^2=a^2$ is $x^2+y^2=2a^2$.

The semi-transverse axis is $\sqrt{3}a$.

Radius of the circle is $\sqrt{2}a$.

Hence, director circle and hyperbola do not intersect.

36. d. The equation of the normal to the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$ at $(2\sec\theta, \tan\theta)$ is $2x\cos\theta + y\cot\theta = 5$

Slope of the normal is $-2\sin\theta = -1$

$$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$Y\text{-intercept of the normal} = \frac{5}{\cot\theta} = \frac{5}{\sqrt{3}}$$

As it touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

We have $\quad a^2+b^2=\frac{25}{9}$

37. b. Equation of normals is given by $ty = t^3x - ct^4 + c = 0$. It passes through $(ct', c/t')$. Hence,

$$\frac{c}{t'} = t^3ct' - ct^4 + c = 0$$

$$t = t^3t'^2 - tt^4 + t'$$

$$t^3t' = -1$$

38. a. Equation of normal at any point $\left(ct, \frac{c}{t}\right)$ is

$$ct^4 - xt^3 + ty - c = 0$$

$\Rightarrow \quad$ Slope of normal $= t^2$

Let P be (h, k)

$$\Rightarrow \quad ct^4 - ht^3 + tk - c = 0$$

$$\Rightarrow \quad \Sigma t_i = \frac{h}{c} \text{ and } \Sigma t_it_j = 0$$

$$\Rightarrow \quad \Sigma t_i^2 = (\Sigma t_i)^2$$

$$\Rightarrow \quad h^2 = c^2\lambda$$

Therefore, the required locus is $x^2 = \lambda c^2$.

39. d. The given hyperbola is

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

$$\Rightarrow \quad e = \frac{5}{4}$$

Its foci are $(\pm 5, 0)$.

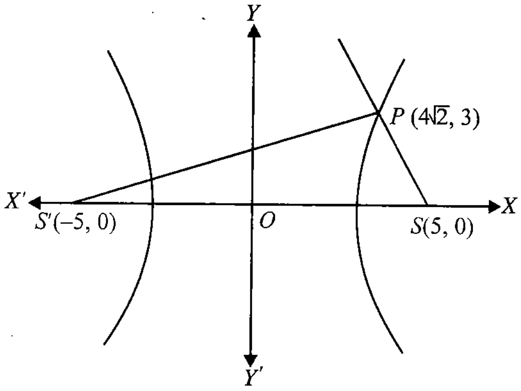


Fig. 5.55

The ray is incident at $P(4\sqrt{2}, 3)$.

The incident ray passes through $(5, 0)$; so the reflected ray must pass through $(-5, 0)$.

Its equation is $\quad \frac{y-0}{x+5} = \frac{3}{4\sqrt{2}+5}$

or $\quad 3x - y(4\sqrt{2}+5) + 15 = 0$

40. b.

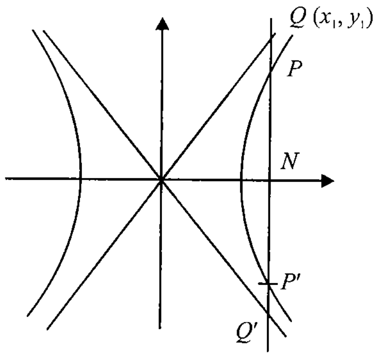


Fig. 5.56

$$NP = \frac{4}{5}\sqrt{x_1^2-25}$$

$$Q \text{ is on } y = \frac{4}{5}x$$

$$NQ = \frac{4}{5}x_1$$

$$PQ = NQ - NP = \frac{4}{5}\left(x_1 - \sqrt{x_1^2-25}\right)$$

$$P'Q = \frac{4}{5}\left(x_1 + \sqrt{x_1^2-25}\right)$$

$$PQ \cdot P'Q = 16$$

41. b. Let the asymptotes be $2x + 3y + \lambda_1 = 0$ and $x + 2y + \lambda_2 = 0$

It will pass through centre (1, 2). Hence,

$$\Rightarrow \quad \lambda_1 = -8, \lambda_2 = -5$$

The equation of the hyperbola is

$(2x + 3y - 8)(x + 2y - 5) + \lambda = 0$

It passes through (2, 4), therefore

$(4 + 12 - 8)(2 + 8 - 5) + \lambda = 0 \Rightarrow \lambda = -40$

Hence, equation of hyperbola is

$(2x + 3y - 8)(x + 2y - 5) = 40$

42. c. Let P be (h, k) be any point. The chord of contact of P w.r.t. the hyperbola is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \qquad \text{(i)}$$

The chord of contact of P w.r.t. the auxiliary circle is

$$hx + ky = a^2 - b^2 \qquad \text{(ii)}$$

Now, $\quad \frac{h}{a^2} \times \frac{b^2}{k} \times \left(-\frac{h}{k}\right) = -1$

$$\Rightarrow \quad \frac{h^2}{a^2} - \frac{k^2}{b^2} = 0$$

Therefore, P lies on one of the asymptotes.

43. c. Slopes of asymptotes are

$$m_1 = \frac{b_1}{a_1}, m_2 = \frac{b_2}{a_2}$$

According to the question,

$$m_1 m_2 = -1$$

$$\Rightarrow \quad a_1 a_2 + b_1 b_2 = 0$$

44. c. For equation $S + K = 0$ to represent a pair of lines,

$$\begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & 1+k \end{vmatrix} = 0$$

$\Rightarrow 3(1 + k) - 1 - 2(2 + 2k + 2) - 2(2 + 6) = 0$

$$\Rightarrow \quad k = -22$$

45. a.

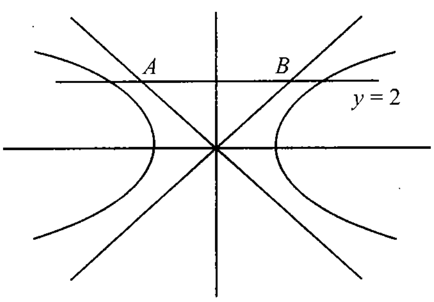


Fig. 5.57

For two distinct tangents on different branches the point should lie on the line $y = 2$ and between A and B (where A and B are the points on the asymptotes).

Equations of asymptotes are $4x = \pm 3y$

Solving with $\quad y = 2$, we have

$$x = \pm\frac{3}{2}$$

$$\therefore \quad -\frac{3}{2} < \alpha < \frac{3}{2}$$

46. d. Transverse axis is the equation of the angle bisector passing containing point (2, 3), which is given by

$$\frac{3x - 4y + 5}{5} = \frac{12x + 5y - 40}{13}$$

$$\Rightarrow \quad 21x + 77y = 265$$

47. d. Let $P(x_1, y_1)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The chord of contact of tangents from P to the hyperbola is given by

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \text{(i)}$$

The equations of the asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0$$

and $\quad \frac{x}{a} + \frac{y}{b} = 0$

The points of intersection of (i) with the two asymptotes are given by

$$x_1 = \frac{2a}{\frac{x_1}{a} - \frac{y_1}{b}}, y_1 = \frac{2b}{\frac{x_1}{a} - \frac{y_1}{b}}$$

$$x_2 = \frac{2a}{\frac{x_1}{a} + \frac{y_1}{b}}, y_2 = \frac{2b}{\frac{x_1}{a} + \frac{y_1}{b}}$$

Area of the said triangle $= \frac{1}{2}(x_1 y_2 - x_2 y_1)$

$$= \frac{1}{2}\left|\left(-\frac{4ab \times 2}{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}}\right)\right| = 4ab$$

48. b. Since $c_1 c_2 (a_1 a_2 + b_1 b_2) < 0$, therefore origin lies in acute angle. $P(1, 2)$ lies in obtuse angle.

Acute angle between the asymptotes is $\frac{\pi}{3}$. Hence,

$$e = \sec\frac{\theta}{2} = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

49. a. Let the equation of asymptotes be

$$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0 \qquad \text{(i)}$$

This equation represents a pair of straight lines.

Therefore, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

We have

$$4\lambda + 25 - \frac{25}{2} - 8 - \lambda\frac{25}{4} = 0$$

$$\Rightarrow \quad -\frac{9\lambda}{4} + \frac{9}{2} = 0$$

$$\Rightarrow \quad \lambda = 2$$

Putting the value of λ in (i), we get

$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

This is the equation of the asymptotes.

50. a. The given hyperbola is

$$xy - hx - ky = 0 \quad \text{(i)}$$

The equation of asymptotes is given by

$$xy - hx - ky + c = 0 \quad \text{(ii)}$$

Equation (ii) gives a pair of straight lines. So,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 0 & \frac{1}{2} & -\frac{h}{2} \\ \frac{1}{2} & 0 & -\frac{k}{2} \\ -\frac{h}{2} & -\frac{k}{2} & c \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{hk}{8} + \frac{hk}{8} - \frac{c}{4} = 0$$

$$\Rightarrow \quad c = hk$$

Hence, asymptotes are

$$xy - hx - ky + hk = 0$$

or $\quad (x - k)(y - h) = 0$

51. a. Any tangent to hyperbola forms a triangle with the asymptotes which has constant area ab.

Given

$$ab = a^2 \tan \lambda$$

$$\Rightarrow \quad \frac{b}{a} = \tan \lambda$$

$$\Rightarrow \quad e^2 - 1 = \tan^2 \lambda$$

$$\Rightarrow \quad e^2 = 1 + \tan^2 \lambda = \sec^2 \lambda$$

$$\Rightarrow \quad e = \sec \lambda$$

52. a. The asymptotes of a rectangular hyperbola are perpendicular to each other.

Given one asymptote,

$$x + y + c = 0$$

Let the other asymptote be

$$x - y + \lambda = 0$$

We also know that the asymptotes pass through centre of the hyperbola. Therefore, the line $2x - y = 0$ and the asymptotes must be concurrent.

Thus, we have

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 1 & c \\ 1 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = -3c$$

53. d. The equation of rectangular hyperbola is

$$(x - 3)(y - 5) + \lambda = 0$$

which passes through (7, 8). Hence,

$$4 \times 3 + \lambda \Rightarrow \lambda = -12$$

$$\therefore \quad xy - 5x - 3y + 15 - 12 = 0$$

$$\Rightarrow \quad xy - 3y - 5x + 3 = 0$$

54. c. Foci of hyperbola lie on $y = x$. So, the major axis is $y = x$.

Major axis of hyperbola bisects the asymptote

⇒ equation of other asymptote is $x = 2y$

⇒ equation of hyperbola is $(y - 2x)(x - 2y) + k = 0$

Given that it passes through (3, 4) ⇒ $k = 10$

Hence, required equation is

$$2x^2 + 2y^2 - 5xy + 10 = 0$$

55. a. If (x_i, y_i) is the point of intersection of given curves, then

$$\frac{\sum_{j=1}^{4} x_i}{4} = \frac{1+1}{2} \text{ and } \frac{\sum_{j=1}^{4} y_i}{4} = 0$$

Now $$\frac{\sum_{i=1}^{3} x_i}{3} = \frac{4 - x_4}{3} \text{ and } \frac{\sum_{i=1}^{3} y_i}{3} = -\frac{y_4}{4}$$

Centroid $\left(\frac{\sum_{j=1}^{3} x_i}{3}, \frac{\sum_{i=1}^{3} y_i}{3} \right)$ lies on the line $y = 3x - 4$.

Hence,

$$\frac{-y_4}{3} = \frac{3(4 - x_4)}{3} - 4$$

$$\Rightarrow \quad y_4 = 3x_4$$

Therefore, the locus of D is $y = 3x$.

56. b.

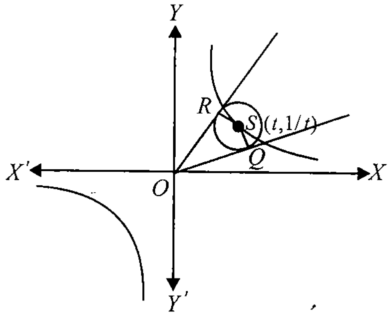


Fig. 5.58

Let S be a point on the rectangular hyperbola $\left[\text{say } \left(t, \frac{1}{t}\right)\right]$

Now, circumcircle of ΔOQR also passes through S. Therefore, circumcentre is the midpoint of OS. Hence,

$$x = \frac{t}{2}, y = \frac{1}{2t}$$

So, the locus of the circum centre is $xy = \frac{1}{4}$

57. a. The points are such that one of the points is the orthocentre of the triangle formed by other three points. When the vertices of a triangle lie on a rectangular hyperbola the orthocentre also lies on the same hyperbola.

58. b.

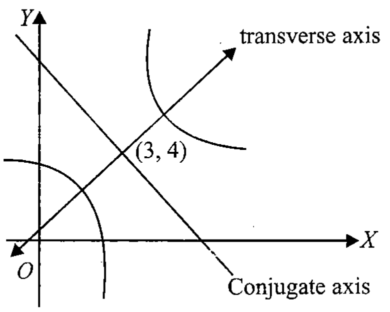


Fig. 5.59

$$(x-3)(y-4) = 5$$

The axes of the hyperbola are $x = 3$ and $y = 4$

Since the hyperbola is rectangular, axes are bisectors of these axes.

Hence, their slopes are ± 1, out of which conjugate axis has slope $m = -1$ and passes through (3, 4).

Hence, its equation is

$$y - 4 = -1(x - 3)$$

59. b.

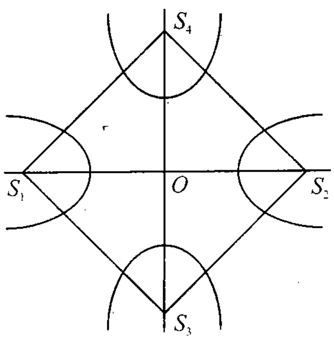


Fig. 5.60

Required area = 4 area $\Delta S_2OS_4 = 4 \times \frac{1}{2} ae \times 8\, be_1$

$- 4 \times \frac{1}{2} \times 2 \times 3 \times ee_1$ (i)

$$b^2 = a^2(e^2 - 1) \Rightarrow e^2 = \frac{9}{4} + 1 = \frac{13}{4}$$

Also $\frac{1}{e_1^2} = 1 - \frac{1}{e^2} = 1 - \frac{4}{13} = \frac{9}{13}$

$$e_1^2 = \frac{13}{9}$$

$$\text{Required area} = 12 \times \frac{\sqrt{13}}{2} \times \frac{\sqrt{13}}{3}$$
$$= 2 \times 13 = 26$$

60. a. For $\lambda = -3$, the equation becomes

$$x^2 + y^2 - 3xy = 0$$

which represents a pair of lines through origin.

61. d. $xy = c^2$

$\Rightarrow$ $$x\frac{dy}{dx} + y = 0$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we have

$$-x\frac{dx}{dy} + y = 0$$

$$\Rightarrow ydy - xdx = 0$$

Integrating, we have

$$x^2 - y^2 = k^2$$

where k is the parameter which represents family of hyperbolas.

62. d.

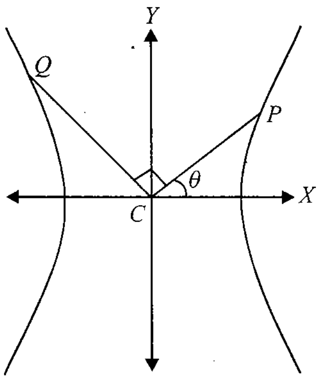


Fig. 5.61

Let $CP = r_1$ be inclined to transverse axis at an angle θ so that P is $(r_1 \cos\theta, r_1 \sin\theta)$ and P lies on the hyperbola. It gives

$$r_1^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1$$

Replacing θ by $90° + \theta$, we have

$$r_2^2\left(\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}\right) = 1$$

$$\Rightarrow \quad \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} + \frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2}$$

$$\Rightarrow \quad \frac{1}{r_1^2}+\frac{1}{r_2^2} = \cos^2\theta\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\sin^2\theta \times\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$\Rightarrow \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad \frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

63. a. The midpoint of the chord is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

The equation of the chord in terms of its midpoint is

$$T = S_1$$

$$\Rightarrow x\left(\frac{y_1+y_2}{2}\right)+y\left(\frac{x_1+x_2}{2}\right)-c^2 = 2\left(\frac{x_1+x_2}{2}\right)\left(\frac{y_1+y_2}{2}\right) - c^2$$

$$\Rightarrow \quad x(y_1+y_2)+y(x_1+x_2) = (x_1+x_2)(y_1+y_2)$$

$$\Rightarrow \quad \frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$$

64. c. A rectangular hyperbola circumscribing a triangle also passes through its orthocentre.

If $\left(ct_i, \frac{c}{t_i}\right)$, where $i = 1, 2, 3$, are the vertices of the triangle then the orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$, where $t_1 t_2 t_3 t_4 = 1$.

Hence, orthocentre is $\left(-ct_4, \frac{-c}{t_4}\right) = (-x_4, -y_4)$.

65. b.

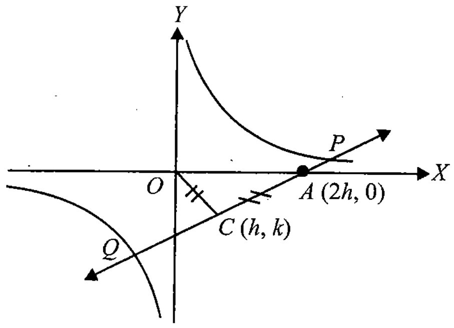


Fig. 5.62

Hyperbola is

$$xy = a^2$$

or $\quad 2xy - 2a^2 = 0$

Chord with a given middle point is given by

$$hy + kx - a^2 = 2hk - a^2$$

$$\Rightarrow \quad \frac{x}{h}+\frac{y}{k}=2$$

From the diagram ΔOCA is isosceles with $OC = CA$.

66. b.

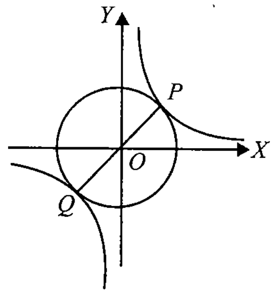


Fig. 5.63

From the diagram PQ = diameter of the circle = 2

67. b. $x - 2 = m$

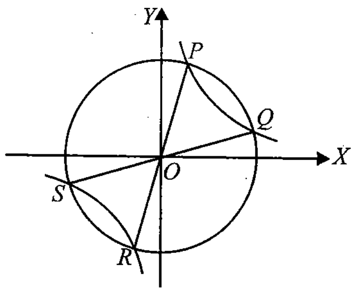


Fig. 5.64

$$y + 1 = \frac{4}{m}$$

$\therefore \quad (x-2)(y+1) = 4$

$\Rightarrow \quad XY = 4$, where $X = x-2, Y = y+1$

$$S \equiv (x-2)^2 + (y+1)^2 = 25$$

$\Rightarrow \quad X^2 + Y^2 = 25$

Curve 'C' and circle S both are concentric.

$\therefore OP^2 + OQ^2 + OR^2 + OS^2 = 4r^2 = 4 \times 25 = 100$

68. a.

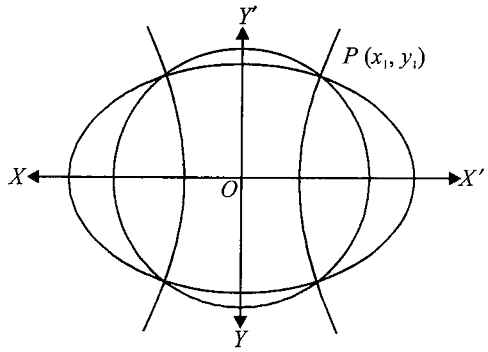


Fig. 5.65

Since ellipse and hyperbola intersect orthogonally, they are confocal.

Hence, $a = 2$ (equating foci).

Let point of intersection in the first quadrant be $P(x_1, y_1)$.

P lies on both the curves. Therefore,

$$4x_1^2 + 9y_1^2 = 36 \text{ and } 4x_1^2 - y_1^2 = 4$$

Adding these two results, we get

$$8(x_1^2 + y_1^2) = 40$$

$\Rightarrow \quad x_1^2 + y_1^2 = 5 \Rightarrow r = \sqrt{5}$

Hence, equation of the circle is

$$x^2 + y^2 = 5$$

69. b. The chord of contact of tangents from (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

It meets the axes at the points $\left(\frac{a^2}{x_1}, 0\right)$ and $\left(0, \frac{b^2}{y_1}\right)$.

Area of the triangle is $\frac{1}{2}\frac{a^2}{x_1}\frac{b^2}{y_1} = k$ (constant)

$\Rightarrow \quad x_1y_1 = \frac{a^2b^2}{2k} = c^2$ (c is constant)

$\Rightarrow \quad xy = c^2$ is the required locus.

70. c.

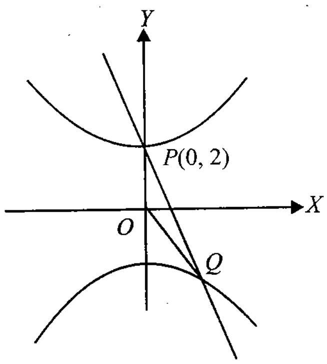


Fig. 5.66

Homogenizing the hyperbola using the straight line, we get pair of straight lines OP and OQ, which are given by

$$y^2 - x^2 = 4\left(\frac{\sqrt{3}x + y}{2}\right)^2$$

$\Rightarrow \quad y^2 - x^2 = 3x^2 + y^2 + 2\sqrt{3}xy$

$\Rightarrow \quad 4x^2 + 2\sqrt{3}\,xy = 0$

$\Rightarrow \quad x = 0$ and $2x + \sqrt{3}\,y = 0$

Angle between the lines is $\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

71. a. Let the variable chord be

$$x\cos\alpha + y\sin\alpha = p \quad \text{(i)}$$

Let this chord intersect the hyperbola at A and B. Then the combined equation of OA and OB is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2$$

$$x^2\left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) - y^2\left(\frac{1}{b^2} + \frac{\sin^2\alpha}{p^2}\right) - \frac{2\sin\alpha\cos\alpha}{p}xy = 0$$

This chord subtends a right angle at centre. Therefore,

Coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \frac{1}{a^2} - \frac{\cos^2\alpha}{p^2} - \frac{1}{b^2} - \frac{\sin^2\alpha}{p^2} = 0$

$\Rightarrow \quad \frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2}$

$\Rightarrow \quad p^2 = \frac{a^2b^2}{b^2 - a^2}$

Hence, p is constant, i.e. it touches the fixed circle.

72. a. Equation of tangent at point $P(\alpha\cos\theta, \sin\theta)$ is

$$\frac{x}{\alpha}\cos\theta + \frac{y}{1}\sin\theta = 1 \quad \text{(i)}$$

Let it cut the hyperbola at points P and Q.

Homogenizing the hyperbola $\alpha^2x^2 - y^2 = 1$ with the help of the above the equation, we get

$$\alpha^2x^2 - y^2 = \left(\frac{x}{\alpha}\cos\theta + y\sin\theta\right)^2$$

This is a pair of straight lines OP and OQ.

Given $\quad \angle POQ = \frac{\pi}{2}$

$\Rightarrow$ Coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - \sin^2\theta = 0$

$\Rightarrow \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - 1 + \cos^2\theta = 0$

$\Rightarrow \quad \cos^2\theta = \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1}$

Now,

$$0 \le \cos^2\theta \le 1$$

$\Rightarrow \quad 0 \le \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1} \le 1$

Solving, we get $\quad \alpha^2 \in \left[\frac{\sqrt{5}+1}{2}, 2\right]$

Multiple Correct Answers Type

1. a. We have,

$$\left|\sqrt{x^2 + (y-1)^2} - \sqrt{x^2 + (y+1)^2}\right| = K$$

which is equivalent to $|S_1P - S_2P|$ = constant, where $S_1 \equiv (0, 1)$, $S_2 \equiv (0, -1)$ and $P \equiv (x, y)$.

The above equation represents a hyperbola. So, we have

$$2a = K$$

and $$2ae = S_1S_2 = 2$$

where $2a$ is the transverse axis and e is the eccentricity.

Dividing, we have

$$e = \frac{2}{K}$$

Since, $e > 1$ for a hyperbola, therefore $K < 2$.

Also, K must be a positive quantity.

So, we have $K \in (0, 2)$.

2. a., b., c.

$3x^4 - 2(19y + 8)x^2 + [(19y)^2 + (10)^2 + (10)^2 + y^4 + y^4 + 8^2]$
$= 2(19 \times 10y + 10y^2 - 8y^2)$

$\Rightarrow 3x^4 - 2(19y + 8)x^2 + (19y - 10)^2 + (10 - y^2)^2 + (y^2 + 8)^2 = 0$

$\Rightarrow 3x^4 - 2(19y - 10 + 10 - y^2 + y^2 + 8)x^2 + (19y - 10)^2 + (10 - y^2)^2 + (y^2 + 8)^2 = 0$

$\Rightarrow [x^2 - (19y - 10)]^2 + [x^2 - (10 - y^2)]^2 + [x^2 - (y^2 + 8)]^2 = 0$

$\Rightarrow x^2 - 19y + 10 = 0$, $x^2 - 10 + y^2 = 0$ and $x^2 - y^2 - 8 = 0$

The three curves represented by the given equation are $x^2 = 19y - 10$ (parabola), $x^2 + y^2 = 10$ (circle) and $x^2 - y^2 = 8$ (hyperbola).

3. a., c. $|PS_1 - PS_2| = 2a$

$$2a = K$$

$$\Rightarrow 2a = \sqrt{(24-0)^2 + (7-0)^2} - \sqrt{12^2 + 5^2} = 12$$

$$\therefore \quad a = 6$$

$$2ae = \sqrt{(24-5)^2 + (12-7)^2}$$

$$= \sqrt{386}$$

$$\therefore \quad e = \frac{\sqrt{386}}{12}$$

$$LR = \frac{2b^2}{a} = \frac{2a^2(e^2 - 1)}{a}$$

$$= 2 \times 6\left(\frac{386}{144} - 1\right) = \frac{121}{6}$$

4. a., d.,

Let A (5, 12) and B (24, 7) be two fixed points.

So, $|OA - OB| = 12$ and $|OA + OB| = 38$.

If the conic is an ellipse, then

$$e = \frac{\sqrt{386}}{38} \quad (\because 2ea = \sqrt{386} \text{ and } a = 19)$$

If the conic is a hyperbola, then

$$e = \frac{\sqrt{386}}{12} \quad (\because 2ae = \sqrt{386} \text{ and } a = 6)$$

5. b. Locus of point of intersection of perpendicular tangents is director circle for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Equation of director circle is $x^2 + y^2 = a^2 - b^2$ which is real if $a > b$.

6. a., b., c., d.

Given hyperbola can be written as

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

$$\Rightarrow \quad \frac{X^2}{16} - \frac{Y^2}{9} = 1$$

(where $X = x - 1$, $Y = y - 1$)

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

Directrices are

$$X = \pm \frac{a}{e}$$

$$\Rightarrow \quad x - 1 = \pm \frac{16}{5} \Rightarrow x = \frac{21}{5} \text{ and } x = -\frac{11}{5}$$

$$\text{Length of latus rectum} = \frac{2b^2}{a} = \frac{9}{2}$$

The foci are given by

$$X = \pm ae, \; Y = 0$$

$\Rightarrow (6, 1)$, $(-4, 1)$

7. a., b.,d.

For the ellipse,

$$a = 5 \text{ and } e = \sqrt{\frac{25 - 9}{25}} = \frac{4}{5}$$

$$\therefore \quad ae = 4$$

Hence, the foci are $(-4, 0)$ and $(4, 0)$.

For the hyperbola,

$$ae = 4, \; e = 2$$

$$\therefore \quad a = 2$$

$$b^2 = 4\,(4 - 1) = 12$$

$$\Rightarrow \quad b = \sqrt{12}$$

8. a., b., c., d.

Solving $xy = c^2$ and $x^2 + y^2 = a^2$, we have

$$x^2 + \frac{c^4}{x^2} = a^2$$

$$\Rightarrow \quad x^4 - a^2x^2 + c^4 = 0$$

$$\Rightarrow \quad \Sigma x_i = 0 \text{ and } x_1 x_2 x_3 x_4 = c^4$$

Similarly, if we eliminate y, then $\Sigma y_i = 0$ and $y_1 y_2 y_3 y_4 = c^4$

9. b., d.

$$\frac{dx}{dy} = \frac{3y}{2x}$$

$$\Rightarrow \quad \int 2x dx = \int 3y dy$$

$$\Rightarrow \quad x^2 = \frac{3y^2}{2} + c$$

or $$\frac{x^2}{3}-\frac{y^2}{2}=\frac{c}{3}$$

If c is positive, then

$$e=\sqrt{1+\frac{2}{3}}=\sqrt{\frac{5}{3}}$$

If c is negative, then

$$e=\sqrt{1+\frac{3}{2}}=\sqrt{\frac{5}{2}}$$

10. a., d.

Circle with points $P(2t_1, 2/t_1)$ and $Q(2t_2, 2/t_2)$ as diameter is given by

$$(x-2t_1)(x-2t_2)+\left(y-\frac{2}{t_1}\right)\left(y-\frac{2}{t_2}\right)=1 \quad \text{(i)}$$

Also, slope of PQ is given by

$$-\frac{1}{t_1t_2}=1 \Rightarrow t_1t_2=-1$$

Hence, from (1), circle is

$(x^2+y^2-8)-(t_1+t_2)(x-y)=0$

which is of the form $S+\lambda L=0$.

Hence, circles pass through the points of intersection of the circle $x^2+y^2-8=0$ and the line $x=y$.

The points of intersection are (2, 2) and (–2, –2).

11. b., c., d.

$(x-\alpha)^2+(\gamma-\beta)^2=k(lx+my+n)^2$

$$\Rightarrow \quad \sqrt{(x-\alpha)^2+(y-\beta)^2}=\sqrt{k}\,\sqrt{l^2+m^2}\,\frac{(lx+my+n)}{\sqrt{l^2+m^2}}$$

$$\Rightarrow \quad \frac{PS}{PM}=\sqrt{k}\,\sqrt{l^2+m^2}$$

where point $P(x, y)$ is any point on the curve.

Fixed point $S(\alpha, \beta)$ is focus and fixed line $lx+my+n=0$ is directrix.

If $k(l^2+m^2)=1$, P lies on parabola.

If $k(l^2+m^2)<1$, P lies on ellipse.

If $k(l^2+m^2)>1$, P lies on hyperbola.

If $k=0$, P lies on a point circle.

12. a., b.

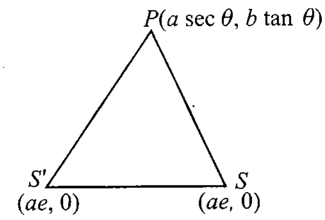


Fig. 5.67

Let (h, k) be excentre, then

$$h=\frac{a(ae\sec\theta+a)-ae(ae\sec\theta-a)-2ae(a\sec\theta)}{2ae(\sec\theta-1)}$$

$h=-a \Rightarrow x=-a$ (for $a\sec\theta>0$)

Similarly, $x=a$ for $a\sec\theta<0$

$\Rightarrow$ locus is $x^2=a^2$

Again let (h, k) be excentre opposite $\angle S'$,

$$\therefore\ h=\frac{2a^2e\sec\theta+a^2e^2\sec\theta+a^2e+a^2e^2\sec\theta-a^2e}{2a+2ae}$$

$$\Rightarrow h=ae\sec\theta,\ k=\frac{2aeb\tan\theta}{2a+2ae}$$

$\Rightarrow$ locus is hyperbola.

13. b., d. Differentiating $xy=1$ w.r.t. x, we have

$$\frac{dy}{dx}=-\frac{1}{x^2}<0$$

Hence, the slope of normal at any point $P(x_1, y_1)$ is

$$x_1^2>0$$

Therefore, slope of the normal must always be positive.

Hence, possible lines are (b) and (d).

14. c., d.

Equations of asymptotes are $4y-3x=0$ and $4y+3x=0$

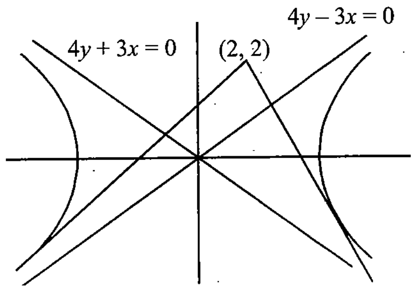
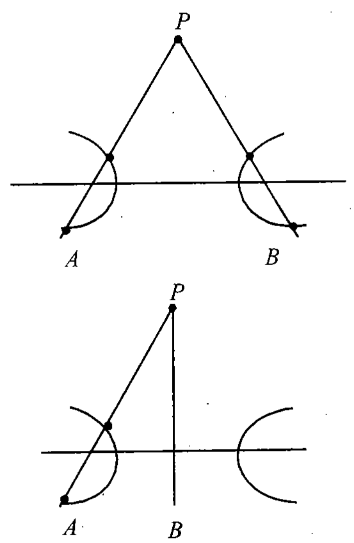


Fig. 5.68

As point (2, 2) lies above the asymptote.

Hence, points of contact of the tangents are in III and IV quadrants.

15. b., c., d.



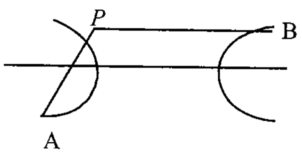


Fig. 5.69

Different cases have been shown in the above diagrams. Therefore, number of points can be 2, 3 or 4.

16. a., b., c.

Locus of point of intersection of perpendicular tangents is director circle $x^2 + y^2 = a^2 - b^2$.

Now,

$$e^2 = 1 + \frac{b^2}{a^2}$$

If $a^2 > b^2$, then there are infinite (or more than 1) points on the circle $\Rightarrow e^2 < 2 \Rightarrow e < \sqrt{2}$.

If $a^2 < b^2$, there does not exist any point on the plane $\Rightarrow e^2 > 2 \Rightarrow e > \sqrt{2}$.

If $a^2 = b^2$, there is exactly one point (centre of the hyperbola) $\Rightarrow e = \sqrt{2}$.

17. a., b., c., d.

Normal at point P (2 sec θ, 2 tan θ) is

$$\frac{2x}{\sec\theta} + \frac{2y}{\tan\theta} = 8$$

It meets the axes at points G (4 sec θ, 0) and g (0, 4 tan θ). Then

$$PG = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$Pg = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$PC = \sqrt{4\sec^2\theta + 4\tan^2\theta}$$

$$Gg = \sqrt{16\sec^2\theta + 16\tan^2\theta}$$

$$= 2\sqrt{4\sec^2\theta + \tan^2\theta} = 2\,PC$$

Reasoning Type

1. b. Statement 1 is correct, see the theory.

Also statement 2 is true as asymptotes are perpendicular, they are bisectors of transverse and conjugate axes of hyperbola.

But statement 2 does not explain statement 1, as in hyperbolas other than rectangular hyperbolas asymptotes are not bisectors of transverse and conjugate axes.

2. a. Tangent to hyperbola having slope m is

$$y = mx \pm \sqrt{4m^2 - 16}$$

which is real line if

$4m^2 - 16 > 0 \Rightarrow m^2 > 4 \Rightarrow m \in (-\infty,-2)\cup(2, \infty)$

Hence, statement 2 is correct.

Also statement 1 is correct and statement 2 is correct explanation of statement 1.

3. a. Let P be the position of the gun and Q be the position of the target. Let u be the velocity of sound, v be the velocity of bullet and R be the position of the man. Then, we have

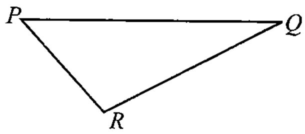


Fig. 5.70

$$\frac{PR}{u} = \frac{QR}{u} + \frac{PQ}{v}$$

$$\Rightarrow \quad \frac{PR}{u} - \frac{QR}{u} = \frac{PQ}{v}$$

$$\Rightarrow \quad PR - QR = \frac{u}{v} PQ = \text{constant}$$

and $\quad \frac{u}{v} PQ < PQ$

Therefore, locus of R is a hyperbola.

4. a. For ellipse $\frac{x^2}{27/12} - \frac{y^2}{27/4} = 1$

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

$$a = 5$$

The foci are (±3, 0).

For hyperbola $\frac{x^2}{27/12} - \frac{y^2}{27/4} = 1$

$$e = \sqrt{1 + \frac{12}{4}} = 2$$

$$a = \frac{3}{2}$$

The foci are (±3, 0).

Therefore, the two conics are confocal. Hence, curves are orthogonal.

5. a. See the theory.

6. d. Statement 2 is true. See the theory.

For the points (2, 2), (4, 1) and (6, 2/3), $t_1 = 1$, $t_2 = 2$ and $t_3 = 3$, respectively.

For the point (1/4, 16), $t_4 = \frac{1}{8}$.

Now $\quad t_1 t_2 t_3 t_4 = \frac{3}{4} \neq 1.$

Hence, statement 1 is false.

7. d.

Statement 1 is false as points in region A lie below the asymptote

$$y = \frac{b}{a}x \Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} > 0$$

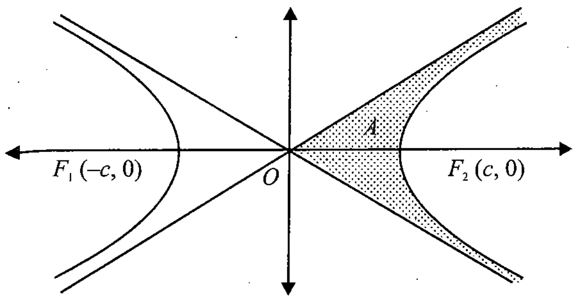


Fig. 5.71

Statement 2 is true (standard result). Indeed for points in region A

$$0 < \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$$

8. b. Given hyperbola is

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

Now line having slope $m = 3$ is tangent to the hyperbola. So, its equation is

$$y = 3x \pm \sqrt{3(3)^2 - 2}$$

or $$y = 3x \pm 5$$

Hence, statement 1 is correct.

Also statement 2 is correct, but information is not enough to get the equations of tangents.

9. d. The locus of point of intersection of two mutually perpendicular tangents drawn on to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is its director circle whose equation is $x^2 + y^2 = a^2 - b^2$.

For $$\frac{x^2}{9} - \frac{y^2}{16} = 1, x^2 + y^2 = 9 - 16$$

So director circle does not exist.

10. b. Chord of contact of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t. point $P(x_1, y_1)$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ (i)

Equation (i) can be written as

$$\frac{x(-x_1)}{a^2} - \frac{y(-y_1)}{b^2} = -1$$

which is tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

at point $(-x_1, -y_1)$.

Obviously, points (x_1, y_1) and $(-x_1, -y_1)$ lie on the different branches of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Hence, statement 1 is correct.

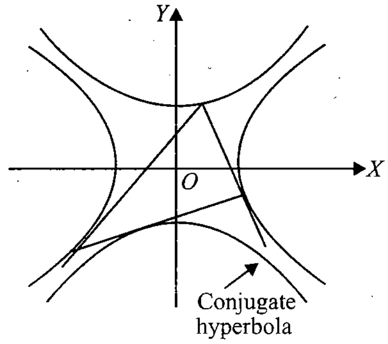


Fig. 5.72

Statement 2 is also correct but does not explain statement I.

11. d. We have

$$\sqrt{(\lambda - 3)^2 + 16} - 4 = 1 \Rightarrow \lambda = 0 \text{ or } 6$$

12. a. Given that

$$\frac{r_2}{r_3} = k \text{ (constant)}$$

$$\frac{\frac{\Delta}{s-b}}{\frac{\Delta}{s-c}} = k$$

where Δ is area of triangle and s is semi-perimeter

$$\Rightarrow \frac{s-c}{s-b} = k$$

$$\Rightarrow \frac{b-c}{2s-(b+c)} = \frac{k-1}{k+1}$$

$$\Rightarrow b - c = a\left(\frac{k-1}{k+1}\right) \text{ (constant) (as } BC = a \text{ is given)}$$

Therefore, the locus of vertex A is a hyperbola.

Linked Comprehension Type

For Problems 1–3

1. b., 2. c., 3. b.

Sol.

Let the curve be $y = f(x)$.

Now tangent at point P to the curve is

$$Y - y = m(X - x)$$

It meets y-axis when

$$X = 0 \Rightarrow Y = y - mx$$

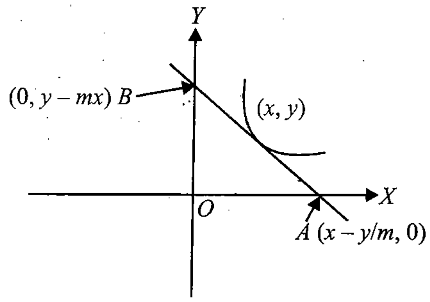


Fig. 5.73

and x-axis when

$$Y = 0 \Rightarrow X = x - \frac{y}{m}$$

Given that P is midpoint of AB. Hence,

$$x - \frac{y}{m} = 2x$$

$$\Rightarrow \quad \frac{y}{m} = -x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\Rightarrow \quad \log_e xy = c$$

$$\Rightarrow \quad xy = c$$

As the curve passes through (2, 4), so

$$xy = 8$$

Solving with $y = x$, we get

$$x = 2\sqrt{2}$$

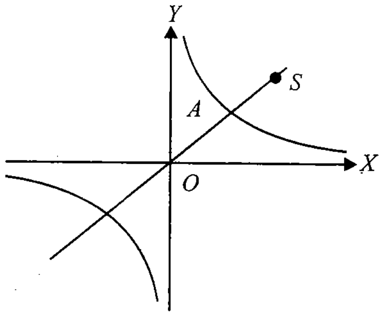


Fig. 5.74

$$\therefore \quad OA = \sqrt{8+8} = 4$$

$$\Rightarrow \quad OS = 4\sqrt{2}$$

Hence, coordinates of S are (4, 4) or (−4, −4).

Directrix is at distance $4/\sqrt{2}$ from origin.

Hence, its equations $x + y = \pm 4$.

For Problems 4–6

4. d., 5. b., 6. c.

Sol. 4. d. Centre $\equiv (1, 2)$

Radius of auxiliary circle $= a = \sqrt{(2-1)^2 + (5-2)^2}$

$$= \sqrt{10}$$

$$2ae = \sqrt{8^2 + 8^2} = 8\sqrt{2} \Rightarrow e = \frac{4}{\sqrt{5}}$$

$$b^2 = a^2e^2 - a^2 = 32 - 10 = 22$$

$$\Rightarrow \quad 2b = 2\sqrt{22}$$

5. b. Directrix is perpendicular to the transverse axis. Let it be $x + y = k$.

Its distance from centre $\frac{a}{e}$

$$\Rightarrow \quad \frac{|1+2-k|}{\sqrt{2}} = \frac{5}{2\sqrt{2}} \Rightarrow k = 3 + \frac{5}{2} = \frac{11}{2}$$

6. c. The tangent is

$$y - 5 = -\frac{5-2}{6-5}(x-2)$$

$$\Rightarrow \quad 3x + y = 11$$

The hyperbola is

$$(x-5)^2 + (y-6)^2 = \frac{16}{5} \times \frac{(2x+2y-11)^2}{8}$$

Solving, we get

$$(x-5)^2 + (5-3x)^2 = \frac{2}{5}(2x + 22 - 6x - 11)^2$$

$$\Rightarrow \quad 5[10x^2 - 40x + 50] = 2(11-4x)^2$$

$$\Rightarrow \quad 9x^2 - 12x + 4 = 0$$

$$\Rightarrow \quad (3x-2)^2 = 0 \Rightarrow x = \frac{2}{3}, y = 9$$

For Problems 7–9

7. c., 8. d., 9. b.

Sol. 7. c. $2a = 3$

Distance between the foci (1, 2) and (5, 5) is 5.

$$\therefore \quad 2ae = 5$$

$$\therefore \quad e = \frac{5}{3}$$

Now if e' is eccentricity of the corresponding conjugate hyperbola, then

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\Rightarrow \quad e' = \frac{5}{4}$$

8. d. Director circle is given by

$$(x-h)^2 + (y-k)^2 = a^2 - b^2$$

where (h, k) is centre, i.e. the

midpoint of foci $\left(\frac{1+5}{2}, \frac{2+5}{2}\right) \equiv \left(3, \frac{7}{2}\right)$.

$$Y - y = m(X - x)$$

$$b^2 = a^2(e^2 - 1) = \left(\frac{3}{2}\right)^2\left(\left(\frac{5}{3}\right)^2 - 1\right)$$
$$= 4$$

Therefore, the director circle is

$$(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{9}{4} - 4$$

$$\Rightarrow \quad (x-3)^2 + \left(\frac{y-7}{2}\right)^2 = -\frac{7}{4}$$

This does not represent any real point.

9. b. Slope of transverse axis is $\frac{3}{4}$.

Therefore, the angle of rotation is

$$\theta = \tan^{-1}\frac{3}{4}.$$

For Problems 10–12

10. a., 11. c., 12. b.

Sol. 10. a. Equation of tangent in parametric form is given by

$$\frac{x-1}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = \pm 3\sqrt{2}$$

$$\Rightarrow \quad A \equiv (4, -2),\ B \equiv (-2, 4)$$

Equations of asymptotes (OA and OB) are given by

$$y + 2 = \frac{-2}{4}(x-4) \Rightarrow 2y + x = 0$$

and

$$y - 4 = \frac{4}{-2}(x+2) \Rightarrow 2x + y = 0$$

Hence, the combined equation of asymptotes is

$$(2x+y)(x+2y) = 0$$

$$\Rightarrow \quad 2x^2 + 2y^2 + 5xy = 0$$

11. c. $m_{OA} = -\frac{1}{2}$, $m_{OB} = -2$

$$\tan\theta = \left|\frac{-1/2+2}{1+1}\right| = \frac{3}{4}$$

$$\Rightarrow \quad \sin\theta = \frac{3}{5} \Rightarrow \theta = \sin^{-1}\left(\frac{3}{5}\right)$$

12. b. Let the equation of the hyperbola be

$$2x^2 + 2y^2 + 5xy + \lambda = 0$$

It passes through (1, 1). Therefore,

$$2 + 2 + 5 + \lambda = 0$$

$$\Rightarrow \quad \lambda = -9$$

So, the hyperbola is

$$2x^2 + 2y^2 + 5xy = 9$$

Equation of the tangent at $\left(-1, \frac{7}{2}\right)$ is given by

$$2x(-1) + 2y\left(\frac{7}{2}\right) + 5\,\frac{x(7/2) + (-1)y}{2} = 9$$

$$\Rightarrow \quad 3x + 2y = 4$$

For Problems 13–15

13. b., 14. c., 15 d.

13. b. Any point on the hyperbola $xy = 16$ is $\left(4t, \frac{4}{t}\right)$

Normal at this point is $y - 4/t = t^2(x - 4t)$.

If the normal passes through $P(h, k)$, then $k - 4/t = t^2(h - 4t)$

$$\Rightarrow \quad 4t^4 - t^3h + tk - 4 = 0$$

This equation has roots t_1, t_2, t_3, t_4 which are parameters of the four feet of normals on the hyperbola. Therefore,

$$\Sigma t_1 = \frac{h}{4}$$

$$\Sigma t_1t_2 = 0$$

$$\Sigma t_1t_2t_3 = -\frac{k}{4}$$

and

$$t_1t_2t_3t_4 = -1$$

$$\therefore \quad \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{4}$$

$$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = k$$

According to the question,

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{h^2}{16} = k$$

Hence, the locus of (h, k) is

$$x^2 = 16y$$

14. c. $x^2 = 16y$

Equation of tangent of P is

$$x\,4t = \frac{16\,(y + t^2)}{2}$$

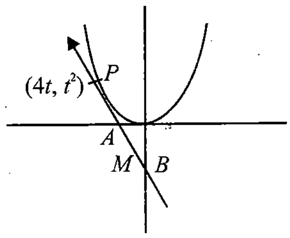


Fig. 5.75

$$\Rightarrow \quad 4tx = 8y + 8t^2$$

$$\Rightarrow \quad tx = 2y + 2t^2$$

$$A = (2t, 0),\ B = (0, -t^2)$$

$M(h, k)$ is the middle point of AB.

$$h = t,\ k = -\frac{t^2}{2} \Rightarrow 2k = -h^2$$

Therefore, the locus of $M(h, k)$ is $x^2 + 2y = 0$.

15. d. $\tan 30° = \frac{4t_1}{t_1^2} = \frac{4}{t_1}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{4}{t_1} \Rightarrow t_1 = 4\sqrt{3}$$

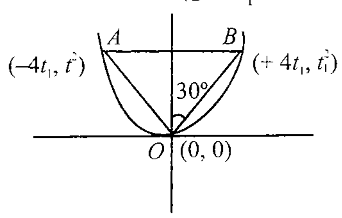


Fig. 5.76

$$AB = 8t_1 = 32\sqrt{3}$$

Area of $\Delta OAB = \frac{\sqrt{3}}{4} \times 32\sqrt{3} \times 32\sqrt{3}$

$$= 768\sqrt{3}\ \text{sq. units}$$

For Problems 16–18

16. b., 17. c., 18. d.

Sol. 16. b. Perpendicular tangents intersect at the centre of rectangular hyperbola. Hence, centre of hyperbola is (1, 1) and equation of asymptotes are $x - 1 = 0$ and $y - 1 = 0$.

17. c. Let the equation of hyperbola is $xy - x - y + 1 + \lambda = 0$. It passes through (3, 2) hence $\lambda = -2$.

So the equation of hyperbola is

$$xy = x + y + 1$$

18. d. From the centre of hyperbola we can draw two real tangents to the rectangular hyperbola.

Matrix-Match Type

1. a → p; b → p, q; c → q; d → s.

We have

$$A = ae_E \text{ and } a = Ae_H$$

$$\Rightarrow \quad e_E\, e_H = 1 \Rightarrow e_E + e_H > 2$$

$$B^2 = A^2\left(e_H^2 - 1\right) = a^2\left(1 - e_E^2\right)$$

$$= b^2$$

$$\Rightarrow \quad \frac{b}{B} = 1$$

Also the angle between the asymptotes is

$$2\tan^{-1}\frac{B}{A} = \frac{2\pi}{3}$$

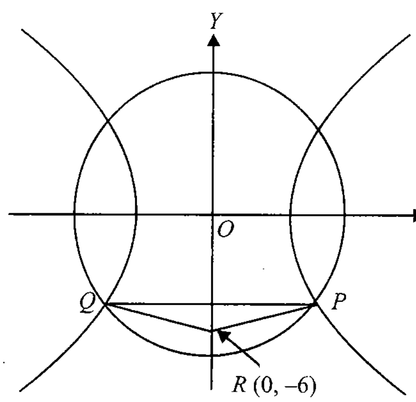
Also, $\quad \frac{B}{A} = \sqrt{3} \Rightarrow \frac{b}{ae_E} = \sqrt{3} \Rightarrow e_E^2 = \frac{1}{4}$

Solving $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2e_E^2} - \frac{y^2}{b^2} = 1$

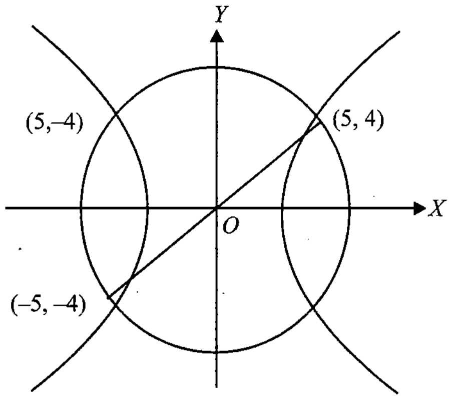
we get

$$x^2 = \frac{2a^2 e_E^2}{b^2(1 - e_E^2)} = 4$$

2. a → p, q, r, s ; b → q, r ; c → p ; d → p, s.



(a)



(b)

Fig. 5.77

a. Obviously all the points in column II are common to the hyperbola and circle.

b. Chord of contact of hyperbola w.r.t. $\left(0, -\frac{9}{4}\right)$ is $0(x) - \left(-\frac{9}{4}\right)y = 9$ or $y = 4$

Solving this with hyperbola we have

$$x^2 - 16 = 9 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$$

Hence, points of contact are (±5, 4).

c. Obviously the required point is (–5, –4).

d. Let the points on the hyperbola be $P(h, k)$ and $Q(-h, k)$.

Then area of triangle is $\frac{1}{2}|2h||-6-k| = 10$

$$\Rightarrow \quad |h||6 + k| = 10 \qquad \text{(i)}$$

Also points P and Q lie on the hyperbola. Hence,

$$h^2 - k^2 = 9 \qquad \text{(ii)}$$

Obviously points (±5, –4) satisfy both Eqs. (i) and (2).

3. a → p; b → s, c → r; d → p.

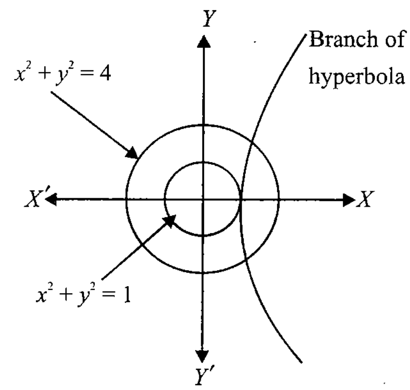


Fig. 5.78

Locus of point P satisfying $PA - PB = 2$ is a branch of the hyperbola $x^2 - y^2/3 = 1$.

For $r = 2$ the circle and the branch of the hyperbola intersect at two points. For $r = 1$ there is no point of intersection.

If m be the slope of the common tangent, then

$$m^2 - 3 = r^2(1 + m^2)$$

$$\Rightarrow \quad m^2 = \frac{r^2 + 3}{1 - r^2}$$

Hence, there are no common tangents for $r > 1$ and two common tangents for $r < 1$.

4. a → r; b → p; c → s; d → q.

a. $\text{Im}(z^2) = 3$

$\Rightarrow \text{Im}((x + iy)^2) = 3$

$\Rightarrow 2xy = 3$, which is a rectangular hyperbola having eccentricity $\sqrt{2}$

b.

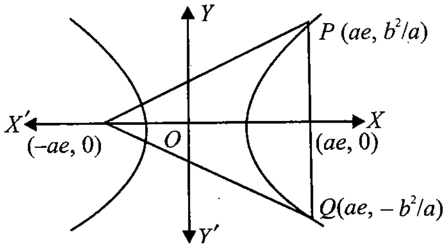


Fig. 5.79

$$\tan 30° = \frac{b^2/a}{2ae}$$

$$\Rightarrow \quad \frac{2}{\sqrt{3}}e = e^2 - 1$$

$$\Rightarrow \quad \sqrt{3}e^2 - 2e - \sqrt{3} = 0$$

$$\Rightarrow \quad e = \frac{2 \pm \sqrt{4 + 12}}{2\sqrt{3}} = \frac{2 \pm 4}{2\sqrt{3}}$$

$$\Rightarrow \quad e = \frac{3}{\sqrt{3}} = \sqrt{3}$$

c. Eccentricity of the hyperbola $= \frac{AB}{PA - PB} = \frac{6}{4} = \frac{3}{2}$

If eccentricity of conjugate hyperbola is e', then $\frac{1}{\left(\frac{3}{2}\right)^2} + \frac{1}{e'^2} = 1$

$$\Rightarrow \quad e' = \frac{3}{\sqrt{5}}$$

d. Angle between the asymptotes is $\tan^{-1}\left|\frac{2ab}{a^2 - b^2}\right| = \frac{\pi}{3}$

$$\Rightarrow \quad \left|\frac{2\frac{a}{b}}{\frac{a^2}{b^2} - 1}\right| = \sqrt{3}$$

$\Rightarrow \frac{2\sqrt{e'^2 - 1}}{|e'^2 - 2|} = \sqrt{3}$ (where e' is eccentricity of conjugate hyperbola)

$$\Rightarrow \quad e' = 2$$

5. a → p., s.; b → q, r; c → r; d → p, s.

a. We must have

$e_1 < 1 < e_2 \Rightarrow f(1) < 0 \Rightarrow 1 - a + 2 < 0 \Rightarrow a > 3$

b. We must have both the roots greater than 1.

i. $D > 0$ or $a^2 - 4 > 0$ or $a \in (-\infty, -2) \cup (2, \infty)$

ii. $1 \cdot f(1) > 0$ or $1 - a + 2 > 0$ or $a < 3$

iii. $\frac{a}{2} >= 1 \Rightarrow a > 2$

From Eqs. (i), (ii) and (iii) we have $a \in (2, 3)$.

c. We must have

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

$$\Rightarrow \quad \frac{(e_1 + e_2)^2 - 2e_1e_2}{e_1^2e_2^2} = 1$$

$$\Rightarrow \quad \frac{a^2 - 4}{4} = 1$$

$$\Rightarrow \quad a = \pm 2\sqrt{2}$$

d. We must have

$$e_2 < \sqrt{2} < e_1$$

$$\Rightarrow \quad f(\sqrt{2}) < 0$$

$$\Rightarrow \quad 2 - a\sqrt{2} + 2 < 0$$

$$\Rightarrow \quad a > 2\sqrt{2}$$

Integer Type

1. (5) For given equation of hyperbola foci are $S(3, 2)$ and $S'(-1, -1)$,

Using definition of hyperbola $|SP - S'P| = 2a$,

We have $SS' = 5$ and $2a = 1$

Hence eccentricity is $\frac{SS'}{2a} = 5$.

2. (5) $e^2 = \frac{b^2}{a^2} + 1 \Rightarrow \frac{b^2}{a^2} = e^2 - 1 = 24$

Now $y = mx + c$ is tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then we must have $a^2m^2 - b^2 \geq 0$

or $m^2 \geq b^2/a^2$ or $m^2 \geq 24$ then least positive integral value of m is 5.

3. (4) We have $y = Ax^2$, $y^2 + 3 = x^2 + 4y$; $A > 0$

Now $y^2 - 4y = x^2 - 3$

$$\Rightarrow \quad (y - 2)^2 = x^2 + 1$$

$$\Rightarrow \quad (y - 2)^2 - x^2 = 1$$

If $x = 0$, $y - 2 = 1$ or $-1 \Rightarrow y = 3$ or 1

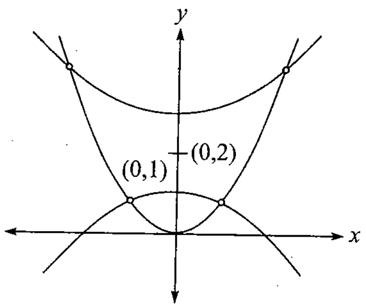


Fig. 5.80

Hence the two graphs of $y = Ax^2$ $(A > 0)$ and the hyperbola $(y-2)^2 - x^2 = 1$ are as shown which intersects in 4 points.

4. **(7)** Given hyperbola is

$$3x^2 - 2y^2 = 6 \text{ or } \frac{x^2}{2} - \frac{y^2}{3} = 1$$

Tangents from the point (α, β)

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

or $(y - mx)^2 = a^2m^2 - b^2$

or $(\beta - m\alpha)^2 = 2m^2 - 3$ ($\because a^2 = 2$ and $b^2 = 3$)

or $m^2\alpha^2 + \beta^2 - 2m\alpha\beta - 2m^2 + 3 = 0$

$m^2(\alpha^2 - 2) - 2\alpha\beta m + \beta^2 + 3 = 0$

$$m_1 . m_2 = \frac{\beta^2 + 3}{\alpha^2 - 3} = 2 = \tan\theta \cdot \tan\phi$$

$\therefore \beta^2 + 3 = 2(\alpha^2 - 2)$

or $2\alpha^2 - \beta^2 = 7$

5. **(3)** Since tangent drawn from the point $A(a, 2)$ are perpendicular then A must lie on the director circle $x^2 + y^2 = 7$. Putting $y = 2$ we get the value of $x^2 = a^2 = 3$

6. **(3)** $(a, 2)$ lies on director circle $x^2 + y^2 = 7$.

$\therefore a^2 = 3$

7. **(8)** Hyperbola is $x^2 - 9y^2 = 9$ or $\frac{x^2}{9} - \frac{y^2}{1} = 1$

Equation of tangent is $y = mx \pm \sqrt{a^2m^2 - b^2}$ (1)

It passes through (3, 2)

$\Rightarrow 2 = 3m \pm \sqrt{9m^2 - 1}$

or $4 + 9m^2 - 12m = 9m^2 - 1$

Solving we get values of m as $m_1 = \frac{5}{12}$ and $m_2 = \infty$

Equation of tangent (1) for $m_1 = \frac{5}{12}$

$$y = \frac{5}{12}x \pm \sqrt{9\left(\frac{5}{12}\right)^2 - 1}$$

or $y = \frac{5}{12}x \pm \frac{3}{4}$

on taking (–)ve sign point $P(3, 2)$ does not satisfy the equation of tangent therefore rejecting (–)ve sign. Hence equation of tangent is $y = \frac{5x}{12} + \frac{3}{4}$ (2)

now equation of tangent (1) for $m_2 = \infty$ is $x \pm 3 = 0$

rejecting (+) sign (since taking (+) sign point $P(3, 2)$ does not satisfy this equation.)

Hence equation of tangent is $x - 3 = 0$ (3)

Now equation of chord of contact w.r.t. point $P(3, 2)$ is $T = 0$

or $3x - 18y = 9$

or $x - 6y = 3$ (4)

Solving (2) and (4); $x = -5, y = -\frac{4}{3}$

Solving (3) and (4); $x = 3, y = 0$

Now vertices of triangle are (3, 2), (3, 0), (–5, –4/3)

$$\therefore \text{Area} = \frac{1}{2}\begin{vmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ -5 & -\frac{4}{3} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \left| 3\left(\frac{4}{3}\right) - 2(3+5) + 1(-4) \right|$$

$$= \frac{1}{2} \,|4 - 16 - 4|$$

$= 8$ sq. units

8. **(2)** Since $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate

$$\therefore \frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\therefore 4 = \frac{e^2e'^2}{e'^2 + e'^2}$$

line passing through the points $(e, 0)$ and $(0, e')$

$e'x + ey - ee' = 0$

It is tangent to the circle $x^2 + y^2 = r^2$

$$\therefore \frac{ee'}{\sqrt{e^2 + e'^2}} = r$$

$$\therefore r^2 = \frac{e^2e'^2}{e^2 + e'^2} = 4$$

$\therefore r = 2$

9. **(3)** Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

then $2a = ae$, i.e. $e = 2$

$$\therefore \frac{b^2}{a^2} = e^2 - 1 = 3$$

$\therefore \dfrac{(2b)^2}{(2a)^2} = 3$

10. (5) Equation of tangents to hyperbola having slope m are

$y = mx \pm \sqrt{9m^2 - 49}$

Distance between tangents is 2

$\Rightarrow \dfrac{2\sqrt{9m^2 - 49}}{\sqrt{1 + m^2}} = 2$

$\Rightarrow 9m^2 - 49 = 1 + m^2$

$\Rightarrow 8m^2 = 50 \Rightarrow m = \pm\dfrac{5}{2}$

11. (4) Equation of hyperbola $(x - 3)(y - 2) = c^2$

or $xy - 2x - 3y + 6 = c^2$

It passes through (4, 6), then

$4 \times 6 - 2 \times 4 - 3 \times 6 + 6 = c^2$

$\Rightarrow c = 2$

$\therefore$ Latus rectum $= 2\sqrt{2}c = 2\sqrt{2} \times 2 = 4\sqrt{2}$

12. (6) Equation of hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{18} = 1$

or $9x^2 - 8y^2 - 144 = 0$

Homogenization of this equation using

$\dfrac{x\cos\alpha + y\sin\alpha}{p} = 1$

we have $9x^2 - 8y^2 - 144\left(\dfrac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$

Since these lines are perpendicular to each other

$\therefore 9p^2 - 8p^2 - 144(\cos^2\alpha + \sin^2\alpha) = 0$

$p^2 = 144$ or $p = \pm 12$

$\therefore$ radius of the circle = 12

$\therefore$ diameter of the circle = 24

13. (8) The point $P\left(\dfrac{\pi}{6}\right)$ is $\left(a\sec\dfrac{\pi}{6}, b\tan\dfrac{\pi}{6}\right)$ or $P\left(\dfrac{2a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}\right)$

$\therefore$ Equation of tangent at P is $\dfrac{x}{\frac{\sqrt{3}a}{2}} - \dfrac{y}{\sqrt{3}b} = 1$

$\therefore$ Area of the triangle $= \dfrac{1}{2} \times \dfrac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2$

$\therefore \dfrac{b}{a} = 4$

$\therefore e^2 = 1 + \dfrac{b^2}{a^2} = 17$

14. (4) Eccentricity of the hyperbola $x^2 - y^2\sec^2\theta = 5$ is

$e_1 = \sqrt{\dfrac{1 + \sec^2\theta}{\sec^2\theta}} = \sqrt{1 + \cos^2\theta}$

Eccentricity of the ellipse $x^2\sec^2\theta + y^2 = 25$ is

$e_2 = \sqrt{\dfrac{\sec^2\theta - 1}{\sec^2\theta}} = |\sin\theta|$

Given $e_1 = \sqrt{3}\, e_2$

$\Rightarrow 1 + \cos^2\theta = 3\sin^2\theta$

$\Rightarrow \cos\theta = \pm\dfrac{1}{\sqrt{2}}$

$\therefore$ least positive value of θ is $\dfrac{\pi}{4}$

$\therefore p = 4$

15. (4) Let the point be (h, k). Then equation of the chord of contact is $hx + ky = 4$

Since $hx + ky = 4$ is tangent to $xy = 1$

$\therefore x\left(\dfrac{4 - hx}{k}\right) = 1$ has two equal roots

i.e. $hx^2 - 4x + k = 0$ has equal roots

$\therefore hk = 4$

$\therefore$ locus of (h, k) is $xy = 4$

i.e. $c^2 = 4$

Archives

Subjective Type

1. Consider any point $P(3\sec\theta, 2\tan\theta)$ on the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$.

Then, equation of chord of contact to the circle $x^2 + y^2 = 9$ w.r.t. point P is

$(3\sec\theta)x + (2\tan\theta)y = 9$ (i)

If (h, k) be the midpoint of chord of contact, then equation of chord of contact will be

$hx + ky - 9 = h^2 + k^2 - 9$ (from $T = S_1$)

or $hx + ky = h^2 + k^2$ (ii)

But Eqs. (i) and (ii) represent the same straight line. Hence,

$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2 + k^2}$$

$\Rightarrow \sec\theta = \dfrac{3h}{h^2 + k^2}$, $\tan\theta = \dfrac{9k}{2(h^2 + k^2)}$

Eliminating θ, we have

$$\frac{9h^2}{(h^2 + k^2)^2} - \frac{81k^2}{4(h^2 + k^2)^2} = 1$$

$\Rightarrow 4h^2 - 9k^2 = \frac{4}{9}(h^2 + k^2)^2$

$\Rightarrow \dfrac{h^2}{9} - \dfrac{k^2}{4} = \left(\dfrac{h^2 + k^2}{9}\right)^2$

Therefore, the locus of (h, k) is

$$\frac{x^2}{9}-\frac{y^2}{4}=\left(\frac{x^2+y^2}{9}\right)^2$$

Objective Type

Fill in the blanks

1.

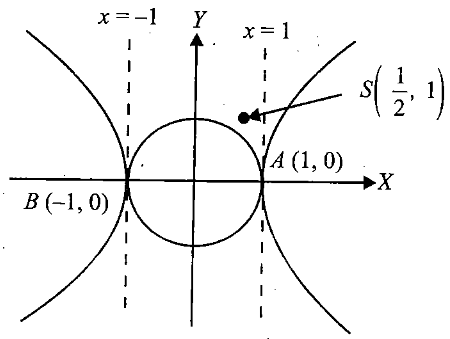


Fig. 5.81

For the circle $x^2 + y^2 = 1$ and rectangular hyperbola $x^2 - y^2 = 1$, one common tangent is evidently $x = 1$, the other being $x = -1$. The required standard form of the ellipse with focus at $S\left(\frac{1}{2}, 1\right)$ and directrix $x = 1$ is

$$\left(x-\frac{1}{2}\right)^2+(y-1)^2=\left(\frac{1}{2}\right)^2(1-x)^2$$

$$\Rightarrow \quad \frac{3x^2}{4}-\frac{x}{2}+(y-1)^2=0$$

$$\Rightarrow \quad \frac{3}{4}\left(x-\frac{1}{3}\right)^2+(y-1)^2=\frac{1}{12}$$

$$\Rightarrow \quad 9\left(x-\frac{1}{3}\right)^2+12(y-1)^2=1$$

Multiple choice questions with one correct answer

1. d. Given that

$$\frac{x^2}{1-r}-\frac{y^2}{1+r}=1$$

As $\quad r > 1$, so $1 - r < 0$ and $1 + r > 0$

Let $\quad 1 - r = -a^2,\ 1 + r = b^2$.

Then we get

$$\frac{x^2}{-a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$$

2. c.

$x^2 + 2y^2 \le 1$ represents interior region of circle, where on taking any two points the midpoint of that segment will also lie inside that circle.

Max $\{|x|, |y|\} \le 1 \Rightarrow |x| \le 1, |y| \le 1 \Rightarrow -1 \le x \le 1$ and $-1 \le y \le 1$

which represents the interior region of a square with its sides $x = \pm 1$ and $y = \pm 1$ in which for any two points, their midpoint also lies inside the region.

$x^2 - y^2 \le 1$ represents the exterior region of hyperbola in which we take two points (4, 3) and (4, –3). Then their midpoint (4, 0) does not lie in the same region (as shown in the figure).

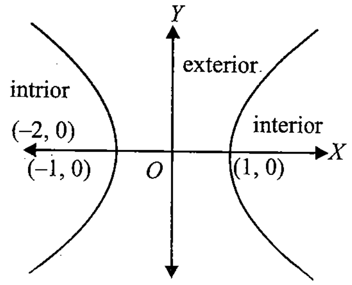


Fig. 5.82

$y^2 \le x$ represents interior region of parabola in which for any two points, their midpoint also lies inside the region.

3. c. We have

$$2x^2 + 3y^2 - 8x - 18y + 35 = k$$
$$\Rightarrow 2(2x^2 - 4x) + 3(y^2 - 6y) + 35 = k$$
$$\Rightarrow \quad 2(x-2)^2 + 3(y-3)^2 = k$$

For $\quad k = 0$, we get

$$2(x-2)^2 + 3(y-3)^2 = 0$$

which represents the point (2, 3).

4. c. We have

$$2x^2 + 3y^2 - 18y + 35 = k$$
$$\Rightarrow 2(2x^2 - 4x) + 3(y^2 - 6y) + 35 = k$$
$$\Rightarrow \quad 2(x-2)^2 + 3(y-3)^2 = k$$

For $\quad k = 0$, we get

$$2(x-2)^2 + 3(y-3)^2 = 0$$

which represents the point (2, 3).

5. d. Normals at $p(\theta)$, $Q(\phi)$ are

$$ax\cos\theta + by\cot\theta = a^2 + b^2$$
$$ax\cos\phi + by\cot\phi = a^2 + b^2$$

where $\varphi = \frac{\pi}{2} - \theta$ and these pass through (h, k). Therefore,

$$ah\cos\theta + bk\cot\theta = a^2 + b^2$$

and

$$ah\sin\theta + bk\tan\theta = a^2 + b^2$$

Eliminating h, we have

$$bk(\cot\theta\sin\theta - \tan\theta\cos\theta) = (a^2 + b^2)(\sin\theta - \cos\theta)$$

$$\Rightarrow \quad k = -\left(\frac{a^2+b^2}{b}\right)$$

6. b. Let a pair of tangents be drawn from point (x_1, y_1) to hyperbola

$$x^2 - y^2 = 9$$

Then chord of contact will be

$$xx_1 - yy_1 = 9 \quad \text{(i)}$$

But the given chord of contact is

$$x = 9 \quad \text{(ii)}$$

As Eqs. (i) and (ii) represent the same line, these equations should be identical and hence

$$\frac{x_1}{1} = -\frac{y_1}{0} = \frac{9}{9} \Rightarrow x_1 = 1, y_1 = 0$$

Therefore, the equation of pair of tangents drawn from (1, 0) to $x^2 - y^2 = 9$ is

$$(x^2 - y^2 - 9)(1^2 - 0^2 - 9) = (x.1 - y.0 - 9)^2$$

(using $SS_1 = T^2$)

$$\Rightarrow \quad (x^2 - y^2 - 9)(-8) = (x - 9)^2$$

$$\Rightarrow \quad -8x^2 + 8y^2 + 72 = x^2 - 18x + 81$$

$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

7. b. $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$

$$a^2 = \cos^2\alpha$$

$$\therefore \quad a^2 e^2 = 1$$

Hence, the foci are $(\pm ae, 0) = (\pm 1, 0)$, which are independent of α.

8. d. Equation of tangent to hyperbola $x^2 - 2y^2 = 4$ at any point (x_1, y_1) is $xx_1 - 2yy_1 = 4$.

Comparing with $2x + \sqrt{6}\,y = 2$ or $4x + 2\sqrt{6}\,y = 4$, we have

$$x_1 = 4 \text{ and } -2y_1 = 2\sqrt{6}$$

$\Rightarrow (4, -\sqrt{6})$ is the required point of contact

9. a. The length of transverse axis is $2\sin\theta = 2a$

$$\Rightarrow \quad a = \sin\theta$$

Also, for ellipse

$$3x^2 + 4y^2 = 12$$

or

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$a^2 = 4, b^2 = 3$$

$$\therefore \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$$

Hence, the focus of ellipse is $\left(2 \times \frac{1}{2}, 0\right) = (1, 0)$.

As hyperbola is confocal with ellipse, focus of hyperbola = (1, 0). Now,

$$ae = 1 \Rightarrow \sin\theta \times e = 1$$

$$\Rightarrow \quad e = \operatorname{cosec}\theta$$

$$\therefore\ b^2 = a^2(e^2 - 1) = \sin^2\theta(\operatorname{cosec}^2\theta - 1) = \cos^2\theta$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$$

or

$$x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$$

10. b. The given hyperbola is

$$\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$$

$$\Rightarrow \quad a = 2, b = \sqrt{2}, e = \sqrt{\frac{3}{2}}$$

Therefore, the required area $= \frac{1}{2} a(e - 1) \times \frac{b^2}{a}$

$$= \frac{1}{2}\frac{(\sqrt{3} - \sqrt{2}) \times 2}{\sqrt{2}}$$

$$= \frac{(\sqrt{3} - \sqrt{2})}{\sqrt{2}}$$

$$= \left(\sqrt{\frac{3}{2}} - 1\right)$$

11. b. $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$

$$\Rightarrow ax^2 + by^2 + c = 0$$

or

$$x^2 - 5xy + 6y^2 = 0$$

$\Rightarrow x^2 + y^2 = \left(-\frac{c}{a}\right)$ iff $a = b$, $x - 2y = 0$ and $x - 3y = 0$

Hence, the given equation represents two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a.

12. b. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xb^2}{ya^2}$$

$\Rightarrow$ slope of normal at (6, 3) is $\frac{-a^2}{2b^2}$

Equation of normal is $(y - 3) = \frac{-a^2}{2b^2}(x - 6)$

It passes through the point (9, 0)

$$\Rightarrow \quad \frac{a^2}{2b^2} = 1 \Rightarrow e = \sqrt{\frac{3}{2}}$$

Multiple choice questions with one or more than one correct answer

1. a, c. For the given ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

Hence, the eccentricity of hyperbola $= \frac{5}{3}$.

Let the hyperbola be

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$$

Then

$$B^2 = A^2\left(\frac{25}{9} - 1\right) = \frac{16}{9}A^2$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{A^2} - \frac{9y^2}{16A^2} = 1$$

As it passes through (3, 0), we get $A^2 = 9$, $B^2 = 16$.

The equation is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Focus of hyperbola is $(\pm ae, 0) \equiv (\pm 5, 0)$.

Vertex of hyperbola is (3, 0).

2. a, b Ellipse and hyperbola will be confocal. So,

$$\left(\pm a \times \frac{1}{\sqrt{2}}, 0\right) \equiv (\pm 1, 0)$$

$\Rightarrow \quad a = \sqrt{2}$ and $e = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad b^2 = a^2(1 - e^2) \Rightarrow b^2 = 1$

Therefore, the equation of ellipse is

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

3. b, d

For ellipse $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$,

$\Rightarrow \quad 1^2 = 2^2(1 - e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$

$\therefore$ eccentricity of the hyperbola is $\frac{2}{\sqrt{3}}$

$\Rightarrow \quad b^2 = a^2\left(\frac{4}{3} - 1\right) \Rightarrow 3b^2 = a^2$

One of the foci of the ellipse is $(\sqrt{3}, 0)$

Hyperbola passes through $(\sqrt{3}, 0)$

$\Rightarrow \quad \frac{3}{a^2} = 1 \Rightarrow a^2 = 3$ and $b^2 = 1$

$\therefore$ Equation of hyperbola is $x^2 - 3y^2 = 3$

Focus of hyperbola is $(ae, 0) \equiv \left(\sqrt{3} \times \frac{2}{\sqrt{3}}, 0\right) \equiv (2, 0)$

Comprehension type

1. b.

A tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is $y = mx + \sqrt{9m^2 - 4}, m > 0$

It is tangent to $x^2 + y^2 - 8x = 0$

$\therefore \quad \frac{4m + \sqrt{9m^2 - 4}}{\sqrt{1 + m^2}} = 4$

$\Rightarrow \quad 495m^4 + 104m^2 - 400 = 0$

$\Rightarrow \quad m^2 = \frac{4}{5}$ or $m = \frac{2}{\sqrt{5}}$

$\Rightarrow$ the tangent is $y = \frac{2}{\sqrt{5}}m + \frac{4}{\sqrt{5}}$

$\Rightarrow \quad 2x - \sqrt{5}y + 4 = 0$.

2. a.

A point on hyperbola is $(3\sec\theta, 2\tan\theta)$

It lies on the circle, so $9\sec^2\theta + 4\tan^2\theta - 24\sec\theta = 0$

$\Rightarrow \quad 13\sec^2\theta - 24\sec\theta - 4 = 0 \Rightarrow \sec\theta = 2, -\frac{2}{13}$

$\therefore \quad \sec\theta = 2 \Rightarrow \tan\theta = \sqrt{3}$.

The point of intersection are $A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$

$\therefore$ The circle with AB as diameter is

$(x - 6)^2 + y^2 = (2\sqrt{3})^2 \Rightarrow x^2 + y^2 - 12x + 24 = 0$

Match the following

1. a → p; b → s, t; c → r; d → q, s.

p. Line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$. Hence,

$$\frac{1}{\sqrt{h^2 + k^2}} = 2$$

$\Rightarrow \quad$ locus is $x^2 + y^2 = \left(\frac{1}{2}\right)^2$ which is a circle

q. If $|z - z_1| - |z - z_2| = k$ where $k < |z_1 - z_2|$, the locus is a hyperbola.

r. Let $t = \tan\alpha$

$\Rightarrow \quad x = \sqrt{3}\cos 2\alpha$ and $y = \sin 2\alpha$

$\Rightarrow \quad \cos 2\alpha = \frac{x}{\sqrt{3}}$ and $\sin 2\alpha = y$

$\Rightarrow \frac{x^2}{3} + y^2 = \sin^2 2\alpha + \cos^2 2\alpha = 1$ which is an ellipse.

s. If eccentricity is $[1, \infty)$, then the conic can be a parabola (if $e = 1$) and a hyperbola if $e \in (1, \infty)$.

t. Let $z = x + iy; x, y \in R$

$\Rightarrow \quad (x + 1)^2 - y^2 = x^2 + y^2 + 1$

$\Rightarrow \quad y^2 = x$; which is a parabola.

Integer type

1. (2)

Substituting $\left(\frac{a}{e}, 0\right)$ in $y = -2x + 1$

$\therefore\ 0 = -\frac{2a}{e} + 1$

$\therefore\ \frac{2a}{e} = 1$

$\therefore\ a = \frac{e}{2}$

Also, $1 = \sqrt{a^2 m^2 - b^2}$

$\Rightarrow\ 1 = a^2 m^2 - b^2$

$\Rightarrow\ 1 = 4a^2 - b^2$

$\Rightarrow\ 1 = \frac{4e^2}{4} - b^2$

$\Rightarrow\ b^2 = e^2 - 1$

Also, $b^2 = a^2 (e^2 - 1)$

$\therefore\ a = 1, e = 2$

Appendix

Solutions to Concept Application Exercises

Chapter 1

Exercise 1.1

1. c. $L = \sqrt{4+12} = 4$

$\Rightarrow \quad p^2 + q^2 = 16$ and $(p-2)^2 + (q - 2\sqrt{3})^2 = 16$

$\Rightarrow \quad p + \sqrt{3}q = 4$

Now from options, only (4, 0) satisfies the equation.

2. c. Distance $= \sqrt{a^2(\cos\alpha - \cos\beta)^2 + a^2(\sin\alpha - \sin\beta)^2}$

$= a\sqrt{\sin^2\alpha + \cos^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$

$= a\sqrt{2\{1 - \cos(\alpha - \beta)\}}$

$= 2a\sin\left(\frac{\alpha - \beta}{2}\right)$

3. In ΔABC, $A \equiv (-3, 0)$; $B \equiv (4, -1)$, and $C \equiv (5, 2)$

$$BC = \sqrt{(5-4)^2 + (2+1)^2} = \sqrt{1+9} = \sqrt{10}$$

and area of $\Delta ABC = \frac{1}{2}[-3(-1-2) + 4(2-0) + (0+1)]$

$= 11$

Therefore, altitude $AL = \frac{2\Delta ABC}{BC} = \frac{2 \times 11}{\sqrt{10}} = \frac{22}{\sqrt{10}}$

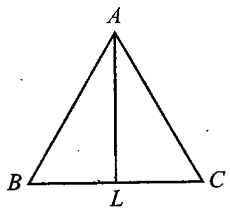


Fig. S–1.1

4. Midpoints of the diagonals must be the same. Therefore,

$$\frac{x-2}{2} = \frac{-3+3}{2}$$

$\Rightarrow \quad x = 2$

and $\quad \frac{-1+3}{2} = \frac{-2+y}{2}$

$\Rightarrow \quad y = 4$

5. Let the vertices of triangle be (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$\Rightarrow \quad \frac{x_1 + x_2}{2} = 2, \frac{x_2 + x_3}{2} = -1, \frac{x_3 + x_1}{2} = 4$

Solving, we get $\quad x_1 = 7, x_2 = -3$ and $x_3 = 1$

Similarly, y_1, y_2, and y_3 can be found.

6. c. $a = \sqrt{(8+2)^2 + (-2-2)^2} = \sqrt{116}$

$b = \sqrt{(-4-8)^2 + (-3+2)^2} = \sqrt{145}$

$c = \sqrt{(-4+2)^2 + (-3-2)^2} = \sqrt{29}$

$\Rightarrow a^2 + c^2 = b$

7. d. The given points are collinear as the point $[(kc + la)/(k + l), (kd + lb)/(k + l)]$ divides the points (a, b) and (c, d) in the ratio of $k : L$.

8. d. $l_1 = \sqrt{(2a)^2 + (2b)^2} = 2\sqrt{a^2 + b^2}$

$l_2 = \sqrt{(a^2 - a)^2 + b^2(a-1)^2} = (a-1)\sqrt{a^2 + b^2}$ (if $a > 1$)

$l_3 = \sqrt{(a^2 + a)^2 + b^2(a+1)^2} = (a+1)\sqrt{a^2 + b^2}$

Now $l_1 + l_2 = l_3$. Hence, points are collinear.

Also when $0 < a < 1$.

$l_2 = (1-a)\sqrt{a^2 + b^2}$,

and hence $\quad l_1 = l_2 + l_3$.

In that case also points are collinear.

9. b. Since circumcenter $P(x, y)$ is equidistant from the vertices of the triangle $A(0, 0)$, $B(-2, -2)$, $C(-4, -8)$

we have $\quad AP = CP$ and $AP = BP$

or $\quad x^2 + y^2 = (x+4)^2 + (y+8)^2$

$\Rightarrow \quad 8x + 16y + 80 = 0 \qquad$ (i)

and $\quad x^2 + y^2 = (x+2)^2 + (y+2)^2$

$\Rightarrow \quad 4x + 4y + 8 = 0 \qquad$ (ii)

Solving (i) and (ii), we get $y = -8$ and $x = 6$.

10. The required area

$$= \frac{1}{2}\left|\begin{array}{cc} 1 & 1 \\ 7 & 21 \\ 7 & -3 \\ 12 & 2 \\ 0 & -3 \\ 1 & 1 \end{array}\right|$$

$= \frac{1}{2}\{(21 - 21 + 14 - 36 + 0) - (7 + 147 - 36 + 0 - 3)\}$

$= 127/2$ sq. units

11. As we know that the centroid of the triangle ABC and that of the triangle formed by joining the middle points of the sides of triangle ABC are same.

Therefore, the required centroid is

$$\left(\frac{4+4-2}{3}, \frac{5-3+3}{3}\right) = \left(2, \frac{5}{3}\right)$$

12.

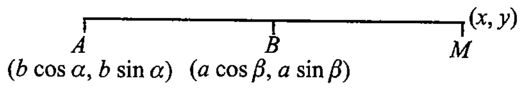


Fig. S–1.2

$$\frac{AM}{BM}=\frac{b}{a}$$

$$\Rightarrow M\left[\frac{ab\cos\alpha-ab\cos\beta}{(a-b)},\frac{ab\sin\alpha-ab\sin\beta}{(a-b)}\right]$$

$$\equiv M(x,y)$$

$$\Rightarrow \quad \frac{x}{y}=\frac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}$$

$$=\frac{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$=-\tan\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \quad x+\tan\left(\frac{\alpha+\beta}{2}\right)y=0$$

13. Let (x, y) be the required point. Therefore,

$$\frac{1}{2}\begin{vmatrix} x & y \\ 1 & 5 \\ 3 & -7 \\ x & y \end{vmatrix}=\pm 21$$

$$\Rightarrow 5x-y-7-15+3y+7x=\pm 42$$

$$\Rightarrow 12x+2y=64 \text{ or } 12x+2y=-20$$

$$\Rightarrow \quad 6x+y=32 \text{ or } 6x+y=-10$$

14. Let $A\equiv(4,-8)$, $B\equiv(-9,7)$, and $G\equiv(1,4)$.

Let $C(x, y)$ be the third vertex of ΔABC. Then,

$$1=(4-9+x)/3 \quad \text{or } x=8$$

and $$4=(-8+7+y)/3 \quad \text{or } y=13$$

Hence, $C\equiv(8,13)$

Now area of ΔABC

$$=\frac{1}{2}\left\|\begin{matrix} 4 & -8 & 1 \\ -9 & 7 & 1 \\ 8 & 13 & 1 \end{matrix}\right\|$$

$$=\frac{1}{2}|[4(7-13)+8(-9-8)+1(-117-56)]|$$

$= 166.5$ sq. units

15. a. Let A (4, 0), B (– 1, – 1), and C (3, 5) be the given points. Then, we get

$$|AB|=\sqrt{(-1-4)^2+(-1-0)^2}$$

$$=\sqrt{25+1}=\sqrt{26}$$

$$|BC|=\sqrt{(3+1)^2+(5+1)^2}$$

$$=\sqrt{16+36}=\sqrt{52}$$

and $$|CA|=\sqrt{(4-3)^2+(0-5)^2}$$

$$=\sqrt{1+25}=\sqrt{26}$$

Clearly, $|AB|=|CA|$

$\Rightarrow$ Triangle is isosceles.

And $BC^2=AB^2+CA^2$ $\quad [\because 52=26+26]$

$\Rightarrow$ Triangle is right angled.

Exercise 1.2

1. Let the point be (h, k). Therefore,

$$(h-a)^2+(k-0)^2=h^2$$

$$\Rightarrow \quad h^2+a^2-2ah+k^2=h^2$$

Hence, locus is $y^2-2ax+a^2=0$.

2. Let the point be (x, y). Then,

$$(x-a)^2+y^2-(x+a)^2-y^2=2k^2$$

$$\Rightarrow \quad -4ax-2k^2=0$$

$$\Rightarrow \quad 2ax+k^2=0$$

The required equation to the line of the point P.

3. Let (h, k) be the centroid of the triangle, then

$$\Rightarrow \quad h=\frac{\cos\alpha+\sin\alpha+1}{3}$$

and $$k=\frac{\sin\alpha-\cos\alpha+2}{3}$$

$$\Rightarrow \quad 3h-1=\cos\alpha+\sin\alpha$$

and $$3k-2=\sin\alpha-\cos\alpha$$

Squaring and adding, we get $(3h-1)^2+(3k-2)^2=2$

$$\Rightarrow 9(h^2+k^2)-6h-12k+3=0$$

$$\Rightarrow \quad 3(h^2+k^2)-2h-4k+1=0$$

Therefore, locus of centroid is $3(x^2+y^2)-2x-4y+1=0$.

4. Let C be (α,β)

The centroid is

$$\left(\frac{2-2+\alpha}{3},\frac{-3+1+\beta}{3}\right),\text{ i.e., }\left(\frac{\alpha}{3},\frac{\beta-2}{3}\right)$$

This lies on $2x+3y=1$, therefore, we get

$$2\left(\frac{\alpha}{3}\right)+3\left(\frac{\beta-2}{3}\right)=1$$

$$\Rightarrow \quad 2\alpha+3\beta=9$$

Hence, the locus of (α,β) is $2x+3y=9$.

5. Let Q be the point (X, Y) and P the point (x, y); the coordinates of Q satisfy the equation $2x + 3y + 4 = 0$, so that $2X + 3Y + 4 = 0$.

Applying the section formula for OQ, O being $(0, 0)$, we get

$$x = \frac{0+3X}{1+3},\ y = \frac{0+3Y}{1+3}$$

from which we get $X = \frac{4}{3}x,\ Y = \frac{4}{3}y$

Substituting these values, then the locus of P is

$$\frac{8}{3}x + 4y + 4 = 0$$

$$\Rightarrow \quad 2x + 3y + 3 = 0$$

6.

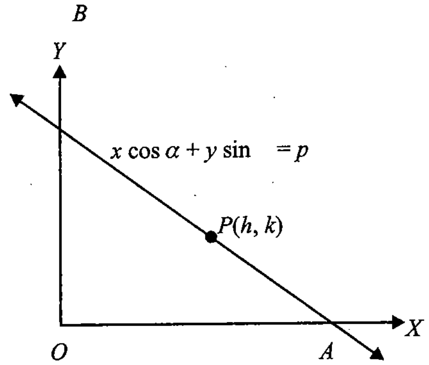


Fig. S–1.3

Equation of the variable line is

$$x\cos\alpha + y\sin\alpha = p \qquad \text{(i)}$$

Here p is a constant and α is the parameter (variable).

Let line Eq. (i) cut x- and y-axes at A and B, respectively, then

Putting $y = 0$ in Eq. (i), we get $A \equiv (p\sec\alpha, 0)$

Putting $x = 0$ in Eq. (i), we get $B \equiv (0, p\operatorname{cosec}\alpha)$

AB is the portion of the Eq. (i) intercepted between the axes.

Let $P(h, k)$ be the midpoint of AB. We have to find the locus of point. $P(h, k)$, For this, we will have to eliminate α and find a relation in h and k. Therefore,

$$h = \frac{p\sec\alpha + 0}{2} = \frac{p}{2}\sec\alpha \qquad \text{(ii)}$$

and

$$k = \frac{0 + p\operatorname{cosec}\alpha}{2} = \frac{p}{2}\operatorname{cosec}\alpha \qquad \text{(iii)}$$

From Eq. (ii), we get

$$\cos\alpha = \frac{p}{2h} \qquad \text{(iv)}$$

From Eq. (iii), we get

$$\sin\alpha = \frac{p}{2k} \qquad \text{(v)}$$

Squaring and adding Eqs. (iv) and (v), we get

$$\cos^2\alpha + \sin^2\alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \text{ or } \frac{1}{h^2} + \frac{1}{k^2} = \frac{4}{p^2}$$

Hence, locus of point $P(h, k)$ is

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$$

7. Let (h, k) be the point of intersection of $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$. Then,

$$h\cos\alpha + k\sin\alpha = a \qquad \text{(i)}$$

$$h\sin\alpha - k\cos\alpha = b \qquad \text{(ii)}$$

Squaring and adding Eqs. (i) and (ii), we get

$$(h\cos\alpha + k\sin\alpha)^2 + (h\sin\alpha - k\cos\alpha)^2 = a^2 + b^2$$

$$\Rightarrow \quad h^2 + k^2 = a^2 + b^2$$

Hence, locus of (h, k) is

$$x^2 + y^2 = a^2 + b^2$$

Exercise 1.3

1.

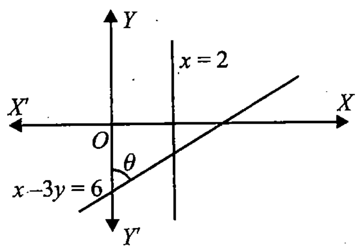


Fig. S-1.4

$$\theta = 90° - \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(3)$$

2. Midpoint of $AB = E\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)$ and midpoint of $CD = F\left(\frac{a'-a}{2}, \frac{b-b'}{2}\right)$.

Hence, equation of line EF is

$$y - \frac{b+b'}{2} = \frac{b-b'-b-b'}{a'-a-a-a'}\left(x - \frac{a+a'}{2}\right)$$

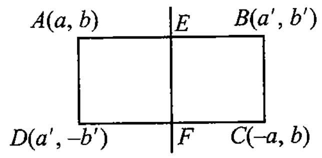


Fig. S–1.5

On simplification, we get

$$2ay - 2b'x = ab - a'b'$$

3. Required equation of median is

$$y + 8 = \frac{-\frac{3}{2} + 8}{-2-5}(x - 5)$$

$$\Rightarrow \quad 13x + 14y + 47 = 0$$

4. The given line is

$$bx - ay = ab$$

Obviously, it cuts x-axis at $(a, 0)$.

The equation of line perpendicular to Eq. (i) is

$$ax + by = k$$

but it passes through $(a, 0)$

$\Rightarrow \quad k = a^2$

Hence, the required equation of line is

$$ax + by = a^2$$

i.e., $\quad \frac{x}{b} + \frac{y}{a} = \frac{a}{b}$

5. Slope of $DE = \frac{7-3}{5-1} = 1$

$\Rightarrow \quad$ Slope of $AB = 1$

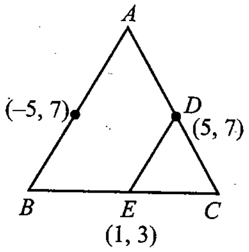


Fig. S–1.6

Hence equation of AB is

$$y - 7 = (x + 5)$$

$\Rightarrow \quad x - y + 12 = 0$

6. Angle between both the lines is

$$\tan^{-1} m \pm \tan^{-1} m = \tan^{-1} \frac{2m}{1-m^2} \text{ or } \tan^{-1} 0$$

Therefore, the equations of the lines are

$$y = 0, y = \frac{2mx}{1-m^2}$$

7. The midpoint of $P(-2, 6)$ and $Q(4, 2)$ is

$\left(\frac{-2+4}{2}, \frac{6+2}{2}\right)$, i.e., $(1, 4)$

and the gradient of line $PQ = \frac{2-6}{4+2} = \frac{-2}{3}$

Therefore, the slope of $L = \frac{3}{2}$.

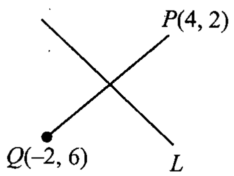


Fig. S–1.7

Hence, the equation of line which passes through point (1, 4) is

$$y - 4 = \frac{3}{2}(x - 1)$$

$\Rightarrow \quad 3x - 2y + 5 = 0$

8. Slope of given lines are $-1/(a-1)$ and $-2/a^2$.

Since the lines are perpendicular, therefore, we get

$$\left(\frac{-1}{a-1}\right)\left(-\frac{2}{a^2}\right) = -1$$

$\Rightarrow \quad \frac{2}{(a-1)a^2} = -1$

$\Rightarrow \quad 2 + a^2(a-1) = 0$

$\Rightarrow \quad a^3 - a^2 + 2 = 0$

$\Rightarrow \quad a = -1$

9. $x + 2|y| = 1$ and $x = 0$ gives

$$|y| = \frac{1}{2}$$

$\therefore \quad y = \pm\frac{1}{2}$

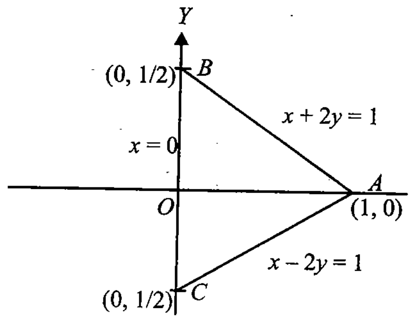


Fig. S–1.8

The required area = area of ΔABC (where AB is $x + 2y = 1$ and AC is $x - 2y = 1$)

$$\text{Required area} = \frac{1}{2}(BC)(OA) = \frac{1}{2}(1)(1) = \frac{1}{2}$$

10. Point of intersection of $x - 2y = 1$ and $x + 3y = 2$ is $(7/5, 1/5)$.

Any line parallel to $3x + 4y = 0$ is $3x + 4y + K = 0$.

As this line passes through $(7/5, 1/5)$, we get

$$\frac{21}{5} + \frac{4}{5} + K = 0$$

$\Rightarrow \quad K = -5$

Therefore, the required line is

$$3x + 4y - 5 = 0$$

11. $\sqrt{3}x + y = 0$ makes an angle of $120°$ with OX, where as $\sqrt{3}x - y = 0$ makes an angle of $60°$ with OX. Therefore, the required line is

$$y - 2 = 0$$

12. Let 'a' and 'b' be the intercepts on the axes. Then the equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$ (i)

Since Eq. (i) passes through (4, 3), we get

$$\frac{4}{a} + \frac{3}{b} = 1 \quad \text{(ii)}$$

Also given that $a + b = -1$ (iii)

From Eq. (iii), we get

$$b = -1 - a \quad \text{(iv)}$$

Putting in Eq. (ii), we get

$$\frac{4}{a} + \frac{3}{-1-a} = 1$$

$\Rightarrow$ $-4 - 4a + 3a = -a - a^2$

$\Rightarrow$ $a^2 = 4$

$\Rightarrow$ $a = \pm 2$

When $a = 2$, then from Eq. (iv), $b = -1 - 2 = -3$

and when $a = -2, b = -1 + 2 = 1$

Therefore, the line is

$$\frac{x}{2} + \frac{y}{-3} = 1 \text{ and } \frac{x}{-2} + \frac{y}{1} = 1$$

13. Let the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

This line meets x-axis at $A(a, 0)$ and y-axis at $B(0, b)$. Therefore, we get

$$(3, 4) = \left(\frac{a}{2}, \frac{b}{2}\right)$$

$\Rightarrow$ $\frac{a}{2} = 3, \frac{b}{2} = 4$

$\Rightarrow$ $a = 6, b = 8.$

Therefore, the required line is

$$\frac{x}{6} + \frac{y}{8} = 1$$

$\Rightarrow$ $4x + 3y = 24$

14. $m_{AC} = \frac{6-2}{5-3} = 2$

$\Rightarrow$ $m_{BD} = -\frac{1}{2}$

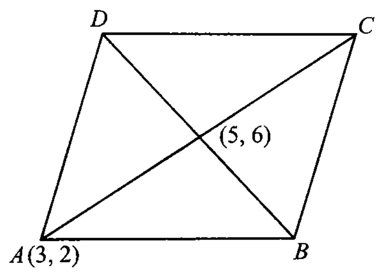


Fig. S–1.9

Thus, the equation of BD is

$$(y - 6) = -\frac{1}{2}(x - 5),$$

i.e., $2y + x - 17 = 0$

15. Let $P(3, -4)$ be the foot of the perpendicular from the origin O on the required line. Then the slope of OP

$$= \frac{-4-0}{3-0} = -\frac{4}{3},$$

and therefore the slope of the required line is 3/4.

Hence, its equation is

$$y + 4 = \frac{3}{4}(x - 3)$$

$\Rightarrow$ $3x - 4y = 25$

16. The given form is $3x + 3y + 7 = 0$.

$\Rightarrow$ $\frac{3}{\sqrt{3^2+3^2}}x + \frac{3}{\sqrt{3^2+3^2}}y + \frac{7}{\sqrt{3^2+3^2}} = 0$

$\Rightarrow$ $\frac{3}{3\sqrt{2}}x + \frac{3}{3\sqrt{2}}y = \frac{-7}{3\sqrt{2}},$

$\therefore p = \left|\frac{-7}{3\sqrt{2}}\right| = \frac{7}{3\sqrt{2}}$

17. Given line is $x + \sqrt{3}\,y + 3\sqrt{3} = 0$. Therefore, we get

$$y = \left(-\frac{1}{\sqrt{3}}\right)x - 3$$

Therefore, slope of Eq. (i) $= -1/\sqrt{3}$.

Let the slope of the required line be m.

Also the angle between these lines is 60°

$\Rightarrow$ $\tan 60° = \left|\frac{m - (-1/\sqrt{3})}{1 + m(1/\sqrt{3})}\right|$

$\Rightarrow$ $\sqrt{3} = \left|\frac{\sqrt{3}\,m + 1}{\sqrt{3} - m}\right|$

$\Rightarrow$ $\frac{\sqrt{3}m + 1}{\sqrt{3} - m} = \sqrt{3}$

$\frac{\sqrt{3}m + 1}{\sqrt{3} - m} = \sqrt{3}$

$\Rightarrow$ $m = \frac{1}{\sqrt{3}}$

Using $y = mx + c$, the equation of the required line is

$$y = \frac{1}{\sqrt{3}}x + 0,$$

i.e., $x - \sqrt{3}\,y = 0$ (as the line passes through the origin, $c = 0$)

$\frac{\sqrt{3}m - 1}{\sqrt{3} - m} = -\sqrt{3}$

$\Rightarrow$ $\sqrt{3}\,m + 1 = -3 + \sqrt{3}m$

$\Rightarrow$ m is not defined.

Therefore, the slope of the required line is not defined. Thus, the required line is a vertical line. This line passes through the origin.

Therefore, the equation of the required line is $x = 0$.

18. Any point A on the first line is $(t, 5t - 4)$. Any point B on the second line is $\left[r, \frac{(3r-4)}{4}\right]$. Hence,

$$1 = \frac{2r + t}{3}$$

and $$5 = \frac{\frac{3r-4}{2} + 5t - 4}{3}$$

$\Rightarrow$ $$2r + t = 3 \text{ and } 3r + 10t = 42$$

On solving, we get $t = \frac{75}{17}$.

Hence A is $\left(\frac{75}{77}, \frac{304}{17}\right)$.

19.

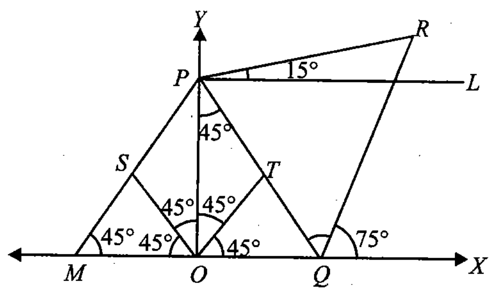


Fig. S-1.10

Equation of line OT: $(0, 0)$ and slope of $OT = \tan 45° = 1$. Therefore, equation of line OT will be

$$y - 0 = 1(x - 0) \text{ or } x - y = 0$$

Equation of line OS: $(0, 0)$ and $\angle SOX = 135°$. Therefore, the slope of line $OS = \tan 135° = -1$.

Equation of line OS will be

$$y - 0 = (-1)(x - 0)$$

or $$y = -x \text{ or } x + y = 0$$

Equation of line SP: Given that $OT = 2\sqrt{2}$. Therefore,

$$OP = OT \sec 45° = 2\sqrt{2}\sqrt{2} = 4$$

$\therefore$ $P \equiv (0, 4)$. Also the slope of $SP = \tan 45° = 1$

Therefore, equation of line SP will be

$$y - 4 = 1(x - 0) \text{ or } x - y + 4 = 0$$

Equation of QR: Given that $OQ = OT \sec 45° = 2\sqrt{2}\sqrt{2} = 4$. Therefore, $Q \equiv (4, 0)$. Also slope of line $QR = \tan 75° = 2 + \sqrt{3}$.

Therefore, equation of line QR will be

$$y - 0 = (2 + \sqrt{3})(x - 4)$$

or $(2 + \sqrt{3})x - y - 8 - 4\sqrt{3} = 0$

Equation of PR: $P \equiv (0, 4)$

Slope of line $PR = \tan 15° = 2 - \sqrt{3}$

Therefore, equation of line PR is

$$y - 4 = (2 - \sqrt{3})(x - 0)$$

or $$y - 4 = (2 - \sqrt{3})x$$

or $(2 - \sqrt{3})x - y + 4 = 0$

Equation of PQ: $P \equiv (0, 4)$ and $Q \equiv (4, 0)$

Therefore, equation of line PQ will be

$$\frac{x}{4} + \frac{y}{4} = 1$$

or $$x + y = 4$$

20. Let m_1 and m_2 be the slopes of the straight lines $x - 2y + 3 = 0$ and $3x + y - 1 = 0$. Then,

$$m_1 = -\frac{1}{-2} = \frac{1}{2} \text{ and } m_2 = -\frac{3}{1} = -3$$

Let $$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = \pm\left(\frac{\frac{1}{2} + 3}{1 - \frac{3}{2}}\right) = \pm 7$$

$\Rightarrow$ $$\theta = \tan^{-1}(\pm 7)$$

Thus, the acute angle between the lines is $\tan^{-1}(7)$ and the obtuse angle is $\pi - \tan^{-1}(7)$.

21. Let $A \equiv (a, 0)$, $B \equiv (0, b)$, $A' \equiv (a', 0)$, $B' \equiv (0, b')$

Equation of $A'B$ is

$$\frac{x}{a'} + \frac{y}{b} = 1 \qquad \text{(i)}$$

and the equation of AB' is

$$\frac{x}{a} + \frac{y}{b'} = 1 \qquad \text{(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get,

$$x\left(\frac{1}{a} - \frac{1}{a'}\right) + y\left(\frac{1}{b'} - \frac{1}{b}\right) = 0$$

$\Rightarrow$ $$\frac{x(a' - a)}{aa'} + \frac{y(b - b')}{bb'} = 0$$

$\Rightarrow$ $$\frac{x}{aa'} + \frac{y}{bb'} = 0 \qquad [\text{using } a' - a = b - b']$$

Exercise 1.4

1. $P(2, -1)$ goes 2 units along $x + y = 1$ up to A and 5 units along $x - 2y = 4$ up to B. Then,

$$\text{slope of } PA = -1 = \tan 135°$$

$$\text{slope of } PB = 1/2 = \tan\theta$$

$\Rightarrow$ $$\sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}$$

The coordinates of B, i.e., $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ are $(2\sqrt{5} + 2, \sqrt{5} - 1)$.

The coordinates of A, i.e., $(x_1 + r\cos 135°, y_1 + r\sin 135°)$ are $(2 - \sqrt{2}, \sqrt{2} - 1)$.

2. Since $m = 3/4$, then $\cos\theta = 4/5$ and $\sin\theta = 3/5$.

Any point on the line through A has the coordinates $(2 + 4r/5, 3 + 3r/5)$.

If this point is also the point of intersection, P, then these coordinates satisfy the equation of the given line. Hence,

$$5\left(2+\frac{4}{5}r\right)+7\left(3+\frac{3}{5}r\right)+40=0$$

or $$r\left(4+\frac{21}{5}\right)+71=0$$

or $$r=-\frac{355}{41}$$

Thus, the distance between A and P is 355/41 units, the vector $\overrightarrow{AP}$ being in the negative direction of the line.

3.

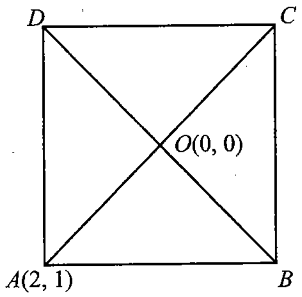


Fig. S–1.11

Let $ABCD$ be the square whose centre is O.

Given $A \equiv (2, 1)$ and $O \equiv (0, 0)$.

Now $AO = \sqrt{5}$

and slope of $AO = \frac{1-0}{2-0} = \frac{1}{2} = \tan\theta$ (say)

$\Rightarrow \cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$

Now coordinates of the points on AC which are at a distance $\sqrt{5}$ from O will be

$(0 \pm \sqrt{5}\cos\theta, 0 \pm \sqrt{5}\sin\theta)$

i.e., $(\pm 2, \pm 1)$ or $(2, 1)$ and $(-2, -1)$

But $A \equiv (2, 1)$, therefore $C \equiv (-2, -1)$.

Again $BD \perp AC$

$\Rightarrow$ slope of $BD = -2 = \tan\alpha$ (say)

$\Rightarrow \cos\alpha = -\frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$

Since B and D are on BD at a distance $\sqrt{5}$ from O, therefore their coordinates (in some order) will be

$(0 \pm \sqrt{5}\cos\alpha, 0 \pm \sqrt{5}\sin\alpha)$.

i.e., $(0 \mp 1, 0 \pm 2)$ or $(-1, 2)$ and $(1, -2)$

Exercise 1.5

1. Since the perpendicular distance between the given lines is $\sqrt{2}$. Therefore, the required line is a straight line perpendicular to the given parallel lines and passes through $(-5, 4)$.

Any line perpendicular to given lines is

$$x - y + k = 0$$

This line passes through $(-5, 4)$, therefore

$$-5 - 4 + k = 0$$

$\therefore$ $$k = 9$$

Hence, the required line is

$$x - y + 9 = 0$$

2. Lines $3x + 4y + 2 = 0$ and $3x + 4y + 5 = 0$ are on the same side of the origin.

The distance between these lines is

$$d_1 = \left|\frac{2-5}{\sqrt{3^2+4^2}}\right| = \frac{3}{5}$$

Lines $3x + 4y + 2 = 0$ and $3x + 4y - 5 = 0$ are on the opposite sides of the origin.

The distance between these lines is

$$d_2 = \left|\frac{2+5}{\sqrt{3^2+4^2}}\right| = \frac{7}{5}$$

Thus, $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ in the ratio $d_1 : d_2$, i.e., 3 : 7.

3. Equation of any line parallel to $3x - 4y - 5 = 0$ is

$$3x - 4y + \lambda = 0 \quad \text{(i)}$$

Distance of (i) from $3x - 4y - 5 = 0$ is 1 unit.

$\Rightarrow \frac{|5+\lambda|}{5} = 1$

$\Rightarrow \lambda = -10$ or 0

$\Rightarrow$ Required line is $2x - 4y - 10 = 0$ or $3x - 4y = 0$

These are the equations of the required lines.

4. Here, $p = \left|\frac{-k}{\sqrt{\sec^2\alpha + \operatorname{cosec}^2\alpha}}\right|, p' = \left|\frac{-k\cos 2\alpha}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right|$

Hence, $4p^2 + p'^2 = \frac{4k^2}{\sec^2\alpha + \operatorname{cosec}^2\alpha} + \frac{k^2(\cos^2\alpha - \sin^2\alpha)^2}{1}$

$= 4k^2\sin^2\alpha\cos^2\alpha + k^2(\cos^4\alpha + \sin^4\alpha) - 2k^2\cos^2\alpha \times \sin^2\alpha$

$= k^2(\sin^2\alpha + \cos^2\alpha)^2 = k^2$

Exercise 1.6

1. **a.** $L_1(8, -9) = 2(8) + 3(-9) - 4 = -15$
   $L_2(8, -9) = 6(8) + 9(-9) + 8 = -25$
   Hence, the point lies on the same side of the lines.

2. **d.** $L(-1,-1) = 3(-1) - 8(-1) - 7 < 0$
   $L(3, 7) = 3 \times 3 - 8 \times 7 - 7 < 0$
   Hence, (–1, –1) and (3, 7) lie on the same side of the line.

3. $L = 2x + 3y - 6$

$$\Rightarrow \quad L(\alpha, 2+\alpha) = 5\alpha$$

$$L\left(\frac{3}{2}\alpha, \alpha^2\right) = 3\alpha + 3\alpha^2 - 6$$

For the given condition, we get

$$5\alpha(3\alpha + 3\alpha^2 - 6) < 0 \text{ or } \alpha(\alpha+2)(\alpha-1) < 0$$

$$\Rightarrow \quad \alpha \in (-\infty, -2) \cup (0, 1)$$

Exercise 1.7

1. $(a - 2b)x + (a + 3b)y + 3a + 4b = 0$
   or $a(x + y + 3) + b(-2x + 3y + 4) = 0$
   which represents a family of straight lines passing through the point of intersection of $x + y + 3 = 0$ and $-2x + 3y + 4 = 0$, i.e., (–1, –2).

2. a, b, c are in H.P., then

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \qquad \text{(i)}$$

Given line is

$$\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0 \qquad \text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$\frac{1}{a}(x-1) + \frac{1}{b}(y+2) = 0$$

Since $\quad a \neq 0, b \neq 0$, we get

$$x - 1 = 0 \text{ and } y + 2 = 20$$

$$\Rightarrow \quad x = 1$$

and $\quad y = -2$

3. **b.** Let the equation of the variable line be

$$ax + by + c = 0$$

Then according to the given condition, we get

$$\frac{2a+c}{\sqrt{a^2+b^2}} + \frac{2b+c}{\sqrt{a^2+b^2}} + \frac{-2a-2b+c}{\sqrt{a^2+b^2}} = 0$$

$$\Rightarrow \quad c = 0$$

which shows that the line passes through (0, 0) for all values of a, b.

4. Lines $5x + 3y - 2 + \lambda_1(3x - y - 4) = 0$ are concurrent at (1, –1) and lines $x - y + 1 + \lambda_2(2x - y - 2) = 0$ are concurrent at (3, 4).
   Thus, the equation of line common to both families is

$$y - 4 = \frac{-1-4}{1-3}(x - 3)$$

i.e., $\quad 5x - 2y - 7 = 0$

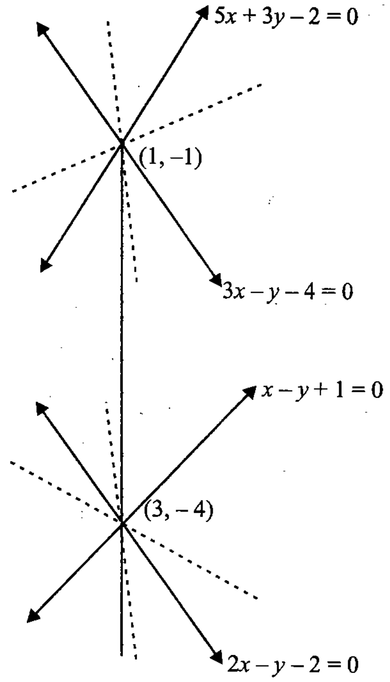


Fig. S–1.12

Exercise 1.8

1. The given pair is $(2x + y)(x - y) = 0$
   So, the required pair is $(2x + y + k)(x - y + k') = 0$,
   where $2x + y + k = 0$ and $x - y + k' = 0$ pass through (1, 0).

$$\therefore \quad k = -2, k' = -1$$

Therefore, the required pair is

$$(2x + y - 2)(x - y - 1) = 0$$

2. If m, $2m$ are the slopes, then

$$m + 2m = -\frac{2/h}{1/b} = -\frac{2b}{h}$$

and $\quad m \times 2m = \frac{1/a}{1/b} = \frac{b}{a}$

Eliminating m, we get

$$1\left(-\frac{2b}{3h}\right)^2 = \frac{b}{a}$$

$$\Rightarrow \quad \frac{ab}{h^2} = \frac{9}{8}$$

3. Let

$$y = mx \quad \text{(i)}$$

be a line through the origin making an angle of 60° with the line

$$Ax + By + C = 0 \quad \text{(ii)}$$

Then, we have

$$\tan 60° = \pm \frac{m - (-A/B)}{1 + m(-A/B)}$$

From Eq. (i) we have, $m = y/x$. Substituting this value of m in the above result,

i.e., $3(B - Am)^2 = (mB + A)^2$, we have

$$(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0 \quad \text{(iii)}$$

These are the straight lines passing through the origin and making angles of 60° with Eq. (ii), i.e., forming an equilateral triangle with Eq. (ii).

Now, OL = length of perpendicular to $Ax + By + C = 0$ from (0, 0)

$$= \frac{C}{\sqrt{(A^2 + B^2)}}$$

So, $$\text{area} = \frac{C^2}{\sqrt{3(A^2 + B^2)}}$$

4. Let $ax^2 + 2hxy + by^2 = b(y \pm x)(y - mx)$

Taking + ve sign, we get

$$ax^2 + 2hxy + by^2 = b(y + x)(y - mx)$$

Equating coefficient of x^2, we get

$$-bm = a$$

$$\Rightarrow \quad m = -\frac{a}{b}$$

Equating coefficient of xy, we get

$$b - bm = 2h$$

5. Given equation is

$$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$$

Writing the Eq. (i) as a quadratic equation in x, we have

$$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$$

$$\therefore x = \frac{-(5y+6) \pm \sqrt{(5y+6)^2 - 4 \times 2(3y^2 + 7y + 4)}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{25y^2 + 60y + 36 - 24y^2 - 56y - 32}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{y^2 + 4y + 4}}{4} = \frac{-(5y+6) + (y+2)}{4}$$

$$\therefore x = \frac{-5y - 6 + y + 2}{4}, \frac{-5y - 6 - y - 2}{4}$$

or $\quad 4x + 4y + 4 = 0$ and $4x + 6y + 8 = 0$

or $\quad x + y + 1 = 0$ and $2x + 3y + 4 = 0$

Hence, Eq. (i) represents a pair of straight lines whose equations are

$$x + y + 1 = 0 \quad \text{(i)}$$

and $$2x + 3y + 4 = 0 \quad \text{(ii)}$$

Solving these two equations, the required point of intersection is (1, – 2).

6. $y - 1 = m(x - 2)$

$$\Rightarrow \quad y - mx = 1 - 2m$$

or $$\frac{y - mx}{1 - 2m} = 1$$

Homogenizing the given pair of straight lines, we get

$$(4x^2 + y^2) - \frac{(x - 4y)(y - mx)}{1 - 2m} - \frac{2(y - mx)^2}{(1 - 2m)^2} = 0$$

$$\Rightarrow (1 - 2m)^2(4x^2 + y^2) - (x - 4y)(y - mx)(1 - 2m) - 2(y - mx)^2 = 0$$

Equating coefficient of xy to 0, we get

$$4m - (1 + 4m)(1 - 2m) = 0 \text{ or } 8m^2 + 2m - 1 = 0$$

or $\quad 8m^2 + 4m - 2m - 1 = 0$

$\Rightarrow 4m(2m + 1) - (2m + 1) = 0$

$$\Rightarrow \quad m = -1/2 \text{ or } m = 1/4$$

Therefore, lines are $y - 1 = -\frac{1}{2}(x - 2)$

and $y - 1 = \frac{1}{4}(x - 2)$

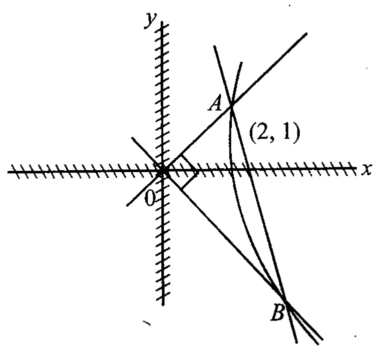


Fig. S–1.13

7. Since lines are parallel

$$\therefore \quad h^2 - ab = 0$$

$$\Rightarrow \quad \frac{(\lambda + \mu)^2}{4} - \lambda\mu = 0$$

$$\Rightarrow \quad (\lambda - \mu)^2 = 0$$

$$\Rightarrow \quad \lambda = \mu$$

8. Bisector of the angle between the positive directions of the axes is $y = x$.

Since it is one of the lines of the given pair of lines $ax^2 + 2hxy + by^2 = 0$, we have

$$x^2(a + 2h + b) = 0 \text{ or } a + b = -2h.$$

9. The given equation of pair of straight lines can be rewritten as

$$(\sqrt{3}x - y)(x - \sqrt{3}y) = 0$$

Their separate equations are

$$y = \sqrt{3}x \text{ and } y = \frac{1}{\sqrt{3}}x$$

or $\quad y = \tan 60°\, x$ and $y = \tan 30°\, x$

After rotation, the separate equations are

$$y = \tan 90°\, x \text{ and } y = \tan 60°\, x$$

or $\quad x = 0$ and $y = \sqrt{3}x$

Therefore, the combined equation in the new position is

$$x(\sqrt{3}x - y) = 0 \text{ or } \sqrt{3}x^2 - xy = 0$$

10. The given equation will represent a pair of real and distinct lines if $h^2 > ab$,

i.e., $\quad \left(-\frac{k}{2}\right) > (2)(2)$ or $k^2 > 16$

or $\quad (k-4)(k+4) > 0$

i.e. $\quad k \in (-\infty, -4) \cup (4, \infty)$.

11. Let $\phi(x, y) = 6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$.

Differentiating with respect to x treating y as constant, we get

$$\frac{d\phi}{dx} = 12x + 5y + 13$$

Differentiating with respect to x treating y as constant, we get

$$\frac{d\phi}{dy} = 5x - 42y + 38$$

Solving equations $12x + 5y + 13 = 0$ and $5x - 42y + 38 = 0$, we get

$$x = -\frac{32}{23} \text{ and } y = \frac{17}{23}$$

Therefore, the point of intersection is $(-32/23, 17/32)$.

12. Let ϕ be the angle between the lines represented by

$$x^2 + 2xy \sec\theta + y^2 = 0 \qquad \text{(i)}$$

Here, $\quad a = 1, b = 1, h = \sec\theta$

Hence,

$$\tan\phi = \frac{2\sqrt{(\sec^2\theta - 1 \times 1)}}{1+1}$$

$$\Rightarrow \quad \tan\phi = \frac{2\sqrt{(\sec^2\theta - 1)}}{2} = \tan\theta$$

$$\therefore \quad \phi = \theta$$

Hence, the angle between the lines represented by Eq. (i) is θ.

Chapter 2

Exercise 2.1

1. Here the centre of circle $(3, -1)$ must lie on the line $x + 2by + 7 = 0$.

Therefore, $3 - 2b + 7 = 0 \Rightarrow b = 5$.

2. Let a triangle has its three vertices as $(0, 0)$, $(a, 0)$, $(0, b)$.

We have the moving point (h, k) such that $h^2 + k^2 + (h - a)^2 + k^2 + h^2 + (k - b)^2 = c$

$$\Rightarrow 3h^2 + 3k^2 - 2ah - 2bk + a^2 + b^2 = c$$

Therefore, locus is $3x^2 + 3y^2 - 2ax - 2by + a^2 + b^2 = c$ which is circle.

3. Here, radius $\sqrt{\left(\frac{\lambda}{2}\right)^2 + \left(\frac{1-\lambda}{2}\right)^2 - 5} \le 5$

$$\Rightarrow 2\lambda^2 - 2\lambda - 119 \le 0$$

$$\Rightarrow \frac{1-\sqrt{239}}{2} \le \lambda \le \frac{1+\sqrt{239}}{2} \Rightarrow -7.2 \le \lambda \le 8.2 \text{ (approx.)}$$

$$\therefore \quad \lambda = -7, -6, \ldots, 8$$

4. Radius = perpendicular distance from $(1, -3)$ to $3x - 4y - 5 = 0$, i.e., $\left|\frac{3 + 12 - 5}{\sqrt{5^2}}\right| = 2$.

5. Centre of the given circle is $(1, 2)$. Let (α, β) be the other end.

$$\therefore \quad \frac{\alpha + 3}{2} = 1;\ \frac{\beta + 2}{2} = 2$$

$$\Rightarrow \quad \alpha = -1, \beta = 2$$

$\Rightarrow$ Other end is $(-1, 2)$.

6. Let (h, k) be the centroid, then $h = \frac{a\cos t + b\sin t + 1}{3}$

and $\quad k = \frac{a\sin t - b\cos t + 0}{3}$

$$\Rightarrow \quad 3h - 1 = a\cos t + b\sin t \qquad \text{(i)}$$

$$\text{and} \quad 3k = a\sin t - b\cos t \qquad \text{(ii)}$$

Squaring and adding Eqs. (i) and (ii), $(3h - 1)^2 + (3k)^2 = a^2 + b^2$.

Hence, the locus of (h, k) is $(3x - 1)^2 + (3y)^2 = a^2 + b^2$.

7. Let centre be (h, k), then $(h - 3)^2 + (k + 2)^2 = (h + 2)^2 + k^2$

$\Rightarrow 10h - 4k - 9 = 0$

Also the centre lies on the given line, so $2h - k = 3$.

On solving $\quad k = -6, h = -\frac{3}{2}$

Radius is $(h - 3)^2 + (k + 2)^2 = \frac{145}{4}$

8. Extremities of diameter are (5, 7) and (1, 4) and radius is half of the distance between them

$$= \frac{1}{2}\sqrt{(5-1)^2 + (7-4)^2} = \frac{5}{2}$$

9.

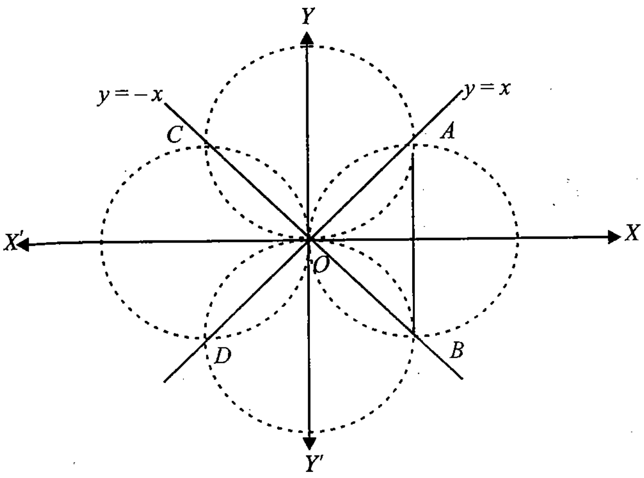


Fig. S-2.1

Coordinates of point A are $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$,

and coordinates of point B are $\left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$.

Now from the geometry, A and B are end points of diameter of the circle.

Then, equation of circle is

$$\left(x - \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0$$

or $\quad x^2 + y^2 - \sqrt{2}ax = 0$

Similarly, circle with C and D as end points of diameter is

$$x^2 + y^2 + \sqrt{2}ax = 0$$

With similar arguments, circles with A and C and B and D as end points of diameter are given by

$$x^2 + y^2 \pm \sqrt{2}ay = 0$$

10. Centre (1, 2) and since circle touches x-axis, therefore, radius is equal to 2.

Hence, the equation is $(x-1)^2 + (y-2)^2 = 2^2$

$\Rightarrow \quad x^2 + y^2 - 2x - 4y + 1 = 0$

11.

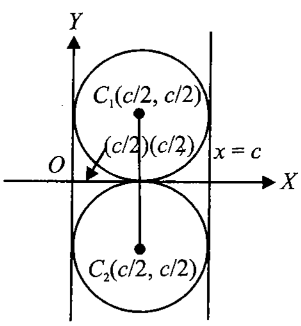


Fig. S–2.2

Since the circle touches both the coordinate axes and the line $x = c$, so there will be in all two such circles with centres C_1 and C_2 in 1st and 4th quadrants.

Hence, diameter of the circle = c,

Therefore, radius of the circle = $c/2$

and the coordinates of the centres are $(c/2, \pm c/2)$.

Hence, the equation of the two circles are $(x - c/2)^2 + (y \pm c/2)^2 = (c/2)^2$,

or $\quad x^2 + y^2 - cx \pm cy + c^2/4 = 0$.

12.

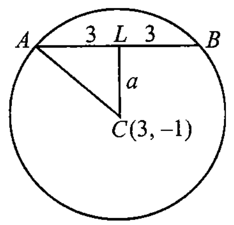


Fig. S–2.3

If the circle cuts off a chord AB of length 6 from the line $2x - 5y + 18 = 0$, then CL = length of ⊥ from centre $C(3, -1)$ on the line

$$= \frac{|2.3 - 5.(-1) + 18|}{\sqrt{(4+25)}} = \sqrt{29}$$

But, $\qquad AL = BL = 3$

In right-angled triangle, CLA, $CA^2 = CL^2 + AL^2 = 29 + 9 = 38$.

Therefore, radius of the circle = $CA = \sqrt{38}$

Hence, the equation of the circle is $(x-3)^2 + (y+1)^2 = \{\sqrt{38}\}^2$.

$\Rightarrow \quad x^2 + y^2 - 6x + 2y - 28 = 0$

13.

$$2\sqrt{g^2 - c} = 2a \qquad \text{(i)}$$

$$2\sqrt{f^2 - c} = 2b \qquad \text{(ii)}$$

On squaring Eqs. (i) and (ii) and then subtracting Eq. (ii) from Eq. (i), we get $g^2 - f^2 = a^2 - b^2$.

Hence, the locus is $x^2 - y^2 = a^2 - b^2$.

14. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through (2, 0).

$\therefore \qquad 4 + 4g + c = 0 \qquad$ (i)

Intercept on x-axis is $2\sqrt{g^2 - c} = 5$

$\therefore \qquad 4(g^2 + 4g + 4) = 25 \qquad$ [by Eq. (i)]

or $\quad (2g + 9)(2g - 1) = 0 \Rightarrow g = -\frac{9}{2}, \frac{1}{2}$

Since centre $(-g, -f)$ lies in first quadrant, we choose

$g = -\frac{9}{2}$ so that $-g = \frac{9}{2}$ (positive)

$\therefore \quad c = 14,$ [from Eq. (i)]

Value of t is variable.

15. Obviously, (3, 0) and (0, 4) are end points of diameter. Then, equation is $(x-3)(x-0)+(y-0)(y-4)=0$ or

$$x^2+y^2-3x-4y=0$$

16. Given, equation of circle is $x^2+y^2-3x-4y+2=0$ and it cuts the x-axis.

$\therefore \quad x^2+0-3x+2=0$

$\Rightarrow \quad x^2-3x+2=0$

$\Rightarrow \quad (x-1)(x-2)=0$

$\Rightarrow \quad x = 1, 2$

Therefore, the points are (1, 0) and (2, 0).

17. Comparing the given equation with $x^2+y^2+2gx+2fy+c=0$, we get $g=5$

$\therefore$ Length of intercept on x-axis

$$= 2\sqrt{g^2-c}$$

$$= 2\sqrt{5^2-9} = 8$$

18. Circle passing through points (1, 0), (0, 1) and (0, 0) is

$$x(x-1)+y(y-1)=0$$

It also passes through the point $(2k, 3k)$

$\Rightarrow 2k(2k-1)+3k(3k-1)=0$

$\Rightarrow \quad k = 0$ or $k = \frac{5}{13}$

19. One end of the diameter is $P(1, 1)$.

Let centre be $Q(\alpha, \beta)$.

Now Q is midpoint of PR where R lies on the line $x+y=3$

Then, point R is $(2\alpha-1, 2\beta-1)$.

This point lies on the line, then $2\alpha-1+2\beta-1=3$

$\Rightarrow \quad 2\alpha+2\beta=5$

Hence, the locus is $2x+2y=5$.

20.

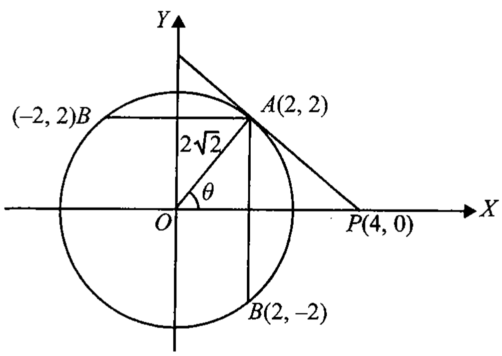


Fig. S–2.4

$$\cos\theta = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

$$\theta = 45°$$

$\therefore \quad A \equiv (2, 2)$

Let $\quad B \equiv (x_1, y_1)$

Given $\quad AB = 4$

$\therefore (x_1-2)^2+(y_1-2)^2=16$

$$x_1^2+y_1^2-4x_1-4y_1=8$$

Also $\quad x_1^2+y_1^2=8$

$\therefore \quad x_1+y_1=0$

$\therefore \quad 2x_1^2=8$

$$x_1 = \pm 2$$

$\therefore \quad B \equiv (2, -2)$ or $(-2, 2)$

21.

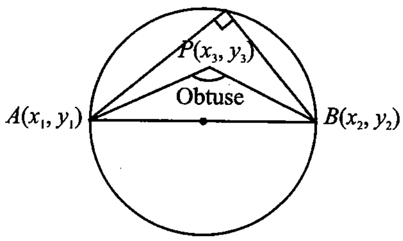


Fig. S–2.5

Equation of circle with AB as diameter, we get

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

since AB subtends obtuse angle at (x_3, y_3)

$\therefore$ P lies inside the circle.

$\therefore (x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)<0$

Exercise 2.2

1. Let the equation of the tangent be $\frac{x}{a}+\frac{y}{a}=1,$

i.e., $\quad x+y=a \qquad$ (i)

$\therefore$ Length of perpendicular from the centre $(-2, 2)$ on (i)

$=$ radius $= \sqrt{4+4-4}$

i.e., $\quad \frac{|-2+2-a|}{\sqrt{1+1}} = 2$

$\Rightarrow \quad a = 2\sqrt{2}$

Hence, the equation of the tangent is $x+y=2\sqrt{2}$.

2. According to the question,

$$\sqrt{(5)^2+(3)^2+2(5)+k(3)+17} = 7$$

$\Rightarrow \qquad 61 + 3k = 49 \Rightarrow k = -4$

3. Line $y = mx + c$ is tangent, if $c = \pm a\sqrt{1+m^2}$

Now $\qquad lx + my + n = 0$

or $\qquad y = -\frac{l}{m}x - \frac{n}{m}$ is tangent, if

$$-\frac{n}{m} = \pm a\sqrt{1+\left(\frac{l}{m}\right)^2}$$

or $\qquad n^2 = a^2(m^2 + l^2)$

4. Equation of pair of tangent is given by $SS_1 = T^2$

Here $S = x^2 + y^2 + 20(x+y) + 20$, $S_1 = 20$

$$T = 10(x+y) + 20$$

$\therefore \qquad SS_1 = T^2$

$\Rightarrow 20\{x^2 + y^2 + 20(x+y) + 20\} = 10^2(x+y+2)^2$

$\Rightarrow \qquad 2x^2 + 2y^2 + 5xy = 0$

5. Normal passes through the centre of the circle.

Hence the equation of normal is $x - y = 0$.

6. Equation of line perpendicular to $5x + 12y + 8 = 0$ is $12x - 5y + k = 0$.

Now it is a tangent to the circle.

If radius of circle = Distance of line from centre of circle

$$\Rightarrow \qquad \sqrt{121+4-25} = \left|\frac{12(11) - 5(2) + k}{\sqrt{144+25}}\right|$$

$$k = 8 \text{ or } -252.$$

Hence, equations of tangents are

$$12x - 5y + 8 = 0 \text{ and } 12x - 5y = 252$$

7. Clearly the point (1, 2) is the centre of the given circle and infinite tangents can only be drawn on a point circle.

Hence, radius should be 0.

$\therefore \qquad \sqrt{1^2 + 2^2 - \lambda} = 0 \Rightarrow \lambda = 5$

8. We must have

Radius of given circle > Perpendicular distance from the centre of circle to the given line

$$\Rightarrow \qquad \sqrt{4+16+5} > \frac{|3(2) - 4(4) - m|}{\sqrt{9+16}}$$

$\Rightarrow \qquad |m + 10| < 25$

$\Rightarrow \qquad -35 < m < 15$

9. The equation of tangents will be $y = mx$ or $y - mx = 0$

Then, applying condition for tangency,

$$\left|\frac{-5-4m}{\sqrt{1+m^2}}\right| = 5$$

$\Rightarrow 25 + 16m^2 + 40m = 25 + 25m^2$

$\Rightarrow \qquad 9m^2 - 40m = 0$

$\Rightarrow \qquad m = 0, \frac{40}{9}$

10. Let $P(x_1, y_1)$ be a point on $x^2 + y^2 = 4$.

Then, the equation of the tangent at P is $xx_1 + yy_1 = 4$ which meets the coordinates axes at $A\left(\frac{4}{x_1}, 0\right)$ and $B\left(0, \frac{4}{y_1}\right)$.

Let (h, k) be the midpoint of AB.

$$\therefore \qquad h = \frac{2}{x_1}, k = \frac{2}{y_1} \text{ i.e., } x_1 = \frac{2}{h}, y_1 = \frac{2}{k}$$

But (x_1, y_1) lies on $x^2 + y^2 = 4$

$$\Rightarrow \qquad \frac{4}{h^2} + \frac{4}{k^2} = 1$$

$$\Rightarrow \qquad 4(x^2 + y^2) = x^2y^2$$

11. Let the point be (x_1, y_1)

According to question, $\dfrac{\sqrt{x_1^2 + y_1^2 + 4x_1 + 3}}{\sqrt{x_1^2 + y_1^2 - 6x_1 + 5}} = \dfrac{2}{3}$

Squaring both sides, $\dfrac{x_1^2 + y_1^2 + 4x_1 + 3}{x_1^2 + y_1^2 - 6x_1 + 5} = \dfrac{4}{9}$

$\Rightarrow \quad 9x_1 + 9y_1^2 + 36x_1 + 27 = 4x_1^2 + 4y_1^2 - 24x_1 + 20$

$\Rightarrow \quad 5x_1^2 + 5y_1^2 + 60x_1 + 7 = 0$

Hence, locus is $5x^2 + 5y^2 + 60x + 7 = 0$.

12. Let (x_1, y_1) be any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$

$\therefore \qquad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1 = 0 \qquad$ (i)

Length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c_2 = 0$ is

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_2} = \sqrt{c_2 - c_1} \qquad$ (using (i))

Exercise 2.3

1. Here the intersection point of chord and circle can be found by solving the equation of circle with the equation of given line. Therefore, the points of intersection are $(-4, -3)$ and $\left(\frac{24}{5}, \frac{7}{5}\right)$. Hence, the midpoint is

$$\left(\frac{-4+\frac{24}{5}}{2}, \frac{-3+\frac{7}{5}}{2}\right) = \left(\frac{2}{5}, -\frac{4}{5}\right).$$

2. **a.** An equation of the chord of contact of P with respect to the given circle is

$$2xh + 2yk - \frac{3}{2}(x + h) + \frac{5}{2}(y + k) - 7 = 0$$

$$\Rightarrow x\left(2h - \frac{3}{2}\right) + y\left(2k + \frac{5}{2}\right) - \frac{3}{2}h + \frac{5}{2}k - 7 = 0 \quad \text{(i)}$$

which should be same as the given line

$$9x + y - 18 = 0 \quad \text{(ii)}$$

Comparing Eqs. (i) and (ii), we get

$$\frac{4h-3}{18} = \frac{4k+5}{2} = \frac{3h-5k+14}{36}$$

Comparing first two ratios, we get

$$h - 9k = 12 \quad \text{(iii)}$$

Comparing first and last ratios, we get

$$h + k = 4 \quad \text{(iv)}$$

Solving (iii) and (iv) for (h, k) we get

$$h = \frac{24}{5}, k = \frac{-4}{5}$$

Hence point P is $\left(\frac{24}{5}, \frac{-4}{5}\right)$

3. Equation of common chord will be

$$3x + 4y + 11 = 0 \quad \text{(i)}$$

Let the point of intersection of the tangents be (α, β).

$\therefore$ Equation of the chord of contact of the tangents drawn from (α, β) to first circle will be

$$x\alpha + y\beta = 9 \quad \text{(ii)}$$

$\because$ Equations (i) and (ii) are identical

$$\therefore \quad \frac{3}{\alpha} = \frac{4}{\beta} = -\frac{11}{9}$$

$$\therefore \quad (\alpha, \beta) = \left(-\frac{27}{11}, -\frac{36}{11}\right)$$

4. Chord of contact from origin $\equiv gx + fy + c = 0$

and from $(g, f) \equiv gx + fy + g(x + g) + f(y + f) + c = 0$

or $2gx + 2fy + g^2 + f^2 + c = 0$

$$\therefore \quad \text{Distance} = \frac{\left|\frac{g^2+f^2+c}{2} - c\right|}{\sqrt{g^2+f^2}} = \frac{|g^2+f^2-c|}{2\sqrt{g^2+f^2}}$$

5. Suppose point be (h, k). Equation of chord of contact is

$$hx + ky - a^2 = 0 \equiv lx + my + n = 0$$

or $$\frac{h}{l} = \frac{k}{m} = \frac{-a^2}{n}$$

or $$h = -\frac{a^2 l}{n}, k = \frac{-a^2 m}{n}$$

Exercise 2.4

1. **a.** Centres of circle $C_1(6, 6)$, $C_2(-3, -3)$

$$\therefore \quad C_1C_2 = \sqrt{(6+3)^2 + (6+3)^2} = 9\sqrt{2}$$

Radius of the circles $= \sqrt{36+36}, \sqrt{9+9} = 6\sqrt{2}, 3\sqrt{2}$

Since $\quad 9\sqrt{2} = 6\sqrt{2} + 3\sqrt{2}$

$\therefore$ Circles touch each other externally.

2. The equation of common chord PQ is $5ax + (c - d).$
$y + a + 1 = 0 \quad$ (i)

Also equation of PQ is $5x + by - a = 0 \quad$ (ii)

$$\therefore \quad \frac{5a}{5} = \frac{c-d}{b} = \frac{a+1}{-a}$$

$$\Rightarrow \quad a = \frac{a+1}{-a}$$

$$\Rightarrow \quad a^2 + a + 1 = 0$$

$\Rightarrow$ No value of a. $\quad [\because D < 0]$

3. **d.** Equation of circles is

$$S_1 = x^2 + y^2 + 2x - 3y + 6 = 0 \quad \text{(i)}$$

$$S_2 = x^2 + y^2 + x - 8y - 13 = 0 \quad \text{(ii)}$$

$\therefore$ Equation of common chord is

$$S_1 - S_2 = 0 \Rightarrow x + 5y + 19 = 0 \quad \text{(iii)}$$

and out of the four given points only point $(1, -4)$ satisfies it.

4. **b.** Centres and radii of the given circles are

Centres: $C_1(0, 1)$, $C_2(1, 0)$; Radii : $r_1 = 3, r_2 = 5$

Clearly, $\quad C_1C_2 = \sqrt{2} < (r_2 - r_1)$

Therefore, one circle lies entirely inside the other.

5. $(C_1C_2)^2 = r_1^2 + r_2^2$

$$\Rightarrow \quad 2a^2 = 18 \Rightarrow a = 3$$

6. Given circles are

$$2x^2 + 2y^2 - 3x + 6y + k = 0$$

or $$x^2 + y^2 - \frac{3}{2}x + 3y + \frac{k}{2} = 0 \quad \text{(i)}$$

and $x^2 + y^2 - 4x + 10y + 16 = 0 \quad$ (ii)

Circle (i) and (ii) cut orthogonally, then

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$2\left(-\frac{3}{4}\right)(-2)+2\left(\frac{3}{2}\right)\cdot 5=\frac{k}{2}+16$$

$$3+15=\frac{k}{2}+16 \Rightarrow k=4$$

7. Clearly, $r_1 - r_2 > C_1C_2$

$$r_1 = R, C_1(0, 0); r_2 = r; C_2(3, 4)$$

$$R - r > \sqrt{(3-0)^2 + (4-0)^2}$$

$$\Rightarrow \quad R - r > 5$$

8. Radical axes are

$$4x + 6y = 10 \text{ or } 2x + 3y = 5 \qquad \text{(i)}$$

and $$2x + 2y = 4 \text{ or } x + y = 2 \qquad \text{(ii)}$$

Point of intersection of (i) and (ii) is (1, 1).

9. Such circle is orthogonal to the given three circles. Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Then, according to the conditions given,

$$g + 2f = c + 3 \qquad \text{(i)}$$

$$2g + 4f = c + 5 \qquad \text{(ii)}$$

$$-7g - 8f = c - 9 \qquad \text{(iii)}$$

$$\Rightarrow \quad g = \frac{2}{3}, f = \frac{2}{3}, c = -1$$

Therefore, the required equation is

$$3(x^2 + y^2) + 4(x + y) - 3 = 0$$

10.

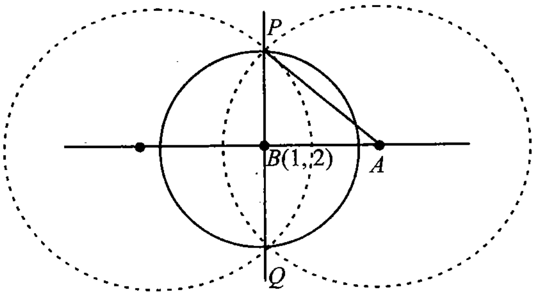


Fig. S–2.6

Clearly, diameter of 'C_1' will be the common chord.

Let the common chord be PQ and centre of C_2 be $A(h, k)$.

We have $AP = 5$, $PB = 3 \Rightarrow AB = 4$ units, where $B \equiv (1, 2)$.

Using parametric equation of line, we get

$$\frac{h-1}{-3/5} = \frac{k-2}{4/5} = \pm 4$$

$$\Rightarrow \quad h = -\frac{7}{5}, k = \frac{26}{5} \text{ or } h = \frac{17}{5}, k = -\frac{6}{5}$$

11.

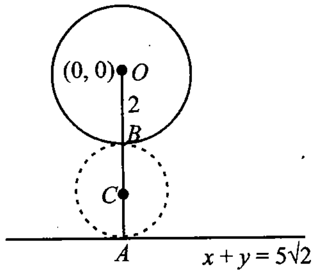


Fig. S–2.7

Here, OB = radius = 2

The distance of (0, 0) from $x + y = 5\sqrt{2}$ is 5.

$\therefore$ The radius of the smallest circle $= \frac{5-2}{2} = \frac{3}{2}$ and $OC = 2 + \frac{3}{2} = \frac{7}{2}$

The slope of $OA = 1 = \tan\theta$

$\therefore \cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow C = (0 + OC.\cos\theta, 0 + OC.\sin\theta) = \left(\frac{7}{2\sqrt{2}}, \frac{7}{2\sqrt{2}}\right)$

12.

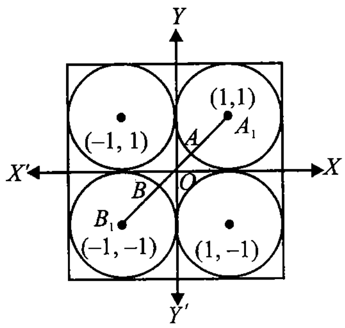


Fig. S–2.8

$$A_1B_1 = \sqrt{4+4} = 2\sqrt{2}$$

$$\Rightarrow \quad AB = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

Thus, equation of required circle is $x^2 + y^2 = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$.

13.

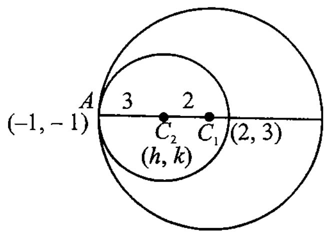


Fig. S–2.9

The given circle is $x^2 + y^2 - 4x - 6y - 12 = 0$, (i)

whose centre is $C_1(2, 3)$ and radius $r_1 = C_1A = 5$.

If $C_2(h, k)$ is the centre of the circle of radius 3 which touches the circle (i) internally at the point $A(-1, -1)$, then $C_2A = 3$.

and $\quad C_1C_2 = C_1A - C_2A = 5 - 3 = 2$

Thus $C_2(h, k)$ divide C_1A in the ratio 2 : 3 internally,

$$\therefore \quad h = \frac{2(-1) + 3.2}{2+3} = \frac{4}{5}$$

$$\text{and} \quad k = \frac{2(-1) + 3.3}{2+3} = \frac{7}{5}$$

Hence, the equation of the required circle is $(x - 4/5)^2 + (y - 7/5)^2 = 3^2$ or $5x^2 + 5y^2 - 8x - 14y - 32 = 0$.

14. The two circles are $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$.

Centres: $C_1(2, 3)$ $C_2(-1, 1)$

radii: $r_1 = 4 \quad r_2 = 1$

We have, $C_1C_2 = 5 = r_1 + r_2$, circles touch externally therefore there are three common tangents to the given circles.

15. Let $A \equiv (0, 0)$ and $B \equiv (r_1 + r_2, 0)$ be the centres of the two given fixed circles.

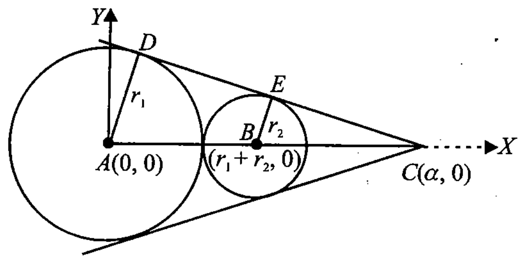


Fig. S–2.10

Let $C \equiv (\alpha, 0)$ be the point of intersection of direct common tangents.

$$\text{Now,} \quad \frac{r_2}{r_1} = \frac{\alpha - (r_1 + r_2)}{\alpha}$$

$$\Rightarrow \quad r_2\alpha = r_1\alpha - r_1^2 - r_1r_2$$

$$\Rightarrow \quad \alpha = \frac{r_1^2 + r_1r_2}{r_1 - r_2}$$

$$\therefore \quad \text{Locus of } C \text{ is } x = \frac{r_1^2 + r_1r_2}{r_1 - r_2} = a$$

which is a straight line.

16.

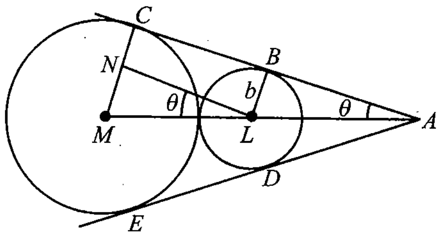


Fig. S–2.11

From Δ MLN

$$\sin\theta = \frac{a-b}{a+b}$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{a-b}{a+b}\right)$$

∴ Angle between AB and AD

$$= 2\theta = 2\sin^{-1}\left(\frac{a-b}{a+b}\right)$$

17. Given circles are

$(x - 1)^2 + (y - 2)^2 = 1$ (i)

and $(x-7)^2 + (y-10)^2 = 4$ (ii)

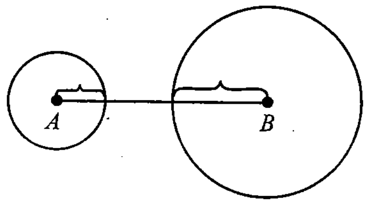


Fig. S–2.12

Let $\quad A \equiv (1, 2), B \equiv (7, 10), r_1 = 1, r_2 = 2$

$AB \equiv 10, r_1 + r_2 = 3$

$AB > r_1 + r_2$, hence the two circles are non-intersecting.

Radius of the two circles at time t are $1 + 0.3t$ and $2 + 0.4t$.

For the two circle to touch each other

$$AB^2 = [(r_1 + 0.3t) \pm (r_2 + 0.4t)]^2$$

$$\Rightarrow \quad 100 = [(1 + 0.3t) \pm (2 + 0.4t)]^2$$

$$\Rightarrow \quad 100 = (3 + 0.7t)^2, ((0.1)t + 1)^2$$

$$\Rightarrow \quad 3 + 0.7t = \pm 10, 0.1t + 1 = \pm 10$$

$$\Rightarrow \quad t = 10, t = 90 \quad [\because t > 0]$$

The two circles will touch each other externally in 10 seconds and internally in 90 seconds.

Exercise 2.5

1.

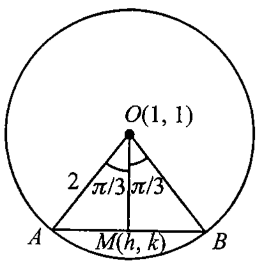


Fig. S–2.13

The coordinates of the centre and radius of the given circle are (1, 1) and 2, respectively. Let AB be the chord subtending an angle of $\frac{2\pi}{3}$ at the centre. Let M be the midpoint of AB and let its coordinates be (h, k).

In ΔOAM, $AM = OA.\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

$\therefore$ $OM^2 = OA^2 - AM^2 = 4 - (\sqrt{3})^2 = 1$

But $OM^2 = (h-1)^2 + (k-1)^2$.

Therefore, $(h-1)^2 + (k-1)^2 = 1$

Hence, the locus of (h, k) is $(x-1)^2 + (y-1)^2 = 1$ or $x^2 + y^2 - 2x - 2y + 1 = 0$

2.

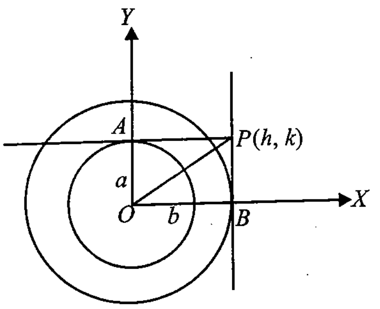


Fig. S–2.14

Let $OA = a, OB = b$

Since tangents at A and B meet at right angles in $P(h, k)$, $OAPB$ is a rectangle.

$\therefore$ $OP^2 = OB^2 + BP^2 = h^2 + k^2 = a^2 + b^2$

$\therefore$ Locus of P is

$$x^2 + y^2 = a^2 + b^2$$

which is concentric circle with given circles.

3.

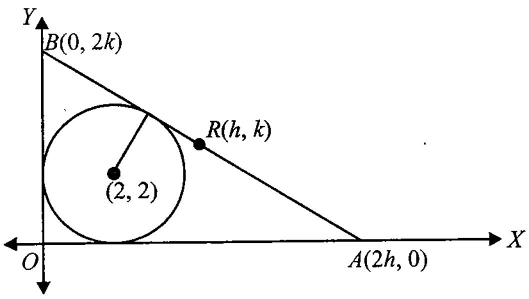


Fig. S–2.15

Let midpoint of AB is $R(h, k)$.

Then, coordinates of A and B are (2h, 0) and $(0, 2k)$, respectively.

Equation of line AB is $\frac{x}{2h} + \frac{y}{2k} = 1$.

Since this line touches given circle, we have

$$\frac{\left|\frac{2}{2h} + \frac{2}{2k} - 1\right|}{\sqrt{\frac{1}{h^2} + \frac{1}{k^2}}} = 2$$

On simplifying, we get locus $x + y - xy + \sqrt{x^2 + y^2} = 0$.

4.

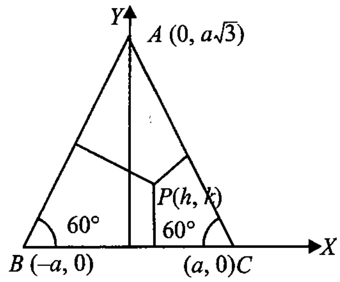


Fig. S–2.16

Taking midpoint of BC as origin, BC as x-axis and perpendicular to BC through O as y-axis, let C be $(a, 0)$, then B is $(-a, 0)$.

Since ΔABC is equilateral, A lies on y-axis.

As $\angle C = 60°$, A will be $(0, a\sqrt{3})$. Let P be (h, k).

Equation of AC and BC are $x\sqrt{3} + y - a\sqrt{3} = 0$ and $x\sqrt{3} - y + a\sqrt{3} = 0$, respectively.

According to the problem, $k^2 + \left(\frac{h\sqrt{3} - k + a\sqrt{3}}{2}\right)^2$

$$+\left(\frac{h\sqrt{3} - k + a\sqrt{3}}{2}\right)^2 = \lambda \text{ (say)}$$

$\Rightarrow 6h^2 + 6k^2 - 4a\sqrt{3}k + 6a^2 - 4\lambda = 0$

Hence, required locus of $P(h, k)$ is $6x^2 + 6y^2 - 4a\sqrt{3}y + 6a^2 - 4\lambda = 0$, which is a circle.

5.

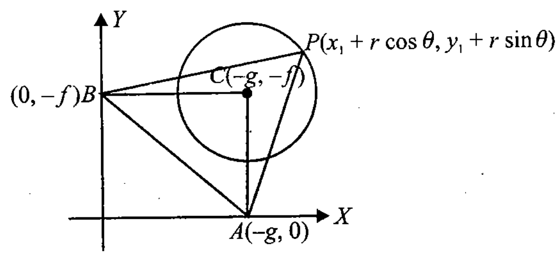


Fig. S–2.17

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Let the centroid of the ΔCAB be (h, k)

$3h = x_1 - g + r\cos\theta$

and $3k = y_1 f + r\sin\theta$

where $x_1 = -g; y_1 = -f$

$\therefore$ $3h = -2g + r\cos\theta$

$3k = -2f + r\sin\theta$

$\Rightarrow (3h + 2g)^2 + (3k + 2f)^2 = r^2$

$\Rightarrow (x + 2g/3)^2 + (y + 2f/3)^2 = r^2/9$

Hence, proved.

6.

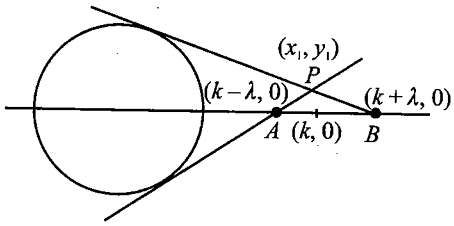


Fig. S–2.18

$$SS_1 = T^2$$

$$\Rightarrow (x_1^2 + y_1^2 - a^2)\,[(k-\lambda)^2 - a^2] = [(k-\lambda)x_1 - a^2]^2 \qquad \text{(i)}$$

and

$$(x_1^2 + y_1^2 - a^2)\,[(k+\lambda)^2 - a^2] = [(k+\lambda)x_1 - a^2]^2 \qquad \text{(ii)}$$

Equation (ii) – (i) gives

$$4(x_1^2 + y_1^2 - a^2)\,k\lambda = [(k+\lambda)x_1 - a^2 + (k-\lambda)x_1 - a^2] \times [(k+\lambda)x_1 - (k-\lambda)x_1]$$

$$= [2(kx_1 - a^2)]\,[2\lambda x_1]$$

$$= 4\lambda x_1\,[kx_1 - a^2]$$

$$\Rightarrow \quad k(x_1^2 + y_1^2 - a^2) = kx_1^2 - a^2x_1$$

$$\Rightarrow \quad ky_1^2 = a^2(k - x_1)$$

$\Rightarrow$ Hence, the locus is $ky^2 = a^2(k - x)$.

7. Let $A(-a, 0)$ and $B(a, 0)$ be two fixed points. Taking AB as x-axis and its right bisector as y-axis. Let the equation of the given line be

$$x\cos\alpha + y\sin\alpha = p \qquad \text{(i)}$$

and line perpendicular to it and passing through $(-a, 0)$ is given by

$$y\cos\alpha - x\sin\alpha = a\sin\alpha \qquad \text{(ii)}$$

Let AN and BM be the perpendiculars from A and B, then

$$AN \times BM = \frac{-a\cos\alpha + 0\sin\alpha - p}{\sqrt{\sin^2\alpha + \cos^2\alpha}} \times$$

$$\frac{a\cos\alpha + 0\sin\alpha - p}{\sqrt{\sin^2\alpha + \cos^2\alpha}} = \lambda^2\text{(contanstant)}$$

$$\Rightarrow \quad p^2 = \lambda^2 + a^2\cos^2\alpha \qquad \text{(iii)}$$

Eliminating p and α from Eqs. (i) and (ii), we get $x^2 + y^2 = \lambda^2 + a^2$ which is the required locus.

By changing a into $-a$, we get the same locus.

Hence, proved.

Chapter 3

Exercise 3.1

1.

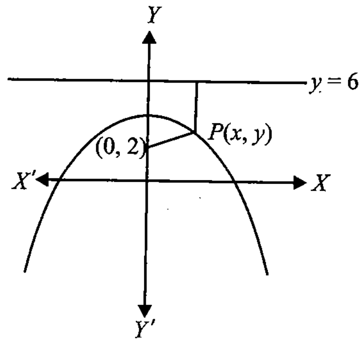


Fig. S–3.1

Since the focus and vertex of the parabola are on y-axis, therefore its directrix is parallel to x-axis and axis of the parabola is y-axis.

Let the equation of the directrix be $y = k$.

The directrix meets the axis of the parabola at $(0, k)$.

But vertex is the midpoint of the line segment joining the focus to the point where directrix meets axis of the parabola.

$$\therefore \quad \frac{k+2}{2} = 4 \Rightarrow k = 6$$

Thus, the equation of the directrix is $y = 6$.

Let (x, y) be a point on the parabola.

Then, by definition

$$(x-0)^2 + (y-2)^2 = (y-6)^2$$

$$\Rightarrow \quad x^2 + 8y = 32$$

2. The equation is

$$(x-0)^2 + (y-1)^2 = \left(\frac{x+2}{\sqrt{1}}\right)^2$$

or $\quad (y-1)^2 = 4(x+1)$

Clearly, $x = t^2 - 1$ and $y = 2t + 1$ satisfy it for all values of t.

3. Solving given parabola and circle, we have

$$x^2 + 4(x+3) + 4x = 0$$

$$\Rightarrow \quad x^2 + 8x + 12 = 0$$

$$\Rightarrow \quad x = -2 \ (x = -6 \text{ is not possible})$$

Since parabola and circle both are symmetrical about x-axis length of common chord is 4.

4. $\quad x^2 = 2(2x + y)$

or $\quad x^2 - 4x + 4 = 2(y+2)$

or $\quad (x-2)^2 = 2(y+2)$.

So, the vertex is $(2, -2)$.

5. The given equation can be written as $(x-2)^2 = 3(y-2)$.
Shifting the origin at (2, 2), this equation reduces to

$$X^2 = 3Y, \text{ where } x = X+2, y = Y+2$$

The directrix of this parabola with reference to new axes is

$$Y = -a, \text{ where } a = \frac{3}{4}$$

$$\Rightarrow \quad y - 2 = \frac{-3}{4}$$

$$\Rightarrow \quad y = \frac{5}{4}$$

6. Shifting the origin at A, equation becomes

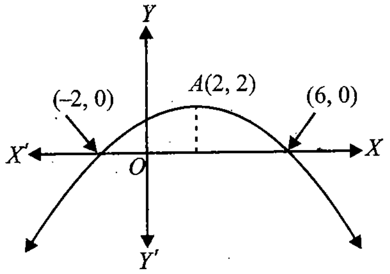


Fig. S–3.2

$$X^2 = -8Y$$

$$\Rightarrow \quad (x-2)^2 = -8(y-2)$$

7. $2a$ = perpendicular distance of the focus from the directrix

$$= \left|-\frac{7}{\sqrt{71}}\right| = \frac{7}{\sqrt{17}}$$

Therefore, latus rectum $= 4a = \frac{14}{\sqrt{17}}$

8.

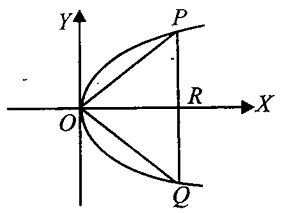


Fig. S–3.3

Let PQ be a double ordinate of length $8a$.

$$\therefore \quad PR = RQ = 4a.$$

Coordinates of P and Q are $(OR, 4a)$ and $(OR, -4a)$, respectively.

Since P lies on the parabola $y^2 = 4ax$,

$$\therefore \quad (4a)^2 = 4a(OR)$$

$$\Rightarrow \quad OR = 4a$$

Thus, the coordinates of P and Q are $(4a, 4a)$ and $(4a, -4a)$, respectively.

Now, $\quad m_1 = \text{slope of } OP = \frac{4a-0}{4a-0} = 1$

and $\quad m_2 = \text{slope of } OQ = \frac{-4a-0}{4a-0} = -1$

Clearly, $\quad m_1 m_2 = -1$

Thus, PQ subtends a right angle at the vertex of the parabola.

9.

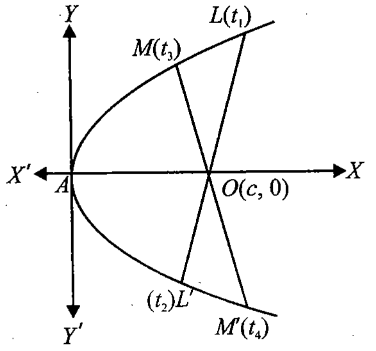


Fig. S–3.4

Equation of LOL': $2x - (t_1+t_2)y + 2at_1t_2 = 0$

Since chord LOL' passes through $(c, 0)$

$$\therefore \quad t_1t_2 = -\frac{c}{a}$$

Similarly, for chord MOM',

$$t_3t_4 = -\frac{c}{a}$$

Now circle with LL' as diameter, the equation is

$$(x - at_1^2)(x - at_2^2) + (y - 2at_1)(y - 2at_2) = 0$$

or $x^2 + y^2 - a(t_1^2 + t_2^2)x - 2a(t_1 + t_2)y + c^2 - 4ac = 0 \quad$ (i)

(as $a^2t_1^2t_2^2 = c^2$ and $4a^2t_1t_2 = -4ac$)

Similarly, circle with MM' as diameter, the equation is

$x^2 + y^2 - a(t_3^2 + t_4^2)x - 2a(t_3 + t_4)y + c^2 - 4ac = 0 \quad$ (ii)

Radical axis of Eqs. (i) and (ii) is (i) – (ii)

or $a(t_1^2 + t_2^2 - t_3^2 - t_4^2)x - 2a(t_1 + t_2 - t_3 - t_4)y = 0$,

which passes through the origin (vertex of the parabola).

10.

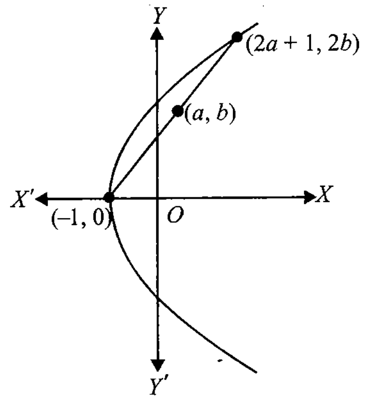


Fig. S–3.5

Since $((2a+1), 2b)$ lies on $y^2 = 4(x+1)$

$\therefore \quad 4b^2 = 4(2a+2)$

$b^2 = 2(a+1)$

11. The parabolas are $y^2 - x = 0$ and $y^2 + x = 0$. The point $(\lambda, -1)$ is an exterior point if

$1 - \lambda > 0$ and $1 + \lambda > 0$

$\Rightarrow \quad \lambda < 1$ and $\lambda > -1$

$\Rightarrow \quad -1 < \lambda < 1$

12. Let the fixed circle is $x^2 + y^2 = a^2$ and fixed line is $x = b$.

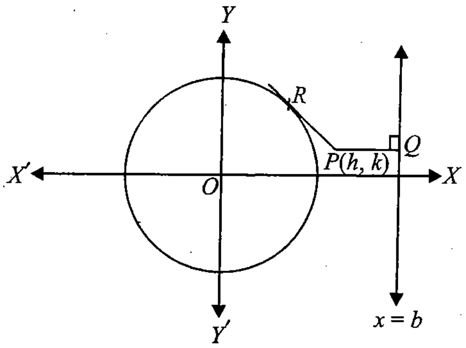


Fig. S–3.6

We have

$PQ = PR$

$\Rightarrow \quad b - h = \sqrt{h^2 + k^2 - a^2}$

$\Rightarrow \quad b^2 + h^2 - 2bh = h^2 + k^2 - a^2$

$\Rightarrow b^2 - 2bx = y^2 - a^2$ which is equation of parabola.

13

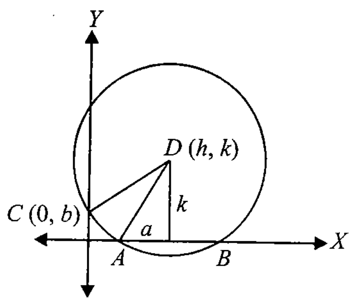


Fig. S–3.7

From the figure, $CD = AD$

$\Rightarrow \quad \sqrt{h^2 + (k-b)^2} = \sqrt{a^2 + k^2}$

$\Rightarrow \quad h^2 + k^2 - 2bk + b^2 = a^2 + k^2$

$\Rightarrow x^2 - 2by + b^2 = a^2$, which is equation of parabola.

Exercise 3.2

1. $x^2 - 4x + 6y + 10 = 0$

$\Rightarrow \quad x^2 - 4x + 4 = -6 - 6y$

$\Rightarrow (x-2)^2 = -6(y+1)$ circle drawn on focal distance as diameter always touches the tangent drawn to parabola at vertex.

Thus, circle will touch the line $y + 1 = 0$.

2. Extremities of the latus rectum are (2, 4) and (2, –4).

Since any circle drawn with any focal chord as it's diameter touches the directrix, radius of the circle is $2a = 4$ ($\because a = 2$).

3. Length of focal chord making an angle 'α' with x-axis is $4a \operatorname{cosec}^2 \alpha$. For $\alpha \in [0, \pi/4]$, its maximum length is $4a \times 2 = 8a$ units.

4. Latus rectum length of $y = ax^2 + bx + c$ is $1/a$. Now, semi-latus rectum is H.M. of SP and SQ.

Then, we have

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{1/2a}$$

$$\Rightarrow \quad 4a = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$\Rightarrow \quad a = \frac{5}{48}$$

5. Line joining $P(x_1, y_1) \equiv (at_1^2, 2at_1)$ and $Q(x_2, y_2) \equiv (at_2^2, 2at_2)$ is a focal chord

$\Rightarrow \quad t_1 t_2 = -1$

Now

$$x_1x_2 + y_1y_2 = a^2t_1^4t_2^4 + 4a^2t_1t_2 = a^2 - 4a^2$$
$$= -3a^2$$

Exercise 3.3

1. The equation of any tangent to the parabola $y^2 = 4ax$ in terms of its slope m is

$$y = mx + \frac{a}{m}$$

Coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Therefore, the equation of tangent to $y^2 = ax$ is

$$y = mx + \frac{a}{4m}$$

and the coordinates of the point of contact are $\left(\frac{a}{4m^2}, \frac{a}{2m}\right)$.

It is given that $m = \tan 45° = 1$.

Therefore, the coordinates of the point of contact are $\left(\frac{a}{4}, \frac{a}{2}\right)$.

2.

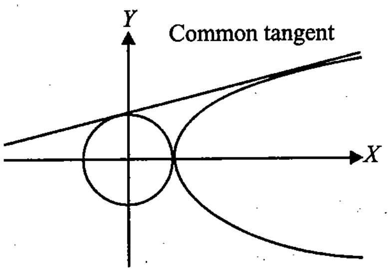


Fig. S–3.8

Equation of tangent to parabola $y^2 = 8ax$ having slope m is

$$y = mx + \frac{2a}{m} \quad \text{(i)}$$

Equation of tangent to circle $x^2 + y^2 = 2a^2$ having slope m is

$$y = mx \pm \sqrt{2}a\sqrt{1 + m^2} \quad \text{(ii)}$$

Equations (i) and (ii) are identical

$$\Rightarrow \quad \frac{2a}{m} = \pm \sqrt{2}a\sqrt{1 + m^2}$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$$

$$\Rightarrow \quad m^2 - 1 = 0$$

$$\Rightarrow \quad m = \pm 1$$

Hence, required tangents are $x \pm y + 2a = 0$.

3. Given parabolas intersect at (16, 18) the slope of the tangent to parabola $y^2 = 4x$ at (16, 8) is given by

$$m_1 = \left(\frac{dy}{dx}\right)_{(16,\,8)} = \left(\frac{4}{2y}\right)_{(16,\,8)} = \frac{2}{8} = \frac{1}{4}$$

The slope of the tangent to parabola $x^2 = 32y$ at (16, 8) is given by

$$m_2 = \left(\frac{dy}{dx}\right)_{(16,\,8)} = \left(\frac{2x}{32}\right)_{(16,\,8)} = 1$$

$$\therefore \quad \tan\theta = \left|\frac{1 - \frac{1}{4}}{1 + \frac{1}{4}}\right| = \frac{3}{5}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{5}\right)$$

4. Here, $\quad S = y^2 - 4x = 0$

and $\quad S(0, -2) \equiv 0^2 - (-2) = 2 > 0.$

Thus, point (0, −2) lies outside parabola, hence two tangents can be drawn.

5. c. We know that the tangents to the parabola $y^2 = 4ax$ at the points $\left(at_1^2, 2at_1\right)$ and $\left(at_2^2, 2at_2\right)$ meet at $\left(at_1t_2, a(t_1 + t_2)\right)$.

Here, $a = 1$, $t_1 = 1$, $t_2 = 2$. So, they meet at (2, 3), which is on the line $y = 3$.

6. Let $P\left(at_1^2, 2at_1\right)$, $Q\left(at_2^2, 2at_2\right)$ be two points on the parabola

$$y^2 = 4ax \quad \text{(i)}$$

Equations of tangents at P and Q on Eq. (i) are $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$.

The point of intersection of these tangents is $T[at_1t_2, a(t_1 + t_2)]$.

The coordinates of the focus S are $(a, 0)$.

$$\therefore (ST)^2 = (at_1t_2 - a)^2 + [a(t_1 + t_2)]^2$$

$$= a^2\left(1 + t_1^2\right)\left(1 + t_2^2\right)$$

$$= \left(a + at_1^2\right)\left(a + at_2^2\right)$$

$$= SP \times SQ$$

7. Eliminating y, we have

$$a - x = x - x^2 \text{ or } x^2 - 2x + a = 0$$

Since the line touches the parabola, we must have equal roots.

$$\therefore \quad 4 - 4a = 0 \text{ or } a = 1$$

8. Equation of tangent to parabola having slope m is

$$y = mx + \frac{1}{m}$$

$\Rightarrow$ It passes through (h, k), therefore $m^2h - mk + 1 = 0$

$$\Rightarrow \quad m_1 + m_2 = \frac{k}{h};\ m_1m_2 = \frac{1}{h}$$

Given $\quad \theta_1 + \theta_2 = \frac{\pi}{4} \Rightarrow \tan(\theta_1 + \theta_2) = 1$

$$\Rightarrow \quad \frac{m_1 + m_2}{1 - m_1m_2} = 1 \Rightarrow \frac{k}{h} = 1 - \frac{1}{h}$$

$$\Rightarrow \quad y = x - 1$$

9. Equation of tangent $y = mx + \frac{1}{m}$ passing through (1, 3)

$$\Rightarrow \quad m^2 - 3m + 1 = 0$$

Hence, the roots are $m_1 = \frac{3 + \sqrt{5}}{2}$, $m_2 = \frac{3 - \sqrt{5}}{2}$.

$$\Rightarrow \quad \tan\theta = \left|\frac{\frac{3 + \sqrt{5}}{2} - \frac{3 - \sqrt{5}}{2}}{1 + \frac{(3 + \sqrt{5})(3 - \sqrt{5})}{4}}\right| = \left|\frac{\sqrt{5}}{2}\right|$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right) \text{ or } \pi - \tan^{-1}\left(\frac{\sqrt{5}}{2}\right)$$

10. Any tangent to the parabola $y^2 = 8x$ is $y = mx + 2/m$, which is normal to the given circle.

Hence, tangents must pass through centre (−3, −4) of the circle.

Then, we have

$$-4 = -3m + \frac{2}{m}$$

$\Rightarrow \quad 3m^2 - 4m - 2 = 0$

$\Rightarrow \quad m = \dfrac{4 \pm \sqrt{40}}{6} = \dfrac{2 \pm \sqrt{10}}{3}$

11. The line $y = x - 1$ passes through $(1, 0)$. That means it is a focal chord. Hence, the required angle is $\frac{\pi}{2}$.

12. We know that perpendicular tangents meet on the directric.

Given parabola is $y^2 + 4y - 6x - 2 = 0$

or $\quad (y + 2)^2 = 6(x + 1)$

Equation of directrix is $x + 1 = -6/4$ or $x = -5/2$ or $2x + 5 = 0$.

13. Diameter of triangle PAB = focal chord

$\Rightarrow$ minimum radius $= \dfrac{\text{length of latus ractum}}{2} = 8$

14. Origin $(0, 0)$ lies on the directrix of given parabola which is $y = 0$. Then, angle between tangents is $90°$.

15. Let P be (h, k). Any tangent $y = mx + a/m$

or $\quad k = mh + a/m$ or $m^2h - mk + a = 0$

Its roots are m_1 and $3m_1$ $\therefore$ $m_1 + 3m_1 = k/h$

$m_1 . 3m_1 = a/h$

Eliminating m_1,

$\Rightarrow \quad$ Locus is $3y^2 = 6x$

Exercise 3.4

1. We have

$$y - x\sqrt{2} + 4a\sqrt{2} = 0$$

or $\quad y = x\sqrt{2} - 4a\sqrt{2} \qquad \text{(i)}$

Comparing Eq. (i) with the equation $y = mx + c$, then

$$m = \sqrt{2}, \; c = -4a\sqrt{2}$$

Since $-2am - am^2 = -2a\sqrt{2} - a(\sqrt{2})^3$

$$= -2a\sqrt{2} - 2a\sqrt{2} = -4a\sqrt{2}$$
$$= c$$

Hence, the given chord is normal to the parabola $y^2 = 4ax$.

The coordinates of the point are $(am^2, -2am) \equiv (2a, -2\sqrt{2}a)$.

2. The equation of a normal to the parabola $y^2 = 24x$ having slope m is $y = mx - 12m - 6m^3$.

It is parallel to $y = 2x + 3$, therefore, $m = 2$

Then equation of the parallel normal is

$$y = 2x - 24 - 48$$

or $\quad y = 2x - 72$

The distance between $y = 2x + 3$ and $y = 2x - 72$ is $\left|\dfrac{72 + 3}{\sqrt{4 + 1}}\right|$ $25\sqrt{5}$.

3. Slope m of the normal $x + y = 6$ is -1 and $a = 2$.

Normal to parabola $y^2 = 4ax$ at point $(am^2, -2am)$ is

$$y = mx - 2am - 2am - am^3$$

$\Rightarrow \quad y = -x + 4 + 2$ at $(2, 4)$

$\Rightarrow \quad x + y = 6$ is normal at $(2, 4)$

4. Normal at $P(at^2, 2at)$ is $y = -tx + 2at + at^3$.

It meets the axis $y = 0$ in $G(2a + at^2, 0)$.

If (x, y) be the midpoint of PG, then

$$2x = 2a + at^2 + at^2, \; 2y = 2at$$

Eliminating t, we have

$$x - a = at^2 = a(y/a)^2$$

or $y^2 = a(x - a)$ which is the required locus.

5.

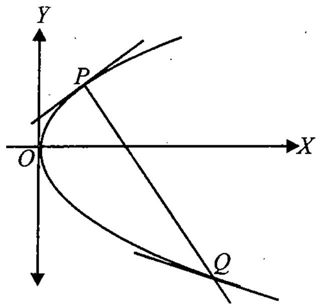


Fig. S–3.9

Slope of normal at point $P(t)$ is $-t$

or slope of line $PQ = -t$

Now, point Q has parameter $-t - \frac{2}{t}$.

Slope of tangent at point $Q = \dfrac{1}{-t - \frac{2}{t}} = \dfrac{-t}{t^2 + 2}$

Now, angle between normal and parabola at Q is equivalent to angle between normal and tangent at point

$$Q \Rightarrow \tan\theta = \left|\dfrac{-t + \dfrac{t}{t^2+2}}{1 + t\dfrac{t}{t^2+2}}\right| = \left|\dfrac{t}{2}\right|$$

Hence, $\quad \theta = \tan^{-1}\left|\dfrac{t}{2}\right|$

6.

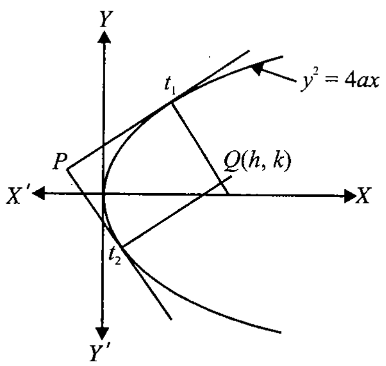


Fig. S–3.10

Let $P\left(at_1t_2, a(t_1+t_2)\right)$; P must satisfy $y^2 = a(x+b)$

Hence $\quad a^2(t_1+t_2)^2 = a\left((at_1t_2)+b\right)$

$$a\left(t_1^2+t_2^2+t_1t_2\right) = b$$

Now coordinates of point of intersection of normals at t_1 and t_2 are

$$h = a\left(t_1^2+t_2^2+t_1t_2+2\right) \quad \text{(i)}$$

and $$k = -at_1t_2(t_1+t_2) \quad \text{(ii)}$$

From Eq. (i), $h = b + 2a$

$\Rightarrow \quad x = b + 2a$

7. If normal at $P(a\alpha^2, 2a\alpha)$ meets the parabola at $Q(a\beta^2, 2a\beta)$

$$\beta = -\alpha - \frac{2}{\alpha}$$

or $\quad 2a(\beta+\alpha) = -\frac{4a}{\alpha},$

$$2a\beta + 2a\alpha = -\frac{4a}{\alpha} = 3$$

$$4a = -3\alpha$$

$\Rightarrow \quad 2a = -1.5\alpha$

8. For $y^2 = 4ax$, normal is

$$y = mx - 2am - am^3 \quad \text{(i)}$$

For $y^2 = 4c(x-b)$, normal is

$$y = m(x-b) - 2cm - cm^3 \quad \text{(ii)}$$

If two parabolas have common normal, then Eq. (i) and (ii) must be identical.

After comparing the coefficients we, get

$$m = \pm\sqrt{\frac{2(a-c)-b}{(c-a)}}$$

which is real if $-2 - \frac{b}{c-a} > 0$

$\Rightarrow \quad \frac{b}{a-c} > 2$

9. Normal having slope m is $y = mx - 2am - am^3$

or $\quad \dfrac{mx-y}{2am+am^3} = 1.$

Make the parabola $y^2 = 4ax$ homogeneous and since the lines are at right angles, sum of the coefficients of x^2 and y^2 is zero.

$\Rightarrow \quad m^2 = 2$ or $m = \sqrt{2}$

or $\quad \tan\theta = \sqrt{2}$

or $\quad \theta = \tan^{-1}(\sqrt{2})$

Exercise 3.5

1. Chord of contact of (h, k) is

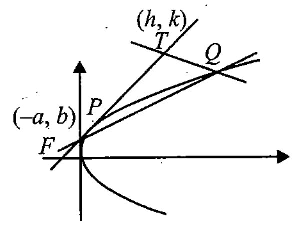


Fig. S–3.11

$$ky = 2a(x+h).$$

It passes through $(-a, b)$

$\therefore \quad bk = 2a(-a+h)$

$\Rightarrow$ Locus is $\quad by = 2a(x-a).$

2. The chord of contact of parabola w.r.t. $(\alpha, 2)$ is

$$2y = 2(x+\alpha)$$

or $\quad x - y + \alpha = 0$

Given $\quad \dfrac{|\alpha - 2 + \alpha|}{\sqrt{2}} = 4$

$\Rightarrow \quad |\alpha - 1| = 2\sqrt{2}$

$\Rightarrow \quad \alpha = 1 \pm 2\sqrt{2}.$

$\Rightarrow \quad \alpha = 1 - 2\sqrt{2}$

(as for $\alpha = 1 + 2\sqrt{2}$, point lies inside paraboia)

3. Any chord PQ which get bisected at point $R(h, k)$ is $T = S_1$ or $ky - 2a(x+h) = k^2 - 4ah.$

Now given that this chord is focal chord, then it must passes through focus $S(a, 0)$.

Then $k(0) - 2a(a+h) = k^2 - 4ah$

$\Rightarrow \quad k^2 = 2ah - 4a^2$

$\Rightarrow \quad y^2 = 2a(x-a)$

4. Chord of contact of $P(0, \lambda)$ is $\lambda y = 2ax$.

$\Rightarrow \quad 2ax - \lambda y = 0$

Line $\lambda x + 2ay = c$ perpendicular to it passes through $(0, \lambda)$

$\Rightarrow \quad c = 2a\lambda$

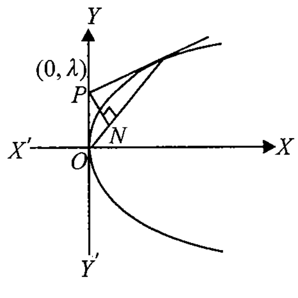


Fig. S–3.12

Hence equation of PN is

$$\lambda x + 2ay = 2a\lambda$$

or $\quad 2ay + \lambda(x - 2a) = 0.$

This passes through $(2a, 0)$.

Chapter 4

Exercise 4.1

1.

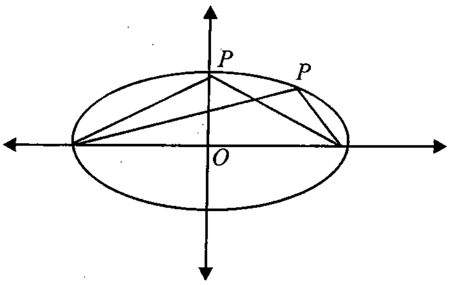


Fig. S–4.1

The maximum area corresponds to P when it is at either end of the minor axis and hence area for such a position of P is $\frac{1}{2}(2a)(b) = ab$.

2. Given $\frac{x}{3} = \cos t + \sin t$ and $\frac{y}{4} = \cos t - \sin t$.

Squaring and adding, we have

$$\frac{x^2}{9} + \frac{y^2}{16} = (1 + \sin 2t) + (1 - \sin 2t)$$

or $$\frac{x^2}{9} + \frac{y^2}{16} = 2$$

which is the equation of the ellipse.

3. Let the equation of the semi-elliptical arch be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (y > 0)$$

Length of the major axis $= 2a = 9 \Rightarrow a = \frac{9}{2}$

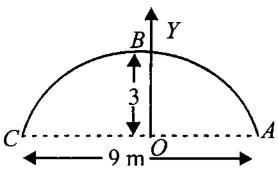


Fig. S–4.2

Length of the semi-minor axis $= b = 3$

So, the equation of the arc becomes $\frac{4x^2}{81} + \frac{y^2}{9} = 1$

If $x = 2$, then $y^2 = \frac{65}{9} \Rightarrow y = \frac{1}{3}\sqrt{65} = \frac{8}{3}$ approximately.

4.

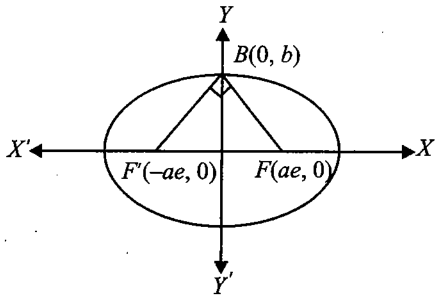


Fig. S–4.3

$$m_{BF} \cdot m_{BF'} = -1$$

$$\Rightarrow \quad \frac{b-0}{0-ae} \times \frac{b-0}{0+ae} = -1$$

$$\Rightarrow \quad \frac{b^2}{a^2e^2} = 1$$

$$\Rightarrow \quad e^2 = 1 - \frac{b^2}{a^2} = 1 - e^2$$

$$\Rightarrow \quad e^2 = 1/2$$

$$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$$

5. Here $ae = 4$ and $e = \frac{4}{5}$. So, a = 5.

Now, $$b^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad b^2 = 25\left(1 - \frac{16}{25}\right) = 9$$

Hence, the equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

6.

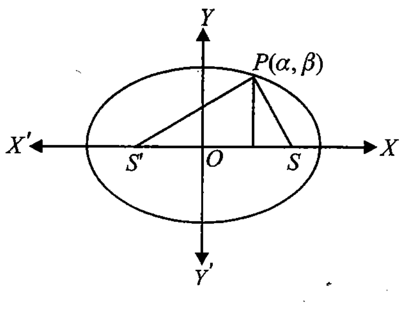


Fig. S–4.4

Since (α, β) lies on the ellipse $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$

$$\Rightarrow \quad \beta = b\sqrt{1 - \frac{\alpha^2}{a^2}}$$

Area of $\Delta SPS'$

$$= \frac{1}{2}\beta . SS'$$

$$= \frac{1}{2}\beta(2ae)$$

$$= bae\sqrt{1-\frac{\alpha^2}{a^2}}$$

$$= be\sqrt{a^2-\alpha^2}$$

7.

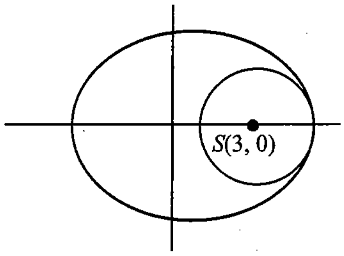


Fig. S–4.5

Given ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \qquad \text{(i)}$$

$$\Rightarrow \qquad e^2 = 1-\frac{16}{25} \Rightarrow e = \frac{3}{5}$$

Then foci are $(\pm ae, 0) \equiv (\pm 3, 0)$.

Now circle having centre (3, 0) is $(x-3)^2+y^2=r^2$ (ii)

Eliminating y^2 from (i) and (ii), we get

$$\frac{x^2}{25}+\frac{r^2-(x-3)^2}{16}=1$$

or $\quad 16x^2+25r^2-25(x^2-6x+9)=400$

or $\quad -9x^2+150x+25r^2-625=0$

Since circle touches ellipse, above equation has equal roots. Hence, $D=0 \Rightarrow 25500+36(25r^2-625)=0$.

which is not possible.

Then circle will touch the ellipse at the end of the major axis (5, 0). Hence, radius is 2.

8. The given equation can be written as

$$\frac{(x+1)^2}{9}+\frac{(y+2)^2}{25}=1$$

which represents an ellipse whose centre is (– 1, – 2) and semi-major and minor axes are 5 and 3, respectively.

The eccentricity of the ellipse is given by

$$9=(1-e^2) \Rightarrow e=\frac{4}{5}$$

Shifting the origin at (– 1, – 2), the given equation reduces to

$$\frac{X^2}{9}+\frac{Y^2}{25}=1 \qquad \text{(i)}$$

where $\qquad x = X-1, y = Y-2 \qquad$ (ii)

Coordinates of the foci of (i) are $(X=0, Y=\pm be)$, where $b=5$, $e=4/5$, i.e., foci of (i) are $(X=0, Y=\pm 4)$.

Therefore, the coordinates of the foci of the given ellipse are (– 1, 2) and (– 1, – 6).

9. Given ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$

Here $\qquad a^2=16, b^2=9 \Rightarrow a=4, b=3$

Sum of the focal distance from any point on the ellipse, $2a = 2(4) = 8$.

10. We have $a^2=16$, $b^2=9$. Then, $e=\sqrt{1-\frac{b^2}{a^2}}=\frac{\sqrt{7}}{4}$.

Coordinates of focus S are $(\sqrt{7}, 0)$.

Therefore $\qquad CS = \sqrt{7}$.

Hence, CS: Major axis $= \sqrt{7} : 2a = \sqrt{7} : 8$.

11.

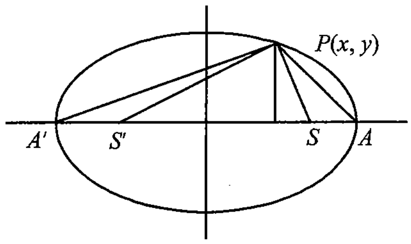


Fig. S–4.6

$$\frac{\Delta PSS'}{\Delta APA'}=\frac{\frac{1}{2}(SS')\,y}{\frac{1}{2}(AA')\,y}$$

$$=\frac{2ae}{2a}=e$$

12. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \; (a<b)$

Since foci are $(0, \pm 1)$

$\Rightarrow \qquad be = 1$ and $2a = 1$

[Since minor axis is of length = 1]

Also $\qquad a^2 = b^2(1-e^2)$

$\Rightarrow \qquad \frac{1}{4} = b^2 - b^2e^2 = b^2 - 1$

$\Rightarrow \qquad b^2 = 1+\frac{1}{4}=\frac{5}{4}$

Hence the equation of the ellipse is $\frac{x^2}{1/4}+\frac{y^2}{5/4}=1$ or $20x^2+4y^2=5$.

Exercise 4.2

1. $(3x-12)^2+(3y+15)^2=\frac{(3x+4y+5)^2}{25}$

$$\Rightarrow \sqrt{(x-4)^2+(y+5)^2}=\frac{1}{3}\,\frac{|3x-4y+5|}{5}$$

Here, ratio of distances of variable point $P(x, y)$ from fixed point (focus) (4, 5) to that from fixed line (directrix

$\equiv 3x - 4y + 5 = 0$) is 1/3

Hence locus is ellipse and its eccentricity is 1/3.

Also length of latus rectum = 2(e)(distance of (4, 5) from the line $3x - 4y + 5 = 0$)

$$= 2 \times \frac{1}{3}\frac{|3 \times 4 - 4 \times 5 + 5|}{5} = \frac{2}{5}$$

2. The given equation is

$$\frac{\left(\frac{3x-4y+2}{5}\right)^2}{\frac{16}{25}} + \frac{\left(\frac{4x-3y-5}{5}\right)^2}{\frac{9}{25}} = 1$$

Hence $\quad a^2 = \frac{16}{25}$ and $b^2 = \frac{9}{25}$

$\Rightarrow$ Length of major axis $= 2a = \frac{8}{5}$ and length of minor axis $= 2b = \frac{6}{5}$

and $\quad e^2 = 1 - \frac{9}{16} \Rightarrow e = \frac{\sqrt{7}}{4}$

Exercise 4.3

1. The coordinates of any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angle is θ are $(a\cos\theta, b\sin\theta)$.

The coordinates of the end points of latus rectum are $\left(ae, \pm\frac{b^2}{a}\right)$.

$\therefore \quad a\cos\theta = ae$ and $b\sin\theta = \pm\frac{b^2}{a}$

$\Rightarrow \quad \tan\theta = \pm\frac{b}{ae}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\pm\frac{b}{ae}\right)$

Hence four points of latus rectum have eccentric angles $\tan^{-1}\left(\frac{b}{ae}\right), \pi - \tan^{-1}\left(\frac{b}{ae}\right), \pi + \tan^{-1}\left(\frac{b}{ae}\right), 2\pi - \tan^{-1}\left(\frac{b}{ae}\right)$ in I[st], II[nd], III[rd], IV[th] quadrants respectively.

2. Here $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$, $S(ae, 0)$ are collinear, then

$$\begin{vmatrix} a\cos\alpha & b\sin\alpha & 1 \\ a\cos\beta & b\sin\beta & 1 \\ ae & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ e & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad e(\sin\alpha - \sin\beta) + \sin(\beta - \alpha) = 0$

$\Rightarrow \quad 2e\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$\Rightarrow \quad e = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \frac{\sin\alpha + \sin\beta}{\sin(\alpha+\beta)}$$

3.

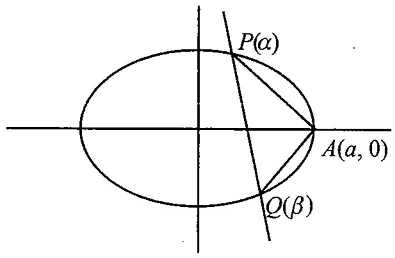


Fig. S–4.7

$$m_{AP} \times m_{QA} = -1$$

$$\Rightarrow \quad \frac{b\sin\alpha}{a\cos\alpha - a} \times \frac{b\sin\beta}{a\cos\beta - a} = -1$$

$$\Rightarrow \quad \frac{4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}}{\left(2\sin^2\frac{\alpha}{2}\right)\left(2\sin^2\frac{\beta}{2}\right)} = \frac{-a^2}{b^2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = -\frac{b^2}{a^2}$$

Exercise 4.4

1. We know that the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

Then comparing the given line $x\cos\alpha + y\sin\alpha = p$ with $y = mx + c$, we have $c = \frac{p}{\sin\alpha}$, $m = -\frac{\cos\alpha}{\sin\alpha}$.

So, the given line will be a tangent if

$$\frac{p^2}{\sin^2\alpha} = a^2\frac{\cos^2\alpha}{\sin^2\alpha} + b^2$$

$$\Rightarrow \quad p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

2. The equation of any tangent to the given ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

If it touches $\quad x^2 + y^2 = r^2$.

Then, $\quad \sqrt{a^2m^2 + b^2} = r\sqrt{1 + m^2}$

$\Rightarrow \quad m^2(a^2 - r^2) = r^2 - b^2$

$$\Rightarrow \quad m = \sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$$

3. The equation of tangent to the given ellipse at point $P(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

Intercept of line on the axes are $\frac{a}{\cos\theta}$ and $\frac{b}{\sin\theta}$.

Given that $\frac{a}{\cos\theta} = \frac{b}{\sin\theta} = l \Rightarrow \cos\theta = \frac{a}{l}$ and $\sin\theta = \frac{b}{l}$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta = \frac{a^2}{l^2} + \frac{b^2}{l^2}$

$\Rightarrow \quad l^2 = a^2 + b^2$

$\Rightarrow \quad l = \sqrt{a^2 + b^2}$

4.

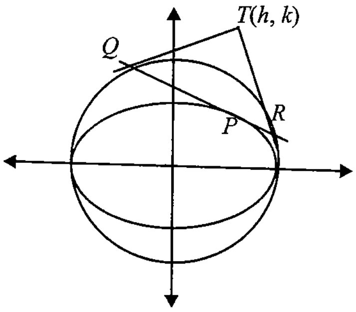


Fig. S–4.8

Equation of tangent to the ellipse at a given point is

$$x\left(\frac{1}{\sqrt{2}}\right) + 2y\left(\frac{1}{2}\right) = 1$$

or $\quad x + \sqrt{2}y = \sqrt{2} \quad$ (i)

Now QR is chord of contact of circle $x^2 + y^2 = 1$ with respect to point $T(h, k)$.

Then,

$$QR \equiv hx + ky = 1. \quad \text{(ii)}$$

Equations (i) and (ii) represent same straight line, then comparing ratio of coefficients, we have $\frac{h}{1} = \frac{k}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

Hence, $(h, k) \equiv \left(\frac{1}{\sqrt{2}}, 1\right)$.

5. For a given ellipse, equation of tangent whose slope is m is $y = mx + \sqrt{18m^2 + 32}$.

For $m = -\frac{4}{3}$, tangent is

$$y = -\frac{4}{3}x \times \sqrt{18\left(\frac{16}{9}\right) + 32}$$

$$= -\frac{4}{3}x + 8$$

or $\quad 4x + 3y = 24$

This intersects major and minor axes at $A(6, 0)$ and $B(0, 8)$.

Then, area of ΔACB is $\frac{1}{2}(6)(8) = 24$.

6. Since the locus of the point of intersection of perpendicular tangents to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the director circle given by

$$x^2 + y^2 = a^2 + b^2.$$

Hence, the perpendicular tangents drawn to $\frac{x^2}{25} + \frac{y^2}{16}$ intersect on the curve $x^2 + y^2 = 25 + 16$, i.e., $x^2 + y^2 = 41$.

7. Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse.

Equation of tangent at $P(\theta)$ is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{(i)}$$

Tangent meets the major axis at $T(a\sec\theta, 0)$.

Applying sine rule in ΔPST, we get

$$\frac{PS}{\sin(\pi - \alpha)} = \frac{ST}{\sin\beta}$$

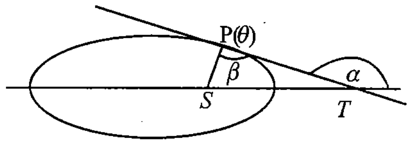


Fig. S–4.9

$$\frac{a(1 - e\cos\theta)}{\sin\alpha} = \frac{a(\sec\theta - e)}{\sin\beta}$$

$$\Rightarrow \quad \frac{1 - e\cos\theta}{\sin\alpha} = \frac{1 - e\cos\theta}{\cos\theta\sin\beta}$$

$$\Rightarrow \quad \cos\theta = \frac{\sin\alpha}{\sin\beta}$$

Slope of tangent is $-\frac{b}{a}\cot\theta = \tan\alpha$

$$\Rightarrow \quad \frac{b^2}{a^2} = \tan^2\alpha(\sec^2\theta - 1)$$

$$\Rightarrow \quad b^2/a^2 = \tan^2\alpha\left(\frac{\sin^2\beta - \sin^2\alpha}{\sin^2\alpha}\right)$$

$$= \frac{\sin^2\beta - \sin^2\alpha}{\cos^2\alpha}$$

$$\Rightarrow \quad e = \sqrt{1 - \frac{\sin^2\beta - \sin^2\alpha}{\cos^2\alpha}}$$

$$= \sqrt{\frac{1 - \sin^2\beta}{\cos^2\alpha}} = \frac{\cos\beta}{\cos\alpha}$$

Exercise 4.5

1. Let the line $lx + my + n = 0$ be normal to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

Now the equation of the normal at $(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{bx}{\sin\theta} = a^2 - b^2 \qquad \text{(i)}$$

Comparing this line with the line $lx + my + n = 0$, we have

$$\frac{a}{l\cos\theta} = \frac{-b}{m\sin\theta} = \frac{a^2-b^2}{-n}$$

$$\Rightarrow \quad \cos\theta = \frac{-an}{l(a^2-b^2)}$$

and $$\sin\theta = \frac{bn}{m(a^2-b^2)}$$

Squaring and adding, we get

$$\Rightarrow \frac{a^2n^2}{l^2(a^2-b^2)^2} + \frac{b^2n^2}{m^2(a^2-b^2)^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{l^2} + \frac{b^2}{m} = \frac{(a^2-b^2)^2}{n^2}$$

2. The equation of the normal at (x_1, y_1) to the given ellipse is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

Here, $x_1 = ae$ and $y_1 = \frac{b^2}{a}$

So, the equation of the normal at positive end of the latus rectum is

$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2e^2 \qquad [\because b^2 = a^2(1-e^2)]$$

$$\Rightarrow \quad \frac{ax}{e} - ay = a^2e^2 \Rightarrow x - ey - e^3a = 0$$

3.

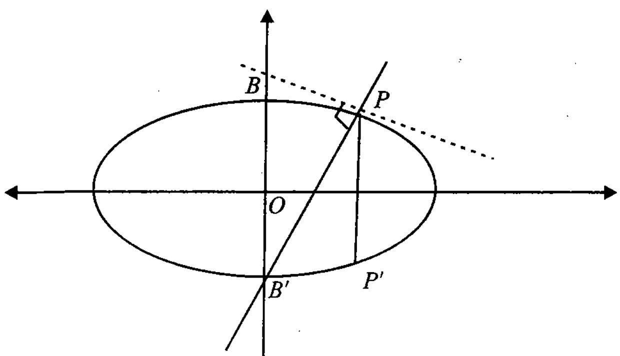


Fig. S–4.10

From the above question, equation of normal at the extremity of latus rectum is $x - ey = ae^3$.

This passes through the extremity of the minor axis, i.e., $B'(0, -b)$

Then $\quad 0 + eb - ae^3 = 0$

$$\Rightarrow \quad b = ae^2 \quad \Rightarrow b^2 = a^2e^4$$

$$\Rightarrow \quad a^2(1-e^2) = a^2e^4 \quad \Rightarrow e^4 + e^2 - 1 = 0$$

4. The equations of the normals at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ to the ellipse are

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

$$\frac{a^2x}{x_2} - \frac{b^2y}{y_2} = a^2 - b^2$$

$$\frac{a^2x}{x_3} - \frac{b^2y}{y_3} = a^2 - b^2, \text{ respectively}$$

These three lines will be concurrent, if

$$\begin{vmatrix} \frac{a^2}{x_1} & \frac{-b^2}{y_1} & a^2-b^2 \\ \frac{a^2}{x_2} & \frac{-b^2}{y_2} & a^2-b^2 \\ \frac{a^2}{x_3} & \frac{-b^2}{y_3} & a^2-b^2 \end{vmatrix} = 0$$

$$\Rightarrow -a^2b^2(a^2-b^2)\begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & 1 \\ \frac{1}{x_2} & \frac{1}{y_2} & 1 \\ \frac{1}{x_3} & \frac{1}{y_3} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} y_1 & x_1 & x_1y_1 \\ y_2 & x_2 & x_2y_2 \\ y_3 & x_3 & x_3y_3 \end{vmatrix} = 0$$

Exercise 4.6

1. The equations of the chords of contact of tangents drawn from (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \text{(i)}$$

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1 \qquad \text{(ii)}$$

It is given that (i) and (ii) are at right angles.

$$\therefore \quad \frac{-b^2}{a^2}\frac{x_1}{y_1} \times \frac{-b^2}{a^2}\frac{x_2}{y_2} = -1$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2} = -\frac{a^4}{b^4}$$

2.

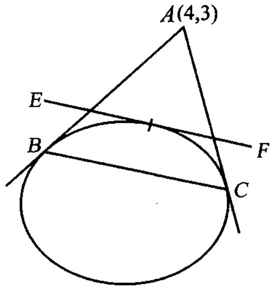


Fig. S–4.11

The equation of the chord of contact is $\frac{x}{4}+\frac{y}{3}=1$.

$\Rightarrow$ Slope of line EF is $\frac{-3}{4}$.

$\Rightarrow$ EF is $y=-\frac{3}{4}x+\sqrt{18}$.

$\Rightarrow$ Distance of (4, 3) from EF is $\frac{|24-4\sqrt{18}|}{5}$.

3. Equation of chord of ellipse which gets bisected at $P(h, k)$ is $\frac{hx}{4}+\frac{ky}{9}=\frac{h^2}{4}+\frac{k^2}{9}$. (i)

Its distance from origin (0, 0) is 2.

$$\Rightarrow \frac{\left|0+0-\left(\frac{h^2}{4}+\frac{k^2}{9}\right)\right|}{\sqrt{\frac{h^2}{16}+\frac{k^2}{81}}}=2$$

$\Rightarrow$ Locus is $4\left(\frac{x^2}{16}+\frac{y^2}{81}\right)=\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2$.

4.

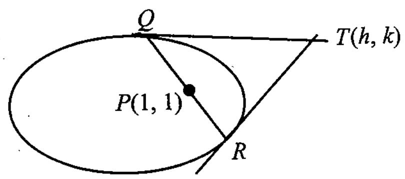


Fig. S–4.12

Equation of chord which gets bisected at point $P(1, 1)$ is

$$\frac{x^2}{16}+\frac{y^2}{9}-1=\frac{1}{16}+\frac{1}{9}-1$$

or $$\frac{x}{16}+\frac{y}{9}=\frac{25}{144}$$

or $$9x+16y=25 \quad \text{(i)}$$

Also line QR is chord of contact with respect to point $T(h, k)$.

Its equation is

$$\frac{hx}{16}+\frac{ky}{9}=1 \text{ or } 9hx+16ky=144 \quad \text{(ii)}$$

Equations (i) and (ii) represent the same straight line.

Hence, $$\frac{9h}{9}=\frac{16k}{16}=\frac{144}{25}$$

$$\Rightarrow (h, k)\equiv\left(\frac{144}{25}, \frac{144}{25}\right)$$

5.

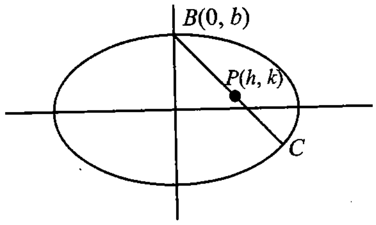


Fig. S–4.13

From the diagram, P is a midpoint of BC

$\Rightarrow$ Coordinates of C are $(2h, 2k-b)$

Now C lies on the ellipse, then

$$\frac{(2h)^2}{a^2}+\frac{(2k-b)^2}{b^2}=1$$

or $$\frac{4x^2}{a^2}+\frac{(2y-b)^2}{b^2}=1$$

which is the ellipse with centre at $(0, b/2)$.

Chapter 5

Exercise 5.1

1.

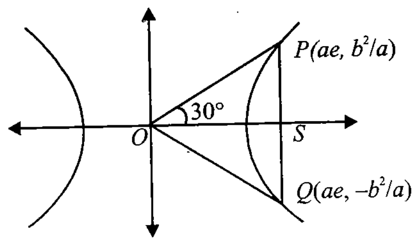


Fig. S–5.1

$$\tan 30°=\frac{b^2/a}{ae}$$

$$\Rightarrow \frac{e}{\sqrt{3}}=e^2-1$$

$$\Rightarrow \sqrt{3}e^2-e-\sqrt{3}=0$$

$$\Rightarrow e=\frac{1\pm\sqrt{13}}{2\sqrt{3}}$$

$$\Rightarrow e=\frac{1+\sqrt{13}}{2\sqrt{3}}$$

2. Distance between the two directrix is $2a/e=10$ units

$$\Rightarrow a=5e$$

Now distance between the foci

$=2ae=10e^2=10(2)=20$ (as rectangular hyperbola, $e=\sqrt{2}$)

3.

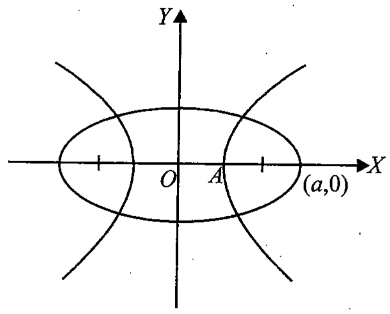


Fig. S–5.2

Let foci of ellipse are $(\pm ae_1, 0)$ and those of hyperbola are $(\pm Ae_2, 0)$.

According to question, we have

$$ae_1 = Ae_2 \qquad \text{(i)}$$

Also it is given that the conjugate axis of hyperbola is equal to the minor axis of the ellipse. Therefore,

$$a^2(1 - e_1^2) = A^2(e_2^2 - 1) \qquad \text{(ii)}$$

From (i) and (ii), we have

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 2$$

4. Let equation of the lines be

$$y = m_1(x - a)$$

and

$$y = m_2(x + a)$$

$$\therefore \quad m_1 m_2 = p$$

$$\therefore \quad y^2 = m_1 m_2(x^2 - a^2) = p(x^2 - a^2)$$

Hence locus of the points of intersection is

$$y^2 = p(x^2 - a^2)$$

or

$$px^2 - y^2 = pa^2$$

which is a hyperbola.

5. Squaring and subtracting the given equations, we get

$$x^2 - y^2 = a^2$$

which is rectangular hyperbola.

6. Let $P \equiv (a \sec\theta, b \tan\theta)$

Then $\quad N \equiv (a \sec\theta, 0)$

Since Q divides AP in the ratio $a^2 : b^2$

Therefore, coordinates of Q are $\dfrac{ab^2 + a^2 \sec\theta}{a^2 + b^2}, \dfrac{a^2\, b \tan\theta}{a^2 + b^2}$.

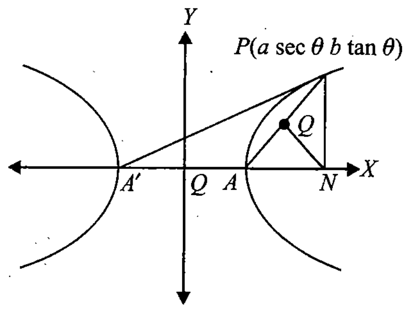


Fig. S–5.3

Slope of $\quad A'P = \dfrac{b \tan\theta}{a(\sec\theta + 1)}$

Slope of $\quad QN = \dfrac{a^2\, b \tan\theta}{ab^2 + a^3 \sec\theta - ab^2 \sec\theta}$

$$= \frac{a^2 b \tan\theta}{ab^2(1 - \sec\theta)}$$

$\therefore$ Slope of $A'P \times$ slope of $QN = \dfrac{ab^2 b^2 \tan^2\theta}{-a^2 b^2 \tan^2\theta} = -1.$

$\therefore QN \perp A'P$.

7. Taking AOB and COD as x- and y-axes and their point of intersection O as origin, clearly O is midpoint of AB and CD. Let A be $(a, 0)$ and C be $(0, c)$. Then B is $(-a, 0)$ and D is $(0, -c)$.

Let $\quad P = (x, y)$.

Given that $\quad PA \cdot PB = PC \cdot PD$

$$\Rightarrow \sqrt{(x-a)^2 + y^2}\,\sqrt{(x+a)^2 + y^2} = \sqrt{x^2 + (y-c)^2}\,\sqrt{x^2 + (y+c)^2}$$

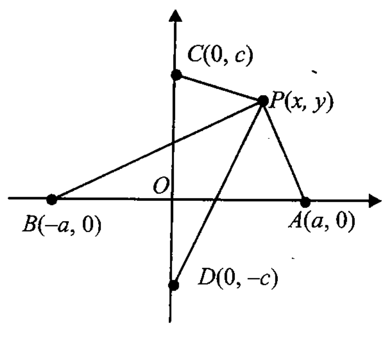


Fig. S–5.4

After simplification, we have

$$x^2 - y^2 = (a^2 - c^2)/2.$$

which is a rectangular hyperbola whose eccentricity is $\sqrt{2}$.

8. Let any point P on the hyperbola

$$x^2 - y^2 = a^2 \text{ be } (x_1, y_1).$$

Now $SP = ex_1 - a$ and $S'P = ex_1 + a$

$$\Rightarrow \quad SP \times S'P = e^2 x_1^2 - a^2$$

$= 2x_1^2 - a^2$
$= 2x_1^2 - (x_1^2 - y_1^2)$
[$\because$ point (x_1, y_1) lies on the hyperbola]
$= x_1^2 + y_1^2$
$= CP^2$

9. The centre of the hyperbola is the midpoint of the line joining the two foci.
So the coordinate of the centre are $\left(\frac{8+0}{2}, \frac{3+3}{2}\right)$, i.e. (4, 3).
Let $2a$ and $2b$ be the lengths of transverse and conjugate axes and let e be the eccentricity.
Then the equation of the hyperbola is

$$\frac{(x-4)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1 \quad \text{(i)}$$

Now, distance between the foci = $2ae$

$\Rightarrow \sqrt{(8-0)^2 + (3-3)^2} = 2ae \Rightarrow ae = 4$

$\Rightarrow \quad a = 3 \left(\because e = \frac{4}{3}\right)$

Now, $\quad b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad b^2 = 9\left(-1 + \frac{16}{9}\right) = 7$

Thus, the equation of the hyperbola is

$$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1$$

Exercise 5.2

1. Equation of tangent is
$\sec\theta x - a\tan\theta y - ab = 0$
$\therefore \quad CN = \dfrac{ab}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$
Equation of normal P is

$ax\cos\theta + by\cot\theta = a^2 + b^2$

$\therefore \quad CM = \dfrac{a^2 + b^2}{\sqrt{a^2\cos^2\theta + b^2\cot^2\theta}}$

$A = \frac{1}{2}CM \times CN = \dfrac{ab(a^2+b^2)}{\sqrt{2a^2b^2 + b^4\operatorname{cosec}^2\theta + a^4\sin^2\theta}}$

A is maximum when $b^4\operatorname{cosec}^2\theta + a^4\sin^2\theta$ is minimum

Now, $b^4\operatorname{cosec}^2\theta + a^4\sin^2\theta \ge 2a^2b^2$.

$A_{max} = \dfrac{ab(a^2+b^2)}{2ab} = \dfrac{a^2+b^2}{2}$, where $\theta = \sin^{-1}\frac{b}{a}$.

2. Given equation of hyperbola is

$$9x^2 - 16y^2 = 144$$

or

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Equation of tangent to hyperbola having slope m is

$$y = mx \pm \sqrt{16m^2 - 9}$$

If it touches the circle, then distance of this line from centre of the circle is radius of the circle. Hence,

$$\frac{\sqrt{16m^2-9}}{\sqrt{m^2+1}} = 3$$

$\Rightarrow \quad 9m^2 + 9 = 16m^2 - 9$

$\Rightarrow \quad 7m^2 = 18$

$\Rightarrow \quad m = \pm 3\sqrt{\frac{2}{7}}$

So, the equation of tangents is

$$y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$$

3. Let m be the slope of the tangent to $4x^2 - 9y^2 = 1$.
Then, m = (slope of the line $4y = 5x + 7$) = 5/4.
We have,

$$\frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $\quad a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$

The equations of the tangents are

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

or

$$y = \frac{5}{4}x \pm \sqrt{\frac{25}{64} - \frac{1}{9}}$$

$\Rightarrow \quad 24y - 30x = \pm\sqrt{161}$

4. The line $y = \alpha x + \beta$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ if } \beta^2 = a^2\alpha^2 - b^2$$

Hence, the locus of (α, β) is

$$y^2 = a^2x^2 - b^2$$

$\Rightarrow \quad a^2x^2 - y^2 = b^2$

$\Rightarrow \quad \dfrac{x^2}{b^2/a^2} - \dfrac{y^2}{b^2} = 1$

which is a hyperbola.

5. Equation of chord of contact of parabola w.r.t. $P(x_1, y_1)$ is given by

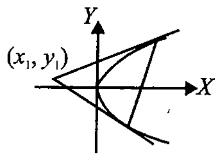


Fig. S–5.5

$$yy_1 = 2a(x + x_1) \quad \text{(i)}$$

(i) touches the curve

$$x^2 - y^2 = c^2 \quad \text{(ii)}$$

Using the condition of tangency on

$$y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

We get, $\left[\because \text{(i) is tangent to } \frac{x^2}{c^2} - \frac{y^2}{c^2} = 1\right]$

$$\frac{4a^2x_1^2}{y_1^2} = c^2\frac{4a^2}{y_1^2} - c^2$$

$$\Rightarrow \quad 4a^2x_1^2 = 4a^2c^2 - c^2y_1^2$$

$$\Rightarrow \quad 4a^2x_1^2 + c^2y_1^2 = 4a^2c^2$$

Hence locus is

$$\frac{x^2}{c^2} + \frac{y^2}{(2a)^2} = 1 \text{ which is an ellipse.}$$

6. Let $P \equiv (a \sec \theta, b \tan \theta)$

Since Q divides AP in the ratio $a^2 : b^2$

Then $N \in (a \sec \theta, 0)$

Therefore, coordinates of Q are $\left(\frac{ab^2 + a^2 \sec \theta}{a^2 + b^2}, \frac{a^2 b \tan \theta}{a^2 + b^2}\right)$.

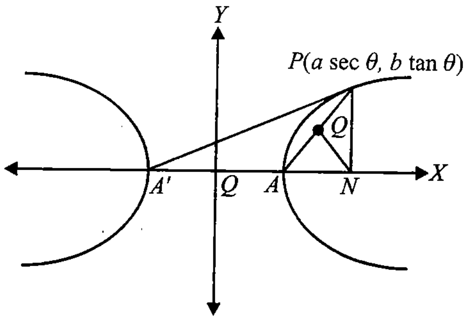


Fig. S–5.6

$$\text{Slope of } A'P = \frac{b \tan \theta}{a(\sec \theta + 1)}$$

$$\text{Slope of } QN = \frac{a^2 b \tan \theta}{ab^2 + a^3 \sec \theta - a^3 \sec \theta - ab^2 \sec \theta}$$

$$= \frac{a^2 b \tan \theta}{ab^2(1 - \sec \theta)}$$

$$\therefore \text{ Slope of } A'P \times \text{slope of } QN = \frac{a^2b^2 \tan^2 \theta}{-a^2b^2 \tan^2 \theta} = -1$$

$\therefore QN \perp A'P$

7. Tangent at $P(a \sec \theta, b \tan \theta)$ is $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$

It meets $bx - ay = 0$ at Q. The point Q is given by

$$\left(\frac{a}{\sec \theta - \tan \theta}, \frac{b}{\sec \theta - \tan \theta}\right)$$

It meets $bx + ay = 0$ at R. The point R is given by

$$\left(\frac{a}{\sec \theta + \tan \theta}, \frac{-b}{\sec \theta + \tan \theta}\right)$$

$$\therefore \quad CQ \cdot CR = \frac{\sqrt{a^2 + b^2}}{(\sec \theta - \tan \theta)} \frac{\sqrt{a^2 + b^2}}{(\sec \theta + \tan \theta)}$$

$$= a^2 + b^2$$

Exercise 5.3

1. Let the perpendicular line cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point $P(x_1, y_1)$ and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at point $Q(x_1, y_2)$.

Normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point P is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \quad \text{(i)}$$

Normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at Q is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_2} = a^2 + b^2 \quad \text{(ii)}$$

In Eqs. (i) and (ii), putting $y = 0$ we get

$$x = \frac{a^2 + b^2}{a^2} x_1$$

Hence both normals meet on x-axis.

2. The equation of normal at the point $Q(a \sec \phi, b \tan \phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

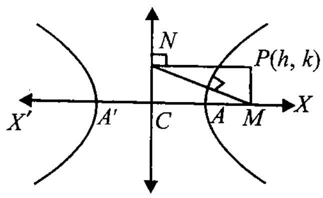


Fig. S–5.7

$$ax \cos \phi + by \cot \phi = a^2 + b^2 \quad \text{(i)}$$

The normal (i) meets the x-axis at

$M\left(\frac{a^2 + b^2}{a} \sec \phi, 0\right)$ and y-axis at $N\left(0, \frac{a^2 + b^2}{b} \tan \phi\right)$

Let point P be (h, k).

From the diagram,

$$h = \frac{a^2 + b^2}{a} \sec \phi$$

and

$$k = \frac{a^2 + b^2}{b} \tan \phi$$

Eliminating ϕ by using the relation $\sec^2 \phi - \tan^2 \phi = 1$, we have

$$\left(\frac{ah}{a^2 + b^2}\right)^2 - \left(\frac{bk}{a^2 + b^2}\right)^2 = 1$$

$\Rightarrow a^2x^2 - b^2y^2 = (a^2 + b^2)^2$ is the required locus of P.

3. Equation of the given hyperbola can be written as

$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$

Therefore, equation of the chord of this hyperbola in terms of the middle point (5, 3) is

$$\frac{5x}{16} - \frac{3y}{25} - 1 = \frac{5^2}{16} - \frac{9}{25} - 1$$

$\Rightarrow \quad 125x - 48y = 481$

4.

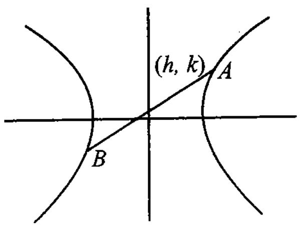


Fig. S–5.8

Equation of chord AB with $T = S_1$ is given by

$$\frac{hx}{3} - \frac{ky}{2} = \frac{h^2}{3} - \frac{k^2}{2}$$

Given that it has slope tan 45° = 1. Hence,

$$\frac{h}{3}\frac{2}{k} = 1$$

$\Rightarrow \quad 2x = 3y$, (as which is the required locus)

5. Normal at a point $(a \sec \theta,\ a \tan \theta)$ is

$$x \cos \theta + y \cot \theta = 2a \quad \text{(i)}$$

If $P(x_1, y_1)$ be the point of intersection of the tangents at the ends of normal chord (i), then (i) must be the chord of contact of $P(h, k)$ whose equation is given by

$$hx - ky = a^2 \quad \text{(ii)}$$

Comparing (i) and (ii) and eliminating θ, we get

$$\frac{a^2}{4h^2} - \frac{a^2}{4k^2} = 1$$

Hence the locus is

$$\frac{1}{x^2} - \frac{1}{y^2} = \frac{4}{a^2}$$

6. **b.** Equation of chord joining α and β is

$$\frac{x}{a} \cos\left(\frac{\alpha - \beta}{2}\right) - \frac{y}{b} \sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\alpha + \beta}{2}\right)$$

$\therefore \quad \alpha + \beta = 3\pi$

$$\frac{x}{a} \cos\left(\frac{\alpha - \beta}{2}\right) + \frac{y}{b} = 0$$

It passes through the centre (0, 0).

Exercise 5.4

1. Angle between asymptotes is given by

$$\tan^{-1}\left(\frac{2ab}{a^2 - b^2}\right) = \tan^{-1}\left(\frac{2(4)(3)}{16 - 9}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right)$$

$$= \pi - 2 \tan^{-1}\left(\frac{3}{4}\right)$$

2.

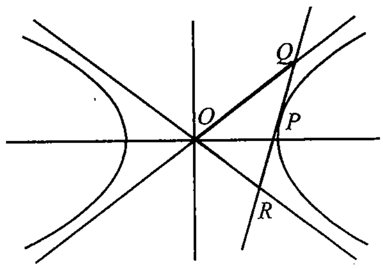


Fig. S–5.9

Consider point $P(a \sec \alpha, b \tan \alpha)$ on the hyperbola.

Tangent at P is

$$\frac{x}{a} \sec \alpha - \frac{y}{b} \tan \alpha = 1 \quad \text{(i)}$$

Asymptotes are

$$y = (b/a)x \quad \text{(ii)}$$

and

$$y = -(b/a)x \quad \text{(iii)}$$

Solving (i) and (ii), we have

$$Q\left(\frac{a}{\sec \alpha - \tan \alpha}, \frac{b}{\sec \alpha - \tan \alpha}\right)$$

Solving (i) and (iii), we get

$$R\left(\frac{a}{\sec \alpha + \tan \alpha}, \frac{-b}{\sec \alpha + \tan \alpha}\right)$$

Then area of $\Delta OQR = \frac{1}{2}$

$$\begin{vmatrix} 0 & 0 & 1 \\ \frac{a}{\sec\alpha - \tan\alpha} & \frac{b}{\sec\alpha - \tan\alpha} & 1 \\ \frac{a}{\sec\alpha + \tan\alpha} & \frac{-b}{\sec\alpha + \tan\alpha} & 1 \end{vmatrix} = ab$$

3. Since equations of a hyperbola and its asymptotes differ in constant terms only, therefore the pair of asymptotes is given by

$$xy - 3y - 2x + \lambda = 0 \quad \text{(i)}$$

where λ is any constant such that it represents two straight lines. Hence,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 0 + 2 \times \left(-\frac{3}{2}\right) \times (-1) \times \left(\frac{1}{2}\right) - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow \lambda = 6$$

From (i), the asymptotes of the given hyperbola are given by

$$xy - 3y - 2x + 6 = 0$$

or

$$(y - 2)(x - 3) = 0$$

Therefore the asymptotes are $x - 3 = 0$ and $y - 2 = 0$.

4. Equation of the asymptotes of the given hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

$$\Rightarrow \quad b^2x^2 - a^2y^2 = 0$$

If θ is an angle between the asymptotes, then

$$\tan\theta = \pm\frac{\sqrt{a^2b^2}}{b^2 - a^2} = \pm\frac{ab}{a^2 - b^2}$$

$$\therefore \quad \tan\theta = \frac{ab}{a^2 - b^2}$$

$$\Rightarrow \quad \cos(\theta/2) = \sqrt{\frac{a^2}{a^2 + b^2}} = \frac{1}{e}$$