CHAPTER

2

Circle

- Definition
- Different Forms of the Equations of a Circle
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DEFINITION

A circle is the locus of a point which moves in a plane such that its distance from a fixed point in plane is always a constant. The fixed point is called the centre and the constant distance is called the radius of the circle.

Equation of Circle with Centre (*h*, *k*) and Radius *r*

The equation of circle is

$$(x-h)^2+(y-k)^2=r^2 \quad \text{(i)}$$

In particular: If the centre is at the origin, the equation of circle is $x^2+y^2=r^2$.

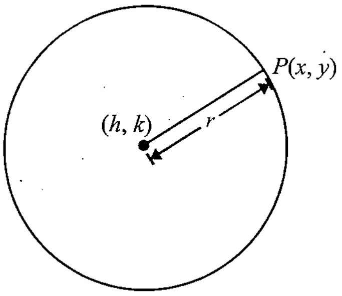


Fig. 2.1

General Equation of a Circle

The general equation of circle is

$$x^2+y^2+2gx+2fy+c=0, \quad \text{(ii)}$$

where g, f and c are constants.

To find the centre and radius: Eq. (ii) can be written as

$$(x+g)^2+(y+f)^2=(\sqrt{g^2+f^2-c})^2$$

Comparing with the equation of circle given in Eq. (i),

$\therefore$ $h=-g, k=-f$

and $r=\sqrt{g^2+f^2-c}$

$\therefore$ Co-ordinates of the centre are $(-g, -f)$ and radius $=\sqrt{g^2+f^2-c}$, $(g^2+f^2 \geq c)$

Note:

1. *A general equation of second degree* $ax^2+2hxy+by^2+2gx+2fy+c=0$ *in x, y represent a circle if*
   - **i.** *coefficient of* x^2 = *coefficient of* y^2, *i.e.* $a=b$.
   - **ii.** *coefficient of xy is zero, i.e.* $h=0$.
   - **iii.** $\Delta=abc+2hgf-af^2-bg^2-ch^2\neq 0$

   However for a point circle (whose radius is zero), $\Delta=0$
2. *Rule to find the centre and radius of a circle whose equation is given*:
   - **i.** *Make the coefficients of* x^2 *and* y^2 *equal to 1 and right hand side equal to zero.*
   - **ii.** *Then co-ordinates of centre will be* (α, β), *where* $\alpha=-\frac{1}{2}$ *(coefficient of x) and* $\beta=-\frac{1}{2}$ *(coefficient of y).*
   - **iii.** *Radius* $=\sqrt{(\alpha^2+\beta^2-\text{constant term})}$
3. *Nature of the circle: radius of the circle* $x^2+y^2+2gx+2fy+c=0$ *is* $\sqrt{(g^2+f^2-c)}$.

   Now the following cases are possible:
   - **i.** *If* $g^2+f^2-c>0$, *then the radius of circle will be real. Hence in this case, real circle is possible.*
   - **ii.** *If* $g^2+f^2-c=0$, *then the radius of circle will be zero. Hence, in this case circle is called a point circle.*
   - **iii.** *If* $g^2+f^2-c<0$, *then the radius of circle will be imaginary number. Hence, in this case, circle is called a virtual circle or imaginary circle.*
4. *Concentric circles: Two circles having the same centre C (h, k) but different radii* r_1 *and* r_2, *respectively, are called concentric circles. Thus, the circles* $(x-h)^2+(y-k)^2=r_1^2$ *and* $(x-h)^2+(y-k)^2=r_2^2$, $r_1\neq r_2$, *are concentric circles. Therefore, the equations of concentric circles differ only in constant terms.*

Example 2.1 **If the lines $3x-4y+4=0$ and $6x-8y-7=0$ are tangents to a circle, then find the radius of the circle.**

Sol. The diameter of circle is perpendicular distance between the parallel lines (tangents) $3x-4y+4=0$ and $3x-4y-\frac{7}{2}=0$ and so it is equal to $\frac{|4-(-7/2)|}{\sqrt{9+16}}=\frac{3}{2}$. Hence, radius is $\frac{3}{4}$.

Example 2.2 **If the equation $px^2+(2-q)xy+3y^2-6qx+30y+6q=0$ represents a circle, then find the values of p and q.**

Sol. In the equation of circle, xy term does not occur and the coefficient of x^2 and y^2 are equal.

Therefore, $2-q=0 \Rightarrow q=2$ and $p=3$.

Also for this value of p and q circle is real.

Example 2.3 **A point P moves in such a way that the ratio of its distance from two coplanar points is always a fixed number ($\neq 1$). Then, identify the locus of the point.**

Sol. Let two coplanar points are $(0, 0)$ and $(a, 0)$. Under given conditions, we get

$$\frac{\sqrt{x^2+y^2}}{\sqrt{(x-a)^2+y^2}}=\lambda, (\lambda\neq 1)$$

(where λ is any fixed number)

$$\Rightarrow x^2+y^2+\left(\frac{\lambda^2}{\lambda^2-1}\right)(a^2-2ax)=0$$

which is the equation of a circle.

Example 2.4 **Prove that the maximum number of points with rational coordinates on a circle whose centre is $(\sqrt{3}, 0)$ is two.**

Sol. There cannot be three points on the circle with rational coordinates as for then the centre of the circle, being the circumcentre of a triangle whose vertices have rational coordinates, must have rational coordinates ($\because$ the coordinates will be obtained by solving two linear equations in x, y having rational coefficients). But the point $(\sqrt{3}, 0)$ does not have rational coordinates. Also, the equation of the circle is

$$(x-\sqrt{3})^2 + y^2 = r^2 \Rightarrow x = \sqrt{3} \pm \sqrt{r^2 - y^2}$$

For suitable r, x, where x is rational, y may have two rational values.

For example, $r = 2$, $x = 0$, $y = 1, -1$, satisfy $x = \sqrt{3} \pm \sqrt{r^2 - y^2}$.

So, we get two points (0, 1), (0, –1) which have rational coordinates.

Example 2.5 **Prove that for all values of θ, the locus of the point of intersection of the lines $x \cos\theta + y \sin\theta = a$ and $x \sin\theta - y \cos\theta = b$ is circle.**

Sol. Since point of intersection satisfies both the given lines we can find locus by eliminating θ from given equation.

Therefore, by squaring and adding, we get equation $x^2 + y^2 = a^2 + b^2$ which is equation of circle.

Example 2.6 **Find the length of the chord cut-off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$.**

Sol.

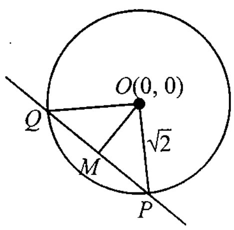


Fig. 2.2

We have, OM = length of the perpendicular from (0, 0) on $y = 2x + 1 \Rightarrow OM = \frac{1}{\sqrt{5}}$ and, OP = radius of the given circle $= \sqrt{2}$.

$$\therefore \quad PQ = 2PM = 2\sqrt{OP^2 - OM^2}$$

$$= 2\sqrt{2 - \frac{1}{5}} = \frac{6}{\sqrt{5}}$$

Example 2.7 **Let $A \equiv (-2, -2)$ and $B \equiv (2, -2)$ be two points and AB substends an angle of 45° at any point P in the plane in such a way that area of ΔAPB is 8 square unit, then find the number of possible position(s) of P.**

Sol.

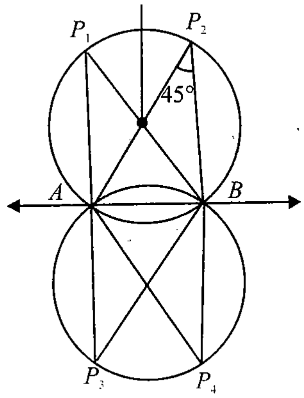


Fig. 2.3

$\because \quad AB = 4$ and area of $\Delta APB = 8$

$\therefore \quad \frac{1}{2} \times 4 \times h = 8 \Rightarrow h = 4;$

h the height of ΔAPB.

From Fig. 2.3, it is clear that P lies on circle of radius $2\sqrt{2}$ unit with AB as its chord so there are four possible positions of P.

Example 2.8 **An acute triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (– 4, 3), respectively, then find $\angle QPR$.**

Sol.

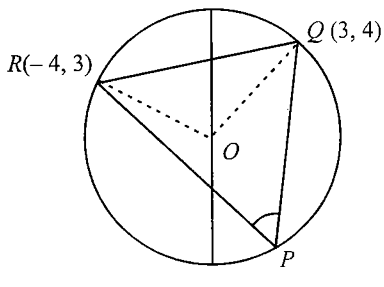


Fig. 2.4

We know that $\angle QPR = \frac{1}{2} \angle QOR$; O being the centre (0, 0) of the given circle $x^2 + y^2 = 25$.

Let $\quad m_1$ = slope of $OQ = \frac{4}{3}$

and $\quad m_2$ = slope of $OR = -\frac{3}{4}$

As $\quad m_1 m_2 = -1, \angle QOR = \frac{\pi}{2}$

$\Rightarrow \quad \angle QPR = \frac{\pi}{4}$

Example 2.9 **Two tangents to the circle $x^2 + y^2 = 4$ at the points A and B meet at $P(-4, 0)$. Then, find the area of the quadrilateral $PAOB$, where O is the origin**

Sol.

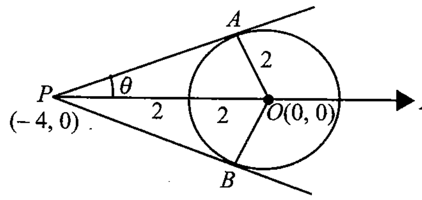


Fig. 2.5

Clearly, $\sin\theta = \frac{2}{4} = \frac{1}{2}$

$\therefore$ $\theta = 30°$

So, $\text{ar}(\Delta POA) = \frac{1}{2} \times 2 \times 4 \times \sin 60°$

$\therefore$ $\text{ar}(\text{quad. } PAOB) = 2 \times \frac{1}{2} \times 2 \times 4 \sin 60°$

$= 8 \times \frac{\sqrt{3}}{2} = 4\sqrt{3}$.

Equation of a Circle Passing Through Three Given Points

The general equation of circle, i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three independent constants g, f and c. Hence, for determining the equation of a circle, three conditions are required.

i. The equation of the circle through three non-collinear points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$:

Let the equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0.$ (i)

If three points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on the circle Eq. (i), their coordinates must satisfy its equation. Hence, solving equations

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ (ii)

$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$ (iii)

$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$ (iv)

g, f, c are obtained from (ii), (iii) and (iv).

Note:

Concyclic quadrilateral: If all the four vertices of a quadrilateral lie on a circle, then the quadrilateral is called a cyclic quadrilateral. The four vertices are said to be concyclic.

Example 2.10 **If a circle passes through the point (0, 0), (a, 0), (0, b), then find its centre.**

Sol: Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Now on passing through the points, we get three equations:

$c = 0$ (i)

$a^2 + 2ga + c = 0$ (ii)

$b^2 + 2fb + c = 0$ (iii)

On solving them, we get $g = -\frac{a}{2}, f = -\frac{b}{2}$

Hence, the centre is $\left(\frac{a}{2}, \frac{b}{2}\right)$.

Example 2.11 **Find the equation of circle which passes through the points (1, − 2), (4, − 3) and whose centre lies on the line $3x + 4y = 7$.**

Sol: Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

If Eq. (i) passes through the points (1, − 2) and (4, − 3), then

$5 + 2g - 4f + c = 0$ (ii)

and $25 + 8g - 6f + c = 0$ (iii)

Since the centre $(-g, -f)$ lies on the line $3x + 4y = 7$,

$\therefore$ $-3g - 4f = 7$ (iv)

Solving Eqs. (ii), (iii) and (iv), we get

$g = -\frac{47}{15}, f = \frac{3}{5}$ and $c = \frac{11}{3}$.

Substituting in Eq. (i), the equation of the circle is

$15x^2 + 15y^2 - 94x + 18y + 55 = 0.$

Example 2.12 **Show that a cyclic quadrilateral is formed by the lines $5x + 3y = 9$, $x = 3y$, $2x = y$ and $x + 4y + 2 = 0$ taken in order. Find the equation of the circumcircle.**

Sol. Solving the given equation in pairs taken in order, the coordinates of the vertices of the quadrilateral $ABCD$ are $A\left(\frac{3}{2}, \frac{1}{2}\right)$, $B(0, 0)$, $C\left(-\frac{2}{9}, -\frac{4}{9}\right)$ and $D\left(\frac{42}{17}, -\frac{19}{17}\right)$.

First, we shall find the equation of the circle passing through A, B and C. Let the equation of this circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

Coordinates of the points A, B and C must satisfy Eq. (i), so substituting in Eq. (i), we have

$3g + f + c + \frac{5}{2} = 0$ (ii)

$c = 0$ (iii)

and $4g + 8f - 9c = \frac{20}{9}$ (iv)

Solving Eqs. (ii), (iii) and (iv), we get $g = -\frac{10}{9}, f = \frac{5}{6}$ and $c = 0$.

Substituting in Eq. (i), the equation of the circle through the vertices A, B, C is

$9x^2 + 9y^2 - 20x + 15y = 0$ (v)

Since the coordinates of the vertex $D\left(\frac{42}{17}, -\frac{19}{17}\right)$ also satisfy Eq. (v), hence a cyclic quadrilateral $ABCD$ is formed by the given lines, and Eq. (v) is the equation of the circumcircle of the quadrilateral.

Equation of Circle on a Given Diameter

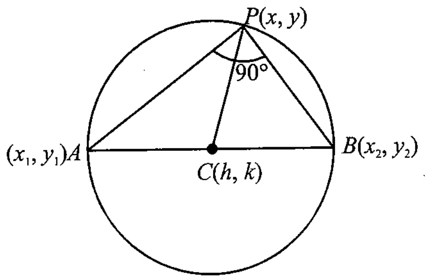


Fig. 2.6

To find the equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter. We will find the locus of point P such that $\angle APB = \pi/2$.

In the figure given above, $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter and let $P(x, y)$ be any point on the circle.

Now, slope of $AP = \frac{y-y_1}{x-x_1}$ and slope of $BP = \frac{y-y_2}{x-x_2}$

Since $\angle APB = 90°$

$\therefore$ Slope of $AP \times$ Slope of $BP = -1$.

$$\Rightarrow \frac{(y-y_1)}{(x-x_1)} \times \frac{(y-y_2)}{(x-x_2)} = -1$$

$\Rightarrow (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

which is the required equation of circle.

Example 2.13 **Find the equation of the circle which passes through (1, 0) and (0, 1) and has its radius as small as possible.**

Sol. The radius will be minimum, if the given points are the end points of a diameter.

Then, equation of circle is

$(x-1)(x-0)+(y-0)(y-1)=0$

$\Rightarrow \quad x^2+y^2-x-y=0.$

Example 2.14 **If the abscissa and ordinates of two points P and Q are the roots of the equations $x^2+2ax-b^2=0$ and $x^2+2px-q^2=0$, respectively, then find the equation of the circle with PQ as diameter.**

Sol. Let x_1, x_2 and y_1, y_2 be roots of $x^2+2ax-b^2=0$ and $x^2+2px-q^2=0$, respectively.

Then, $\quad x_1+x_2=-2a, x_1x_2=-b^2$

and $\quad y_1+y_2=-2p, y_1y_2=-q^2$

The equation of the circle with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as the end points of diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$\Rightarrow x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+x_1x_2+y_1y_2=0$$

$$\Rightarrow x^2+y^2+2ax+2py-b^2-q^2=0.$$

Example 2.15 **Tangents PA and PB are drawn to $x^2+y^2=a^2$ from the point $P(x_1, y_1)$. Then find the equation of the circumcircle of triangle PAB.**

Sol. Clearly the points O, A, P and B are concycllic, and OP is the diameter of circle.

Thus, equation of circumcircle of triangle PAB is

$$x(x-x_1)+y(y-y_1)=0$$

or $\quad x^2+y^2-xx_1-yy_1=0$

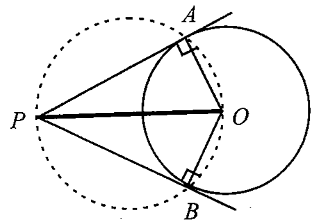


Fig. 2.7

Parametric Form of Circle

i.

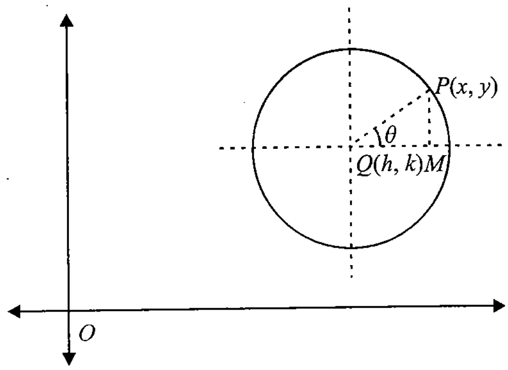


Fig. 2.8

The parametric coordinates of any point on the circle $(x-h)^2+(y-k)^2=r^2$ are given by $(h+r\cos\theta, k+r\sin\theta)$, where θ is parameter $(0 \le \theta \le 2\pi)$.

As from Fig. 2.8, $\cos\theta = \frac{QM}{PQ} = \frac{x-h}{r}$ and $\sin\theta = \frac{PM}{PQ} = \frac{y-k}{r}$

$\Rightarrow \quad x = h + r\cos\theta$ and $y = k + r\sin\theta$

In particular, coordinates of any point on the circle $x^2 + y^2 = r^2$ are $(r\cos\theta, r\sin\theta)$ $(0 \le \theta < 2\pi)$.

ii. The parametric coordinates of any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + \sqrt{g^2 + f^2 - c}\cos\theta$ and $y = -f + \sqrt{g^2 + f^2 - c}\sin\theta$ $(0 \le \theta < 2\pi)$.

Example 2.16 **Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$.**

Sol. Equation of the circle can be rewritten in the form

$$\left(x + \frac{p}{2}\right)^2 + \left(y + \frac{p}{2}\right)^2 = \frac{p^2}{2}$$

Therefore, the parametric form of the equation of the given circle is

$$x = -\frac{p}{2} + \frac{p}{\sqrt{2}}\cos\theta$$
$$= \frac{p}{2}(-1 + \sqrt{2}\cos\theta)$$

and
$$y = -\frac{p}{2} + \frac{p}{\sqrt{2}}\sin\theta$$
$$= \frac{p}{2}(-1 + \sqrt{2}\sin\theta),$$

where $0 \le \theta < 2\pi$.

Example 2.17 **Find the centre of the circle $x = -1 + 2\cos\theta$, $y = 3 + 2\sin\theta$.**

Sol. Given that $\frac{x+1}{2} = \cos\theta$ and $\frac{y-3}{2} = \sin\theta$

$$\Rightarrow \quad \left(\frac{x+1}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

$$\Rightarrow \quad (x+1)^2 + (y-3)^2 = 4$$

whose centre is $(-1, 3)$.

Intercepts Made on the Axes by a Circle

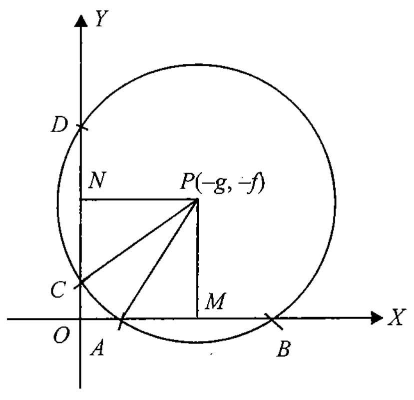


Fig. 2.9

From the diagram $PM = |f|$ and $PN = |g|$

Also $AP = CP$, radius $= \sqrt{g^2 + f^2 - c}$

$$\therefore \quad AB = 2AM = 2\sqrt{AP^2 - PM^2}$$
$$= 2\sqrt{(g^2 + f^2 - c) - f^2} = 2\sqrt{g^2 - c}$$

Similarly, $CD = 2\sqrt{f^2 - c}$

Note:

- *Intercepts are always positive.*
- *If circle touches x-axis then $|AB| = 0$ $\therefore$ $c = g^2$ and if circle touches y-axis then $|CD| = 0$ $\therefore$ $c = f^2$.*
- *If circle touches both axes, then $|AB| = 0 = |CD|$ $\therefore$ $c = g^2 = f^2$.*

Example 2.18 **Find the length of intercept, the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the x-axis.**

Sol. Comparing the given equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get $g = 5$, $f = -3$ and $c = 9$.

$\therefore$ Length of intercept on x-axis $= 2\sqrt{g^2 - c}$

$$= 2\sqrt{(5)^2 - 9} = 8$$

Example 2.19 **If the intercepts of the variable circle on the x-axis and y-axis are 2 units 4 units respectively, then find the locus of the centre of the variable circle.**

Sol.

Given that $2\sqrt{g^2 - c} = 2$ and $2\sqrt{f^2 - c} = 4$

$\Rightarrow$ $g^2 - c = 1$ and $f^2 - c = 4$

$\Rightarrow$ $f^2 - g^2 = 3$

Hence, locus is $y^2 - x^2 = 3$.

DIFFERENT FORMS OF THE EQUATIONS OF A CIRCLE

When the circle passes through the origin (0, 0) and has intercepts α and β on the x-axis and y-axis, respectively:

Clearly, A and B are end points of diameter.
Hence, equation of circle is

$$(x - \alpha)(x - 0) + (y - 0)(y - \beta) = 0$$

or
$$x^2 + y^2 - \alpha x - \beta y = 0$$

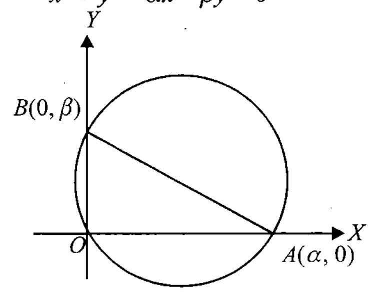


Fig. 2.10

When the circle touches x-axis:

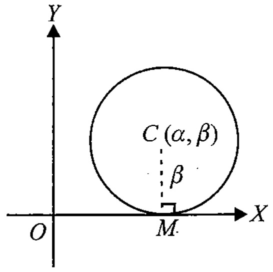


Fig. 2.11

$$(x-\alpha)^2+(y-\beta)^2=\beta^2$$

When the circle touches y-axis:

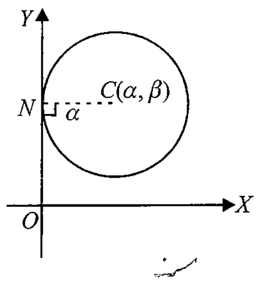


Fig. 2.12

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2$$

When the circle touches both axes:

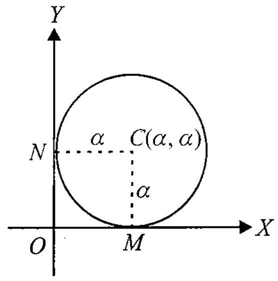


Fig. 2.13

$$(x-\alpha)^2+(y-\alpha)^2=\alpha^2$$

When the circle touches x-axis at $(\alpha, 0)$ and cuts off intercept on y-axis of length $2l$:

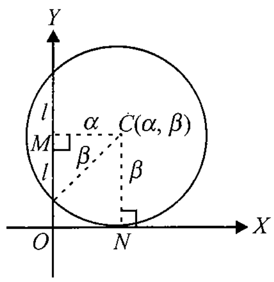


Fig. 2.14

From the figure, $\beta=\sqrt{\alpha^2+l^2}$

Hence, equation of circle is $(x-\alpha)^2+(y-\beta)^2=\beta^2$.

When the circle touches y-axis at $(0, \beta)$ and cuts off intercept on x-axis of length $2k$:

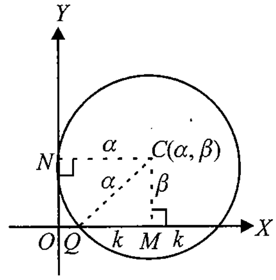


Fig. 2.15

From the figure, $\alpha=\sqrt{\beta^2+k^2}$

Hence, equation of circle is $(x-\alpha)^2+(y-\beta)^2=\alpha^2$.

Example 2.20 **Find the equation of the circle which touches both the axes and the straight line $4x + 3y = 6$ in the first quadrant and lies below it.**

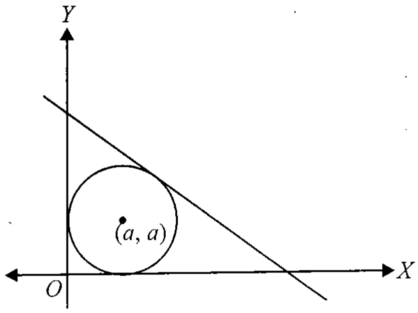


Fig. 2.16

Sol. Since the circle touches both the axes and the straight line $4x + 3y = 6$ in first quadrant, therefore coordinates of its centre are (a, a) and radius $= a$, where $a > 0$. Since $4x + 3y - 6 = 0$ touches the circle.

$$\therefore \qquad \frac{7a-6}{\sqrt{16+9}}=\pm a$$

$$\Rightarrow \qquad 7a-6=\pm 5a \Rightarrow a=3, \frac{1}{2}$$

Since $(0, 0)$ and $(1/2, 1/2)$ lie on the same side of the line $4x + 3y = 6$, where $(0, 0)$ and $(3, 3)$ lie on the opposite side of the line.

Therefore, for the required circle, $a=\frac{1}{2}\cdot$ Hence, equation of the required circle is

$$(x-1/2)^2+(y-1/2)^2=(1/2)^2$$

$$\text{or} \qquad 4x^2+4y^2-4x-4y+1=0$$

Example 2.21 **A circle touches the y-axis at the point (0, 4) and cuts the x-axis in a chord of length 6 units. Then find the radius of the circle.**

Sol. O' is centre and from the figure given below,

$$\text{Radius } (r) = \sqrt{(4)^2 + (3)^2} = 5.$$

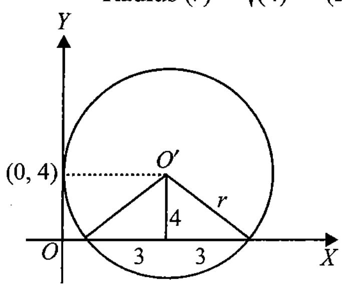


Fig. 2.17

Example 2.22 **Find the equation of the circle which is touched by $y = x$, has its centre on the positive direction of the x-axis and cuts off a chord of length 2 units along the line $\sqrt{3}y - x = 0$.**

Sol. Since the required circle has its centre on x-axis. So, let the coordinates of the centre be $(a, 0)$. The circle touches $y = x$.

Therefore, radius = length of the $\perp$ from $(a, 0)$ on $x - y = 0$ is $a/\sqrt{2}$.

The circle cuts off a chord of length 2 units along $x - \sqrt{3}y = 0$.

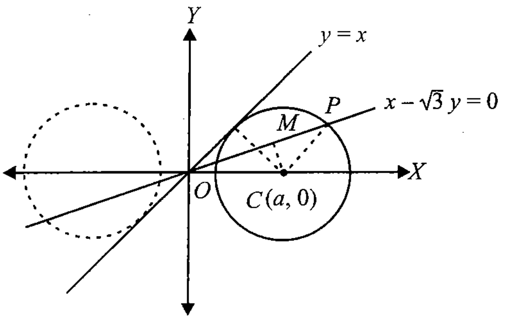


Fig. 2.18

From diagram,

$$CP^2 = CM^2 + MP^2$$

$$\left(\frac{a}{\sqrt{2}}\right)^2 = \left(\frac{a - \sqrt{3} \times 0}{\sqrt{1^2 + (\sqrt{3})^2}}\right)^2 + 1^2$$

$$\Rightarrow \qquad \frac{a^2}{2} = 1 + \frac{a^2}{4} \Rightarrow a = 2$$

Thus, centre of the circle is at (2, 0) and radius $= \frac{a}{\sqrt{2}} = \sqrt{2}$

So, its equation is $x^2 + y^2 - 4x + 2 = 0$.

Example 2.23 **Find the equations of the circles passing through the point $(-4, 3)$ and touching the lines $x + y = 2$ and $x - y = 2$.**

Sol.

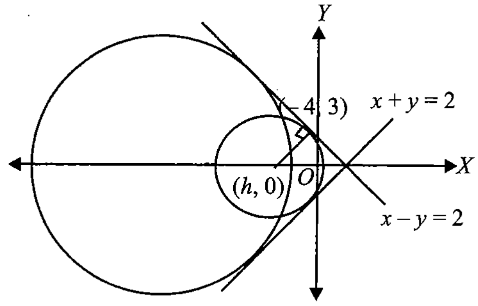


Fig. 2.19

Since the circle touches both the lines $x + y = 2$ and $x - y = 2$, its centre must lie on the x-axis. Let the centre of the circle be $(h, 0)$.

Now radius of the circle = perpendicular distance of point $(h, 0)$ from the line $x + y - 2 = 0$

$$= \frac{|h + 0 - 2|}{\sqrt{2}} = \frac{|h - 2|}{\sqrt{2}}$$

Then, equation of the circle is $(x - h)^2 + (y - 0)^2 = \frac{(h - 2)^2}{2}$

Since the circle passes through the point $(-4, 3)$,

$$(-4 - h)^2 + (3 - 0)^2 = \frac{(h - 2)^2}{2}$$

$$\Rightarrow \qquad h^2 + 20h + 46 = 0$$

$$\Rightarrow \qquad h = \frac{-20 \pm \sqrt{400 - 184}}{2}$$

$$= -10 \pm 3\sqrt{6}$$

POSITION OF A POINT WITH RESPECT TO A CIRCLE

Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

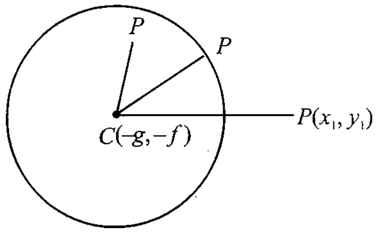


Fig. 2.20

Point P lies outside, on or inside the circle accordingly as CP >, =, < radius

or $\sqrt{(x_1 + g)^2 + (y_1 + f)^2}$ >, =, < $\sqrt{g^2 + f^2 - c}$

or $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ >, =, < 0

Maximum and Minimum Distance of a Point from the Circle

Let any point $P(x_1, y_1)$ and circle $x^2 + y^2 + 2gx + 2fy + c = 0$

The centre and radius of the circle are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$, respectively.

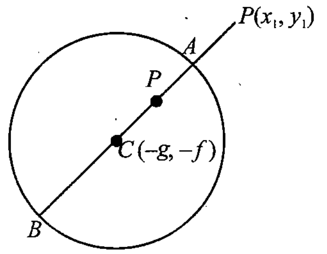


Fig. 2.21

The maximum and minimum distance from $P(x_1, y_1)$ to the circle are

$$PB = CB + PC = r + PC$$

and $$PA = |CP - CA| = |PC - r|$$ (*P* inside or outside)

where $$r = \sqrt{g^2 + f^2 - c}$$

Note: *If* $PC < r$ *then P inside,* $PC > r$ *then P outside.*

Example 2.24 **Find the greatest distance of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$.**

Sol. Since $S_1 = 10^2 + 7^2 - 4 \times 10 - 2 \times 7 - 20 > 0$. So, *P* lies outside the circle. Join *P* with the centre *C*(2, 1) of the given circle. Suppose *PC* cuts the circle at *A* and *B* (where *A* is nearer to *C*). Then, *PB* is the greatest distance of *P* from the circle.

We have: $$PC = \sqrt{(10-2)^2 + (7-1)^2} = 10$$

and $$CB = \text{radius} = \sqrt{4 + 1 + 20} = 5$$

$$\therefore \quad PB = PC + CB = 10 + 5 = 15$$

Example 2.25 **Find the values of α for which the point $(\alpha - 1, \alpha + 1)$ lies in the larger segment of the circles $x^2 + y^2 - x - y - 6 = 0$ made by the chord whose equation is $x + y - 2 = 0$.**

Sol.

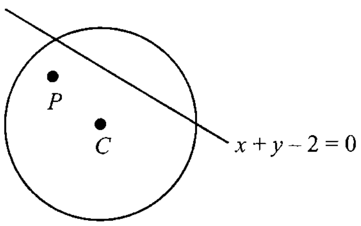


Fig. 2.22

The given circle $S(x, y) \equiv x^2 + y^2 - x - y - 6 = 0$ (i)

has centre at $$C \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$$

According to the required conditions, the given point $P(\alpha - 1, \alpha + 1)$ must lie inside the given circle i.e.

$$S(\alpha - 1, \alpha + 1) < 0$$

$$\Rightarrow (\alpha - 1)^2 + (\alpha + 1)^2 - (\alpha - 1) - (\alpha + 1) - 6 < 0$$

$$\Rightarrow \quad \alpha^2 - \alpha - 2 < 0,$$

i.e. $$(\alpha - 2)(\alpha + 1) < 0$$

$$\Rightarrow \quad -1 < \alpha < 2 \quad \text{(ii)}$$

and also *P* and *C* must lie on the same side of the line (see Fig. 2.22)

$$L(x, y) \equiv x + y - 2 = 0 \quad \text{(iii)}$$

i.e. $L\left(\frac{1}{2}, \frac{1}{2}\right)$ and $L(\alpha - 1, \alpha + 1)$ must have the same sign.

Now, since $$L\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} - 2 < 0$$

Therefore, we have $L(\alpha - 1, \alpha + 1) = (\alpha - 1) + (\alpha + 1) - 2 < 0$,

i.e. $$\alpha < 1. \quad \text{(iv)}$$

Inequalities (ii) and (iv) together give the possible values of α as $-1 < \alpha < 1$.

Example 2.26 **Find the number of points (x, y) having integral coordinates satisfying the condition $x^2 + y^2 < 25$.**

Sol. Since $x^2 + y^2 < 25$ and *x* and *y* are integers, the possible values of *x* and *y* are $(0, \pm 1, \pm 2, \pm 3, \pm 4)$.

Thus, *x* and *y* can be chosen in nine ways each and (x, y) can be chosen in $9 \times 9 = 81$ ways.

However, we have to exclude cases $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$ and $(\pm 4, \pm 4)$, i.e., $3 \times 4 = 12$ cases (as these points lies either on the circle or outside circle).

Hence, the number of points are $81 - 12 = 69$.

Example 2.27 **The circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect the coordinate axes, and the point (1, 4) is inside the circle. Find the range of the values of k.**

Sol.

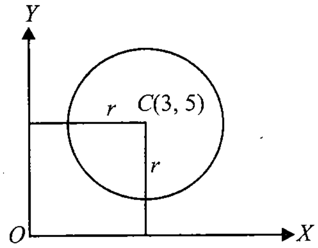


Fig. 2.23

The equation of the circle is

$$x^2 + y^2 - 6x - 10y + k = 0 \quad \text{(i)}$$

whose centre is $C(3, 5)$ and radius $r = \sqrt{34 - k}$

If the circle does not touch or intersect the x-axis, then radius $r < y$-coordinate of centre C,

or $\sqrt{34 - k} < 5$

$\Rightarrow$ $34 - k < 25$

$\Rightarrow$ $k > 9$ (ii)

Also if the circle does not touch or intersect the y-axis, then the radius $r < x$-coordinate of centre C

or $\sqrt{34 - k} < 3$

$\Rightarrow$ $34 - k < 9$

$\Rightarrow$ $k > 25$ (iii)

If the point (1, 4) is inside the circle, then its distance from centre $C < r$ (radius),

or $\sqrt{[(3-1)^2 + (5-4)^2]} < \sqrt{34 - k}$

$\Rightarrow$ $5 < 34 - k$

$\Rightarrow$ $k < 29$ (iv)

Now all the conditions (ii), (iii) and (iv) are satisfied if $25 < k < 29$ which is the required range of the values of k.

Concept Application Exercise 2.1

1. If the line $x + 2by + 7 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$, then find the value of b.
2. Prove that the locus of a point that moves such that the sum of the square of its distances from the three vertices of a triangle is constant is a circle.
3. Find the number of integral values of λ for which $x^2 + y^2 + \lambda x + (1 - \lambda)y + 5 = 0$ is the equation of a circle whose radius does not exceed 5.
4. If a circle whose centre is $(1, -3)$ touches the line $3x - 4y - 5 = 0$, then find its radius.
5. If one end of a diameter of the circle $2x^2 + 2y^2 - 4x - 8y + 2 = 0$ is (3, 2) the find the other end of the diameter.
6. Prove that the locus of centroid of the triangle whose vertices are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$ and (1, 0), where 't' is a parameter is circle.
7. Find the equation of the circle which passes through the points $(3, -2)$ and $(-2, 0)$ and centre lies on the line $2x - y = 3$.
8. Find the radius of the circle $(x - 5)(x - 1) + (y - 7)(y - 4) = 0$.
9. Find the equations of the circles which pass through the origin and cut off chords of length a from each of the lines $y = x$ and $y = -x$.
10. Find the equation of the circle which touches x-axis and whose centre is (1, 2).
11. Find the equation of circle which touches both the axes and the line $x = c$.
12. Find the equation of the circle with centre at $(3, -1)$ and which cuts off an intercept of length 6 from the line $2x - 5y + 18 = 0$.
13. Find the locus of the centre of the circle which cuts off intercepts of length $2a$ and $2b$ from x-axis and y-axis, respectively.
14. Circles are drawn through the point (2, 0) to cut intercept of length 5 units on the x-axis. If their centres lie in the first quadrant, then find their equation.
15. Find the equation of the circle passing through the origin and cutting intercepts of length 3 and 4 units from the positive axes.
16. Find the point of intersection of the circle $x^2 + y^2 - 3x - 4y + 2 = 0$ with the x-axis.
17. Find the length of intercept, the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the x-axis.
18. Find the values of k for which points $(2k, 3k)$, (1, 0), (0, 1) and (0, 0) lie on a circle.
19. If one end of the diameter is (1, 1) and other end lies on the line $x + y = 3$, then find the locus of centre of circle.
20. Tangent drawn from the point $P(4, 0)$ to the circle $x^2 + y^2 = 8$ touches it at the point A in the first quadrant. Find the coordinates of another point B on the circle such that $AB = 4$.
21. If the join of (x_1, y_1) and (x_2, y_2) makes an obtuse angle at (x_3, y_3), then prove that $(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) < 0$.

INTERSECTION OF A LINE AND A CIRCLE

Let the equation of the circle be

$$x^2 + y^2 = a^2 \quad \text{(i)}$$

and the equation of the line be

$$y = mx + c \quad \text{(ii)}$$

Solving Eqs. (i) and (ii),

$$x^2 + (mx + c)^2 = a^2$$

or $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ (iii)

Case I: When points of intersection are real and distinct. In this case Eq. (iii) has two distinct roots.

$$\therefore \qquad B^2 - 4AC > 0$$

$$\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

$$\Rightarrow \qquad a^2 > \frac{c^2}{1 + m^2}$$

$\Rightarrow \quad a > \dfrac{|c|}{\sqrt{(1 + m^2)}}$ = length of perpendicular from (0, 0) to $y = mx + c$

$\Rightarrow \quad a >$ length of perpendicular from (0, 0) to $y = mx + c$

Thus, a line intersects a given circle at two distinct points if radius of circle is greater than the length of perpendicular from centre of the circle to the line.

Case II: When the points of intersection are coincident. In this case, Eq. (iii) has two equal roots.

$$\therefore \qquad B^2 - 4AC = 0$$

$$\Rightarrow \qquad a = \frac{|c|}{\sqrt{(1 + m^2)}}$$

a = length of the perpendicular from the point (0, 0) to $y = mx + c$

Thus, a line touches the circle if radius of circle is equal to the length of perpendicular from centre of the circle to the line.

Case III: When the points of intersection are imaginary. In this case, Eq. (iii) has imaginary roots.

$$\therefore \qquad B^2 - 4AC < 0$$

$$\Rightarrow \qquad a^2 < \frac{c^2}{1 + m^2}$$

$\Rightarrow \quad a < \dfrac{|c|}{\sqrt{(1 + m^2)}}$ = length of perpendicular from (0, 0) to $y = mx + c$

or $a <$ length of perpendicular from (0, 0) to $y = mx + c$

Thus, a line does not intersect a circle if the radius of circle is less than the length of perpendicular from centre of the circle to the line.

Example 2.28 **Find the range of values of m for which the line $y = mx + 2$ cuts the circle $x^2 + y^2 = 1$ at distinct or coincident points.**

Sol. The length of the perpendicular from the centre (0, 0) to the line $= \dfrac{2}{\sqrt{1 + m^2}}$.

The radius of the circle = 1

For the line to cut the circle at distinct or coincident points, $\dfrac{2}{\sqrt{1 + m^2}} \le 1$ or $1 + m^2 \ge 4$ or $m^2 \ge 3$.

Segments of Secants, Chords and Tangent

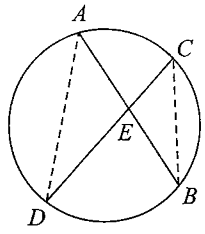


Fig. 2.24(i)

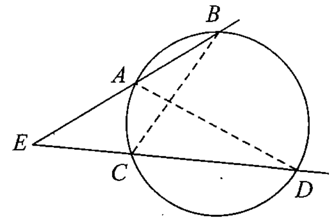


Fig. 2.24(ii)

Secants AB and CD intersect insider the circle in Fig. 2.24(i) and outside the circle in Fig. 2.24(ii).

From the figure, we have $\angle DCB = \angle DAB$ and $\angle ADC = \angle ABC$.

Hence $\Delta ADE \sim \Delta CBE$.

$$\therefore \frac{AE}{CE} = \frac{DE}{BE} \quad \therefore AE \times BE = CE \times DE$$

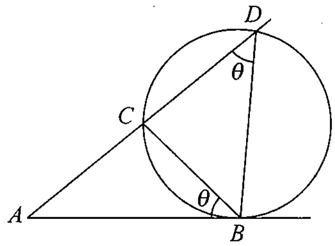


Fig. 2.24(iii)

In Fig. 2.24(iii), AD is secant and AB is tangent.

From Fig. 2.24(iii), $\Delta ABD \sim \Delta ACB$

$$\therefore \frac{AB}{AC} = \frac{AD}{AB} \quad \therefore AB^2 = AC \times AD$$

Example 2.29 **If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then prove that $|a_1a_2| = |b_1b_2|$.**

Sol. Let the given lines be $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$. Suppose L_1 meets the coordinate axes at P and Q and L_2 meets at R and S. Then, coordinates of P, Q, R, S are

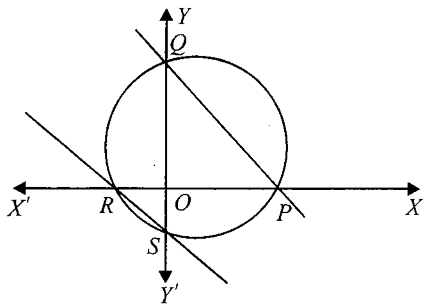


Fig. 2.25

$P\left(-\frac{c_1}{a_1}, 0\right), Q\left(0, -\frac{c_1}{b_1}\right)$

$R\left(-\frac{c_2}{a_2}, 0\right)$, and $S\left(0, -\frac{c_2}{b_2}\right)$

Since P, Q, R, S are concyclic,

$$\therefore \quad OP \cdot OR = OQ \cdot OS$$

$$\Rightarrow \quad \left|\left(-\frac{c_1}{a_1}\right)\left(-\frac{c_2}{a_2}\right)\right| = \left|\left(-\frac{c_1}{b_1}\right)\left(-\frac{c_2}{b_2}\right)\right|$$

$$\Rightarrow \quad |a_1a_2| = |b_1b_2|$$

Example 2.30 **If a line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2 + y^2 = a^2$ at A and B, then find the value of $PA \cdot PB$.**

Sol. From the figure, $PA \cdot PB$ = constant

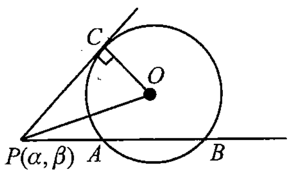


Fig. 2.26

Also $\quad PA \cdot PB = PC^2$

But $\quad PC^2 = OP^2 - OC^2$

$$= \alpha^2 + \beta^2 - a^2$$

$$\Rightarrow \quad PA \cdot PB = \alpha^2 + \beta^2 - a^2$$

TANGENT TO A CIRCLE AT A GIVEN POINT

Let PQ be *a* chord and AB be a secant passing through P.

Let P be the fixed point and Q move along the circle towards P, then the secant PQ turns about P. In the limit, when Q coincides with P, then the secant AB becomes *a* tangent to a circle at the point P.

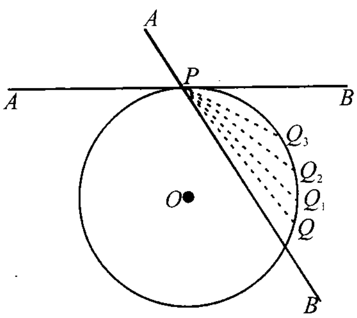


Fig. 2.27

Different Forms of the Equations of Tangents

Point Form

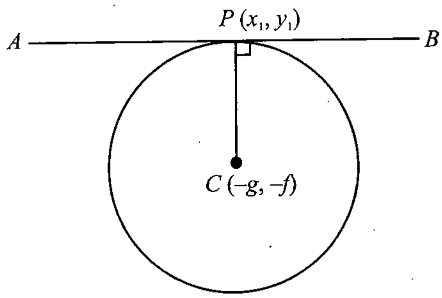


Fig. 2.38

To find the equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point $P(x_1, y_1)$ on it.

CP is perpendicular of tangent at P

$$\text{Slope of } CP = \frac{y_1 + f}{x_1 + g}$$

$$\therefore \quad \text{Slope of tangent} = -\frac{x_1 + g}{y_1 + f}$$

Slope of tangent at point P can be obtained by differentiation also.

Differentiating equation of circle w.r.t. to x

we get $2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)(x_1, y_1) = -\frac{x_1 + g}{y_1 + f}$$

$\therefore$ equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

$$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

or $\quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(since point (x_1, y_1) lies on the circle $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$)

or $\quad T = 0,$

where $\quad T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Note:

1. For equation of tangent of circle at (x_1, y_1) substitute xx_1 for x^2, yy_1 for y^2, $\frac{x+x_1}{2}$ for x, $\frac{y+y_1}{2}$ for y and keep the constant as such.
2. For circle $x^2 + y^2 = a^2$ equation of tangent at point (x_1, y_1) is given by $xx_1 + yy_1 = a^2$
3. For circle $(x-h)^2 + (y-k)^2 = a^2$ equation of tangent at point (x_1, y_1) is given by $(x-h)(x-x_1) + (y-k)(y-y_1) = a^2$
4. Since parametric coordinates of circle $x^2 + y^2 = a^2$ are $(a \cos \theta, a \sin \theta)$, then equation of tangent at $(a \cos \theta, a \sin \theta)$ is $x.a \cos \theta + y.a \sin \theta = a^2$ or $x \cos \theta + y \sin \theta = a$

Slope Form

Let $y = mx + c$ is the tangent of the circle $x^2 + y^2 = a^2$

$\therefore$ Length of perpendicular from centre of circle (0, 0) on $(y = mx + c)$ = radius of circle

$$\therefore \quad \frac{|c|}{\sqrt{1+m^2}} = a \Rightarrow c = \pm a\sqrt{1+m^2}$$

substituting this value of c in $y = mx + c$, we get $y = mx \pm a\sqrt{1+m^2}$ which are the required equations of tangents.

Corollary : It also follows that $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ which is the condition of tangency.

Note:

1. If slope of tangent is given, then two parallel tangents can be drawn the circle at the ends of diameter.
2. Equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is $y + f = m(x + g) \pm \sqrt{g^2 + f^2 - c}\sqrt{1+m^2}$

Points of Contact

If circle $x^2 + y^2 = a^2$ and tangent in terms of slope $y = mx \pm a\sqrt{(1+m^2)}$

Solving $x^2 + y^2 = a^2$ and $y = mx \pm a\sqrt{1+m^2}$, simultaneously, we get

$$x = \pm \frac{am}{\sqrt{(1+m^2)}}$$

and

$$y = \mp \frac{a}{\sqrt{(1+m^2)}}$$

Thus, the coordinates of the points of contact are $\left(\pm \frac{am}{\sqrt{(1+m^2)}}, \mp \frac{a}{\sqrt{(1+m^2)}}\right)$.

Alternative Method

Let point of contact be (x_1, y_1), then tangent at (x_1, y_1) of $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

Since $xx_1 + yy_1 = a^2$ and $y = mx \pm a\sqrt{1+m^2}$ are identical

$$\Rightarrow \quad \frac{x_1}{m} = \frac{y_1}{-1} = \frac{+a^2}{\pm a\sqrt{1+m^2}}$$

$$\Rightarrow \quad x_1 = \pm \frac{am}{\sqrt{1+m^2}}$$

$$\text{and} \quad y_1 = \mp \frac{a}{\sqrt{1+m^2}}$$

Thus, the coordinates of the points of contact are $\left(\pm \frac{am}{\sqrt{(1+m^2)}}, \mp \frac{a}{\sqrt{(1+m^2)}}\right)$.

Note:

1. *If the line $y = mx + c$ is the tangent to the circle $x^2 + y^2 = r^2$, then point of contact is given by* $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$.
2. *If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then point of contact is given by* $\left(-\frac{ar^2}{c}, -\frac{br^2}{c}\right)$.

Tangents from a Point Outside the Circle

If circle is $x^2 + y^2 = a^2$ (i)

any tangent to the circle Eq. (i) is

$$y = mx + a\sqrt{(1+m^2)} \quad \text{(ii)}$$

If outside point is (x_1, y_1), then $y_1 = mx_1 + a\sqrt{1+m^2}$ or $(y_1 - mx_1)^2 = a^2(1 + m^2)$

or $y_1^2 + m^2x_1^2 - 2xm_1y_1 = a^2 + a^2m^2$

$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0$

which is quadratic in m which given two values of m.

Substituting these values of m in Eq. (ii), we get the equation of tangents.

Example 2.31 **Find the angle between the two tangents from the origin to the circle $(x-7)^2 + (y+1)^2 = 25$.**

Sol. Any line through (0, 0) be $y - mx = 0$ and it is a tangent to circle $(x-7)^2 + (y+1)^2 = 25$, if

$$\frac{|-1-7m|}{\sqrt{1+m^2}} = 5$$

$$\Rightarrow \quad m = \frac{3}{4}, -\frac{4}{3}$$

Therefore, the product of both the slopes is – 1 i.e.,

$$\frac{3}{4} \times \left(-\frac{4}{3}\right) = -1$$

Hence, the angle between the two tangents is $\frac{\pi}{2}$.

Example 2.32 **Find the equation of the tangents to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the straight line $4x + 3y + 5 = 0$.**

Sol. Let equation of tangent be $4x + 3y + k = 0$, then radius = distance of centre from the line

$$\Rightarrow \quad \sqrt{9+4+12} = \left|\frac{4(3)+3(-2)+k}{\sqrt{16+9}}\right|$$

$$\Rightarrow \quad 6 + k = \pm 25$$

$$\Rightarrow \quad k = 19 \text{ and } -31$$

Hence, the tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$.

Example 2.33 **Tangent to circle $x^2 + y^2 = 5$ at (1, – 2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$. Find the coordinate of the corresponding point of contact**

Sol. Equation of tangent to $x^2 + y^2 = 5$ at (1, – 2) is $x - 2y - 5 = 0$.

Putting $x = 2y + 5$ in second circle, we get $(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$

$$\Rightarrow \quad 5y^2 + 10y + 5 = 0$$

$$\Rightarrow \quad y = -1$$

$$\Rightarrow \quad x = -2 + 5 = 3$$

Thus, point of contact is (3, –1).

Example 2.34 **Prove that the equation of any tangent to the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ is of the form $y = m(x-1) + 3\sqrt{1+m^2} - 2$.**

Sol. The circle is $(x-1)^2 + (y+2)^2 = 3^2$.

As any tangent to $x^2 + y^2 = 3^2$ is given by $y = mx + 3\sqrt{1+m^2}$, any tangent to the given circle will be

$$y + 2 = m(x-1) + 3\sqrt{1+m^2}.$$

Example 2.35 **If $a > 2b > 0$ then find the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$, and $(x-a)^2 + y^2 = b^2$.**

Sol. $y = mx - b\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = b^2$ for all values of m.

If it also touches the circle $(x-a)^2 + y^2 = b^2$, then the length of the perpendicular from its centre $(a, 0)$ on this line is equal to the radius b of the circle, which gives

$$\frac{ma - b\sqrt{1+m^2}}{\sqrt{1+m^2}} = \pm b.$$

Taking negative value of R.H.S., we get $m = 0$, so we neglect it.

Taking the positive value of R.H.S., we get

$$ma = 2b\sqrt{1+m^2}$$

$$\Rightarrow \quad m^2(a^2 - 4b^2) = 4b^2$$

$$\Rightarrow \quad m = \frac{2b}{\sqrt{a^2-4b^2}}.$$

Length of the Tangent from a Point to a Circle

Let circle be $x^2 + y^2 + 2gx + 2fy + c = 0$, then centre and radius of circle are $(-g, -f)$ and $\sqrt{(g^2+f^2-c)}$, respectively, and let $P(x_1, y_1)$ be any point outside the circle.

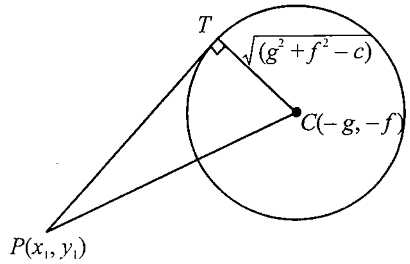


Fig. 2.29

In ΔPCT,

$$PT = \sqrt{(PC)^2 - (CT)^2}$$

$$= \sqrt{(x_1+g)^2 + (y_1+f)^2 - g^2 - f^2 + c}$$

$$= \sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}$$

$$= \sqrt{S_1},$$

where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note:

1. *For S_1, first write the equation of circle in standard form and coefficient of x^2 = coefficient of $y^2 = 1$ and making R.H.S. of circle is zero, then let L.H.S. be S.*

Example 2.36 **Prove that the angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right)$, where $S_1 = \alpha^2 + \beta^2 - a^2$.**

Sol.

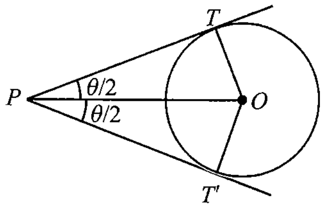


Fig. 2.30

Let PT and PT' be the tangents drawn from $P(\alpha, \beta)$ to the circle $x^2 + y^2 = a^2$ and let $\angle TPT' = \theta$. If O is the centre of the circle, then $\angle TPO = \angle T'PO = \frac{\theta}{2}$.

$$\therefore \quad \tan\frac{\theta}{2} = \frac{OT}{PT} = \frac{a}{\sqrt{S_1}}$$

$$\Rightarrow \quad \theta = 2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right).$$

Example 2.37 **Find the ratio of the length of the tangents from any point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$ to the two circles $5x^2 + 5y^2 - 24x + 32y + 75 = 0$, $5x^2 + 5y^2 - 48x + 64y + 300 = 0$.**

Sol. Let $P(h, k)$ be a point on the circle $15x^2 + 15y^2 - 48x + 64y = 0$.

Then, the lengths PT_1 and PT_2 of the tangent from $P(h, k)$ to $5x^2 + 5y^2 - 24x + 32y + 75 = 0$ and $5x^2 + 5y^2 - 48x + 64y + 300 = 0$, respectively, are

$$PT_1 = \sqrt{h^2 + k^2 - \frac{24}{5}h + \frac{32}{5}k + 15}$$

and $$PT_2 = \sqrt{h^2 + k^2 - \frac{48}{5}h + \frac{64}{5}k + 60}$$

Since (h, k) lies on $15x^2 + 5y^2 - 48x + 64y = 0$

$$\therefore \quad h^2 + k^2 - \frac{48}{15}h + \frac{64}{15}k = 0$$

$$\therefore \quad PT_1 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{24}{5}h + \frac{32}{5}k + 15}$$

$$= \sqrt{\frac{32}{15}k - \frac{24}{15}h + 15}$$

and, $$PT_2 = \sqrt{\frac{48}{15}h - \frac{64}{15}k - \frac{48}{5}h + \frac{64}{5}k + 60}$$

$$= \sqrt{-\frac{96}{15}h + \frac{128}{15}k + 60}$$

$$= 2\sqrt{-\frac{24}{15}h + \frac{32}{15}k + 15} = 2(PT_1)$$

$\therefore PT_1 : PT_2 = 1 : 2.$

Example 2.38 **If from any point P on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, tangents are drawn to the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$, then find the angle between the tangents.**

Sol.

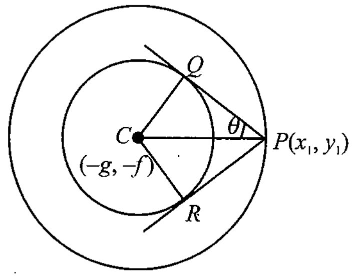


Fig. 2.31

Let $P(x_1, y_1)$ be a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{(i)}$$

and the length of the tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$ is $PQ = PR$

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha}$$

$$= \sqrt{-c + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha}$$

$$= (\sqrt{g^2 + f^2 - c})\cos\alpha$$

The radius of the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$ is

$$CQ = CR$$

$$= \sqrt{g^2 + f^2 - c\sin^2\alpha - (g^2 + f^2)\cos^2\alpha}$$

$$= (\sqrt{g^2 + f^2 - c})\sin\alpha$$

In ΔCPQ,

$$\tan\theta = \frac{CQ}{PQ}$$

$$= \frac{\sqrt{g^2 + f^2 - c}\,\sin\alpha}{\sqrt{g^2 + f^2 - c}\,\cos\alpha} = \tan\alpha \Rightarrow \theta = \alpha$$

Example 2.39 **If the distance from the origin of the centres of three circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$ ($i = 1, 2, 3$) are in G.P., then prove that the lengths of the tangents drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in G.P.**

Sol. The centres of the given circles are $(-\lambda_i, 0)$ ($i = 1, 2, 3$). The distances from the origin of the centres are λ_i, ($i = 1, 2, 3$). It is given that $\lambda_2^2 = \lambda_1\lambda_3$. Let $P(h, k)$ be any point on the circle $x^2 + y^2 = c^2$. Then $h^2 + k^2 = c^2$.

Now, L_i = length of the tangent from (h, k) to $x^2 + y^2 + 2\lambda_i x - c^2 = 0$

$$= \sqrt{h^2 + k^2 + 2\lambda_i h - c^2}$$

$$= \sqrt{c^2 + 2\lambda_i h - c^2} \qquad [\because h^2 + k^2 = c^2]$$

$$= \sqrt{2\lambda_i h},\ i = 1, 2, 3$$

Therefore,

$$L_2^2 = 2\lambda_2 h = 2h(\sqrt{\lambda_1\lambda_3}) \ [\because \lambda_2^2 = \lambda_1\lambda_3]$$

$$= \sqrt{2\lambda_1 h}\ \sqrt{2\lambda_3 h} = L_1 L_3$$

Hence, L_1, L_2, L_3 are in G.P.

Pair of Tangents

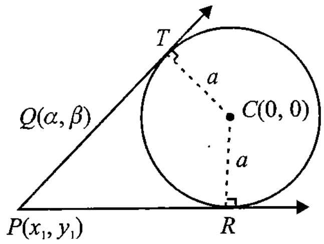


Fig. 2.32

Let the circle be $x^2 + y^2 = a^2$

Its centre and radius are $C(0, 0)$ and a, respectively. Let the given external point be $P(x_1, y_1)$.

From point $P(x_1, y_1)$, two tangents PT and PR can be drawn to the circle, touching circle at T and R, respectively.

Let $Q(\alpha, \beta)$ on PT, then equation of PQ is

$$y - y_1 = \frac{\beta - y_1}{\alpha - x_1}(x - x_1)$$

or $\quad y(\alpha - x_1) - (\beta - y_1)x - \alpha y_1 + \beta x_1 = 0$

Length of perpendicular from $C(0, 0)$ on $PT = a$(radius)

$$\Rightarrow \frac{|\beta x_1 - \alpha y_1|}{\sqrt{(\alpha - x_1)^2 + (\beta - y_1)^2}} = a$$

or $\quad (\beta x_1 - \alpha y_1)^2 = a^2\{(\alpha - x_1)^2 + (\beta - y_1)^2\}$

$\therefore$ Locus of $Q(\alpha, \beta)$ is $(yx_1 - xy_1)^2 = a^2\{(x - x_1)^2 + (y - y_1)^2\}$

$$\Rightarrow y^2x_1^2 + x^2y_1^2 - 2xy\,x_1y_1 = a^2\{x^2 + x_1^2 - 2xx_1 + y^2 + y_1^2 - 2yy_1\}$$

$$\Rightarrow y^2(x_1^2 + y_1^2 - a^2) + x^2(x_1^2 + y_1^2 - a^2) - a^2(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

$$\Rightarrow (x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

$$\Rightarrow SS_1 = T^2$$

This is the required equation of pair of tangents drawn from (x_1, y_1) to circle $x^2 + y^2 = a^2$.

where $S = x^2 + y^2 - a^2$, $S_1 = x_1^2 + y_1^2 - a^2$,
and $T = xx_1 + yy_1 - a^2$.

NORMAL TO A CIRCLE AT A GIVEN POINT

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point of contact. Normal always passes through the centre of the circle.

Point Form

The normal of a circle at any point is a straight line which is perpendicular to the tangent at the point of contact.

The normal of the circle always passes through the centre of the circle.

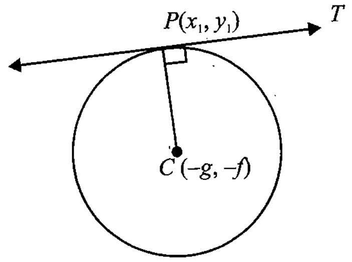


Fig. 2.33

To find the equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) on it.

Since normal passes through the centre, we have slope of normal $CP = \dfrac{y_1 + f}{x_1 + g}$

Hence, equation of normal is $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$

or $\quad \dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$

Example 2.40 **Find the equations of the normals to the circle $x^2 + y^2 - 8x - 2y + 12 = 0$ at the points whose ordinate is -1.**

Sol. The abscissa of point is found by substituting the ordinates and solving for abscissa

$\Rightarrow \quad x^2 - 8x + 15 = 0$

$\Rightarrow \quad x = 5$ or 3

$\Rightarrow$ Points are $(5, -1)$ and $(3, -1)$.

Normal is given by $\dfrac{x - 5}{5 - 4} = \dfrac{y + 1}{-1 - 1}$

$\Rightarrow \quad 2x + y - 9 = 0$

and $\quad \dfrac{x-3}{3-4} = \dfrac{y+1}{-1-1}$

$\Rightarrow \quad 2x - y - 7 = 0$

Example 2.41 **Find the equation of the normal to the circle $x^2 + y^2 - 2x = 0$ parallel to the line $x + 2y = 3$.**

Sol. Any line parallel to $x + 2y = 3$ is $x + 2y + \lambda = 0$ and for this to be a normal to the given circle, must pass through its centre (1, 0), i.e., $\lambda = -1$.

So, normal is $x + 2y - 1 = 0$.

Concept Application Exercise 2.2

1. Find the equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which makes equal intercepts on the positive co-ordinates axes.
2. If the length of tangent drawn from the point (5, 3) to the circle $x^2 + y^2 + 2x + ky + 17 = 0$ be 7, then find the value of k.
3. If the line $lx + my + n = 0$ is tangent to the circle $x^2 + y^2 = a^2$, then find the condition.
4. A pair of tangents are drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$. Then find its equations.
5. Find the equation of the normal to the circle $x^2 + y^2 = 9$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
6. Find the equations of tangents to the circle $x^2 + y^2 - 22x - 4y + 25 = 0$ which are perpendicular to the line $5x + 12y + 8 = 0$.
7. An infinite number of tangents can be drawn from (1, 2) to the circle $x^2 + y^2 - 2x - 4y + \lambda = 0$, then find the value of λ.
8. If circle $x^2 + y^2 - 4x - 8y - 5 = 0$ intersect the line $3x - 4y = m$ in two distinct points, then find the values of m.
9. If a line passing through origin touches the circle $(x - 4)^2 + (y + 5)^2 = 25$, then find its slope.
10. The tangent at any point P on the circle $x^2 + y^2 = 4$, meets the coordinate axes in A and B, then find the locus of the midpoint of AB.
11. Find the locus of a point which moves so that the ratio of the of the length of the tangents to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is 2:3.
12. Find the length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_2 = 0$.

CHORD OF CONTACT

From any external point $A(x_1, y_1)$ draw pair of tangents AP and AQ touching the circle at $P(x', y')$ and $Q(x'', y'')$, respectively. Then PQ is the chord of contact with P, Q as its points of contact.

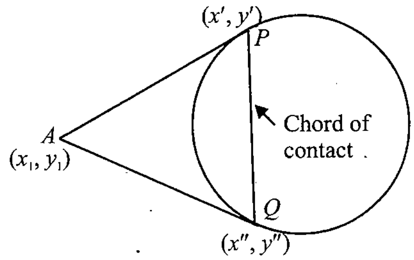


Fig. 2.34

Let the circle be $x^2 + y^2 = a^2$.

Then equations of tangents AP and AQ are $xx' + yy' = a^2$ and $xx'' + yy'' = a^2$, respectively.

Since both tangents AP and AQ pass through $A(x_1, y_1)$, then $x_1x' + y_1y' = a^2$ and $x_1x'' + y_1y'' = a^2$.

Points $P(x', y')$ and $Q(x'', y'')$ lie on $xx_1 + yy_1 = a^2$.

$\therefore$ Equation of chord of contact PQ is $xx_1 + yy_1 = a^2$.

or $T = 0$, where $T = xx_1 + yy_1 - a^2$

- *Equation of chord of contact at (x_1, y_1) with circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ or $T = 0$.*

Example 2.42 **If the chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then prove that a, b, c are in G.P.**

Sol. Let (h, k) be a point on $x^2 + y^2 = a^2$. Then

$$h^2 + k^2 = a^2 \quad \text{(i)}$$

The equation of the chord of contact of tangents drawn from (h, k) to $x^2 + y^2 = b^2$ is

$$hx + ky = b^2 \quad \text{(ii)}$$

This touches the circle $x^2 + y^2 = c^2$. Therefore $\left|\dfrac{-b^2}{\sqrt{h^2 + k^2}}\right| = c$

$\Rightarrow \quad \left|\dfrac{-b^2}{\sqrt{a^2}}\right| = c \quad$ [Using Eq. (i)]

$\Rightarrow \quad b^2 = ac \Rightarrow a, b, c$ are in G.P.

Example 2.43 **If the straight line $x - 2y + 1 = 0$ intersects the circle $x^2 + y^2 = 25$ in points P and Q, then find the coordinates of the point of intersection of tangents drawn at P and Q to the circle $x^2 + y^2 = 25$.**

Sol.

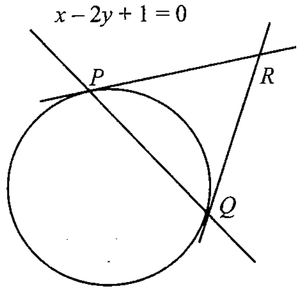


Fig. 2.35

Let $R(h, k)$ be the point of intersection of tangents drawn at P and Q to the given circle. Then PQ is the chord of contact of tangents drawn from R to $x^2 + y^2 = 25$.

So, its equation is

$$hx + ky - 25 = 0 \qquad \text{(i)}$$

It is given that the equation of PQ is $x - 2y + 1 = 0$ (ii)

Since Eq. (i) and (ii) represent the same line, therefore

$$\frac{h}{1} = \frac{k}{-2} = \frac{-25}{1}$$

$$\Rightarrow \qquad h = -25, k = 50$$

Hence, the required point is (–25, 50).

Example 2.44 **If the chord of contact of tangents drawn from the point (h, k) to the circle $x^2 + y^2 = a^2$ subtends a right angle at the centre, then prove that $h^2 + k^2 = 2a^2$.**

Sol.

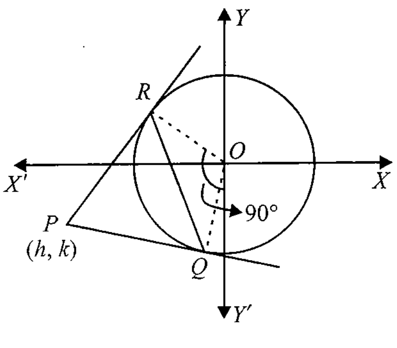


Fig. 2.36

As shown in diagram $\angle ROQ = \pi/2$ also $\angle PRO = \angle PQO = \pi/2$.

Then quadrilateral $PROQ$ is square and hence $PR = RO$

$$\Rightarrow \quad \sqrt{h^2 + k^2 - a^2} = a \Rightarrow h^2 + k^2 = 2a^2$$

Example 2.45 **Tangents are drawn to $x^2 + y^2 = 1$ from any arbitrary point P on the line $2x + y - 4 = 0$. The corresponding chord of contact passes through a fixed point whose coordinates are**

a. $\left(\frac{1}{4}, \frac{1}{2}\right)$ b. $\left(\frac{1}{2}, 1\right)$

c. $\left(\frac{1}{2}, \frac{1}{4}\right)$ d. $\left(1, \frac{1}{2}\right)$

Sol.

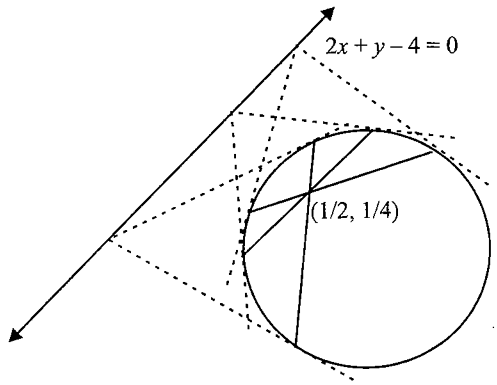


Fig. 2.37

Let any point on the line $2x + y - 4 = 0$ be $P \equiv (a, 4 - 2a)$.

Equation of chord of contact of the circle $x^2 + y^2 = 1$ with respect to point P is

$$x.\, a + y.\, (4 - 2a) = 1$$

$$\Rightarrow (4y - 1) + a(x - 2y) = 0.$$

This line always passes through a point of intersection of the lines $4y - 1 = 0$ and $x - 2y = 0$ which is fixed point whose coordinate are $y = \frac{1}{4}$ and $x = 2y = \frac{1}{2}$.

Example 2.46 **Find the area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their points of contact.**

Sol.

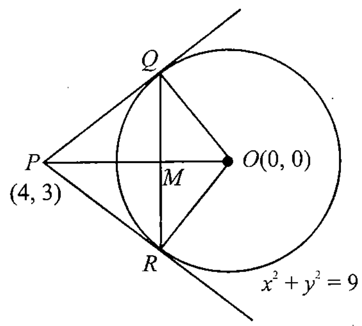


Fig. 2.38

The equation of the chord of contact of tangents drawn from $P(4, 3)$ to $x^2 + y^2 = 9$ is $4x + 3y = 9$.

The equation of PO is $y = \frac{3}{4}x$.

Now, OM = (length of the $\perp$ from (0, 0) on $4x + 3y - 9 = 0$ is $\frac{9}{5}$

$$QR = 2.\,QM = 2\sqrt{OQ^2 - OM^2}$$

$$= 2\sqrt{9 - \frac{81}{25}} = \frac{24}{5}$$

Now, $$PM = OP - OM = 5 - \frac{9}{5} = \frac{16}{5}$$

So, area of $$\Delta PQR = \frac{1}{2}\left(\frac{24}{5}\right)\left(\frac{16}{5}\right).$$

$$= \left(\frac{192}{25}\right) \text{ sq. units.}$$

EQUATION OF THE CHORD BISECTED AT A GIVEN POINT

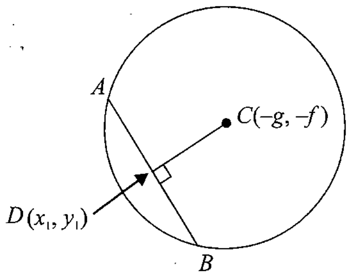


Fig. 2.39

Let any chord AB of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ be bisected at $D(x_1, y_1)$.

Then slope of $DC = \frac{y_1 + f}{x_1 + g}$

Therefore, slope of the chord AB is $-\frac{x_1 + g}{y_1 + f}$

then equation of AB is $y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$

$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow T = S_1$.

where $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$ and $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Note:

Chord bisected at point (x_1, y_1) is the farthest from centre among all the chords passing through the point (x_1, y_1). Also for such chord, the length of the chord is minimum.

Example 2.47 **Find the equation to the chord of the circle $x^2 + y^2 = 9$ whose middle point is (1, –2).**

Sol. The required equation is

$$x - 2y - 9 = 1 + 4 - 9$$

[Using $S' = T$]

or $$x - 2y - 5 = 0$$

Example 2.48 **A variable chord is drawn through the origin to the circle $x^2 + y^2 - 2ax = 0$. Find the locus of the centre of the circle drawn on this chord as diameter.**

Sol. Let (h, k) be the coordinates of the centre of the circle of which the given chord is the diameter. Then (h, k) is the mid-point of the chord and so its equation is

$$T = S_1$$

i.e. $$h^2 + k^2 - 2ah = hx + ky - a(x + h)$$

$\Rightarrow$ $$x(h - a) + ky = h^2 + k^2 - ah$$

It passes through (0, 0), therefore $h^2 + k^2 - ah = 0$

So, the locus of (h, k) is $x^2 + y^2 - ax = 0$.

Alternative Method:

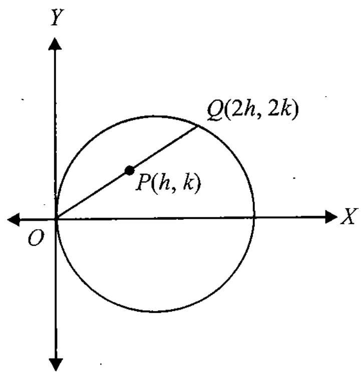


Fig. 2.40

Since point P is midpoint of chord OQ, Q has coordinates $(2h, 2k)$, which lies on the given circle.

$\therefore$ $(2h)^2 + (2k)^2 - 2a(2h) = 0$ or $x^2 + y^2 - ax = 0$.

Example 2.49 **Find the equation of chord of the circle $x^2 + y^2 = a^2$ passing through the point (2, 3) farthest from the centre.**

Sol.

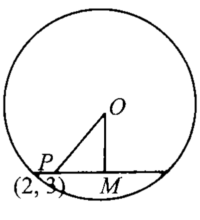


Fig. 2.41

Let $P(2, 3)$ be given point, M be the middle point of a chord of the circle $x^2 + y^2 = a^2$ through P.

Then the distance of the centre O of the circle from the chord is OM.

and $(OM)^2 = (OP)^2 - (PM)^2$ which is maximum when PM is minimum,

i.e. P coincides with M, which is the middle point of the chord.

Hence, the equation of the chord is $T = S_1$, i.e. $2x + 3y - a^2 = (2)^2 + (3)^2 - a^2 \Rightarrow 2x + 3y = 13$

Concept Application Exercise 2.3

1. Find the middle point of the chord of the circle $x^2 + y^2 = 25$ intercepted on the line $x - 2y = 2$.
2. The line $9x + y - 18 = 0$ is the chord of contact of the point $P(h, k)$ with respect to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$, for

   a. $\left(\frac{24}{5}, \frac{-4}{5}\right)$ b. $P(3, 1)$

   c. $P(-3, 1)$ d. $\left(-\frac{2}{5}, \frac{12}{5}\right)$
3. Tangents are drawn to the circle $x^2 + y^2 = 9$ at the points where it is met by the circle $x^2 + y^2 + 3x + 4y + 2 = 0$. Find the point of intersection of these tangents.
4. Find the distance between the chords of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f).
5. A line $lx + my + n = 0$ meets the circle $x^2 + y^2 = a^2$ at the points P and Q. If the tangents drawn at the points P and Q meet at R, then find the coordinates of R.

DIRECTOR CIRCLE AND ITS EQUATION

Director Circle: The locus of the point of intersection of two perpendicular tangents to a given circle is known as its director circle.

Equation of Director Circle: The equation of any tangent to the circle $x^2 + y^2 = a^2$ is

$$y = mx + a\sqrt{(1 + m^2)} \quad \text{(i)}$$

Let $P(h, k)$ be the point of intersection of tangents, then $P(h, k)$ lies on Eq. (i)

$$\therefore \quad k = mh + a(\sqrt{1 + m^2})$$

or $\quad (k - mh)^2 = a^2(1 + m^2)$

or $m^2(h^2 - a^2) - 2mkh + k^2 - a^2 = 0$

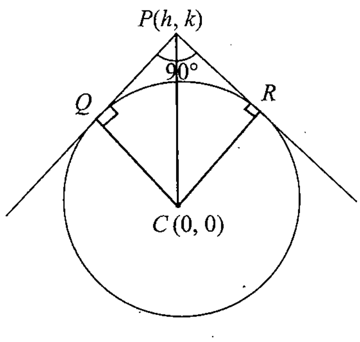


Fig. 2.42

This is quadratic equation in m, let two roots are m_1 and m_2.

But tangents are perpendiculars, therefore $m_1 m_2 = -1$

$$\Rightarrow \frac{k^2 - a^2}{h^2 - a^2} = -1 \Rightarrow k^2 - a^2 = -h^2 + a^2 \Rightarrow h^2 + k^2 = 2a^2$$

Hence, locus of $P(h, k)$ is $x^2 + y^2 = 2a^2$.

Equation of director circle for circle $(x - p)^2 + (y - q)^2 = a^2$ is given by $(x - p)^2 + (y - q)^2 = 2a^2$.

Alternative Method:

From the figure,

$CRPQ$ is a square

$\therefore \quad CQ = CP \cos 45°$

or $\quad 2a^2 = h^2 + k^2$

or $\quad x^2 + y^2 = 2a^2$ which is the required locus.

INTERSECTION OF TWO CIRCLES

Different cases of intersection of two circles:

Let the two circles be $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ (i)

and $(x - x_2)^2 + (y - y_2)^2 = r_2^2$ (ii)

with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2, respectively. Then following cases may arise:

Case I:

When $|C_1C_2| > r_1 + r_2$, i.e., the distance between the centres is greater than the sum of radii, then two circles neither intersect not touch each other .

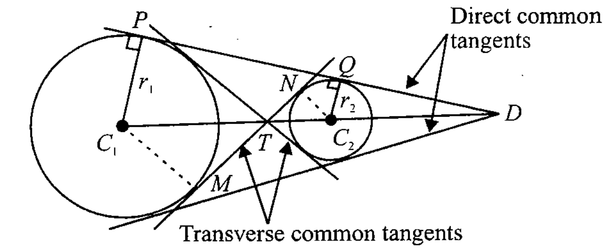


Fig. 2.43

In this case, four common tangents can be drawn to two circles, in which two are direct common tangents and the other two are transverse common tangents.

From Fig. 2.43, ΔC_1MT and ΔC_2NT are similar. Hence, $\frac{C_1T}{C_2T} = \frac{C_1M}{C_2N} = \frac{r_1}{r_2}$. Using this, we can find point T.

Similarly, ΔC_1PD and C_2QD are similar. Hence, $\frac{C_1D}{C_2D} = \frac{C_1P}{C_2Q} = \frac{r_1}{r_2}$

To find equations of common tangents:

Now assume the equation of tangent of any circle in the form of the slope $(y + f) = m(x + g) + a\sqrt{(1 + m^2)}$ (where a is the radius of the circle).

T and D will satisfy the assumed equation. Thus 'm' is obtained. We can find the equation of common tangent if we substitute the value of m in the assumed equation.

Case II:

When $|C_1C_2| = r_1 + r_2$, i.e, the distance between the centres is equal to the sum of radii, then two circles touch externally.

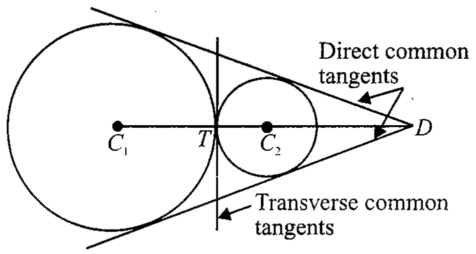


Fig. 2.44

In this case, two direct common tangents are real and distinct while the transverse tangents are coincident.

In such cases, the point of contact T divides the line joining C_1 and C_2 internally in the ratio $r_1 : r_2 \Rightarrow \frac{C_1T}{C_2T} = \frac{r_1}{r_2}$.

Then coordinates of T are $\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$

The equation of tangent at point T is $S_1 - S_2 = 0$, where $S_1 = 0$ and $S_2 = 0$ are equations of circles.

Case III:

When $|r_1 - r_2| < |C_1C_2| < r_1 + r_2$, i.e., the distance between the centres is less than sum of radii then two circles intersect at two distinct points.

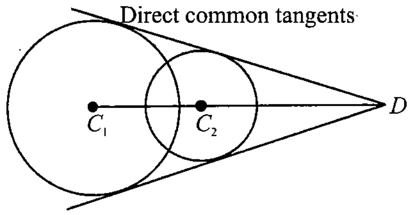


Fig. 2.45

In this case, two direct common tangents are real and distinct while the transverse tangents are imaginary.

Here, also point D divides C_1C_2 externally, $\frac{C_1D}{C_2D} = \frac{r_1}{r_2}$

Case IV:

When $|C_1C_2| = |r_1 - r_2|$, i.e., the distance between the centres is equal to the difference of the radii, then two circles touch internally.

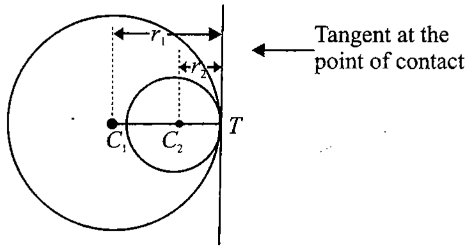


Fig. 2.46

In this case, there is only one common tangent.

If circles are represented by $S_1 = 0$ and $S_2 = 0$, then equation of common tangent is $S_1 - S_2 = 0$.

In such cases, the point of contact T divides the line joining C_1 and C_2 externally in the ratio $r_1 : r_2 \Rightarrow \frac{C_1T}{C_2T} = \frac{r_1}{r_2}$.

Then coordinates of T are $\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$.

Case V:

When $|C_1C_2| < |r_1 - r_2|$, i.e., the distance between the centres is less than the difference of the radii.

In this case, all the four common tangents are imaginary.

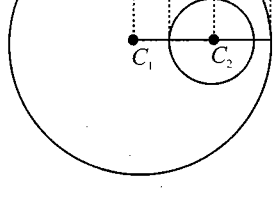


Fig. 2.47

Length of an External Common Tangent and Internal Common Tangent to Two Circles

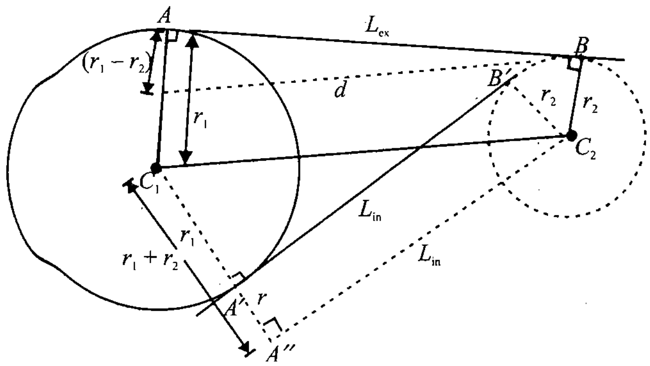


Fig. 2.48

Length of external common tangent $L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$

and length of internal common tangent

$L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$

[Applicable only when $d > (r_1 + r_2)$]

where d is the distance between the centres of two circles

and r_1 and r_2 are the radii of two circles where $|C_1C_2| = d$.

Example 2.50 **Find the number of common tangents to circles $x^2 + y^2 + 2x + 8y - 23 = 0$ and $x^2 + y^2 - 4x - 10y + 9 = 0$.**

Sol. For $x^2 + y^2 + 2x + 8y - 23 = 0$

$\therefore \quad C_1(-1, -4), r_1 = 2\sqrt{10}$

For $x^2 + y^2 - 4x - 10y + 9 = 0 \therefore C_2(2, 5), r_2 = 2\sqrt{5}$

Now $\quad C_1C_2$ = distance between centres

$\therefore \quad C_1C_2 = \sqrt{9 + 81} = 3\sqrt{10} = 9.486$

and $\quad r_1 + r_2 = 2(\sqrt{10} + \sqrt{5}) = 10.6$

$r_1 - r_2 = 2\sqrt{5}(\sqrt{2} - 1)$

$= 2 \times 2.2 \times 0.4$

$= 4.4 \times 0.4 = 1.76$

$\Rightarrow \quad r_1 - r_2 < C_1C_2 < r_1 + r_2$

$\Rightarrow$ Two circles intersect at two distinct points

$\Rightarrow$ Two tangents can be drawn.

Example 2.51 **Find the equation of a circle with centre (4, 3) touching the circle $x^2 + y^2 = 1$.**

Sol. Let the circle be $x^2 + y^2 - 8x - 6y + k = 0$ touching the circle $x^2 + y^2 = 1$. Then the equation of the common tangent is $S_1 - S_2 = 0 \Rightarrow 8x + 6y - 1 - k = 0$

This is a tangent to the circle $x^2 + y^2 = 1$. Therefore,

$\pm 1 = \frac{k+1}{\sqrt{8^2 + 6^2}} \Rightarrow k + 1 = \pm 10 \Rightarrow k = -11$ or 9

Hence, the circles are $x^2 + y^2 - 8x - 6y + 9 = 0$ and, $x^2 + y^2 - 8x - 6y - 11 = 0$.

Alternative Solution:

The given circle is $x^2 + y^2 = 1$, which has centre $C_1(0, 0)$ and radius $r_1 = 1$. The required circle has centre $C_2(4, 3)$ and radius r_2

If the circle are touching externally then, $r_1 + r_2 = C_1C_2$

$\Rightarrow r_2 = 5 - 1 = 4$ If circles are touching internally then

$r_2 - r_1 = C_1C_2 \Rightarrow r_2 = 6$

Thus, required circles are $(x - 4)^2 + (y - 3)^2 = 16$ or $(x - 4)^2 + (y - 3)^2 = 36$.

Example 2.52 **Find the condition if circles whose equations are $x^2 + y^2 + c^2 = 2ax$ and $x^2 + y^2 + c^2 - 2by = 0$ will touch one another externally.**

Sol. The two circles are $x^2 + y^2 - 2ax + c^2 = 0$ and $x^2 + y^2 - 2by + c^2 = 0$

Centres: $\quad C_1(a, 0) \quad C_2(0, b)$

Radii: $\quad r_1 = \sqrt{a^2 - c^2}, r_2 = \sqrt{b^2 - c^2}$

Since the two circles touch each other externally, therefore $C_1C_2 = r_1 + r_2$

$\Rightarrow \quad \sqrt{a^2 + b^2} = \sqrt{a^2 - c^2} + \sqrt{b^2 - c^2}$

$\Rightarrow a^2 + b^2 = a^2 - c^2 + b^2 - c^2 + 2\sqrt{a^2 - c^2}\sqrt{b^2 - c^2}$

$\Rightarrow \quad c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$

$\Rightarrow \quad a^2b^2 = c^2(a^2 + b^2) \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Example 2.53 **Find the equation of the smaller circle that touches the circle $x^2 + y^2 = 1$ and passes through the point (4, 3).**

Sol.

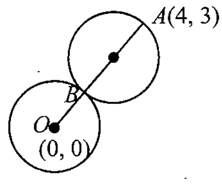


Fig. 2.49

For smallest circle OA will become common normal $OA = 5$ $\Rightarrow AB = 4$.

Equation of line OA is $y = \frac{3}{4}x$. Putting this value of 'y' in $x^2 + y^2 = 1$, we get

$$x^2 + \frac{9x^2}{16} = 1 \Rightarrow x = \pm\frac{4}{5}$$

$\Rightarrow B \equiv \left(\frac{4}{5}, \frac{3}{5}\right)$. Thus, required circle is $\left(x - \frac{4}{5}\right)(x - 4) + (y - 3)\left(y - \frac{3}{5}\right) = 0$

or $x^2 + y^2 - \frac{24}{5}x - \frac{18}{5}y + 5 = 0.$

Example 2.54 **Show that the circles $x^2 + y^2 - 10x + 4y - 20 = 0$ and $x^2 + y^2 + 14x - 6y + 22 = 0$ touches each other. Find the coordinates of the point of contact and the equation of the common tangent at the point of contact.**

Sol.

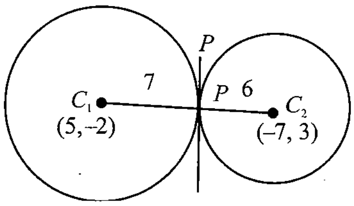


Fig. 2.50

$$C_1C_2 = \sqrt{[(5+7)^2 + (-2-3)^2]}$$
$$= 13 = r_1 + r_2$$

Hence, the two circles touch externally.

Coordinates of the point of contact:

If P is the point of contact of the two circle, then P will divide C_1C_2 internally in the ratio $r_1 : r_2$, i.e. 7 : 6.

$\therefore$ Coordinates of P are $\left(\frac{7.(-7) + 6.5}{7+6}, \frac{7.3 + 6.(-2)}{7+6}\right)$ or $(-19/13, 9/13)$

Equation of the common tangent:

Since the two circles touch each other, $S_1 - S_2 = 0$ is the common tangent at the point of contact which is $-24x + 10y - 42 = 0$ or $12x - 5y + 21 = 0$.

Example 2.55 **Find all the common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.**

Sol. Here for the first circle, centre is $C_1(1, 3)$ and radius $r_1 = 1$;

and for second circles, centre is $C_2(-3, 1)$ and radius $r_2 = 3$.

Thus, $C_1C_2 = 2\sqrt{5}$ and $r_1 + r_2 = 4 \Rightarrow C_1C_2 > r_1 + r_2, r_1 \neq r_2$; thus we have 4 common tangents.

To find direct common tangents:

The coordinates of the point P dividing line C_1C_2 in the ratio $r_1 : r_2$, i.e. 1 : 3 externally, are

$$\left(\frac{1.(-3) - 3.1}{1-3}, \frac{1.1 - 3.3}{1-3}\right) \text{ or } (3, 4)$$

Therefore, equation of any line through point $P(3, 4)$ is

$y - 4 = m_1(x - 3)$

or $m_1x - y + 4 - 3m_1 = 0.$ (i)

If Eq. (i) is tangent to first circle, then length of $\perp$ from centre $C_1(1, 3)$ of (i) $= r_1$ (radius)

$$\Rightarrow \frac{|m_1 - 3 + 4 - 3m_1|}{\sqrt{(m_1^2 + 1)}} = 1 \Rightarrow (1 - 2m_1)^2 = m_1^2 + 1$$

$$\Rightarrow 3m_1^2 - 4m_1 = 0 \Rightarrow m_1 = 0, 4/3$$

Substituting $m_1 = 0$ and 4/3 in Eq. (i), the equation of direct common tangents are $y = 4$ and $4x - 3y = 0$.

To find transverse common tangents:

The coordinates of the point Q dividing the line C_1C_2 in the ratio $r_1 : r_2$, i.e. 1 : 3 internally, are (0, 5/2).

$\therefore$ Equation of any line through $Q(0, 5/2)$ is $y - 5/2 = m_2(x - 0)$ or $m_2x - y + 5/2 = 0.$ (ii)

If Eq. (ii) is tangent to first circles, then length of $\perp$ from centre $C_1(1, 3)$ on Eq. (ii) $= r_1$ (radius)

$$\Rightarrow \frac{|m_2 - 3 + 5/2|}{\sqrt{(m_1^2 + 1)}} = 1 \Rightarrow (2m_2 - 1)^2 = 4(m_2^2 + 1)$$

$$\Rightarrow -4m_2 - 3 = 0$$

$\therefore$ $m_2 = -3/4$ and ∞, as coeff. of $m_2^2 = 0$.

Substituting $m_2 = -3/4$ and ∞ in Eq. (ii), the equations of transverse common tangents are

$$3x + 4y - 10 = 0 \text{ and } x = 0.$$

ANGLE OF INTERSECTION OF TWO CIRCLES

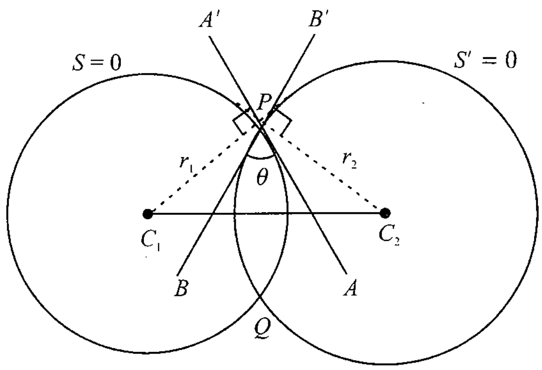


Fig. 2.51

Let the two circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ intersect each other at the point P and Q. The angle θ between two circles $S = 0$ and $S' = 0$ is defined as the angle between the tangents to the two circles at the point of intersection. θ must be taken acute angle.

C_1 and C_2 are the centres of circles $S = 0$ and $S' = 0$, then $C_1 \equiv (-g, -f)$ and $C_2 \equiv (-g_1, -f_1)$ and radii of circles $S = 0$ and $S' = 0$ are $r_1 = \sqrt{g^2 + f^2 - c}$ and $r_2 = \sqrt{g_1^2 + f_1^2 - c_1}$

Let $\quad d = |C_1C_2|$ = Distance between their centres

$$= \sqrt{(-g + g_1)^2 + (-f + f_1)^2}$$

$$= \sqrt{(g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1)}$$

$$C_1P \perp AA' \Rightarrow \angle C_1PB = \frac{\pi}{2} - \theta$$

$$C_2P \perp BB' \Rightarrow \angle C_2PA = \frac{\pi}{2} - \theta$$

$$\Rightarrow \quad \angle C_1PC_2 = \frac{\pi}{2} - \theta + \frac{\pi}{2} - \theta + \theta = \pi - \theta$$

Now, in ΔPC_1C_2, $\cos\theta = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right| \quad$ (i)

If the angle between the circles is 90°, i.e., $\theta = 90°$, then the circles are said to be **orthogonal circles** or we say that the cirlces cut each other **orthogonally.**

Then, from Eq. (i),

$$0 = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

$$\Rightarrow \quad r_1^2 + r_2^2 - d^2 = 0$$

$$\Rightarrow \quad r_1^2 + r_2^2 = d^2$$

$$\Rightarrow \quad g^2 + f^2 - c + g_1^2 + f_1^2 - c_1$$

$$= g^2 + f^2 + g_1^2 + f_1^2 - 2gg_1 - 2ff_1$$

$$\Rightarrow \quad 2gg_1 + 2ff_1 = c + c_1$$

Example 2.56 **Find the angle at which the circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ intersect.**

Sol. The angle of intersection of two circles is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - C_1C_2}{2r_1r_2},$$

where r_1, r_2 are radii of two circles and C_1C_2 is the distance between their centres.

Here, $\quad r_1 = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = r_2$ and $C_1C_2 = 1$

$$\Rightarrow \quad \cos\theta = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

Example 2.57 **If the circles $x^2 + y^2 + 2a'x + 2b'y + c' = 0$ and $2x^2 + 2y^2 + 2ax + 2by + c = 0$ intersect orthogonally, then prove that $aa' + bb' = c + \frac{c'}{2}$**

Sol. The given circles are $x^2 + y^2 + 2a'x + 2b'y + c' = 0$ and $x^2 + y^2 + ax + by + \frac{c}{2} = 0$

These two intersect orthogonally,

$$\therefore \quad 2\left(a'.\frac{a}{2} + b'.\frac{b}{2}\right) = c' + \frac{c}{2} \Rightarrow aa' + bb' = c' + \frac{c}{2}$$

Example 2.58 **A circle passes through the origin and has its centre on $y = x$. If it cuts $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, then find the equation of the circle.**

Sol. Let the required circle is $x^2 + y^2 + 2gx + 2fy + c = 0 \quad$ (i)

This passes through (0, 0), therefore $c = 0$

The centre $(-g, -f)$ of Eq. (i) lies on $y = x$, therefore $g = f$.

Since Eq. (i) cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, therefore

$$2(-2g - 3f) = c + 10 \Rightarrow -10g = 10$$

$$[\because g = f \text{ and } c = 0]$$

$$\Rightarrow \quad g = f = -1.$$

Hence, the required circle is $x^2 + y^2 - x - y = 0$

RADICAL AXIS

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

Consider, $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad$ (i)

and $\quad S' \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad$ (ii)

Let $P(x_1, y_1)$ be a point such that $|PA| = |PB|$

$$\Rightarrow \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2gy_1 + c}$$
$$= \sqrt{x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1}$$

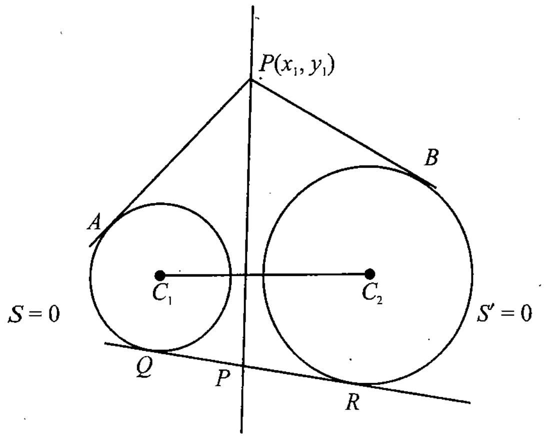


Fig. 2.52

On squaring, we get

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$$

$$\Rightarrow 2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$$

Therefore, locus of $P(x_1, y_1)$ is

$$2(g - g_1)x + 2(f - f_1)y + c - c_1 = 0$$

which is the required equation of radical axis of the given circles. Clearly, this is a straight line.

Properties of the Radical Axis

1. Radical axis is perpendicular to the line joining the centres of the given circles.

   Slope of $C_1C_2 = \dfrac{-f_1 + f}{-g_1 + g} = \dfrac{f - f_1}{g - g_1} = m_1$ (say)

   Slope of radical axis is $-\dfrac{(g - g_1)}{(f - f_1)} = m_2$ (say)

   $\therefore \; m_1m_2 = -1$

   Hence, C_1C_2 and radical axis are perpendicular to each other.

2. The radical axis bisects common tangents of two circles:

   Let QR be the common tangent. If it meets the radical axis P, then PQ and PR are two tangents to the circles. Hence, $PQ = PR$ since length of tangents are equal from any point on radical axis. Hence, radical axis bisects the common tangent QR.

Note:

Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

3. If two circles cut a third circle orthogonally, then the radical axis of the two circles will pass through the centre of the third circle, or the locus of the centre of a circle cutting two given circles orthogonally is the radical axis of the given two circles.

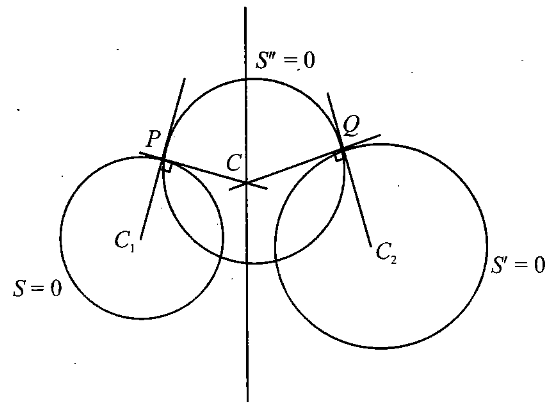


Fig. 2.53

Since circle $S'' = 0$ intersects the circle $S = 0$ and $S' = 0$ orthogonally

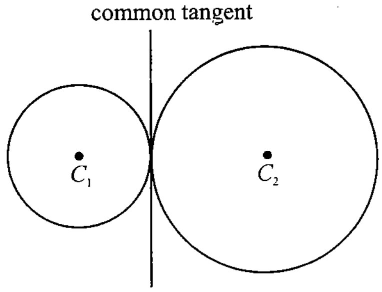
$C_1P \perp CP$ and $C_2Q \perp CQ$. (where C is centre of the circle $S'' = 0$)

But $CP = CQ =$ radius of the circle $S'' = 0$

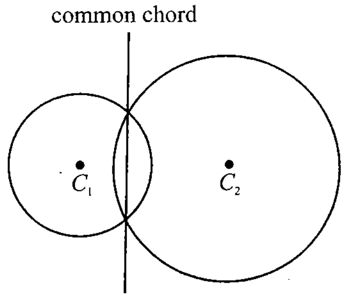
Hence, C lies on the radical axis of the circles $S = 0$ and $S' = 0$, as CP and CQ are also the length of tangents from C to the given circles.

4. The position of the radical axis of the two circles geometrically is shown below:

Radical axis is common tangent



Radical axis is common chord



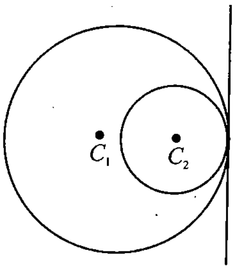


Fig. 2.54

Radical Centre

The radical axes of three circles, taken in pairs, meet in a point, which is called their radical centre. Let the three circles be

$$S_1 = 0 \quad \text{(i)}$$
$$S_2 = 0 \quad \text{(ii)}$$
$$S_3 = 0 \quad \text{(iii)}$$

Let OL, OM and ON be radical axes of the pair sets of circles $\{S_1 = 0, S_2 = 0\}$, $\{S_3 = 0, S_1 = 0\}$ and $\{S_2 = 0, S_3 = 0\}$ respectively.

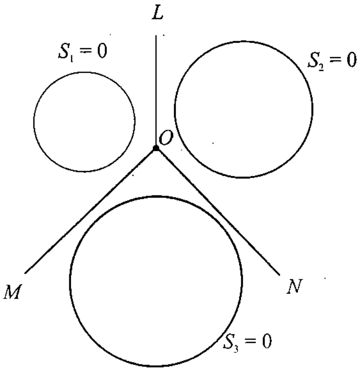


Fig. 2.55

Equations of OL, OM and ON are respectively

$$S_1 - S_2 = 0 \quad \text{(iv)}$$
$$S_3 - S_1 = 0 \quad \text{(v)}$$
$$S_2 - S_3 = 0 \quad \text{(vi)}$$

Let the straight lines (iv) and (v) i.e., OL and OM meet in O. The equation of any straight line passing through O is

$$(S_1 - S_2) + \lambda (S_3 - S_1) = 0$$

where λ is any constant.

For $\lambda = 1$, this equation becomes $S_2 - S_3 = 0$ which is, by (vi), equation of ON.

Thus the third radical axis also passes through the point where (iv) and (v) meet. In the above figure, O is the **radical centre**.

Properties of Radical Centre

1. Coordinates of radical centre can be found by solving the equations $S_1 = S_2 = S_3 = 0$.
2. The radical centre of three circles described on the sides of a triangle as diameters is the orthocentre of the triangle. Draw perpendicular from A on BC.

   $\therefore \quad \angle ADB = \angle ADC = \pi/2$

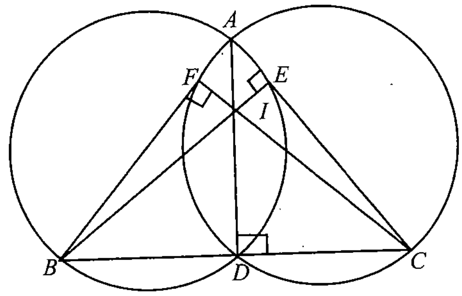


Fig. 2.56

   Therefore, the circles whose diameters are AB and AC passes through D and A. Hence, AD is their radical axis. Similarly, the radical axis of the circles on AB and BC as diameter is the perpendicular line from B on CA and radical axis of the circles on BC and CA as diameter is the perpendicular line from C on AB. Hence, the radical axis of three circles meet in a point. This point I is radical centre but here radical centre is the point of intersection of altitudes, i.e., AD, BE and CF. Hence, radical centre = orthocentre.
3. The radical centre of three given circles will be the centre of a fourth circle which cuts all the three circles orthogonally and the radius of the fourth circle is the length of tangent drawn from radical centre of the three given circles to any of these circles.

   Let the fourth circle be $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is centre of this circle and r be the radius. The centre of circle is the radical centre of the given circles and r is the length of tangent from (h, k) to any of the given three circles.

Example 2.59 **If the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, show that either $g = 3/4$ or $f = 2$.**

Sol. The radical axis of the given circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + (3/2)x + 4y + c = 0$,

is $(2g - 3/2)x + (2f - 4)y = 0$ or $(4g - 3)x + 4(f - 2)y = 0$ (i)

This radical axis (i) touches the circles $x^2 + y^2 + 2x + 2y + 1 = 0$, (ii)

if the length of $\perp$ from centre $(-1, -1)$ on the line (i) = radius of circle (ii),

i.e. $$\frac{(4g-3)(-1) + 4(f-2)(-1)}{\sqrt{[(4g-3)^2 + 16(f-2)^2]}} = \pm\sqrt{(1+1-1)}$$

$$\Rightarrow \quad [(4g-3)+4(f-2)]^2 = (4g-3)^2 + 16(f-2)^2$$

$$\Rightarrow \quad 8(4g-3)(f-2) = 0$$

$$\Rightarrow \quad g = 3/4 \text{ or } f = 2$$

Example 2.60 **The equation of the three circles are given $x^2+y^2=1$, $x^2+y^2-8x+15=0$, $x^2+y^2+10y+24=0$.**

Determine the coordinates of the point P such that the tangents drawn from it to the circles are equal in length.

Sol. We know that the point from which lengths of tangents are equal in length is radical centre of the given three circles. Now radical axis of the first two circles is

$(x^2+y^2-1)-(x^2+y^2-8x+15)=0$,

i.e., $x-2=0$, (i)

and radical axis of the second and third circles is

$(x^2+y^2-8x+15)-(x^2+y^2+10y+24)=0$,

i.e., $8x+10y+9=0$ (ii)

Solving Eqs. (i) and (ii), the coordinates of the radical centre, i.e. of point P are $P(2, -5/2)$.

Example 2.61 **The line $Ax+By+C=0$ cuts the circle $x^2+y^2+ax+by+c=0$ in P and Q. The line $A'x+B'x+C'=0$ cuts the circle $x^2+y^2+a'x+b'y+c'=0$ in R and S. If P, Q, R, S are concyclic, then show that**

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

Sol. P and Q are the points of intersection of the line $L_1 \equiv Ax + By + C = 0$ (i)

and the circles $\quad S_1 \equiv x^2+y^2+ax+by+c=0$, (ii)

R and S are the points of intersection of the line $L_2 \equiv A'x+B'y+C'=0$ (iii)

and the circle $\quad S_2 \equiv x^2+y^2+a'x+b'y+c'=0$ (iv)

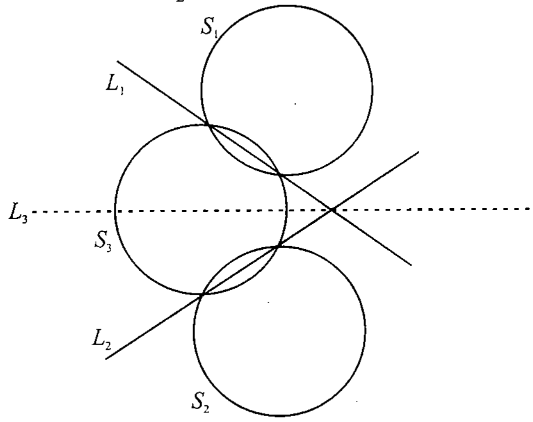


Fig. 2.57

Radical axis of circle $S_1=0$ and $S_2=0$ is $S_1-S_2=0$,

i.e. $L_3 \equiv (a-a')x+(b-b')y+(c-c')=0$ (v)

If P, Q, R, S are concyclic and $S_3=0$ is the equation of this circle through P, Q, R, S, line (i) is the radical axis of circles $S_1=0$ and $S_3=0$ and line (ii) is the radical axis of the circles $S_2=0$ and $S_3=0$.

Thus, the straight lines given by Eqs. (v), (i) and (iii) are the radical axes of circles $S_1=0$, $S_2=0$ and $S_3=0$ taken in pairs.

Since the radical axes of three circles taken in pairs are concurrent or parallel, $\therefore$ we have

$$\therefore \begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

Example 2.62 **Find the equation of a circle which cuts the three circles $x^2+y^2-3x-6y+14=0$, $x^2+y^2-x-4y+8=0$, $x^2+y^2+2x-6y+9=0$ orthogonally.**

Sol. The circle having centre at the radical centre of three given circles and radius equal to the length of the tangent from it to any one of three circles cuts all the three circles orthogonally. The given circles are

$$x^2+y^2-3x-6y+14=0 \quad \text{(i)}$$

$$x^2+y^2-x-4y+8=0 \quad \text{(ii)}$$

$$x^2+y^2+2x-6y+9=0 \quad \text{(iii)}$$

The radical axes (i), (ii) and (iii) are, respectively

$$x+y-3=0 \quad \text{(iv)}$$

and $\quad 3x-2y+1=0$ (v)

Solving Eqs. (iv) and (v), we get $x=1, y=2$

Thus, the coordinates of the radical centre are $(1, 2)$.

The length of the tangent from $(1, 2)$ to Eq. (i) is $r = \sqrt{1+4-3-12+14} = 2$

Hence, the required circle is $(x-1)^2+(y-2)^2=2^2$ or $x^2+y^2-2x-4y+1=0$.

COMMON CHORD OF TWO CIRCLES

The common chord joining the point of intersection of two given circles is called their common chord.

If $S=0$ and $S'=0$ be two intersecting circles, the equation of their common chord is

$$S - S' = 0$$

Let $\quad S \equiv x^2+y^2+2gx+2fy+c=0$

and $\quad S' \equiv x^2+y^2+2g'x+2f'y+c'=0$

be two circles intersecting at P and Q.

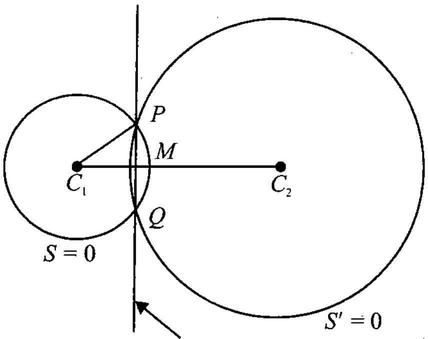


Fig. 2.58

Then PQ is their common chord.

$\therefore \quad S - S' = 0$

$\Rightarrow 2(g-g')x + 2(f-f')y + c - c' = 0$

is the common chord of two circles $S = 0$ and $S' = 0$.

Length of the Common Chord

$$PQ = 2(PM) = 2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$$

where C_1P = radius of the circle $S = 0$ and C_1M is the length of perpendicular form C_1 on common chord PQ.

Note:

1. *The length of common chord PQ of two circles is maximum when it is a diameter of the smaller circle.*
2. *If circle is described on the common chord as a diameter then centre of the circle passing through P and Q lie on the common chord of two circles i.e., $S - S' = 0$.*
3. *If the length of common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.*

Example 2.63 **If the tangents are drawn to the circle $x^2 + y^2 = 12$ at the point where it meets the circle $x^2 + y^2 - 5x + 3y - 2 = 0$, then find the point of intersection of these tangents.**

Sol.

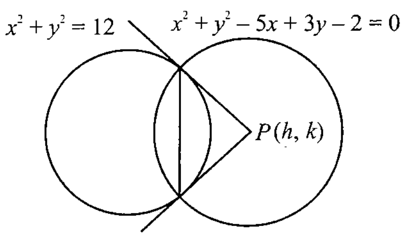


Fig. 2.59

Let (h, k) be the point of intersection of the tangents.

Then the chord of contact of tangents is the common chord of the circle $x^2 + y^2 = 12$

and $x^2 + y^2 - 5x + 3y - 2 = 0$

i.e. $\quad 5x - 3y - 10 = 0$

Also, the equation of the chord of contact w.r.t. P is is $hx + ky - 12 = 0$

Equations $hx + ky - 12 = 0$ and $5x - 3y - 10 = 0$ represent the same line, $\therefore \frac{h}{5} = \frac{k}{-3} = \frac{-12}{-10} \Rightarrow h = 6, k = \frac{-18}{5}$

Hence, the required point is $(6, -18/5)$.

Example 2.64 **Find the angle which the common chord of $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ subtends at the origin.**

Sol.

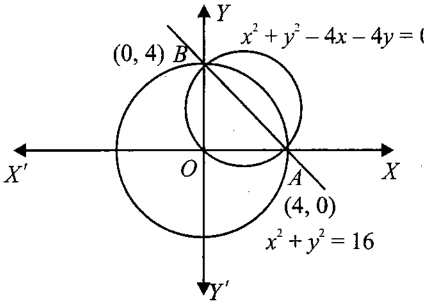


Fig. 2.60

The equation of the common chord of the circles $x^2 + y^2 - 4x - 4y = 0$

and $x^2 + y^2 = 16$ is $x + y = 4$ which meets $x^2 + y^2 = 16$ at $A(4, 0)$ and $B(-4, 0)$. Obviously $OA \perp OB$.

Hence, the common chord AB makes a right angle at the centre of the circle $x^2 + y^2 = 16$.

Example 2.65 **Find the length of the common chord of the circles $x^2 + y^2 + 2x + 6y = 0$ and $x^2 + y^2 - 4x - 2y - 6 = 0$**

Sol.

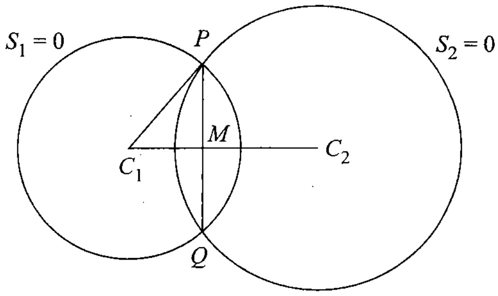


Fig. 2.61

The equation of common chord PQ of the circles

$$S_1 : x^2 + y^2 + 2x + 6y = 0$$

and $\quad S_2 : x^2 + y^2 - 4x - 2y - 6 = 0$

is $\quad S_1 - S_2 = 0$ or $6x + 8y + 6 = 0$ or $3x + 4y + 3 = 0$

centre of S_1 is $(-1, -3)$, radius $= \sqrt{1+9} = \sqrt{10}$

C_1M = length of the $\perp$ from $(-1, -3)$ to $3x + 4y + 3 = 0$

$$= \frac{|-3-12+3|}{\sqrt{9+16}} = \frac{12}{5}$$

Now $\quad PQ = 2PM = 2\sqrt{C_1P^2 - C_1M^2}$

$$= 2\sqrt{10 - \frac{144}{25}}$$

$$= 2\sqrt{\left(\frac{106}{25}\right)}$$

$$= \frac{2\sqrt{106}}{5}$$

Example 2.66 **If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisect the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then prove that**

$$2g'(g - g') + 2f'(f - f') = c - c'$$

Sol. It is given that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle

$x^2 + y^2 + 2g'x + 2f'y + c' = 0$, therefore the common chord of these two circles passes through the centre

$(-g', -f')$ of $x^2 + y^2 + 2g'x + 2f'y + c' = 0$

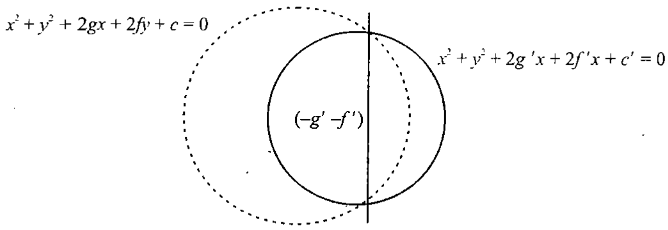


Fig. 2.62

The equation of the common chord of the two given circles is $2x(g - g') + 2y(f - f') + c - c' = 0$.

This passes through $(-g', -f')$

$\therefore -2g'(g - g') - 2f'(f - f') + c - c' = 0$

$\Rightarrow 2g'(g - g') + 2f'(f - f') = c - c'$

Concept Application Exercise 2.4

1. The circles $x^2 + y^2 - 12x - 12y = 0$ and $x^2 + y^2 + 6x + 6y = 0$

   a. touch each other externally

   b. touch each other internally

   c. intersect in two points

   d. none of these

2. If the circle $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct point P and Q, then find the values of a for which the line $5x + by - a = 0$ passes through P and Q.

3. Which of the following is a point on the common chord of the circle $x^2 + y^2 + 2x - 3y + 6 = 0$ and $x^2 + y^2 + x - 8y - 31 = 0$

   a. $(1, -2)$ **b.** $(1, 4)$

   c. $(1, 2)$ **d.** $(1, -4)$

4. Consider the circles $x^2 + (y - 1)^2 = 9$, $(x - 1)^2 + y^2 = 25$. They are such that

   a. These circles touch each other

   b. One of these circles lies entirely inside the other

   c. Each of these circles lies outside the other

   d. They intersect in two points

5. If the circles of same radius a and centres at (2, 3) and (5, 6) cut orthogonally, then find a.

6. If the two circles $2x^2 + 2y^2 - 3x + 6y + k = 0$ and $x^2 + y^2 - 4x + 10y + 16 = 0$ cut orthogonally, then find the value of k.

7. Find the condition that the circle $(x - 3)^2 + (y - 4)^2 = r^2$ lies entirely within the circle $x^2 + y^2 = R^2$.

8. Find the radical centre of the circles $x^2 + y^2 + 4x + 6y = 19$, $x^2 + y^2 = 9$ and $x^2 + y^2 - 2x - 2y = 5$.

9. Find the equation of the circle which intersects circles $x^2 + y^2 + x + 2y + 3 = 0$, $x^2 + y^2 + 2x + 4y + 5 = 0$ and $x^2 + y^2 - 7y - 8y - 9 = 0$ at right angle.

10. Two circles 'C_2' and 'C_1' intersect in such a way that their common chord is of maximum length. Centre of C_1 is (1, 2) and it's radius is 3 units. Radius of C_2 is 5 units. If slope of common chord is $\frac{3}{4}$, then find the centre of C_2.

11. The equation of a circle is $x^2 + y^2 = 4$. Find the centre of the smallest circle touching the circle and the line $x + y = 5\sqrt{2}$.

12. Consider four circles $(x \pm 1)^2 + (y \pm 1)^2 = 1$. Find the equation of smaller circle touching these four circles.

13. Find the equation of the circle whose radius is 3 and which touches internally the circle

    $x^2 + y^2 - 4x - 6y - 12 = 0$ at the point $(-1, -1)$.

14. Find the number of common tangents that can be drawn to the circles $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$.

15. Two fixed circles with radii r_1 and $r_2 (r_1 > r_2)$, respectively, touch each other externally. Then

identify the locus of the point of intersection of their direct common tangents.

16. Two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents ($a > b \geq 2$), then prove that $\theta = 2\sin^{-1}\left(\frac{a-b}{a+b}\right)$.

17. If the radius of the circle $(x-1)^2 + (y-2)^2 = 1$ and $(x-7)^2 + (y-10)^2 = 4$ are increasing uniformly w.r.t. time as 0.3 and 0.4 unit/sec, then at what value of t will they touch each other?

FAMILY OF CIRCLES

1. The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as
$S + \lambda S' = 0$ (where λ is a parameter, $\lambda \neq -1$)

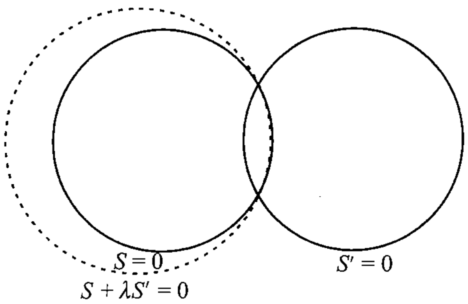


Fig. 2.63

2. The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as $S + \lambda L = 0$
(where λ is a parameter)

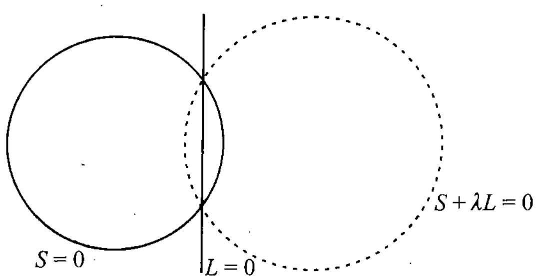


Fig. 2.64

3. The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is $S + \lambda L = 0$
(where λ is a parameter)

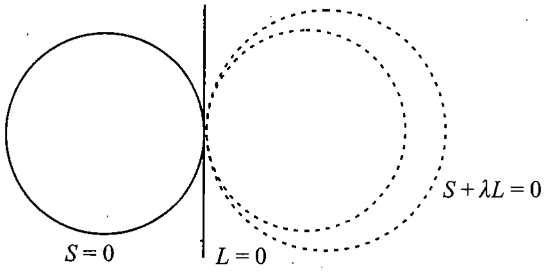


Fig. 2.65

4. The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$S + \lambda L = 0$

(where λ is a parameter)

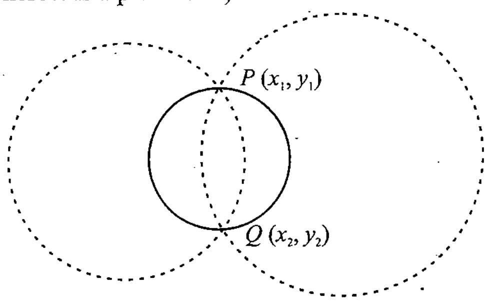
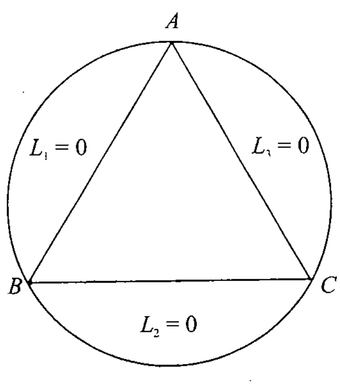


Fig. 2.66

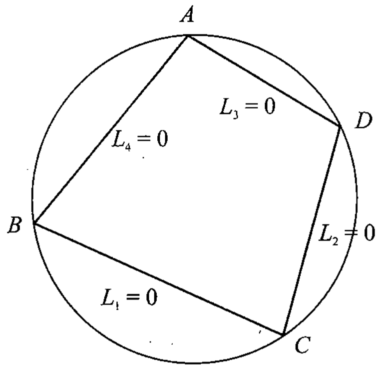
Here $S = 0$ is equation of circle with P and Q as end point of diameter and $L = 0$ is line through points P and Q.

5. The equation of family of circles which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is
$(x - x_1)^2 + (y - y_1)^2 + \lambda\{(y - y_1) - m(x - x_1)\} = 0$ and if m is infinite, the family of cirlces is
$(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$
(where λ is a parameter).
Here $(x - x_1)^2 + (y - y_1)^2 = 0$ is point circle at point (x_1, y_1)

6.



(a)



(b)

Fig. 2.67

Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ and $L_3 = 0$ is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of $xy = 0$ and coefficient of x^2 = coefficient of y^2.

Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ is given by $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

Example 2.67 **If the circle $x^2 + y^2 + 2x + 3y + 1 = 0$ cuts $x^2 + y^2 + 4x + 3y + 2 = 0$ in A and B, then find the equation of the circle on AB as diameter.**

Sol. The equation of the common chord AB of the two circles is $2x + 1 = 0$. [Using $S_1 - S_2 = 0$]

The equation of the required circle is $(x^2 + y^2 + 2x + 3y + 1) + \lambda(2x + 1) = 0$. [Using $S_1 + \lambda(S_2 - S_1) = 0$]

$\Rightarrow x^2 + y^2 + 2x(\lambda + 1) + 3y + \lambda + 1 = 0$

Since AB is a diameter of this circle, therefore centre lies on it.

So, $-2\lambda - 2 + 1 = 0 \Rightarrow \lambda = -1/2$

Thus, the required circle is $x^2 + y^2 + x + 3y + 1/2 = 0$

or $2x^2 + 2y^2 + 2x + 6y + 1 = 0$

Example 2.68 **Show that the equation of the circle passing through (1, 1) and the points of intersection of the circles $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is $4x^2 + 4y^2 + 30x - 13y - 25 = 0$.**

Sol. Equation of the circle passing through the points of intersection of the given circle is

$$(x^2 + y^2 + 13x - 3y) + \lambda(2x^2 + 2y^2 + 4x - 7y - 25) = 0 \quad \text{(i)}$$

If this circle passes through the point (1, 1), then

$(1 + 1 + 13 - 3) + \lambda(2 + 2 + 4 - 7 - 25) = 0$

$\Rightarrow \quad \lambda = 1/2$

Substituting $\lambda = 1/2$ in Eq. (i), the equation of the required circle is $4x^2 + 4y^2 + 30x - 13y - 25 = 0$.

Example 2.69 **Find the equation of the smallest circle passing through the intersection of the line $x + y = 1$ and the circle $x^2 + y^2 = 9$.**

Sol. Any circle passing through the points of intersection of the given line and circle has the equation

$x^2 + y^2 - 9 + \lambda(x + y - 1) = 0$. Its centre $= \left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$

The circle is the smallest if $\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$ is on the chord $x + y = 1$.

$\Rightarrow \quad -\frac{\lambda}{2} - \frac{\lambda}{2} = 1 \Rightarrow \lambda = -1$

Putting this value for λ, the equation of the smallest circle is $x^2 + y^2 - 9 - (x + y - 1) = 0$.

Example 2.70 **C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0), respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Find the equation of the common tangent to C_1 and C_3 which does not pass through C_2.**

Sol.

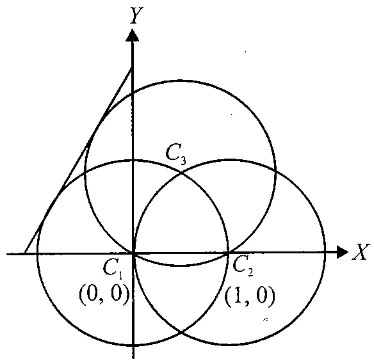


Fig. 2.68

Equation of any circle through (0, 0) and (1, 0) is

$$(x-1)(x-0) + (y-0)(y-0) + \lambda \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad x^2 + y^2 - x + \lambda y = 0$

If it represents C_3, its radius = 1

$\Rightarrow \quad 1 = (1/4) + (\lambda^2/4)$

$\Rightarrow \quad \lambda = \pm\sqrt{3}$

As the centre of C_3, lies above the x-axis, we take $\lambda = -\sqrt{3}$ and thus an equation of C_3

is $\quad x^2 + y^2 - x - \sqrt{3}y = 0$.

Since C_1 and C_3 intersect and are of unit radius, their common tangents are parallel to the line joining their centres (0, 0) and $(1/2, \sqrt{3}/2)$.

So, let the equation of a common tangent be $\sqrt{3}x - y + k = 0$.

It will touch C_1, if $\frac{|k|}{\sqrt{3+1}} = 1 \Rightarrow k = \pm 2$

From the figure, we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation is $\sqrt{3}x - y + 2 = 0$.

Example 2.71 **Find the radius of the smallest circle which touches the straight-line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$. Compute upto one place of decimal only.**

Sol. Equation of the circle touching the line

$3x - y - 6 = 0$ at $(1, -3)$ is given by $(x - 1)^2 + (y + 3)^2 + \lambda(3x - y - 6) = 0$

i.e. $x^2 + y^2 + (3\lambda - 2)x + (6 - \lambda)y + 10 - 6\lambda = 0$ (i)

Equation (i) touches $y = x$,

$\therefore\ 2x^2 + (2\lambda + 4)x + 10 - 6\lambda = 0$

i.e. $x^2 + (\lambda + 2)x + 5 - 3\lambda = 0$ (ii)

has equal roots

$\Rightarrow (\lambda + 2)^2 - 4(5 - 3\lambda) = \lambda^2 + 16\lambda - 16 = 0$

$\Rightarrow \lambda = \frac{1}{2}[-16 \pm \sqrt{(256 + 64)}] = -8 \pm \sqrt{(80)}$

Now (radius)$^2 = R^2 = \frac{1}{4}[(3\lambda - 2)^2 + (6 - \lambda)^2 - (10 - 6\lambda)4]$ $= \frac{1}{4}[10\lambda^2]$

$\Rightarrow R = \frac{\sqrt{10}\lambda}{2} = \frac{\sqrt{10}}{2}(-8 \pm 4\sqrt{5}) = |-4\sqrt{10} \pm 10\sqrt{2}|$

$\Rightarrow$ radius of smaller circle $= |-4\sqrt{10} + 10\sqrt{2}| = 1.5$ **approx.**

Example 2.72 **A variable circle which always touches the line $x + y - 2 = 0$ at $(1, 1)$ cuts the circle $x^2 + y^2 + 4x + 5y - 6 = 0$. Prove that all the common chords of intersection pass through a fixed point. Find that points.**

Sol. Any circle which touches the line $x + y - 2 = 0$ at $(1, 1)$ will be of the form

$$(x - 1)^2 + (y - 1)^2 + \lambda(x + y - 2) = 0$$

or $x^2 + y^2 + (\lambda - 2)x + (\lambda - 2)y + 2 - 2\lambda = 0$

The common chord of this circle and $x^2 + y^2 + 4x + 5y - 6 = 0$ will be

$(\lambda - 6)x + (\lambda - 7)y + 8 - 2\lambda = 0$ or $(-6x - 7y + 8) + \lambda(x + y - 2) = 0$

which is a family of lines, each member of which will be passing through a fixed point, which is the point of intersection of the lines $-6x - 7y + 8 = 0$ and $x + y - 2 = 0$ which is $(6, -4)$.

Example 2.73 **Let S_1 be a circle passing through $A(0, 1)$, $B(-2, 2)$ and S_2 is a circle of radius $\sqrt{10}$ units such that AB is common chord of S_1 and S_2. Find the equation of S_2.**

Sol. Equation of line AB is

$$y - 2 = \frac{2-1}{-2-0}(x + 2) = -\frac{1}{2}(x + 2)$$

$\Rightarrow x + 2y - 2 = 0$ (i)

Equation of circle whose diagonally opposite points are A and B:

$$(x - 0)(x + 2) + (y - 1)(y - 2) = 0$$

$\Rightarrow$ $x^2 + y^2 + 2x - 3y + 2 = 0$ (ii)

Family of circles passing through the points of intersection of Eqs. (i) and (ii)

$$x^2 + y^2 + 2x - 3y + 2 + \lambda(x + 2y - 2) = 0$$

$\Rightarrow$ $x^2 + y^2 + (2 + \lambda)x + (2\lambda - 3)y + 2 - 2\lambda = 0$ (iii)

Equation (iii), represents a circle of radius $\sqrt{10}$ units

$\Rightarrow \sqrt{\left(-\frac{2+\lambda}{2}\right)^2 + \left(-\frac{2\lambda - 3}{2}\right)^2 - 2 + 2\lambda} = \sqrt{10}$

$\Rightarrow (4 + 4\lambda + \lambda^2) + (4\lambda^2 + 9 - 12\lambda) + 8\lambda - 8 = 40$ $\Rightarrow \lambda = \pm\sqrt{7}$

Hence, required circles are

$$x^2 + y^2 + 2x - 3y + 2 \pm \sqrt{7}(x + 2y - 2) = 0$$

There are two such circles possible.

Example 2.74 **If C_1, C_2 and C_3 belong to a family of circles through the points (x_1, y_1) and (x_2, y_2), prove that the ratio of the lengths of the tangent from any point on C_1 to the circles C_2 and C_3 is constant.**

Sol. Equations of the circles through (x_1, y_1) and (x_2, y_2) are

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda_r \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ $(r = 1, 2, 3)$

Let (h, k) be a point on C_1.

$\Rightarrow$ $\phi(h, k) + \lambda_1 \psi(h, k) = 0$

where $\phi(h, k) = (h - x_1)(h - x_2) + (k - y_1)(k - y_2)$

and $\psi(h, k) = \begin{vmatrix} h & k & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$

Let T_2 be the length of the tangent from (h, k) to C_2 and T_3 be the length of the tangent from (h, k) to C_3.

$\Rightarrow$ $T_2 = \sqrt{\phi(h, k) + \lambda_2 \psi(h, k)}$, $T_3 = \sqrt{\phi(h, k) + \lambda_3 \psi(h, k)}$

$\Rightarrow \frac{T_2}{T_3} = \frac{\sqrt{\phi(h, k) + \lambda_2 \psi(h, k)}}{\sqrt{\phi(h, k) + \lambda_3 \psi(h, k)}} = \frac{\sqrt{(\lambda_2 - \lambda_1)\psi(h, k)}}{\sqrt{(\lambda_3 - \lambda_1)\psi(h, k)}}$

$= \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}}$ which is independent of the choice of (h, k) and hence a constant.

PROBLEMS BASED ON LOCUS

Example 2.75 **If a line segment $AM = a$ moves in the plane XOY remaining parallel to OX so that the left end point A slides along the circle $x^2 + y^2 = a^2$, then find the locus of M.**

Sol.

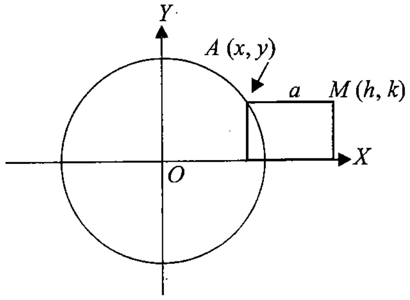


Fig. 2.69

Let the coordinates of A be (x, y) and M be (h, k)

Since AM is parallel to OX,

$$h = x + a \text{ and } k = y$$

$$\Rightarrow \quad x = h - a \text{ and } y = k$$

As $A(x, y)$ lies on the circle

$$x^2 + y^2 = a^2, \text{ we have}$$

$$(h - a)^2 + k^2 = a^2 \Rightarrow h^2 - 2ah + k^2 = 0$$

$\Rightarrow$ Locus of $M(h, k)$ is

$$x^2 + y^2 = 2ax.$$

Example 2.76 **The tangents to $x^2 + y^2 = a^2$ having inclinations α and β intersect at P. If $\cot\alpha + \cot\beta = 0$, then find the locus of P.**

Sol. Let the coordinates of P be (h, k). Let the equation of a tangent from $P(h, k)$ to the circle $x^2 + y^2 = a^2$ be

$$y = mx + a\sqrt{1 + m^2}..$$

Since $P(h, k)$ lies on $\quad y = mx + a\sqrt{1 + m^2}.$

$$\therefore \quad k = mh + a\sqrt{1 + m^2}$$

$$\Rightarrow \quad (k - mh)^2 = a\sqrt{(1 + m^2)}$$

$$\Rightarrow m^2(h^2 - a^2) - 2mkh + k^2 - a^2 = 0$$

This is a quadratic in m. Let the two roots be m_1 and m_2. Then, $m_1 + m_2 = \dfrac{2hk}{h^2 - a^2}$

But $\tan\alpha = m_1$, $\tan\beta = m_2$ and it is given that $\cot\alpha + \cot\beta = 0$

$$\therefore \quad \frac{1}{m_1} + \frac{1}{m_2} = 0 \Rightarrow m_1 + m_2 = 0$$

$$\Rightarrow \quad \frac{2hk}{h^2 - a^2} = 0 \Rightarrow hk = 0$$

Hence, the locus of (h, k) is

$$xy = 0.$$

Example 2.77 **Find the locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ at points whose parametric angles differ by $\pi/3$.**

Sol.

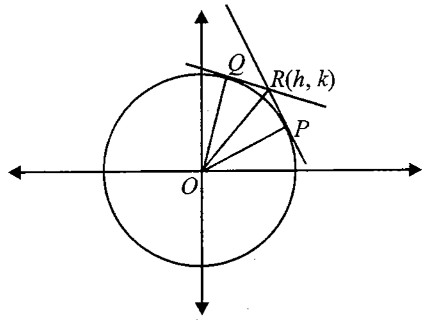


Fig. 2.70

Let the parametric angles of two points on the circle

$$x^2 + y^2 = a^2 \text{ be } \theta \text{ and } \pi/3 + \theta.$$

Then, the two points are $P(a\cos\theta, a\sin\theta)$ and $Q(a\cos(\pi/3 + \theta), a\sin(\pi/3 + \theta))$

In the figure $\quad \angle POQ = \pi/3$ and $\angle POR = \pi/6$.

In ΔOPR, $\quad OP = OR\cos 30°$

$$\Rightarrow \quad a = \sqrt{h^2 + k^2}\,\frac{\sqrt{3}}{2}$$

$\Rightarrow$ Locus of $R(h, k)$ is

$$3(x^2 + y^2) = 4a^2.$$

Example 2.78 **Find the locus of the centre of a circle touching the circle $x^2 + y^2 - 4y - 2x = 4$ internally and tangents on which from (1, 2) is making a 60° angle with each other.**

Sol.

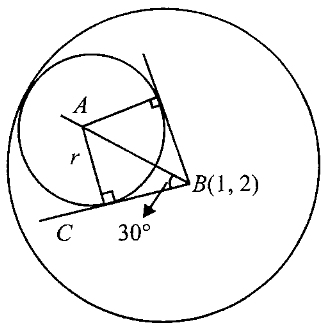


Fig. 2.71

Let r and R be radius of required and given circle respectively and let centre is (h, k).

By given condition $\sqrt{(h-1)^2+(k-2)^2} = R - r$

Now, $\frac{r}{AB} = \sin 30°$

$\Rightarrow \quad r = AB \sin 30° = (R-r)\frac{1}{2}$ $\quad (AB = R - r)$

$\Rightarrow \quad \sqrt{(h-1)^2+(k-2)^2} = R - \frac{R}{3} = \frac{2R}{3}$

Now, $R = 3$

$\Rightarrow \quad \sqrt{(h-1)^2+(k-2)^2} = 2$

$\Rightarrow$ Locus is

$$(x-1)^2+(y-2)^2 = 4$$

Example 2.79 **Two rods of lengths a and b slide along the x-axis and y-axis, respectively, in such a manner that their ends are concyclic. Find the locus of the centre of the circle passing through the end points.**

Sol.

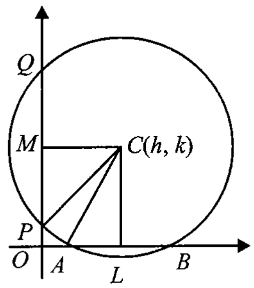


Fig. 2.72

Let $C(h, k)$ be the centre of the circle passing through the end points of the rod AB and PQ of lengths a and b, respectively,

CL and CM be perpendiculars from C on AB and PQ, respectively.

Then $\quad AL = (1/2)\,AB = a/2,$

$PM = (1/2)\,PQ = b/2$

and $\quad CA = CP$ (radii of the same circle)

$\Rightarrow \quad k^2 + \frac{a^2}{4} = h^2 + \frac{b^2}{4}$

$\Rightarrow \quad 4(h^2 - k^2) = a^2 - b^2$

So that locus of (h, k) is $4(x^2 - y^2) = a^2 - b^2$.

Example 2.80 **A circle with centre at the origin and radius equal to a meets the axis of x at A and B. $P(\alpha)$ and $Q(\beta)$ are two points on the circle so that $\alpha - \beta = 2\gamma$, where γ is a constant. Find the locus of the point of intersection of AP and BQ.**

Sol.

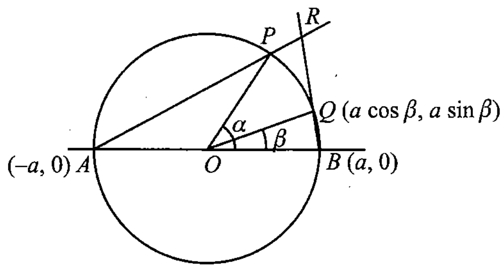


Fig. 2.73

Coordinates of A are $(-a, 0)$ and of P are $(a\cos\alpha, a\sin\alpha)$

$\therefore$ Equation of AP is $\quad y = \frac{a\sin\alpha}{a(\cos\alpha+1)}(x+a)$

or $\quad y = \tan(\alpha/2)(x+a) \quad$ (i)

Similarly equation of BQ is

$$y = \frac{a\sin\beta}{a(\cos\beta-1)}(x-a)$$

or $\quad y = -\cot(\beta/2)(x-a) \quad$ (ii)

We now eliminate α, β from Eqs. (i) and (ii)

$\therefore \quad \tan(\alpha/2) = \frac{y}{a+x},\ \tan(\beta/2) = \frac{a-x}{y}$

Now $\quad \alpha - \beta = 2\gamma$

$$\Rightarrow \quad \tan\gamma = \frac{\tan(\alpha/2)-\tan(\beta/2)}{1+\tan(\alpha/2)\tan(\beta/2)}$$

$$= \frac{\frac{y}{a+x} - \frac{a-x}{y}}{1+\frac{y}{a+x}\cdot\frac{a-x}{y}}$$

$$\Rightarrow \quad \tan\gamma = \frac{y^2-(a^2-x^2)}{(a+x)y+(a-x)y}$$

$$= \frac{x^2+y^2-a^2}{2ay}$$

$\Rightarrow \quad x^2 + y^2 - 2ay\tan\gamma = a^2.$

Example 2.81 **Find the locus of the midpoint of the chords of the circle $x^2 + y^2 = a^2$ which subtend a right angle at the point $(c, 0)$.**

Sol.

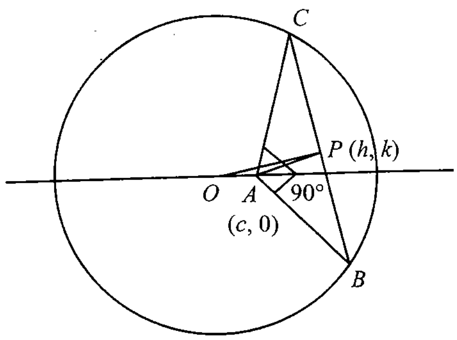


Fig. 2.74

Let $P(h, k)$ be the midpoint of a chord BC which subtends a right angle at $A(c, 0)$.

Then, clearly

$$AP = PC = PB = \sqrt{[(h-c)^2 + k^2]} \quad \text{(i)}$$

Also $$PC = \sqrt{(a^2 - OP^2)}$$

$$= \sqrt{[a^2 - (h^2 + k^2)]} \quad \text{(ii)}$$

From Eq. (i) and (ii), generalizing (h, k), we get the locus of P as

$$(x-c)^2 + y^2 = a^2 - (x^2 + y^2)$$

i.e. $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0.$

Example 2.82 **A variable circle passes through the point $A(a, b)$ and touches the x-axis. Show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.**

Sol.

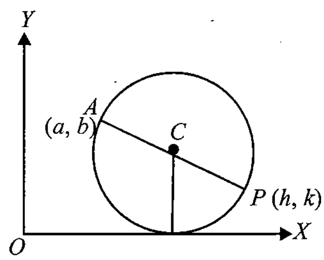


Fig. 2.75

$|y\text{-coord. of centre}| = \text{radius}$

$$\Rightarrow \quad \left(\frac{k+b}{2}\right)^2 = \frac{(h-a)^2 + (k-b)^2}{4}$$

$\Rightarrow$ Locus of $P(h, k)$ is

$$(x-a)^2 = 4by.$$

Concept Application Exercise 2.5

1. Find the locus of the midpoint of the chord of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$, which makes an angle of 120° at the centre
2. A tangent is drawn to each of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$. Show that if the two tangents are mutually perpendicular, the locus of their point of intersection is a circle concentric with the given circles.
3. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a variable triangle OAB. Sides OA and OB lie along the x-and y-axis, respectively, where 'O' is the origin. Find the locus of the midpoint of side AB.
4. A point moves so that the sum of the squares of the perpendiculars let fall from it on the sides of an equilateral triangle is constant, prove that its locus is a circle.
5. 'P' is the variable point on the circle with centre at C. CA and CB are perpendicular from C on x-axis and y-axis respectively. Show that the locus of the centroid of triangle PAB is a circle with centre at the centroid of triangle CAB and radius equal to the one third of the radius of the given circle.
6. Tangents are drawn to the circle $x^2 + y^2 = a^2$ from two points on the axis of x, equidistant from the point $(k, 0)$. Show that the locus of their intersection is $ky^2 = a^2(k - x)$.
7. A straight line moves so that the product of length of the perpendiculars on it from two fixed points is constant. Prove that the locus of the feet of the perpendiculars from each of these points upon the straight-line is a unique circle.

EXERCISES

Subjective Type

Solutions on page 2.56

1. A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at M and N, respectively. If the distance from C to the line-segment MN is equal to 5 units, find the area of the rectangle $ABCD$.
2. Let ABC be a triangle right angled at A and S be its circumcircle. Let S_1 be the circle touching the lines AB, AC and the circle S internally. Further, let S_2 be the circle touching the lines AB and AC produce and the circle S externally. If r_1 and r_2 be the radii of the circles S_1 and S_2, respectively, show that $r_1 r_2 = 4$ area(ΔABC).
3. Find the range of parameter 'a' for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.
4. Find the locus of the centres of the circles $x^2 + y^2 - 2ax - 2by + 2 = 0$, where '$a$' and '$b$' are parameters, if the tangents from the origin to each of the circles are orthogonal.
5. Three concentric circles, of which biggest is $x^2 + y^2 = 1$, have their radii in A.P. If the line $y = x + 1$ cuts all the circles in real and distinct points, then find the interval in which the common difference of A.P. will lie.

6. Let $A = (-1, 0)$, $B = (3, 0)$ and PQ be any line passing through (4, 1) having slope m. Find the range of 'm' for which there exist two points on PQ at which AB subtends a right angle.

7. The equation of radical axis of two circles is $x + y = 1$. One of the circles has the ends of a diameter at the points $(1, -3)$ and $(4, 1)$ and the other passes through the point $(1, 2)$. Find the equations of these circles.

8. S is a circle having centre at $(0, a)$ and radius $b(b < a)$. A variable circle centred at $(\alpha, 0)$ and touching circle S, meets the X-axis at M and N. Find a point P on the Y-axis, such that $\angle MPN$ is a constant for any choice of α.

9. $S(x, y) = 0$ represents a circle. The equation $S(x, 2) = 0$ gives two identical solutions $x = 1$ and the equation $S(1, y) = 0$ gives two solutions $y = 0, 2$. Find the equation of the circle.

10. Find the equation of a family of circles touching the lines $x^2 - y^2 + 2y - 1 = 0$.

11. A and B are two points in xy-plane, which are $2\sqrt{2}$ unit distance apart and subtend an angle of 90° at the point $C(1, 2)$ on the line $x - y + 1 = 0$, which is larger than any angle subtended by the line segment AB at any other point on the line. Find the equation(s) of the circle through the points A, B and C.

12. From the variable point A on a circle $x^2 + y^2 = 2a^2$, two tangents are drawn to the circle $x^2 + y^2 = a^2$ which meet the curve at B and C. Find the locus of the circumcentre of ΔABC.

13. Find the circle of minimum radius which passes through the point (4, 3) and touches the circle $x^2 + y^2 = 4$ externally.

14. Two variable chords AB and BC of a circle $x^2 + y^2 = r^2$ are such that $AB = BC = r$. Find the locus of point of intersection of tangents at 'A' and 'C'.

15. If $3x + y = 0$ is a tangent to a circle whose centre is $(2, -1)$, then find the equation of the other tangent to the circle from the origin.

16. Find the length of the chord of contact with respect to the point on the director circle of circle $x^2 + y^2 + 2ax - 2by + a^2 - b^2 = 0$.

17. A circle $x^2 + y^2 + 4x - 2\sqrt{2}\,y + c = 0$ is the director circle of circle S_1, and S_2, is the director circle of circle S_1 and so on. If the sum of radii of all these circles is 2, then find the value of c.

18. Consider three circles C_1, C_2 and C_3 such that C_2 is the director circle of C_1 and C_3 is the director circle of C_2. Tangents to C_1 from any point on C_3 intersect C_2 at P and Q. Find the angle between the tangents to C_2 at P and Q. Also identify the locus of the point of intersection of tangents at P and Q.

19. From a point P on the normal $y = x + c$ of the circle $x^2 + y^2 - 2x - 4y + 5 - \lambda^2 = 0$, two tangents are drawn to the same circle touching it at point B and C. If area of quadrilateral $OBPC$ (where O is the centre of the circle) is 36 sq. units. Find the possible values of λ, it is given that point P is at a distance $|\lambda|(\sqrt{2} - 1)$ from the circle.

20. Find the centre of the smallest circle which cut circles $x^2 + y^2 = 1$ and $x^2 + y^2 + 8x + 8y - 33 = 0$ orthogonally.

21. Perpendiculars are drawn, respectively, from the points P and Q to the chords of contact of the points Q and P with respect to a circle. Prove that the ratio of the lengths of perpendiculars is equal to the ratio of the distances of the points P and Q from the centre of the circles.

22. Find the number of such points $(a + 1, \sqrt{3}\,a)$, where $a \in Z$, lying inside the region bounded by the circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 - 2x - 15 = 0$.

23. If eight distinct points can be found on the curve $|x| + |y| = 1$ such that from each point two mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$, then find the range of a.

24. A circle of radius 5 units has diameter along the angle bisector of the lines $x + y = 2$ and $x - y = 2$. If chord of contact from origin makes an angle of 45° with the positive direction of x-axis, find the equation of the circle.

25. Let AB be the chord of contact of the point $(5, -5)$ w.r.t. the circle $x^2 + y^2 = 5$, then find the locus of the orthocentre of the triangle PAB, where P be any point moving on the circle.

26. Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$. AB be the chord of contact of this point w.r.t. the circle $x^2 + y^2 - 2x = 0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle.

27. AB is a diameter of a circle, CD is a chord parallel to AB and $2CD = AB$. The tangent at B meets the line AC produced at E. Prove that $AE = 2AB$.

28. Two parallel tangents to a given circle are cut by a third tangent in the points R and Q. Show that the lines from R and Q to the centre of the circle are mutually perpendicular.

29. A circle of radius 1 unit touches positive x-axis and positive y-axis at A and B, respectively. A variable line passing through origin intersects the circle in two points D and E. If the area of the triangle DEB is maximum when the slope of the line is m, then find the value of m^{-2}.

Objective Type

Solutions on page 2.62

Each question has four choices a, b, c and d, out of which *only one* answer is correct.

1. The number of rational point(s) (a point (a, b) is called rational, if a and b both are rational numbers) on the circumference of a circle having centre (π, e) is

   a. at most one **b.** at least two
   c. exactly two **d.** infinite

2. If the equation of any two diagonals of a regular pentagon belongs to family of lines $(1 + 2\lambda)y - (2 + \lambda)x + 1 - \lambda = 0$ and their lengths are sin 36°, then locus of centre of circle circumscribing the given pentagon (the triangles formed by these diagonals with sides of pentagon have no side common) is

a. $x^2 + y^2 - 2x - 2y + 1 + \sin^2 72° = 0$

b. $x^2 + y^2 - 2x - 2y + \cos^2 72° = 0$

c. $x^2 + y^2 - 2x - 2y + 1 + \cos^2 72° = 0$

d. $x^2 + y^2 - 2x - 2y + \sin^2 72° = 0$

3. If OA and OB are equal perpendicular chord of the circles $x^2 + y^2 - 2x + 4y = 0$, then equations of OA and OB are where O is origin.

a. $3x + y = 0$ and $3x - y = 0$

b. $3x + y = 0$ or $3y - x = 0$

c. $x + 3y = 0$ and $y - 3x = 0$

d. $x + y = 0$ or $x - y = 0$

4. Equation of chord of the circle $x^2 + y^2 - 3x - 4y - 4 = 0$, which passes through the origin such that the origin divides it in the ratio 4 : 1, is

a. $x = 0$ **b.** $24x + 7y = 0$

c. $7x + 24y = 0$ **d.** $7x - 24y = 0$

5. The line $2x - y + 1 = 0$ is tangent to the circle at the point (2, 5) and the centre of the circles lies on $x - 2y = 4$. The radius of the circle is

a. $3\sqrt{5}$ **b.** $5\sqrt{3}$ **c.** $2\sqrt{5}$ **d.** $5\sqrt{2}$

6. In a triangle ABC, right angled at A, on the leg AC as diameter, a semicircle is described. If a chord joins A with the point of intersection D of the hypotenuse and the semicircle, then the length of AC equals to

a. $\frac{AB.AD}{\sqrt{AB^2 + AD^2}}$ **b.** $\frac{AB.AD}{AB + AD}$

c. $\sqrt{AB.AD}$ **d.** $\frac{AB \cdot AD}{\sqrt{AB^2 - AD^2}}$

7. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is

a. $8\sqrt{3}$ sq. units **b.** $4\sqrt{3}$ sq. units

c. $6\sqrt{3}$ sq. units **d.** None

8. The locus of the centre of the circles such that the point (2, 3) is the midpoint of the chord $5x + 2y = 16$ is

a. $2x - 5y + 11 = 0$ **b.** $2x + 5y - 11 = 0$

c. $2x + 5y + 11 = 0$ **d.** None

9. Two congruent circles with centres at (2, 3) and (5, 6), which intersect at right angles, have radius equal to

a. $2\sqrt{2}$ **b.** 3 **c.** 4 **d.** None

10. A circle of radius unity is centred at origin. Two particles start moving at the same time from the point (1, 0) and move around the circle in opposite direction. One of the particle moves counterclockwise with constant speed v and the other moves clockwise with constant speed $3v$. After leaving (1, 0), the two particles meet first at a point P, and continue until they meet next at point Q. The coordinates of the point Q are

a. (1, 0) **b.** (0, 1) **c.** (0, –1) **d.** (–1, 0)

11. The value of 'c' for which the set $\{(x, y) \mid x^2 + y^2 + 2x \le 1\} \cap \{(x, y) \mid x - y + c \ge 0\}$ contains only one point in common is

a. $(-\infty, -1] \cup [3, \infty)$ **b.** $\{-1, 3\}$

c. $\{-3\}$ **d.** $\{-1\}$

12. A circle is inscribed into a rhombous $ABCD$ with one angle 60°. The distance from the centre of the circle to the nearest vertex is equal to 1. If P is any point of the circle, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to

a. 12 **b.** 11

c. 9 **d.** None of these

13. Two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents $(a > b \ge 2)$, then

a. $\theta = 2\cos^{-1}\left(\frac{a-b}{a+b}\right)$

b. $\theta = 2\tan^{-1}\left(\frac{a+b}{a-b}\right)$

c. $\theta = 2\sin^{-1}\left(\frac{a+b}{a-b}\right)$

d. $\theta = 2\sin^{-1}\left(\frac{a-b}{a+b}\right)$

14. B and C are fixed points having co-ordinates (3, 0) and (–3, 0), respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation

a. $x^2 + y^2 = 1$ **b.** $x^2 + y^2 = 2$

c. $9(x^2 + y^2) = 1$ **d.** $9(x^2 + y^2) = 4$

15. $ABCD$ is a square of unit area. A circle is tangent to two sides of $ABCD$ and passes through exactly one of its vertices. The radius of the circle is

a. $2 - \sqrt{2}$ **b.** $\sqrt{2} - 1$

c. $\frac{1}{2}$ **d.** $\frac{1}{\sqrt{2}}$

16. A pair of tangents are drawn to a unit circle with centre at the origin and these tangents intersect at A enclosing an angle of 60°. The area enclosed by these tangents and the arc of the circle is

a. $\frac{2}{\sqrt{3}} - \frac{\pi}{6}$ **b.** $\sqrt{3} - \frac{\pi}{3}$

c. $\frac{\pi}{3} - \frac{\sqrt{3}}{6}$ **d.** $\sqrt{3}\left(1 - \frac{\pi}{6}\right)$

17. A straight line with slope 2 and y-intercept 5 touches the circle, $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the co-ordinates of Q are

a. (–6, 11) b. (–9, –13)
c. (–10, –15) d. (–6, –7)

18. A circle of constant radius 'a' passes through origin 'O' and cuts the axes of co-ordinates in points P and Q, then the equation of the locus of the foot of perpendicular from O to PQ is

a. $(x^2 + y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$

b. $(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = a^2$

c. $(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$

d. $(x^2 + y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = a^2$

19. A line meets the co-ordinate axes in A and B. A circle is circumscribed about the triangle OAB. If d_1 and d_2 are the distances of the tangent to the circle at the origin O from the points A and B, respectively, then the diameter of the circle is

a. $\frac{2d_1 + d_2}{2}$ b. $\frac{d_1 + 2d_2}{2}$
c. $d_1 + d_2$ d. $\frac{d_1 d_2}{d_1 + d_2}$

20. If a circle of constant radius $3k$ passes through the origin 'O' and meets co-ordinate axes at A and B, then the locus of the centroid of the triangle OAB is

a. $x^2 + y^2 = (2k)^2$ b. $x^2 + y^2 = (3k)^2$
c. $x^2 + y^2 = (4k)^2$ d. $x^2 + y^2 = (6k)^2$

21. The equation of a line inclined at an angle $\pi/4$ to the X-axis, such that the two circles $x^2 + y^2 = 4$, $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal lengths on it, is

a. $2x - 2y - 3 = 0$ b. $2x - 2y + 3 = 0$
c. $x - y + 6 = 0$ d. $x - y - 6 = 0$

22. Let C be a circle with two diameters intersecting at an angle of 30°. A circle S is tangent to both the diameters and to C, and has radius unity. The largest radius of C is

a. $1 + \sqrt{6} + \sqrt{2}$ b. $1 + \sqrt{6} - \sqrt{2}$
c. $\sqrt{6} + \sqrt{2} - 11$ d. None of these

23. A straight line l_1 with equation $x - 2y + 10 = 0$ meets the circle with equation $x^2 + y^2 = 100$ at B in the first quadrant. A line through B, perpendicular to l_1 cuts y-axis at $P(0, t)$. The value of 't' is

a. 12 b. 15 c. 20 d. 25

24. Let a and b represent the length of a right triangle's legs. If d is the diameter of a circle inscribed into the triangle, and D is the diameter of a circle circumscribed on the triangle, then $d + D$ equals

a. $a + b$ b. $2(a + b)$
c. $\frac{1}{2}(a + b)$ d. $\sqrt{a^2 + b^2}$

25. If the chord $y = mx + 1$ of the circles $x^2 + y^2 = 1$ subtends an angle 45° at the major segment of the circle, then value of m is

a. 2 b. –2
c. –1 d. None of these

26. A variable chord of circle $x^2 + y^2 = 4$ is drawn from the point $P(3, 5)$ meeting the circle at the points A and B. A point Q is taken on this chord such that $2PQ = PA + PB$. Locus of 'Q' is

a. $x^2 + y^2 + 3x + 4y = 0$
b. $x^2 + y^2 = 36$
c. $x^2 + y^2 = 16$
d. $x^2 + y^2 - 3x - 5y = 0$

27. In triangle ABC, equation of side BC is $x - y = 0$. Circumcentre and orthocentre of the triangle are (2, 3) and (5, 8), respectively. Equation of circumcircle of the triangle is

a. $x^2 + y^2 - 4x + 6y - 27 = 0$
b. $x^2 + y^2 - 4x - 6y - 27 = 0$
c. $x^2 + y^2 + 4x + 6y - 27 = 0$
d. $x^2 + y^2 + 4x + 6y - 27 = 0$

28. The range of values of r for which the point $\left(-5 + \frac{r}{\sqrt{2}}, -3 + \frac{r}{\sqrt{2}}\right)$ is an interior point of the major segment of the circle $x^2 + y^2 = 16$, cut-off by the line $x + y = 2$, is

a. $(-\infty, 5\sqrt{2})$
b. $(4\sqrt{2} - \sqrt{14}, 5\sqrt{2})$
c. $(4\sqrt{2} - \sqrt{14}, 4\sqrt{2} + \sqrt{14})$
d. None of these

29. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the coordinate axis. The co-ordinates of its vertices are

a. (–6, –9), (–6, 5), (8, –9), (8, 5)
b. (–6, 9), (–6, –5), (8, –9), (8, 5)
c. (–6, –9), (–6, 5), (8, 9), (8, 5)
d. (–6, –9), (–6, 5), (8, –9), (8, –5)

30. (– 6, 0), (0, 6) and (– 7, 7) are the vertices of a $\Delta\, ABC$. The incircle of the triangle has the equation

a. $x^2 + y^2 - 9x - 9y + 36 = 0$
b. $x^2 + y^2 + 9x - 9y + 36 = 0$
c. $x^2 + y^2 + 9x + 9y - 36 = 0$
d. $x^2 + y^2 + 18x - 18y + 36 = 0$

31. If O is the origin and OP, OQ are the tangents from the origin to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$, then circumcenter of the triangle OPQ is

a. $(3, -2)$ **b.** $\left(\frac{3}{2}, -1\right)$

c. $\left(\frac{3}{4}, -\frac{1}{2}\right)$ **d.** $\left(-\frac{3}{2}, 1\right)$

32. The locus of the midpoint of a line segment that is drawn from a given external point P to a given circle with centre O (where O is origin) and radius r, is

a. a straight line perpendicular to PO

b. a circle with centre P and radius r

c. a circle with centre P and radius $2r$

d. a circle with centre at the midpoint PO and radius $r/2$

33. The difference between the radii of the largest and the smallest circles which have their centre on the circumference of the circle $x^2 + y^2 + 2x + 4y - 4 = 0$ and pass through the point (a, b) lying outside the given circle, is

a. 6 **b.** $\sqrt{(a+1)^2 + (b+2)^2}$

c. 3 **d.** $\sqrt{(a+1)^2 + (b+2)^2} - 3$

34. An isosceles triangles ABC is inscribed in a circle $x^2 + y^2 = a^2$ with the vertex A at $(a, 0)$ and the base angle B and C each equal to 75°, then coordinates of an end point of the base are

a. $\left(\frac{-\sqrt{3}a}{2}, \frac{a}{2}\right)$ **b.** $\left(-\frac{\sqrt{3}a}{2}, a\right)$

c. $\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ **d.** $\left(\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$

35. The equations of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. The radius of a circle touching all the four circles is

a. $(\sqrt{2} + 2)a$ **b.** $2\sqrt{2}a$

c. $(\sqrt{2} + 1)a$ **d.** $(2 + \sqrt{2})a$

36. The locus of a point which moves such that the sum of the squares of its distance from three vertices of a triangle is constant is a/an

a. circle **b.** straight line

c. ellipse **d.** None of these

37. A circle passes through the points $A(1, 0)$, $B(5, 0)$ and touches the y-axis at $C(0, h)$. If $\angle ACB$ is maximum then

a. $h = 3\sqrt{5}$ **b.** $h = 2\sqrt{5}$ **c.** $h = \sqrt{5}$ **d.** $h = 2\sqrt{10}$

38. A circle with centre (a, b) passes through the origin. The equation of the tangent to the circle at the origin is

a $ax - by = 0$ **b.** $ax + by = 0$

c. $bx - ay = 0$ **d.** $bx + ay = 0$

39. The area of the triangle formed by joining the origin to the points of intersection of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is

a. 3 **b.** 4 **c.** 5 **d.** 6

40. If (α, β) is a point on the circle whose centre is on the x-axis and which touches the line $x + y = 0$ at $(2, -2)$, then the greatest values of α is

a. $4 - \sqrt{2}$ **b.** 6 **c.** $4 + 2\sqrt{2}$ **d.** $4 + \sqrt{2}$

41. A region in the x–y plane is bounded by the curve $y = \sqrt{25 - x^2}$ and the line $y = 0$. If the point $(a, a + 1)$ lies in the interior of the region, then

a. $a \in (-4, 3)$ **b.** $a \in (-\infty, -1) \in (3, \infty)$

c. $a \in (-1, 3)$ **d.** None of these

42. There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0$, $n \in Z$. If the two circles have exactly two common tangents then the number of possible values of n is

a. 2 **b.** 8

c. 9 **d.** None of these

43. C_1 is a circle of radius 1 touching the x-axis and the y-axis. C_2 is another circle of radius > 1 and touching the axes as well as the circle C_1. Then, the radius of C_2 is

a. $3 - 2\sqrt{2}$ **b.** $3 + 2\sqrt{2}$

c. $3 + 2\sqrt{3}$ **d.** None of these

44. Equation of incircle of equilateral triangle ABC where $B \equiv (2, 0)$ $C \equiv (4, 0)$ and A lies in fourth quadrant is

a. $x^2 + y^2 - 6x + \frac{2y}{\sqrt{3}} + 9 = 0$

b. $x^2 + y^2 - 6x - \frac{2y}{\sqrt{3}} + 9 = 0$

c. $x^2 + y^2 + 6x + \frac{2y}{\sqrt{3}} + 9 = 0$

d. None of these

45. $f(x, y) = x^2 + y^2 + 2ax + 2Ly + c = 0$ represents a circle. If $f(x, 0) = 0$ has equal roots, each being 2 and $f(0, y) = 0$ has 2 and 3 as it's roots, then centre of circle is

a. $(2, 5/2)$ **b.** Data are not sufficient

c. $(-2, -5/2)$ **d.** Data are inconsistent

46. The area bounded by the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ and the pair of lines $\sqrt{3}(x^2 + y^2) = 4xy$, is equal to

a. $\frac{\pi}{2}$ **b.** $\frac{5\pi}{2}$ **c.** 3π **d.** $\frac{\pi}{4}$

47. The straight line $x \cos\theta + y \sin\theta = 2$ will touch the circle $x^2 + y^2 - 2x = 0$, if

a. $\theta = n\pi, n \in I$ **b.** $A = (2n + 1)\pi, n \in I$

c. $\theta = 2n\pi, n \in I$ **d.** None of these

48. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the side, then its radius is

a. 3 **b.** 2 **c.** $\frac{3}{2}$ **d.** 1

49. Consider a family of circle which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the co-ordinates of the centre of the circles, then the set of values of k is given by the interval

a. $k \geq \frac{1}{2}$ **b.** $-\frac{1}{2} \leq k \leq \frac{1}{2}$

c. $k \leq \frac{1}{2}$ **d.** $0 < k < \frac{1}{2}$

50. If the conics whose equations are $S \equiv \sin^2\theta\, x^2 + 2hxy + \cos^2\theta\, y^2 + 32x + 16y + 19 = 0$, $S' \equiv \cos^2\theta\, x^2 + 2h'xy + \sin^2\theta\, y^2 + 16x + 32y + 19 = 0$ intersects in four concyclic points then, (where $\theta \in R$)

a. $h + h' = 0$ **b.** $h = h'$

c. $h + h' = 1$ **d.** None of these

51. The range of values of $\lambda(\lambda > 0)$ such that the angle θ between the pair of tangents drawn from $(\lambda, 0)$ to the circle $x^2 + y^2 = 4$ lies in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ is

a. $\left(\frac{4}{\sqrt{3}}, 2\sqrt{2}\right)$ **b.** $(0, \sqrt{2})$

c. $(1, 2)$ **d.** None of these

52. The circles which can be drawn, to pass through $(1, 0)$ and $(3, 0)$ and to touch the y-axis, intersect at an angle θ, then $\cos\theta$ is equal to

a. $\frac{1}{2}$ **b.** $-\frac{1}{2}$ **c.** $\frac{1}{4}$ **d.** $-\frac{1}{4}$

53. Locus of midpoints of the chords of contact of $x^2 + y^2 = 2$ from the points on the line $3x + 4y = 10$ is a circle with centre P. If O be the origin then OP is equal to

a. 2 **b.** 3 **c.** $\frac{1}{2}$ **d.** $\frac{1}{3}$

54. Suppose $ax + by + c = 0$, where a, b, c are in A.P. be normal to a family of circles. The equation of the circle of the family intersects the circle $x^2 + y^2 - 4x - 4y - 1 = 0$ orthogonally is

a. $x^2 + y^2 - 2x + 4y - 3 = 0$

b. $x^2 + y^2 + 2x - 4y - 3 = 0$

c. $x^2 + y^2 - 2x + 4y - 5 = 0$

d. $x^2 + y^2 - 2x - 4y + 3 = 0$

55. If a circle of radius r is touching the lines $x^2 - 4xy + y^2 = 0$ in the first quadrant at points A and B, then area of triangle OAB (O being the origin) is

a. $\frac{3\sqrt{3}\, r^2}{4}$ b. $\frac{\sqrt{3}\, r^2}{4}$

c. $\frac{3r^2}{4}$ d. r^2

56. Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$. AB be the chord of contact of this point w.r.t. the circle $x^2 + y^2 - 2x = 0$. The locus of the circumcnetre of the triangle CAB (C being centre of the circles) is

a. $2x^2 + 2y^2 - 4x + 1 = 0$

b. $x^2 + y^2 - 4x + 2 = 0$

c. $x^2 + y^2 - 4x + 1 = 0$

d. $2x^2 + 2y^2 - 4x + 3 = 0$

57. Two circles of radii 'a' and 'b' touching each other externally, are inscribed in the area bounded by $y = \sqrt{1 - x^2}$ and the x-axis. if $b = \frac{1}{2}$, then a is equal to

a. $\frac{1}{4}$ **b.** $\frac{1}{8}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{\sqrt{2}}$

58. From an arbitrary point 'P' on the circle $x^2 + y^2 = 9$, tangents are drawn to the circle $x^2 + y^2 = 1$, which meet $x^2 + y^2 = 9$ at A and B. Locus of the point of intersection of tangents at A and B to the circle $x^2 + y^2 = 9$ is

a. $x^2 + y^2 = \left(\frac{27}{7}\right)^2$ **b.** $x^2 - y^2 = \left(\frac{27}{7}\right)^2$

c. $y^2 - x^2 = \left(\frac{27}{7}\right)^2$ **d.** None of these

59. If r_1 and r_2 are the radii of smallest and largest circles which passes through $(5, 6)$ and touches the circle $(x - 2)^2 + y^2 = 4$, then $r_1 r_2$ is

a. $\frac{4}{41}$ **b.** $\frac{41}{4}$ **c.** $\frac{5}{41}$ **d.** $\frac{41}{6}$

60. Minimum radius of circle which is orthogonal with both the circles $x^2 + y^2 - 12x + 35 = 0$ and $x^2 + y^2 + 4x + 3 = 0$ is

a. 4 **b.** 3 **c.** $\sqrt{15}$ **d.** 1

61. If $C_1 : x^2 + y^2 = (3 + 2\sqrt{2})^2$ be a circle and PA and PB are pair of tangents on C_1, where P is any point on the director circle of C_1, then the radius of smallest circle which touch C_1 externally and also the two tangents PA and PB is

a. $2\sqrt{3} - 3$ **b.** $2\sqrt{2} - 1$ **c.** $2\sqrt{2} - 1$ **d.** 1

62. From a point $R(5, 8)$ two tangents RP and RQ are drawn to a given circle $S = 0$ whose radius is 5. If circumcentre of the triangle PQR is $(2, 3)$, then the equation of circle $S = 0$ is

a. $x^2 + y^2 + 2x + 4y - 20 = 0$

b. $x^2 + y^2 + x + 2y - 10 = 0$

c. $x^2 + y^2 - x - 2y - 20 = 0$

d. $x^2 + y^2 - 4x - 6y - 12 = 0$

63. On the line segment joining $(1, 0)$ and $(3, 0)$, an equilateral triangle is drawn having its vertex in the fourth quadrant, then radical centre of the circles described on its sides as diameter is

a. $\left(3, -\frac{1}{\sqrt{3}}\right)$ **b.** $(3, -\sqrt{3})$

c. $\left(2, -\frac{1}{\sqrt{3}}\right)$ **d.** $(2, -\sqrt{3})$

64. If the tangents are drawn from any point on the line $x+y=3$ to the circle $x^2+y^2=9$, then the chord of contact passes through the point

a. (3, 5) **b.** (3, 3)
c. (5, 3) **d.** None of these

65. If the radius of the circumcircle of the triangle TPQ, where PQ is chord of contact corresponding to point T with respect to circle $x^2+y^2-2x+4y-11=0$, is 12 units, then minimum distance of T from the director circle of the given circle is

a. 6 **b.** 12
c. $6\sqrt{2}$ **d.** $12-4\sqrt{2}$

66. P is a point (a, b) in the first quadrant. If the two circles which pass through P and touch both the co-ordinates axes cut at right angles, then

a. $a^2-6ab+b^2=0$ **b.** $a^2+2ab-b^2=0$
c. $a^2-4ab+b^2=0$ **d.** $a^2-8ab+b^2=0$

67. The number of common tangent(s) to the circles $x^2+y^2+2x+8y-23=0$ and $x^2+y^2-4x-10y-19=0$ is

a. 1 **b.** 2 **c.** 3 **d.** 4

68. Two circles of radii 4 cm and 1 cm touch each other externally and θ is the angle contained by their direct common tangents. Then $\sin\theta$ is equal to

a. $\frac{24}{25}$ **b.** $\frac{12}{25}$
c. $\frac{3}{4}$ **d.** None of these

69. The locus of the midpoints of the chords of the circle $x^2+y^2-ax-by=0$ which subtend a right angle at $\left(\frac{a}{2}, \frac{b}{2}\right)$ is

a. $ax+by=0$
b. $ax+by=a^2+b^2$
c. $x^2+y^2-ax-by+\frac{a^2+b^2}{8}=0$
d. $x^2+y^2-ax-by-\frac{a^2+b^2}{8}=0$

70. From points (3 , 4), chords are drawn to the circle $x^2+y^2-4x=0$. The locus of the midpoints of the chords is

a. $x^2+y^2-5x-4y+6=0$
b. $x^2+y^2+5x-4y+6=0$
c. $x^2+y^2-5x+4y+6=0$
d. $x^2+y^2-5x-4y-6=0$

71. The locus of the centers of the circles which cut the circles $x^2+y^2+4x-6y+9=0$ and $x^2+y^2-5x+4y-2=0$ orthogonally is

a. $9x+10y-7=0$ **b.** $x-y+2=0$
c. $9x-10y+11=0$ **d.** $9x+10y+7=0$

72. The angle at which the circles $(x-1)^2+y^2=10$ and $x^2+(y-2)^2=5$ intersect is

a. $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{2}$

73. A tangent at a point on the circle $x^2+y^2=a^2$ intersects a concentric circle C at two points P and Q. The tangents to the circle X at P and Q meet at a point on the circle $x^2+y^2=b^2$, then the equation of circle is

a. $x^2+y^2=ab$ **b.** $x^2+y^2=(a-b)^2$
c. $x^2+y^2=(a+b)^2$ **d.** $x^2+y^2=a^2+b^2$

74. Tangents are drawn to the circle $x^2+y^2=1$ at the points where it is met by the circles, $x^2+y^2-(\lambda+6)x+(8-2\lambda)y-3=0$, λ being the variable. The locus of the point of intersection of these tangents is

a. $2x-y+10=0$ **b.** $x+2y-10=0$
c. $x-2y+10=0$ **d.** $2x+y-10=0$

75. If point A and B are (1, 0) and B (0, 1).If point C is on the circle $x^2+y^2=1$, then locus of the orthocentre of the triangle ABC is

a. $x^2+y^2=4$
b. $x^2+y^2-x-y=0$
c. $x^2+y^2-2x-2y+1=0$
d. $x^2+y^2+2x-2y+1=0$

76. If the line $x\cos\theta+y\sin\theta=2$ is the equation of a transverse common tangent to the circles $x^2+y^2=4$ and $x^2+y^2-6\sqrt{3}x-6y+20=0$, then the value of θ is

a. $5\pi/6$ **b.** $2\pi/3$ **c.** $\pi/3$ **d.** $\pi/6$

77. Any circle through the point of intersection of the lines $x+\sqrt{3}y=1$ and $\sqrt{3}x-y=2$ if intersects these lines at points P and Q, then the angle subtended by the arc PQ at its centre is

a. 180°
b. 90°
c. 120°
d. Depends on centre and radius

78. If the angle between tangents drawn to $x^2+y^2+2gx+2fy+c=0$ form (0, 0) is $\pi/2$, then

a. $g^2+f^2=3c$ **b.** $g^2+f^2=2c$
c. $g^2+f^2=5c$ **d.** $g^2+f^2=4c$

79. The common chord of the circle $x^2+y^2+6x+8y-7=0$ and a circle passing through the origin, and touching the line $y=x$, always passes through the point

a. (−1/2, 1/2) **b.** (1, 1)
c. (1/2, 1/2) **d.** None of these

80. The chord of contact of tangents from three points A, B, C to the circle $x^2+y^2=a^2$ are concurrent, then A, B, C will

a. be concyclic
b. be collinear
c. form the vertices of a triangle
d. None of these

81. The chord of contact of tangents from a point P to a circle passes through Q. If l_1 and l_2 are the lengths of the tangents from P and Q to the circle, then PQ is equal to

a. $\frac{l_1 + l_2}{2}$ **b.** $\frac{l_1 - l_2}{2}$

c. $\sqrt{l_1^2 + l_2^2}$ **d.** $2\sqrt{l_1^2 + l_2^2}$

82. If $C_1: x^2 + y^2 - 20x + 64 = 0$ and $C_2: x^2 + y^2 + 30x + 144 = 0$. Then the length of the shortest line segment PQ which touches C_1 at P and to C_2 at Q is

a. 20 **b.** 15 **c.** 22 **d.** 27

83. The ends of a quadrant of a circle have the coordinates (1, 3) and (3, 1). Then the centre of such a circle is

a. (2, 2) **b.** (1,1) **c.** (4, 4) **d.** (2, 6)

84. If the line $ax + by = 2$ is a normal to the circle $x^2 + y^2 - 4x - 4y = 0$ and a tangent to the circle $x^2 + y^2 = 1$, then

a. $a = \frac{1}{2}, b = \frac{1}{2}$

b. $a = \frac{1+\sqrt{7}}{2}, b = \frac{1-\sqrt{7}}{2}$

c. $a = \frac{1}{4}, b = \frac{3}{4}$

d. $a = 1, b = \sqrt{3}$

85. Radius of the tangent circle that can be dawn to pass through the point (0, 7), (0, 6) and touching the x-axis is

a. $\frac{5}{2}$ **b.** $\frac{3}{2}$ **c.** $\frac{7}{2}$ **d.** $\frac{9}{2}$

86. The equation of the tangent to the circle $x^2 + y^2 = a^2$, which makes a triangle of area a^2 with the coordinate axes, is

a. $x \pm y = a\sqrt{2}$ **b.** $x \pm y = \pm a\sqrt{2}$

c. $x \pm y = 2a$ **d.** $x + y = \pm 2a$

87. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is touched by $y = x$ at P such that $OP = 6\sqrt{2}$, then the value of c is

a. 36 **b.** 144

c. 72 **d.** None of these

88. The circle $x^2 + y^2 = 4$ cuts the line joining the points $A(1, 0)$ and $B(3, 4)$ in two points P and Q. Let $\frac{BP}{PA} = \alpha$ and $\frac{BQ}{QA} = \beta$. Then α and β are roots of the quadratic equation

a. $3x^2 + 2x - 21 = 0$ **b.** $3x^2 + 2x + 21 = 0$

c. $2x^2 + 3x - 21 = 0$ **d.** None of these

89. The area of the triangle formed by the positive x-axis and the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is

a. $2\sqrt{3}$ sq. units **b.** $3\sqrt{2}$ sq. units

c. $\sqrt{6}$ sq. units **d.** None of these

90. Let P be point on the circle $x^2 + y^2 = 9$, Q a point on the line $7x + y + 3 = 0$, and the perpendicular bisector of PQ be the line $x - y + 1 = 0$. Then the coordinate of P are

a. (0, −3) **b.** (0, 3)

c. $\left(\frac{72}{25}, -\frac{21}{25}\right)$ **d.** $\left(-\frac{72}{25}, \frac{21}{25}\right)$

91. A straight line moves such that the algebraic sum of the perpendicular drawn to it from two fixed points is equal to $2k$. Then, the straight line always touches a fixed circle of radius

a. $2k$ **b.** $k/2$

c. k **d.** None of these

92. The coordinates of the middle point of the chord cut-off by $2x - 5y + 18 = 0$ by the circle

$x^2 + y^2 - 6x + 2y - 54 = 0$ are

a. (1, 4) **b.** (2, 4) **c.** (4, 1) **d.** (1, 1)

93. The locus of a point from which the lengths of the tangents to the circles $x^2 + y^2 = 4$ and $2(x^2 + y^2) - 10x + 3y - 2 = 0$ are equal to

a. a straight line inclined at $\pi/4$ with the line joining the centres of the circles

b. a circle

c. an ellipse

d. a straight line perpendicular to the line joining the centres of the circles

94. The equation of circumcircle of an equilateral triangle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and one vertex of the triangle is (1, 1). The equation of incircle of the triangle is

a. $4(x^2 + y^2) = g^2 + f^2$

b. $4(x^2 + y^2) + 8gx + 8fy = (1 - g)(1 + 3g) + (1 - f)(1 + 3f)$

c. $4(x^2 + y^2) + 8gx + 8fy = g^2 + f^2$

d. None of these

95. A light ray gets reflected from the $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and point of incident is (−2, −4), then equation of incident ray is

a. $4y + 3x + 22 = 0$

b. $3y + 4x + 20 = 0$

c. $4y + 2x + 20 = 0$

d. $y + x + 6 = 0$

96. Tangents PA and PB drawn to $x^2 + y^2 = 9$ from any arbitrary point 'P' on the line $x + y = 25$. Locus of midpoint of chord AB is

a. $25(x^2 + y^2) = 9(x + y)$

b. $25(x^2 + y^2) = 3(x + y)$

c. $5(x^2 + y^2) = 3(x + y)$

d. None of these

97. The circles having radii r_1 and r_2 intersect orthogonally. Length of their common chord is

a. $\frac{2r_1r_2}{\sqrt{r_1^2+r_2^2}}$ **b.** $\frac{\sqrt{r_1^2+r_2^2}}{2r_1r_2}$

c. $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$ **d.** $\frac{\sqrt{r_1^2+r_2^2}}{r_1r_2}$

98. If the pair of straight line $xy\sqrt{3}-x^2=0$ is tangent to the circle at P and Q from origin O such that area of smaller sector formed by CP and CQ is 3π sq. unit, where C is the centre of circle, then OP equals to

a. $(3\sqrt{3})/2$ **b.** $3\sqrt{3}$ **c.** 3 **d.** $\sqrt{3}$

99. The two circles which passes through $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$ will intersect each other at right angle, if

a. $a^2 = c^2(2m+1)$ **b.** $a^2 = c^2(2+m^2)$

c. $c^2 = a^2(2+m^2)$ **d.** $c^2 = a^2(2m+1)$

100. The condition that the chord $x\cos\alpha + y\sin\alpha - p = 0$ of $x^2+y^2-a^2=0$ may subtend a right angle at the centre of the circle is

a. $a^2 = 2p^2$ **b.** $p^2 = 2a^2$ **c.** $a = 2p$ **d.** $p = 2a$

101. The locus of the midpoints of a chord of the circle $x^2 + y^2 = 4$, which subtends a right angle at the origin, is

a. $x + y = 2$ **b.** $x^2 + y^2 = 1$

c. $x^2 + y^2 = 2$ **d.** $x + y = 1$

102. If the chord of contact of tangents from a point P to a given circle passes through Q, then the circle on PQ as diameter

a. cuts the given circle orthogonally
b. touches the given circle externally
c. touches the given circle internally
d. None of these

103. Let these base AB of a triangle ABC be fixed and the vertex C lie on a fixed circle of radius r. Lines through A and B are drawn to intersect CB and CA, respectively, at E and F such that $CE : EB = 1 : 2$ and $CF : FA = 1 : 2$. If the point of intersection P of these lines lies on the median through AB for all positions of AB then the locus of P is

a. a circle of radius $\frac{r}{2}$
b. a circle of radius $2r$
c. a parabola of latus rectum $4r$
d. a rectangular hyperbola

104. The number of integral values of y for which the chord of the circle $x^2 + y^2 = 125$ passing through the point $P(8, y)$ gets bisected at the point $P(8, y)$ and has integral slope is

a. 8 **b.** 6 **c.** 4 **d.** 2

105. Consider a square $ABCD$ of side length 1. Let P be the set of all segments of length 1 with end points on adjacent sides of square $ABCD$. The midpoints of segments in P enclose a region with area A, the value of A is

a. $\frac{\pi}{4}$ **b.** $1-\frac{\pi}{4}$

c. $4-\frac{\pi}{4}$ **d.** None of these

106. The range of values of α for which the line $2y = gx + \alpha$ is a normal to the circle $x^2 + y^2 + 2gx + 2gy - 2 = 0$ for all values of g is

a. $[1, \infty)$ **b.** $[-1, \infty)$ **c.** $(0, 1)$ **d.** $(-\infty, 1]$

107. Six points (x_i, y_i), $i = 1, 2, \ldots, 6$ are taken on the circle $x^2 + y^2 = 4$ such that $\sum_{i=1}^{6} x_i = 8$ and $\sum_{i=1}^{6} y_i = 4$. The line segment joining orthocentre of a triangle made by any three points and the centroid of the triangle made by other three points passes through a fixed points (h, k), then $h + k$ is

a. 1 **b.** 2 **c.** 3 **d.** 4

108. A circle with radius $|a|$ and centre on y-axis slides along it and a variable lines through $(a, 0)$ cuts the circle at points P and Q. The region in which the point of intersection of tangents to the circle at points P and Q lies is represented by

a. $y^2 \geq 4(ax - a^2)$ **b.** $y^2 \leq 4(ax - a^2)$

c. $y \geq 4(ax - a^2)$ **d.** $y \leq 4(ax - a^2)$

109. If the angle of intersection of the circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ is θ, then equation of the line passing through $(1, 2)$ and making an angle θ with the y-axis is

a. $x = 1$ **b.** $y = 2$

c. $x + y = 3$ **d.** $x - y = 3$

110. The centres of a set of circle, each of radius 3, lies on the circle $x^2 + y^2 = 25$. The locus of any point in the set is

a. $4 \leq x^2 + y^2 \leq 64$ **b.** $x^2 + y^2 \leq 25$

c. $x^2 + y^2 \geq 25$ **d.** $3 \leq x^2 + y^2 \leq 9$

111. The co-ordinates of two points P and Q are (x_1, y_1) and (x_2, y_2) and O is the origin. If circles be described on OP and OQ as diameters then length of their common chord is

a. $\frac{|x_1y_2 + x_2y_1|}{PQ}$ **b.** $\frac{|x_1y_2 - x_2y_1|}{PQ}$

c. $\frac{|x_1x_2 + y_1y_2|}{PQ}$ **d.** $\frac{|x_1x_2 + y_1y_2|}{PQ}$

112. Consider a circle $x^2 + y^2 + ax + by + c = 0$ lying completely in first quadrant. If m_1 and m_2 are the maximum and minimum values of y/x for all ordered pairs (x, y) on the circumference of the circle, then the value of $(m_1 + m_2)$ is

a. $\frac{a^2-4c}{b^2-4c}$ b. $\frac{2ab}{b^2-4c}$

c. $\frac{2ab}{4c-b^2}$ d. $\frac{2ab}{b^2-4ac}$

113. If the circumference of the circle $x^2+y^2+8x+8y-b=0$ is bisected by the circle $x^2+y^2-2x+4y+a=0$, then $a+b$ equals to

a. 50 b. 56 c. −56 d. −34

114. The equation of the circle passing through the point of intersection of the circle $x^2+y^2=4$ and the line $2x+y=1$ and having minimum possible radius is

a. $5x^2+5y^2+18x+6y-5=0$
b. $5x^2+5y^2+9x+8y-15=0$
c. $5x^2+5y^2+4x+9y-5=0$
d. $5x^2+5y^2-4x-2y-18=0$

115. The locus of the centre of the circle touching the line $2x-y=1$ at (1, 1) is

a. $x+3y=2$ b. $x+2y=0$
c. $x+y=2$ d. None of these

116. The distance from the centre of the circle $x^2+y^2=2x$ to the common chord of the circles $x^2+y^2+5x-8y+1=0$ and $x^2+y^2-3x+7y-25=0$ is

a. 2 b. 4 c. $\frac{34}{13}$ d. $\frac{26}{17}$

117. The equation of the circle passing through the point of intersection of the circles $x^2+y^2-4x-2y=8$ and $x^2+y^2-2x-4y=8$ and the point (−1, 4) is

a. $x^2+y^2+4x+4y-8=0$
b. $x^2+y^2-3x+4y+8=0$
c. $x^2+y^2+x+y-8=0$
d. $x^2+y^2-3x-3y-8=0$

118. If the radius of the circle $(x-1)^2+(y-2)^2=1$ and $(x-7)^2+(y-10)^2=4$ are increasing uniformly w.r.t. time as 0.3 and 0.4 unit/sec, then they will touch each other at t equal to

a. 45 sec b. 90 sec c. 11 sec d. 135 sec

119. The equation of a circle which has normals $(x-1)\times(y-2)=0$ and a tangent $3x+4y=6$ is

a. $x^2+y^2-2x-4y+4=0$
b. $x^2+y^2-2x-4y+5=0$
c. $x^2+y^2=5$
d. $(x-3)^2+(y-4)^2=5$

Multiple Correct Answers Type

Solutions on page 2.82

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. The circle $x^2+y^2+2a_1x+c=0$ lies completely inside the circle $x^2+y^2+2a_2x+c=0$, then

a. $a_1a_2>0$ b. $a_1a_2<0$ c. $c>0$ d. $c<0$

2. Consider the circle $x^2+y^2-10x-6y+30=0$. Let O be the centre of the circle and tangent at $A(7, 3)$ and $B(5, 1)$ meet at C. Let $S=0$ represents family of circles passing through A and B, then

a. area of quadrilateral $OACB=4$
b. the radical axis for the family of circles $S=0$ is $x+y=10$
c. the smallest possible circle of the family $S=0$ is $x^2+y^2-12x-4y+38=0$
d. the coordinates of point C are (7, 1)

3. If the circle $x^2+y^2+2x+2ky+6=0$ and $x^2+y^2+2ky+k=0$ intersect orthogonally, then k is

a. 2 b. −2 c. $-\frac{3}{2}$ d. $\frac{3}{2}$

4. A$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a point on the circle $x^2+y^2=1$ and B is another point on the circle such that arc length $AB=\frac{\pi}{2}$ units. Then, co-ordinates of B can be

a. $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ b. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

c. $\left(-\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ d. None of these

5. Let x, y be real variable satisfying the $x^2+y^2+8x-10y-40=0$. Let $a=\max\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$ and $b=\min\left\{\sqrt{(x+2)^2+(y-3)^2}\right\}$, then

a. $a+b=18$ b. $a+b=\sqrt{2}$
c. $a-b=4\sqrt{2}$ d. $a\cdot b=73$

6. Three sides of a triangle have the equations $L_i\equiv y-m_ix=0$; $i=1, 2, 3$. Then $L_1L_2+\lambda L_2L_3+\mu L_3L_1=0$, where $\lambda\neq 0$, $\mu\neq 0$, is the equation of the circumcircle of the triangle if

a. $1+\lambda+\mu=m_1m_2+\lambda m_2m_3+\lambda m_3m_1$
b. $m_1(1+\mu)+m_2(1+\lambda)+m_3(\mu+\lambda)=0$
c. $\frac{1}{m_3}+\frac{1}{m_1}+\frac{1}{m_2}=1+\lambda+\mu$
d. None of these

7. If equation $x^2+y^2+2hxy+2gx+2fy+c=0$ represents a circle, then the condition for that circle to pass through three quadrants only but not passing through the origin is

a. $f^2>c$ b. $g^2>c$
c. $c>0$ d. $h=0$

8. The points on the line $x=2$ from which the tangents drawn to the circle $x^2+y^2=16$ are at right angles is (are)

a. $(2, 2\sqrt{7})$ b. $(2, 2\sqrt{5})$

c. $(2, -2\sqrt{7})$ **d.** $(2, -2\sqrt{5})$

9. Co-ordinates of the centre of a circle, whose radius is 2 unit and which touches the line pair $x^2 - y^2 - 2x + 1 = 0$, are

a. $(4, 0)$ **b.** $(1 + 2\sqrt{2}, 0)$

c. $(4, 1)$ **d.** $(1, 2\sqrt{2})$

10. If the circles $x^2 + y^2 - 9 = 0$ and $x^2 + y^2 + 2ax + 2y + 1 = 0$ touch each other, then a is

a. $-\frac{4}{3}$ **b.** 0 **c.** 1 **d.** $\frac{4}{3}$

11. Point M moved on the circle $(x - 4)^2 + (y - 8)^2 = 20$. Then it broke away from it and moving along a tangent to the circle, cuts the x-axis at the point $(-2, 0)$. The co-ordinates of a point on the circle at which the moving point broke away is

a. $\left(\frac{42}{5}, \frac{36}{5}\right)$ **b.** $\left(-\frac{2}{5}, \frac{44}{5}\right)$

c. $(6, 4)$ **d.** $(2, 4)$

12. The equation of a tangent to the circle $x^2 + y^2 = 25$ passing through $(-2, 11)$ is

a. $4x + 3y = 25$ **b.** $3x + 4y = 38$

c. $24x - 7y + 125 = 0$ **d.** $7x + 24y = 250$

13. If the area of the quadrilateral formed by the tangents from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the radii corresponding to the points of contact is 15, then a value of c is

a. 9 **b.** 4 **c.** 5 **d.** 25

14. The equation of the circle which touches the axes of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centre lies in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is

a. 1 **b.** 2 **c.** 3 **d.** 6

15. The equations of tangents to the circle $x^2 + y^2 - 6x - 6y + 9 = 0$ drawn from the origin are

a. $x = 0$ **b.** $x = y$ **c.** $y = 0$ **d.** $x + y = 0$

16. If a circle passes through the point of intersection of the lines $x + y + 1 = 0$ and $x + \lambda y - 3 = 0$ with the co-ordinates axes, then

a. $\lambda = -1$

b. $\lambda = 1$

c. $\lambda = 2$

d. λ can have any real value

17. Which of the following lines have the intercepts of equal lengths on the circle, $x^2 + y^2 - 2x + 4y = 0$?

a. $3x - y = 0$

b. $x + 3y = 0$

c. $x + 3y + 10 = 0$

d. $3x - y - 10 = 0$

18. The equation of the lines parallel to $x - 2y = 1$ which touches (touch) the circle $x^2 + y^2 - 4x - 2y - 15 = 0$ is (are)

a. $x - 2y + 2 = 0$

b. $x - 2y - 10 = 0$

c. $x - 2y - 5 = 0$

d. $x - 2y + 10 = 0$

19. The circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 4x + 4y - 1 = 0$

a. touch internally

b. touch externally

c. have $3x + 4y - 1 = 0$ as the common tangent at the point of contact

d. have $3x + 4y + 1 = 0$ as the common tangent at the point of contact

20. The circles $x^2 + y^2 + 2x + 4y - 20 = 0$ and $x^2 + y^2 + 6x - 8y + 10 = 0$

a. are such that the number of common tangents on them is 2

b. are orthogonal

c. are such that the length of their common tangent is $5(12/5)^{1/4}$

d. are such that the length of their common chord is $5\sqrt{\frac{3}{2}}$

21. A point $P(\sqrt{3}, 1)$ moves on the circle $x^2 + y^2 = 4$ and after covering a quarter of the circle leaves it tangentially. The equation of a line along which the point moves after leaving the circle is

a. $y = \sqrt{3}\,x + 4$ **b.** $\sqrt{3}\,y = x + 4$

c. $y = \sqrt{3}x - 4$ **d.** $\sqrt{3}y = x - 4$

22. The equation of a circle of radius 1 touching the circles $x^2 + y^2 - 2|x| = 0$ is

a. $x^2 + y^2 + 2\sqrt{2}\,x + 1 = 0$

b. $x^2 + y^2 - 2\sqrt{3}\,y + 2 = 0$

c. $x^2 + y^2 + 2\sqrt{3}\,y + 2 = 0$

d. $x^2 + y^2 - 2\sqrt{2} + 1 = 0$

23. If the circles $x^2 + y^2 - 9 = 0$ and $x^2 + y^2 + 2ax + 2y + 1 = 0$ touch each other, then $a =$

a. $-4/3$ **b.** 0 **c.** 1 **d.** 4/3

24. The range of values of 'a' such that angle θ between the pair of tangent drawn from $(a, 0)$ to the circle $x^2 + y^2 = 1$ satisfies $\frac{\pi}{2} < \theta < \pi$, lies in

a. $(1, 2)$ **b.** $(1, \sqrt{2})$

c. $(-\sqrt{2}, -1)$ **d.** $(-2, -1)$

25. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1

a. $x + y = 0$ **b.** $x - y = 0$

c. $x + 7y = 0$ **d.** $x - 7y = 0$

26. The centre(s) of the circle(s) passing through the points (O, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is/are

a. $\left(\frac{3}{2}, \frac{1}{2}\right)$ **b.** $\left(\frac{1}{2}, \frac{3}{2}\right)$

c. $\left(\frac{1}{2}, 2^{1/2}\right)$ d. $\left(\frac{1}{2}, -2^{1/2}\right)$

Reasoning Type

Solutions on page 2.86

Each question has four choices a, b, c and d, out of which only one is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are True and Statement 2 is the correct explanation of Statement 1.

b. Both the statements are True but Statement 2 is not the correct explanation of Statement 1.

c. Statement 1 is True and Statement 2 is False.

d. Statement 1 is False and Statement 2 is True.

1. Statement 1 : The number of circles that pass through the points (1, – 7) and (– 5, 1) and of radius 4, is two.

Statement 2: The centre of any circle that pass through the points A and B lies on the perpendicular bisector of AB.

2. Statement 1: The chord of contact of tangent from three points A, B, C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B, C will be collinear.

Statement 2: Lines $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$ always pass through a fixed point for $k \in R$.

3. Statement 1: Circles $x^2 + y^2 = 144$ and $x^2 + y^2 - 6x - 8y = 0$ do not have any common tangent.

Statement 2: If two circles are concentric, then they do not have common tangents.

4. Statement 1: The least and greatest distances of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ are 5 and 15 units, respectively.

Statement 2: A point (x_1, y_1) lies outside a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, if $S_1 > 0$, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

5. Statement 1: Number of circles passing through (1, 2), (4, 8) and (0, 0) is one.

Statement 2: Every triangle has one circumcircle.

6. Let C_1 be the circle with centre O_1 (0, 0) and radius 1 and C_2 be the circle with centre O_2 $(t, t^2 + 1)$ $(t \in R)$ and radius 2.

Statement 1: Circles C_1 and C_2 always have at least one common tangent for any value of t.

Statement 2: For the two circles, $O_1O_2 \geq |r_1 - r_2|$, where r_1 and r_2 are their radii for any value of t.

7. Tangents are drawn from the origin to the circle $x^2 + y^2 - 2hx - 2hy + h^2 = 0$ $(h \geq 0)$.

Statement 1: Angle between the tangents is $\pi/2$.

Statement 2: The given circle is touching the co-ordinate axes.

8. Consider two circles $x^2 + y^2 - 4x - 6y - 8 = 0$ and $x^2 + y^2 - 2x - 3 = 0$.

Statement 1: Both circles intersect each other at two distinct points.

Statement 2: Sum of radii of two circles in greater than distance between the centres of two circles.

9. From the point $P(\sqrt{2}, \sqrt{6})$, tangents PA and PB are drawn to the circle $x^2 + y^2 = 4$.

Statement 1: Area of the quadrilateral $OAPB$ (being origin) is 4.

Statement 2: Area of square is a^2 where a is length of side.

10. Statement 1: Centre of the circle having $x + y = 3$ and $x - y = 1$ as its normal is (1, 2).

Statement 2: Normals to the circle always passes through its centre.

11. Statement 1: The circle having equation $x^2 + y^2 - 2x + 6y + 5 = 0$ intersects both the coordinate axes.

Statement 2: The lengths of x and y intercepts made by the circle having equation $x^2 + y^2 + 2gx + 2fy + c = 0$ are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$, respectively.

12. Statement 1: Number of circles touching lines $x + y = 1$, $2x - y = 5$ and $3x + 5y - 1 = 0$ is four.

Statement 2: In any triangle, four circles can be drawn touching all the three sides of triangle.

13. Statement 1: Chord of contact of the circle $x^2 + y^2 = 1$ w.r.t. points (2, 3). (3, 5) and (1, 1) are concurrent.

Statement 2: Points (1, 1), (2, 3) and (3, 5) are collinear.

14. Statement 1: The equation $x^2 + y^2 - 2x - 2ay - 8 = 0$ represents, for different values of 'a', a system of circles passing through two fixed points lying on the x-axis.

Statement 2: $S = 0$ is a circle and $L = 0$ is a straight line, then $S + \lambda L = 0$ represents the family of circles passing through the points of intersection of circle and straight line (where λ is arbitrary parameter).

15. **Statement 1:** The chord of contact of tangent from three points A, B, C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B, C will be collinear.

    Statement 2: A, B, C always lies on the normal to the circle $x^2 + y^2 = a^2$

16. **Statement 1:** Points $A(1, 0)$, $B(2, 3)$, $C(5, 3)$and $D(6, 0)$ are concyclic.

    Statement 2: Points A, B, C, D forms isosceles trapezium or AB and CD meet in E, then $EA \cdot EB = EC \cdot ED$.

17. **Statement 1:** The equations of the straight lines joining origin to the points of intersection of $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 2x - 4y - 4 = 0$ is $x - y = 0$.

    Statement 2: $y + x = 0$ is common chord of $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 2x - 4y - 4 = 0$.

18. **Statement 1:** Two orthogonal circles intersect to generate a common chord which subtends complimentary angles at their circumferences.

    Statement 2: Two orthogonal circles intersect to generate a common chord which subtends supplementary angles at their centres.

19. **Statement 1:** The point $(a, -a)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$ whenever $a \in (-1, 4)$

    Statement 2: Point (x_1, y_1) lies inside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, if $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$.

20. **Statement 1:** If circle with centre $P(t, 4 - 2t)$, $t \in R$ cuts the circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2x - y - 12 = 0$; then both the intersections are orthogonal.

    Statement 2: Length of tangent from P for $t \in R$ is same for both the given circles.

21. **Statement 1:** Let $S_1 : x^2 + y^2 - 10x - 12y - 39 = 0$

    $S_2 : x^2 + y^2 - 2x - 4y + 1 = 0$

    and $S_3 : 2x^2 + 2y^2 - 20x - 24y + 78 = 0$

    The radical centre of these circles taken pairwise as $(-2, -3)$.

    Statement 2: Point of intersection of three radical axis of three circles taken in pairs is known as radical centre.

22. **Statement 1:** The equation of chord through the point $(-2, 4)$ which is farthest from the centre of the circle $x^2 + y^2 - 6x + 10y - 9 = 0$ is $x + y - 2 = 0$.

    Statement 2: In notations, the equation of such chord of the circle $S = 0$ bisected at (x_1, y_1) must be $T = S_1$.

23. **Statement 1:** If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then $f'g = fg'$.

    Statement 2: Two circles touch other, if line joining their centres is perpendicular to all possible common tangents.

24. **Statement 1:** The circles $x^2 + y^2 + 2px + r = 0$, $x^2 + y^2 + 2qy + r = 0$ touch, if $\frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{r}$.

    Statement 2: Two circles with centre C_1, C_2 and radii r_1, r_2 touch each other if $|r_1 \pm r_2| = c_1c_2$.

Linked Comprehension Type

Solutions on page 2.88

Based upon each paragraph, there multiple choice questions have to be answered. Each face equation has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1–3

Each side of a square has length 4 units and its centre is at (3, 4). If one of the diagonals is parallel to the line $y = x$, then answer the following questions.

1. Which of the following is not the vertex of the square?

   a. (1, 6) **b.** (5, 2) **c.** (1, 2) **d.** (4, 6)

2. The radius of the circle inscribed in the triangle formed by any three vertices is

   a. $2\sqrt{2}(\sqrt{2} + 1)$ **b.** $2\sqrt{2}(\sqrt{2} - 1)$

   c. $2(\sqrt{2} + 1)$ **d.** None of these

3. The radius of the circle inscribed in the triangle formed by any two vertices of square and the centre is

   a. $2(\sqrt{2} - 1)$ **b.** $2(\sqrt{2} + 1)$

   c. $\sqrt{2}(\sqrt{2} - 1)$ **d.** None of these

For Problems 4–6

Tangents PA and PB are drawn to the circle $(x - 4)^2 + (y - 5)^2 = 4$ from the point P on the curve $y = \sin x$, where A and B lie on the circle. Consider the function $y = f(x)$ represented by the locus of the center of the circumcircle of triangle PAB, then answer the following questions.

4. Range of $y = f(x)$ is

   a. $[-2, 1]$ **b.** $[-1, 4]$ **c.** $[0, 2]$ **d.** $[2, 3]$

5. Period of $y = f(x)$ is

   a. 2π **b.** 3π

   c. π **d.** Not defined

6. Which of the following is true?

   a. $f(x) = 4$ has real roots

   b. $f(x) = 1$ has real roots

   c. Range of $y = f^{-1}(x)$ is $\left[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2\right]$

   d. None of these

For Problems 7–9

Consider a family of circles passing through the points (3, 7) and (6, 5). Answer the following questions.

7. Number of circles which belong to the family and also touching x-axis are

a. 0 b. 1 c. 2 d. Infinite

8. If each circle in the family cuts the circle $x^2+y^2-4x-6y-3=0$, then all the common chords pass through the fixed point which is

a. (1, 23) b. (2, 23/2)

c. (– 3, 3/2) d. None of these.

9. If circle which belong the given family cuts the circle $x^2+y^2=29$ orthogonally then centre of that circle is

a. (1/2, 3/2) b. (9/2, 7/2)

c. (7/2, 9/2) d. (3, –7/9)

For Problems 10–12

Consider the relation $4l^2-5m^2+6l+1=0$, where $l, m \in R$, then the line $lx+my+1=0$ touches a fixed circle whose

10. Centre and radius of circle one

a. (2, 0), 3 b. (–3, 0), $\sqrt{3}$

c. (3, 0), $\sqrt{5}$ d. None of these

11. Tangents PA and PB are drawn to the above fixed circle from the points P on the line $x+y-1=0$. Then chord of contact AB passes though the fixed point

a. $\left(\frac{1}{2},-\frac{5}{2}\right)$ b. $\left(\frac{1}{3},\frac{4}{3}\right)$

c. $\left(-\frac{1}{2},\frac{3}{2}\right)$ d. None of these

12. Number of tangents which can be drawn from the point (2, – 3) are

a. 0 b. 1 c. 2 d. 1 or 2

For Problems 13–15

A circle C whose radius is 1 unit, touches x-axis at point A. The centre Q of C lies in first quadrant. The tangent from origin O to the circle touches it at T and a point P lies on it such that ΔOAP is a right-angled triangle at A and its perimeter is 8 units.

13. The length of PQ is

a. $\frac{1}{2}$ b. $\frac{4}{3}$

c. $\frac{5}{3}$ d. None of these

14. Equation of circle C is

a. $\{x-(2+\sqrt{3})\}^2+(y-1)^2=1$

b. $\{x-(\sqrt{3}-\sqrt{2})\}^2+(y-1)^2=1$

c. $(x-\sqrt{3})^2+(y-1)^2=1$

d. None of these

15. Equation of tangent OT is

a. $x-\sqrt{3}y=0$ b. $x-\sqrt{2}y=0$

c. $y-\sqrt{3}x=0$ d. None of these

For Problems 16–18

P is a variable point on the line $L=0$. Tangents are drawn to the circles $x^2+y^2=4$ from P to touch it at Q and R. The parallelogram $PQSR$ is completed.

16. If $L \equiv 2x+y-6=0$, then the locus of circumcentre of ΔPQR is

a. $2x-y=4$ b. $2x+y=3$

c. $x-2y=4$ d. $x+2y=3$

17. If $P \equiv (6, 8)$, then the area of ΔQRS is

a. $\frac{(6)^{3/2}}{25}$ sq. units b. $\frac{(24)^{3/2}}{25}$ sq. units

c. $\frac{48\sqrt{6}}{25}$ sq. units d. $\frac{192\sqrt{6}}{25}$ sq. units

18. If $P \equiv (3, 4)$, then coordinates of S are

a. $\left(-\frac{46}{25},\frac{63}{25}\right)$ b. $\left(\frac{-51}{25},\frac{-68}{25}\right)$

c. $\left(-\frac{46}{25},\frac{68}{25}\right)$ d. $\left(-\frac{68}{25},\frac{51}{25}\right)$

For Problems 19–21

To the circle $x^2+y^2=4$, two tangents are drawn from $P(-4, 0)$, which touches the circle at T_1 and T_2, a rhombus $PT_1P'T_2$ is completed.

19. Circumcentre of the triangle PT_1T_2 is at

a. (– 2, 0) b. (2, 0)

c. $\left(\frac{\sqrt{3}}{2},0\right)$ d. None of these

20. Ratio of the area of triangle PT_1P' to that the $P'T_1T_2$ is

a. 2 : 1 b. 1 : 2

c. $\sqrt{3}$: 2 d. None of these

21. If P is taken to be at $(h, 0)$ such that P' lies on the circle, the area of the rhombus is

a. $6\sqrt{3}$ b. $2\sqrt{3}$

c. $3\sqrt{3}$ d. None of these

For Problems 22–24

Let α chord of a circle be that chord of the circle which subtends an angle α at the centre.

22. If $x+y=1$ is a chord of $x^2+y^2=1$, then α is equal to

a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$

c. $\frac{\pi}{6}$ d. $x+y=1$ is not a chord

23. If slope of a $\frac{\pi}{3}$ chord of $x^2+y^2=4$ is 1, then its equation is

a. $x-y+\sqrt{6}=0$ b. $x-y=2\sqrt{3}$

c. $x-y=\sqrt{3}$ d. $x-y+\sqrt{3}=0$

24. Distance of $\frac{2\pi}{3}$ chord of $x^2 + y^2 + 2x + 4y + 1 = 0$ from the centre is

a. 1 **b.** 2 **c.** $\sqrt{2}$ **d.** $\frac{1}{\sqrt{2}}$

For Problems 25–27

Two variable chords AB and BC of a circles $x^2 + y^2 = a^2$ are such that $AB = BC = a$, and M and N are the midpoints of AB and BC, respectively, such that line joining MN intersect the circles at P and Q, where P is closer to AB and O is the centre of the circle.

25. $\angle OAB$ is

a. 30° **b.** 60° **c.** 45° **d.** 15°

26. Angles between tangents at A and C is

a. 90° **b.** 120° **c.** 60° **d.** 150°

27. Locus of point of intersection of tangents at A and C is

a. $x^2 + y^2 = a^2$ **b.** $x^2 + y^2 = 2a^2$

c. $x^2 + y^2 = 4a^2$ **d.** $x^2 + y^2 = 8a^2$

For Problems 28–30

Given two circles intersecting orthogonally having length of common chord 24/5 units. Radius of one of the circles is 3 units.

28. Radius of other circle is

a. 6 units **b.** 5 units **c.** 2 units **d.** 4 units

29. Angle between direct common tangent is

a. $\sin^{-1}\frac{24}{25}$ **b.** $\sin^{-1}\frac{4\sqrt{6}}{25}$

c. $\sin^{-1}\frac{4}{5}$ **d.** $\sin^{-1}\frac{12}{25}$

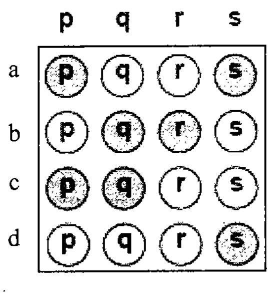
30. Length of direct common tangent is

a. $\sqrt{12}$ **b.** $4\sqrt{3}$ **c.** $2\sqrt{6}$ **d.** $3\sqrt{6}$

Matrix-Match Type

Solutions on page 2.91

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a → p, a → s, , b → q, b → r, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. Number of circles touching given three non-concurrent lines	**p.** 1
b. Number of circles touching $y = x$ at (2, 2) and also touching line $x + 2y - 4 = 0$	**q.** 2
c. Number of circles touching lines $x \pm y = 2$ and passing thorough the point (4, 3)	**r.** 4
d. Number of circle intersecting given three circles orthogonally	**s.** Infinite

2. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be an equation of circle.

Column I	Column II
a. If circle lie in first quadrant, then	**p.** $g < 0$
b. If circle lie above x-axis, then	**q.** $g > 0$
c. If circle lie on the left of y-axis, then	**r.** $g^2 - c < 0$
d. If circle touches positive x-axis and does not intersect y-axis, then	**s.** $c > 0$

3.

Column I	Column II
a. If $ax + by - 5 = 0$ is the equation of the chord of the circle $(x - 3)^2 + (y - 4)^2 = 4$, which passes through (2, 3) and at the greatest distance from the centre of the circle, then $\lvert a + b \rvert$ is equal to	**p.** 6
b. Let O be the origin and P be a variable point on the circle $x^2 + y^2 + 2x + 2y = 0$. If the locus of midpoint of OP is $x^2 + y^2 + 2gx + 2fy + c = 0$, then $(g + f)$ is equal to	**q.** 3
c. The x-coordinates of the centre of the smallest circle which cuts the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x + 12y + 52 = 0$ orthogonally is	**r.** 2
d. If θ be the angle between two tangents which are drawn to the circles $x^2 + y^2 - 6\sqrt{3}x - 6y + 27 = 0$ from the origin, then $2\sqrt{3}\tan\theta$ equals to	**s.** 1

4.

Column I	Column II
a. The length of the common chord of two circles of radii 3 and 4 units which intersect orthogonally is $\frac{k}{5}$, then k equals to	**p.** 1

b.	The circumference of the circle $x^2 + y^2 + 4x + 12y + p = 0$ is bisected by the circle $x^2 + y^2 - 2x + 8y - q = 0$, then $p + q$ is equal to	**q.**	24
c.	Number of distinct chords of the circle $2x(x - \sqrt{2}) + y(2y - 1) = 0$; chords are passing through the point $\left(\sqrt{2}, \frac{1}{2}\right)$ and are bisected on x-axis is	**r.**	32
d.	One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points $(-3, 4)$ and $(5, 4)$, respectively, then the area of the rectangle is	**s.**	36

5. Let C_1 and C_2 be two circles whose equations are $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 2x = 0$. $P(\lambda_1, \lambda)$ is a variable point. Then match the following.

Column I	Column II
a. P lies inside C_1 but outside C_2	**p.** $\lambda \in (-\infty, -1) \cup (0, \infty)$
b. P lies inside C_2 but outside C_1	**q.** $\lambda \in (-\infty, -1) \cup (1, \infty)$
c. P lies outside C_1 and outside C_2	**r.** $\lambda \in (-1, 0)$
d. P does not lie inside C_2	**s.** $\lambda \in (0, 1)$

6.

Column I	Column II
a. If two circles $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2x + b = 0$ touch each other then triplet (a_1, a_2, b) can be	**p.** $(2, 2, 2)$
b. If two circles $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2y + b = 0$ touch each other then triplet (a_1, a_2, b) can be	**q.** $\left(1, 1, \frac{1}{2}\right)$
c. If the straight line $a_1x - by + b^2 = 0$ touches the circle $x^2 + y^2 = a_2x + by$, then triplet (a_1, a_2, b) can be	**r.** $(2, 1, 0)$
d. If the line $3x + 4y - 4 = 0$ touches the circle $(x - a_1)^2 + (y - a_2)^2 = b^2$, then triplet (a_1, a_2, b) can be	**s.** $\left(1, 1, \frac{3}{5}\right)$

Integer Type

Solutions on page 2.93

1. Let the lines $(y - 2) = m_1(x - 5)$ and $(y + 4) = m_2(x - 3)$ intersect at right angles at P (where m_1 and m_2 are parameters). If locus of P is $x^2 + y^2 + gx + fy + 7 = 0$, then the value of $|f + g|$ is

2. Consider the family of circles $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$ passing through two fixed points A and B. Then the distance between the points A and B is

3. The number of points $P(x, y)$ lying inside or on the circle $x^2 + y^2 = 9$ and satisfying the equation $\tan^4 x + \cot^4 x + 2 = 4\sin^2 y$, is

4. If real numbers x and y satisfy $(x + 5)^2 + (y - 12)^2 = (14)^2$, then the minimum value of $\sqrt{x^2 + y^2}$ is

5. The line $3x + 6y = k$ intersect the curve $2x^2 + 2xy + 3y^2 = 1$ at points A and B. The circle on AB as diameter passes through the origin. Then the value of k^2 is

6. The sum of the slopes of the lines tangent to both circles $x^2 + y^2 = 1$ and $(x - 6)^2 + y^2 = 4$ is

7. A circle $x^2 + y^2 + 4x - 2\sqrt{2}\,y + c = 0$ is the director circle of circle S_1 and S_1 is the director circle of circle S_2 and so on. If the sum of radii of all these circles is 2, then the value of $c = k\sqrt{2}$, where value of k is

8. Two circles are externally tangent. Lines PAB and $PA'B'$ are common tangents with A and A' on the smaller circle and B and B' on the larger circle. If $PA = AB = 4$, then the square of radius of circle is

9. The length of a common internal tangent to two circles is 7 and a common external tangent is 11. If the product of the radii of the two circles is p, then the value of $p/2$ is

10. Line segments AC and BD are diameters of circle of radius one. If $\angle BDC = 60°$, the length of line segment AB is

11. As shown in Fig. 2.76, three circles which have the same radius r, have centres at (0,0), (1,1) and (2,1). If they have a common tangent line, as shown, then the value of $10\sqrt{5}r$ is

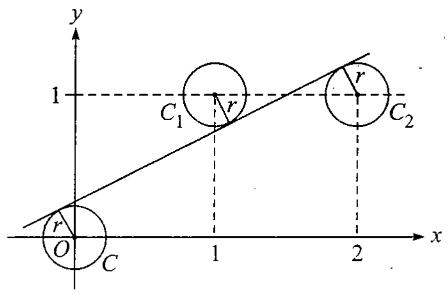


Fig. 2.76

12. The acute angle between the line $3x - 4y = 5$ and the circle $x^2 + y^2 - 4x + 2y - 4 = 0$ is θ, then $9\cos\theta$

13. If two perpendicular tangents can be drawn from the origin to the circle $x^2 - 6x + y^2 - 2py + 17 = 0$, then the value of $|p|$ is

14. Let $A(-4, 0)$ and $B(4, 0)$. If the number of points on the circle $x^2 + y^2 = 16$ such that the area of the triangle whose vertices are A, B and C is a positive integer, is N then the value of $[N/7]$ is, where N represents greatest integer function

15. If the circle $x^2 + y^2 + (3 + \sin \beta)x + (2 \cos \alpha)y = 0$ and $x^2 + y^2 + (2 \cos \alpha)x + 2cy = 0$ touches each other then the maximum value of 'c' is

16. Two circles C_1 and C_2 both passes through the points $A(1, 2)$ and $E(2, 1)$ and touch the line $4x - 2y = 9$ at B and D respectively. The possible coordinates of a point C such that the quadrilateral $ABCD$ is a parallelogram is (a, b) then the value of $|ab|$ is

17. Difference in values of radius of a circle whose centre is at the origin and which touches the circle $x^2 + y^2 - 6x - 8y + 21 = 0$ is

Archives

Solutions on page 2.96

Subjective Type

1. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5). **(IIT-JEE, 1978)**

2. Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2 y = 1$ as c tends to 1.

3. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at the point C. Find the area of the quadrilateral $ABCD$. **(IIT-JEE, 1981)**

4. Find the equations of the circle passing through $(-4, 3)$ and touching the lines $x + y = 2$ and $x - y = 2$. **(IIT-JEE, 1982)**

5. Through a fixed point (h, k) secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of the midpoints of the secants by the circle is $x^2 + y^2 = hx + ky$. **(IIT-JEE, 1983)**

6. The abscissa of the two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation and the radius of the circle with AB as diameter. **(IIT-JEE, 1984)**

7. Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines. **(IIT-JEE, 1986)**

8. Let a given line L_1 intersect the x and y axes at P and Q, respectively. Let another line L_2, perpendicular to L_1, cut the x and y-axis at R and S, respectively. Show that the locus of the point of intersection of the lines PS and QR is a circle passing through the origin. **(IIT-JEE, 1987)**

9. The circle $x^2 + y^2 - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the co-ordinate axes. The locus of the circumcnetre of the triangle is $x + y - xy + k(x^2 + y^2)^{1/2} = 0$. Find k. **(IIT-JEE,1987)**

10. Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends a right angle at the origin. **(IIT-JEE, 1988)**

11. If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0$, i = 1, 2, 3, 4 are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$. **(IIT-JEE, 1989)**

12. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle. **(IIT-JEE, 1990)**

13. Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$, find the equation of the circles. **(IIT-JEE, 1991)**

14. Let a circle be given by $2x(x - a) + y(2y - b) = 0$, $(a \neq 0, b \neq 0)$. Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from $\left(a, \frac{b}{2}\right)$. **(IIT-JEE, 1992)**

15. Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point. **(IIT-JEE, 1993)**

16. Find the coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y = -4$ and $x^2 + y^2 - 12x - 8y = -36$ touch each other. Also find equations common tangents touching the circles in the distinct points. **(IIT-JEE, 1993)**

17. Find the intervals of values of a for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$. **(IIT-JEE, 1996)**

18. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line from the point P intersects the curve at points Q and R. If the product $PQ \cdot PR$ is independent of the slope of the line, then show that the curve is a circle. **(IIT-JEE, 1997)**

19. Let C be any circle with centre $(0, \sqrt{2})$. Prove that at the most two rational points can be there on C (a rational point is a point both of whose coordinates are irrational numbers). **(IIT-JEE, 1997)**

20. C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. **(IIT-JEE, 1998)**

21. Let T_1, T_2 be two tangents drawn from $(-2, 0)$ onto the circle $C: x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time.

22. Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA. **(IIT-JEE, 2001)**

23. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum. **(IIT-JEE, 2003)**

24. Find the equation of circle touching the line $2x + 3y + 1 = 0$ at $(1, -1)$ and cutting orthogonally the circle having line segment joining $(0, 3)$ and $(-2, -1)$ as diameter. **(IIT-JEE, 2004)**

25. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C. (IIT-JEE, 2009)

Objective Type

Fill in the blanks

1. If A and B are points in the plane such that $\frac{PA}{PB} = k$ (constant) for all P on a given circle, then the value of k cannot be equal to____________. **(IIT-JEE, 1982)**

2. The points of intersection of the line $4x - 3y - 10 = 0$ and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ are _____ and_____. **(IIT-JEE, 1983)**

3. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle. The radius of this circle is_________. **(IIT-JEE, 1984)**

4. Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from the point (4, 5) with a pair of radii form a quadrilateral of area____________. **(IIT-JEE, 1985)**

5. From the origin chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. The equation of the locus of the midpoints of these chord is________________. **(IIT-JEE, 1985)**

6. The equation of the line passing through the points of intersection of the circles $3x^2 + 3y^2 - 2x + 12y - 9 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$ is_________. **(IIT-JEE, 1986)**

7. From the point $A(0, 3)$ on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that $AM = 2AB$. The equation of the locus of M is________. **(IIT-JEE, 1986)**

8. The area of the triangle formed by the tangents from the point (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their points of contact is___________. **(IIT-JEE, 1988)**

9. If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that common chord is of maximum length and has a slope equal to $\frac{3}{4}$, then the coordinates of the centre of C_2 are____. **(IIT-JEE, 1988)**

10. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is,______________. **(IIT-JEE, 1989)**

11. If a circle passes through points of intersection of the coordinates axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is___. **(IIT-JEE, 1991)**

12. The equation of the locus of the midpoints of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\frac{2\pi}{3}$ at its centre is_____________. **(IIT-JEE, 1993)**

13. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is________________. **(IIT-JEE, 1996)**

14. Two vertices of an equilateral triangle are (–1, 0) and (1, 0), and its third vertex lies above the x-axis. The equation of its circumcircle is___________. **(IIT-JEE, 1997)**

15. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to circle $x^2 + y^2 = 1$ pass through point________________. **(IIT-JEE, 1997)**

True or false

1. No tangent can be drawn from the point (5/2, 1) to the circumcircle of the triangle with vertices $(1, \sqrt{3}), (1, -\sqrt{3}, (3, -\sqrt{3})$. **(IIT-JEE, 1985)**

2. The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$. Its sides are parallel to the coordinate axes. The one vertex of the square is **(IIT-JEE, 1980)**
   a. $(1 + \sqrt{2}, -2)$ b. $(1 - \sqrt{2}, -2)$
   c. $(1, -2 + \sqrt{2})$ d. None of these

2. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Then, the equation of the circle through their points of intersection and the point (1, 1) is
   a. $x^2 + y^2 - 6x + 4 = 0$
   b. $x^2 + y^2 - 3x + 1 = 0$
   c. $x^2 + y^2 - 4y + 2 = 0$
   d. None of these **(IIT-JEE, 1980)**

3. The equation of the circle passing through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is
   a. $4x^2 + 4y^2 - 30x - 10y - 25 = 0$
   b. $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
   c. $4x^2 + 4y^2 - 17x - 10y + 25 = 0$
   d. None of these **(IIT-JEE, 1983)**

4. The locus of the midpoint of a chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin is
   a. $x + y = 2$ b. $x^2 + y^2 = 1$
   c. $x^2 + y^2 = 2$ d. $x + y = 1$
   (IIT-JEE, 1984)

5. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = k^2$ orthogonally, then the equation of the locus of its centre is **(IIT-JEE, 1988)**
   a. $2ax + 2by - (a^2 + b^2 + k^2) = 0$
   b. $2ax + 2by - (a^2 - b^2 + k^2) = 0$
   c. $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$
   d. $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - k^2) = 0$

6. If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then
   a. $2 < r < 8$ b. $r < 2$
   c. $r = 2$ d. $r > 2$

7. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. Then the equation of this circle is
   a. $x^2 + y^2 + 2x - 2y = 62$
   b. $x^2 + y^2 + 2x - 2y = 47$
   c. $x^2 + y^2 - 2x + 2y = 47$
   d. $x^2 + y^2 - 2x + 2y = 62$

8. The centre of the circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is
   (IIT-JEE, 1992)
   a. $\left(\frac{3}{2}, \frac{1}{2}\right)$ b. $\left(\frac{1}{2}, \frac{3}{2}\right)$
   c. $\left(\frac{1}{2}, \frac{1}{2}\right)$ d. $\left(\frac{1}{2}, -2^{\frac{1}{2}}\right)$

9. The locus of the centre of a circle, which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis, is given by equation **(IIT-JEE, 1993)**
   a. $x^2 - 6x - 10y + 14 = 0$
   b. $x^2 - 10x - 6y + 14 = 0$
   c. $y^2 - 6x - 10y + 14 = 0$
   d. $y^2 - 10x - 6y + 14 = 0$

10. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9 \sin^2 \alpha + 13 \cos^2 \alpha = 0$ is 2α. Then equation of the locus of the point P is
    (IIT-JEE, 1996)
    a. $x^2 + y^2 + 4x - 6y + 4 = 0$
    b. $x^2 + y^2 + 4x - 6y - 9 = 0$
    c. $x^2 + y^2 + 4x - 6y - 4 = 0$
    d. $x^2 + y^2 + 4x - 6y + 9 = 0$

11. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 - px - qy = 0$ (where $pq \neq 0$) are bisected by the x-axis, then
    a. $p^2 = q^2$ b. $p^2 = 8q^2$
    c. $p^2 < 8q^2$ d. $p^2 > 8q^2$
    (IIT-JEE, 1999)

12. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) and (− 4, 3) respectively, then $\angle QPR$ is equal to
    a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{6}$
    (IIT-JEE, 2000)

13. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$, $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is
    (IIT-JEE, 2000)
    a. 2 or $-\frac{3}{2}$ b. -2 or $-\frac{3}{2}$
    c. 2 or $\frac{3}{2}$ d. -2 or $\frac{3}{2}$

14. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of the triangle PAB as P moves on the circle is
    (IIT-JEE, 2001)
    a. a parabola b. a circle

c. an ellipse d. a pair of straight lines

15. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and PQ intersect at a point X on the circumference of the circle, then $2r$ equals **(IIT-JEE, 2001)**

a. $\sqrt{PQ.RS}$ b. $\frac{(PQ+RS)}{2}$

c. $\frac{2PQ \times RS}{PQ+RS}$ d. $\frac{\sqrt{(PQ^2+RS^2)}}{2}$

16. If the tangent at the point P on the circle $x^2+y^2+6x+6y = 2$ meets a straight lines $5x-2y+6=0$ at a point Q on the y-axis, then the length of PQ is **(IIT-JEE, 2002)**

a. 4 b. $2\sqrt{5}$ c. 5 d. $3\sqrt{5}$

17. The centre of circle inscribed in square formed by the lines $x^2-8x+12=0$ and $y^2-14y+45=0$, is

a. (4, 7) b. (7, 4) c. (9, 4) d. (4, 9)

(IIT-JEE, 2003)

18. If one of the diameters of the circle $x^2+y^2-2x-6y+6 = 0$ is a chord to the circle with centre (2, 1), then the radius of the circle is **(IIT-JEE, 2004)**

a. $\sqrt{3}$ b. $\sqrt{2}$ c. 3 d. 2

19. A circle is given by $x^2+(y-1)^2=1$, another circle C touches it externally and also the x-axis, then the locus of its centre is

a. $\{(x,y): x^2=4y\} \cup \{(x,y): y \le 0\}$

b. $\{(x,y): x^2+(y-1)^2=4\} \cup \{(x,y): y \le 0\}$

c. $\{(x,y): x^2=y\} \cup \{(0,y): y \le 0\}$

d. $\{(x,y): x^2=4y\} \cup \{(0,y): y \le 0\}$

(IIT-JEE, 2005)

20. Tangents drawn from the point $P(1, 8)$ to the circle $x^2+y^2-6x-4y-11=0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is

a. $x^2+y^2+4x-6y+19=0$

b. $x^2+y^2-4x-10y+19=0$

c. $x^2+y^2-2x+6y-20$

d. $x^2+y^2-6x-4y+19=0$

21. The circle passing through the point $(-1, 0)$ and touching the y–axis at (0, 2) also passes through the point

(A) $\left(-\frac{3}{2},0\right)$ (B) $\left(-\frac{5}{2},2\right)$

(C) $\left(-\frac{3}{2},\frac{5}{2}\right)$ (D) (–4, 0) **(IIT-JEE, 2010)**

Multiple choice questions with one or more than one correct answer

1. The equations of the tangents drawn from the origin to the circle $x^2+y^2-2rx-2hy+h^2=0$, are

a. $x=0$

b. $y=0$

c. $(h^2-r^2)x-2rhy=0$

d. $(h^2-r^2)x+2rhy=0$

(IIT-JEE, 1988)

2. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then **(IIT-JEE, 1998)**

a. $x_1+x_2+x_3+x_4=0$

b. $y_1+y_2+y_3+y_4=0$

c. $x_1x_2x_3x_4=c^4$

d. $y_1y_2y_3y_4=c^4$

Comprehension Type

For Problems 1–3

Let $ABCD$ be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of square $ABCD$. L is a line through A.

1. If P is a point on C_1 and Q in another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to

a. 0.75 b. 1.25 c. 1 d. 0.5

(IIT-JEE, 2006)

2. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

a. ellipse b. hyperbola
c. parabola d. pair of straight line

3. A line M through A is drawn parallel to BD. Point S moves such that its distance from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is **(IIT-JEE, 2006)**

a. 1/2 sq. units b. 2/3 sq. units
c. 1 sq. units d. 2 sq. units

For Problems 4–6

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}x+y-6=0$ and the point D is $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ. **(IIT-JEE, 2008)**

4. The equation of circle C is

a. $(x-2\sqrt{3})^2+(y-1)^2=1$

b. $(x-2\sqrt{3})^2+\left(y+\frac{1}{2}\right)^2=1$

c. $(x-\sqrt{3})^2+(y+1)^2=1$

d. $(x-\sqrt{3})^2+(y-1)^2=1$

5. Points E and F are given by

a. $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),(\sqrt{3},0)$

b. $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right),(\sqrt{3},0)$

c. $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

d. $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

6. Equation of the sides QR, RP are

a. $y=\frac{2}{\sqrt{3}}x+1, y=-\frac{2}{\sqrt{3}}x-1$

b. $y=\frac{1}{\sqrt{3}}x, y=0$

c. $y=\frac{\sqrt{3}}{2}x+1, y=-\frac{\sqrt{3}}{2}x-1$

d. $y=\sqrt{3}x, y=0$

Assertion and reasoning

1. Tangents are drawn from the point (17, 7) to the circle $x^2+y^2=169$.

Statement 1: The tangents are mutually perpendicular.

Statement 2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2+y^2=338$. **(IIT-JEE, 2007)**

a. Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

b. Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

c. Statement 1 is true, Statement 2 is false.

d. Statement 1 is false, Statement 2 is true.

2. Consider

$L_1: 2x+3y+p-3=0$

$L_2: 2x+3y+p+3=0,$

where p is a real number, and $C: x^2+y^2+6x-10y+30=0$.

Statement 1: If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

and

Statement 2: If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

a. Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

b. Statement 1 is true, Statement 2 is true; Statement 2 is NOT a correct explanation for statement 1.

c. Statement 1 is true, Statement 2 is false

d. Statement 1 is false, statement 2 is true

(IIT-JEE, 2008)

Matrix-match

1. This question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q, and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a.	(p)	(q)	(r)	(s)
b.	(p)	(q)	(r)	(s)
c.	(p)	(q)	(r)	(s)
d.	(p)	(q)	(r)	(s)

(IIT-JEE, 2007)

	Column I		Column II
a.	Two intersecting circles	**p.**	have a common tangent
b.	Two mutually external circles	**q.**	have a common normal
c.	Two circles, one strictly inside the other	**r.**	do not have a common tangent
d.	Two branches of a hyperbola	**s.**	do not have a common normal

Integer type

1. The straight line $2x-3y=1$ divides the circular region $x^2+y^2\le 6$ into two parts. If

$$S=\left\{\left(2,\frac{3}{4}\right),\left(\frac{5}{2},\frac{3}{4}\right),\left(\frac{1}{4},-\frac{1}{4}\right),\left(\frac{1}{8},\frac{1}{4}\right)\right\}.$$

then the number of point(s) in S lying inside the smaller part is **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1.

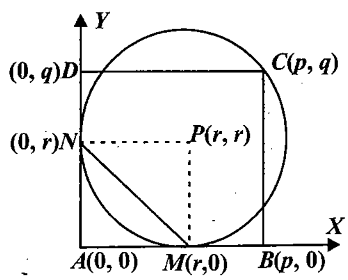


Fig. 2.77

Let us take AB and AD as coordinate axes

If r be the radius of circle, then its centre is $P(r, r)$.

Equation of circle is $(x - r)^2 + (y - r)^2 = r^2$

$\Rightarrow x^2 + y^2 - 2rx - 2ry + r^2 = 0$

Let the coordinates of $C \equiv (p, q)$

Equation of MN is $x + y = r$

Its distance from C is 5 units $\Rightarrow \dfrac{|p+q-r|}{\sqrt{2}} = 5$

$\Rightarrow \quad (p + q - r)^2 = 50$

Since (p, q) lie on the circle,

$p^2 + q^2 - 2rp - 2rq + r^2 = 0$

$\Rightarrow \quad (p + q - r)^2 - 2pq = 0$

$\Rightarrow \quad 50 - 2pq = 0 \Rightarrow pq = 25$

Area of rectangle = 25 sq. units

2.

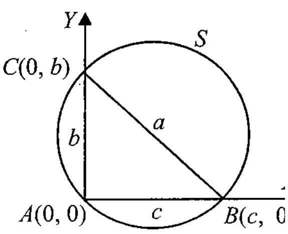


Fig. 2.78

Take AB as the x-axis and AC as the y-axis

Let $\quad BC = a$

Centre of S is $\left(\frac{c}{2}, \frac{b}{2}\right)$ and radius is $\frac{a}{2}$

If a circle S' touches the rays AB and AC, its centre must be (r, r) and its radius must be r, for some $r > 0$.

Circle S and S' touch each other if

$$\sqrt{\left(r-\frac{c}{2}\right)^2 + \left(r-\frac{b}{2}\right)^2} = \frac{a}{2} \pm r$$

Squaring both sides and using the fact that $a^2 = b^2 + c^2$, we get

$$r = b + c \pm a$$

$\Rightarrow \quad r_1 = b + c - a,\ r_2 = b + c + a$

$\Rightarrow \quad r_1 \cdot r_2 = (b + c)^2 - a^2 = 2bc$ (as $a^2 = b^2 + c^2$)

$= 4$ area (ΔABC)

3. The given circles are

$C_1 : (x - 1)^2 + (y - 1)^2 = 1,\ r_1 = 2$

and $C_2 : (x - 8)^2 + (y - 1)^2 = 4,\ r_2 = 2$

The line $y - 2x - a = 0$ will lie between these circles if centre of the circles lie on opposite sides of the line,

i.e. $(1 - 2 - a)(1 - 16 - a) < 0$

$\Rightarrow \quad a \in (-15, -1)$

Line wouldn't touch or intersect the circles, if

$$\frac{|1-2-a|}{\sqrt{5}} > 1,\ \frac{|1-16-a|}{\sqrt{5}} > 2$$

$\Rightarrow \quad |1 + a| > \sqrt{5},\ |15 + a| > 2\sqrt{5}$

$\Rightarrow a > \sqrt{5} - 1$ or $a < \sqrt{5} - 1$, $a > 2\sqrt{5} - 15$

or $a < -2\sqrt{5} - 15$

Hence, common values of 'a' are $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

4. The given circle is:

$x^2 + y^2 - 2ax - 2by + 2 = 0$

or $\quad (x - a)^2 + (y - b)^2 = a^2 + b^2 - 2$

it's director circle is

$(x - a)^2 + (y - b)^2 = 2(a^2 + b^2 - 2)$

Given that tangents drawn from the origin to the circle are orthogonal, it implies that director circle of the circle must pass through the origin,

$\Rightarrow \quad a^2 + b^2 = 2(a^2 + b^2 - 2)$

$\Rightarrow \quad a^2 + b^2 = 4$

Thus, the locus of the centre of the given circle is,

$$x^2 + y^2 = 4$$

5. If 'd' be the common difference of A.P., then radius of the smallest circle is $1 - 2d$. If the given line $y - x - 1 = 0$ cuts the smallest circle in real and distinct points, then it will definitely cut the remaining circles in real and distinct points.

$\Rightarrow \quad \dfrac{|0-0-1|}{\sqrt{2}} < (1 - 2d)$

$\Rightarrow \quad 1 - 2d > \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad d < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$

Hence, $\quad d \in \left(0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right)$

6. Equation of line PQ is

$$(y - 1) = m(x - 4)$$

or $\quad y - mx + 4m - 1 = 0$

For the required 'm', we have to make sure that the line PQ meets the circle, with diameter AB, at real and distinct points.

Equation of circle having AB as diameter is

$$x^2 + y^2 - 2x - 3 = 0$$

$$\Rightarrow \quad \frac{|0 - m + 4m - 1|}{\sqrt{1 + m^2}} < 2$$

$$\Rightarrow \quad 5m^2 - 6m - 3 > 0$$

$$\Rightarrow \quad m \in \left(\frac{3 - 2\sqrt{6}}{5}, \frac{3 + 2\sqrt{6}}{5}\right)$$

7. Equation of the circle having the ends of diameter at $(1, -3)$ and $(4, 1)$ is

$(x - 1)(x - 4) + (y + 3)(y - 1) = 0$

$\Rightarrow \quad x^2 + y^2 - 5x + 2y + 1 = 0$

Other circle will be $x^2 + y^2 - 5x + 2y + 1 + \lambda(x + y - 1) = 0$

It passes through $(1, 2)$

$$\Rightarrow \quad \lambda = -\frac{5}{2}$$

$\Rightarrow$ circle is; $x^2 + y^2 - \frac{15}{2}x - \frac{y}{2} + \frac{7}{2} = 0$

8. Let radius $= r$

$\therefore$ From figure $\quad \sqrt{\alpha^2 + a^2} = b + r \quad$ (i)

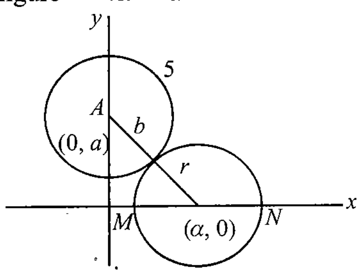


Fig. 2.79

Consider a point $P(0, k)$ on the y-axis

$M(\alpha - r, 0)$ and $N(\alpha + r, 0)$

Now, slope of $\quad MP = \frac{-k}{\alpha - r}$, slope $NP = \frac{-k}{\alpha + r}$

If $\quad \angle MPN = \theta$

$$\tan\theta = \left|\frac{\frac{k}{\alpha - r} - \frac{k}{\alpha + r}}{1 + \frac{k^2}{\alpha^2 - r^2}}\right|$$

$$= \left|\frac{2kr}{\alpha^2 - r^2 + k^2}\right|$$

According to the given condition, θ is a constant for any choice α

$$\frac{2kr}{\alpha^2 - r^2 + k^2} = \text{constant}$$

i.e., $\quad \frac{r}{\alpha^2 - r^2 + k^2} = \text{constant}$

i.e., $\quad \frac{\sqrt{\alpha^2 + a^2} - b}{\alpha^2 - \left(\sqrt{\alpha^2 + a^2} - b^2\right) + k^2} = \text{constant}$

(From Eq. (1))

i.e., $\quad \frac{\sqrt{\alpha^2 + a^2} - b}{2b\sqrt{\alpha^2 + a^2} - a^2 - b^2 + k^2} = \text{constant}$

$$\frac{\sqrt{\alpha^2 + a^2} - b}{\sqrt{\alpha^2 + a^2} - \lambda} = \text{constant}$$

where $\quad \left\{\frac{a^2 + b^2 - k^2}{2b}\right\} = \lambda$

which is possible only, if $\lambda = b$

$$\frac{a^2 + b^2 - k^2}{2b} = b \Rightarrow k = \pm\sqrt{a^2 - b^2}$$

$$\therefore \quad P \equiv \left(0, \pm\sqrt{a^2 - b^2}\right)$$

9. $S(x, 2) = 0$ given two identical solutions $x = 1$

$\Rightarrow$ line $y = 2$ is a tangent to the circle $S(x, y) = 0$ at the point $(1, 2)$ and $S(1, y) = 0$ gives two distinct solutions $y = 0, 2$.

$\Rightarrow$ Line $x = 1$ cuts the circle $S(x, y) = 0$ at points $(1, 0)$ and $(1, 2)$.

$\therefore A(1, 2)$ and $B(1, 0)$ are diametrically opposite points.

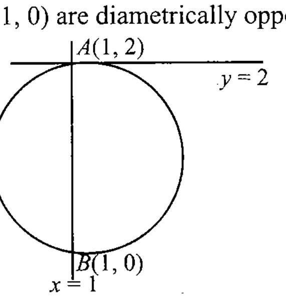


Fig. 2.80

$\therefore$ Equation of the circle is $(x - 1)^2 + y(y - 2) = 0$

$$x^2 + y^2 - 2x - 2y + 1 = 0$$

10. $x^2 - y^2 + 2y - 1 = 0$

$$(x - y + 1)(x + y - 1) = 0$$

The centre of family of circles touching the above lines will lie on the angle bisectors of the above lines.

Equations of the angle bisectors of the above lines are given by

$$\frac{x - y + 1}{\sqrt{2}} = \pm\frac{x + y - 1}{\sqrt{2}}$$

$\Rightarrow x = 0$ and $y = 1$

Case 1:

Let $A(h, 1)$ be a point on the line $y = 1$, $AP = \frac{h}{\sqrt{2}}$

Equation of circle touching the given line is $(x - h)^2 + (y - 1)^2 = \frac{h^2}{2}$

i.e., $x^2 + y^2 - 2hx - 2y + \frac{h^2}{2} = 0$

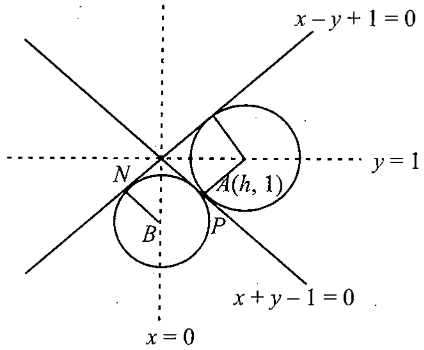


Fig. 2.81

Case 2:

Let $B(0, k)$ be a point on the line $x = 0$.

Now, $$BN = \frac{k}{\sqrt{2}}$$

Equation of circle touching the given lines is

$$(x - 0)^2 + (y - k)^2 = \frac{k^2}{2}$$

or $$x^2 + y^2 - 2ky + \frac{k^2}{2} = 0$$

11. AB subtends the greatest angle at C, so the line $x - y + 1 = 0$ touches the circle at C and hence AB is the diameter.

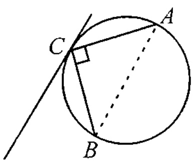


Fig. 2.82

Family of circle touching line $x - y + 1 = 0$ at point (1, 2) is $(x - 1)^2 + (y - 2)^2 + \lambda(x - y + 1) = 0$

Its radius $= \sqrt{\left(\frac{\lambda - 2}{2}\right)^2 + \left(\frac{\lambda + 4}{2}\right)^2 - (5 + \lambda)}$

$= \sqrt{2}$

$\Rightarrow$ $$\lambda = \pm 2$$

Therefore, The equations of circle are $x^2 + y^2 - 64 + 7 = 0$

and $x^2 + y^2 - 4x - 2y + 3 = 0$.

12.

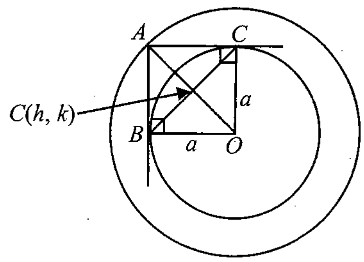


Fig. 2.83

Clearly, $x^2 + y^2 = 2a^2$ is director circle of the circle $x^2 + y^2 = a^2$

Hence, in the diagram $ABCO$ is a square and circumcentre $P(h, k)$ of ΔABC is midpoint of OA

Hence, $\sqrt{h^2 + k^2} = \frac{\sqrt{2}a}{2}$

or locus is $x^2 + y^2 = \frac{a^2}{2}$

13. Circle of minimum radius that touches the given circle $x^2 + y^2 = 4$ will be possible only if (4, 3) is the end point of the diameter of the required circle. Let the centre of the required circle be O' and point of contact of circles B.

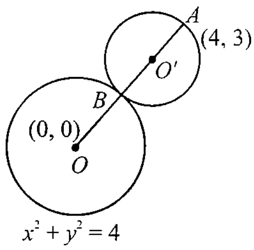


Fig. 2.84

Now $$OA = 5 \text{ and } OB = 2$$

$$BA = 5 - 2 = 3$$

$\Rightarrow$ Point B divides OA in the ratio 2 : 3 internally

$\Rightarrow$ Point $$B = \left(\frac{8}{5}, \frac{6}{5}\right)$$

$\Rightarrow$ Point O' is $\left(\frac{14}{5}, \frac{21}{10}\right)$.

Hence, the required circle is $\left(x - \frac{14}{5}\right)^2 + \left(y - \frac{21}{10}\right)^2 = \frac{9}{4}$

14. Let the point of intersection of tangents at A and C is $P = (x, y)$.

Since $$AB = AO = BO = r$$

$\Rightarrow$ $$\angle AOB = 60°$$

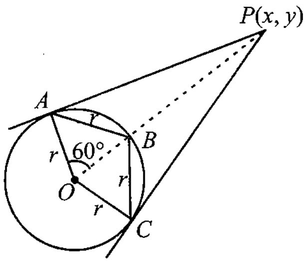


Fig. 2.85

$\Rightarrow \quad \frac{PA}{r} = \tan 60° = \sqrt{3}$

$\Rightarrow \quad \sqrt{x^2 + y^2 - r^2} = \sqrt{3}r$

$\Rightarrow x^2 + y^2 = 4r^2$ is the locus of the point P.

15. Angle between $3x + y = 0$ and the line joining $(2, -1)$ to $(0, 0)$ is

$$\theta = \tan^{-1}\left|\frac{-3 + \frac{1}{2}}{1 + (-3)\left(-\frac{1}{2}\right)}\right|$$

$$= \tan^{-1}|-1| = \frac{\pi}{4}$$

$\Rightarrow$ Other tangent is perpendicular to $3x + y = 0$.

$\Rightarrow$ Equation of the other tangent is $x - 3y = 0$.

16.

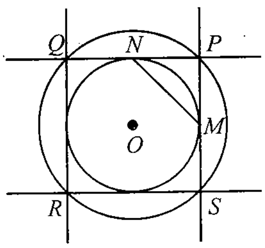


Fig. 2.86

From the diagram, $OMPN$ is square and P lies on director circle.

$MN = OP$

= radius of the direction of given circle

$= \sqrt{2}\,|\sqrt{2}\,b| = 2b$

17. Radius of given circle $= \sqrt{4 + 2 - c} = \sqrt{6 - c} = a$ (say)

Radius of circle $\quad S_1 = \frac{a}{\sqrt{2}}$

Radius of circle $\quad S_2 = \frac{a}{2}$ and so on.

$\therefore \quad a + \frac{a}{\sqrt{2}} + \frac{a}{2} + \cdots + \infty = 2$

$\Rightarrow \quad a\left[\frac{1}{1 - 1/\sqrt{2}}\right] = 2 \Rightarrow a\left(\frac{\sqrt{2}}{\sqrt{2} - 1}\right) = 2.$

$\Rightarrow \quad a = 2 - \sqrt{2} \Rightarrow \sqrt{6 - c} = 2 - \sqrt{2}$

$\Rightarrow \quad 6 - c = 4 + 2 - 4\sqrt{2} \Rightarrow c = 4\sqrt{2}$

18.

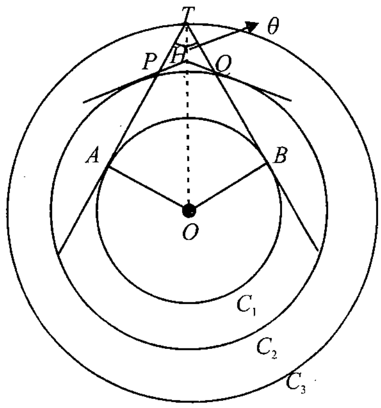


Fig. 2.87

Let C_1 be a circle of radius r.

So radius of $\quad C_2 = \sqrt{2}\,r$ and that of $C_3 = 2r$.

In ΔAOT, $\quad \sin\frac{\theta}{2} = \frac{r}{2r} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow \quad \angle ATB = \frac{\pi}{3}$

$\Rightarrow$ In ΔAOP, $\quad \sin \angle APO = \frac{r}{r\sqrt{2}} \Rightarrow \angle APO = \frac{\pi}{4}$

In ΔOPT, $\quad \angle POT = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

In ΔOPH,

$\angle OPH = \frac{\pi}{2}$,

$\angle PHO = \pi - \left(\frac{\pi}{2} + \frac{\pi}{12}\right) = \frac{5\pi}{12}$

$\Rightarrow \quad \angle PHQ = \frac{5\pi}{6}$. As $\angle POQ = \frac{\pi}{6}$

$OPHQ$ is a cyclic quadrilateral. Hence, locus of H is a circle.

19.

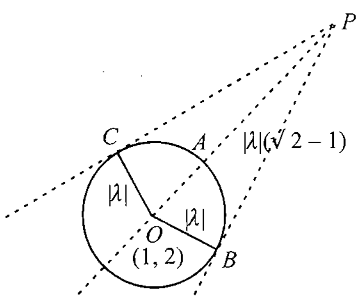


Fig. 2.88

Since line $y = x + c$ is normal to the given circle

$\Rightarrow$ $c = 1$

$\Rightarrow$ Equation of line is

$$y = x + 1 \quad \text{(i)}$$

Also, radius of the circle = $|\lambda|$

Given $AP = |\lambda| (\sqrt{2} - 1)$

$\Rightarrow$ $OP = \sqrt{2} |\lambda|$

$\Rightarrow$ $PC = |\lambda|$

$\Rightarrow$ Area of quadrilateral $OBPC$

$$= 2 \times \frac{1}{2} |\lambda|^2 = 36 \Rightarrow \lambda = \pm 6$$

20. Let the equation of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

This is orthogonal to given circle

$\Rightarrow$ $2g \,.\, 0 + 2f \,.\, 0 = -1 + c \Rightarrow c = 1$

and $2 \cdot 4 \cdot g + 2 \cdot 4 \cdot f = -33 + c = -32$

$\Rightarrow$ $g + f = -4$

Hence, radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{g^2 + (g + 4)^2 - 1}$

i.e., $r = \sqrt{2g^2 + 8g + 15}$

$= \sqrt{2(g + 2)^2 + 7}$

For minimum $r, g + 2 = 0$

$\Rightarrow$ $g = -2 \Rightarrow$ the centre is (2, 2).

21.

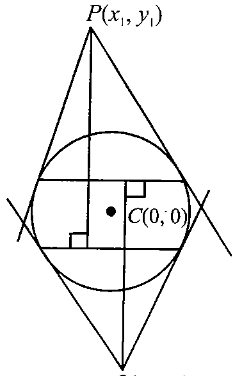


Fig. 2.89

Let the circle be $x^2 + y^2 = a^2$. Then the chord of contact of $Q(x_2, y_2)$ w.r.t. the circle is

$$xx_2 + yy_2 = a^2 \quad \text{(i)}$$

Its distance from $P(x_1, y_1)$ is $\dfrac{|a^2 - (x_1x_2 + y_1y_2)|}{\sqrt{x_2^2 + y_2^2}}$

Similarly, distance of Q from the chord of contact of P is $\dfrac{|a^2 - (x_1x_2 + y_1y_2)|}{\sqrt{x_1^2 + y_1^2}}$.

Hence, ratio of the lengths of perpendiculars $= \dfrac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}}$

$= \dfrac{\text{distance of the point } P \text{ from the centre of the circle}}{\text{distance of the point } Q \text{ from the centre of the circle}}$

22. The given circles are $(x - 1)^2 + y^2 = 4$ and $(x - 1)^2 + y^2 = 16$

The points $(a + 1, \sqrt{3}a)$ lie on the line

$\Rightarrow$ $x = a + 1, y = \sqrt{3}a$

$\Rightarrow$ $y = \sqrt{3}(x - 1)$ [eliminating a]

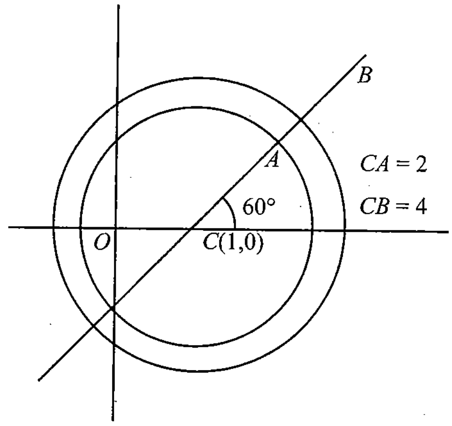


Fig. 2.90

This line makes an angle 60° with the +ve direction of the x-axis.

Hence, we have

$A \equiv (1 + 2 \cos 60°, 2 \sin 60°) \equiv (2, \sqrt{3})$

and $B \equiv (1 + 4 \cos 60°, 4 \sin 60°)(3, 2\sqrt{3})$

Hence, there is no point on the line segment AB whose abscissa is an integer since abscissa of A is 2 and that of B is 3.

23.

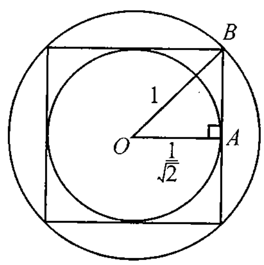


Fig. 2.91

The director circle $x^2 + y^2 = 2a^2$ must intersect the square formed by the lines $|x| + |y| = 1$ in eight points.

for which $\frac{1}{\sqrt{2}} < \sqrt{2}\, a < 1$ or $\frac{1}{2} < a < \frac{1}{\sqrt{2}}$

24. Obviously angle bisectors are $x = 2$ and $y = 0$

Now centre cannot lies on $y = 0$ because their chord of contact from origin will always be parallel to y-axis. So let the centre $(2, \alpha)$, then equation of circle will be

$$(x-2)^2 + (y-\alpha)^2 = 5^2$$

$$\Rightarrow x^2 + y^2 - 4x - 2\alpha y - 21\alpha^2 = 0$$

Now chord of contact is

$$-4\frac{x}{2} - 2\alpha\frac{y}{2} + \alpha^2 - 21 = 0 \Rightarrow 2x + \alpha y - \alpha^2 + 21 = 0$$

Now, $-\frac{2}{\alpha} = 1 \Rightarrow \alpha = -2$

So equation of circle is $(x-2)^2 + (y+2)^2 = 5^2$.

25. Equation of chord of contact AB is $5x - 5y = 5$

Solving it with $x^2 + y^2 = 5$ we get,

$$x^2 + (x-1)^2 = 5$$

$$\Rightarrow \quad x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow \quad x = -1, 2 \text{ and } y = -2, 1$$

So, A and B are $(-1, -2)$ and $(2, 1)$.

Let $\quad P \equiv (\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$

Now as circumcentre of the triangle PAB is origin, orthocentre would be $h = x_1 + x_2 + x_3$ and $k = y_1 + y_2 + y_3$ (as centroid divides line joining orthocentre and circumcentre in 2 : 1).

i.e. $\quad h = \sqrt{5}\cos\theta + 1$

and $\quad k = \sqrt{5}\sin\theta - 1$

So the required locus is

$$(x-1)^2 + (y+1)^2 = 5.$$

26.

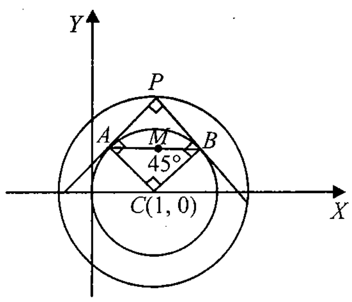


Fig. 2.92

The two circles are

$$(x-1)^2 + y^2 = 1 \quad \text{(i)}$$

$$(x-1)^2 + y^2 = 2 \quad \text{(ii)}$$

So the second one is the director circle of the first circle.

So $\angle APB = \frac{\pi}{2}$

$$\Rightarrow \quad \angle ACB = \frac{\pi}{2}$$

Now, circumcentre of the right angled triangle CAB would lie on the midpoint of AB.

So, let that point be $\quad M \equiv (h, k)$

Now, $\quad CM = CB\sin 45° = \frac{1}{\sqrt{2}}$

So, $\quad (h-1)^2 + k^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

So, locus of M is $(x-1)^2 + y^2 = \frac{1}{2}$.

27.

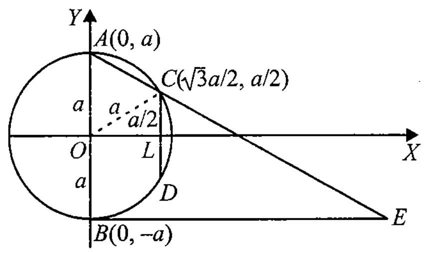


Fig. 2.93

Let origin be taken at the centre of the circle and y-axis along the diameter AB.

Then, equation of the circle is $x^2 + y^2 = a^2 \quad$ (i)

and coordinates of A and B are $(0, a)$ and $(0, -a)$, respectively.

If CD is parallel to AB such that $2CD = AB$, then $CD = \frac{1}{2}AB = a$

Obviously, CD is bisected by x-axis say at L, then $CL = a/2$.

In ΔOLC, $\quad OL = \sqrt{(OC^2 - CL^2)}$

$$= \sqrt{(a^2 - a^2/4)} = \sqrt{3}a/2$$

$\Rightarrow$ The coordinates of the point C are $(\sqrt{3}a/2, a/2)$.

$\Rightarrow$ Equation of line AC is $y - a = \frac{a/2 - a}{\sqrt{3}a/2 - 0}(x - 0)$ or

$$y = -\frac{1}{\sqrt{3}}x + a \quad \text{(ii)}$$

Equation of the tangent to the circle at the point $B(0, -a)$ on it is

$$x.0 + y.(-a) = a \text{ or } y = -a \quad \text{(iii)}$$

Solving Eqs. (ii) and (iii), their point of intersection is $E(2\sqrt{3}\,a, -a)$.

$$\Rightarrow \quad AE = \sqrt{[(2\sqrt{3}a - 0)^2 + (-a-a)^2]}$$

$$= 4a = 2AB$$

28. Equation of tangent to the circle at the point (h, k) is

$$hx + ky = a^2 \quad \text{(i)}$$

Pair of lines $y = a$ and $y = -a$ is $y^2 - a^2 = 0$

Pair of lines OR and OQ is given by making $y^2 - a^2 = 0$ homogeneous with the help of line (i)

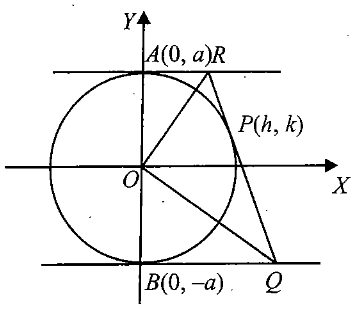


Fig. 2.94

$\therefore \quad y^2 - a^2\left(\frac{hx+ky}{a^2}\right)^2 = 0$

or $\quad a^2y^2 - (hx+ky)^2 = 0$

now, coefficient of x^2 + coefficient of y^2

$= -h^2 - k^2 + a^2$

$= 0$ (as (h, k) lies on the circle)

Hence, lines OR and OQ are perpendicular.

29.

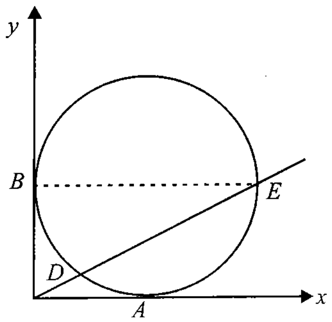


Fig. 2.95

The equation of the circle is $(x-1)^2 + (y-1)^2 = 1$

$\Rightarrow \quad x^2 + y^2 - 2x - 2y + 1 = 0$ (i)

Let the equation of the variable straight line be $y = mx$ (ii)

Solving Eqs. (i) and (ii), we get

$(1+m^2)x^2 - 2x(1+m) + 1 = 0$

$\therefore$ Length $\quad DE = \sqrt{\frac{8m}{1+m^2}}$

Area of ΔDEB,

$A = \frac{1}{2} DE \times$ distance of B from DE

$\therefore \quad A^2 = \frac{1}{4}\cdot\frac{8m}{1+m^2}\cdot\frac{1}{1+m^2}$

$= \frac{2m}{(1+m^2)^2}$

$\Rightarrow \quad A = \frac{\sqrt{2m}}{1+m^2}$

$$\frac{dA}{dm} = \frac{1-3m^2}{\sqrt{m}(1+m^2)^2} = 0$$

$\Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$

$$\frac{d^2A}{dm^2} < 0 \text{ if } m = \frac{1}{\sqrt{3}}$$

$\therefore$ Area is maximum for $m = \frac{1}{\sqrt{3}}$.

Objective Type

1. a. If there are more than one rational points on the circumference of the circle $x^2 + y^2 - 2\pi x - 2ey + c = 0$ (as (π, e) is the centre), then e will be a rational multiple of π, which is not possible. Thus, the number of rational points on the circumference of the circle is at most one.

2. a. Point of intersection of diagonals lie on circumcircle i.e. (1, 1), since $(y - 2x + 1) + \lambda(2y - x - 1) = 0$

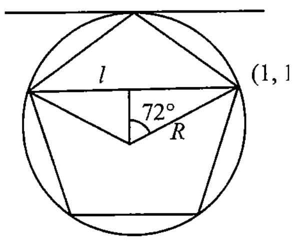


Fig. 2.96

$l = 2R \sin 72°$

$R = \frac{\sin 36°}{2 \sin 72°} = \cos 72°$

$\Rightarrow$ Locus is $(x-1)^2 + (y-1)^2 = \cos^2 72°$

$\Rightarrow x^2 + y^2 - 2x - 2y + 1 + \sin^2 72° = 0$

3. c. Let the equation of the chord OA of the circle

$x^2 + y^2 - 2x + 4y = 0$ (i)

be $\quad y = mx$ (ii)

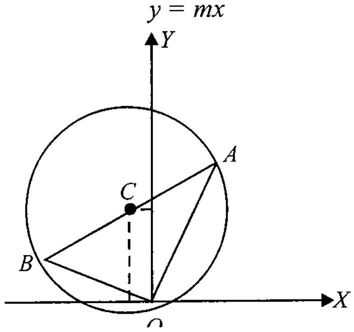


Fig. 2.97

Solving (i) and (ii), we get

$\Rightarrow \quad x^2 + m^2x^2 - 2x + 4mx = 0$

$\Rightarrow (1+m^2)x^2 - (2-4m)x = 0$

$\Longrightarrow \qquad x = 0 \text{ and } x = \dfrac{2-4m}{1+m^2}$

Hence, the points of intersection are

$(O, 0)$ and $A\left(\dfrac{2-4m}{1+m^2}, \dfrac{m(2-4m)}{1+m^2}\right)$

$$\Longrightarrow \qquad OA^2 = \left(\frac{2-4m}{1+m^2}\right)^2 (1+m^2)$$

$$= \frac{(2-4m)^2}{1+m^2}$$

Since OAB is an isosceles right-angled triangle $OA^2 = \frac{1}{2} AB^2$

where AB is a diameter of the given circle $OA^2 = 10$

$$\Longrightarrow \qquad \frac{(2-4m)^2}{1+m^2} = 10$$

$$\Longrightarrow \qquad 4 - 16m + 16m^2 = 10\,(1+m)^2$$

$$\Longrightarrow \qquad 3m^2 - 8m - 3 = 0$$

$$\Longrightarrow \qquad m = 3 \text{ or } -\frac{1}{3}$$

Hence, the required equations are $y = 3x$ or $x + 3y = 0$.

4. b. $\qquad y = mx$ be chord.

Then point of intersection are given by

$x^2(1+m^2) - x(3+4m) - 4 = 0$

$$\therefore \qquad x_1 + x_2 = \frac{3+4m}{1+m^2} \text{ and } x_1x_2 = \frac{-4}{1+m^2}$$

Since $(0, 0)$ divides chord in the ratio $1 : 4$

$$\therefore \qquad x_2 = -4x_1$$

$$\therefore \qquad -3x_1 = \frac{3+4m}{1+m^2} \text{ and } 4x_1^2 = -\frac{-4}{1+m^2}$$

$$\therefore \qquad 9 + 9m^2 = 9 + 16m^2 + 24m$$

$$\text{i.e.} \qquad m = 0, -\frac{24}{7}$$

Therefore, the lines are $\quad y = 0$ and $y + 24x = 0$.

5. a.

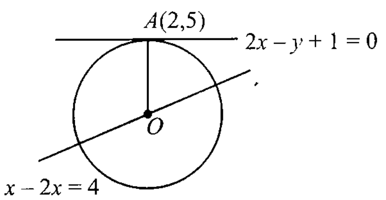


Fig. 2.98

$2x - y + 1 = 0$ is tangent

Slope of line $\qquad OA = -\dfrac{1}{2}$

Equation of OA, $\qquad (y-5) = -\dfrac{1}{2}(x-2)$

or $\qquad x + 2y = 12$

$\therefore$ Intersection with $x - 2y = 4$ will give coordinates of centre which are $(8, 2)$

$$\therefore \qquad r = OA = \sqrt{(8-2)^2 + (2-5)^2} = 3\sqrt{5}$$

6. d.

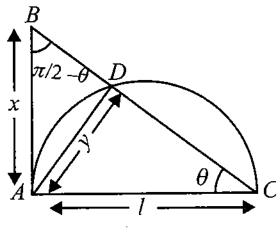


Fig. 2.99

$\Delta\, ABC$ and $\Delta\, DBA$ are similar

$$\Rightarrow \qquad l \cdot x = y\sqrt{l^2 + x^2}$$

$$\Rightarrow \qquad l^2x^2 = y^2(l^2 + x^2)$$

$$\Rightarrow \qquad l^2(x^2 - y^2) = x^2y^2$$

$$\Rightarrow \qquad l = \frac{xy}{\sqrt{x^2 - y^2}} = \frac{AB.AD}{\sqrt{AB^2 - AD^2}}$$

7. a.

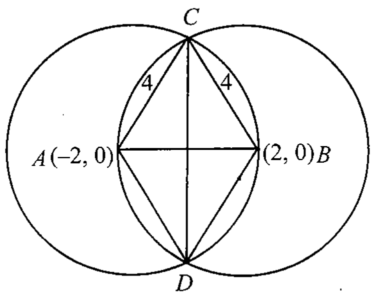


Fig. 2.100

Circles with centre $(2, 0)$ and $(-2, 0)$ each with radius 4.

$\Rightarrow$ y-axis is their common chord.

$\Delta\, ABC$ is equilateral. Hence, area of $ADBC$ is $\dfrac{2.\sqrt{3}}{4}(4)^2 = 8\sqrt{3}$.

8. a.

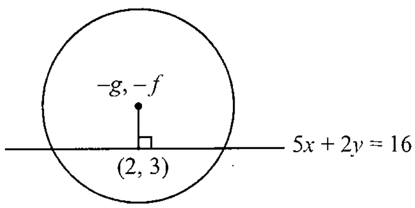


Fig. 2.101

Slope of the given line $= -\dfrac{5}{2}$

$\Rightarrow \quad \left(\frac{5}{2}\right)\left(\frac{3+f}{2+g}\right) = -1$

$\Rightarrow \quad 15 + 5f = 4 + 2g$

$\Rightarrow$ Locus is $\quad 2x - 5y + 11 = 0$

9. b.

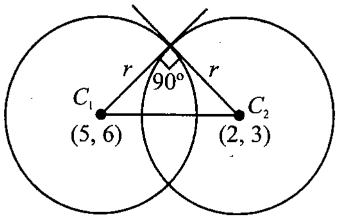


Fig. 2.102

$2r^2 = 3^2 + 3^2 = 18 \Rightarrow r^2 = 9$

$\Rightarrow \quad r = 3$

10. d.

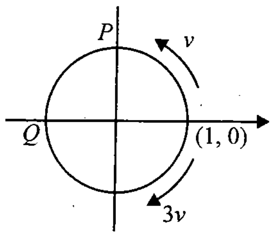


Fig. 2.103

The particle which moves clockwise is moving three times as fast as the particle moving anticlockwise.

This means the clockwise particle travels (3/4)th of the way around the circle, the anticlockwise particle will travel (1/4)th of the way around the circle and so the second particle will meet at $P(0, 1)$.

Using the same logic they will meet at $Q(-1, 0)$ when they meet the second time.

11. d.

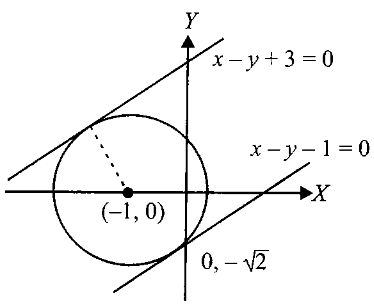


Fig. 2.104

$x^2 + y^2 + 2x - 1 = 0$

Centre $(-1, 0)$ and radius $= \sqrt{2}$

Line $x - y + c = 0$ must be tangent to the circle.

$\Rightarrow \quad \left|\frac{-1+c}{\sqrt{2}}\right| = \sqrt{2}$

$\Rightarrow \quad |c - 1| = 2$

$\Rightarrow \quad c - 1 = \pm 2$

$\Rightarrow \quad c = 3$ or -1

$\Rightarrow \quad c = 1$ ($\because$ for $c = 3$ there will be infinite points common lying inside circle)

12. b.

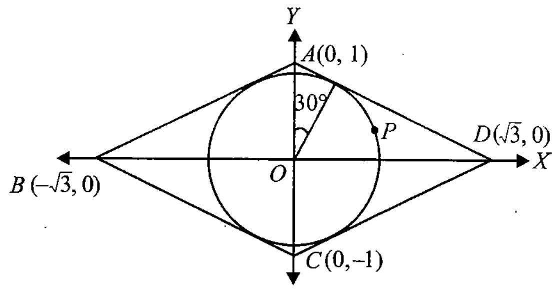


Fig. 2.105

$OA = 1$

$r = OA \cos 30° = \frac{\sqrt{3}}{2}$

Equation of circle is $x^2 + y^2 = 3/4$

$PA^2 + PB^2 + PC^2 + PD^2$

$= x_1^2 + (y_1 - 1)^2 + (x_1 + \sqrt{3})^2 + y_1^2 + x_1^2 + (y_1 + 1)^2 + (x_1 - \sqrt{3})^2 + y_1^2$

$= 4x_1^2 + 4y_1^2 + 8 = 4(x_1^2 + y_1^2) + 8$

$= 4 \times \frac{3}{4} + 8$

$= 11$

13. d.

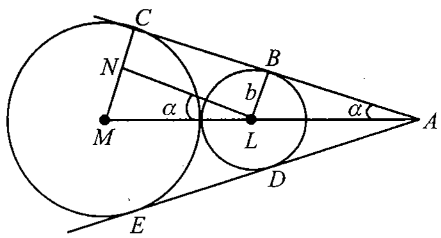


Fig. 2.106

From $\Delta\, MLN$

$\sin \alpha = \frac{a-b}{a+b}$

$\therefore \quad a = \sin^{-1}\left(\frac{a-b}{a+b}\right)$

$\therefore$ Angle between AB and AD

$= 2\alpha = \sin^{-1}\left(\frac{a-b}{a+b}\right)$

14. a.

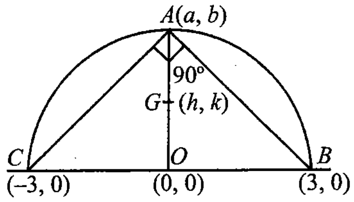


Fig. 2.107

Let $A(a, b)$ and $G(h, k)$

Now A, G, O are collinear with $AG : GO \equiv 2 : 1$

$\Rightarrow \qquad h = \dfrac{2.0 + a}{3}$

$\Rightarrow \qquad a = 3h$ and similarly $b = 3k$

Now (a, b) lies on the circle $x^2 + y^2 = 9$

Therefore, locus of (h, k) is $x^2 + y^2 = 1$.

15. a.

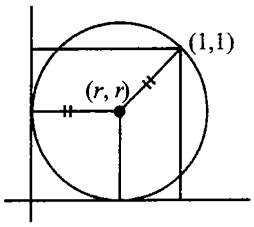


Fig. 2.108

From Fig. 2.108,

$$2(1 - r)^2 = r^2$$

$\Rightarrow \qquad \sqrt{2}(1 - r) = r$

$\Rightarrow \qquad r(\sqrt{2} + 1) = \sqrt{2}$

$\Rightarrow \qquad r = \dfrac{\sqrt{2}}{\sqrt{2} + 1} = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$

16. b. $\qquad r = 1; L = \sqrt{3}$

Area of quadrilateral = $\sqrt{3}$

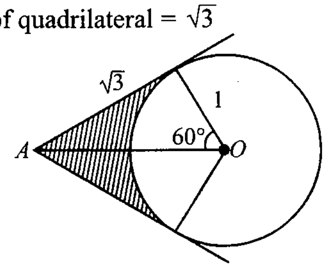


Fig. 2.109

$$\text{Sector} = \frac{1}{2}.1.\frac{2\pi}{3} = \frac{\pi}{3}$$

$$\text{Shaded region} = \sqrt{3} - \frac{\pi}{3}$$

17. d.

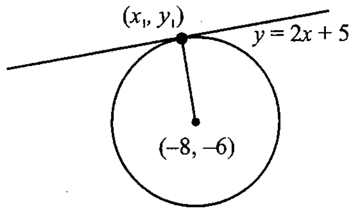


Fig. 2.110

From the figure, we have

$$y_1 = 2x_1 + 5 \text{ and } \frac{y_1 + 6}{x_1 + 8} \times 2 = -1$$

$\Rightarrow \qquad x_1 = -6$ and $y_1 = -7$

18. c.

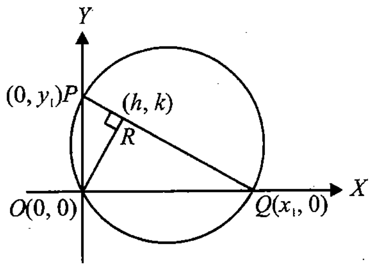


Fig. 2.111

Equation of line PQ is $y - k = -\dfrac{h}{k}(x - h)$

or $\qquad hx + ky = h^2 + k^2$

$\Rightarrow$ Points $Q\left(\dfrac{h^2 + k^2}{h}, 0\right)$ and $P\left(0, \dfrac{h^2 + k^2}{k}\right)$

Also $\qquad 2a = \sqrt{x_1^2 + y_1^2}$

$\Rightarrow \qquad x_1^2 + y_1^2 = 4a^2$

Eliminating x_1 and y_1 we have

$$(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$$

19. c.

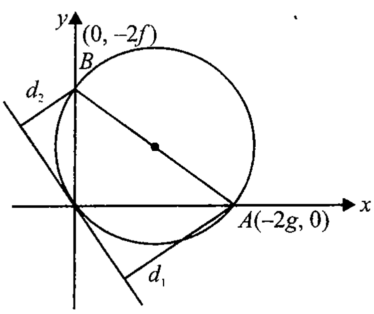


Fig. 2.112

Let the circle be $x^2 + y^2 + 2gx + 2fy = 0$

Tangent at the origin is

$$gx + fy = 0$$

$$d_1 = \frac{2g^2}{\sqrt{g^2 + f^2}} \text{ and } d^2 = \frac{2f^2}{\sqrt{g^2 + f^2}}$$

$$\Rightarrow \quad d_1 + d_2 = 2\sqrt{g^2 + f^2}$$

$$= \text{diameter of the circle}$$

20. a.

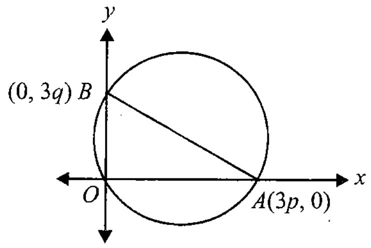


Fig. 2.113

Let the centroid of triangle OAB is (p, q).

Hence, points A and B are $(3p, 3q)$.

But diameter of triangle $AB = 6k$

Hence, $\sqrt{9p^2 + 9q^2} = 6k$

Therefore, locus of (p, q) is $x^2 + y^2 = 4k^2$.

21. a.

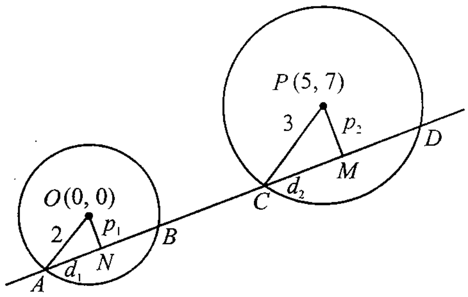


Fig. 2.114

Let equation of line be $y = x + c$ or $y - x = c$ (i)

Perpendicular from (0, 0) on line (i) is $\left|\frac{-c}{\sqrt{2}}\right| = \frac{c}{\sqrt{2}}$

In ΔAON, $\sqrt{2^2 - \left(\frac{c}{\sqrt{2}}\right)^2} = AN$

and in ΔCPM, $\sqrt{3^2 - \left(\frac{2-c}{\sqrt{2}}\right)^2} = CM$.

Given $AN = CM \Rightarrow 4 - \frac{c^2}{2} = 9 - \frac{(2-c)^2}{2}$

$$\Rightarrow \quad c = -\frac{3}{2}$$

Therefore, Equation of line $y = x - \frac{3}{2}$ or $2x - 2y - 3 = 0$

22. a.

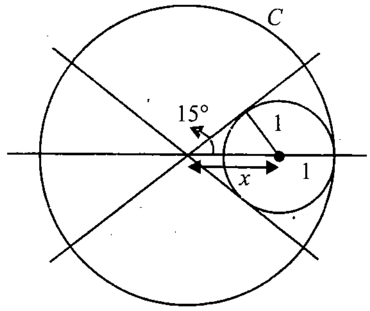


Fig. 2.115

$$\operatorname{cosec} 15° = \frac{x}{1}$$

$$\Rightarrow \quad x = \operatorname{cosec} 15°$$

$$\Rightarrow \quad R = x + 1 = 1 + \operatorname{cosec} 15°$$

$$= 1 + \frac{2\sqrt{2}}{\sqrt{3} - 1}$$

$$= 1 + \frac{4}{\sqrt{6} - \sqrt{2}}$$

$$= 1 + \sqrt{6} + \sqrt{2}$$

23. c.

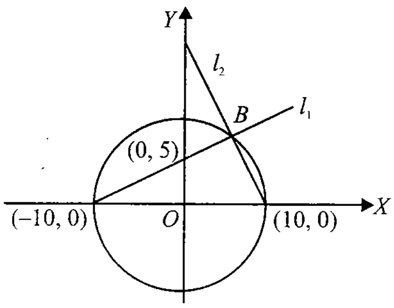


Fig. 2.116

Slope of $l_1 = \frac{1}{2}$

Slope of $l_2 = -2$

Equation of l_2, $y = -2(x - 10)$

$$\Rightarrow \quad y + 2x = 20$$

Hence, $t = 20$

24. a. $AB = \sqrt{a^2 + b^2}$

Hence, $D = \sqrt{b^2 + a^2}$ (i)

Now, $\frac{d}{2} = \frac{\Delta}{s} = \frac{ab}{2s}$ (where s is semi-perimeter)

$$\therefore \quad \frac{d}{2} = \frac{ab}{a + b + \sqrt{a^2 + b^2}}$$

or $$d = \frac{2ab}{a+b+\sqrt{a^2+b^2}} \quad \text{(ii)}$$

From Eqs. (i) and (ii)

$$d+D = \frac{\sqrt{a^2+b^2}\left[(a+b)+\sqrt{a^2+b^2}\right]+2ab}{a+b+\sqrt{a^2+b^2}}$$

$$= \frac{(a+b)^2+(a+b)\sqrt{a^2+b^2}}{a+b+\sqrt{a^2+b^2}} = a+b$$

25. c.

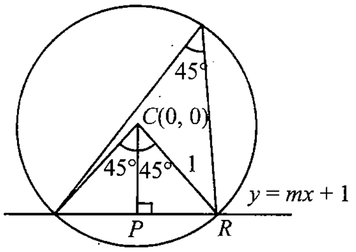


Fig. 2.117

Given circle is $x^2+y^2=1$.

$C(0, 0)$ and radius = 1 and chord is $y = mx+1$

$$\cos 45° = \frac{CP}{CR}$$

CP = perpendicular distance from (0, 0) to chord $y = mx + 1$

$$CP = \frac{1}{\sqrt{m^2+1}} \quad (CR = \text{radius} = 1)$$

$$\Rightarrow \quad \cos 45° = \frac{1/\sqrt{m^2+1}}{1}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{m^2+1}}$$

$$\Rightarrow \quad m^2+1 = 2$$

$$\Rightarrow \quad m = \pm 1$$

26. d.

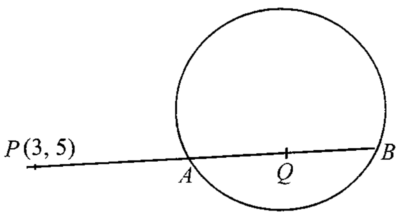


Fig. 2.118

$$2PQ = PA + PB$$

$$\Rightarrow \quad PQ - PA = PB - PQ$$

$$\Rightarrow \quad AQ = QB$$

$\Rightarrow Q$ is midpoint of AB.

Let Q has coordinates (h, k).

Then equation of chord AB is given by $T = S_1$

or $$hx + ky - 4 = h^2 + k^2 - 4$$

This variable chord passes through the point $P(3, 5)$.

$$\Rightarrow \quad 3h + 5k = h^2 + k^2$$

$$\Rightarrow \quad x^2 + y^2 - 3x - 5y = 0$$

which is required locus.

27. b.

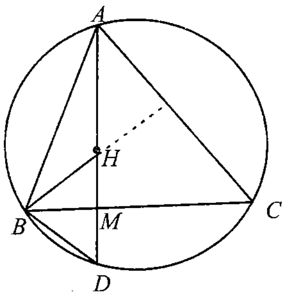


Fig. 2.119

Let orthocenter be $H(5, 8)$.

Now, $\angle HBM = \pi/2 - C$

Also, $\angle DBC = \angle DAC = \pi/2 - C$

Hence, ΔBMH and ΔBMD are congruent.

$$\Rightarrow \quad HM = MD$$

$\Rightarrow D$ is image of H in the line $x - y = 0$ which is $D(8, 5)$.

Thus, equation of circumcircle is

$$(x-2)^2 + (y-3)^2 = (8-2)^2 + (5-3)^2$$

i.e. $x^2 + y^2 - 4x - 6y - 27 = 0$

28. b. The given point is an interior point

$$\Rightarrow \quad \left(-5+\frac{r}{\sqrt{2}}\right)^2 + \left(-3+\frac{r}{\sqrt{2}}\right)^2 - 16 < 0$$

$$\Rightarrow \quad r^2 - 8\sqrt{2}\,r + 18 < 0$$

$$\Rightarrow \quad 4\sqrt{2} - \sqrt{14} < r < 4\sqrt{2} + \sqrt{14} \quad \text{(i)}$$

The point is on the major segment

$\Rightarrow$ The centre and the point are on the same side of the line $x + y = 2$

$$\Rightarrow \quad -5 + \frac{r}{\sqrt{2}} - 3 + \frac{r}{\sqrt{2}} - 2 < 0$$

$$\Rightarrow \quad r < 5\sqrt{2} \quad \text{(ii)}$$

From Eqs. (i) and (ii). $4\sqrt{2} - \sqrt{14} < r < 5\sqrt{2}$

29. a. Let $x = a, x = b, y = c, y = d$ be the sides of the square.

The length of each diagonal of the square is equal to the diameter of the circle, i.e., $2\sqrt{98}$.

Let l be the length of each side of the square. Then, $2l^2 = (\text{Diagonal})^2$

$\Rightarrow \quad l = 14$

Therefore, each side of the square is at a distance 7 from the centre (1, −2) of the given circle. This implies that $a = -6, b = 8, c = -9, d = 5$

Hence, the vertices of the square are (−6, −9), (−6, 5), (8, −9), (8, 5).

30. b.

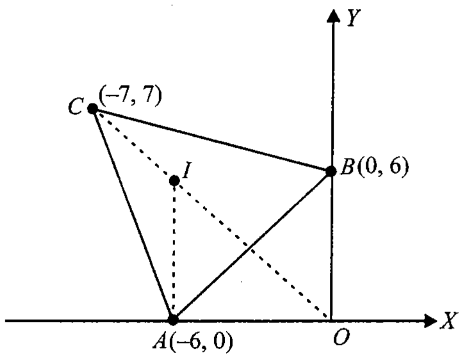


Fig. 2.120

The triangle is evidently isosceles and therefore the median through C is the angle bisector of $\angle C$.

The equation of the angle bisector is $y = -x$ and in centre $I = (-a, a)$ where a is positive.

Equation of AC is $y - 0 = -7(x + 6)$ or $7x + y + 42 = 0$ and equation of AB is $x - y + 6 = 0$.

The length of the perpendicular from I to AB and AC are equal.

$\therefore \left|\frac{-7a + a + 42}{\sqrt{50}}\right| = \left|\frac{-a - a + 6}{\sqrt{2}}\right|$

$\therefore \quad a = \frac{9}{2} \ (\because a > 0)$

$\therefore$ Centre is $\left(-\frac{9}{2}, \frac{9}{2}\right)$ and radius $= \frac{3}{\sqrt{2}}$

$\therefore$ The equation of the circle is $\left(x + \frac{9}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{9}{2}$

$\therefore \quad x^2 + y^2 + 9x - 9y + 36 = 0$

31. b.

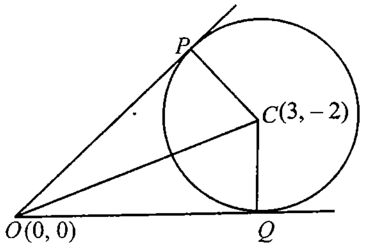


Fig. 2.121

Clearly, $OPCQ$ is cyclic quadrilateral, then circumcircle of ΔOPQ passes through the point C.

For this circle, OC is diameter, then centre is midpoint of OC which is $\left(\frac{3}{2}, -1\right)$.

32. d.

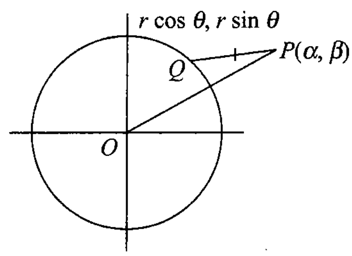


Fig. 2.122

$$2h = \alpha + r\cos\theta$$

$$2k = \beta + r\sin\theta$$

$\Rightarrow \quad (2h - \alpha)^2 + (2k - \beta)^2 = r^2$

$$\left(h - \frac{\alpha}{2}\right)^2 + \left(k - \frac{\beta}{2}\right)^2 = \left(\frac{r}{2}\right)^2$$

locus is $\left(x - \frac{\alpha}{2}\right)^2 + \left(y - \frac{\beta}{2}\right)^2 = \left(\frac{r}{2}\right)^2$

which is a circle with centre as midpoint of OP and radius $r/2$

33. a.

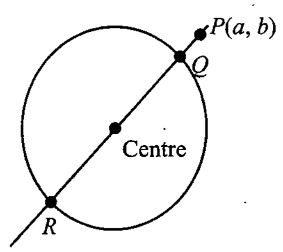


Fig. 2.123

The given circle is $(x + 1)^2 = (y + 2)^2 = 9$, which has radius = 3.

The points on the circle which are nearest and farthest to the point $P(a, b)$ are Q and R, respectively.

Thus, the circle centred at Q having radius PQ will be the smallest circle while the circle centred at R having radius PR will be the largest required circle.

Hence, difference between their radii $= PR - PQ = QR = 6$

34. a.

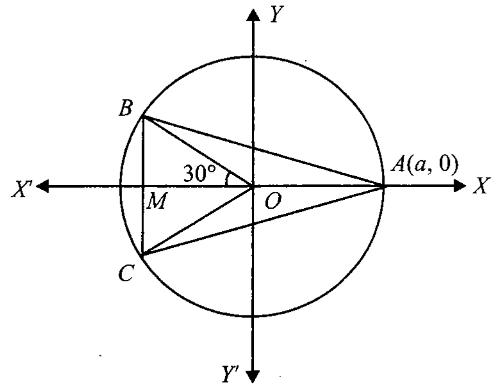


Fig. 2.124

Since $\angle B = \angle C = 75°$

$\Rightarrow \angle BAC = 30°$

$\Rightarrow \angle BOC = 60°$

$\Rightarrow$ B has coordinates $(-a\cos 30°, a\sin 30°)$

or $\left(\frac{-\sqrt{3}a}{2}, \frac{a}{2}\right)$ and those of C are $\left(-\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$

35. c.

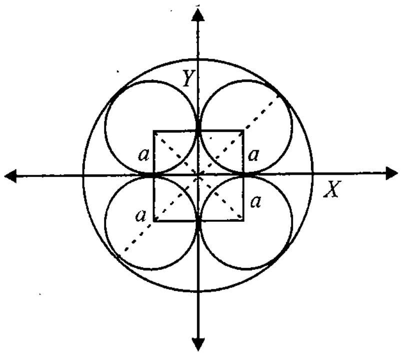


Fig. 2.125

The four circles are as shown in the Fig. 2.125.

The smallest circle touching all of them has the radius $= \sqrt{2}\,a - a$ and the greatest circle touching all of them has the radius $= \sqrt{2}\,a + a$.

36. a. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC, and let $P(h, k)$ be any point on the locus.

Then, $PA^2 + PB^2 + PC^2 = c$ (constant)

$$\Rightarrow \sum_{i=1}^{3} (h - x_i)^2 + (k - y_i)^2 = c$$

$$\Rightarrow h^2 + k^2 - \frac{2h}{3}(x_1 + x_2 + x_3) - \frac{2k}{3}(y_1 + y_2 + y_3) + \sum_{i=1}^{3}(x_i^2 + y_i^2) - c = 0$$

So, locus of (h, k) is

$$x^2 + y^2 - \frac{2x}{3}(x_1 + x_2 + x_3) - \frac{2y}{3}(y_1 + y_2 + y_3) + \lambda = 0,$$

where $\lambda = \sum_{i=1}^{3}(x_i^2 + y_i^2) - c = 0$ constant

37. c. Equation of any circles passing through (1, 0) and (5, 0) is

$$y^2 + (x - 1)(x - 5) + \lambda y = 0$$

i.e. $x^2 + y^2 + \lambda y - 6x + 5 = 0$

If $\angle ACB$ is maximum, then this circle must touch the y-axis at $(0, h)$. Putting $x = 0$ in the equation of circle, we get $y^2 + \lambda y + 5 = 0$. It should have $y = h$ as it is a repeated root.

$\Rightarrow h^2 = 5$ and $\lambda = -2h$

$\Rightarrow |h| = \sqrt{5}$

38. b.

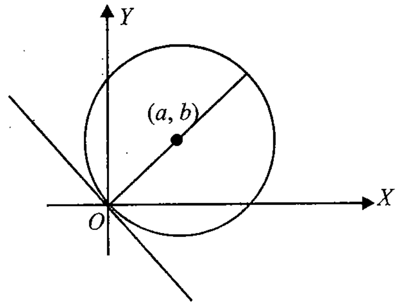


Fig. 2.126

Obviously, the slope of the tangent will be $-\left(\frac{1}{b/a}\right)$, i.e., $-\frac{a}{b}$.

Hence, the equation of the tangent is $y = -\frac{a}{b}x$, i.e., $by + ax = 0$.

39. c.

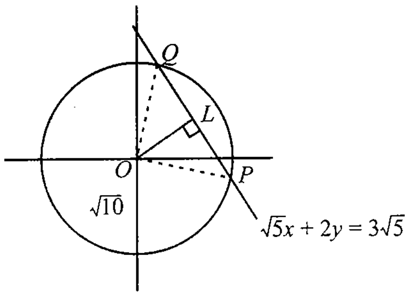


Fig. 2.127

Length of perpendicular from origin to the line $x\sqrt{5} + 2y = 3\sqrt{5}$ is

$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2 + 2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$

Radius of the given circle $= \sqrt{10} = OQ = OP$

$$PQ = 2QL = 2\sqrt{OQ^2 - OL^2}$$
$$= 2\sqrt{10 - 5} = 2\sqrt{5}$$

Thus, area of $\Delta OPQ = \frac{1}{2} \times PQ \times OL$

$$= \frac{1}{2} \times 2\sqrt{5} \times \sqrt{5} = 5$$

40. c. If $(a, 0)$ is the centre C and P is $(2, -2)$, then $\angle COP = 45°$

Since the equation of OP is $x + y = 0$

$\therefore$ $OP = 2\sqrt{2} = CP$. Hence, $OC = 4$

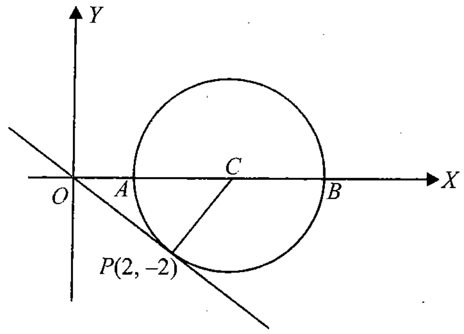


Fig. 2.128

The point on the circle with the greatest x coordinates is B.

$$a = OB = OC + CB = 4 + 2\sqrt{2}$$

41. c.

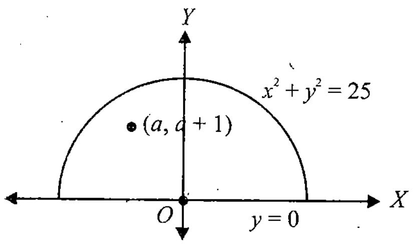


Fig. 2.129

$y = \sqrt{25 - x^2}$, $y = 0$ bound the semicircle above the x-axis.

$\therefore \quad a + 1 > 0 \quad$ (i)

and $\quad a^2 + (a + 1)^2 - 25 < 0 \Rightarrow 2a^2 + 2a - 24 < 0$

$\Rightarrow \quad a^2 + a - 12 < 0$

$\Rightarrow \quad -4 < a < 3 \quad$ (ii)

From Eqs.(i) and (ii)

$$-1 < a < 3$$

42. c. For $x^2 + y^2 = 9$, the centre = (0, 0) and the radius = 3

For $x^2 + y^2 - 8x - 6y + n^2 = 0$,

the centre = (4, 3) and the radius = $\sqrt{4^2 + 3^2 - n^2}$

$\therefore\ 4^2 + 3^2 - n^2 > 0$ or $n^2 < 5^2$ or $-5 < n < 5$

Circles should cut to have exactly two common tangents.

So, $\quad r_1 + r_2 > d$ (distance between centres)

$\therefore \quad 3 + \sqrt{25 - n^2} > \sqrt{4^2 + 3^2}$

or $\quad \sqrt{25 - n^2} > 2$

or $\quad 25 - n^2 > 4$

$\therefore \quad n^2 < 21$ or $-\sqrt{21} < n < \sqrt{21}$

Therefore, common values of n should satisfy $-\sqrt{21} < n < \sqrt{21}$.

But $n \in Z$. So, $n = -4, -3, \ldots, 4$

43. b.

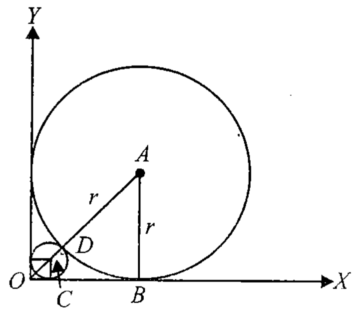


Fig. 2.130

or $\quad 25 - n^2 > 4$

From the diagram $\quad \sin 45° = \dfrac{AB}{OA}$

$$= \frac{r}{OC + CD + DA}$$

$$= \frac{r}{\sqrt{2} + 1 + r}$$

$\Rightarrow \quad \sqrt{2} + 1 + r = \sqrt{2}\, r$

$\Rightarrow \quad r = \dfrac{\sqrt{2} + 1}{\sqrt{2} - 1} = 3 + 2\sqrt{2}$

44. a.

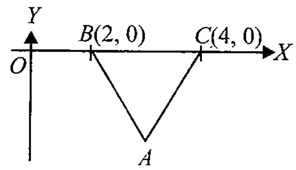


Fig. 2.131

Clearly, $\quad A \equiv (3, -\sqrt{3})$.

Centroid of triangle ABC is $\left(3, -\frac{1}{\sqrt{3}}\right)$, thus equation of incircle is

$$(x - 3)^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$\Rightarrow \quad x^2 + y^2 - 6x + \dfrac{2y}{\sqrt{3}} + 9 = 0$

45. d. $x^2 + 2ax + c = (x - 2)^2$

$\Rightarrow \quad -2a = 4, c = 4$

$\Rightarrow \quad a = -2, c = 4,$

$$y^2 + 2by + c = (y - 2)(y - 3)$$

$$\Rightarrow \quad -2b = 5, c = 6$$

$\Rightarrow \quad b = -\frac{5}{2}, c = 6$ clearly the data are not consistent.

46. d.

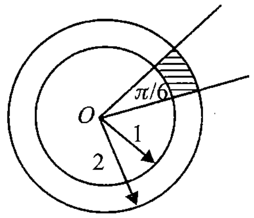


Fig. 2.132

The angle θ between the lines represented by $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ is given by

$$\theta = \tan^{-1}\frac{\sqrt{h^2 - ab}}{|a+b|}$$

$$= \tan^{-1}\frac{2\sqrt{2^2-3}}{\sqrt{3}+\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Gives $\quad \theta = \frac{\pi}{6}$

Hence, the shaded area $= \frac{\pi/6}{2\pi} \times \pi(2^2 - 1^2) = \frac{\pi}{4}$

47. **c.** Centre of circle is (1, 0) and radius is 1. Line will touch the circle if $|\cos\theta - 2| = 1 \Rightarrow \cos\theta = 1, 3$.

Thus, $\quad \cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$

48. b.

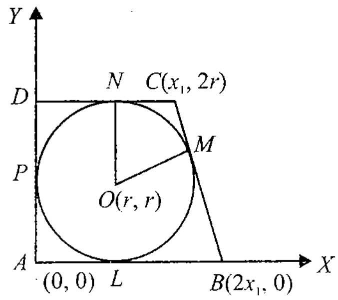


Fig. 2.133

Here $\quad \frac{1}{2} \times 3x_1 \times 2r = 18$

$$\Rightarrow \quad x_1 \times r = 6 \qquad \text{(i)}$$

Equation of BC is: $y = -\frac{2r}{x_1}(x - 2x_1)$

BC is tangent to the circle $(x - r)^2 + (y - r)^2 = r^2$

$\therefore$ Perpendicular distance of BC from centre = radius

$$\Rightarrow \quad \frac{\left|r + \frac{2r}{x_1}(r - 2x_1)\right|}{\sqrt{1 + \frac{4r^2}{x_1^2}}} = r$$

$$\Rightarrow \quad \frac{2r^2}{x_1} - 3r = r\sqrt{1 + 4\frac{r^2}{x_1^2}}$$

$$\Rightarrow \quad (2r - 3x_1)^2 = x_1^2 + 4r^2$$

$$\Rightarrow \quad r \cdot x_1 = \frac{2}{3}x_1^2 \Rightarrow 3r = 2x_1 \qquad \text{(ii)}$$

From Eqs. (i) and (ii), $\quad r = 2$ units

49. a.

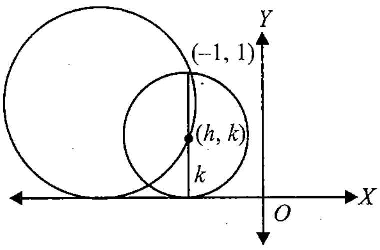


Fig. 2.134

From the figure, $k \geq \frac{1}{2}$

50. **a.** Curve passing through point of intersection of S and S' is

$$\Rightarrow \quad S + \lambda S' = 0$$

$\Rightarrow x^2(\sin^2\theta + \lambda\cos^2\theta) + y^2(\cos^2\theta + \lambda\sin^2\theta) + 2xy(h + \lambda h') + x(32 + 16\lambda) + y(16 + 32\lambda) + 19(1 + \lambda) = 0$

for this equation to be a circle

$$\sin^2\theta + \lambda\cos^2\theta = \cos^2\theta + \lambda\sin^2\theta \Rightarrow \lambda = 1$$

and $\quad h + \lambda h' = 0 \Rightarrow h + h' = 0$

51. a.

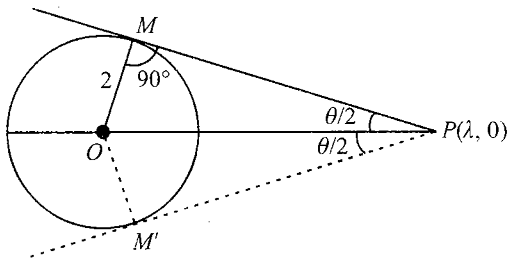


Fig. 1.235

We have $\quad \frac{\pi}{2} < \theta < \frac{2\pi}{3}$, i.e., $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{3}$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} < \sin\frac{\theta}{2} < \frac{\sqrt{3}}{2}$$

But, $\sin\frac{\theta}{2} = \frac{2}{\lambda} \Rightarrow \frac{1}{\sqrt{2}} < \frac{2}{\lambda} < \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \frac{4}{\sqrt{3}} < \lambda < 2\sqrt{2}$

52. a.

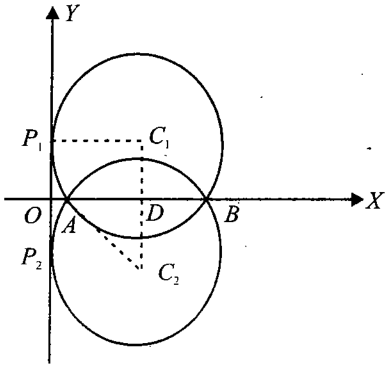


Fig. 2.136

Let $A \equiv (1, 0)$, $B \equiv (3, 0)$ and C_1, C_2 be the centre of circles passing through A, B and touching the y-axis at P_1 and P_2. If r be the radius of circle (here radius of both circles will be same), $C_1A = C_2A = r = OD = 2$ and $C_1 \equiv (2, h)$

where $h^2 = AC_1^2 - AD^2 = 4 - 1 = 3$

$\Rightarrow \quad C_1 \equiv (2, \sqrt{3}), C_2 \equiv (2, -\sqrt{3})$

If $\angle C_1AC_2 = \theta$

$\Rightarrow \quad \cos\theta = \frac{AC_1^2 + AC_2^2 - C_1C_2^2}{2AC_1 \cdot AC_2} = \frac{1}{2}$

53. c. Chord with midpoint (h, k) is

$$hx + ky = h^2 + k^2 \qquad \text{(i)}$$

Chord of contact of (x_1, y_1) is

$$xx_1 + yy_1 = 2 \qquad \text{(ii)}$$

Comparing, we get

$$x_1 = \frac{2h}{h^2+k^2} \text{ and } y_1 = \frac{2k}{h^2+k^2}$$

(x_1, y_1) lies on $3x + 4y = 10 \Rightarrow 6h + 8k = 10(h^2 + k^2)$

$\therefore$ Locus of (h, k) is $x^2 + y^2 - \frac{3}{5}x - \frac{4}{5}y = 0$

which is circle with centre $P\left(\frac{3}{10}, \frac{4}{10}\right)$

$\therefore \quad OP = \frac{1}{2}$

54. a. a, b, c are in A.P., so $ax + by + c = 0$ represents a family of lines passing through the point $(1, -2)$. So, the family of circles (concentric) will be given by $x^2 + y^2 - 2x + 4y + c = 0$. It intersects given circle orthogonally.

$\Rightarrow 2(-1 \times -2) + (2 \times -2) = -1 + c \Rightarrow c = -3$

55. d.

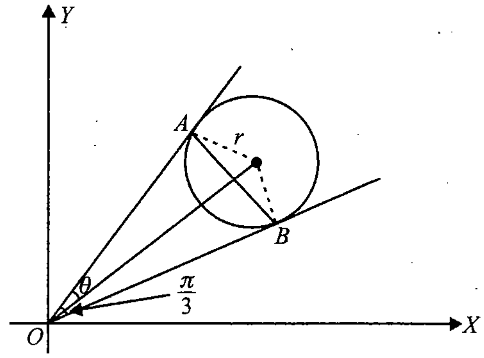


Fig. 2.137

Here $\tan 2\theta = \frac{2\sqrt{4-1}}{2} = \sqrt{3}$

$\Rightarrow \quad \theta = \pi/6$

Area of $\Delta OAB = \frac{1}{2}(r\cot\theta)^2(\sin 2\theta)$

$= \frac{1}{2}(r\sqrt{3})^2\frac{\sqrt{3}}{2}$

56. a. Let P be $(1 + \sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$ and C is $(1, 0)$. Circumcentre of triangle ABC is midpoint of PC.

$\Rightarrow \quad 2h = 1 + \sqrt{2}\cos\theta + 1$

and $\quad 2k = \sqrt{2}\sin\theta$

$\Rightarrow \quad [2(h-1)]^2 + (2k)^2 = 2$

$\Rightarrow \quad 2(h-1)^2 + k^2 - 1 = 0$

$\Rightarrow \quad 2x^2 + 2y^2 - 4x + 1 = 0$

57. a.

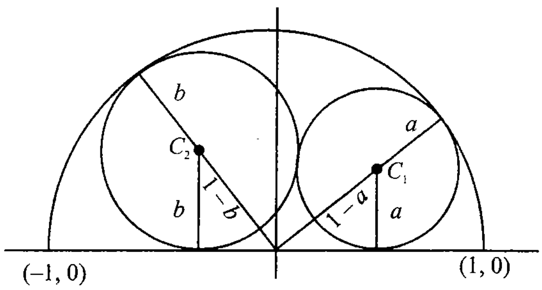


Fig. 2.138

Let centre of the circles be C_1 and C_2.

$\Rightarrow C_1$ is $(\sqrt{1-2a}, a)$ and C_2 is $(\sqrt{1-2b}, b)$

Now $\quad C_1C_2 = a + b = a + \frac{1}{2}$

$\Rightarrow \quad 1-2a+\left(a-\frac{1}{2}\right)^2=\left(a+\frac{1}{2}\right)^2$

$\Rightarrow \quad a=\frac{1}{4}$

58. a. Since ΔPOQ and ΔAOQ are congruent

Hence, $\quad \angle POQ = \angle QOA = \theta$

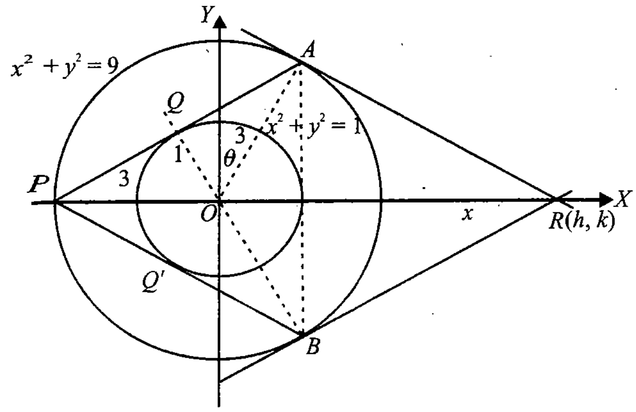


Fig. 2.139

$$\cos\theta=\frac{1}{3},\ \text{since } \angle POR=180°$$

$\Rightarrow \quad \angle AOR = \pi - 2\theta$

Now in triangle AOR, $\angle AOR = \pi - 2\theta$ and $AO = 3$ unit

$\Rightarrow \quad \cos(\pi-2\theta)=\frac{OA}{OR}=\frac{3}{\sqrt{h^2+k^2}}$

$\Rightarrow \quad \sqrt{h^2+k^2}=\frac{27}{7}$

$\Rightarrow \quad x^2+y^2=\left(\frac{27}{7}\right)^2$

59. b.

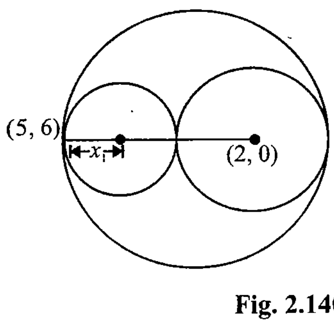


Fig. 2.140

Given circle is $(x-2)^2+y^2=4$

Centre is (2, 0) and radius = 2

Therefore, distance between (2, 0) and (5, 6) is $\sqrt{9+36}=3\sqrt{5}$

$\Rightarrow \quad r_1=\frac{3\sqrt{5}-2}{2}$

and $\quad r_2=\frac{3\sqrt{5}+2}{2}$

$$=r_1r_2=\frac{41}{4}$$

60. c. $\quad x^2+y^2-12x+35=0 \qquad$ (i)

$x^2+y^2+4x+3=0 \qquad$ (ii)

Equation of radical axis of circles (i) and (ii) is

$$-16x+32=0 \Rightarrow x=2$$

It intersects the line joining the centres, i.e. $y=0$ at the point (2, 0)

$\therefore$ Required radius $=\sqrt{4-24+35}=\sqrt{15}$ (length of tangent from (2, 0)

61. d.

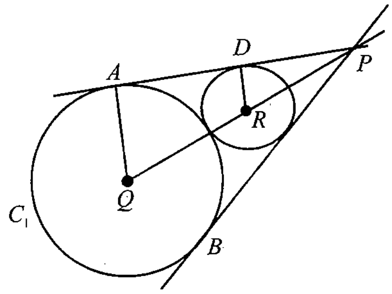


Fig. 2.141

$$AQ=3+2\sqrt{2}$$
$$PQ=3\sqrt{2}+4$$

Let r be required radius

$\therefore \quad 3\sqrt{2}+4=3+2\sqrt{2}+r+r\sqrt{2}$

$\left(\because \angle RPD=\frac{\pi}{4}\right)$

$$\sqrt{2}+1=r(1+\sqrt{2}) \Rightarrow r=1$$

62. a.

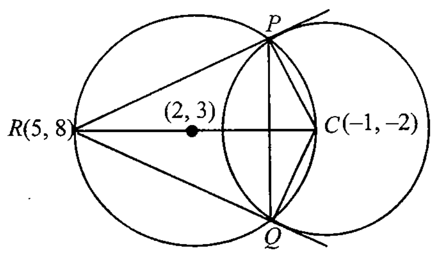


Fig. 2.142

Let C be the centre of the given circle.

Then, circumcircle of the ΔRPQ passes through C.

$\therefore$ (2, 3) is the midpoint of RC.

$\therefore$ Coordinates of C are (−1, −2).

$\therefore$ Equation of the circle is $x^2+y^2+2x+4y-20=0$.

63. c.

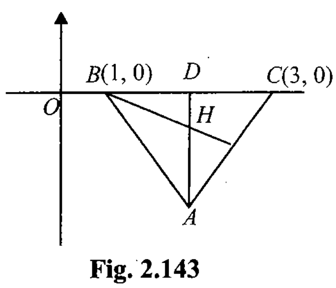


Fig. 2.143

Radical centre of the circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

$\therefore$ $D = (2, 0)$

$$DH = -BD \tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

$\therefore$ Coordinates of H are $\left(2, -\frac{1}{\sqrt{3}}\right)$.

64. b. Let $(\alpha, 3-\alpha)$ be any point on $x + y = 3$.

$\therefore$ Equation of chord of contact is $\alpha x + (3-\alpha)y = 9$

i.e., $\alpha(x-y) + 3y - 9 = 0$

$\therefore$ The chord passes though the point (3, 3) for all values of α.

65. d. Given circle $(x-1)^2 + (y+2)^2 = 16$

Its director circle is $(x-1)^2 + (y+2)^2 = 32$

$\Rightarrow$ $OS = 4\sqrt{2}$

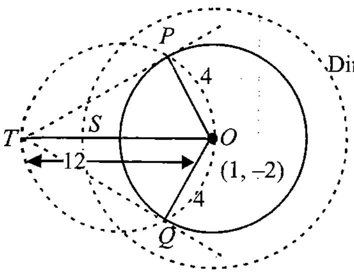


Fig. 2.144

Therefore, required distance, $TS = OT - SO = 12 - 4\sqrt{2}$

66. c. Equation of the two circles be $(x-r)^2 + (y-r)^2 = r^2$ i.e. $x^2 + y^2 - 2rx - 2ry + r^2 = 0$, where $r = r_1$ and r_2. Condition of orthogonality gives

$$2r_1r_2 + 2r_1r_2 = r_1^2 + r_2^2 \Rightarrow 4r_1r_2 = r_1^2 + r_2^2$$

Circle passes through (a, b)

$\Rightarrow$ $a^2 + b^2 - 2ra - 2rb + r^2 = 0$

i.e. $r^2 - 2r(a+b) + a^2 + b^2 = 0$

$$r_1 + r_2 = 2(a+b) \text{ and } r_1r_2 = a^2 + b^2$$

$\therefore$ $4(a^2 + b^2) = 4(a+b)^2 - 2(a^2 + b^2)$

i.e. $a^2 - 4ab + b^2 = 0$

67. c. $C_1 = (-1, -4)$; $C_2 = (2, 5)$;

$$r_1 = \sqrt{1 + 16 + 23} = 2\sqrt{10};$$

$$r_2 = \sqrt{4 + 25 + 19} = \sqrt{10};$$

$$C_1C_2 = \sqrt{9 + 18} = 3\sqrt{10}$$

$\Rightarrow$ $C_1C_2 = r_1 + r_2$.

Hence, circles touch externally.

68. a.

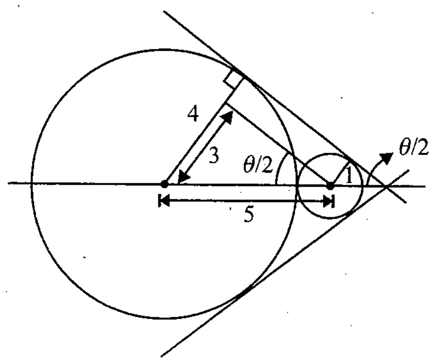


Fig. 2.145

$$\sin\frac{\theta}{2} = \frac{3}{5}$$

and $\cos\frac{\theta}{2} = \frac{4}{5}$

$\therefore$ $\sin\theta = 2.\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$

69. c.

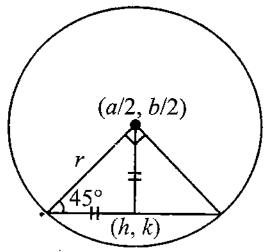


Fig. 2.146

$$r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

$$\sin 45° = \frac{\sqrt{\left(h - \frac{a}{2}\right)^2 + \left(h - \frac{b}{2}\right)^2}}{\frac{\sqrt{a^2 + b^2}}{2}}$$

$\Rightarrow$ $$\frac{1}{2} = 4\left[\frac{\frac{(2h-a)^2}{4} + \frac{(2k-b)^2}{4}}{a^2 + b^2}\right]$$

Simplify to get locus $x^2 + y^2 - ax - by - \frac{a^2 + b^2}{8} = 0$.

70. a.

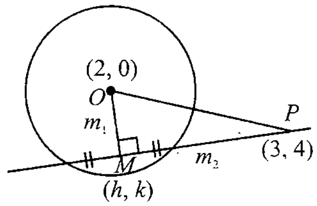


Fig. 2.147

$\Rightarrow$ $m_1m_2 = -1$

$$\Rightarrow \quad \left(\frac{4-k}{3-h}\right)\left(\frac{k-0}{h-2}\right) = -1$$

Hence, locus is $x^2 + y^2 - 5x - 4y + 6 = 0$

71. c. Locus of the centre of the circle cutting $S_1 = 0$ and $S_2 = 0$ orthogonally is the radical axis between $S_1 = 0$ and $S_2 = 0$, i.e., $S_1 - S_2 = 0$ or $9x - 10y + 11 = 0$.

72. b. For given $r_1 = \sqrt{10}, C_1(1, 0)$

and $\quad r_2 = \sqrt{5}, C_2(0, 2)$

$$d = C_1C_2 = \sqrt{5}$$

If θ is the angle between the circle, then

$$\cos\theta = \frac{|d^2 - r_1^2 - r_2^2|}{2r_1r_2}$$

$$= \frac{|5 - 10 - 5|}{2\sqrt{10}\sqrt{5}}$$

$$= \frac{1}{\sqrt{2}}$$

Hence, $\quad \theta = \frac{\pi}{4}$

73. a.

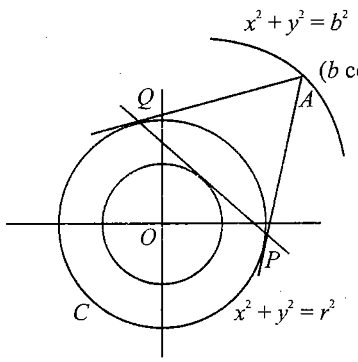


Fig. 2.148

Chord of contact of the point A w.r.t. $x^2 + y^2 = r^2$ is

$$xb\cos\theta + yb\sin\theta = r^2 \quad \text{(i)}$$

This must be a tangent to the circle $x^2 + y^2 = a^2$

$$\Rightarrow \left[\frac{r^2}{\sqrt{b^2\cos^2\theta + b^2\sin^2\theta}}\right] = a \Rightarrow r^2 = ab$$

Hence, equation of circle is $x^2 + y^2 = ab$.

74. a. Locus of point of intersection of tangents chord of contact of (x_1, y_1) w.r.t.

$$x^2 + y^2 = 1 \text{ is } xx_1 + yy_1 = 1 \ (AB) \quad \text{(i)}$$

AB is also common chord between two circles

$\therefore -1 + (\lambda + 6)x - (8 - 2\lambda)y + 3 = 0$

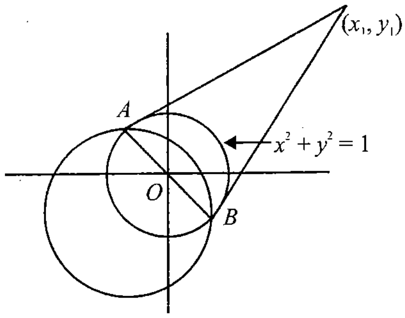


Fig. 2.149

$\Rightarrow (\lambda + 6)x - (8 - 2\lambda)y + 2 = 0 \quad$ (ii)

Comparing Eqs. (i) and (ii), we get

$$\frac{x_1}{\lambda + 6} = \frac{y_1}{2\lambda - 8} = \frac{-1}{2}$$

Eliminate $\lambda \Rightarrow 2x - y + 10 = 0$ which is required locus.

75. c.

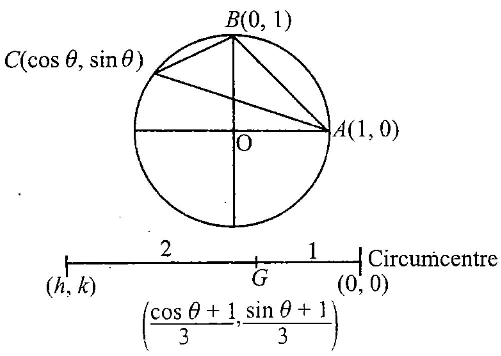


Fig. 2.150

Let $C(\cos\theta, \sin\theta)$; $H(h, k)$ is the orthocentre of the ΔABC.

Since circumcentre of the triangle is (0, 0), for orthocentre $h = 1 + \cos\theta$ and $k = 1 + \sin\theta$.

Eliminating θ, $(x - 1)^2 + (y - 1)^2 = 1$

$$\therefore \quad x^2 + y^2 - 2x - 2y + 1 = 0$$

76. d.

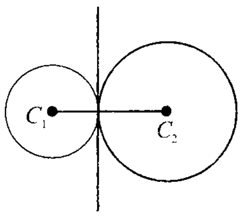


Fig. 2.151

$$C_1C_2 = r_1 + r_2$$

$$C_1 = (0, 0); C_2 = (3\sqrt{3}, 3)$$

and $\qquad r_1 = 2, r_2 = 4$

$\Rightarrow$ Circles touch each other externally.

Equation of common tangent is $\sqrt{3}x + y - 4 = 0 \qquad$ (i)

Comparing it with $x\cos\theta + y\sin\theta = 2$, we get

$$\theta = \frac{\pi}{6}$$

77. a.

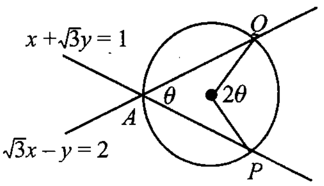


Fig. 2.152

Let the point of intersection of two lines is A.

$\therefore$ The angle subtended by PQ on centre C

$= 2 \times$ the angle subtended by PQ on point A.

For $x + \sqrt{3}y = 1$, $m_1 = \frac{-1}{\sqrt{3}}$ and for $\sqrt{3}x - y = 2$, $m_2 = \sqrt{3}$

$\because \qquad m_1 \times m_2 = \frac{-1}{\sqrt{3}} \times \sqrt{3} = -1,$

$\therefore \qquad \angle A = 90°$

$\therefore$ The angle subtended by arc PQ at its centre

$$= 2 \times 90° = 180°$$

78. b. Clearly (0, 0) lies on director circle of the given circle.

Now, equation of director circle is

$$(x+g)^2 + (y+f)^2 = 2(g^2 + f^2 - c)$$

If (0, 0) lies on it, then

$$g^2 + f^2 = 2(g^2 + f^2 - c)$$

$$\Rightarrow \qquad g^2 + f^2 = 2c$$

79. c. Let the second circle be $x^2 + y^2 + 2gx + 2fy = 0$.

But $y = x$ touches the circle.

Hence, $x^2 + x^2 + 2gx + 2fx = 0$ has equal roots, i.e., $f + g = 0$

Therefore, the equation of the common chord is $2(g - 3)x + 2(-g - 4)y + 7 = 0$

or $(-6x - 8y + 7) + g(2x - 2y) = 0$, which passes through the point of intersection of

$-6x - 8y + 7 = 0$ and $2x - 2y = 0$ which is (1/2, 1/2).

80. b. Let the coordinates of A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. Then, the chords of contact of tangents drawn from A, B and C are

$xx_1 + yy_1 = a^2$, $xx_2 + yy_2 = a^2$ and $xx_3 + yy_3 = a^2$, respectively. These three lines will be concurrent, if

$$\begin{vmatrix} x_1 & y_1 & -a^2 \\ x_2 & y_2 & -a^2 \\ x_3 & y_3 & -a^2 \end{vmatrix} = 0$$

$$\Rightarrow \qquad -a^2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \qquad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ Points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

81. c. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points and $x^2 + y^2 = a^2$ be the given circle. Then, the chord of contact of tangents drawn from P to the given circle is $xx_1 + yy_1 = a^2$.

It will pass through $Q(x_2, y_2)$, if

$x_1x_2 + y_1y_2 = a^2 \qquad$ (i)

Now, $\quad l_1 = \sqrt{x_1^2 + y_1^2 - a^2}$,

$\qquad l_2 = \sqrt{x_2^2 + y_2^2 - a^2}$

and $\quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\qquad = \sqrt{(x_2^2 + y_2^2) + (x_1^2 + y_1^2) - 2(x_1x_2 + y_1y_2)}$

$\therefore \quad PQ = \sqrt{[(x_2^2 + y_2^2) + (x_1^2 + y_1^2) - 2a^2]}$

[Using Eq. (i)]

$\Rightarrow \quad PQ = \sqrt{(x_1^2 + y_1^2 - a^2) + (x_2^2 + y_2^2 - a^2)}$

$\Rightarrow \quad PQ = \sqrt{l_1^2 + l_2^2}$

82. a. Centres are (10, 0) and (– 15, 0)

and radii are $\qquad r_1 = 6; r_2 = 9$

Also $\qquad d = 25$

$$r_1 + r_2 < d$$

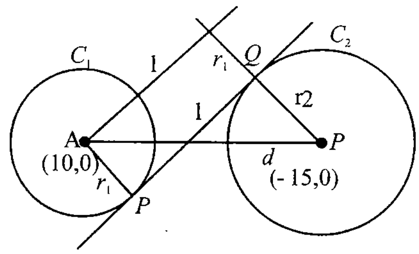


Fig. 2.153

$\Rightarrow$ circles are neither intersecting nor touching

$$PQ = \sqrt{d^2 - (r_1 + r_2)^2}$$

$= \sqrt{625-225}$

$= 20$

83. b. If (α, β) is the centre

then $(\alpha-1)^2+(\beta-3)^2=(\alpha-3)^2+(\beta-1)^2$ (i)

and $\frac{\beta-3}{\alpha-1}\cdot\frac{\beta-1}{\alpha-3}=-1$

or $(\alpha-1)(\alpha-3)+(\beta-1)(\beta-3)=0$ (ii)

(i) $\Rightarrow 4\alpha-4\beta=0 \quad \therefore \alpha=\beta$

(ii) $\Rightarrow 2(\alpha-1)(\alpha-3)=0 \quad \therefore \alpha=1, 3$

$\therefore$ $(\alpha, \beta)=(1, 1), (3, 3)$.

84. b. The centre of $x^2+y^2-4x-4y=0$ is (2, 2).

It is $ax+by=2$

$\therefore$ $2a+2b=2$ or $a+b=1$

$ax+by=2$ touches $x^2+y^2=1$.

So, $1=\left|\frac{-2}{\sqrt{a^2+b^2}}\right|$

$\therefore$ $a^2+b^2=4$ or $a^2+(1-a)^2=4$

or $2a^2-2a-3=0$

$\therefore$ $a=\frac{2\pm\sqrt{4+24}}{4}=\frac{1\pm\sqrt{7}}{2}$

$\therefore$ $b=1-a=1-\frac{1\pm\sqrt{7}}{2}=\frac{1\mp\sqrt{7}}{2}$

85. c. Equation of any circles through (0, 1) and (0, 6) is

$x^2+(y-1)(y-6)+\lambda x=0$

$\Rightarrow$ $x^2+y^2+\lambda x-7y+6=0$

If it touches x-axis, then $x^2+\lambda x+6=0$ should have equal roots

$\Rightarrow$ $\lambda^2=24 \Rightarrow \lambda=\pm\sqrt{24}$

Radius of these circles $=\sqrt{6+\frac{49}{4}-6}=\frac{7}{2}$ units.

That means we can draw two circles but radius of both circles is $\frac{7}{2}$.

86. b. Let the tangent be of form $\frac{x}{x_1}+\frac{y}{y_1}=1$ and area of Δ formed by it with coordinate axes is

$\frac{1}{2}x_1y_1=a^2$ (i)

Again, $y_1x+x_1y-x_1y_1=0$

Applying conditions of tangency

$\frac{|-x_1y_1|}{\sqrt{x_1^2+y_1^2}}=a$ or $(x_1^2+y_1^2)=\frac{x_1^2y_1^2}{a^2}$ (ii)

From Eqs. (i) and (ii), we get x_1, y_1, which gives equation of tangent as $x\pm y=\pm a\sqrt{2}$.

87. c. The equation of the line $y=x$ in distance form is $\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=r$, where $\theta=\frac{\pi}{4}$.

For point P, $r=6\sqrt{2}$. Therefore, coordinates of P are given by $\frac{x}{\cos\frac{\pi}{4}}=\frac{y}{\sin\frac{\pi}{4}}=6\sqrt{2} \Rightarrow x=6, y=6$.

Since $P(6, 6)$ lies on $x^2+y^2+2gx+2fy+c=0$, therefore

$72+12(g+f)+c=0$ (i)

Since $y=x$ touches the circle, therefore the equation $2x^2+2x(g+f)+c=0$ has equal roots

$\Rightarrow$ $4(g+f)^2=8c$

$\Rightarrow$ $(g+f)^2=2c$ (ii)

From (i), we get

$[12(g+f)]^2=[-(c+72)]^2$

$\Rightarrow$ $144(g+f)^2=(c+72)^2$

$\Rightarrow$ $144(2c)=(c+72)^2$

$\Rightarrow$ $(c-72)^2=0 \Rightarrow c=72$

88. a.

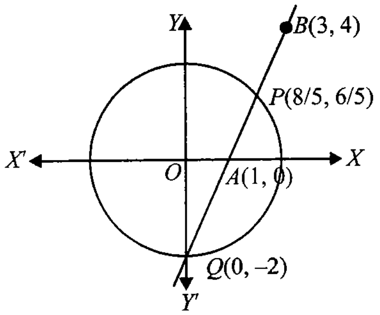


Fig. 2.154

The equation of the line joining $A(1, 0)$ and $B(3, 4)$ is $y=2x-2$.

This cuts the circle $x^2+y^2=4$ at $Q(0, -2)$ and $P\left(\frac{8}{5}, \frac{6}{5}\right)$.

We have $BQ=3\sqrt{5}$, $QA=\sqrt{5}$, $BP=\frac{7}{\sqrt{5}}$ and $PA=\frac{3}{\sqrt{5}}$

$\therefore$ $\alpha=\frac{BP}{PA}=\frac{7/\sqrt{5}}{3/\sqrt{5}}=\frac{7}{3}$

and $\beta=\frac{BQ}{QA}=\frac{3\sqrt{5}}{-\sqrt{5}}=-3$

$\therefore \alpha, \beta$ are roots of the equation $x^2-x(\alpha+\beta)+\alpha\beta=0$

i.e., $x^2-x\left(\frac{7}{3}-3\right)+\frac{7}{3}(-3)=0$

or $\quad 3x^2 + 2x - 21 = 0$

89. a.

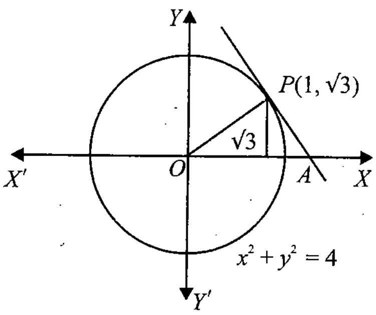


Fig. 2.155

The equations of the tangent and normal to $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ are $x + \sqrt{3}y = 4$ and $y = \sqrt{3}x$

The tangent meets x-axis at (4, 0).

Therefore, area of $\Delta OAP = \frac{1}{2}(4)\sqrt{3} = 2\sqrt{3}$ sq. units

90. d. Any point on the line $7x + y + 3 = 0$ is $Q(t, -3 - 7t)$, $t \in R$

Now $P(h, k)$ is image of point Q in the line $x - y + 1 = 0$

Then, $\quad \frac{h-t}{1} = \frac{k-(-3-7t)}{-1}$

$$= -\frac{2(t-(-3-7t)+1)}{1+1}$$

$$= -8t - 4$$

$\Rightarrow \quad (h, k) \equiv (-7t - 4, t + 1)$

This point lies on the circle $x^2 + y^2 = 9$

$\Rightarrow \quad (-7t - 4)^2 + (t + 1)^2 = 9$

$\Rightarrow \quad 50t^2 + 58t + 8 = 0$

$\Rightarrow \quad 25t^2 + 29t + 4 = 0$

$\Rightarrow \quad (25t + 4)(t + 1) = 0$

$\Rightarrow \quad t = -4/25, t = -1$

$\Rightarrow \quad (h, k) \equiv \left(-\frac{72}{25}, \frac{21}{25}\right)$ or (3, 0)

91. c. Let the coordinates be $A(a, 0)$ and $B(-a, 0)$ and let the straight line be $y = mx + c$. Then,

$$\frac{mx+c}{\sqrt{1+m^2}} + \frac{-mx+c}{\sqrt{1+m^2}} = 2k$$

$\Rightarrow \quad c = k\sqrt{1+m^2}$

So, the straight line is $\quad y = mx + k\sqrt{1+m^2}$.

Clearly, it touches the circle $x^2 + y^2 = k^2$ of radius k.

92. a. The midpoint is the intersection of the chord and perpendicular line to it from the centre (3, –1).

The equation of perpendicular line is $5x + 2y - 13 = 0$.

Solving this with the given line, we get the point (1, 4).

93. d. The locus is the radical axis which is perpendicular to the line joining the centres of the circles.

94. b. In an equilateral triangle, circumcentre and in centre are coincident.

$\therefore \quad$ Incentre $= (-g, -f)$

(1, 1) lies on the circle

$\Rightarrow \quad 1^2 + 1^2 + 2g + 2f + c = 0$

$\Rightarrow \quad c = -2(g + f + 1)$

Also, in an equilateral triangle,

Circumradius = 2 × inradius

Therefore, inradius $= \frac{1}{2} \times \sqrt{g^2 + f^2 - c}$

it is continuous single word.

$\therefore$ The equation of the incircle is

$$(x+g)^2 + (y+f)^2 = \frac{1}{4}(g^2 + f^2 - c)$$

$$= \frac{1}{4}(g^2 + f^2 + 2(g + f + 1))$$

95. a.

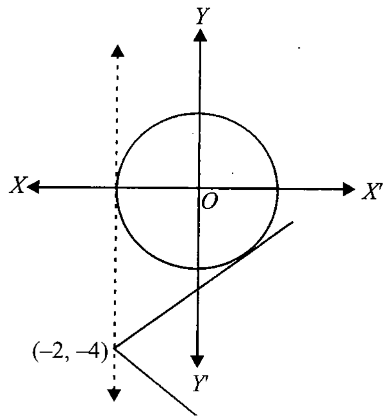


Fig. 2.156

Any tangent of $x^2 + y^2 = 4$ is $y = mx \pm 2\sqrt{1+m^2}$ if it passes through (–2, –4) then $(2m - 4)^2 = 4(1 + m^2)$

$\Rightarrow \quad 4m^2 + 16 - 16m = 4 + 4m^2$

$\Rightarrow \quad m = \infty, m = \frac{3}{4}$.

Hence, slope of reflected ray is $\frac{3}{4}$.

Thus, equation of incident ray is $(y + 4) = -\frac{3}{4}(x + 2)$, i.e., $4y + 3x + 22 = 0$.

96. a. Any point on line $x + y = 25$ is $P \equiv (a, 25 - a), a \in R$

Equation of chord AB is $T = 0$,

i.e., $$xa + y(25 - a) = 9 \quad \text{(i)}$$

If midpoint of chord AB is $C(h, k)$, then equation of chord AB is

$$T = S_1, \text{ i.e., } xh + yk = h^2 + k^2 \quad \text{(ii)}$$

Comparing the ratio of coefficients of Eqs. (i) and (ii), we get

$$\frac{a}{h} = \frac{25-a}{k} = \frac{9}{h^2+k^2}$$

$$\Rightarrow \quad \frac{a+25-a}{h+k} = \frac{9}{h^2+k^2}$$

Thus, locus of 'C' is $25(x^2 + y^2) = 9(x + y)$.

97. a.

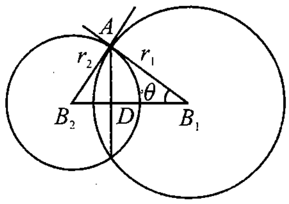


Fig. 2.157

Let $$\angle AB_1B_2 = \theta$$

$$\Rightarrow \quad AD = r_1 \sin\theta$$

and $$AD = r_2 \cos\theta$$

$$\Rightarrow \quad AD^2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) = 1$$

$$\Rightarrow \quad AD = \frac{r_1r_2}{\sqrt{r_1^2 + r_2^2}}$$

Thus, length of common chord $= \dfrac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$

98. b.

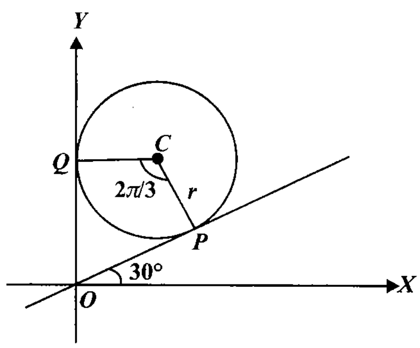


Fig. 2.158

$$\tan 30° = \frac{r}{OP}$$

$$\Rightarrow \quad OP = r\sqrt{3}$$

Also $$r^2 \frac{1}{2}\frac{2\pi}{3} = 3\pi$$

$$\therefore \quad r = 3$$

$$\therefore \quad OP = 3\sqrt{3}$$

99. c.

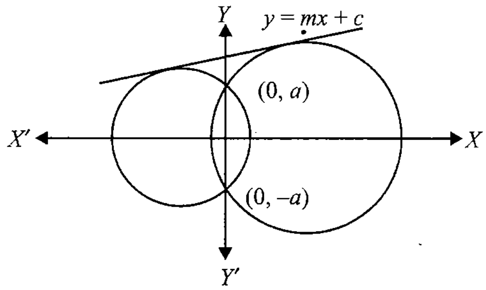


Fig. 2.159

Equation of family circles through $(0, a)$ and $(0, -a)$ is

$$[x^2 + (y - a)(y + a)] + \lambda x = 0, \lambda \in R$$

$$\Rightarrow \quad x^2 + y^2 + \lambda x - a^2 = 0$$

and $$\sqrt{\left(\frac{\lambda}{2}\right)^2 + a^2} = \frac{-\frac{m\lambda}{2} + c}{\sqrt{1+m^2}}$$

$$\Rightarrow \quad (1 + m^2)\left[\frac{\lambda^2}{4} + a^2\right] = \left(\frac{m\lambda}{2} - c\right)^2$$

$$\Rightarrow \quad (1 + m^2)\left[\frac{\lambda^2}{4} + a^2\right] = \frac{m^2\lambda^2}{4} - mc\lambda + c^2$$

$$\Rightarrow \lambda^2 + 4mc\lambda + 4a^2(1 + m^2) - 4c^2 = 0$$

$$\therefore \quad \lambda_1\lambda_2 = 4[a^2(1 + m^2) - c^2]$$

$$\Rightarrow \quad g_1g_2 = [a^2(1 + m^2) - c^2]$$

and $$g_1g_2 + f_1f_2 = \frac{c_1 + c_2}{2}$$

$$\Rightarrow \quad a^2(1 + m^2) - c^2 = -a^2$$

Hence, $$c^2 = a^2(2 + m^2)$$

100. a. Distance of given line from the centre of the circle is $|p|$.

Now line subtends right angle at the centre.

Hence, $$\text{radius} = \sqrt{2}\,|p|$$

$$\Rightarrow \quad a = \sqrt{2}\,|p|$$

$\Rightarrow \qquad a^2 = 2p^2$

101.c. Let the midpoint of the chord be $P(h, k)$.

Then $CP = \sqrt{h^2 + k^2}$, where C is centre of the circle.

Since chord subtends right angle at the centre.

$$\text{Radius} = \sqrt{2}\sqrt{h^2 + k^2}$$

$$\Rightarrow \qquad 2 = \sqrt{2}\sqrt{h^2 + k^2}$$

$\Rightarrow$ locus of P is $\quad x^2 + y^2 = 2$

102.a. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points and $x^2 + y^2 = a^2$ be the circle.

The chord of contact of tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

If it passes through $Q(x_2, y_2)$, then

$$x_1x_2 + y_1y_2 = a^2 \qquad \text{(i)}$$

The equation of the circle on PQ as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$$

This circle will cut the given circle orthogonally, if

$$0(x_1 + x_2) + 0(y_1 + y_2) = -a^2 + x_1x_2 + y_1y_2$$

$\Rightarrow \quad x_1x_2 + y_1y_2 - a^2 = 0$, which is true by Eq. (i).

103. a.

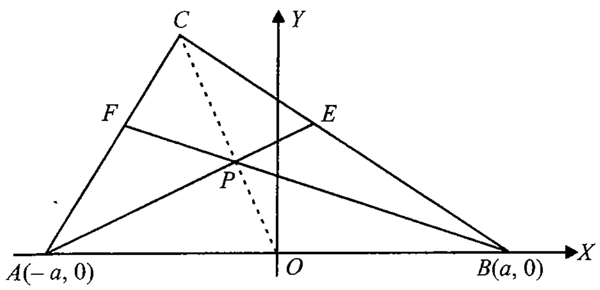


Fig. 2.160

Let A and B be $(-a, 0)$ and $(a, 0)$. Also let P be (h, k).

Then by geometry, we know $\frac{CP}{PO} = \frac{CF}{FA} + \frac{CE}{EB}$

$$\therefore \qquad \frac{CP}{PO} = 1$$

If $C(\alpha, \beta)$ lies on $x^2 + y^2 + 2gx + 2fy + c = 0$, then $\alpha = 2h$ and $\beta = 2k$

$\Rightarrow 4(h^2 + k^2 + gh + fk) + c = 0$

$\therefore$ Locus of $P(h, k)$ is $x^2 + y^2 + gx + fy + \frac{c}{4} = 0$ which is a circle of radius $= \sqrt{\left(\frac{g}{2}\right)^2 + \left(\frac{f}{2}\right)^2 - \frac{c}{4}}$

$$= \frac{1}{2}\sqrt{g^2 + f^2 - c}$$

$$= \frac{r}{2}$$

104. b. The slope of the chord is $m = -\frac{8}{y}$

$$\Rightarrow \qquad y = \pm 1, \pm 2, \pm 4, \pm 8$$

But $(8, y)$ must also lie inside the circle $x^2 + y^2 = 125$

$\Rightarrow y$ can be equal to $\pm 1, \pm 2, \pm 4 \Rightarrow 6$ values.

105. b.

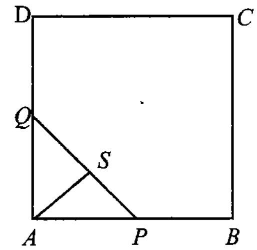


Fig. 2.161

Let S be the midpoint of PQ.

Since $\angle PAQ = \frac{\pi}{2}$, we get $AS = SP = SQ = \frac{1}{2}$

$\Rightarrow S$ lies on the quarter circle of radius $\frac{1}{2}$ with centre at A.

Similarly S can also lie on quarter circle of radius $\frac{1}{2}$ with centre at B, C or D.

$$\Rightarrow \qquad \text{area } A = 1 - \frac{\pi}{4}$$

106.b. The line $2y = gx + \alpha$ should pass through $(-g, -g)$, so $-2g = -g^2 + \alpha \Rightarrow \alpha = g^2 - 2g = (g-1)^2 - 1 \geq -1$.

107.c. Let $\sum_{i=1}^{6} x_i = \alpha$ and $\sum_{i=1}^{6} y_i = \beta$

Let O be the orthocentre of the triangle made by (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$\Rightarrow O$ is $(x_1 + x_2 + x_3, y_1 + y_2 + y_3) \equiv (\alpha_1, \beta_1)$

Similarly let G be the centroid of the triangle made by other three points

$$\Rightarrow \quad G \text{ is } \left(\frac{x_4 + x_5 + x_6}{3}, \frac{y_4 + y_5 + y_6}{3}\right)$$

$\Rightarrow \quad G$ is $\left(\frac{\alpha-\alpha_1}{3}, \frac{\beta-\beta_1}{3}\right)$

The point dividing OG is the ratio 3 : 1 is $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right) \equiv (2, 1)$

$\Rightarrow \qquad h + k = 3$

108. a. Let the centre be $(0, \alpha)$ equation of circle $x^2 + (y - \alpha)^2 = |a|^2$

$\therefore$ Equation of chord of contact for $P(h, k)$ is
$xh + yk - \alpha(y + k) + \alpha^2 - a^2 = 0$

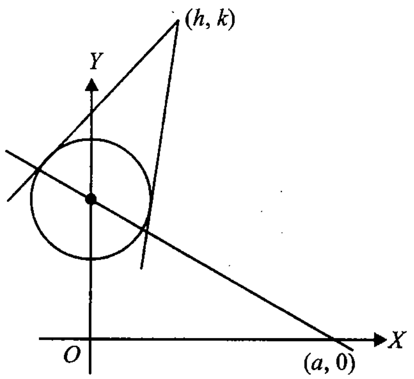


Fig. 2.162

It passes through $(a, 0)$

$\Rightarrow \qquad \alpha^2 - \alpha k + ah - a^2 = 0$

As α is real

$\Rightarrow \qquad k^2 - 4(ah - a^2) \geq 0$

109. b. Let A and B be the centres and r_1 and r_2 the radii of the two circles, then

$$A = \left(-\frac{1}{2}, -\frac{1}{2}\right), B \equiv \left(-\frac{1}{2}, \frac{1}{2}\right),$$

$$r_1 = \frac{1}{\sqrt{2}}, r_2 = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{r_1^2 + r_2^2 - AB^2}{2r_1 r_2}$$

$$= \frac{\frac{1}{2} + \frac{1}{2} - 1}{2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = 0$$

$$\therefore \qquad \theta = \frac{\pi}{2}$$

$\therefore$ Required line is parallel to x-axis and since it passes through $(1, 2)$, therefore its equation will be $y = 2$.

110. a.

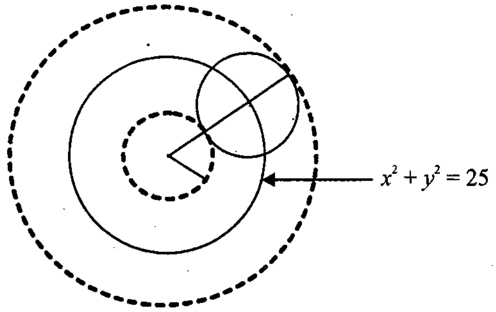


Fig. 2.163

Let (h, k) be any point in the set, then equation of circle is

$$(x - h)^2 + (y - k)^2 = 9$$

But $\quad (h, k)$ lies on $x^2 + y^2 = 25$,

then $\qquad h^2 + k^2 = 25$

$\therefore\ 2 \leq$ Distance between the centres of two circles ≤ 8

$$4 \leq h^2 + k^2 \leq 64$$

Therefore, locus of (h, k) is $4 \leq x^2 + y^2 \leq 64$.

111. b. $\qquad OR = \dfrac{2 \text{ area of } \Delta OPQ}{PQ}$

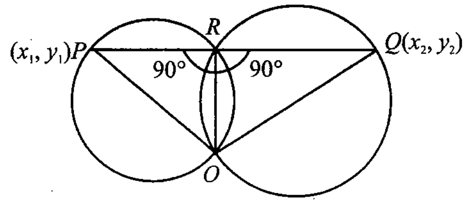


Fig. 2.164

$$= \frac{2 \cdot \left|\frac{1}{2}(x_1y_2 - x_2y_1)\right|}{PQ}$$

$$= \frac{|x_1y_2 - x_2y_1|}{PQ}$$

112. c. Substituting $y = mx$ in the equation of circle we get $x^2 + m^2x^2 + ax + bmx + c = 0$ (y/x denotes the slope of the tangent from the origin on the circle)

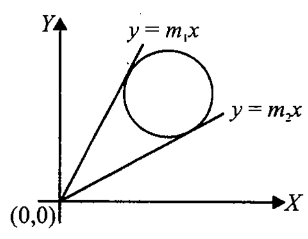


Fig. 2.165

Since line is touching the circle, we must have discriminant

$\Rightarrow \quad (a + bm)^2 - 4c(1 + m^2) = 0$

$\Rightarrow a^2 + b^2m^2 + 2abm - 4c - 4cm^2 = 0$

$\Rightarrow \quad m^2 (b^2 - 4c) + 2abm + a^2 - 4c = 0$

This equation has two roots m_1 and m_2,

$$\Rightarrow \quad m_1 + m_2 = -\frac{2ab}{b^2 - 4c} = \frac{2ab}{4c - b^2}$$

113. c. Equation of radical axis (i.e. common chord) of the two circles is

$$10x + 4y - a - b = 0 \quad \text{(i)}$$

Centre of first circle is $H(-4, -4)$.

Since second circle bisects the circumference of the first circle, therefore, centre $H(-4, -4)$ of the first circle must lie on the common chord Eq. (i).

$\therefore \quad -40 - 16 - a - b = 0$

$\Rightarrow \quad a + b = -56$

114. d. Let the equation of circle be

$$x^2 + y^2 - 4 + k(2x + y - 1) = 0$$

where k is a real number

$$\text{Radius} = \sqrt{\frac{5k^2}{4} + 4 + k}$$

Radius is minimum when $k = -\frac{2}{5}$

$\therefore$ The required equation will be

$$5x^2 + 5y^2 - 4x - 2y - 18 = 0$$

115. b.

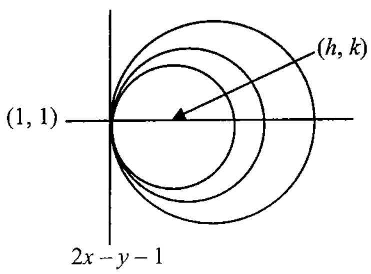


Fig. 2.166

Obviously, locus of centre is line perpendicular to the given line.

Hence, locus is $\frac{k-1}{h-1} = -\frac{1}{2}$ or $x + 2y = 0$.

116. a. Centre of the circle $x^2 + y^2 = 2x$ is $(1, 0)$.

Common chord of the other two circles is

$$8x - 15y + 26 = 0$$

Distance from $(1, 0)$ to $8x - 15y + 26 = 0$

$$= \frac{|8 + 26|}{\sqrt{15^2 + 8^2}} = 2$$

117. d. Equation of any circle though the points of intersection of given circles is

$$x^2 + y^2 - 4x - 2y - 8 + k(x^2 + y^2 - 2x - 4y - 8) = 0 \quad \text{(i)}$$

Since circle Eq. (i) passes through $(-1, 4)$

$\therefore \quad k = 1$

$\therefore$ Required circle is

$$x^2 + y^2 - 3x - 3y - 8 = 0$$

118. b. Given circles are

$$(x - 1)^2 + (y - 2)^2 = 1 \quad \text{(i)}$$

and $\quad (x - 7)^2 + (y - 10)^2 = 4 \quad$ (ii)

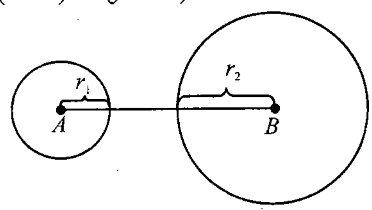


Fig. 2.167

Let $A \equiv (1, 2)$, $B \equiv (7, 10)$, $r_1 = 1$, $r_2 = 2$

$AB \equiv 10$, $r_1 + r_2 = 3$

$AB > r_1 + r_2$, hence the two circles are separated.

Radius of the two circles at time t are $(1 + 0.3t)$ and $(2 + 0.4t)$

For the two circle to touch each other

$$AB^2 = [(r_1 + 0.3t) \pm (r_2 + 0.4t)]^2$$

or $\quad 100 = [(1 + 0.3t) \pm (2 + 0.4t)]^2$

or $\quad 100 = (3 + 0.7t)^2, [(0.1)t + 1]^2$

or $\quad 3 + 0.7t = \pm 10, 0.1t + 1 = \pm 10$

$\therefore \quad t = 10, t = 90 \quad [\because t > 0]$

The two circles will touch each other externally in 10 seconds and internally in 90 seconds.

119. a. The two normals are $x = 1$ and $y = 2$

Their point of intersection $(1, 2)$ is the centre of the required circle

$$\text{Radius} = \frac{|3 + 8 - 6|}{5} = 1$$

$\therefore$ Required circle is

$$(x - 1)^2 + (y - 2)^2 = 1$$

i.e., $\quad x^2 + y^2 - 2x - 4y + 4 = 0$

Multiple Correct Answers Type

1. a., c. Equation of radical axis of the given circle is $x = 0$.

If one circle lies completely inside the other, centre of both circles should lie on the same side of radical axis and radical axis should not intersect the circles.

$\Rightarrow \quad (-a_1)(-a_2) > 0$

$\Rightarrow a_1a_2 > 0$ and $y^2 + c = 0$ should have imaginary roots

$\Rightarrow \quad c > 0.$

2. a., c., d. Coordinates of O are (5, 3) and radius = 2

Equation of tangent at $A(7, 3)$ is $7x + 3y - 5(x + 7) - 3(y + 3) + 30 = 0$

i.e., $\quad 2x - 14 = 0$, i.e., $x = 7$

Equation of tangent at $B(5, 1)$ is $5x + y - 5(x + 5) - 3(y + 1) + 30 = 0$, i.e., $-2y + 2 = 0$, i.e. $y = 1$

$\therefore$ Coordinate of C are (7, 1)

$\therefore$ Area of $\quad OACB = 4$

Equation of AB is $x - y = 4$ (radical axis)

Equation of the smallest circles is

$(x - 7)(x - 5) + (y - 3)(y - 1) = 0$

i.e., $\quad x^2 + y^2 - 12x - 4y + 38 = 0$

3. a., c. $\quad 2gg' + 2ff' = c + c'$

$\Rightarrow \quad 2 \times 1 \times 0 + 2 \cdot k \cdot k = 6 + k$

$\Rightarrow \quad 2k^2 - k - 6 = 0$

$\Rightarrow \quad (2k + 3)(k - 2) = 0$

$\therefore \quad k = 2, -\frac{3}{2}$

4. a., b. Let $O \equiv (0, 0)$ be the centre of the circle.

$\because$ Arc length $AB = \frac{\pi}{2} = \frac{1}{4}$ (circumference of the circle)

$\therefore \quad \angle AOB = \frac{\pi}{2}$

$\therefore \quad$ Slope of $OB = -\dfrac{1}{\text{slope of } OA}$

$\Rightarrow \quad$ slope of $OB = -\frac{1}{1} = -1 \quad$ (i)

Let $\quad B \equiv \left(\alpha, \pm\sqrt{1-\alpha^2}\right)$

$\therefore \quad \pm\dfrac{\sqrt{1-\alpha^2}}{\alpha} = -1 \quad$ [from Eq. (i)]

$\therefore \quad B$ can be $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ but possible points are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

5. a., c., d.

$x^2 + y^2 + 8x - 10y - 40 = 0$

Centre of the circle is $(-4, 5)$

Its radius = 9

Distance of the centre $(-4, 5)$ from the point $(-2, 3)$ is $\sqrt{4+4} = 2\sqrt{2}$

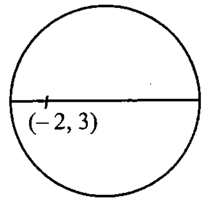


Fig. 2.168

$\therefore \quad a = 2\sqrt{2} + 9$ and $b = -2\sqrt{2} + 9$

$\therefore \quad a + b = 18$

$a - b = 4\sqrt{2}$

$a \cdot b = 81 - 8 = 73$

6. a., b. $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$

$\Rightarrow (y - m_1x)(y - m_2x) + \lambda(y - m_2x)(y - m_3x)$
$+ \mu(y - m_3x)(y - m_1x) = 0 \quad$ (i)

Clearly Eq. (i) represents a curve passing through points of intersection of lines L_1, L_2 and L_3.

Equation (i) will represent a circle if coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$

$\therefore \quad 1 + \lambda + \mu = m_1m_2 + \lambda m_2m_3 + \mu m_1m_3$

and $m_1(1 + \mu) + m_2(1 + \lambda) + m_3(\mu + \lambda) = 0$

7. a., b., c., d. Given circle is

$x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0 \quad$ (i)

For Eq. (i) to represent a circle, $h = 0$

$\therefore$ Given circle is

$x^2 + y^2 + 2gx + 2fy + c = 0 \quad$ (ii)

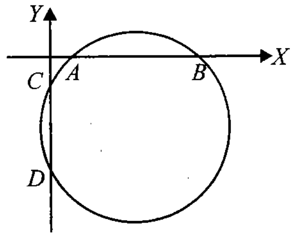


Fig. 2.169

For circle Eq. (ii) to pass thorough three quadrants only

i. $AB > 0 \therefore g^2 - c > 0$

ii. $CD > 0 \therefore g^2 - c > 0$

iii. Origin should lie outside circle Eq. (ii)

$\therefore \quad c > 0$

Therefore, required conditions are $g^2 > c, f^2 > c, c > 0, h = 0$

8. a., c. The point from which the tangents drawn are at right angle lie on the director circle.

Equation of director circle is $x^2 + y^2 = 2 \times 16 = 32$

Putting $\quad x = 2$, we get

$y^2 = 28$

$\Rightarrow \quad y = \pm 2\sqrt{7}$

$\therefore$ The points can be $(2, 2\sqrt{7})$ or $(2, -2\sqrt{7})$.

9. b., d. Line pair is $(x-1)^2 - y^2 = 0$, i.e. $x + y - 1 = 0$, $x - y - 1 = 0$.

Let the centre be $(\alpha, 0)$, then its distance from $x + y - 1 = 0$ is

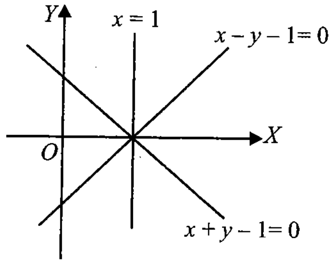


Fig. 2.170

$$\left|\frac{\alpha - 1}{\sqrt{2}}\right| = 2(\text{radius})$$

i.e. $\alpha = 1 \pm 2\sqrt{2}$

$\therefore$ Centre may be $(1 + 2\sqrt{2}, 0)$, $(1 - 2\sqrt{2}, 0)$

Now let the centre be $(1, \beta)$, then

$$\left|\frac{1 + \beta - 1}{\sqrt{2}}\right| = 2$$

$\Rightarrow$ $\beta = \pm 2\sqrt{2}$

$\therefore$ Centre may be $(1, 2\sqrt{2})$, $(1, -2\sqrt{2})$

10. a., d. Equation of the radical axis is

$$2\alpha x + 2y + 10 = 0$$

i.e. $\alpha x + y + 5 = 0$ (i)

Putting the value of y from Eq. (i) in the circle $x^2 + y^2 = 9$, we get

$$(1 + \alpha^2)x^2 + 10\alpha x + 16 = 0$$

$\because$ Radical axis is tangent

$\therefore$ $D = 0$

$\Rightarrow$ $36\alpha^2 - 64 = 0$

$\Rightarrow$ $\alpha = \pm\frac{4}{3}$

11. b., c.

$$x^2 + y^2 - 8x - 16y + 60 = 0 \quad \text{(i)}$$

Equation of chord of contact from $(-2, 0)$ is $-2x - 4(x - 2) - 8y + 60 = 0$

$$3x + 4y - 34 = 0 \quad \text{(ii)}$$

Solving Eqs. (i) and (ii)

$$x^2 + \left(\frac{34 - 3x}{4}\right)^2 - 8x - 16\left(\frac{34 - 3x}{4}\right) + 60 = 0$$

$\Rightarrow$ $16x^2 + 1156 - 204x + 9x^2 - 128x - 2176 + 192x + 960 = 0$

$\Rightarrow$ $5x^2 - 28x - 12 = 0$

$\Rightarrow$ $(x - 6)(5x + 2) = 0$

$\Rightarrow$ $x = 6, -\frac{2}{5}$

$\Rightarrow$ Points are $(6, 4)$, $\left(-\frac{2}{5}, \frac{44}{5}\right)$.

12. a., c. Equation of any tangent to the circle $x^2 + y^2 = 25$ is of the form

$$y = mx + 5\sqrt{1 + m^2}$$

(where m is the slope)

$\because$ It passes through $(-2, 11)$.

$\therefore$ $11 = -2m + 5\sqrt{1 + m^2}$

$\Rightarrow$ $(11 + 2m)^2 = 25(1 + m^2)$

$\Rightarrow$ $m = \frac{24}{7}, -\frac{4}{3}$

Therefore, equation of the tangents are

$$24x - 7y + 125 = 0$$

or $4x + 3y = 25$

13. a., d. Area of the quadrilateral $= \sqrt{c} \times \sqrt{9 + 25 - c} = 15$

$\therefore$ $c = 9, 25$

14. a., d.

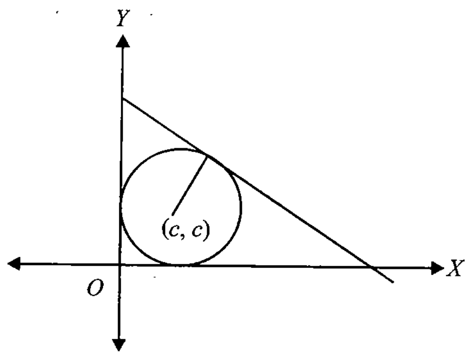


Fig. 2.171

We must have $\left|\dfrac{\frac{c}{3} + \frac{c}{4} - 1}{\sqrt{\frac{1}{3^2} + \frac{1}{4^2}}}\right| = c$

$\Rightarrow$ $c = 6, 1$

15. a.c. Since the given circle is $(x - 3)^2 + (y - 3)^2 = 9$ is touching both the axis, tangents from the origin are x-axis and y-axis or $y = 0$ and $x = 0$.

16. b. Since A, B, C, D are concyclic

$\therefore$ $OA \cdot OC = OB \cdot OD$

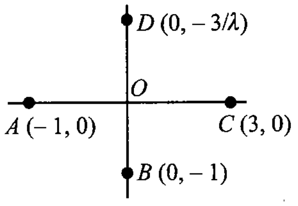


Fig. 2.172

$\Rightarrow \quad 1 \times 3 = 1 \times \left|\frac{3}{\lambda}\right|$

$\Rightarrow \quad \lambda = \pm 1$

But when $\lambda = -1$, B and D will not lie on the circle simultaneously

$\therefore \quad \lambda = 1$

17. a., b., c., d. Chords equidistance from the centre are equal.

18. b., d. Let the equation of the tangent be

$$x - 2y = k \quad \text{(i)}$$

$\because$ Line Eq. (i) touches the circle

$\therefore$ Distance from centre to line Eq. (i)

= radius of the circle

$$\therefore \quad \frac{|2 - 2 - k|}{\sqrt{5}} = \sqrt{4 + 1 + 15}$$

$$|k| = 10 \Rightarrow k = \pm 10$$

$\therefore$ The tangents can be $x - 2y \pm 10 = 0$

19. b., c. For given circle S_1: $x^2 + y^2 - 2x - 4y + 1 = 0$ and S_2: $x^2 + y^2 + 4x + 4y - 1 = 0$

$$C_1(1, 2), r_1 = 2 \text{ and } C_2(-2, -2), r_2 = 3$$

Now $\quad r_1 + r_2 = 5$ and $C_1C_2 = 5$

Hence, circles touch externally. Also comon tangent at point of contact is $S_1 - S_2 = 0$ or $3x + 4y - 1 = 0$

20. a., b., c., d.

$$r_1 = 5; r_2 = \sqrt{15}; C_1C_2 = \sqrt{40}$$

$$\Rightarrow \quad r_1 + r_2 > C_1C_2 > r_1 - r_2$$

Hence, circles intersect in two distinct points.

There are two common tangents.

Also $\quad 2g_1g_2 + 2f_1f_2 = 2(1)(3) + 2(2)(-4) = -10$

and $\quad c_1 + c_2 = -20 + 10 = -10$

Thus, two circle are orthogonal.

Length of common chord is $\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}} = 5\sqrt{\frac{3}{2}}$

Length of common tangent is $\sqrt{C_1C_2^2 - (r_1 - r_2)^2} = 5\left(\frac{12}{5}\right)^{\frac{1}{4}}$.

21. b., d.

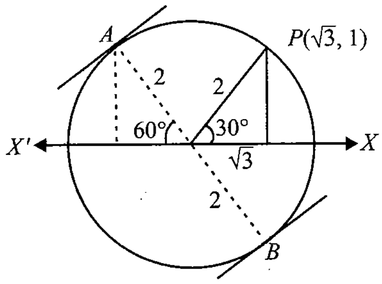


Fig. 2.173

Clearly, $A = (-2\cos 60°, 2\sin 60°)$ and $B = (2\cos 60°, -2\sin 60°)$

The tangent at A is $x(-2\cos 60°) + y(2\sin 60°) = 4$ and the tangent at B is $x(2\cos 60°) + y(-2\sin 60°) = 4$.

22. b., c.

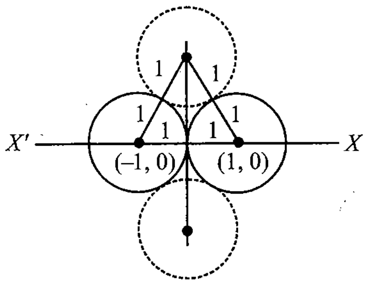


Fig. 2.174

The given circles are $x^2 + y^2 - 2x = 0$, $x > 0$, and $x^2 + y^2 + 2x = 0$, $x < 0$

From the above figure, the centres of the required circles will be $(0, \sqrt{3})$ and $(0, -\sqrt{3})$

$\therefore$ The equations of the circles are $(x - 0)^2 + (y \mp \sqrt{3})^2 = 1^2$

23. a., d. When two circle touch each other externally, then

$$r_1 + r_2 = \sqrt{\{0 - (-a)\}^2 + \{0 - (-1)\}^2}$$

$$\Rightarrow \quad 3 + a = \sqrt{a^2 + 1}$$

$$\Rightarrow \quad a = -\frac{4}{3}$$

When two circle touch each other internally, then

$$|r_1 - r_2| = \sqrt{\{0 - (-a)\}^2 + \{0 - (-1)\}^2}$$

$$\Rightarrow \quad |3 - a| = \sqrt{a^2 + 1}$$

$$\Rightarrow \quad a = \frac{4}{3}$$

24. b., c. Equation of pair of tangent by $SS' = T^2$ is

$$(ax + 0 - 1)^2 = (x^2 + y^2 - 1)(a^2 + 0 - 1)$$

or $(a^2 - 1)y^2 - x^2 + 2ax - a^2 = 0$

If θ be the angle between the tangents, then

$$\tan\theta = \frac{2\sqrt{H^2 - AB}}{A+B}$$

$$= \frac{2\sqrt{-(a^2-1)(-1)}}{a^2-2}$$

$$= \frac{2\sqrt{a^2-1}}{a^2-2}$$

If θ lies in II quadrant, then $\tan\theta < 0$

$\therefore \quad \frac{2\sqrt{a^2-1}}{a^2-2} < 0$

$\Rightarrow \quad a^2 - 1 > 0$

and $\quad a^2 - 2 < 0$

$\Rightarrow \quad |a| > 1$ and $|a| < 2$

$\Rightarrow \quad a \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$

25. b., c.

Distance of line $x + y - 1 = 0$ from the centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$ is $\frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}} = \sqrt{2}$

Now distance of line in options (b) and (c) from the centre is also $\sqrt{2}$.

Hence, given lines are $x - y = 0$ and $x + 7y = 0$.

26. c., d.

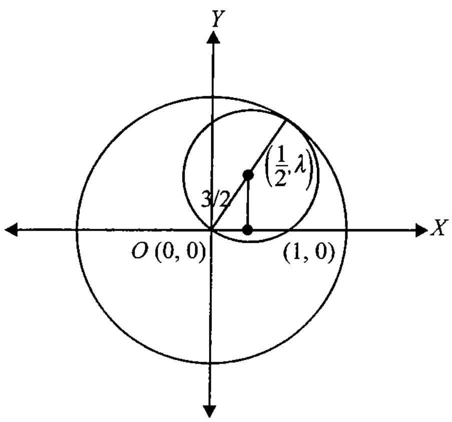


Fig. 2.175

From the diagram

$$\sqrt{\left(\frac{1}{2}\right)^2 + \lambda^2} = \frac{3}{2} \Rightarrow 1 = \pm\sqrt{2}$$

Hence, centres of the circle are $\left(\frac{1}{2}, \pm\sqrt{2}\right)$.

Reasoning Type

1. **d.** Statement 2 is true as the centre is equidistant from A and B, hence lies on the perpendicular bisector of AB.

   Statement 1 is false as the distance between the given points is 10 and hence any circle through A and B has radius more than or equal to 5, and hence there is no circle of radius 4 through A and B is possible.

2. **a.** We know that chords of contact of given circle generated by any point on given line passes through the fixed point, as they form family of straight lines, hence both the statements are true and statement 2 is the correct explanation of statement 1.

3. **b.** For circle $x^2 + y^2 = 144$, centre $C_1(0, 0)$ and radius $r_1 = 12$.

   For circle $x^2 + y^2 - 6x - 8y = 0$, centre $C_2 = (3, 4)$ and radius $r_2 = 5$.

   Now $C_1C_2 = 5$ and $r_1 - r_2 = 7$, thus $C_1C_2 < r_1 - r_2$, hence one circle is completely lying inside other without touching it, hence there is no common tangent. Therefore, statement 1 is true. Therefore, both the statements are true but statement 2 is not correct explanation of statement 1.

4. **b.** Centre of the circle $C(2, 1)$ and radius $r = 5$.

   Distance of $P(10, 7)$ from $C(2, 1)$ is 10 units, hence required distances are 5, 15, respectively. Therefore, Statement 1 is true. Statement 2 is true but not the correct explanation of statement 1, as the information is not sufficient to get distance said in Statement 1.

5. **d.** Given points are collinear, hence circle is not possible. Hence, statement 1 is false, however statement 2 is true.

6. **a.** Here $(O_1O_2)^2 = t^2 + (t^2+1)^2 = t^4 + 3t^2 + 1 \geq 0$

   $\Rightarrow \quad O_1O_2 \geq 1$ and $|r_1 - r_2| = 1$

   $\Rightarrow O_1O_2 \geq |r_1 - r_2|$ hence the two circles have at least one common tangent.

7. **a.** The centre of circle is (h, h) and radius $= h$

   $\Rightarrow$ The circle is touching the co-ordinate axes.

8. **b.** Circles $S_1: x^2 + y^2 - 4x - 6y - 8 = 0$ and $S_2: x^2 + y^2 - 2x - 3 = 0$

   $C_1(2, 3), r_1 = \sqrt{21}, C_2(1, 0), r_2 = 2$

   $C_1C_2 = \sqrt{10}, r_1 + r_2 = 2 + \sqrt{21}, r_2 - r_1 = \sqrt{21} - 2$

   Here $r_2 - r_1 < C_1C_2 < r_1 + r_2$. Hence, two circle intersect at two distinct points. Statement 2 is true, but does not explain statement 1.

9. **a.** Clearly $(\sqrt{2}, \sqrt{6})$ lies on $x^2 + y^2 = 8$, which is the director circle of $x^2 + y^2 = 4$.

$\Rightarrow$ Tangents PA and PB are perpendicular to each other.

$\therefore$ $(OAPB)$ is a square.

$\therefore$ Area of $OAPB = 4$.

10. **d.** Point of intersection of $x + 7 = 3$ and $x - y = 1$ is $(2, 1)$.

11. **d.** Statement 2 is correct (a known fact).

Using statement 2, x intercept made by $x^2 + y^2 - 2x + 6y + 5 = 0$ is $2\sqrt{(-1)^2 - 5}$ an imaginary number. Thus, $x^2 + y^2 - 2x + 6y + 5 = 0$ is away from x-axis. Hence, statement 1 is false.

12. **d.** Statement 2 is true as in any triangle in-circle and three ex-circles touches the three sides of the triangle. But Statement 1 is false as given lines are concurrent, hence triangle is not formed.

13. **a.** Given points are $A(1, 1)$, $B(2, 3)$ and $C(3, 5)$ which are collinear as slope AB = slope $BC = 2$. Hence, statement 2 is true.

Chord of contact are concurrent then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Hence, point (x_1, y_1), (x_1, y_1) and (x_3, y_3) are collinear.

14. **a.** $x^2 + y^2 - 2x - 2ay - 8 = 0$

$\Rightarrow (x^2 + y^2 - 2x - 8) - 2a(y) = 0$

$$S + \lambda L = 0$$

Solving circle $x^2 + y^2 - 2x - 8 = 0$ and line $y = 0$

$\therefore$ $x^2 - 4x + 2x - 8 = 0$

$\therefore$ $x = 4, x = -2$

So, $(4, 0)$, $(-2, 0)$ are the points of intersection which lie on x-axis.

15. **c.** Equation of chord of contact from $A(x_1, y_1)$ is

$$xx_1 + yy_1 - a^2 = 0$$
$$xx_2 + yy_2 - a^2 = 0$$
$$xx_3 + yy_3 - a^2 = 0$$

i.e., $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow A, B, C$ are collinear.

16. **a.**

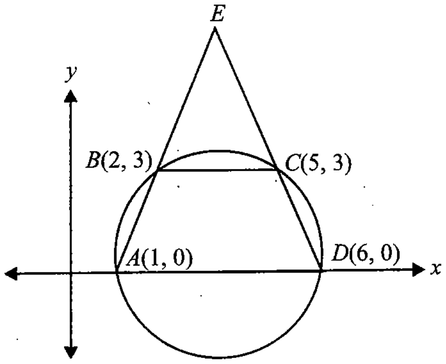


Fig. 2.176

From the figure, it is clear that $ABCD$ is isosceles trapezium as $AB = CD$. Also ΔEAD is isosceles $\Rightarrow EA \times EB = EC \times ED$

Hence, both the statements are correct and statement 1 is correct explanation of statement 1.

17. **c.** Statement 1 is true because common chord itself passes through origin.

Statement 2 is false (common chord is $x - y = 0$).

18. **a.** Common chord of two orthogonal circles subtend supplementary angles at the centre and so complemtary angles on the circumferences of the two circle.

$\therefore$ Both the statements are correct and statement 2 is the correct explanation of statement 1.

19. **a.** Since point lies inside the circle

$\Rightarrow$ $a^2 + a^2 - 4a - 2a - 8 < 0$

$\Rightarrow$ $a^2 - 3a - 4 < 0$

$\Rightarrow$ $-1 < a < 4$

20. **a.** We know that the radical axis of the circle is the locus of point from which length of tangents to given two circles is same, also it the locus of the centre of the circle which intersect the given two circles orthogonally.

Now radical axis of the given two circles is $2x + y - 4 = 0$. Any point on this line is $(t, 4 - 2t)$, $t \in R$.

Hence, both the statements are true and statement 2 is correct explanation of statement.

21. **d.** Since $S_1 = 0$ and $S_3 = 0$ has no radical axis

$\therefore$ Radical centre does not exist.

22. **d.** The statement 2 is well-known result, but if applied to the data given in statement 1 will yield $5x - 9y + 46 = 0$.

$\Rightarrow$ Statement 1 is false, statement 2 is true.

23. **c.** Statement 2 is false because line joining centres may not be parallel to common tangents.

Statement 1 can be proved easily by using distance between centres = sum of radii.

24. **a.** Two circles touch each other $C_1C_2 = |r_1 \pm r_2|$

$\Rightarrow$ $$\sqrt{p^2 + q^2} = \sqrt{p^2 - r} = \sqrt{q^2 - r}$$

$$\Rightarrow \quad p^2 + q^2 = p^2 - r + q^2 - r + 2\sqrt{(p^2 - r)(q^2 - r)}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{p^2} + \frac{1}{q^2}$$

Linked Comprehension Type

For Problems 1–3

1. d, 2. b, 3. a.

Sol. It is given that one of the diagonals of the square is parallel to the line $y = x$.

Also the length of the diagonal of the square is $4\sqrt{2}$.

Hence, the equation of the one of diagonals is

$$\frac{x-3}{\frac{1}{\sqrt{2}}} = \frac{y-4}{\frac{1}{\sqrt{2}}} = r = \pm 2\sqrt{2}$$

Hence, $\quad x - 3 = y - 4 = \pm 2$

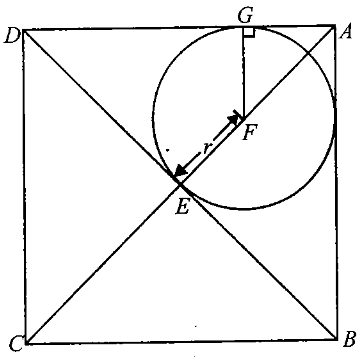
$\Rightarrow \quad x = 5, 1$ and $y = 6, 2$

Hence, two of the vertices are (1, 2) and (5, 6).

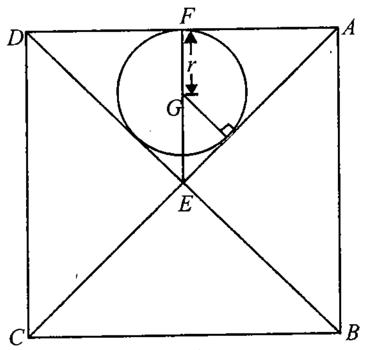
The other diagonal is parallel to the line $y = -x$, so that its equation is

$$\frac{x-3}{\frac{1}{\sqrt{2}}} = \frac{y-4}{-\frac{1}{\sqrt{2}}} = r = \pm 2\sqrt{2}$$

Hence, the two vertices on this diagonal are (1, 6) and (5, 2).



(a)



(b)

Fig. 2.177

$AB = 4, AC = 4\sqrt{2}$

$\Rightarrow \quad AE = 2\sqrt{2}$

In Fig. (a), $\quad EF + FA = AE$

$\Rightarrow \quad r + \sqrt{2}r = 2\sqrt{2}$

$\Rightarrow \quad r = \dfrac{2\sqrt{2}}{\sqrt{2}+1} = 2\sqrt{2}(\sqrt{2} - 1)$

In Fig. (b), $\quad EG + GF = EF$

$\Rightarrow \quad \sqrt{2}\,r + r = 2$

$\Rightarrow \quad r = \dfrac{2}{\sqrt{2}+1} = 2(\sqrt{2} - 1)$

For Problems 4–6

4. d., 5.c., 6. c.,

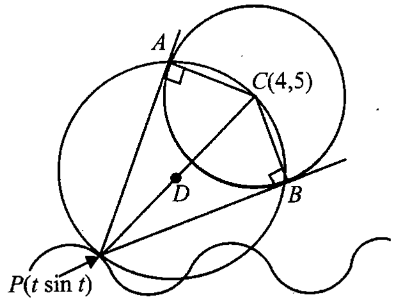


Fig. 2.178

Sol. Centre of the given circle is $C(4, 5)$. Points P, A, C, B are concyclic such that PC is diameter of the circle. Hence, centre D of the circumcircle of ΔABC is midpoint of PC, then we have

$$h = \frac{t+4}{2} \text{ and } k = \frac{\sin t - 5}{2}.$$

Eliminating t, we have $\quad k = \dfrac{\sin(2h-4)+5}{2}$

or $\quad y = \dfrac{\sin(2x-4)+5}{2}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{\sin^{-1}(2x-5)+4}{2}$

Thus range of $y = \dfrac{\sin(2x-4)+5}{2}$ is [2, 3] and period is π.

Also $f(x) = 4 \Rightarrow \sin(2x - 4) = 3$ which has no real solutions.

For $f(x) = 1 \Rightarrow \sin(2x - 4) = -3$ which has no real solutions.

But range of $y = \dfrac{\sin^{-1}(2x-5)+4}{2}$ is $\left[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2\right]$

For Problems 7–9

7. c., 8. d., 9. c.

Sol. Equation of line passing through the points $A(3, 7)$ and $B(6, 5)$ is

$$y - 7 = -\frac{2}{3}(x - 3)$$

or $\quad 2x + 3y - 27 = 0$

Also equation of circle with A and B as diameter end points is

$(x-3)(x-6)+(y-7)(y-5)=0$

Now family of circle through A and B is

$(x-3)(x-6)+(y-7)(y-5)+\lambda(2x+3y-27)=0$ (i)

If circle belonging to this family touches the x-axis, then equation $(x-3)(x-6)+(0-7)(0-5)+\lambda(2x+3(0)-27)=0$ has two equal roots, for which Discriminant $D=0$, which gives two values of λ.

Equation of common chord of (i) and $x^2+y^2-4x-6y-3=0$ is radical axis, which is

$[(x-3)(x-6)+(y-7)(y-5)+\lambda(2x+3y-27)]-[x^2+y^2-4x-6y-3]=0$

or $(2\lambda-5)x+(3\lambda-6)y+(-27\lambda+56)=0$

or $(-5x-6y+56)+\lambda(2x+3y-27)=0$

This is family of lines which passes through the point of intersection of $-5x-6y+56=0$ and $2x+3y-27=0$ which is $(2, 23/3)$.

If circle (i) cuts $x^2+y^2=29$ orthogonally, then $0+0=-29+56-27\lambda=0 \Rightarrow \lambda=1$

$\Rightarrow$ Required circle is $x^2+y^2-7x-9y+26=0$, centre is $(7/2, 9/2)$

For Problems 10–12

10. c., 11. a., 12.c.

Sol. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{(i)}$$

The line $lx+my+1=0$, will touch circle (i), if the length of $\perp$ from the centre $(-g,-f)$ of the circle on the line is equal to its radius,

i.e., $$\frac{|-gl-mf+1|}{\sqrt{(l^2+m^2)}}=\sqrt{(g^2+f^2-c)}$$

$$(gl+mf-1)^2=(l^2+m^2)(g^2+f^2-c)$$

$$\Rightarrow (c-f^2)l^2+(c-g^2)m^2-2gl-2fm+2gflm+1=0 \quad \text{(ii)}$$

But the given condition of tangency is

$$4l^2-5m^2+6l+1=0 \quad \text{(iii)}$$

$\therefore$ Comparing Eqs. (ii) and (iii), we get $c-f^2=4$, $c-g^2=-5$, $-2g=6$, $-2f=0$, $2gf=0$.

Solving, we get $f=0$, $g=-3$, $c=4$

Substituting these values in Eq. (i), the equation of the circle is $x^2+y^2-6x+4=0$. Any point on the line $x+y-1=0$ is $(t, 1-t)$, $t \in R$.

Chord of contact generated by this point for the circle is $tx+y(1-t)-3(t+x)+4=0$ or $t(x-y-3)+(-3x+y+4)=0$, which are concurrent at point of intersection of the lines $x-y-3=0$ and $-3x+y+4=0$ for all values of t. Hence, lines are concurrent at $\left(\frac{1}{2},-\frac{5}{2}\right)$.

Also point $(2,-3)$ lies outside the circle from which two tangents can be drawn.

For Problems 13–15

13. c., 14. a., 15. a.

Sol. 13. c. Given $QT=QA=1$

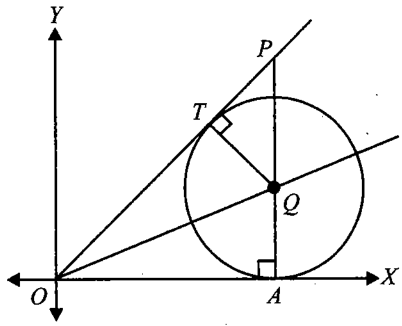


Fig. 1.179

Let $PQ=x$, then $PT=\sqrt{x^2-1}$

ΔTQP and ΔAPO are similar triangles

Then, $$OT=OA=\frac{x+1}{\sqrt{x^2-1}}$$

$$\Rightarrow 1+x+\frac{2(x+1)}{\sqrt{x^2-1}}+\sqrt{x^2-1}=8$$

$$\Rightarrow x=\frac{5}{3}$$

14. a. $$AP=\frac{8}{3}, OP=\frac{16}{3}$$

Let $\angle AOP=2\theta$, then $\sin 2\theta=\frac{1}{2}$

From ΔOAQ, $$\tan\theta=\frac{1}{OA}$$

$$\Rightarrow OA=\frac{1}{\tan\theta}$$

From $$\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{1}{2}$$

$$\Rightarrow \tan\theta=2-\sqrt{3}$$

Hence, $$OA=2+\sqrt{3}$$

Hence, equation of circle is $(x-(2+\sqrt{3}))^2+(y-1)^2=1$

15. a. Equation of tangent OT is

$$\frac{x-0}{\cos 2\theta}=\frac{y-0}{\sin 2\theta}$$

$$\Rightarrow x-\sqrt{3}y=0.$$

For Problems 16–18

16. b., 17.d., 18.b.

Sol. 16. b. $\because$ $PQ=PR$, i.e., parallelogram $PQRS$ is a rhombus

$\therefore$ Midpoint of QR = midpoint of PS and $QR \perp PS$

$\therefore$ S is the mirror image of P w.r.t. QR

$\because$ $L \equiv 2x+y=6$

Let $P \equiv (k, 6-2k)$

$\because$ $\angle PQO=\angle PRO=\frac{\pi}{2}$

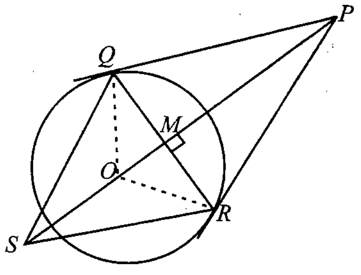


Fig. 1.180

$\therefore$ OP is diameter of circumcircle PQR, then centre is $\left(\frac{k}{2}, 3-k\right)$

$\therefore$ $x = \frac{k}{2} \Rightarrow k = 2x$ and $y = 3 - k$

$\therefore$ Required locus is $2x + y = 3$.

17. d. $P(6, 8)$

$\therefore$ Equation of QR(chord of contact) is $6x + 8y = 4$

$\Rightarrow$ $3x + 4y - 2 = 0$

$\therefore$ $PM = \frac{48}{5}$ and $PQ = \sqrt{96}$

$$QM = \sqrt{96 - \frac{48^2}{25}} = \sqrt{\frac{96}{25}}$$

$\therefore$ $QR = 2\sqrt{\frac{96}{25}}$

$\therefore$ Area of $\Delta PQR = \frac{1}{2} \cdot PM \times QR = \frac{192\sqrt{6}}{25}$

$\because$ $PQRS$ is a rhombus

$\therefore$ Area of ΔQRS = Area of ΔPQR

$= \frac{192\sqrt{6}}{25}$ sq. units

18. b. As $P \equiv (3, 4)$

$\therefore$ Equation of QR is $3x + 4y = 4$ (i)

Let $S \equiv (x_1, y_1)$

$\because$ S is the mirror image of P w.r.t. Eq. (i)

Then $\frac{x_1 - 3}{3} = \frac{y_1 - 4}{4}$

$$= \frac{-2(3 \times 3 + 4 \times 4 - 4)}{3^2 + 4^2}$$

$$= -\frac{42}{25}$$

$\therefore$ $x_1 = -\frac{51}{25}, y_1 = -\frac{68}{25}$

$$S\left(-\frac{51}{25}, -\frac{68}{25}\right)$$

For Problems 19–21

19. a., 20. d., 21. a.

Sol. 19. a.

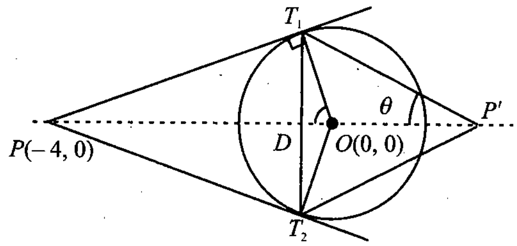


Fig. 2.181

$$PT_2 = PT_1 = \sqrt{(-4)^2 + 0^2 - 4} = 2\sqrt{3}$$

Circumcentre of triangle PT_1T_2 is midpoint of PO as

$$\angle PT_1O = \angle PT_2O = 90°$$

So, $\left(\frac{-4+0}{2}, \frac{0+0}{2}\right) = (-2, 0)$

20. d.

21. a. Let P' be a point on the circle

$$3\theta = \frac{\pi}{2}$$

$\Rightarrow$ $\theta = \frac{\pi}{6}$

Area of the rhombus = $(2\sqrt{3})(2\sqrt{3})\sin\frac{\pi}{3} = 6\sqrt{3}$

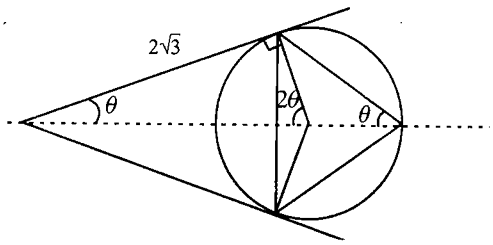


Fig. 2.182

For Problems 22–24

22. b., 23.a., 24. a.

Sol. 22. b.

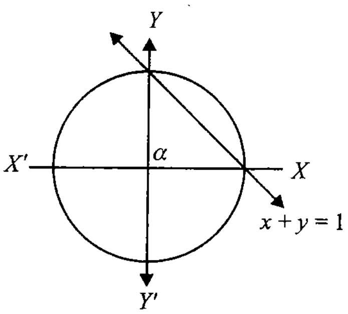


Fig. 2.183

23. a. Slope of chord = 1