Chapter 11

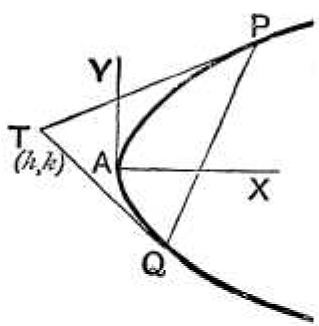
THE PARABOLA

(continued)

Some examples of Loci connected with the Parabola.

235. Ex. 1. *Find the locus of the intersection of tangents to the parabola $y^2=4ax$, the angle between them being always a given angle α.*

The straight line $y=mx+\frac{a}{m}$ is always a tangent to the parabola.



If it pass through the point T (h, k) we have

$$m^2h-mk+a=0\text{............}(1).$$

If m_1 and m_2 be the roots of this equation we have (by Art. 2)

$$m_1+m_2=\frac{k}{h}\text{.............}(2),$$

and

$$m_1m_2=\frac{a}{h}\text{..............}(3),$$

and the equations to TP and TQ are then

$$y=m_1x+\frac{a}{m_1} \text{ and } y=m_2x+\frac{a}{m_2}.$$

Hence, by Art. 66, we have

$$\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$$

$$=\frac{\sqrt{\frac{k^2}{h^2}-\frac{4a}{h}}}{1+\frac{a}{h}}=\frac{\sqrt{k^2-4ah}}{a+h}, \text{ by (2) and (3).}$$

$$\therefore\ k^2 - 4ah = (a+h)^2 \tan^2 \alpha.$$

Hence the coordinates of the point T always satisfy the equation

$$y^2 - 4ax = (a+x)^2 \tan^2 \alpha.$$

We shall find in a later chapter that this curve is a hyperbola.

As a particular case let the tangents intersect at right angles, so that $m_1 m_2 = -1$.

From (3) we then have $h = -a$, so that in this case the point T lies on the straight line $x = -a$, which is the directrix.

Hence the locus of the point of intersection of tangents, which cut at right angles, is the directrix.

Ex. 2. *Prove that the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$ is the curve*

$$y^2(x+2a) + 4a^3 = 0.$$

Let PQ be a chord which is normal at P. Its equation is then

$$y = mx - 2am - am^3 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let the tangents at P and Q intersect in T, whose coordinates are h and k, so that we require the locus of T.

Since PQ is the polar of the point (h, k) its equation is

$$yk = 2a(x+h) \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now the equations (1) and (2) represent the same straight line, so that they must be equivalent. Hence

$$m = \frac{2a}{k}, \text{ and } -2am - am^3 = \frac{2ah}{k}.$$

Eliminating m, *i.e.* substituting the value of m from the first of these equations in the second, we have

$$-\frac{4a^2}{k} - \frac{8a^4}{k^3} = \frac{2ah}{k},$$

i.e.
$$k^2(h+2a) + 4a^3 = 0,$$

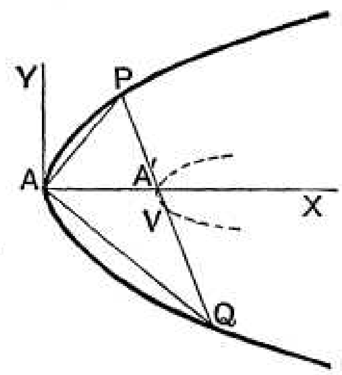
The locus of the point T is therefore

$$y^2(x+2a) + 4a^3 = 0.$$

Ex. 3. *Find the locus of the middle points of chords of a parabola which subtend a right angle at the vertex, and prove that these chords all pass through a fixed point on the axis of the curve.*

First Method. Let PQ be any such chord, and let its equation be

$$y = mx + c \quad \text{..............} (1).$$



The lines joining the vertex with the points of intersection of this straight line with the parabola

$$y^2 = 4ax \quad \text{..............} (2),$$

are given by the equation

$$y^2c = 4ax(y - mx). \quad \text{(Art. 122)}$$

These straight lines are at right angles if

$$c + 4am = 0. \quad \text{(Art. 111)}$$

Substituting this value of c in (1), the equation to PQ is

$$y = m(x - 4a) \quad \text{..............} (3).$$

This straight line cuts the axis of x at a constant distance $4a$ from the vertex, *i.e.* $AA' = 4a$.

If the middle point of PQ be (h, k) we have, by Art. 220,

$$k = \frac{2a}{m} \quad \text{..............} (4).$$

Also the point (h, k) lies on (3), so that we have

$$k = m(h - 4a) \quad \text{..............} (5).$$

If between (4) and (5) we eliminate m, we have

$$k = \frac{2a}{k}(h - 4a),$$

i.e.

$$k^2 = 2a(h - 4a),$$

so that (h, k) always lies on the parabola

$$y^2 = 2a(x - 4a).$$

This is a parabola one half the size of the original, and whose vertex is at the point A' through which all the chords pass.

Second Method. Let P be the point $(at_1^2, 2at_1)$ and Q be the point $(at_2^2, 2at_2)$.

The tangents of the inclinations of AP and AQ to the axis are

$$\frac{2}{t_1} \text{ and } \frac{2}{t_2}.$$

Since AP and AQ are at right angles, therefore

$$\frac{2}{t_1} \cdot \frac{2}{t_2} = -1,$$

i.e.

$$t_1 t_2 = -4 \quad \text{..............} (6).$$

As in Art. 229 the equation to PQ is

$$(t_1 + t_2)y = 2x + 2at_1t_2 \quad \text{..............} (7).$$

This meets the axis of x at a distance $-at_1t_2$, *i.e.*, by (6), $4a$, from the origin.

Also, (h, k) being the middle point of PQ, we have

$$2h = a(t_1^2 + t_2^2),$$

and
$$2k = 2a(t_1 + t_2).$$

Hence
$$k^2 - 2ah = a^2(t_1 + t_2)^2 - a^2(t_1^2 + t_2^2)$$
$$= 2a^2t_1t_2 = -8a^2,$$

so that the locus of (h, k) is, as before, the parabola

$$y^2 = 2a(x - 4a).$$

Third Method. The equation to the chord which is bisected at the point (h, k) is, by Art. 221,

$$k(y - k) = 2a(x - h),$$

i.e.
$$ky - 2ax = k^2 - 2ah \qquad (8).$$

As in Art. 122 the equation to the straight lines joining its points of intersection with the parabola to the vertex is

$$(k^2 - 2ah)y^2 = 4ax(ky - 2ax).$$

These lines are at right angles if

$$(k^2 - 2ah) + 8a^2 = 0.$$

Hence the locus as before.

Also the equation (8) becomes

$$ky - 2ax = -8a^2.$$

This straight line always goes through the point $(4a, 0)$.

EXAMPLES XXIX

From an external point P tangents are drawn to the parabola; find the equation to the locus of P when these tangents make angles θ_1 and θ_2 with the axis, such that

1. $\tan\theta_1 + \tan\theta_2$ is constant $(= b)$.
2. $\tan\theta_1 \tan\theta_2$ is constant $(= c)$.
3. $\cot\theta_1 + \cot\theta_2$ is constant $(= d)$.
4. $\theta_1 + \theta_2$ is constant $(= 2\alpha)$.
5. $\tan^2\theta_1 + \tan^2\theta_2$ is constant $(= \lambda)$.
6. $\cos\theta_1 \cos\theta_2$ is constant $(= \mu)$.

7. Two tangents to a parabola meet at an angle of 45°; prove that the locus of their point of intersection is the curve

$$y^2 - 4ax = (x+a)^2.$$

If they meet at an angle of 60°, prove that the locus is

$$y^2 - 3x^2 - 10ax - 3a^2 = 0.$$

8. A pair of tangents are drawn which are equally inclined to a straight line whose inclination to the axis is α; prove that the locus of their point of intersection is the straight line

$$y = (x-a)\tan 2\alpha.$$

9. Prove that the locus of the point of intersection of two tangents which intercept a given distance $4c$ on the tangent at the vertex is an equal parabola.

10. Shew that the locus of the point of intersection of two tangents, which with the tangent at the vertex form a triangle of constant area c^2, is the curve $x^2(y^2 - 4ax) = 4c^4a^2$.

11. If the normals at P and Q meet on the parabola, prove that the point of intersection of the tangents at P and Q lies either on a certain straight line, which is parallel to the tangent at the vertex, or on the curve whose equation is $y^2(x+2a) + 4a^3 = 0$.

12. Two tangents to a parabola intercept on a fixed tangent segments whose product is constant; prove that the locus of their point of intersection is a straight line.

13. Shew that the locus of the poles of chords which subtend a constant angle α at the vertex is the curve

$$(x+4a)^2 = 4\cot^2\alpha\,(y^2 - 4ax).$$

14. In the preceding question if the constant angle be a right angle the locus is a straight line perpendicular to the axis.

15. A point P is such that the straight line drawn through it perpendicular to its polar with respect to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$. Prove that its locus is the straight line

$$2ax + by + 4a^2 = 0.$$

16. Two equal parabolas, A and B, have the same vertex and axis but have their concavities turned in opposite directions; prove that the locus of poles with respect to B of tangents to A is the parabola A.

17. Prove that the locus of the poles of tangents to the parabola $y^2 = 4ax$ with respect to the circle $x^2 + y^2 = 2ax$ is the circle $x^2 + y^2 = ax$.

18. Shew the locus of the poles of tangents to the parabola $y^2 = 4ax$ with respect to the parabola $y^2 = 4bx$ is the parabola

$$y^2 = \frac{4b^2}{a}x.$$

Find the locus of the middle points of chords of the parabola which

19. pass through the focus.

20. pass through the fixed point (h, k).

21. are normal to the curve.

22. subtend a constant angle α at the vertex.

23. are of given length l.

24. are such that the normals at their extremities meet on the parabola.

25. Through each point of the straight line $x = my + h$ is drawn the chord of the parabola $y^2 = 4ax$ which is bisected at the point; prove that it always touches the parabola

$$(y - 2am)^2 = 8a(x - h).$$

26. Two parabolas have the same axis and tangents are drawn to the second from points on the first; prove that the locus of the middle points of the chords of contact with the second parabola all lie on a fixed parabola.

27. Prove that the locus of the feet of the perpendiculars drawn from the vertex of the parabola upon chords, which subtend an angle of 45° at the vertex, is the curve

$$r^2 - 24ar\cos\theta + 16a^2\cos 2\theta = 0.$$

ANSWERS

1. $y = bx$. **2.** $cx = a$. **3.** $y = ad$.

4. $y = (x - a)\tan 2\alpha$. **5.** $y^2 - \lambda x^2 = 2ax$.

6. $x^2 = \mu^2[(x - a)^2 + y^2]$. **19.** $y^2 = 2a(x - a)$.

20. $y^2 - ky = 2a(x - h)$. **21.** $y^2(y^2 - 2ax + 4a^2) + 8a^4 = 0$.

22. $(8a^2 + y^2 - 2ax)^2 \tan^2\alpha = 16a^2(4ax - y^2)$.

23. $y^4 + 4ay^2(a - x) - 16a^3x + a^2l^2 = 0$.

24. The parabola $y^2 = 2a(x + 2a)$.

SOLUTIONS/HINTS

In Exs. 1—8, by Art. 235, Ex. 1, Equations (2) and (3), we have

$$\tan\theta_1 + \tan\theta_2 = \frac{y}{x}, \ldots\text{(i)} \qquad \tan\theta_1 \tan\theta_2 = \frac{a}{x}, \ldots\text{(ii)}$$

where (x, y) are the coordinates of P.

1. $y = bx$, by (i).

2. $cx = a$, by (ii).

3. $\cot\theta_1 + \cot\theta_2 = \dfrac{\tan\theta_1 + \tan\theta_2}{\tan\theta_1 \tan\theta_2} = \dfrac{y}{a}$; $\therefore\ y = ad.$

4. $\tan(\theta_1 + \theta_2) = \dfrac{\frac{y}{x}}{1 - \frac{a}{x}}$; $\therefore\ y = (x - a)\tan 2\alpha.$

5.
$$\tan^2\theta_1 + \tan^2\theta_2 = (\tan\theta_1 + \tan\theta_2)^2 - 2\tan\theta_1\tan\theta_2$$
$$= \frac{y^2}{x^2} - \frac{2a}{x}.$$

Hence $\lambda x^2 = y^2 - 2ax.$

6.
$$\mu = \cos\theta_1\cos\theta_2.$$
$$\therefore\ \frac{1}{\mu^2} = \sec^2\theta_1\sec^2\theta_2 = (1 + \tan^2\theta_1)(1 + \tan^2\theta_2)$$
$$= 1 + \tan^2\theta_1 + \tan^2\theta_2 + \tan^2\theta_1\tan^2\theta_2$$
$$= 1 + \frac{y^2 - 2ax}{x^2} + \frac{a^2}{x^2}, \text{ by (i) and (ii).}$$
$$\therefore\ x^2 = \mu^2\{y^2 + (x - a)^2\}.$$

7.
$$\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} = \tan 45^\circ = 1.$$
$$\therefore\ \frac{\tan^2\theta_1 + \tan^2\theta_2 - 2\tan\theta_1\tan\theta_2}{(1 + \tan\theta_1\tan\theta_2)^2} = 1.$$

(i) $\therefore$ by (i) and (ii),
$$\frac{y^2}{x^2} - \frac{4a}{x} = \left(1 + \frac{a}{x}\right)^2, \text{ i.e. } y^2 - 4ax = (x + a)^2.$$

(ii) Similarly,
$$\frac{\frac{y^2}{x^2} - \frac{4a}{x}}{\left(1 + \frac{a}{x}\right)^2} = \tan^2 60^\circ = 3;$$
$$\therefore\ y^2 - 3x^2 - 10ax - 3a^2 = 0.$$

8. Using the same equation, it is easily seen from a figure that $\theta_1 + \theta_2 = 2\alpha$ or $2\alpha \pm \pi$. Hence, in either case,

$$\tan 2\alpha = \tan(\theta_1 + \theta_2) = \frac{y}{x-a}, \text{ by (i) and (ii).}$$

9. Using the equations of Ex. 1, Art. 235, we have

$$\frac{a}{m_1} - \frac{a}{m_2} = 4c; \quad \therefore \frac{a^2(m_1 - m_2)^2}{m_1^2 m_2^2} = 16c^2,$$

or $$\frac{a^2\{(m_1+m_2)^2 - 4m_1m_2\}}{m_1^2m_2^2} = 16c^2; \quad \therefore y^2 - 4ax = 16c^2,$$

which is the equation of an equal parabola.

10. Using the same equations, the base of the $\triangle$ = the intercept on the axis of $y = \frac{a}{m_1} - \frac{a}{m_2}$; and the perpendicular from opposite vertex $= \frac{a}{m_1 m_2}$.

Hence $\frac{a^2}{m_1^2m_2^2}\left\{\frac{a}{m_1} - \frac{a}{m_2}\right\}^2 = 4c^4. \quad \therefore x^2(y^2 - 4ax) = 4c^4.$

11. Using the equations of Art. 229, the coordinates of the intersection of tangents at P and Q are

$$x = at_1t_2 \text{ and } y = a(t_1 + t_2)\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

(1) If the normal at P passes through Q,

$$t_1 + t_2 = -\frac{2}{t_1}.\ \ldots\ldots\text{(ii)} \quad \text{(Ex. XXVIII. 6.)}$$

Whence $t_1 = -\frac{2a}{y}$, and $t_2 = -\frac{xy}{2a^2}$, by (i).

Substitute in (ii), and we have $y^2(x+2a) + 4a^3 = 0$.

(2) If the normals at P and Q intersect at the point $R(t_3)$ on the curve, then

$$t_1 + \frac{2}{t_1} = -t_3 = t_2 + \frac{2}{t_2}, \text{ (Ex. XXVIII. 6),}$$

whence $\qquad t_1t_2 = 2. \quad \therefore x = 2a,$

which is a straight line parallel to $x = 0$.

12. Using the equations of Art. 229, let the points of contact of the variable tangents be t_1, t_2 and t that of the fixed tangent. The projections of the segments on the axis of x are $a(t^2 - tt_1)$ and $a(t^2 - tt_2)$ and their product is constant.

$$\therefore \; t^2 - (t_1 + t_2)\,t + t_1 t_2 = \text{cons.},$$

substituting $\qquad x = at_1 t_2, \;\; y = a(t_1 + t_2),$

we have $at^2 - yt + x = \text{cons.}$, which is a straight line.

13. Take $yy' = 2a(x + x')$, the polar of (x', y'), for the equation of the chord.

The lines joining the origin to common points of $y^2 = 4ax$ and $yy' = 2a(x + x')$ are

$$y^2 . 2ax' = 4ax(yy' - 2ax), \quad [\text{Art. } 122],$$

or

$$8a^2x^2 - 4ay' . xy + 2ax' . y^2 = 0.$$

If these intersect at an angle α, then, by Art. 110,

$$4 \cot^2\alpha\,(4a^2y'^2 - 16a^3x') = (2ax' + 8a^2)^2.$$

$\therefore$ the required locus is $4\cot^2\alpha\,(y^2 - 4ax) = (x + 4a)^2$.

14. From Ex. 13, if $\cot\alpha = 0$, the locus becomes

$$x + 4a = 0.$$

15. The line through (x', y') perpendicular to its polar

$$yy' = 2a(x + x') \;\text{ is }\; y'(x - x') + 2a(y - y') = 0,$$

or

$$x = -\frac{2a}{y'}y + x' + 2a.$$

This touches $x^2 = 4by$ if $x' + 2a = \dfrac{b}{-2a/y'}$. [Art. 206.]

$\therefore$ the required locus is $2ax + by + 4a^2 = 0$.

16. Let $y^2 - 4ax = 0$ and $y^2 + 4ax = 0$ be equations to A and B respectively. The polar of (x', y') with respect to B is $yy' + 2a(x + x') = 0$.

This touches A if $\left(-\dfrac{2a}{y'}\right)\left(-\dfrac{2ax'}{y'}\right) = a$. [Art. 206.]

$\therefore$ the required locus is $y^2 - 4ax = 0$, *i.e.* A.

17. The polar of (x', y') with respect to the circle $x^2+y^2=2ax$, viz. $xx'+yy'=a(x+x')$, touches $y^2=4ax$, if

$$\frac{a-x'}{y'}\cdot\frac{ax'}{y'}=a. \quad \text{[Art. 206.]}$$

$\therefore$ the required locus is $x^2+y^2=ax$.

18. The polar of (x', y') with respect to the parabola $y^2=4bx$, viz. $yy'=2b(x+x')$, touches $y^2=4ax$, if

$$\frac{2b}{y'}\cdot\frac{2bx'}{y'}=a. \quad \text{[Art. 206.]}$$

$\therefore$ the required locus is $y^2=\dfrac{4b^2}{a}\cdot x$.

In Nos. 19—24, t_1 and t_2 are the extremities of the chord, so that the coordinates of the middle point are

$$x=\frac{a}{2}(t_1^2+t_2^2), \quad \dots\dots\dots\dots\text{(i)}$$

$$y=a(t_1+t_2). \dots\dots\dots\dots\text{(ii)}$$

19. Since the chord passes through the focus,

$$\therefore\ t_1t_2=-1. \quad \text{[Art. 233.]}$$

From (i) and (ii),

$$y^2-2ax=2a^2t_1t_2=-2a^2; \quad \therefore\ y^2=2a(x-a).$$

20. Since the chord goes through (h, k),

$$\therefore\ k(t_1+t_2)=2h+2at_1t_2. \quad \text{[Art. 229.]}$$

Hence, from (i) and (ii),

$$\frac{ky}{a}=2h+2a\cdot\frac{y^2-2ax}{2a^2}; \quad \therefore\ y^2-ky=2a(x-h).$$

21. Since the chord is normal at t_1,

$$\therefore\ t_1+t_2=-\frac{2}{t_1}. \quad \text{[Ex. XXVIII. 6.]}$$

$\therefore$ from (ii), $t_1=-\dfrac{2a}{y}$, and, from (i) and (ii),

$$y^2-2ax=2a^2t_1t_2=-\frac{4a^3t_2}{y};$$

$$\therefore \quad t_2 = \frac{y(2ax - y^2)}{4a^3}, \text{ and } \frac{y}{a} = t_1 + t_2 = \frac{y(2ax - y^2)}{4a^3} - \frac{2a}{y};$$

$$\therefore \quad y^2(y^2 - 2ax + 4a^2) + 8a^4 = 0.$$

22. $$\alpha = \tan^{-1}\left(\frac{2at_1}{at_1^2}\right) \sim \tan^{-1}\left(\frac{2at_2}{at_2^2}\right).$$

$$\therefore \quad \tan\alpha = \frac{\frac{2}{t_2} - \frac{2}{t_1}}{1 + \frac{4}{t_1 t_2}} = \frac{2(t_1 - t_2)}{4 + t_1 t_2}.$$

$$\therefore \quad 4(t_1 - t_2)^2 = \tan^2\alpha\,(4 + t_1 t_2)^2.$$

Substitute from equations (i) and (ii).

$$\therefore \quad 4(4ax - y^2) = a^2\tan^2\alpha\left(4 + \frac{y^2 - 2ax}{2a^2}\right)^2,$$

or $$16a^2(4ax - y^2) = \tan^2\alpha\,(y^2 - 2ax + 8a^2)^2.$$

23. From (i) and (ii),

$$4ax - y^2 = a^2(t_1 - t_2)^2, \text{ and } y^2 = a^2(t_1 + t_2)^2;$$

$$\therefore \quad \frac{(4ax - y^2)\,y^2}{a^2} = a^2(t_1^2 - t_2^2)^2.$$

Also $$l^2 = a^2(t_1^2 - t_2^2)^2 + 4a^2(t_1 - t_2)^2.$$

Substituting $l^2 = \dfrac{y^2(4ax - y^2)}{a^2} + 4(4ax - y^2);$

$$\therefore \quad (y^2 - 4ax)(y^2 + 4a^2) + a^2 l^2 = 0.$$

24. If the normals at t_1 and t_2 both pass through t_3,

then $$t_1 + \frac{2}{t_1} = -t_3 = t_2 + \frac{2}{t_2}, \quad \text{[Ex. XXVIII. 6.]}$$

$\therefore \; t_1 t_2 = 2.$ From (i) and (ii)

$$y^2 - 2ax = 2a^2 t_1 t_2 = 4a^2; \quad \therefore \quad y^2 = 2a(x + 2a).$$

25. The chord of the parabola which is bisected at (x_1, y_1) is $y_1(y - y_1) = 2a(x - x_1)$ [Art. 221]

$$= 2a(x - my_1 - h), \text{ since } x_1 = my_1 + h.$$

This equation may be written

$$y + 2am = \frac{2a}{y_1}(x-h) + \frac{2a}{\frac{2a}{y_1}},$$

which shews that it always touches the parabola

$$(y+2am)^2 = 8a(x-h). \quad \text{[Art. 206.]}$$

26. Let $y^2 = 4ax$ ...(i) and $y^2 = 4b(x+c)$ ...(ii) be the equations of the parabolas.

The pole of the chord joining t_1 and t_2 of (i) is

$$[a(t_1t_2),\ a(t_1+t_2)].$$

If this point lies on (ii), then

$$a^2(t_1+t_2)^2 = 4b(at_1t_2+c). \quad \text{...........(iii)}$$

If (x, y) be the coordinates of the middle point of the chord, then $2x = a(t_1^2 + t_2^2)$, and $y = a(t_1+t_2)$,

$$\therefore\ y^2 - 2ax = 2a^2t_1t_2.$$

Substitute in (iii); $\therefore\ ay^2 = 2b(y^2 - 2ax + 2ac)$, which is the equation of a parabola.

27. The lines joining the origin to the common points of $y^2 = 4ax$ and $x\cos\alpha + y\sin\alpha = p$, are

$$y^2p = 4ax(x\cos\alpha + y\sin\alpha), \quad \text{[Art. 122]},$$

or $$4a\cos\alpha \,.\, x^2 + 4a\sin\alpha \,.\, xy - py^2 = 0.$$

If the angle between these lines is 45°, then

$$2\sqrt{4a^2\sin^2\alpha + 4ap\cos\alpha} = 4a\cos\alpha - p. \quad \text{[Art. 110.]}$$

But (p, α) are the polar coordinates of the foot of the perpendicular from the vertex on the chord. Hence the required locus is

$$16a(a\sin^2\theta + r\cos\theta) = 16a^2\cos^2\theta + r^2 - 8ar\cos\theta,$$

or $$r^2 - 24ar\cos\theta + 16a^2\cos 2\theta = 0.$$

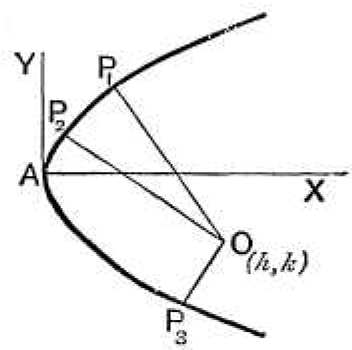
236. *To prove that, in general, three normals can be drawn from any point to the parabola and that the algebraic sum of the ordinates of the feet of these three normals is zero.*

The straight line

$$y = mx - 2am - am^3 \text{ ....(1)}$$

is, by Art. 208, a normal to the parabola at the points whose coordinates are

$$am^2 \text{ and } -2am\text{.......(2).}$$



If this normal passes through the fixed point O, whose coordinates are h and k, we have

$$k = mh - 2am - am^3,$$

i.e.

$$am^3 + (2a - h)m + k = 0 \text{ ....(3),}$$

This equation, being of the third degree, has three roots, real or imaginary. Corresponding to each of these roots, we have, on substitution in (1), the equation to a normal which passes through the point O.

Hence three normals, real or imaginary, pass through any point O.

If m_1, m_2, and m_3 be the roots of the equation (3), we have

$$m_1 + m_2 + m_3 = 0.$$

If the ordinates of the feet of these normals be y_1, y_2, and y_3, we then have, by (2),

$$y_1 + y_2 + y_3 = -2a(m_1 + m_2 + m_3) = 0.$$

Hence the second part of the proposition.

We shall find, in a subsequent chapter, that, for certain positions of the point O, all three normals are real; for other positions of O, one normal only will be real, and the other two imaginary.

237. Ex. *Find the locus of a point which is such that* (α) *two of the normals drawn from it to the parabola are at right angles,* (β) *the three normals through it cut the axis in points whose distances from the vertex are in arithmetical progression.*

Any normal is $y = mx - 2am - am^3$, and this passes through the point (h, k), if

$$am^3 + (2a - h)\, m + k = 0 \quad \text{.........................(1).}$$

If then m_1, m_2, and m_3 be the roots, we have, by Art. 2,

$$m_1 + m_2 + m_3 = 0, \quad \text{..............................(2),}$$

$$m_2m_3 + m_3m_1 + m_1m_2 = \frac{2a - h}{a}, \quad \text{................. (3),}$$

and

$$m_1m_2m_3 = -\frac{k}{a} \quad \text{..............................(4).}$$

(α) If two of the normals, say m_1 and m_2, be at right angles, we have $m_1m_2 = -1$, and hence, from (4), $m_3 = \frac{k}{a}$.

The quantity $\frac{k}{a}$ is therefore a root of (1) and hence, by substitution, we have

$$\frac{k^3}{a^2} + (2a - h)\frac{k}{a} + k = 0,$$

i.e.

$$k^2 = a\,(h - 3a).$$

The locus of the point (h, k) is therefore the parabola $y^2 = a\,(x - 3a)$ whose vertex is the point $(3a, 0)$ and whose latus rectum is one-quarter that of the given parabola.

The student should draw the figure of both parabolas.

(β) The normal $y = mx - 2am - am^3$ meets the axis of x at a point whose distance from the vertex is $2a + am^2$. The conditions of the question then give

$$(2a + am_1^2) + (2a + am_3^2) = 2\,(2a + am_2^2),$$

i.e.

$$m_1^2 + m_3^2 = 2m_2^2 \quad \text{............................(5).}$$

If we eliminate m_1, m_2, and m_3 from the equations (2), (3), (4), and (5) we shall have a relation between h and k.

From (2) and (3), we have

$$\frac{2a - h}{a} = m_1m_3 + m_2\,(m_1 + m_3) = m_1m_3 - m_2^2 \quad \text{............ (6).}$$

Also, (5) and (2) give

$$2m_2^2 = (m_1 + m_3)^2 - 2m_1m_3 = m_2^2 - 2m_1m_3,$$

i.e.

$$m_2^2 + 2m_1m_3 = 0 \quad \text{..............................(7).}$$

Solving (6) and (7), we have

$$m_1 m_3 = \frac{2a-h}{3a}, \text{ and } m_2^2 = -2 \times \frac{2a-h}{3a}.$$

Substituting these values in (4), we have

$$\frac{2a-h}{3a}\sqrt{-2\frac{2a-h}{3a}} = -\frac{k}{a},$$

i.e.

$$27ak^2 = 2(h-2a)^3,$$

so that the required locus is

$$27ay^2 = 2(x-2a)^3.$$

238. Ex. *If the normals at three points P, Q, and R meet in a point O and S be the focus, prove that SP . SQ . SR = a . SO².*

As in the previous question we know that the normals at the points $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$ meet in the point (h, k) if

$$m_1 + m_2 + m_3 = 0 \quad\text{.............................(1)},$$

$$m_2 m_3 + m_3 m_1 + m_1 m_2 = \frac{2a-h}{a} \quad\text{....................(2)},$$

and

$$m_1 m_2 m_3 = -\frac{k}{a} \quad\text{.............................(3)}.$$

By Art. 202 we have

$$SP = a(1+m_1^2),\ SQ = a(1+m_2^2),\ \text{and}\ SR = a(1+m_3^2).$$

Hence

$$\frac{SP \,.\, SQ \,.\, SR}{a^3} = (1+m_1^2)(1+m_2^2)(1+m_3^2)$$

$$= 1 + (m_1^2 + m_2^2 + m_3^2) + (m_2^2 m_3^2 + m_3^2 m_1^2 + m_1^2 m_2^2) + m_1^2 m_2^2 m_3^2.$$

Also, from (1) and (2), we have

$$m_1^2 + m_2^2 + m_3^2 = (m_1 + m_2 + m_3)^2 - 2(m_2 m_3 + m_3 m_1 + m_1 m_2)$$

$$= 2\frac{h-2a}{a},$$

and

$$m_2^2 m_3^2 + m_3^2 m_1^2 + m_1^2 m_2^2 = (m_2 m_3 + m_3 m_1 + m_1 m_2)^2 - 2m_1 m_2 m_3(m_1 + m_2 + m_3)$$

$$= \left(\frac{h-2a}{a}\right)^2, \text{ by (1) and (2).}$$

Hence $$\frac{SP \cdot SQ \cdot SR}{a^3} = 1 + 2\frac{h-2a}{a} + \left(\frac{h-2a}{a}\right)^2 + \frac{k^2}{a^2}$$

$$= \frac{(h-a)^2 + k^2}{a^2} = \frac{SO^2}{a^2},$$

i.e. $$SP \cdot SQ \cdot SR = SO^2 \cdot a.$$

EXAMPLES XXX

Find the locus of a point O when the three normals drawn from it are such that

1. two of them make complementary angles with the axis.

2. two of them make angles with the axis the product of whose tangents is 2.

3. one bisects the angle between the other two.

4. two of them make equal angles with the given line $y = mx + c$.

5. the sum of the three angles made by them with the axis is constant.

6. the area of the triangle formed by their feet is constant.

7. the line joining the feet of two of them is always in a given direction.

The normals at three points P, Q, and R of the parabola $y^2 = 4ax$ meet in a point O whose coordinates are h and k; prove that

8. the centroid of the triangle PQR lies on the axis.

9. the point O and the orthocentre of the triangle formed by the tangents at P, Q, and R are equidistant from the axis.

10. if OP and OQ make complementary angles with the axis, then the tangent at R is parallel to SO.

11. the sum of the intercepts which the normals cut off from the axis is $2(h+a)$.

12. the sum of the squares of the sides of the triangle PQR is equal to $2(h-2a)(h+10a)$.

13. the circle circumscribing the triangle PQR goes through the vertex and its equation is $2x^2 + 2y^2 - 2x(h+2a) - ky = 0$.

14. if P be fixed, then QR is fixed in direction and the locus of the centre of the circle circumscribing PQR is a straight line.

15. Three normals are drawn to the parabola $y^2=4ax\cos\alpha$ from any point lying on the straight line $y=b\sin\alpha$. Prove that the locus of the orthocentre of the triangles formed by the corresponding tangents is the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the angle α being variable.

16. Prove that the sum of the angles which the three normals, drawn from any point O, make with the axis exceeds the angle which the focal distance of O makes with the axis by a multiple of π.

17. Two of the normals drawn from a point O to the curve make complementary angles with the axis; prove that the locus of O and the curve which is touched by its polar are parabolas such that their latera recta and that of the original parabola form a geometrical progression. Sketch the three curves.

18. Prove that the normals at the points, where the straight line $lx+my=1$ meets the parabola, meet on the normal at the point $\left(\frac{4am^2}{l^2}, \frac{4am}{l}\right)$ of the parabola.

19. If the normals at the three points P, Q, and R meet in a point and if PP', QQ', and RR' be chords parallel to QR, RP, and PQ respectively, prove that the normals at P', Q', and R' also meet in a point.

20. If the normals drawn from any point to the parabola cut the line $x=2a$ in points whose ordinates are in arithmetical progression, prove that the tangents of the angles which the normals make with the axis are in geometrical progression.

21. PG, the normal at P to a parabola, cuts the axis in G and is produced to Q so that $GQ=\frac{1}{2}PG$; prove that the other normals which pass through Q intersect at right angles.

22. Prove that the equation to the circle, which passes through the focus and touches the parabola $y^2=4ax$ at the point $(at^2, 2at)$, is

$$x^2+y^2-ax(3t^2+1)-ay(3t-t^3)+3a^2t^2=0.$$

Prove also that the locus of its centre is the curve

$$27ay^2=(2x-a)(x-5a)^2.$$

23. Shew that three circles can be drawn to touch a parabola and also to touch at the focus a given straight line passing through the focus, and prove that the tangents at the point of contact with the parabola form an equilateral triangle.

24. Through a point P are drawn tangents PQ and PR to a parabola and circles are drawn through the focus to touch the parabola in Q and R respectively; prove that the common chord of these circles passes through the centroid of the triangle PQR.

25. Prove that the locus of the centre of the circle, which passes through the vertex of a parabola and through its intersections with a normal chord, is the parabola $2y^2 = ax - a^2$.

26. A circle is described whose centre is the vertex and whose diameter is three-quarters of the latus rectum of a parabola; prove that the common chord of the circle and parabola bisects the distance between the vertex and the focus.

27. Prove that the sum of the angles which the four common tangents to a parabola and a circle make with the axis is equal to $n\pi + 2\alpha$, where α is the angle which the radius from the focus to the centre of the circle makes with the axis and n is an integer.

28. PR and QR are chords of a parabola which are normals at P and Q. Prove that two of the common chords of the parabola and the circle circumscribing the triangle PRQ meet on the directrix.

29. The two parabolas $y^2 = 4a(x - l)$ and $x^2 = 4a(y - l')$ always touch one another, the quantities l and l' being both variable; prove that the locus of their point of contact is the curve $xy = 4a^2$.

30. A parabola, of latus rectum l, touches a fixed equal parabola, the axes of the two curves being parallel; prove that the locus of the vertex of the moving curve is a parabola of latus rectum $2l$.

31. The sides of a triangle touch a parabola, and two of its angular points lie on another parabola with its axis in the same direction; prove that the locus of the third angular point is another parabola.

ANSWERS

1. $y^2 = a(x - a)$, 2. $y^2 = 4ax$. 3. $27ay^2 = (2x - a)(x - 5a)^2$.
4. A parabola. 5. A straight line.
6. $27ay^2 - 4(x - 2a)^3 = \text{constant}$.
7. A straight line, itself a normal.

SOLUTIONS/HINTS

1. Using the equations of Art. 237, we have $m_1 m_2 = 1$.

$\therefore\ m_3 = -\frac{k}{a}$, from (4).

Substitute in (i); $\therefore\ \frac{k^3}{a^2} + (2a - h)\frac{k}{a} - k = 0$, whence the locus is $y^2 = a(x - a)$.

2. Using the same equations,

$$m_1m_2 = 2. \quad \therefore \ m_3 = -\frac{k}{2a}, \text{ from (4).}$$

Substitute in (i); $\therefore \ \frac{k^3}{8a^2} + (2a-h)\frac{k}{2a} - k = 0$, whence the locus is $y^2 = 4ax$.

3. Using the same equations, since $2\theta_2 = \theta_1 + \theta_3$,

$$\therefore \ \frac{2m_2}{1-m_2^2} = \frac{m_1+m_3}{1-m_1m_3} = \frac{-m_2}{1-m_1m_3};$$

$\therefore \ m_2^2 + 2m_1m_3 = 3$, and from (3) $m_2^2 - m_1m_3 = \frac{h}{a} - 2.$

$$\therefore \ 3m_1m_3 = \frac{5a-h}{a}, \text{ and } 3m_2^2 = \frac{2h-a}{a}.$$

$$\therefore \ (h-5a)^2(2h-a) = 27a^3m_1^2m_2^2m_3^2 = 27ak^2.$$

$\therefore$ the required locus is $(x-5a)^2(2x-a) = 27ay^2$.

4. The condition gives $\theta_1 + \theta_2 = 2\tan^{-1} m$;

$$\therefore \ \frac{m_1+m_2}{1-m_1m_2} = \text{cons.} = \frac{1}{\lambda} \text{ say.}$$

$$\therefore \ \frac{1}{\lambda} = \frac{m_1+m_2}{1-m_1m_2} = \frac{-m_3}{1+\frac{k}{am_3}} = -\frac{am_3^2}{k+am_3}.$$

$$\therefore \ \lambda am_3^2 + am_3 + k = 0; \ \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Also $\qquad am_3^3 + (2a-h)m_3 + k = 0.$

Subtracting, $\quad am_3^2 - \lambda am_3 + a - h = 0. \ \ldots\ldots\ldots\ldots\ldots$(ii)

From (i) and (ii),

$$\frac{m_3^2}{\lambda k - (h-a)} = \frac{m_3}{k + \lambda(h-a)} = -\frac{1}{a(\lambda^2+1)}.$$

Hence the locus is

$$\{y + \lambda(x-a)\}^2 + a(\lambda^2+1)\{\lambda y - (x-a)\} = 0.$$

5. Using the same equations,

$$\frac{\Sigma m_1 - m_1 m_2 m_3}{1 - \Sigma m_1 m_2} = \text{cons.} = \lambda.$$

$$\therefore \lambda = \frac{k/a}{1 + \dfrac{h-2a}{a}}. \quad \therefore \text{ locus is } \lambda(x-a) = y.$$

6. By Ex. II. 7 we have $\Pi(m_1 - m_2)^2 = \text{cons.} = \lambda.$

$$\therefore \lambda = \Pi(m_1 - m_2)^2 = \Pi(m_1^2 + m_2^2 + 2m_1m_2 - 4m_1m_2)$$

$$= \Pi\left(m_3^2 + \frac{4\dfrac{y}{a}}{m_3}\right) = \Pi\left(\frac{m_3^3 + 4\dfrac{y}{a}}{m_3}\right). \quad \text{[Art. 237.]}$$

$$\therefore \lambda a^3 m_1 m_2 m_3 = 64y^3 + 16ay^2\Sigma m_1^3 + 4a^2y\Sigma m_1^3m_2^3 + a^3m_1^3m_2^3m_3^3.$$

Now $\Sigma m_1^3 = 3m_1m_2m_3$, (since $m_1 + m_2 + m_3 = 0$,) $= -\dfrac{3y}{a}$.

$$\Sigma m_1^3 m_2^3 = \{\Sigma m_1 m_2\}^3 - 3\Pi(m_2m_3 + m_3m_1)$$

$$= \left(\frac{2a-x}{a}\right)^3 + 3m_1^2m_2^2m_3^2 = \frac{(2a-x)^3 + 3ay^2}{a^3}.$$

Hence, substituting, we have

$$\therefore -\lambda a^2 y = 64y^3 - 48y^3 + \frac{4y}{a}\{(2a-x)^3 + 3ay^2\} - y^3.$$

$$\therefore 27ay^2 + 4(2a-x)^3 = \text{a constant.}$$

7. From the equation of the chord joining the points $(am_1^2, -2am_1)$ and $(am_2^2, -2am_2)$, we have $m_1 + m_2 = \text{cons.}$

$$\therefore m_3 = \text{cons.} = \lambda.$$

$\therefore$ the required locus is, by (i) of Art. 237,

$$a\lambda^3 + (2a-x)\lambda + y = 0,$$

a straight line which is the normal at the point $(a\lambda^2, -2a\lambda)$.

8. The ordinate of the centroid

$$= \tfrac{1}{3}\Sigma y_1 = -\frac{2a}{3}\Sigma m_1 = 0.$$

9. By Art. 234, the ordinate of the orthocentre

$$= -a\{m_1 + m_2 + m_3 + m_1m_2m_3\} = k.$$

10. We have $m_1m_2 = 1$, and from equations (2), (3), (4), of Art. 237, $m_3^2 = \frac{h-a}{a}$, and $-m_3 = \frac{k}{a}$.

$\therefore \frac{k}{h-a} = -\frac{1}{m_3}$, *i.e.* the "$m$" of the line SO = the "m" of the tangent at R.

11. We have $\Sigma(2a + am_1^2) = 6a + 2h - 4a$ [Art. 238]

$$= 2(h+a).$$

12. $a^2\Sigma[(m_1^2 - m_2^2)^2 + 4(m_1 - m_2)^2] = a^2\Sigma[(m_1 - m_2)^2(m_3^2 + 4)]$

$= a^2\Sigma[m_1^2m_3^2 + m_2^2m_3^2 - 2m_1m_2m_3^2 + 4m_1^2 + 4m_2^2 - 8m_1m_2]$

$= a^2\Sigma(2m_1^2m_2^2 + 8m_1^2 - 8m_1m_2)$

$= 2(h-2a)^2 + 16a(h-2a) + 8a(h-2a)$ [Art. 238]

$= 2(h-2a)(h+10a).$

13. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle passing through P, Q, R.

Then $\quad a^2m_1^4 + 4a^2m_1^2 + 2agm_1^2 - 4afm_1 + c = 0.$

$$\therefore\ am_1[am_1^3 + 4am_1 + 2gm_1 - 4f] + c = 0.$$

$\therefore am_1[(h-2a)m_1 - k + 4am_1 + 2gm_1 - 4f] + c = 0$, by Art. 237.

$$\therefore\ a[(h + 2a + 2g)m_1^2 - (k + 4f)m_1] + c = 0. \quad \ldots\text{(i)}$$

Subtract from the similar equation in m_2;

$$\therefore\ (h + 2a + 2g)(m_1 + m_2) - (k + 4f) = 0.$$

Add to two similar equations; $\therefore k + 4f = 0$, and hence $h + 2a + 2g = 0$, and from (i) $c = 0$.

Substituting, we have the required equation.

14. If $m_3 = \text{cons.} = \lambda$, then $m_1 + m_2 = \text{cons.}$; also

$$a\lambda^3 + (2a - h)\lambda + k = 0. \quad \text{[Art. 237.]}$$

Substitute for h, k from equations (2) of the last Ex.

$$\therefore\ a\lambda^3 + (4a + 2g)\lambda - 4f = 0.$$

Also the centre of the circle is $(-g, -f)$ so that its locus is a straight line.

15. By Art. 234, if (x, y) be the coordinates of the orthocentre,

$$x = -a\cos\alpha, \text{ and } y = -a\cos\alpha\, m_1 m_2 m_3 = k = b\sin\alpha.$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

16. If SO is inclined at α to the axis, $\tan\alpha = \frac{k}{h-a}$,

and $$\tan(\theta_1+\theta_2+\theta_3) = \frac{\Sigma m_1 - m_1 m_2 m_3}{1 - \Sigma m_1 m_2} = \frac{k}{a - 2a + h}$$

$$= \frac{k}{h-a} = \tan\alpha.$$

$$\therefore \theta_1+\theta_2+\theta_3 = n\pi + \alpha.$$

17. If $m_1 m_2 = 1$, then $m_3 = -\frac{k}{a}$.

$\therefore \frac{k^3}{a^2} + (2a-h)\frac{k}{a} - k = 0$, from (i) of Art. 237.

Hence the required locus of O is $y^2 = a(x-a)$.

The equation of the polar of (h, k) is

$$yk = 2a(x+h) = 2a\left(x + \frac{k^2+a^2}{a}\right),$$

or $$y = \frac{2a}{k}(x+a) + \frac{4a}{2a/k},$$

which always touches $y^2 = 16a(x+a)$. Also $a \,.\, 16a = (4a)^2$.

18. If $lx + my = 1$ is identical with

$$2x + (t_1+t_2)y = -2at_1t_2,$$

which, by Art. 236, is the straight line joining the feet of the normals t_1 and t_2, then $\frac{l}{2} = \frac{m}{t_1+t_2} = -\frac{m}{t_3}$, by Art. 237.

$$\therefore -t_3 = \frac{2m}{l}, \text{ and } at_3^2 = \frac{4am^2}{l^2}, \ -2at_3 = \frac{4am}{l}.$$

19. Let P, Q, R be the points m_1, m_2, m_3 and P', Q', R' be the points t_1, t_2, t_3.

Then, since PP' is parallel to QR,

$$\therefore\ t_1 + m_1 = m_2 + m_3 = -m_1. \quad \text{[Arts. 229 and 237.]}$$

$$\therefore\ t_1 = -2m_1. \qquad \therefore\ \Sigma t_1 = -2\Sigma m_1 = 0.$$

$\therefore$ the normals at the points P', Q', R' meet in a point also.

20. From the equation of the normals, the condition gives $2m_3^3 = m_1^3 + m_2^3$; also $-m_3 = m_1 + m_2$.

$$\therefore\ -2m_3^2 = m_1^2 + m_2^2 - m_1 m_2, \text{ and } m_3^2 = m_1^2 + m_2^2 + 2m_1 m_2.$$

$$\therefore\ m_3^2 = m_1 m_2.$$

21. If (h, k) be the coordinates of Q,

$$k = -\tfrac{1}{2}(-2am_3) = am_3, \text{ and } m_1 m_2 m_3 = -\frac{k}{a}.$$

$$\therefore\ m_1 m_2 = -1.$$

22. Let (h, k) be the coordinates of the centre and r the radius; then, if $N\hat{P}G = \theta$, from a figure we have, since the centre must lie on the normal PG,

$$h = a\cot^2\theta + r\sin\theta = at^2 + \frac{r}{\sqrt{1+t^2}},$$

and
$$k = 2a\cot\theta - r\cos\theta = 2at - \frac{rt}{\sqrt{1+t^2}}.$$

Also $r^2 = (h-a)^2 + k^2$.

Substituting for h and k, we obtain $r = \frac{1}{2}a(t^2+1)^{\frac{3}{2}}$, and $2h = a(3t^2+1)$, $2k = a(3t - t^3)$.

$\therefore$ the equation of the circle is

$$x^2 + y^2 - ax(3t^2+1) - ay(3t - t^3) + c = 0;$$

since it passes through $(a, 0)$, $\therefore\ c = 3a^2t^2$.

If (x, y) are the coordinates of the centre,

$$2x = a(3t^2+1), \quad \text{.....................(i)}$$

and
$$2y = a(3t - t^3). \quad \text{.....................(ii)}$$

Multiplying (i) by t and (ii) by 3, and adding, we have

$$2tx + 6y = 10at; \quad \therefore\ t = \frac{3y}{5a - x}.$$

Substitute in (i); $\therefore\ (2x - a)(x - 5a)^2 = 27ay^2$.

23. Let $x \cos \alpha - y \sin \alpha = a \cos \alpha$ be the equation of the given line through the focus.

The perpendicular to it through the focus is

$$x \sin \alpha + y \cos \alpha = a \sin \alpha.$$

Since this passes through the centre of the circle we have, from Ex. 22, $(3t^2 + 1) \sin \alpha + (3t - t^3) \cos \alpha = 2 \sin \alpha$.

$$\therefore\ \tan \alpha = \frac{3t - t^3}{1 - 3t^2} = \tan 3\theta, \text{ if } t = \tan \theta.$$

Hence there are three values of θ, viz. θ_1, θ_2, θ_3 and

$$3\theta_1 = 3\theta_2 + \pi = 3\theta_3 + 2\pi. \quad \therefore\ \theta_1 - \theta_2 = \frac{\pi}{3} = \theta_2 - \theta_3.$$

Hence the three normals, and therefore also the three tangents, are inclined at $60°$.

24. If (h, k) be the coordinates of the centroid of points, $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $\{a(t_1t_2), a(t_1 + t_2)\}$, *i.e.* of Q, R and P respectively, then $3h = a(t_1^2 + t_2^2 + t_1t_2)$, and $k = a(t_1 + t_2)$. Using the equation of Ex. 22, the common chord of the circles is

$$3ax(t_1 + t_2) + ay(3 - t_1^2 - t_2^2 - t_1t_2) = 3a^2(t_1 + t_2),$$

i.e. $kx + y(a - h) = ak$, which passes through the point (h, k).

25. The chord joining the points t_1, t_2 is a normal if

$$t_1 + t_2 = -\frac{2}{t_1}. \quad \text{.........(i)} \quad [\text{Ex. XXVIII. 6.}]$$

Let $x^2 + y^2 - 2gx - 2fy = 0$ be the equation of the circle.

Then $$at_1^3 + 4at_1 - 2gt_1 - 4f = 0, \quad \text{...........(ii)}$$

and $$at_2^3 + 4at_2 - 2gt_2 - 4f = 0. \quad \text{...........(iii)}$$

Subtracting, $a(t_1^2 + t_2^2 + t_1 t_2) + 4a - 2g = 0$,

or from (i) $2g = at_2^2 + 2a$.

Multiplying (ii) by t_1, and (i) by t_2 and subtracting, we have, $at_1 t_2 (t_1 + t_2) + 4f = 0$, or from (i) $2f = at_2$.

Eliminating t_2, we have, for the required locus of the centre, $2y^2 = a(x - a)$.

26. The common points of $y^2 = 4ax$ and the circle

$$x^2 + y^2 = \frac{9a^2}{4},$$

are given by $x^2 + 4ax - \frac{9a^2}{4} = 0$, *i.e.* $\left(x - \frac{a}{2}\right)\left(x + \frac{9a}{2}\right) = 0$.

Hence the common chord is $x = \frac{a}{2}$.

27. The intersections of $y = mx + \frac{a}{m}$ and

$x^2 + y^2 - 2gx - 2fy + c = 0$ are given by

$$x^2(1 + m^2) + 2x(a - g - fm) + \frac{a^2}{m^2} - 2f\frac{a}{m} + c = 0,$$

which has equal roots if

$$(a - g - fm)^2 = (1 + m^2)\left(\frac{a^2}{m^2} - 2f\frac{a}{m} + c\right),$$

or $m^4(f^2 - c) + 2gfm^3 + (g^2 - 2ag - c)m^2 + 2fam - a^2 = 0$.

If $\tan\theta_1$, $\tan\theta_2$, etc. be the roots, then

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{s_1 - s_3}{1 - s_2 + s_4} = \frac{2f(g - a)}{(g - a)^2 - f^2} = \tan 2\alpha,$$

if $\tan\alpha = \frac{f}{g - a}$. $\therefore\ \theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi + 2\alpha$.

28. If R be the point m_3, we have

$$am_3^2 = h, \text{ and } -2am_3 = k.$$

But, from the equations of Art. 237,

$$-am_1 m_2 m_3 = k. \quad \therefore\ m_1 m_2 = 2.$$

Also $\frac{k}{2a} = -m_3 = m_1 + m_2$.

The common chords are PQ and the line through R equally inclined to the axis [Ex. XXVIII. 25],

i.e. $$-y(m_1+m_2)=2x+2am_1m_2,$$

and $$-(m_1+m_2)(y-k)+2(x-h)=0\,;$$

or $yk+4a(x+2a)=0$, and $k(y-k)-4a(x-h)=0$;

which intersect where $x+a=0$, since $k^2=4ah$.

29. $(l+at_1^2,\ 2at_1)$ is any point on 1st parabola.

$(2at_2,\ l'+at_2^2)$ " " 2nd "

Hence if $(x,\ y)$ be a common point, then $xy=4a^2t_1t_2$.

Again $x-l-t_1y+at_1^2=0$, and $y-l'-t_2x+at_2^2=0$,

i.e. the tangents at t_1 and t_2, will be coincident if

$$-\frac{1}{t_2}=-\frac{t_1}{1},\ \ i.e.\ \text{if}\ \ t_1t_2=1.$$

Hence the required locus is $xy=4a^2$.

30. Let $(h,\ k)$ be the vertex of the moving curve, and its equation $(y-k)^2+l(x-h)=0$, and $y^2=lx$, the equation of the fixed curve.

The tangents at the point $(lt^2,\ lt)$ to the two curves are

$$2yt=x+lt^2,\ \text{and}$$

$$y(lt-k)+\frac{l}{2}x+\frac{l^2t^2}{2}+k^2-lh-klt=0.\quad \text{[See Art. 152.]}$$

These are coincident if

$$\frac{k-lt}{2t}=\frac{l}{2}=\frac{l^2t^2+2k^2-2lh-2klt}{2lt^2},$$

whence $t=\dfrac{k}{2l}$, and $k^2-lh=klt$.

Eliminating t, we have $k^2-2lh=0$.

Hence the locus of the vertex is the parabola $y^2=2lx$.

31. Let t_1, t_2, t_3 be the points of contact of the sides of the triangle whose sides touch the parabola $y^2=4ax$. If two of the angular points lie on the curve

$$(y-k)^2=4b(x-h),$$

then $$\{a(t_2+t_3)-k\}^2=4b(at_2t_3-h),\ \ \ldots\ldots\ldots\ldots\text{(i)}$$

and $$\{a(t_3+t_1)-k\}^2 = 4b(at_3t_1 - h), \quad \ldots\ldots\ldots\ldots\text{(ii)}$$
also $$x = at_1t_2, \quad y = a(t_1+t_2).$$

Subtracting (ii) from (i) we have,

$$a(t_1+t_2+2t_3) - 2k = 4bt_3. \quad \therefore \; t_3 = \frac{y-2k}{4b-2a}.$$

Multiplying (i) by t_1 and (ii) by t_2 and subtracting,

$$-a^2t_1t_2 + a^2t_3^2 + k^2 - 2akt_3 = -4bh.$$

$$\therefore \; -ax + a^2\left(\frac{y-2k}{4b-2a}\right)^2 + k^2 - ak\,.\frac{y-2k}{2b-a} = -4bh,$$

which is the equation of a parabola.

239. In Art. 197 we obtained the simplest possible form of the equation to a parabola.

We shall now transform the origin and axes in the most general manner.

Let the new origin have as coordinates (h, k), and let the new axis of x be inclined at θ to the original axis, and let the new angle between the axes be ω'.

By Art. 133 we have for x and y to substitute

$$x\cos\theta + y\cos(\omega'+\theta) + h,$$

and $$x\sin\theta + y\sin(\omega'+\theta) + k$$

respectively.

The equation of Art. 197 then becomes

$$\{x\sin\theta + y\sin(\omega'+\theta) + k\}^2 = 4a\{x\cos\theta + y\cos(\omega'+\theta) + h\},$$

i.e.

$$\{x\sin\theta + y\sin(\omega'+\theta)\}^2 + 2x\{k\sin\theta - 2a\cos\theta\}$$
$$+ 2y\{k\sin(\omega'+\theta) - 2a\cos(\omega'+\theta)\} + k^2 - 4ah = 0$$
$$\ldots\ldots\ldots\ldots(1).$$

This equation is therefore the most general form of the equation to a parabola.

We notice that in it the terms of the second degree always form a perfect square.

240. *To find the equation to a parabola, any two tangents to it being the axes of coordinates and the points of contact being distant a and b from the origin.*

By the last article the most general form of the equation to any parabola is

$$(Ax + By)^2 + 2gx + 2fy + c = 0 \text{............} (1).$$

This meets the axis of x in points whose abscissae are given by

$$A^2x^2 + 2gx + c = 0 \text{..................} (2).$$

If the parabola touch the axis of x at a distance a from the origin, this equation must be equivalent to

$$A^2 (x - a)^2 = 0 \text{.....................} (3).$$

Comparing equations (2) and (3), we have

$$g = -A^2a, \text{ and } c = A^2a^2 \text{...............} (4).$$

Similarly, since the parabola is to touch the axis of y at a distance b from the origin, we have

$$f = -B^2b, \text{ and } c = B^2b^2 \text{...............} (5).$$

From (4) and (5), equating the values of c, we have

$$B^2b^2 = A^2a^2,$$

so that

$$B = \pm A\frac{a}{b} \text{.....................} (6).$$

Taking the negative sign, we have

$$B = -A\frac{a}{b}, \quad g = -A^2a, \quad f = -A^2\frac{a^2}{b}, \text{ and } c = A^2a^2.$$

Substituting these values in (1) we have, as the required equation,

$$\left(x - \frac{a}{b}y\right)^2 - 2ax - 2\frac{a^2}{b}y + a^2 = 0,$$

i.e. $$\left(\frac{x}{a}-\frac{y}{b}\right)^2-\frac{2x}{a}-\frac{2y}{b}+1=0 \text{ .... } (7).$$

This equation can be written in the form

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2-2\left(\frac{x}{a}+\frac{y}{b}\right)+1=\frac{4xy}{ab},$$

i.e. $$\frac{x}{a}+\frac{y}{b}-1=\pm 2\sqrt{\frac{xy}{ab}},$$

i.e. $$\left(\sqrt{\frac{x}{a}}\mp\sqrt{\frac{y}{b}}\right)^2=1,$$

i.e. $$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1 \text{ .... } (8).$$

[The radical signs in (8) can clearly have both the positive and negative signs prefixed. The different equations thus obtained correspond to different portions of the curve. In the figure of Art. 243, the abscissa of any point on the portion PAQ is $<a$, and the ordinate $<b$, so that for this portion of the curve we must take both signs positive. For the part beyond P the abscissa is $>a$, and $\frac{x}{a}>\frac{y}{b}$, so that the signs must be $+$ and $-$. For the part beyond Q the ordinate is $>b$, and $\frac{y}{b}>\frac{x}{a}$, so that the signs must be $-$ and $+$. There is clearly no part of the curve corresponding to two negative signs.]

241. If in the previous article we took the positive sign in (6), the equation would reduce to

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2-2\frac{x}{a}-\frac{2y}{b}+1=0,$$

i.e. $$\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=0.$$

This gives us (Fig., Art. 243) the pair of coincident straight lines PQ. This pair of coincident straight lines is also a conic meeting the axes in two coincident points at P and Q, but is not the parabola required.

242. *To find the equation to the tangent at any point (x', y') of the parabola*

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1.$$

Let (x'', y'') be any point on the curve close to (x', y'). The equation to the line joining these two points is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \dots\dots\dots\dots\dots (1).$$

But, since these points lie on the curve, we have

$$\sqrt{\frac{x'}{a}} + \sqrt{\frac{y'}{b}} = 1 = \sqrt{\frac{x''}{a}} + \sqrt{\frac{y''}{b}} \dots\dots\dots (2),$$

so that
$$\frac{\sqrt{y''} - \sqrt{y'}}{\sqrt{x''} - \sqrt{x'}} = -\frac{\sqrt{b}}{\sqrt{a}} \dots\dots\dots\dots\dots\dots (3).$$

The equation (1) is therefore

$$y - y' = \frac{\sqrt{y''} - \sqrt{y'}}{\sqrt{x''} - \sqrt{x'}} \, \frac{\sqrt{y''} + \sqrt{y'}}{\sqrt{x''} + \sqrt{x'}}(x - x'),$$

or, by (3),

$$y - y' = -\frac{\sqrt{b}}{\sqrt{a}} \, \frac{\sqrt{y''} + \sqrt{y'}}{\sqrt{x''} + \sqrt{x'}}(x - x') \dots\dots\dots (4).$$

The equation to the tangent at (x', y') is then obtained by putting $x'' = x'$ and $y'' = y'$, and is

$$y - y' = -\frac{\sqrt{b}}{\sqrt{a}} \, \frac{\sqrt{y'}}{\sqrt{x'}}(x - x'),$$

i.e.
$$\frac{x}{\sqrt{ax'}} + \frac{y}{\sqrt{by'}} = \sqrt{\frac{x'}{a}} + \sqrt{\frac{y'}{b}} = 1 \dots\dots\dots (5).$$

This is the required equation.

[In the foregoing we have assumed that (x', y') lies on the portion PAQ (Fig., Art. 243). If it lie on either of the other portions the proper signs must be affixed to the radicals, as in Art. 240.]

Ex. *To find the condition that the straight line* $\frac{x}{f}+\frac{y}{g}=1$ *may be a tangent.*

This line will be the same as (5), if

$$f=\sqrt{ax'} \text{ and } g=\sqrt{by'},$$

so that
$$\sqrt{\frac{x'}{a}}=\frac{f}{a}, \text{ and } \sqrt{\frac{y'}{b}}=\frac{g}{b}.$$

Hence
$$\frac{f}{a}+\frac{g}{b}=1.$$

This is the required condition; also, since $x'=\frac{f^2}{a}$ and $y'=\frac{g^2}{b}$, the point of contact of the given line is $\left(\frac{f^2}{a}, \frac{g^2}{b}\right)$.

Similarly, the straight line $lx+my=n$ will touch the parabola if $\frac{n}{al}+\frac{n}{bm}=1$.

243. *To find the* **focus** *of the parabola*

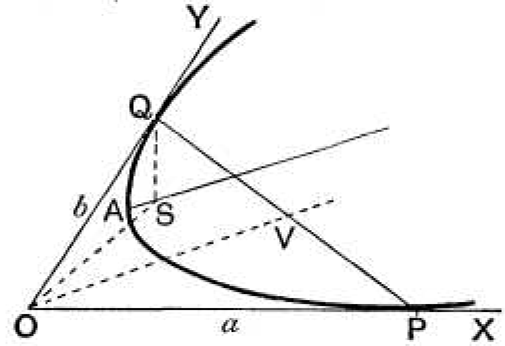
$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1.$$

Let S be the focus, O the origin, and P and Q the points of contact of the parabola with the axes.

Since, by Art. 230, the triangles OSP and QSO are similar, the angle $SOP=$ angle SQO.

Hence if we describe a circle through O, Q, and S, then, by Euc. III. 32, OP is the tangent to it at O.

Hence S lies on the circle passing through the origi O, the point Q, $(0, b)$, and touching the axis of x at tl origin.



The equation to this circle is

$$x^2 + 2xy \cos \omega + y^2 = by \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Similarly, since $\angle SOQ = \angle SPO$, S will lie on the circle through O and P and touching the axis of y at the origin, *i.e.* on the circle

$$x^2 + 2xy \cos \omega + y^2 = ax \quad \ldots\ldots\ldots\ldots\ldots (2).$$

The intersections of (1) and (2) give the point required.

On solving (1) and (2), we have as the focus the point

$$\left(\frac{ab^2}{a^2 + 2ab \cos \omega + b^2},\ \frac{a^2 b}{a^2 + 2ab \cos \omega + b^2}\right).$$

244. *To find the equation to the* **axis.**

If V be the middle point of PQ, we know, by Art. 223, that OV is parallel to the axis.

Now V is the point $\left(\frac{a}{2},\ \frac{b}{2}\right)$.

Hence the equation to OV is $y = \frac{b}{a} x$.

The equation to the axis (a line through S parallel to OV) is therefore

$$y - \frac{a^2 b}{a^2 + 2ab \cos \omega + b^2} = \frac{b}{a}\left(x - \frac{ab^2}{a^2 + 2ab \cos \omega + b^2}\right).$$

i.e. $$ay - bx = \frac{ab(a^2 - b^2)}{a^2 + 2ab \cos \omega + b^2}.$$

245. *To find the equation to the* **directrix.**

If we find the point of intersection of OP and a tangent perpendicular to OP, this point will (Art. 211, γ) be on the directrix.

Similarly we can obtain the point on OQ which is on the directrix.

A straight line through the point $(f, 0)$ perpendicular to OX is

$$y = m(x - f), \text{ where (Art. 93) } 1 + m \cos \omega = 0.$$

The equation to this perpendicular straight line is then

$$x + y\cos\omega = f \text{ ....(1).}$$

This straight line touches the parabola if (Art. 242)

$$\frac{f}{a} + \frac{f}{b\cos\omega} = 1, \quad \textit{i.e.} \text{ if } f = \frac{ab\cos\omega}{a + b\cos\omega}.$$

The point $\left(\dfrac{ab\cos\omega}{a + b\cos\omega}, 0\right)$ therefore lies on the directrix.

Similarly the point $\left(0, \dfrac{ab\cos\omega}{b + a\cos\omega}\right)$ is on it.

The equation to the directrix is therefore

$$x(a + b\cos\omega) + y(b + a\cos\omega) = ab\cos\omega \text{ ......(2).}$$

The latus rectum being twice the perpendicular distance of the focus from the directrix = twice the distance of the point

$$\left(\frac{ab^2}{a^2 + 2ab\cos\omega + b^2}, \frac{a^2b}{a^2 + 2ab\cos\omega + b^2}\right)$$

from the straight line (2)

$$= \frac{4a^2b^2\sin^2\omega}{(a^2 + 2ab\cos\omega + b^2)^{\frac{3}{2}}},$$

by Art. 96, after some reduction.

246. *To find the coordinates of the* **vertex** *and the equation to the tangent at the vertex.*

The vertex is the intersection of the axis and the curve, *i.e.* its coordinates are given by

$$\frac{y}{b} - \frac{x}{a} = \frac{a^2 - b^2}{a^2 + 2ab\cos\omega + b^2} \text{ ....(1).}$$

and by $$\left(\frac{x}{a} - \frac{y}{b}\right)^2 - \frac{2x}{a} - \frac{2y}{b} + 1 = 0 \text{ ......(Art. 240),}$$

i.e. by $$\left(\frac{x}{a} - \frac{y}{b} + 1\right)^2 = \frac{4x}{a} \text{ ....(2).}$$

From (1) and (2), we have

$$x = \frac{a}{4}\left[1 - \frac{a^2 - b^2}{a^2 + 2ab\cos\omega + b^2}\right]^2 = \frac{ab^2(b + a\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}.$$

Similarly $$y = \frac{a^2 b(a + b\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}.$$

These are the coordinates of the vertex.

The tangent at the vertex being parallel to the directrix, its equation is

$$(a + b\cos\omega)\left[x - \frac{ab^2(b + a\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}\right] + (b + a\cos\omega)\left[y - \frac{a^2b(a + b\cos\omega)^2}{(a^2 + 2ab\cos\omega + b^2)^2}\right] = 0,$$

i.e. $$\frac{x}{b + a\cos\omega} + \frac{y}{a + b\cos\omega} = \frac{ab}{a^2 + 2ab\cos\omega + b^2}.$$

[The equation of the tangent at the vertex may also be written down by means of the example of Art. 242.]

EXAMPLES XXXI

1. If a parabola, whose latus rectum is $4c$, slide between two rectangular axes, prove that the locus of its focus is $x^2y^2 = c^2(x^2 + y^2)$, and that the curve traced out by its vertex is

$$x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = c^2.$$

2. Parabolas are drawn to touch two given rectangular axes and their foci are all at a constant distance c from the origin. Prove that the locus of the vertices of these parabolas is the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

3. The axes being rectangular, prove that the locus of the focus of the parabola $\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 = \frac{4xy}{ab}$, a and b being variables such that $ab = c^2$, is the curve $(x^2 + y^2)^2 = c^2xy$.

4. Parabolas are drawn to touch two given straight lines which are inclined at an angle ω; if the chords of contact all pass through a fixed point, prove that

(1) their directrices all pass through another fixed point, and (2) their foci all lie on a circle which goes through the intersection of the two given straight lines.

5. A parabola touches two given straight lines at given points; prove that the locus of the middle point of the portion of any tangent which is intercepted between the given straight lines is a straight line.

6. TP and TQ are any two tangents to a parabola and the tangent at a third point R cuts them in P' and Q'; prove that

$$\frac{TP'}{TP}+\frac{TQ'}{TQ}=1, \text{ and } \frac{QQ'}{Q'T}=\frac{TP'}{P'P}=\frac{Q'R}{RP'}.$$

7. If a parabola touch three given straight lines, prove that each of the lines joining the points of contact passes through a fixed point.

8. A parabola touches two given straight lines; if its axis pass through the point (h, k), the given lines being the axes of coordinates, prove that the locus of the focus is the curve

$$x^2-y^2-hx+ky=0.$$

9. A parabola touches two given straight lines, which meet at O, in given points and a variable tangent meets the given lines in P and Q respectively; prove that the locus of the centre of the circumcircle of the triangle OPQ is a fixed straight line.

10. The sides AB and AC of a triangle ABC are given in position and the harmonic mean between the lengths AB and AC is also given; prove that the locus of the focus of the parabola touching the sides at B and C is a circle whose centre lies on the line bisecting the angle BAC.

11. Parabolas are drawn to touch the axes, which are inclined at an angle ω, and their directrices all pass through a fixed point (h, k). Prove that all the parabolas touch the straight line

$$\frac{x}{h+k\sec\omega}+\frac{y}{k+h\sec\omega}=1.$$

SOLUTIONS/HINTS

1. (i) The focus is given by

$$x^2+y^2=ax=by \quad [\text{Art. } 243].$$

Also $$c^2(a^2+b^2)^3=a^4b^4 \quad [\text{Art. } 245].$$

Eliminating a and b, we have

$$c^2(x^2+y^2)^6\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^3=\frac{(x^2+y^2)^8}{x^4y^4}. \quad \therefore\ c^2(x^2+y^2)=x^2y^2.$$

(ii) The coordinates of the vertex are, by Art. 246,

$$x=\frac{ab^4}{(a^2+b^2)^2},\quad y=\frac{a^4b}{(a^2+b^2)^2}. \quad \text{...........(i)}$$

$$\therefore\ \frac{b}{x^{\frac{1}{3}}}=\frac{a}{y^{\frac{1}{3}}}=\lambda \text{ (say)}.$$

Substitute for a and b in $c^2(a^2+b^2)^3=a^4b^4$;

$$\therefore\ c^2=\frac{\lambda^2 . x^{\frac{4}{3}}y^{\frac{4}{3}}}{\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3}, \text{ and, from (i), } 1=\frac{\lambda x^{\frac{1}{3}}y^{\frac{1}{3}}}{\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}.$$

Eliminating λ, we have $c^2=x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})$.

2. As in the previous example, $b=\lambda x^{\frac{1}{3}}$, $a=\lambda y^{\frac{1}{3}}$,

and
$$\lambda^2=\frac{\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^4}{x^{\frac{2}{3}}y^{\frac{2}{3}}} \quad \text{.....................(1)}$$

Also, from Art. 243, $c^2=\dfrac{a^2b^4}{(a^2+b^2)^2}+\dfrac{a^4b^2}{(a^2+b^2)^2}=\dfrac{a^2b^2}{a^2+b^2}$

$$=\frac{\lambda^2x^{\frac{2}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}. \quad \text{..................(2)}$$

Eliminating λ from (1) and (2), $c^{\frac{2}{3}}=x^{\frac{2}{3}}+y^{\frac{2}{3}}$.

3. Eliminating a and b from $ax=by=x^2+y^2$, and $ab=c^2$, we obtain $(x^2+y^2)^2=c^2xy$.

4. The equation of the directrix may be written

$$\frac{x+y\sec\omega}{a}+\frac{y+x\sec\omega}{b}=1, \quad \text{...........(i)}$$

and, since the chord of contact passes through a fixed point (h, k),
$$\therefore\ \frac{h}{a}+\frac{k}{b}=1. \quad \text{....................(ii)}$$

Hence the directrix passes through the fixed point given by $x + y \sec\omega = h$, and $y + x\sec\omega = k$.

The focus is given by $ax = by = x^2 + y^2 + 2xy\cos\omega$.

Substitute for a and b in (ii), and we have

$$xh + yk = x^2 + y^2 + 2xy\cos\omega,$$

which is a circle passing through the origin.

5. By Art. 242, $\frac{x}{f} + \frac{y}{g} = 1$ will touch the parabola if $\frac{f}{a} + \frac{g}{b} = 1$. But the middle point is $\left(\frac{f}{2}, \frac{g}{2}\right)$.

$\therefore$ the required locus is $\frac{2x}{a} + \frac{2y}{b} = 1$.

6. The straight line $\frac{x}{f} + \frac{y}{g} = 1$ ....(i)

touches the parabola if $\frac{f}{a} + \frac{g}{b} = 1$. ....(ii)

Let TP, TQ be the axes and (i) the equation of the third tangent.

By (ii) $$\frac{TP'}{TP} + \frac{TQ'}{TQ} = 1.$$

Again from (ii) $fg = ab - bf - ag + fg = (b-g)(a-f)$.

$$\therefore \frac{b-g}{g} = \frac{f}{a-f}.$$

Now $$\frac{QQ'}{Q'T} = \frac{b-g}{g} = \frac{f}{a-f} = \frac{TP'}{P'P}.$$

Draw RH parallel to TQ, then $TH = \frac{f^2}{a}$ (see Ex. of Art. 242).

$$\frac{Q'R}{RP'} = \frac{TH}{HP'} = \frac{f^2/a}{f - f^2/a} = \frac{f}{a-f} = \frac{TP'}{P'P}.$$

7. Take any two of the lines as axes, so that the equation of the parabola is $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, and let

$\frac{x}{f}+\frac{y}{g}=1$ be the equation of the third tangent, so that

$$\frac{f}{a}+\frac{g}{b}=1. \quad \text{[Art. 242.]}$$

$\therefore\ \frac{x}{a}+\frac{y}{b}=1$ passes through the point (f, g).

8. The equations for the focus are

$$ax = by = x^2 + y^2 + 2xy\cos\omega.$$

Also $$x = \frac{ab^2}{a^2+b^2+2ab\cos\omega}. \quad \text{[Art. 243.]}$$

Since the axis goes through (h, k), we have, by Art. 244,

$$ak - bh = \frac{ab(a^2-b^2)}{a^2+b^2+2ab\cos\omega} = \frac{x(a^2-b^2)}{b}.$$

$$\therefore\ \frac{k}{x}-\frac{h}{y} = yx\left(\frac{1}{x^2}-\frac{1}{y^2}\right). \quad \therefore\ x^2-y^2+ky-hx=0.$$

9. Take the given lines as axes, and $\frac{x}{f}+\frac{y}{g}=1$ as the equation of the variable tangent, so that $\frac{f}{a}+\frac{g}{b}=1$.

The required locus is the intersection of the lines bisecting OP and OQ at right angles, *i.e.* of

$$x + y\cos\omega = \frac{f}{2}, \text{ and } y + x\cos\omega = \frac{g}{2}.$$

Eliminating f, g we obtain $\frac{x+y\cos\omega}{a}+\frac{y+x\cos\omega}{b}=\frac{1}{2}$.

10. Take AB, AC as axes, putting $AB=a$, $AC=b$, and let k be the given harmonic mean, so that

$$\frac{1}{a}+\frac{1}{b}=\frac{2}{k}. \quad \text{......................... (i)}$$

By Art. 243, the foci are given by

$$ax = by = x^2 + 2xy\cos\omega + y^2.$$

Eliminating a and b, we have

$$\frac{2}{k} = \frac{x + y}{x^2 + 2xy \cos\omega + y^2},$$

so that the locus is the circle

$$x^2 + 2xy \cos\omega + y^2 - \frac{k}{2}(x + y) = 0.$$

Clearly, by Art. 175, the coordinates of the centre are equal, so that it lies on the bisector of the angle.

11. The equation of the directrix may be written

$$\frac{x + y \sec\omega}{a} + \frac{y + x \sec\omega}{b} = 1.$$

$$\therefore \quad \frac{h + k \sec\omega}{a} + \frac{k + h \sec\omega}{b} = 1.$$

But this is the condition that the parabola may touch the given line [Art. 242].