03

Matrix

3.01 Introduction

In 1857 a mathematician Aurther kelly studied to find the solutions of system of equations and came to know about the concept of Matrix. In this method arrange the quantities or objects in a rectangular arrangement.

3.02 Definition and notation

A *matrix* is an ordered rectangular array of same numbers or quantities. It may be real or complex. The numbers of functions are called the elements or the entries of the matirx. We denote matrices by capital letters A, B, C. . .

The following are some examples of matrices.

$$\begin{pmatrix} 1 & 3 \\ 5 & -2 \\ 0 & 7 \end{pmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 5 & -2 \\ 0 & 7 \end{bmatrix}, \quad \begin{Vmatrix} 1 & 3 \\ 5 & -2 \\ 0 & 7 \end{Vmatrix}$$

Note: Matrix is an arrangement, its value cannot be found out.

3.03 Order of matrix

A matrix having m rows and n columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as a m by n matrix).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & & a_{mn} \end{bmatrix}$$

This is a general form of matrix.

Here $a_{11}, a_{12}, ..., a_{mn}$ are the elements of the matrix. a_{ij} represents the i th row and j th column of the element. In short we represent it as $A = \left[a_{ij}\right]_{m\times n}$

Note: In a_{ij}, i always represents the row and j represents the column.

3.04 Type of matrix

1. Row matrix

A matrix is said to be a *row matrix* if it has only one row. For example

(i) $\begin{bmatrix} 2 & 5 & 3 \end{bmatrix}_{1\times 3}$ (ii) $\begin{bmatrix} 3 & -4 & 0 & 7 & 1 \end{bmatrix}_{1\times 5}$

2. Column matrix

A matrix is said to be a *column matrix* if it has only one column. For example its order have $m \times 1$ where m number of rows and column is 1.

(i) $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}_{3\times 1}$ (ii) $\begin{bmatrix} -4 \\ 1 \\ 5 \\ 7 \\ 6 \end{bmatrix}_{5\times 1}$

3. Zero or Null matrix

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero. We denote zero matrix by O. **For example**

(i) $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ (ii) $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$

4. Square matrix

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus a $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.

In general, $A = \left[a_{ij}\right]_{m\times n}$ is a square matrix of order m.

(i) $\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 0 & 1 & 5 \\ 2 & 3 & 7 \\ 6 & -4 & 8 \end{bmatrix}_{3\times 3}$ (iii) $\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}_{n\times n}$

Elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal elements and also termed as Principal diagonal as the subscripts of all the elements are equal.

5. Diagonal matrix

A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be a *diagonal matrix* if all its elements are zero except element of principal diagonal, that is a matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be a diagonal matrix if $a_{ij} = 0$ when $i \neq j$.

For Example

(i) $[5]_{1\times 1}$ (ii) $\begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2}$ (iii) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}_{3\times 3}$

6. Scalar matrix

A diagonal matrix is said to be a *scalar matrix* if its principal diagonal elements are equal, that is, a square matrix

$$A = [a_{ij}]_{m\times m} \quad \text{if } a_{ij} = \begin{cases} 0 & when \quad i \neq j \\ 1 & when \quad i = j \end{cases}$$

For Example (i) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}_{3\times 3}$

7. Unit or Identity matrix

A square matrix in which elements in the principal diagonal are all 1 and rest are all zero is called an *identity matrix.* In other words, the square matirx $A = [a_{ij}]_{m\times m}$ is an identity matrix, if

$$I_n = [a_{ij}]_{n\times n}, \quad a_{ij} = \begin{cases} 0 & when \quad i \neq j \\ 1 & when \quad i = j \end{cases}$$

For Example (i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

8. Triangular matrix

(*i*) Upper triangular matrix

A square matrix in which all the elements below the principal diagonal elements are zero then it is called as an *Upper Triangular Matrix.*

Therefore, in $A = [a_{ij}]_{n\times n}$, $a_{ij} = 0$ when $i > j$

For Example (i) $\begin{bmatrix} 9 & 5 \\ 0 & 2 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 1 & -2 & 6 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$

(*ii*) Lower triangular matrix

A square matrix in which all the elements above the principal diagonal elements are zero then it is called as an *Lower Triangular Matrix.* Therefore, in $A = [a_{ij}]_{n\times n}$, $a_{ij} = 0$ when $i < j$

For Example (i) $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}_{2\times 2}$ (ii) $\begin{bmatrix} 5 & 0 & 0 \\ 6 & 7 & 0 \\ 9 & 2 & -4 \end{bmatrix}_{3\times 3}$

3.05 Properties of matrx

1. Transpose of a matrix

If $A=[a_{ij}]_{m\times n}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose fo the matrix A is denoted by A^T or A' . In other words, if $A=[a_{ij}]_{m\times n}$ then $A^T = A' = [a_{ji}]_{n\times m}$. For example,

(i) $A=\begin{bmatrix}1 & 0 & 2\\ 5 & 3 & -4\\ 3 & 8 & 6\end{bmatrix}_{3\times3} \Rightarrow A^T=\begin{bmatrix}1 & 5 & 3\\ 0 & 3 & 8\\ 2 & -4 & 6\end{bmatrix}_{3\times3}$ (ii) $A=\begin{bmatrix}1 & 3\\ 5 & 0\\ 2 & -4\end{bmatrix}_{3\times2} \Rightarrow A^T=\begin{bmatrix}1 & 5 & 2\\ 3 & 0 & -4\end{bmatrix}_{2\times3}$

2. Symmetric and skew symmetric matrix

(*i*) Symmetric Matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be *symmetric* if $A = A^T$, for example:

(i) $A=\begin{bmatrix}2 & -3\\ -3 & 4\end{bmatrix}_{2\times2}$; $A^T=\begin{bmatrix}2 & -3\\ -3 & 4\end{bmatrix}_{2\times2}$

$\therefore$ A is Symmetric matrix

(ii) $A=\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}_{3\times3}$; $A^T=\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}_{3\times3}$

Note: In Symmetric Matrix, All elements are equal at equidisant with respect to principal diagonal means, $a_{ij} = a_{ji}$.

(*ii*) Skew-Symmetric Matrix

A square matrix $A=[a_{ij}]_{m\times n}$ is said to be skew *symmetric* if $A^T = -A$, for example:

(i) $A=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$; $A^T=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}=-A$

(ii) $A=\begin{bmatrix}0 & h & g\\ -h & 0 & -f\\ -g & f & 0\end{bmatrix}$; $A^T=\begin{bmatrix}0 & -h & -g\\ h & 0 & f\\ g & -f & 0\end{bmatrix}=-A$

Note: (a) In Skew-Symmetric Matrix, $a_{ij} = -a_{ji}$ for all possible values of *i* and *j*.

(b) All the diagonal elements of a skew symmetric matrix are zero. If

$a_{ij} = -a_{ji}$ and if $i = 1, j = 1$ then

$a_{11} = -a_{11}$

$\Rightarrow$ $2a_{11} = 0$

$\therefore$ $a_{11} = 0 = a_{22} = ...a_{nn}$

(c) For any matrices A and B of suitable orders, for addition and multiplication, then

(i) $(A \pm B)^T = A^T \pm B^T$ (ii) $(KA)^T = KA^T$, (where k is any constant) (iii) $(AB)^T = B^T A^T$

(d) If A is a square matrix then

(i) $A + A^T$ is a Symmetric matrix (ii) $A - A^T$ is a Skew-Symmetric matrix

(iii) AA^T and $A^T A$ is a symmetric matrix (iv) $(A^T)^T = A$

(e) Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$

where A is a square matrix

$A + A^T$ is a symmetric Matrix

and $A - A^T$ is a skew-symmetric matrix

(f) A matrix is said to be equal if their corresponding elements are equal,

For example: $A = \begin{bmatrix} 2 & -2 & 0 \\ 3 & -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$\Rightarrow$ $b_{11} = 2, \quad b_{12} = -2, \quad b_{13} = 0$

$b_{21} = 3, \quad b_{22} = -4, \quad b_{23} = 2$

Illustrative Examples

Example 1. The order of A is 3 × 5 and R is a row matrix of A then write the order of R.

Solution : $\because$ order of matrix A is 3 × 5

$\therefore$ has 5 elements in each row A

$\therefore$ order of matrix R is 1 × 5

Example 2. Find a matrix of order 2 × 3, $A = [a_{ij}]$ whose elements are (i) $a_{ij} = 2i + j$; (ii) $a_{ij} = i^2 - j^2$

Solution : (i) $a_{ij} = 2i + j$ Here $i = 1, 2$ and $j = 1, 2, 3$ as the matrix is of order 2×3

$\therefore$ $a_{11} = 2 + 1 = 3, \; a_{12} = 2 + 2 = 4, \; a_{13} = 2 + 3 = 5$

$a_{21} = 4 + 1 = 5, \; a_{22} = 4 + 2 = 6, \; a_{23} = 4 + 3 = 7$

$\therefore$ Required matrix is $A = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$

(ii) $a_{ij} = i^2 - j^2$ given matrix is of the order 2×3 thus $i = 1, 2$ and $j = 1, 2, 3$.

$\therefore$ $a_{11} = 1^2 - 1^2 = 0, \; a_{12} = 1^2 - 2^2 = -3, \; a_{13} = 1^2 - 3^2 = -8$

$a_{21} = 2^2 - 1^2 = 3, \; a_{22} = 2^2 - 2^2 = 0, \; a_{23} = 2^2 - 3^2 = -5$

$\therefore$ Required matrix $A = \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \end{bmatrix}$

Example 3. For what vlaues of x, y and z matrices A and B are equal

$$A=\begin{bmatrix}2 & 0 & x+3\\ y-4 & 4 & 6\end{bmatrix};\quad B=\begin{bmatrix}2 & 0 & 6\\ -2 & 4 & 2z\end{bmatrix}$$

Solution : $\because$ A and B are equal matrices, hence their corresponding elemetns are also equal

$\therefore\quad x+3=6,\ \ y-4=-2,$ and $2z=6$

$\Rightarrow\quad x=3,\ \ y=2$ and $z=3$

Example 4. If $\begin{bmatrix}2x+y & 3 & x-2y\\ a-b & 2a+b & -5\end{bmatrix}=\begin{bmatrix}3 & 3 & 4\\ 4 & -1 & -5\end{bmatrix}$ then find the vlaues of x, y, a and b

Solution : $\because$ Both are equal matrices, hence their corresponding elements are also equal

$\therefore\quad 2x+y=3$ (1)

$x-2y=4$ (2)

Solving (1) and (2) we have

$$x=2,\ \ y=-1$$

again $a-b=4$ (3)

$2a+b=-1$ (4)

Solving (3) and (4) we have

$$a=1,\ \ b=-3$$

$$\therefore\quad x=2,\ \ y=-1,\ \ a=1,\ \ b=-3$$

Exercise 3.1

1. If the matrix $A=[a_{ij}]_{2\times 4}$, then find the number of elements of A
2. Find out the unit matrix of order 4×4
3. If $\begin{bmatrix}k+4 & -1\\ 3 & k-6\end{bmatrix}=\begin{bmatrix}a & -1\\ 3 & -4\end{bmatrix}$ then find the vlaue of a
4. Find the possible orders of matrix with 6 elements.
5. Find a matrix $A=[a_{ij}]$ of order 2×2 whose elements

   (i) $a_{ij}=\dfrac{2i-j}{3i+j}$ (ii) $a_{ij}=\dfrac{(i+2j)^2}{2i}$ (iii) $a_{ij}=2i-3j$
6. Find a matrix $A=a_{ij}$ of order 2×3 whose elements are $a_{ij}=\dfrac{1}{2}|2i-3j|$.
7. If $\begin{bmatrix}a+b & 2\\ 7 & ab\\ -3 & 4\end{bmatrix}=\begin{bmatrix}6 & 2\\ 7 & 8\\ -3 & 4\end{bmatrix}$ then find the vlaues of a and b,

8. If $\begin{bmatrix} 2x & 3x+y \\ -x+z & 3y-2p \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -4 & -3 \end{bmatrix}$ then find the vlaues of x, y, z and p.

9. For what values of a, b and c , matrices A and B are equal matrices where.

$$A = \begin{bmatrix} a-2 & 3 & 2c \\ 12c & b+2 & bc \end{bmatrix}; \quad B = \begin{bmatrix} b & c & 6 \\ 6b & a & 3b \end{bmatrix}$$

3.06 Operations on matrix

1. Addition

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order, say m x n. then, the sum of the two matrices A and B is defined as a matrix $A + B = [a_{ij} + b_{ij}]_{m\times n}$, for all possible values for i and j.

For example: (i) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times2}$ then

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}_{2\times2}$$

(ii) If $A = \begin{bmatrix} 2 & 5 & -3 \\ 4 & 0 & 6 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 3 & 5 \end{bmatrix}_{2\times3}$ then

$$A + B = \begin{bmatrix} 2+4 & 5+2 & -3-1 \\ 4+1 & 0+3 & 6+5 \end{bmatrix} = \begin{bmatrix} 6 & 7 & -4 \\ 5 & 3 & 11 \end{bmatrix}_{2\times3}$$

2. Subtraction

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order, say $m \times n$. Then, the subtraction of the two matrices A and B is defined as a matrix $A - B = [a_{ij} - b_{ij}]_{m\times n}$, for all possible values of i and j.

For example: (i) if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times2}$ then

$$A - B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{bmatrix}_{2\times2}$$

(ii) If $A = \begin{bmatrix} 5 & 3 & 7 \\ 6 & 2 & 1 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 4 & 1 \end{bmatrix}_{2\times3}$ then

$$A - B = \begin{bmatrix} 5-2 & 3-4 & 7-6 \\ 6-3 & 2-4 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 3 & -2 & 0 \end{bmatrix}_{2\times3}$$

3. Multiplication

For multiplication of two matrices A and B, the number of columnes in A should be equal to the number of rows in B. Further more for getting the elements of the product matrix, we take rows of A and columns of B, multiply them elements-wise and take the sum. The product of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B. Let $A=[a_{ij}]_{m\times p}$ be a $m\times p$ matrix and $B=[b_{ij}]_{p\times n}$ be a $p\times n$ matrix. Then the product of the matrices A and B is the matrix C of order $m\times n$.

Order of matrix AB = No . of rows in A × No. of columns in B

$\therefore$ $A=[a_{ij}]_{m\times p}$ and $B=[b_{ij}]_{p\times n}$ then

order of AB will be $m\times \boxed{p \qquad p}\times n = m\times n$

For example : (i) If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$ and $B=\begin{bmatrix} b_{11} & b_{12} & b_{12} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}_{2\times 3}$ then

order of AB will be $AB \ \ 2\times \boxed{2 \qquad 2}\times 3 = 2\times 3$

$$\therefore \quad AB=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} & a_{11}b_{13}+a_{12}b_{23} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} & a_{21}b_{13}+a_{22}b_{23} \end{bmatrix}_{2\times 3}$$

(ii) If $A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}_{2\times 2}$ then $B=\begin{bmatrix} 5 & 4 \\ 6 & 0 \end{bmatrix}_{2\times 2}$

order of AB will be $AB \ \ 2\times \boxed{2 \qquad 2}\times 2 = 2\times 2$

$$\therefore \quad AB=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\times\begin{bmatrix} 5 & 4 \\ 6 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 2\times 5+3\times 6 & 2\times 4+3\times 0 \\ -1\times 5+4\times 6 & -1\times 4+4\times 0 \end{bmatrix}$$

$$=\begin{bmatrix} 10+18 & 8+0 \\ -5+24 & -4+0 \end{bmatrix}=\begin{bmatrix} 28 & 8 \\ 19 & -4 \end{bmatrix}_{2\times 2}$$

4. Scalar Multiplication

In general, we may define multiplication of a matrix by a scalar as follows: if $A=[a_{ij}]_{m\times n}$ is a matrix and n is a scalar, then nA is another matrix which is obtained

$\therefore$ $A=[a_{ij}]_{m\times n}$ then $nA=[na_{ij}]_{m\times n}$

For example (i) if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$ then

$$nA = n\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} na_{11} & na_{12} & na_{13} \\ na_{21} & na_{22} & na_{23} \end{bmatrix}$$

(ii) If $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}_{2\times 2}$ तो $3A = \begin{bmatrix} 6 & 9 \\ 3 & -15 \end{bmatrix}_{2\times 2}$

and $-5A = \begin{bmatrix} -10 & -15 \\ -5 & 25 \end{bmatrix}_{2\times 2}$

3.07 Properties of matrix addition

(i) Commutativity

If A and B are matrices of same order then $A + B = B + A$

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then clearly $A + B$ and $B + A$ are matrices of same order

$$[A + B]_{m\times n} = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$$

$$= [a_{ij} + b_{ij}]_{m\times n}$$

$$= [b_{ij} + a_{ij}]_{m\times n} \quad \text{(Commutative law of addition)}$$

$$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$$

$$= [B + A]_{m\times n}$$

$\therefore \quad A + B = B + A$

(ii) Associativity

If A, B and C are matrices of same order then $(A + B) + C = A + (B + C)$

Let $A = [a_{ij}]_{m\times n}$; $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$ then clearly $(A + B) + C$ and $A + (B + C)$ are matrices of same order

$$[(A + B) + C]_{m\times n} = ([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}) + [c_{ij}]_{m\times n}$$

$$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$$

$$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}$$

$$= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n} \quad \text{(Associative law of addition)}$$

$$= [a_{ij}]_{m\times n} + ([b_{ij}]_{m\times n} + [c_{ij}]_{m\times n})$$

$$= [A + (B + C)]_{m\times n}$$

$\therefore \quad (A + B) + C = A + (B + C)$

(iii) Additive identity

A zero matrix O, $m \times n$ is known as the identity matrix of A as

$$A + O = A = O + A$$

(iv) Additive inverse

For matrix $A=[a_{ij}]_{m\times n}$, if $-A=[-a_{ij}]_{m\times n}$ then $-A$ is the additive inverse of matrix A

as $A+(-A)=O=(-A)+A$, where O is the zero matrix of order $m\times n$

Let $A=[a_{ij}]_{m\times n}$ then $-A=-[a_{ij}]_{m\times n}=[-a_{ij}]_{m\times n}$

$\therefore$ $A+(-A)=[a_{ij}]_{m\times n}+[-a_{ij}]_{m\times n}=0$

and $(-A)+A=A+(-A)$ (Commutative law of addition)

$A+(-A)=O=(-A)+A$

(v) Cancellation law

If A, B and C are three matrices of same order then

$A+B=A+C\Rightarrow B=C$ (Left cancellation law)

and $B+A=C+A\Rightarrow B=C$ (Right cancellation law)

3.08 Properties of Matrix Multiplication

(i) Commutativity

Generally matrix multiplication does not obey Commutative law due to conditions given below:

(a) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ then AB and BA can be found out but they are not necessarily equal.

for example let $A=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ then

$$AB=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$$

and
$$BA=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

$\therefore$ $AB\neq BA$

(b) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then matrix AB can be found but BA cannot be found so no question of proving commutative law.

(c) If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times m}$ then AB and BA can be found out but their order will not be same so $AB\neq BA$

Note: Under certain conditions $AB=BA$ is possible.

(ii) Associativity

If matrix A, B and C are favourable for AB and BC then associative law is verified

i.e. $(AB)C=A(BC)$

(iii) Identity

If I is an unit matrix and A is a matrix of order m × n then

$$I_m A=A=AI_n$$

where I_m, m is the unit matrix of order m and I_n, n is the unit matrix of order n

Note: For square matrix, A, $AI = A = IA$ where I has same order as A.

(iv) Distributivity

If matrices A, B and C are favourable for addition and multiplication then they obey distrbutive law.

(a) $A(B+C) = AB + AC$

(b) $(A+B)C = AC + BC$

3.09 Properties of scalar multiplication of a matirx

If A and B are two matrices of same order and let k and ℓ are two constants then

(i) $(k+\ell)A = kA + \ell A$ (ii) $k(A+B) = kA + kB$

(iii) $k(\ell A) = \ell(kA) = (\ell k)A$ (iv) $1.A = A$

(v) $(-1)A = -A$

3.10 Multiplicative Inverse Matrix

If the product of two square matrices of same order A and B is a Unit matrix then B is known as the multiplicative inverse matrix of A and A is known as the multiplicative inverse matrix of B i.e.

If $AB = I = BA$ then A and B are multiplicative inverse matrix of invertible matrices, for example:

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 3 & 7 & 5 \end{bmatrix}_{3\times 3}$ and $B = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}_{3\times 3}$ then

$$AB = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4+2 & -4+2+2 & 2+0-2 \\ 6-10+4 & -8+5+4 & 4+0-4 \\ 9-14+5 & -12+7+5 & 6+0-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I_3$$

and $$BA = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 3 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-8+6 & 6-20+14 & 6-16+10 \\ -2+2+0 & -4+5+0 & -4+4+0 \\ 1+2-3 & 2+5-7 & 2+4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I_3$$

$\therefore$ $AB = I_3 = BA$ thus A amd B are multiplicative inverse matrix of each other.

3.11 Zero Divisors

If the product of two non-zero matrices A and B is a zero matrix then A and B are divisors of zero

$\therefore$ $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ are divisors of zero

$$AB = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1 & 1-1 \\ -3+3 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$\therefore$ A and B are divisors of zero.

3.12 Positive Integral Power of a Square Matrix

If a square matrix A is multiplied by itself then we get A^2 , again if A^2 is multiplied with A then we get A^3 similarly when A^{n-1} is multiplied with A then we get A^n i.e.

$$AA = A^2 \qquad A^2A = A^3$$

and $A^{n-1}A = A^n$

If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ then

$$A^2 = AA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+6 & 3+12 \\ 2+8 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$$

and $A^3 = A^2A = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7+30 & 21+60 \\ 10+44 & 30+88 \end{bmatrix} = \begin{bmatrix} 37 & 81 \\ 54 & 118 \end{bmatrix}$

Illustrative Examples

Example 5. If $A=\begin{bmatrix}2 & 4 & -1\\3 & 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1 & 0\\-1 & 3 & 4\end{bmatrix}$ then find $2A-3B$

Solution: $\because\quad A=\begin{bmatrix}2 & 4 & -1\\3 & 2 & 5\end{bmatrix}$

$$\therefore\quad 2A=2\begin{bmatrix}2 & 4 & -1\\3 & 2 & 5\end{bmatrix}=\begin{bmatrix}4 & 8 & -2\\6 & 4 & 10\end{bmatrix} \qquad (1)$$

and $$3B=3\begin{bmatrix}2 & 1 & 0\\-1 & 3 & 4\end{bmatrix}=\begin{bmatrix}6 & 3 & 0\\-3 & 9 & 12\end{bmatrix}$$

$$-3B=\begin{bmatrix}-6 & -3 & 0\\3 & -9 & -12\end{bmatrix}$$

$$\therefore\quad 2A-3B=2A+(-3B)$$

$$=\begin{bmatrix}4 & 8 & -2\\6 & 4 & 10\end{bmatrix}+\begin{bmatrix}-6 & -3 & 0\\3 & -9 & -12\end{bmatrix}$$

$$=\begin{bmatrix}4-6 & 8-3 & -2+0\\6+3 & 4-9 & 10-12\end{bmatrix}=\begin{bmatrix}-2 & 5 & -2\\9 & -5 & -2\end{bmatrix}$$

Example 6. If $B=\begin{bmatrix}-2 & 2 & 0\\3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\7 & 1 & 6\end{bmatrix}$ then find A where $2A-3B+5C=O$, and O is a zero matrix of order 2×3.

Solution : $\because\quad 2A-3B+5C=O$

$$\therefore\quad 2A=3B-5C+O$$

$$=3\begin{bmatrix}-2 & 2 & 0\\3 & 1 & 4\end{bmatrix}+(-5)\begin{bmatrix}2 & 0 & -2\\7 & 1 & 6\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-6 & 6 & 0\\9 & 3 & 12\end{bmatrix}+\begin{bmatrix}-10 & 0 & 10\\-35 & -5 & -30\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-6-10+0 & 6+0+0 & 0+10+0\\9-35+0 & 3-5+0 & 12-30+0\end{bmatrix}$$

$$=\begin{bmatrix}-16 & 6 & 10\\-26 & -2 & -18\end{bmatrix}$$

$\therefore \qquad A=\frac{1}{2}\begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$

$$=\begin{bmatrix}-8 & 3 & 5\\ -13 & -1 & -9\end{bmatrix}$$

Example 7. If $A=\begin{bmatrix}4 & 2 & -5\\ 1 & 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}6 & -7 & 0\\ -1 & 2 & 5\\ 1 & 0 & 3\end{bmatrix}$ then find AB, and BA if exists.

Solution : $\because$ order of A is 2×3 and order of B is 3×3

$\therefore$ AB exists but BA does not

$\therefore \qquad AB=\begin{bmatrix}4 & 2 & -5\\ 1 & 0 & 3\end{bmatrix}\begin{bmatrix}6 & -7 & 0\\ -1 & 2 & 5\\ 1 & 0 & 3\end{bmatrix}$

$$=\begin{bmatrix}24-2-5 & -28+4+0 & 0+10-15\\ 6-0+3 & -7+0+0 & 0+0+9\end{bmatrix}=\begin{bmatrix}17 & -24 & -5\\ 9 & -7 & 9\end{bmatrix}_{2\times 3}$$

Example 8. Find the value of x for which

$$\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=O$$

where O is a zero matrix of order 1×1

Solution : $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=O$

or $\qquad \begin{bmatrix}1+2x+15 & 3+5x+3 & 2+x+2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=O$

or $\qquad \begin{bmatrix}2x+16 & 5x+6 & x+4\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}=O$

or $\qquad \begin{bmatrix}2x+16+10x+12+x^2+4x\end{bmatrix}=O$

or $\left[x^2+16x+28\right]=0$

or $x^2+16x+28=0$

or $(x+2)(x+14)=0$

$\Rightarrow x+2=0$ or $x+14=0$

$\Rightarrow x=-2$ or $x=-14$

Example 9. If $A-2I=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}$ then find AA^T where I is the identity matrix of order 3×3.

Solution : $\because$ $A-2I=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}$

$$\therefore \quad A=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}+2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -2 & 3\\ 2 & 1 & -1\\ -3 & 1 & 0\end{bmatrix}+\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$$

$$A^T=\begin{bmatrix}1 & 2 & -3\\ -2 & 3 & 1\\ 3 & -1 & 2\end{bmatrix}$$

$$\therefore \quad AA^T=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -3\\ -2 & 3 & 1\\ 3 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9 & 2-6-3 & -3-2+6\\ 2-6-3 & 4+9+1 & -6+3-2\\ -3-2+6 & -6+3-2 & 9+1+4\end{bmatrix}=\begin{bmatrix}14 & -7 & 1\\ -7 & 14 & -5\\ 1 & -5 & 14\end{bmatrix}$$

Example 10. If $A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ then verify the following:

(i) $A^2=2A$ (ii) $A^3=4A$

Solution : (i) L.H.S. $A^2 = AA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2A = \text{R.H.S.}$$

(ii) L.H.S. $A^3 = A^2A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2+2 & -2-2 \\ -2-2 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4A = \text{R.H.S.}$$

Example 11. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & 1 \end{bmatrix}$ then verify the following.

$$A(B+C) = AB + AC$$

Solution : L.H.S. $= A(B+C)$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \times \left\{\begin{bmatrix} 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 8 \\ -4 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 6-12 & 2+18 & 16+15 \\ 3-8 & 1+12 & 8+10 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 20 & 31 \\ -5 & 13 & 18 \end{bmatrix} \qquad (1)$$

R.H.S. $= AB + AC$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-9 & -4+6 & 4+12 \\ 1-6 & -2+4 & 2+8 \end{bmatrix} + \begin{bmatrix} 4-3 & 6+12 & 12+3 \\ 2-2 & 3+8 & 6+2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 & 16 \\ -5 & 2 & 10 \end{bmatrix} + \begin{bmatrix} 1 & 18 & 15 \\ 0 & 11 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 20 & 31 \\ -5 & 13 & 18 \end{bmatrix} \qquad (2)$$

from (1) and (2) L.H.S. = R.H.S.

Exericse 3.2

1. If $A=\begin{bmatrix}-3 & 2 & 1\\ 1 & -4 & 7\end{bmatrix}$ and $B=\begin{bmatrix}3 & 5 & -2\\ -1 & 4 & -2\end{bmatrix}$ then find $A+B$ and $A-B$.

2. If $A+B=\begin{bmatrix}-7 & 0\\ 2 & -5\end{bmatrix}$ and $A-B=\begin{bmatrix}3 & -2\\ 0 & 3\end{bmatrix}$ then find matrices A and B.

3. If $A=\begin{bmatrix}1 & 3\\ 2 & 1\\ 3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 1 & 2\\ -1 & 0\end{bmatrix}$ then find matrix C where $A+2B+C=O$ and O is a zero matrix.

4. If $A=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & 4\\ -1 & 7\end{bmatrix}$ then find the value of $3A^2-2B$.

5. If $A=\begin{bmatrix}0 & 1 & 2 & 3\\ 3 & 2 & 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 3\\ 1 & 2\\ 2 & 1\\ 3 & 0\end{bmatrix}$ then show that $AB\neq BA$.

6. If $f(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$ then show that $f(A)f(B)=f(A+B)$.

7. If $A=\begin{bmatrix}4 & 2 & -5\\ 1 & 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}6 & -7 & 0\\ -1 & 2 & 5\\ 1 & 0 & 2\end{bmatrix}$ then prove that: $(AB)^T=B^TA^T$

8. Prove that $\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}ax^2+by^2+cz^2+2hxy+2fyz+2gzx\end{bmatrix}$.

9. If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}$ and I is the identity matrix of third order then prove that

$$A^2-3A+9I=\begin{bmatrix}-6 & 1 & 2\\ 5 & 4 & 4\\ 2 & 8 & -3\end{bmatrix}$$

10. If $\begin{bmatrix} a & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} a \\ 4 \\ 1 \end{bmatrix} = O$, where O is a zero matrix then find the value of a.

11. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$ then find the values of a and b

12. If $A = \begin{bmatrix} 0 & -\tan\frac{x}{2} \\ \tan\frac{x}{2} & 0 \end{bmatrix}$ and I is a unit matrix of order 2×2 then prove that

$$I + A = (I - A)\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

13. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find the value of K where $A^2 = 8A + KI$.

14. If $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} 1 & -4 & 3 \\ -2 & -10 & 6 \\ 13 & 20 & -9 \end{bmatrix}$ then find the value of A.

15. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ then prove that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, where n is a positive integer

Miscellaneous Exericse - 3

1. If matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ then find A^2.

2. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ then find $(A - 2I).(A - 3I)$.

3. If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ then find AB .

4. If $A = \begin{bmatrix} -i & o \\ o & i \end{bmatrix}$ and $B = \begin{bmatrix} o & i \\ i & o \end{bmatrix}$, where $i = \sqrt{-1}$ then find BA .

5. If $A - B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $A + B = \begin{bmatrix} 3 & 5 & -7 \\ -1 & 1 & 4 \\ 11 & 8 & 0 \end{bmatrix}$ then find matrices A and B.

6. If $\begin{bmatrix} -2 & -3 & 1 \\ -y-2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} x+2 & -3 & 1 \\ 5 & -1 & 4 \end{bmatrix}$ then find the values of x and y.

7. The order of matrix A is 3×4 and B is a matrix such that $A^T B$ and AB^T both are defined then find the order of B.

8. If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ is a symmetric matrix then find the value of x.

9. Write a 3×3 matrix $B = [b_{ij}]$ whose elements are $b_{ij} = (i)(j)$.

10. If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$ then find $A + B^T$.

11. Express the matrix A as the sum of symmetric and skew-symmetric matrix where $A = \begin{bmatrix} 6 & 2 \\ 5 & 4 \end{bmatrix}$.

12. If $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ then prove that

(i) $(A^T)^T = A$.

(ii) $A + A^T$ is a symmetric matrix.

(iii) $A - A^T$ is a skew-symmetric matrix.

(iv) AA^T and $A^T A$ are symmetric matrices.

13. If $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}; B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$ and $3A - 2B + C$ is a zero matrix then find matrix C.

14. Write a matrix $B = [b_{ij}]$ of order 2×3 whose elements are $b_{ij} = \dfrac{(i+2j)^2}{2}$.

15. If $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & -2 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 5 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ then find the elements of first row of the matrix ABC.

16. If matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then find AA^T .

17. If $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = O$ then find the value of x

18. If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that $B^2 - (a+d)B = (bc-ad)I_2$, where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

19. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then write $(aA+bB)(aA-bB)$ in the form of matrix A.

20. If $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$ then prove that $(A-B)^2 \neq A^2 - 2AB + B^2$.

21. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $A^2 = kA - 2I_2$, then find the value of k.

22. If $A = \begin{bmatrix} i & o \\ o & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} o & i \\ i & o \end{bmatrix}$ where $i = \sqrt{-1}$ then verifty the following expression.

(i) $A^2 = B^2 = C^2 = -I_2$.

(ii) $AB = -BA = -C$.

23. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $f(A) = A^2 - 5A + 7I$ then find $f(A)$.

24. Prove that

$$\begin{bmatrix} \cos^2 \propto & \cos \propto \sin \propto \\ \cos \propto \sin \propto & \sin^2 \propto \end{bmatrix} \begin{bmatrix} \cos^2 \beta & \cos \beta \sin \beta \\ \cos \beta \sin \beta & \sin^2 \beta \end{bmatrix} = O$$

where $\propto - \beta = (2m-1)\frac{\pi}{2}; m \in N$.

IMPORTANT POINTS

1. A matrix is an ordered rectangular array of number or functions.
2. **Types of Matrices:** Row Matrix, Column Matrix, Zero Matrix, Square Matrix, Diagonal Matrix, Scalar Matrix. Unit Matrix, Upper Triangular Matrix, Lower Triangular Matrix, Symmetric and Skew-Symmetric Matrices.
3. Addition and substraction of matrices. Addition and substration of two matrices of same order is obtained by addtiion and substraction of their respective elements.
4. **Multiplication of Matrices :** Let two matrices A and B, their multiplication AB is possible when number of column in A is equal to number of row in B and element of AB is obtained by sum of product of element of i^{th} column in A with element of j^{th} row in B.
5. **Scalar Multiplication :** When a non zero scalar is multiplied with matrices A then we have new matrices nA in which all elements is n^{th} time of element of A.
6. Addition of matrices follows commutative and associative law while substraction is not.
7. Multiplication of matrices follows associative law but it doesn't follow commutative law.
8. A matrix having m rows and n columns is called a matrix of order $m \times n$.
9. A $m \times n$ matrix is a square matrix if $m = n$.
10. **Transpose Matrix:** If $A = [a_{ij}]_{m\times n}$ then $A^T = [a_{ji}]_{n\times m}$
11. **Symmetric Matrix:** $A^T = A$
12. **Skew-Symmetric Matrix:** $A^T = -A$
13. If A is a square matrix then

    (i) $A + A^T$ is a symmetric matrix

    (ii) $A - A^T$ is skew symmetric matrix

    (iii) AA^T and $A^T A$ are symmetric matrices

    (iv) $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$

14. If A and B are two matrices then

    (i) $\left(A \pm B\right)^T = A^T \pm B^T$

    (ii) $(A^T)^T = A$

    (iii) $\left(AB\right)^T = B^T.A^T$

    (iv) $\left(kA\right)^T = k.A^T$, where $k \neq 0$

Answers

Exericse 3.1

1. 8

2. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. $a = 8$

4. $1 \times 6,\ 6 \times 1,\ 2 \times 3,\ 3 \times 2$

5. (i) $\begin{bmatrix} 1/4 & 0 \\ 3/7 & 1/4 \end{bmatrix}$; (ii) $\begin{bmatrix} 9/2 & 25/2 \\ 4 & 9 \end{bmatrix}$; (iii) $\begin{bmatrix} -1 & -4 \\ 1 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1/2 & 2 & 7/2 \\ 1/2 & 1 & 5/2 \end{bmatrix}$

7. $a = 4, b = 2$ or $a = 2, b = 4$

8. $x = 2,\ y = -1,\ z = -2,\ p = 0$

9. $a = 1, b = 6, c = 3$

Exericse 3.2

1. $A + B = \begin{bmatrix} 0 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix}$, $A - B = \begin{bmatrix} -6 & -3 & 3 \\ 2 & -8 & 9 \end{bmatrix}$

2. $A = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 1 \\ 1 & -4 \end{bmatrix}$

3. $\begin{bmatrix} -5 & -5 \\ -4 & -5 \\ -1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -20 \\ 38 & -11 \end{bmatrix}$

10. $a = -2, -3$

11. $a = 1, b = 4$

13. $k = -7$ 1 4 .

$A = \begin{bmatrix} 1 & -4 & 3 \\ 4 & 2 & 0 \end{bmatrix}$

Miscellaneous Exercise - 3

1. $A^2 = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

2. O

3. $\begin{bmatrix} 3 \\ -11 \end{bmatrix}$

4. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & 2 \\ 5 & 4 & 0 \end{bmatrix}$

6. $x = -4,\ y = -7$

7. 3×4

8. -4

9. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 6 & -9 \\ 1 & 6 & -3 \end{bmatrix}$

11. $\begin{bmatrix} 6 & 7/2 \\ 7/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -3/2 \\ 3/2 & 0 \end{bmatrix}$

13. $\begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$

14. $\begin{bmatrix} 9/2 & 25/2 & 49/2 \\ 8 & 18 & 32 \end{bmatrix}$

15. 8

16. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or I_2

17. $-9/8$

19. $\left(a^2 + b^2\right)A$

21. $k = 1$

23. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$