Chapter 6

Averages, Mixtures, and Alligations

CHAPTER HIGHLIGHTS

- *Averages*
- *Mixtures*
- *Alligations*

Averages

'Average is a very simple but effective way of representing an entire group by a single value'.

'Average' of a group is defined as:

$$\text{Average} = \frac{\text{Sum of all items in the group}}{\text{Number of items in the group}}$$

'Sum of all the items in the group' means 'sum of the values of all the items in the group'.

A batsman's performance can be expressed as the average number of runs scored per innings rather than giving the scores in individual innings. For example, let us say a cricketer scored the following runs in 9 different innings in a year: 35, 56, 124, 29, 0, 87, 98, 45, and 75. Then his average score (per innings) for the year is

$$\frac{35 + 56 + 124 + 29 + 0 + 87 + 98 + 45 + 75}{9} = 61$$

Similarly, if there are 60 students in a class, instead of talking of the height of each individual student, we can talk of 'average' height of the class. The average height of the class of students is equal to the sum of the heights of all the students of the class divided by the number of students in the class.

Average is also called the 'mean' or mean value of all the values.

Points to Remember

1. If the value of each item is increased by the same value p, then the average of the group or items will also increase by p.
2. If the value of each item is decreased by the same value p, then the average of the group or items will also decrease by p.
3. If the value of each item is multiplied by the same value p, then the average of the group or items will also get multiplied by p.
4. If the value of each item is divided by the same value p ($p \neq 0$), then the average of the group or items will also get divided by p.
5. The average of a group of items will always lie between the smallest value in the group and largest value in the group, i.e. the average will be greater than the smallest value and less than the largest value in the group.

An Easy Method to Calculate Averages

As already discussed, the average of a group of items whose values are given can be found out by the rule given at the beginning of this section. However, in most of the cases, we do not need to perform such elaborate additions and divisions. The calculation of averages can be simplified greatly by taking some arbitrary number (P) as a starting point, take

the deviations (differences) of the given items (Q_i) from this arbitrary number, find the average of all these deviations ($Q_i - P$) and algebracially add it to the arbitrary number (P) to give the correct average of the given items.

If there are n items and they are denoted by Q_1, Q_2, Q_3, …, Q_n, then the average of these n items is given by

$$\text{Average} = P + \frac{1}{n}\sum_{i=1}^{n}(Q_i - P)$$

For example, the cricketer that we considered above earlier scored the following runs in seven innings: 35, 56, 45, 43, 67, 70, and 48. Now, to find his average, we take an arbitrary figure, say 50, and first find the deviations of each of the scores from this figure. The deviations of the scores from 50 are –15, +6, –5, –7, +17, +20 and –2. The sum of these deviations is +14.

Hence, the average of the cricketer's scores is

$$50 + \frac{14}{7} = 52$$

Please note that the number P (= 50 above) can be any value. Let us work out the same example taking a different value for P. Let us take P equal to 45. The deviations of the scores from P are –10, +11, 0, –2, +22, +25 and +3. The sum of these deviations is 49. Hence, the average is 45 + 49/7 = 45 + 7 = 52.

Weighted Average

When two groups of items are combined together, then we can talk of the average of the entire group. However, if we know only the average of the two groups individually, we cannot find out the average of the combined group of items.

For example, there are two sections A and B of a class where the average height of section A is 150 cm and that of section B is 160 cm. On the basis of this information alone, we cannot find the average of the entire class (of the two sections). As discussed earlier, the average height of the entire class is

$$\frac{\text{Total height of the entire class}}{\text{Total number of students in the entire class}}$$

Since we do not have any information regarding the number of students in the two sections, we cannot find the average of the entire class. Now, suppose that we are given that there are 60 students in the section A and 40 students in section B, then we can calculate the average height of the entire class which, in this case will be equal to $\frac{60\times150+40\times160}{60+40} = 154$ cm.

This average height 154 cm of the entire class is called '**weighted average**' of the class.

The aforementioned step in calculating the weighted average of the class can be rewritten:

$$\frac{60\times150+40\times160}{60+40} = \frac{60}{100}150 + \frac{40}{100}160 = \frac{3}{5}150 + \frac{2}{5}160$$

It is clear from the aforementioned step that we would have been able to calculate the average height of the entire class even if we had not been given the number of students in the individual sections but only the **ratio** of the number of students in the two sections (which in this case is 3 : 2).

Even if there are more than two groups of items to be combined, then also the weighted average can be calculated by the same method. For example, if three sections in a class have their average marks as 75, 76, and 79, respectively, and their respective strengths are 30, 35, and 35, then the average mark of the entire class is given by

$$\frac{30\times75+35\times76+35\times79}{30+35+35} = 76.75$$

The method of deviations we used for calculating averages can be applied to calculate weighted average also. Here, that method will involve finding out deviations from the arbitrarily chosen number and calculating the weighted average of these deviations. In the aforementioned example, if we take 70 as the arbitrary figure, then the deviations of the three observed values given from 70 are +5, +6, and +9. The weighted average of these deviations is

$$\frac{30\times5+35\times6+35\times9}{30+35+35} = \frac{675}{100} = 6.75$$

Hence, the weighted average will be 70 + 6.75 = 76.75.

The arbitrary figure chosen can be any figure, and if it is selected, as in the previous case, between the smallest and largest observed figures, some of the deviations will be positive and some negative making the final division relatively simpler. For example, in the aforementioned case, if we take with 76 as the arbitrary figure, the deviations are –1, 0, and +3. Then, the weighted average will be

$$\frac{30-(-1)+35-0+35-(+3)}{30+35+35} = \frac{75}{100} = 0.75$$

Hence, the weighted average will be 76 + 0.75 = 76.75.

'Weighted average' can be defined or calculated for any MIXTURE.

Mixtures

Mixing of two or more qualities of things produces a mixture. When two items of different qualities are thus mixed, the quality of the resultant mixture lies in between the qualities of the original constituent items, i.e. it will be higher than the lowest quality and lower than the highest quality of the items being mixed.

In the aforementioned example that we took, the 'quality' that we looked at was the height of the students. We could also have taken their weights or the marks scored by them or any other 'quality' or 'parameter' and calculated the 'weighted average' value of that particular 'quality' for the entire group.

Similarly, if two types of a product of different prices per unit are mixed, the unit price of the resultant mixture will lie between the prices of the two types that form the mixture.

Here, the average quality is essentially the **weighted average** of the two constituent items.

If q_1 is the quantity (or number of items) of one particular item of quality p_1, and q_2 be the quantity (or number of items) of the second item of quality p_2 are mixed together to give a new mixture, then the **weighted average** value (p) of the quality of the mixture is given by

$$p = \frac{p_1 q_1 + p_2 q_2}{q_1 + q_2}$$

Even if there are more than two groups of items mixed, the weighted average rule can be applied. We will only have to take figures (as shown in the formula for the two groups) for all the groups in the numerator as well as the denominator and calculate the weighted average. For example, if there are four groups of quantities q_1, q_2, q_3, and q_4 whose respective qualities are p_1, p_2, p_3, and p_4, then the weighted average quality of the group can be written as

$$p = \frac{p_1 q_1 + p_2 q_2 + p_3 q_3 + p_4 q_4}{q_1 + q_2 + q_3 + q_4}$$

A mixture can also be a solution—that is, a liquid mixed with another liquid which is normally water. The concentration of the solution is expressed as the proportion (or percentage) of the liquid in the total solution.

For example, if 10 litres of pure alcohol is mixed with 40 litres of water, then in a total solution of 50 litres, there is 10 litres of alcohol. Hence, the concentration of this solution is 0.2 (= 10/50) or 20%.

Similarly, if 30 litres of pure milk is mixed with 10 litres of water, the concentration of this solution can be expressed as 75% (= 30/40) milk or 25% water.

We can also have two solutions mixed together to give a new solution. Such problems can also be handled in the same manner as other mixtures. In the weighted average rule, the quality of the constituents (p_1, p_2, etc.) will then be the concentrations of various solutions mixed together.

Solved Examples

Example 1

Rajiv purchased three dozen mangoes at ₹10 per dozen, two dozen mangoes at ₹15 per dozen, and five dozen mangoes at ₹16 per dozen. Find the average cost per dozen of the mangoes that he purchased.

Solution

The cost of first three dozen mangoes = (3) (10) = ₹30

The cost of next two dozen mangoes = (2) (15) = ₹30

The cost of next five dozen mangoes = (5) (16) = ₹80

Total cost of the mangoes purchased = ₹140.

Average cost per dozen

$$= \frac{\text{Total cost of mangoes}}{\text{Number of dozens}} = \frac{140}{10} = ₹14$$

Example 2

The average age of 5 men is 20 years. Their average age increased by 1 year when a new man joined them. Find the age of the new man.

Solution

Total age of 5 men = (5) (20) = 100 years

Total age of 6 men = (6) (21) = 126 years

The age of the new man = 126 – 100

i.e. 26 years

Example 3

Six kilograms of wheat costing of ₹18 per kg is mixed with nine kilograms of wheat costing of ₹12 per kg. Find the price per kg of the mixture.

Solution

Total cost of 6 kg = (6) (18) = ₹108

Total cost of 9 kg = (9) (12) = ₹108

$$\text{Average cost of the mixture} = \frac{\text{Total cost}}{\text{Total quantity}}$$

$$= \frac{108\,(2)}{15} = ₹\,14.40 \text{ per kg}$$

Example 4

The average marks of three sections in the tenth class were 90, 120, and 150. If the number of students in these sections are 30, 40, and 50, respectively, find the average mark of the tenth class.

Solution

Total mark of the first section = (90) (30) = 2700

Total mark of the second section

= (120) (40) = 4800

Total mark of the third section = (150) (50) = 7500

$$\text{Average mark of class } X = \frac{\text{Total mark}}{\text{Number of students}}$$

$$= \frac{2700 + 4800 + 7500}{30 + 40 + 50} = \frac{15000}{120} = 125$$

Example 5

Tarun earned an average of ₹1500 per month from January to April in a year. He earned an average of ₹1600 per month from May to October in that year. His earning in the month of December of that year was ₹300 more than his earning in

the month of November of that year. His average monthly earnings for that year was ₹1675. Find his earnings in the month of November.

Solution

Total earnings of Tarun from January to April = (1500) (4) = ₹6000

Total earnings of Tarun from May to October = (1600) (6) = ₹9600

Let his earnings in November be ₹x

His earnings in December = ₹ $(x + 300)$

$$\frac{6000+9600+x+x+300}{12} = 1675$$

$$x = 2100$$

Example 6

The average age of a group of friends is 37 years. If 6 new friends whose average age is 35 years join them, the average age of the entire group becomes 36 years. How many people were there in the group initially?

Solution

Let the initial number of people in the group be n.

The total age of the initial group of friends = $37n$ years

The total age of the six friends who joined the group = 35×6 i.e., 210 years.

Given that, $37n + 210 = 36\ (n + 6)$

$$\Rightarrow \quad 37n - 36n = 216 - 210$$

$$\therefore \quad n = 6$$

Alligations

We will take the weighted average rule discussed in the previous section and rewrite the formula such that the quantity terms come on one side and the price terms come on the other side. If we do this we get the rule $\frac{q_1}{q_2} = \frac{p - p_2}{p_1 - p}$

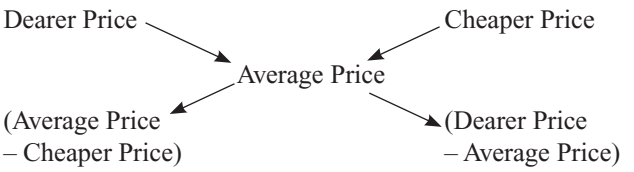
This is called the RULE OF ALLIGATION. This rule connects quantities and prices in mixtures. This can also be written as $\frac{q_1}{q_2} = \frac{p_2 - p}{p - p_1}$

In a descriptive manner, the Rule of Alligation can be written as $\frac{\text{Quantity of Cheaper}}{\text{Quantity of Dearer}}$

$$= \frac{\text{Rate of Dearer} - \text{Average Rate}}{\text{Average Rate} - \text{Rate of Cheaper}}$$

This rule is a very powerful rule and is useful in problems on weighted averages and mixtures. This rule is also useful in a number of problems which can be treated as mixtures and applied to parameters other than price also. We will take examples where alligation rule can be applied.

In actual practice, to apply alligation rule, we do not need to remember the aforementioned formula at all. It can be made very simple by representing the rule pictorially. The formula can be represented as follows:



{We write the dearer and cheaper prices in one line with some gap in between them. Then, we write the average price in between these two but in the line below the line in which dearer and cheaper prices are written. Then, take the differences of quantities as shown along the arrows and write along the same direction of the arrows continued, i.e. in a diagonally opposite corner. The difference between the top left hand quantity (dearer price) and average price has to be written at the bottom right hand corner. Similarly, the difference between the top right hand corner (cheaper price) and the average price has to be written at the bottom left hand corner. Now the ratio of the two quantities in the bottommost line will give us the ratio of the quantities of dearer and cheaper varieties. Please note that since we took dearer price on the top left corner, the ratio of the bottom left figure to that of the bottom right figure will give the ratio of dearer quantity to cheaper quantity.}

Example 7

A bag contains a total of 120 coins in the denominations of 50 p and ₹1. Find the number of 50 p coins in the bag if the total value of the coins is ₹100.

Solution

Let the number of 50 p coins be x.

$$\text{Total value of the coins} = ₹\left[\frac{50x}{100} + 1\,(120 - x)\right]$$

$$\frac{50x}{100} + 120 - x = 100$$

$$\Rightarrow \quad x = 40$$

Alternative method:

$$\text{Average value per coin} = \frac{(100)(100)}{120}$$

$$= \frac{500}{6} \text{ paise}$$

$$\text{Using allegation rule, } \frac{x}{120 - x}$$

$$= \frac{100 - \frac{500}{6}}{\frac{500}{6} - 50} = \frac{1}{2}$$

$$\Rightarrow \quad 2x = 120 - x$$

$$\Rightarrow \quad x = 40$$

Example 8

A vessel has 60 litres of solution of milk and water having 80% milk. How much water should be added to it to make it a solution in which milk forms 60%?

Solution

Let the quantity of water to be added be x litres.

Quantity of milk in the vessel

$$= \left(\frac{80}{100}\right)(60) = 48 \text{ litres}$$

$$\Rightarrow \quad 48 = 0.6\,(60 + x)$$

$$\therefore \quad x = 20$$

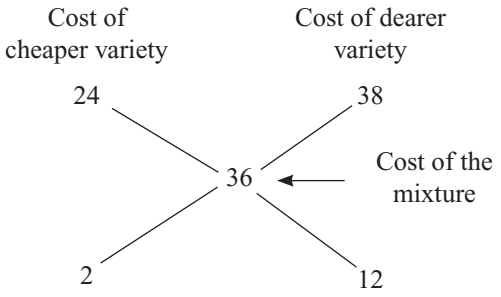
Example 9

In what ratio must Anand mix two varieties of barley costing ₹24 per kg and ₹38 per kg so that by selling the mixture at ₹40 per kg he would make $11\frac{1}{9}\%$ profit?

Solution

$$\text{Cost price of the mixture} = \frac{40(100)}{100 + 11\frac{1}{9}}$$

$$= ₹36 \text{ per kg}$$



$\therefore$ The ratio of the costs of the cheaper to the dearer variety $= 2 : 12 = 1 : 6$.

If there is P volume of pure liquid initially and, in each operation, Q volume is taken out and replaced by Q volume of water, then at the end of n such operations, the concentration (k) of the liquid in the solution is given by $\left\{\frac{P-Q}{P}\right\}^n = k$

This gives the concentration (k) of the liquid as a Proportion of the total volume of the solution.

If the concentration has to be expressed as a percentage, then it will be equal to 100 k.

If the volume of the liquid is to be found out at the end of n operations, it is given by kP, i.e. the concentration k multiplied by the total volume P of the solution.

Example 10

A vessel has 400 litres of pure milk. 40 litres of milk is removed from the vessel and replaced by water. 40 litres of the mixture thus formed is replaced by water. This procedure is repeated once again. Find the percentage of milk in the resultant solution.

Solution

Let v litre be volume of milk with a concentration of $c_1\%$.

If x litres of the solution is removed and replaced with water, the new concentration is $\left(\frac{v-x}{v} \times c_1\right)\%$

Given that initial concentration is 100%. (pure milk), $v = 400$, $x = 40$ and the replacement is done thrice.

$\therefore$ Concentration of milk in the resultant solution

$$= \left(\frac{400-40}{400}\right)^3 \times 100 = 72.9\%$$

EXERCISES

Direction for questions 1 to 25: Select the correct alternative from the given choices.

1. Find the average of all the two digit numbers divisible by 10.
(A) 40 (B) 50 (C) 45 (D) 60

2. Find the average of all odd numbers less than 50.
(A) 26.5 (B) 25.5 (C) 26 (D) 25

3. Find the average of all the multiples of 12 less than 100.
(A) 48 (B) 54 (C) 60 (D) 66

4. The average salary per month of a man for the first four months, next four months, and the last four months of a year are ₹6000, ₹8000, and ₹13000, respectively. Find his average salary per month in that year (in ₹).
(A) 7500 (B) 9000
(C) 10500 (D) 6600

5. In an office there are 20 employees. The average heights of the male employees is 180 cm. The average height of the female employees is 170 cm. Find the average height of all the employees (in cm).
(A) 172 (B) 174
(C) 176 (D) Cannot be determined

6. The average age of the boys in a class is ten years. The average age of the girls in the class is eight years. There are 50% more boys than girls in the class. Find the average age of the class (in years).
(A) 8.4 (B) 8.8 (C) 9.2 (D) 9.6

7. A vessel has 20 litres of a mixture of milk and water having 60% milk. Five litres of pure milk is added to the vessel. Find the percentage of milk in the new solution.
(A) 34% (B) 51%
(C) 68% (D) None of these

8. In what ratio must two kinds of coffee which cost ₹80 per kg and ₹108 per kg be mixed such that the resultant mixture costs ₹96 per kg?
(A) 1 : 2 (B) 2 : 3
(C) 3 : 4 (D) 2 : 1

9. Vessel *A* has 20 litres of a mixture of milk and water having 75% milk. Vessel *B* has *x* litres of a mixture of milk and water having 60% milk. The contents of the vessels are mixed to form a mixture having 66% milk. Find *x*.
(A) 25 (B) 30 (C) 20 (D) 40

10. A milkman has 15 litres of pure milk. How many litres of water have to be added to it so that he gets a 60% profit by selling at cost price?
(A) 9 (B) 10 (C) 8 (D) 12

11. From 90 litres of pure milk, 9 litres is withdrawn and replaced by water. 9 litres of the mixture is then withdrawn and replaced by water. Find the ratio of milk and water in the present mixture.
(A) 19 : 81 (B) 19 : 100
(C) 81 : 19 (D) 81 : 100

12. Just before the last match in a season, the total number of runs scored by Sachin Tendulkar added up to 2100. In his last match, he scored 101 runs. As a result his average score for the season went up by one run. Find the total number of matches he played in that season if he got out in every match.
(A) 31 (B) 5
(C) 71 (D) Either 31 or 71

13. The average weight of all the students of classes I and II equals the average of the average weight of the students of the two classes. There are twice as many students in class II as in class I. The sum of twice the average weight of the students of class I and the average weight of the students of class II is 60 kg. Find the average weight of class I (in kg).
(A) 10 (B) 15 (C) 20 (D) 25

14. Two varieties of wheat are mixed in the proportion of 3 : 4 and the mixture is sold at ₹28 per kg at a profit of 40%. If the second variety of wheat costs ₹3 more than the first variety of wheat, find the cost price of the first variety of wheat.
(A) ₹128/7 per kg (B) ₹120/7 per kg
(C) ₹141/7 per kg (D) ₹149/7 per kg

15. A man buys milk at ₹4 per litre, mixes it with water and sells the mixture at the same price. If his profit is 25%, find the amount of water mixed with each litre of milk.
(A) 0.25 litres (B) 0.5 litres
(C) 0.75 litres (D) 0.6 litres

16. In what proportion can three varieties of sugar priced at ₹10 per kg, ₹12 per kg, and ₹18 per kg, be mixed so that the price of the mixture is ₹14 per kg?
(A) 2 : 2 : 5 (B) 2 : 3 : 4
(C) 1 : 3 : 4 (D) 3 : 4 : 5

17. The ratio of alcohol and water in three mixtures of alcohol and water is 3 : 2, 4 : 1, and 7 : 3. If equal quantities of the mixtures are drawn and mixed, the concentration of alcohol in the resulting mixture will be_____.
(A) 65% (B) 70% (C) 75% (D) 80%

18. In what proportion should milk and water be mixed to reduce the cost of litre of milk from ₹18 per litre to ₹16?
(A) 8 : 1 (B) 6 : 1 (C) 10 : 1 (D) 7 : 1

19. *A*'s weight equals the average weight of *B*, *C,* and *D*. *B*'s weight equals the average weight of *A*, *C,* and *D*. The average weight of *C* and *D* is 30 kg. Find the average weight of *A* and *B*.
(A) 15 kg (B) 30 kg (C) 60 kg (D) 45 kg

20. Of five numbers, the first number is thrice the third, the fourth number is two less than the first, the fifth number is one-seventh of the second and the second number is three less than thrice the first. Find the fifth number, if the average of the numbers is 16.2.
(A) 3 (B) 4 (C) 5 (D) 6

21. There are nine two-digit numbers with distinct tens digits. The units digit of each number is one less than its tens digit. Find the average of the units digits.
(A) 3 (B) 4 (C) 5 (D) 6

22. A sum of ₹7.75 is made up of 100 coins, which are in the denominations of 5 paise and 10 paise. Find the number of 5 paise coins.
(A) 50 (B) 55 (C) 75 (D) 45

23. A businessman lends ₹1800 in two parts, one at 10% and the other at 12% interest. At the end of the year, the average interest he obtained worked out to be 10.5%. Find the interest earned by the businessman from the part which was lent at 10%.
(A) ₹135 (B) ₹150 (C) ₹200 (D) ₹250

24. A vessel is full of a mixture of milk and water, with 9% milk. Nine litres are withdrawn and then replaced with pure water. If the milk is now 6%, how much does the vessel hold?
(A) 27 litres (B) 18 litres
(C) 36 litres (D) 40 litres

25. Three varieties of rice, *A*, *B,* and *C* costing ₹6/kg, ₹9/kg and ₹12/kg are mixed together in a certain ratio. The mixture is sold at $66\frac{2}{3}$% profit for ₹15 / kg. Of the total of 100 kg of the mixture, 50 kg is variety *B*. Find the quantity of variety *A* (in kgs)
(A) 15 (B) 25 (C) 20 (D) 10

Answer Keys

1. B	**2.** D	**3.** B	**4.** B	**5.** D	**6.** C	**7.** C	**8.** C	**9.** B	**10.** A
11. C	**12.** D	**13.** C	**14.** A	**15.** A	**16.** D	**17.** B	**18.** A	**19.** B	**20.** D
21. B	**22.** D	**23.** A	**24.** A	**25.** B					