Chapter 6

ON EQUATIONS REPRESENTING TWO OR MORE STRAIGHT LINES

106. SUPPOSE we have to trace the locus represented by the equation

$$y^2 - 3xy + 2x^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This equation is equivalent to

$$(y - x)(y - 2x) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

It is satisfied by the coordinates of all points which make the first of these brackets equal to zero, and also by the coordinates of all points which make the second bracket zero, *i.e.* by all the points which satisfy the equation

$$y - x = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and also by the points which satisfy

$$y - 2x = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

But, by Art. 47, the equation (3) represents a straight line passing through the origin, and so also does equation (4).

Hence equation (1) represents the two straight lines which pass through the origin, and are inclined at angles of $45°$ and $\tan^{-1} 2$ respectively to the axis of x.

107. Ex. 1. *Trace the locus* $xy = 0$. This equation is satisfied by all the points which satisfy the equation $x = 0$ and by all the points which satisfy $y = 0$, *i.e.* by all the points which lie either on the axis of y or on the axis of x.

The required locus is therefore the two axes of coordinates.

Ex. 2. *Trace the locus* $x^2 - 5x + 6 = 0$. This equation is equivalent to $(x-2)(x-3) = 0$. It is therefore satisfied by all points which satisfy the equation $x - 2 = 0$ and also by all the points which satisfy the equation $x - 3 = 0$.

But these equations represent two straight lines which are parallel to the axis of y and are at distances 2 and 3 respectively from the origin (Art. 46).

Ex. 3. *Trace the locus* $xy - 4x - 5y + 20 = 0$. This equation is equivalent to $(x-5)(y-4) = 0$, and therefore represents a straight line parallel to the axis of y at a distance 5 and also a straight line parallel to the axis of x at a distance 4.

108. Let us consider the general equation

$$ax^2 + 2hxy + by^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

On multiplying it by a it may be written in the form

$$(a^2x^2 + 2ahxy + h^2y^2) - (h^2 - ab)\, y^2 = 0,$$

i.e. $\{(ax + hy) + y\sqrt{h^2 - ab}\}\{(ax + hy) - y\sqrt{h^2 - ab}\} = 0.$

As in the last article the equation (1) therefore represents the two straight lines whose equations are

$$ax + hy + y\sqrt{h^2 - ab} = 0 \ \ldots\ldots\ldots\ldots\ldots(2),$$

and

$$ax + hy - y\sqrt{h^2 - ab} = 0 \ \ldots\ldots\ldots\ldots\ldots(3),$$

each of which passes through the origin.

For (1) is satisfied by *all* the points which satisfy (2), and also by *all* the points which satisfy (3).

These two straight lines are real and different if $h^2 > ab$, real and coincident if $h^2 = ab$, and imaginary if $h^2 < ab$.

[For in the latter case the coefficient of y in each of the equations (2) and (3) is partly real and partly imaginary.]

In the case when $h^2 < ab$, the straight lines, though themselves imaginary, intersect in a real point. For the origin lies on the locus given by (1), since the equation (1) is always satisfied by the values $x = 0$ and $y = 0$.

109. An equation such as (1) of the previous article, which is such that in each term the sum of the indices of x and y is the same, is called a homogeneous equation. This equation (1) is of the second degree; for in the first term the index of x is 2; in the second term the index of both x and y is 1 and hence their sum is 2; whilst in the third term the index of y is 2.

Similarly the expression

$$3x^3 + 4x^2y - 5xy^2 + 9y^3$$

is a homogeneous expression of the third degree.

The expression

$$3x^3 + 4x^2y - 5xy^2 + 9y^3 - 7xy$$

is not however homogeneous; for in the first four terms the sum of the indices is 3 in each case, whilst in the last term this sum is 2.

From Art. 108 it follows that a homogeneous equation of the second degree represents two straight lines, real and different, coincident, or imaginary.

110. *The axes being rectangular, to find the angle between the straight lines given by the equation*

$$ax^2 + 2hxy + by^2 = 0 \qquad\qquad (1).$$

Let the separate equations to the two lines be

$$y - m_1x = 0 \text{ and } y - m_2x = 0 \qquad\qquad (2),$$

so that (1) must be equivalent to

$$b(y - m_1x)(y - m_2x) = 0 \qquad\qquad (3).$$

Equating the coefficients of xy and x^2 in (1) and (3), we have

$$-b(m_1 + m_2) = 2h, \text{ and } bm_1m_2 = a,$$

so that $$m_1 + m_2 = -\frac{2h}{b}, \text{ and } m_1m_2 = \frac{a}{b}.$$

If θ be the angle between the straight lines (2) we have, by Art. 66,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{\mathbf{2\sqrt{h^2 - ab}}}{\mathbf{a + b}} \quad \text{............(4).}$$

Hence the required angle is found.

111. *Condition that the straight lines of the previous article may be* (1) *perpendicular, and* (2) *coincident.*

(1) If $a + b = 0$ the value of $\tan\theta$ is ∞ and hence θ is $90°$; the straight lines are therefore perpendicular.

Hence two straight lines, represented by one equation, are at right angles if the algebraic sum of the coefficients of x^2 and y^2 be zero.

For example, the equations

$$x^2 - y^2 = 0 \text{ and } 6x^2 + 11xy - 6y^2 = 0$$

both represent pairs of straight lines at right angles.

Similarly, whatever be the value of h, the equation

$$x^2 + 2hxy - y^2 = 0,$$

represents a pair of straight lines at right angles.

(2) If $h^2 = ab$, the value of $\tan\theta$ is zero and hence θ is zero. The angle between the straight lines is therefore zero and, since they both pass through the origin, they are therefore coincident.

This may be seen directly from the original equation. For if $h^2 = ab$, *i.e.* $h = \sqrt{ab}$, it may be written

$$ax^2 + 2\sqrt{ab}\, xy + by^2 = 0,$$

i.e.

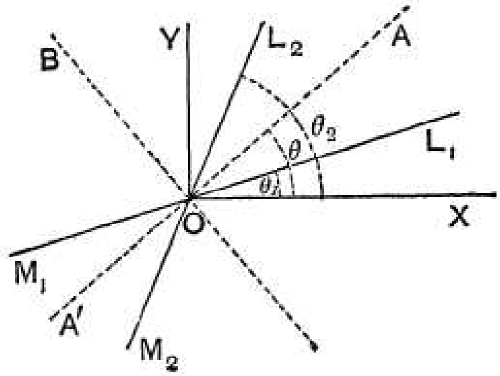
$$(\sqrt{a}x + \sqrt{b}y)^2 = 0,$$

which is two coincident straight lines.

112. *To find the equation to the straight lines bisecting the angle between the straight lines given by*

$$ax^2 + 2hxy + by^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let the equation (1) represent the two straight lines



L_1OM_1 and L_2OM_2 inclined at angles θ_1 and θ_2 to the axis of x, so that (1) is equivalent to

$$b(y - x\tan\theta_1)(y - x\tan\theta_2) = 0.$$

Hence

$$\tan\theta_1 + \tan\theta_2 = -\frac{2h}{b}, \text{ and } \tan\theta_1\tan\theta_2 = \frac{a}{b} \ldots(2).$$

Let OA and OB be the required bisectors.

Since $$\angle AOL_1 = \angle L_2OA,$$

$$\therefore \angle AOX - \theta_1 = \theta_2 - \angle AOX.$$

$$\therefore 2\angle AOX = \theta_1 + \theta_2.$$

Also $$\angle BOX = 90^\circ + \angle AOX.$$

$$\therefore 2\angle BOX = 180^\circ + \theta_1 + \theta_2.$$

Hence, if θ stand for *either* of the angles AOX or BOX, we have

$$\tan 2\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = -\frac{2h}{b - a},$$

by equations (2).

But, if (x, y) be the coordinates of any point on either of the lines OA or OB, we have

$$\tan\theta = \frac{y}{x}.$$

$$\therefore\ -\frac{2h}{b-a}=\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$=\frac{2\dfrac{y}{x}}{1-\dfrac{y^2}{x^2}}=\frac{2xy}{x^2-y^2},$$

i.e. $$\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}}=\frac{\mathbf{xy}}{\mathbf{h}}.$$

This, being a relation holding between the coordinates of *any* point on *either* of the bisectors, is, by Art. 42, the equation to the bisectors.

113. The foregoing equation may also be obtained in the following manner:

Let the given equation represent the straight lines

$$y-m_1x=0 \text{ and } y-m_2x=0 \qquad\text{.....................(1)},$$

so that $$m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b} \qquad\text{................. (2)}.$$

The equations to the bisectors of the angles between the straight lines (1) are, by Art. 84,

$$\frac{y-m_1x}{\sqrt{1+m_1^2}}=\frac{y-m_2x}{\sqrt{1+m_2^2}} \text{ and } \frac{y-m_1x}{\sqrt{1+m_1^2}}=-\frac{y-m_2x}{\sqrt{1+m_2^2}},$$

or, expressed in one equation,

$$\left\{\frac{y-m_1x}{\sqrt{1+m_1^2}}-\frac{y-m_2x}{\sqrt{1+m_2^2}}\right\}\left\{\frac{y-m_1x}{\sqrt{1+m_1^2}}+\frac{y-m_2x}{\sqrt{1+m_2^2}}\right\}=0,$$

i.e. $$\frac{(y-m_1x)^2}{1+m_1^2}-\frac{(y-m_2x)^2}{1+m_2^2}=0,$$

i.e. $$(1+m_2^2)(y^2-2m_1xy+m_1^2x^2)-(1+m_1^2)(y^2-2m_2xy+m_2^2x^2)=0,$$

i.e. $$(m_1^2-m_2^2)(x^2-y^2)+2(m_1m_2-1)(m_1-m_2)xy=0,$$

i.e. $$(m_1+m_2)(x^2-y^2)+2(m_1m_2-1)xy=0.$$

Hence, by (2), the required equation is

$$\frac{-2h}{b}(x^2-y^2)+2\left(\frac{a}{b}-1\right)xy=0,$$

i.e. $$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

EXAMPLES XII

Find what straight lines are represented by the following equations and determine the angles between them.

1. $x^2-7xy+12y^2=0$.
2. $4x^2-24xy+11y^2=0$.
3. $33x^2-71xy-14y^2=0$.
4. $x^3-6x^2+11x-6=0$.
5. $y^2-16=0$.
6. $y^3-xy^2-14x^2y+24x^3=0$.
7. $x^2+2xy\sec\theta+y^2=0$.
8. $x^2+2xy\cot\theta+y^2=0$.
9. Find the equations of the straight lines bisecting the angles between the pairs of straight lines given in examples 2, 3, 7, and 8.
10. Shew that the two straight lines

$$x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$$

make with the axis of x angles such that the difference of their tangents is 2.

11. Prove that the two straight lines

$$(x^2+y^2)(\cos^2\theta\sin^2\alpha+\sin^2\theta)=(x\tan\alpha-y\sin\theta)^2$$

include an angle 2α.

12. Prove that the two straight lines

$$x^2\sin^2\alpha\cos^2\theta+4xy\sin\alpha\sin\theta+y^2[4\cos\alpha-(1+\cos\alpha)^2\cos^2\theta]=0$$

meet at an angle α.

ANSWERS

1. $(x-3y)(x-4y)=0$; $\tan^{-1}\frac{1}{13}$.
2. $(2x-11y)(2x-y)=0$; $\tan^{-1}\frac{4}{3}$.
3. $(11x+2y)(3x-7y)=0$; $\tan^{-1}\frac{83}{19}$.
4. $x=1$; $x=2$; $x=3$.
5. $y=\pm4$.
6. $(y+4x)(y-2x)(y-3x)=0$; $\tan^{-1}(-\frac{6}{7})$; $\tan^{-1}(\frac{1}{7})$.
7. $x(1-\sin\theta)+y\cos\theta=0$; $x(1+\sin\theta)+y\cos\theta=0$; θ.
8. $y\sin\theta+x\cos\theta=\pm x\sqrt{\cos 2\theta}$; $\tan^{-1}(\operatorname{cosec}\theta\sqrt{\cos 2\theta})$.
9. $12x^2-7xy-12y^2=0$; $71x^2+94xy-71y^2=0$; $x^2-y^2=0$; $x^2-y^2=0$.

SOLUTIONS/HINTS

1. The equation is $(x-3y)(x-4y)=0$.

$\therefore$ $m_1=\frac{1}{3}$, $m_2=\frac{1}{4}$, and the required angle

$$=\tan^{-1}\frac{\frac{1}{3}-\frac{1}{4}}{1+\frac{1}{3}\cdot\frac{1}{4}}=\text{etc.}$$

2. The equation is $(2x-y)(2x-11y)=0$.

$\therefore\ m_1=2,\ m_2=\frac{2}{11}$, and the required angle

$$=\tan^{-1}\frac{2-\frac{2}{11}}{1+2\times\frac{2}{11}}=\text{etc.}$$

3. The equation is $(3x-7y)(11x+2y)=0$.

$\therefore\ m_1=\frac{3}{7},\ m_2=-\frac{11}{2}$ and the angle

$$=\tan^{-1}\frac{\frac{3}{7}-(-\frac{11}{2})}{1+\frac{3}{7}(-\frac{11}{2})}=\text{etc.}$$

6. The equation is

$$(y+4x)(y-3x)(y-2x)=0.$$

Here $\qquad m_1=-4,\ \ m_2=3,\ \ m_3=2.$

The angle between the first and third

$$=\tan^{-1}\frac{-4-2}{1+(-4)\times 2}=\tan^{-1}\frac{6}{7}.$$

The angle between the second and third

$$=\tan^{-1}\frac{3-2}{1+3\times 2}=\tan^{-1}\frac{1}{7}.$$

7. Solving as a quadratic in $\frac{y}{x}$, we have

$$\frac{y}{x}=-\sec\theta\pm\sqrt{\sec^2\theta-1}=\frac{-1\pm\sin\theta}{\cos\theta}.$$

$\therefore$ the lines are $x(1+\sin\theta)+y\cos\theta=0$,

and $x(1-\sin\theta)+y\cos\theta=0$,

and the angle between them, by Art. 110,

$$=\tan^{-1}\left\{\frac{2\sqrt{\sec^2\theta-1}}{2}\right\}=\theta.$$

8. Solving as a quadratic in $\frac{y}{x}$,

$$\frac{y}{x}=-\cot\theta\pm\sqrt{\cot^2\theta-1}=\frac{-\cos\theta\pm\sqrt{\cos 2\theta}}{\sin\theta}.$$

$\therefore$ the lines are $x\cos\theta+y\sin\theta=\pm x\sqrt{\cos 2\theta}$, and the angle between them, by Art. 110,

$$= \tan^{-1} \frac{2\sqrt{\cot^2\theta - 1}}{2} = \tan^{-1}(\operatorname{cosec}\theta\sqrt{\cos 2\theta}).$$

9. Use the result of Art. 112.

For the third pair the equation becomes $\frac{x^2 - y^2}{0} = \frac{xy}{\sec\theta}$; hence $x^2 - y^2 = 0$ is the required equation.

10. As in Art. 112,

$$\tan\theta_1 + \tan\theta_2 = \frac{2\tan\theta}{\sin^2\theta} = \frac{2}{\sin\theta \,.\, \cos\theta}$$

$$= \frac{2(\sin^2\theta + \cos^2\theta)}{\sin\theta \,.\, \cos\theta} = 2(\tan\theta + \cot\theta),$$

and $\tan\theta_1 - \tan\theta_2 = \frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta} = \sec^2\theta + \cot^2\theta$

$$= 1 + \tan^2\theta + \cot^2\theta;$$

$$\therefore \ (\tan\theta_1 - \tan\theta_2)^2 = 4\{(\tan\theta + \cot\theta)^2 - (1 + \tan^2\theta + \cot^2\theta)\}$$

$$= 4. \quad \therefore \text{ etc.}$$

11. Since

$$\cos^2\theta\sin^2\alpha + \sin^2\theta = \cos^2\theta\sin^2\alpha + \sin^2\theta(\sin^2\alpha + \cos^2\alpha)$$

$$= \sin^2\alpha + \sin^2\theta\cos^2\alpha,$$

the equation may be written

$$(x^2 + y^2)(\sin^2\theta\cos^2\alpha + \sin^2\alpha) = (x\tan\alpha - y\sin\theta)^2;$$

$$\therefore \ (x^2 + y^2)(\sin^2\theta + \tan^2\alpha) = (1 + \tan^2\alpha)(x\tan\alpha - y\sin\theta)^2;$$

$$\therefore \ x^2\sin^2\theta + y^2\tan^2\alpha + 2xy\tan\alpha\sin\theta$$

$$= \tan^2\alpha(x\tan\alpha - y\sin\theta)^2;$$

$$\therefore \ x\sin\theta + y\tan\alpha = \pm\tan\alpha(x\tan\alpha - y\sin\theta).$$

The m's of these lines are

$$\frac{\tan^2\alpha - \sin\theta}{\tan\alpha(1 + \sin\theta)} \text{ and } \frac{-\tan^2\alpha - \sin\theta}{\tan\alpha(1 - \sin\theta)}.$$

The angle between them

$$= \tan^{-1}\frac{\dfrac{\tan^2\alpha - \sin\theta}{\tan\alpha(1 + \sin\theta)} + \dfrac{\tan^2\alpha + \sin\theta}{\tan\alpha(1 - \sin\theta)}}{1 - \dfrac{\tan^4\alpha - \sin^2\theta}{\tan^2\alpha(1 - \sin^2\theta)}}$$

$$= \tan^{-1} \frac{2 \tan \alpha (\tan^2 \alpha + \sin^2 \theta)}{(1 - \tan^2 \alpha)(\tan^2 \alpha + \sin^2 \theta)} = 2\alpha.$$

12. Taking the terms in x^2, y^2, and xy separately,

$$x^2 \sin^2 \alpha \cos^2 \theta = x^2 \sin^2 \alpha - x^2 \sin^2 \alpha \sin^2 \theta,$$

$$y^2 [4 \cos \alpha - (1 + \cos \alpha)^2 \cos^2 \theta]$$
$$= y^2 [(1 + \cos \alpha)^2 - (1 - \cos \alpha)^2 - (1 + \cos \alpha)^2 \cos^2 \theta]$$
$$= y^2 (1 + \cos \alpha)^2 \sin^2 \theta - y^2 (1 - \cos \alpha)^2,$$

$$4xy \sin \alpha \sin \theta$$
$$= 2xy \sin \alpha (1 + \cos \alpha) \sin \theta + 2xy \sin \alpha (1 - \cos \alpha) \sin \theta.$$

Hence the given equation is equivalent to

$$[x \sin \alpha + y (1 + \cos \alpha) \sin \theta]^2 = [x \sin \alpha \sin \theta - y (1 - \cos \alpha)]^2,$$

i.e. $x \sin \alpha + y (1 + \cos \alpha) \sin \theta = \pm [x \sin \alpha \sin \theta - y (1 - \cos \alpha)]$,

i.e. $x \tan \frac{\alpha}{2} + y \sin \theta = \pm \left[x \tan \frac{\alpha}{2} \sin \theta - y \tan^2 \frac{\alpha}{2} \right].$

$\therefore$ the two lines are

$$y \left(\tan^2 \frac{\alpha}{2} + \sin \theta \right) = x \tan \frac{\alpha}{2} (\sin \theta - 1),$$

and $$y \left(\tan^2 \frac{\alpha}{2} - \sin \theta \right) = x \tan \frac{\alpha}{2} (\sin \theta + 1).$$

The angle between them

$$= \tan^{-1} \frac{\dfrac{\tan \frac{\alpha}{2} (\sin \theta - 1)}{\tan^2 \frac{\alpha}{2} + \sin \theta} \sim \dfrac{\tan \frac{\alpha}{2} (\sin \theta + 1)}{\tan^2 \frac{\alpha}{2} - \sin \theta}}{1 + \dfrac{\tan^2 \frac{\alpha}{2} (\sin^2 \theta - 1)}{\tan^4 \frac{\alpha}{2} - \sin^2 \theta}}$$

$$= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \left(\tan^2 \frac{\alpha}{2} + \sin^2 \theta \right)}{\left(1 - \tan^2 \frac{\alpha}{2} \right) \left(\tan^2 \frac{\alpha}{2} + \sin^2 \theta \right)} = \alpha.$$

GENERAL EQUATION OF THE SECOND DEGREE

114. The most general expression, which contains terms involving x and y in a degree not higher than the second, must contain terms involving x^2, xy, y^2, x, y, and a constant.

The notation which is in general use for this expression is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \quad \ldots\ldots\ldots (1).$$

The quantity (1) is known as the general expression of the second degree, and when equated to zero is called the **general equation of the second degree.**

The student may better remember the seemingly arbitrary coefficients of the terms in the expression (1) if the reason for their use be given.

The most general expression involving terms only of the second degree in x, y, and z is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \quad \ldots\ldots (2),$$

where the coefficients occur in the order of the alphabet.

If in this expression we put z equal to unity we get

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy,$$

which, after rearrangement, is the same as (1).

Now in Solid Geometry we use three coordinates x, y, and z. Also many formulæ in Plane Geometry are derived from those of Solid Geometry by putting z equal to unity.

We therefore, in Plane Geometry, use that notation corresponding to which we have the standard notation in Solid Geometry.

115. In general, as will be shown in Chapter 15, the general equation represents a Curve-Locus.

If a certain condition holds between the coefficients of its terms it will, however, represent a pair of straight lines.

This condition we shall determine in the following article.

116. *To find the condition that the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots(1)$$

may represent two straight lines.

If we can break the left-hand members of (1) into two factors, each of the first degree, then, as in Art. 108, it will represent two straight lines.

If a be not zero, multiply equation (1) by a and arrange in powers of x; it then becomes

$$a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac.$$

On completing the square on the left hand we have

$$a^2x^2 + 2ax(hy + g) + (hy + g)^2 = y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac,$$

i.e.

$$(ax + hy + g) = \pm\sqrt{y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac} \ldots(2).$$

From (2) we cannot obtain x in terms of y, involving only terms of the *first* degree, unless the quantity under the radical sign be a perfect square.

The condition for this is

$$(gh - af)^2 = (h^2 - ab)(g^2 - ac),$$

i.e. $$g^2h^2 - 2afgh + a^2f^2 = g^2h^2 - abg^2 - ach^2 + a^2bc.$$

Cancelling and dividing by a, we have the required condition, viz.

$$\mathbf{abc + 2fgh - af^2 - bg^2 - ch^2 = 0} \ldots\ldots(3).$$

117. The foregoing condition may be otherwise obtained thus:

The given equation, multiplied by (a), is

$$a^2x^2 + 2ahxy + aby^2 + 2agx + 2afy + ac = 0 \text{ ....(4).}$$

The terms of the second degree in this equation break up, as in Art. 108, into the factors

$$ax + hy - y\sqrt{h^2 - ab} \text{ and } ax + hy + y\sqrt{h^2 - ab}.$$

If then (4) break into factors it must be equivalent to

$$\{ax + (h - \sqrt{h^2 - ab})\, y + A\}\{ax + (h + \sqrt{h^2 - ab})\, y + B\} = 0,$$

where A and B are given by the relations

$$a(A + B) = 2ga \text{ .... (5),}$$

$$A(h + \sqrt{h^2 - ab}) + B(h - \sqrt{h^2 - ab}) = 2fa \text{ ....(6),}$$

and

$$AB = ac \text{.................................. (7).}$$

The equations (5) and (6) give

$$A + B = 2g, \text{ and } A - B = \frac{2fa - 2gh}{\sqrt{h^2 - ab}}.$$

The relation (7) then gives

$$4ac = 4AB = (A + B)^2 - (A - B)^2$$

$$= 4g^2 - 4\frac{(fa - gh)^2}{h^2 - ab},$$

i.e.

$$(fa - gh)^2 = (g^2 - ac)(h^2 - ab),$$

which, as before, reduces to

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Ex. If a be zero, prove that the general equation will represent two straight lines if

$$2fgh - bg^2 - ch^2 = 0.$$

If both a and b be zero, prove that the condition is $2fg - ch = 0$.

118. The relation (3) of Art. 116 is equivalent to the expression

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

This may be easily verified by writing down the value of the determinant by the rule of Art. 5.

A geometrical meaning to this form of the relation (3) will be given in a later chapter. [Art. 355.]

The quantity on the left-hand side of equation (3) is called the **Discriminant** of the General Equation.

The general equation therefore represents two straight lines if its discriminant be zero.

119. Ex. 1. *Prove that the equation*

$$12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$$

represents two straight lines, and find the angle between them.

Here

$$a = 12, \quad h = \tfrac{7}{2}, \quad b = -10, \quad g = \tfrac{13}{2}, \quad f = \tfrac{45}{2}, \text{ and } c = -35.$$

Hence $\qquad abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 12 \times (-10) \times (-35) + 2 \times \tfrac{45}{2} \times \tfrac{13}{2} \times \tfrac{7}{2} - 12 \times (\tfrac{45}{2})^2 - (-10) \times (\tfrac{13}{2})^2 - (-35)(\tfrac{7}{2})^2$$

$$= 4200 + \tfrac{4095}{4} - 6075 + \tfrac{1690}{4} + \tfrac{1715}{4}$$

$$= -1875 + \tfrac{7500}{4} = 0.$$

The equation therefore represents two straight lines.

Solving it for x, we have

$$x^2 + x\frac{7y+13}{12} + \left(\frac{7y+13}{24}\right)^2 = \frac{10y^2 - 45y + 35}{12} + \left(\frac{7y+13}{24}\right)^2$$

$$= \left(\frac{23y - 43}{24}\right)^2.$$

$$\therefore \quad x + \frac{7y+13}{24} = \pm \frac{23y-43}{24},$$

i. e.

$$x = \frac{2y-7}{3} \text{ or } \frac{-5y+5}{4}.$$

The given equation therefore represents the two straight lines

$$3x = 2y - 7 \text{ and } 4x = -5y + 5.$$

The "m's" of these two lines are therefore $\frac{3}{2}$ and $-\frac{4}{5}$, and the angle between them, by Art. 66,

$$=\tan^{-1}\frac{\frac{3}{2}-\left(-\frac{4}{5}\right)}{1+\frac{3}{2}\left(-\frac{4}{5}\right)}=\tan^{-1}\left(-\tfrac{23}{2}\right).$$

Ex. 2. *Find the value of h so that the equation*

$$6x^2+2hxy+12y^2+22x+31y+20=0$$

may represent two straight lines.

Here

$$a=6,\quad b=12,\quad g=11,\quad f=\tfrac{31}{2},\text{ and } c=20.$$

The condition (3) of Art. 116 then gives

$$20h^2-341h+\tfrac{2907}{2}=0,$$

i.e. $$\left(h-\tfrac{17}{2}\right)(20h-171)=0.$$

Hence $$h=\tfrac{17}{2}\text{ or }\tfrac{171}{20}.$$

Taking the first of these values, the given equation becomes

$$6x^2+17xy+12y^2+22x+31y+20=0,$$

i.e. $$(2x+3y+4)(3x+4y+5)=0.$$

Taking the second value, the equation is

$$20x^2+57xy+40y^2+\tfrac{220}{3}x+\tfrac{310}{3}y+\tfrac{200}{3}=0,$$

i.e. $$\left(4x+5y+\tfrac{20}{3}\right)(5x+8y+10)=0.$$

EXAMPLES XIII

Prove that the following equations represent two straight lines; find also their point of intersection and the angle between them.

1. $6y^2-xy-x^2+30y+36=0$. **2.** $x^2-5xy+4y^2+x+2y-2=0$.

3. $3y^2-8xy-3x^2-29x+3y-18=0$.

4. $y^2+xy-2x^2-5x-y-2=0$.

5. Prove that the equation

$$x^2+6xy+9y^2+4x+12y-5=0$$

represents two parallel lines.

Find the value of k so that the following equations may represent pairs of straight lines :

6. $6x^2+11xy-10y^2+x+31y+k=0$.

7. $12x^2-10xy+2y^2+11x-5y+k=0$.

8. $12x^2+kxy+2y^2+11x-5y+2=0$.

9. $6x^2+xy+ky^2-11x+43y-35=0$.

10. $kxy - 8x + 9y - 12 = 0.$

11. $x^2 + \frac{10}{3}xy + y^2 - 5x - 7y + k = 0.$

12. $12x^2 + xy - 6y^2 - 29x + 8y + k = 0.$

13. $2x^2 + xy - y^2 + kx + 6y - 9 = 0.$

14. $x^2 + kxy + y^2 - 5x - 7y + 6 = 0.$

15. Prove that the equations to the straight lines passing through the origin which make an angle α with the straight line $y + x = 0$ are given by the equation

$$x^2 + 2xy \sec 2\alpha + y^2 = 0.$$

16. What relations must hold between the coefficients of the equations

(i) $ax^2 + by^2 + cx + cy = 0,$

and (ii) $ay^2 + bxy + dy + ex = 0,$

so that each of them may represent a pair of straight lines?

17. The equations to a pair of opposite sides of a parallelogram are

$$x^2 - 7x + 6 = 0 \text{ and } y^2 - 14y + 40 = 0;$$

find the equations to its diagonals.

ANSWERS

1. $(\frac{6}{5}, -\frac{12}{5})$; $45°$. **2.** $(2, 1)$; $\tan^{-1}\frac{3}{5}$. **3.** $(-\frac{3}{2}, -\frac{5}{2})$; $90°$.
4. $(-1, 1)$; $\tan^{-1}3$. **6.** -15. **7.** 2. **8.** -10 or $-17\frac{1}{2}$.
9. -12. **10.** 6. **11.** 6. **12.** 14. **13.** -3.
14. $\frac{5}{2}$ or $\frac{10}{3}$. **16.** (i) $c(a+b) = 0$; (ii) $e = 0$, or $ae = bd$.
17. $5y + 6x = 56$; $5y - 6x = 14$.

SOLUTIONS/HINTS

It is easily seen in Nos. 1—5 that the condition of Art. 116 is satisfied.

1. If $(3y + x + A)(2y - x + B) \equiv 6y^2 - xy - x^2 + 30y + 36$, then $B - A = 0,\ 2A + 3B = 30,\ AB = 36.$

Whence $A = B = 6.$

Solving the resulting equations $3y + x + 6 = 0$ and $2y - x + 6 = 0$, we have $x = \frac{6}{5}$; $y = -\frac{12}{5}$.

The angle $= \tan^{-1} \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \tan^{-1} 1 = 45°.$

2. If $(x-4y+A)(x-y+B) \equiv x^2-5xy+4y^2+x+2y-2$, then $A+B=1,\ A+4B=-2,\ AB=-2.$

Whence $A=2,\ B=-1.$

Solving the resulting separate equations we have

$$x=2,\ y=1.$$

The angle $= \tan^{-1} \dfrac{1-\frac{1}{4}}{1+1\cdot\frac{1}{4}} = \tan^{-1}\frac{3}{5}.$

3. If

$$(3y+x+A)(y-3x+B) \equiv 3y^2-8xy-3x^2-29x+3y-18,$$

then $3A-B=29,\ 3B+A=3,\ AB=-18.$

Whence $A=9,\ B=-2.$

The two straight lines are therefore

$$3y+x+9=0 \text{ and } y-3x-2=0.$$

Solving $x=-\frac{3}{2},\quad y=-\frac{5}{2}.$

The angle $= \tan^{-1} \dfrac{3+\frac{1}{3}}{1-3\cdot\frac{1}{3}} = \tan^{-1}(\infty) = 90°.$

4. If $(y+2x+A)(y-x+B) \equiv y^2+xy-2x^2-5x-y-2$, then $2B-A=-5,\ A+B=-1,\ AB=-2.$

Whence $A=1,\ B=-2.$

Solving $x=-1;\ y=1.$ Also the m's of the two straight lines are -2 and 1.

The angle $= \tan^{-1} \dfrac{-2-1}{1-2} = \tan^{-1} 3.$

5. Since the condition of Art. 116 is satisfied, and the terms of the second degree are $(x+3y)^2$, the two lines must be parallel. In fact the two lines are

$$x+3y+5=0 \text{ and } x+3y-1=0.$$

6. The condition of Art. 116 gives

$$6\times(-10)\times k+2\times\tfrac{31}{2}\times\tfrac{1}{2}\times\tfrac{11}{2}-6\times\left(\tfrac{31}{2}\right)^2-(-10)\times\left(\tfrac{1}{2}\right)^2 - k\left(\tfrac{11}{2}\right)^2=0,$$

i.e. $-60k+\frac{341}{4}-\frac{5766}{4}+\frac{10}{4}-k\frac{121}{4}=0,$

i.e. $\frac{361}{4}k=-\frac{5415}{4}$, *i.e.* $k=-15.$

7. Here

$$12 \times 2 \times k + 2\left(-\tfrac{5}{2}\right) \times \tfrac{11}{2} \times (-5) - 12 \times \left(-\tfrac{5}{2}\right)^2 - 2 \times \left(\tfrac{11}{2}\right)^2 - k(-5)^2 = 0,$$

etc. giving k.

8. Here

$$12 \times 2 \times 2 + 2\left(-\frac{5}{2}\right)\left(\frac{11}{2}\right)\left(\frac{k}{2}\right) - 12\left(-\frac{5}{2}\right)^2 - 2\left(\frac{11}{2}\right)^2 - 2\left(\frac{k}{2}\right)^2 = 0,$$

i.e. $$48 - \frac{55k}{4} - 75 - \frac{121}{2} - \frac{k^2}{2} = 0,$$

i.e. $2k^2 + 55k + 350 = 0$, *i.e.* $(k+10)(2k+35) = 0$.

10. Here $0 + 2\left(\frac{9}{2}\right)(-4)\left(\frac{k}{2}\right) - (-12)\frac{k^2}{4} = 0.$

$$\therefore\ 3k^2 - 18k = 0,\ \textit{i.e.}\ k = 6.$$

The value $k = 0$ is clearly inadmissible.

14. Here

$$1 \cdot 1 \cdot 6 + 2 \cdot \left(-\frac{7}{2}\right)\left(-\frac{5}{2}\right) \cdot \frac{k}{2} - 1 \cdot \left(-\frac{7}{2}\right)^2 - 1 \cdot \left(-\frac{5}{2}\right)^2 - 6 \times \frac{k^2}{4} = 0,$$

i.e. $6k^2 - 35k + 50 = 0$, *i.e.* $(2k-5)(3k-10) = 0$. $\therefore$ etc.

15. The line $y + x = 0$ is inclined at $135°$ to the axis. Hence one of the straight lines is

$$y = x \tan(135° + a) = x\frac{-1 + \tan a}{1 + \tan a},$$

i.e. $$(\cos a + \sin a)\,y + x(\cos a - \sin a) = 0.$$

So the other is $y = x\tan(135° - a)$,

i.e. $$(\cos a - \sin a)\,y + x(\cos a + \sin a) = 0.$$

$\therefore$ by multiplication, the equation to the two lines is

$$y^2 \cos 2a + 2xy + x^2 \cos 2a = 0. \quad \therefore \text{ etc.}$$

16. (i) The condition of Art. 116 becomes

$$a\frac{c^2}{4} + b\frac{c^2}{4} = 0, \text{ or } c(a+b) = 0.$$

(ii) The condition becomes $e(db - ae) = 0$.

17. The equations to the sides are

$$x-1=0,\ x-6=0,\ y-4=0, \text{ and } y-10=0.$$

Hence the coordinates of the corners are

$$(1,\ 4);\ (6,\ 4);\ (6,\ 10);\ (1,\ 10).$$

Hence the diagonals are $\frac{y-4}{10-4}=\frac{x-1}{6-1}$; $\frac{y-4}{10-4}=\frac{x-6}{1-6}$,

or $$5y-6x=14 \text{ and } 6x+5y=56.$$

120. *To prove that a homogeneous equation of the nth degree represents n straight lines, real or imaginary, which all pass through the origin.*

Let the equation be

$$y^n + A_1 x y^{n-1} + A_2 x^2 y^{n-2} + A_3 x^3 y^{n-3} + \dots + A_n x^n = 0.$$

On division by x^n, it may be written

$$\left(\frac{y}{x}\right)^n + A_1 \left(\frac{y}{x}\right)^{n-1} + A_2 \left(\frac{y}{x}\right)^{n-2} + \dots + A_n = 0 \text{ ....(1).}$$

This is an equation of the nth degree in $\frac{y}{x}$, and hence must have n roots.

Let these roots be $m_1, m_2, m_3, \dots m_n$. Then (C. Smith's Algebra, Art. 89) the equation (1) must be equivalent to the equation

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\left(\frac{y}{x} - m_3\right) \dots \left(\frac{y}{x} - m_n\right) = 0 \text{...(2).}$$

The equation (2) is satisfied by *all* the points which satisfy the separate equations

$$\frac{y}{x} - m_1 = 0, \ \frac{y}{x} - m_2 = 0, \dots \frac{y}{x} - m_n = 0,$$

i.e. by all the points which lie on the n straight lines

$$y - m_1 x = 0, \quad y - m_2 x = 0, \dots y - m_n x = 0,$$

all of which pass through the origin. Conversely, the coordinates of all the points which satisfy these n equations satisfy equation (1). Hence the proposition.

121. Ex. 1. The equation

$$y^3 - 6xy^2 + 11x^2 y - 6x^3 = 0,$$

which is equivalent to

$$(y - x)(y - 2x)(y - 3x) = 0,$$

represents the three straight lines

$$y - x = 0, \quad y - 2x = 0, \quad \text{and} \quad y - 3x = 0,$$

all of which pass through the origin.

Ex. 2. The equation $y^3 - 5y^2 + 6y = 0$,

i.e. $$y(y-2)(y-3) = 0,$$

similarly represents the three straight lines

$$y = 0, \quad y = 2, \quad \text{and} \quad y = 3,$$

all of which are parallel to the axis of x.

122. *To find the equation to the two straight lines joining the origin to the points in which the straight line*

$$lx + my = n \quad \text{.......................}(1)$$

meets the locus whose equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........}(2).$$

The equation (1) may be written

$$\frac{lx + my}{n} = 1 \quad \text{.......................}(3).$$

The coordinates of the points in which the straight line meets the locus satisfy both equation (2) and equation (3), and hence satisfy the equation

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\frac{lx + my}{n} + c\left(\frac{lx + my}{n}\right)^2 = 0 \quad \text{......}(4).$$

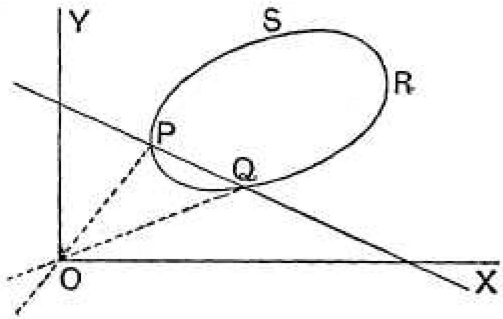
[For at the points where (3) and (4) are true it is clear that (2) is true.]

Hence (4) represents *some locus* which passes through the intersections of (2) and (3).

But, since the equation (4) is homogeneous and of the second degree, it represents two straight lines passing through the origin (Art. 108).

It therefore must represent the two straight lines joining the origin to the intersections of (2) and (3).

123. The preceding article may be illustrated geometrically if we assume that the equation (2) represents some such curve as $PQRS$ in the figure.



Let the given straight line cut the curve in the points P and Q.

The equation (2) holds for all points on the curve $PQRS$.

The equation (3) holds for all points on the line PQ.

Both equations are therefore true at the points of intersection P and Q.

The equation (4), which is derived from (2) and (3), holds therefore at P and Q.

But the equation (4) represents two straight lines, each of which passes through the point O.

It must therefore represent the two straight lines OP and OQ.

124. Ex. *Prove that the straight lines joining the origin to the points of intersection of the straight line $x-y=2$ and the curve*

$$5x^2+12xy-8y^2+8x-4y+12=0$$

make equal angles with the axes.

As in Art. 122 the equation to the required straight lines is

$$5x^2+12xy-8y^2+(8x-4y)\frac{x-y}{2}+12\left(\frac{x-y}{2}\right)^2=0\ldots\ldots(1).$$

For this equation is homogeneous and therefore represents two straight lines through the origin; also it is satisfied at the points where the two given equations are satisfied.

Now (1) is, on reduction,

$$y^2 = 4x^2,$$

so that the equations to the two lines are

$$y = 2x \text{ and } y = -2x.$$

These lines are equally inclined to the axes.

125. It was stated in Art. 115 that, *in general*, an equation of the second degree represents a curve-line, including (Art. 116) as a particular case two straight lines.

In some cases however it will be found that such equations only represent isolated points. Some examples are appended.

Ex. 1. *What is represented by the locus*

$$(x - y + c)^2 + (x + y - c)^2 = 0? \qquad \dots\dots\dots\dots(1).$$

We know that the sum of the squares of two real quantities cannot be zero unless each of the squares is separately zero.

The only real points that satisfy the equation (1) therefore satisfy both of the equations

$$x - y + c = 0 \text{ and } x + y - c = 0.$$

But the only solution of these two equations is

$$x = 0, \text{ and } y = c.$$

The only real point represented by equation (1) is therefore $(0, c)$.

The same result may be obtained in a different manner. The equation (1) gives

$$(x - y + c)^2 = -(x + y - c)^2,$$

i.e.

$$x - y + c = \pm\sqrt{-1}\,(x + y - c).$$

It therefore represents the two imaginary straight lines

$$x(1 - \sqrt{-1}) - y(1 + \sqrt{-1}) + c(1 + \sqrt{-1}) = 0,$$

and

$$x(1 + \sqrt{-1}) - y(1 - \sqrt{-1}) + c(1 - \sqrt{-1}) = 0.$$

Each of these two straight lines passes through the real point $(0, c)$. We may therefore say that (1) represents two imaginary straight lines passing through the point $(0, c)$.

Ex. 2. *What is represented by the equation*

$$(x^2 - a^2)^2 + (y^2 - b^2)^2 = 0\ ?$$

As in the last example, the only real points on the locus are those that satisfy *both* of the equations

$$x^2 - a^2 = 0 \quad \text{and} \quad y^2 - b^2 = 0,$$

i.e. $$x = \pm a, \quad \text{and} \quad y = \pm b.$$

The points represented are therefore

$$(a, b), \ (a, -b), \ (-a, b), \ \text{and} \ (-a, -b).$$

Ex. 3. *What is represented by the equation*

$$x^2 + y^2 + a^2 = 0\ ?$$

The only real points on the locus are those that satisfy all three of the equations

$$x = 0, \quad y = 0, \quad \text{and} \quad a = 0.$$

Hence, unless a vanishes, there are no such points, and the given equation represents nothing real.

The equation may be written

$$x^2 + y^2 = -a^2,$$

so that it represents points whose distance from the origin is $a\sqrt{-1}$. It therefore represents the *imaginary* circle whose radius is $a\sqrt{-1}$ and whose centre is the origin.

126. Ex. 1. *Obtain the condition that one of the straight lines given by the equation*

$$ax^2 + 2hxy + by^2 = 0 \quad \text{..........................(1)}$$

may coincide with one of those given by the equation

$$a'x^2 + 2h'xy + b'y^2 = 0 \quad \text{........................(2)}.$$

Let the equation to the common straight line be

$$y - m_1 x = 0 \quad \text{............................... (3)}.$$

The quantity $y - m_1 x$ must therefore be a factor of the left-hand of both (1) and (2), and therefore the value $y = m_1 x$ must satisfy both (1) and (2).

We therefore have

$$bm_1^2+2hm_1+a=0\ldots\ldots\ldots\ldots(4),$$

and

$$b'm_1^2+2h'm_1+a'=0\ldots\ldots\ldots\ldots(5).$$

Solving (4) and (5), we have

$$\frac{m_1^2}{2(ha'-h'a)}=\frac{m_1}{ab'-a'b}=\frac{1}{2(bh'-b'h)}.$$

$$\therefore \frac{ha'-h'a}{bh'-b'h}=m_1^2=\left\{\frac{ab'-a'b}{2(bh'-b'h)}\right\}^2,$$

so that we must have

$$(ab'-a'b)^2=4(ha'-h'a)(bh'-b'h).$$

Ex. 2. *Prove that the equation*

$$m(x^3-3xy^2)+y^3-3x^2y=0$$

represents three straight lines equally inclined to one another.

Transforming to polar coordinates (Art. 35) the equation gives

$$m(\cos^3\theta-3\cos\theta\sin^2\theta)+\sin^3\theta-3\cos^2\theta\sin\theta=0,$$

i.e. $$m(1-3\tan^2\theta)+\tan^3\theta-3\tan\theta=0,$$

i.e. $$m=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\tan 3\theta.$$

If $m=\tan\alpha$, this equation gives

$$\tan 3\theta=\tan\alpha,$$

the solutions of which are

$$3\theta=\alpha,\ \text{or}\ 180^\circ+\alpha,\ \text{or}\ 360^\circ+\alpha,$$

i.e. $$\theta=\frac{\alpha}{3},\ \text{or}\ 60^\circ+\frac{\alpha}{3},\ \text{or}\ 120^\circ+\frac{\alpha}{3}.$$

The locus is therefore three straight lines through the origin inclined at angles

$$\frac{\alpha}{3},\quad 60^\circ+\frac{\alpha}{3},\quad \text{and}\quad 120^\circ+\frac{\alpha}{3}$$

to the axis of x.

They are therefore equally inclined to one another.

Ex. 3. *Prove that two of the straight lines represented by the equation*

$$ax^3+bx^2y+cxy^2+dy^3=0\ \ldots\ldots\ldots\ldots\ (1)$$

will be at right angles if

$$a^2+ac+bd+d^2=0.$$

Let the separate equations to the three lines be

$$y-m_1x=0,\quad y-m_2x=0,\quad \text{and}\quad y-m_3x=0,$$

so that the equation (1) must be equivalent to

$$d(y-m_1x)(y-m_2x)(y-m_3x)=0,$$

and therefore

$$m_1+m_2+m_3=-\frac{c}{d} \quad\ldots\ldots\ldots\ldots(2),$$

$$m_2m_3+m_3m_1+m_1m_2=\frac{b}{d} \quad\ldots\ldots\ldots\ldots(3),$$

and

$$m_1m_2m_3=-\frac{a}{d} \quad\ldots\ldots\ldots\ldots(4).$$

If the first two of these straight lines be at right angles we have, in addition,

$$m_1m_2=-1 \quad\ldots\ldots\ldots\ldots(5).$$

From (4) and (5), we have

$$m_3=\frac{a}{d},$$

and therefore, from (2),

$$m_1+m_2=-\frac{c}{d}-\frac{a}{d}=-\frac{c+a}{d}.$$

The equation (3) then becomes

$$\frac{a}{d}\left(-\frac{c+a}{d}\right)-1=\frac{b}{d},$$

i.e.

$$a^2+ac+bd+d^2=0.$$

EXAMPLES XIV

1. Prove that the equation

$$y^3-x^3+3xy(y-x)=0$$

represents three straight lines equally inclined to one another.

2. Prove that the equation

$$y^2(\cos\alpha+\sqrt{3}\sin\alpha)\cos\alpha-xy(\sin 2\alpha-\sqrt{3}\cos 2\alpha)+x^2(\sin\alpha-\sqrt{3}\cos\alpha)\sin\alpha=0$$

represents two straight lines inclined at 60° to each other.

Prove also that the area of the triangle formed with them by the straight line

$$(\cos\alpha-\sqrt{3}\sin\alpha)\,y-(\sin\alpha+\sqrt{3}\cos\alpha)\,x+a=0$$

is

$$\frac{a^2}{4\sqrt{3}},$$

and that this triangle is equilateral.

3. Shew that the straight lines

$$(A^2-3B^2)\,x^2+8ABxy+(B^2-3A^2)\,y^2=0$$

form with the line $Ax+By+C=0$ an equilateral triangle whose area is $\dfrac{C^2}{\sqrt{3}(A^2+B^2)}$.

4. Find the equation to the pair of straight lines joining the origin to the intersections of the straight line $y=mx+c$ and the curve

$$x^2+y^2=a^2.$$

Prove that they are at right angles if

$$2c^2=a^2(1+m^2).$$

5. Prove that the straight lines joining the origin to the points of intersection of the straight line

$$kx+hy=2hk$$

with the curve $\qquad (x-h)^2+(y-k)^2=c^2$

are at right angles if $\qquad h^2+k^2=c^2.$

6. Prove that the angle between the straight lines joining the origin to the intersection of the straight line $y=3x+2$ with the curve

$$x^2+2xy+3y^2+4x+8y-11=0 \text{ is } \tan^{-1}\frac{2\sqrt{2}}{3}.$$

7. Shew that the straight lines joining the origin to the other two points of intersection of the curves whose equations are

$$ax^2+2hxy+by^2+2gx=0$$

and $\qquad a'x^2+2h'xy+b'y^2+2g'x=0$

will be at right angles if

$$g(a'+b')-g'(a+b)=0.$$

What loci are represented by the equations

8. $x^2-y^2=0.$ **9.** $x^2-xy=0.$ **10.** $xy-ay=0.$

11. $x^3-x^2-x+1=0.$ **12.** $x^3-xy^2=0.$ **13.** $x^3+y^3=0.$

14. $x^2+y^2=0.$ **15.** $x^2y=0.$ **16.** $(x^2-1)(y^2-4)=0.$

17. $(x^2-1)^2+(y^2-4)^2=0.$ **18.** $(y-mx-c)^2+(y-m'x-c')^2=0.$

19. $(x^2-a^2)^2(x^2-b^2)^2+c^4(y^2-a^2)^2=0.$ **20.** $(x-a)^2-y^2=0.$

21. $(x+y)^2-c^2=0.$ **22.** $r=a\sec(\theta-\alpha).$

23. Shew that the equation

$$bx^2-2hxy+ay^2=0$$

represents a pair of straight lines which are at right angles to the pair given by the equation

$$ax^2+2hxy+by^2=0.$$

24. If pairs of straight lines

$$x^2-2pxy-y^2=0 \text{ and } x^2-2qxy-y^2=0$$

be such that each pair bisects the angles between the other pair, prove that $pq=-1$.

25. Prove that the pair of lines

$$a^2x^2+2h(a+b)xy+b^2y^2=0$$

is equally inclined to the pair

$$ax^2+2hxy+by^2=0.$$

26. Shew also that the pair

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$$

is equally inclined to the same pair.

27. If one of the straight lines given by the equation

$$ax^2 + 2hxy + by^2 = 0$$

coincide with one of those given by

$$a'x^2 + 2h'xy + b'y^2 = 0,$$

and the other lines represented by them be perpendicular, prove that

$$\frac{ha'b'}{b' - a'} = \frac{h'ab}{b - a} = \sqrt{-aa'bb'}.$$

28. Prove that the equation to the bisectors of the angle between the straight lines $ax^2 + 2hxy + by^2 = 0$ is

$$h(x^2 - y^2) + (b - a)xy = (ax^2 - by^2)\cos\omega,$$

the axes being inclined at an angle ω.

29. Prove that the straight lines

$$ax^2 + 2hxy + by^2 = 0$$

make equal angles with the axis of x if $h = a\cos\omega$, the axes being inclined at an angle ω.

30. If the axes be inclined at an angle ω, shew that the equation

$$x^2 + 2xy\cos\omega + y^2\cos 2\omega = 0$$

represents a pair of perpendicular straight lines.

31. Shew that the equation

$$\cos 3\alpha\,(x^3 - 3xy^2) + \sin 3\alpha\,(y^3 - 3x^2y) + 3a(x^2 + y^2) - 4a^3 = 0$$

represents three straight lines forming an equilateral triangle.

Prove also that its area is $3\sqrt{3}a^2$.

32. Prove that the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents two parallel straight lines if

$$h^2 = ab \text{ and } bg^2 = af^2.$$

Prove also that the distance between them is

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}}.$$

33. If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points where these meet the axes is

$$ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0.$$

34. If the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represent two straight lines, prove that the square of the distance of their point of intersection from the origin is

$$\frac{c(a+b)-f^2-g^2}{ab-h^2}.$$

35. Shew that the orthocentre of the triangle formed by the straight lines

$$ax^2+2hxy+by^2=0 \text{ and } lx+my=1$$

is a point (x', y') such that

$$\frac{x'}{l}=\frac{y'}{m}=\frac{a+b}{am^2-2hlm+bl^2}.$$

36. Hence find the locus of the orthocentre of a triangle of which two sides are given in position and whose third side goes through a fixed point.

37. Shew that the distance between the points of intersection of the straight line

$$x\cos\alpha+y\sin\alpha-p=0$$

with the straight lines $\quad ax^2+2hxy+by^2=0$

is

$$\frac{2p\sqrt{h^2-ab}}{b\cos^2\alpha-2h\cos\alpha\sin\alpha+a\sin^2\alpha}.$$

Deduce the area of the triangle formed by them.

38. Prove that the product of the perpendiculars let fall from the point (x', y') upon the pair of straight lines

$$ax^2+2hxy+by^2=0$$

is

$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$

39. Shew that two of the straight lines represented by the equation

$$ay^4+bxy^3+cx^2y^2+dx^3y+ex^4=0$$

will be at right angles if

$$(b+d)(ad+be)+(e-a)^2(a+c+e)=0.$$

40. Prove that two of the lines represented by the equation

$$ax^4+bx^3y+cx^2y^2+dxy^3+ay^4=0$$

will bisect the angles between the other two if

$$c+6a=0 \text{ and } b+d=0.$$

41. Prove that one of the lines represented by the equation

$$ax^3+bx^2y+cxy^2+dy^3=0$$

will bisect the angle between the other two if

$$(3a+c)^2(bc+2cd-3ad)=(b+3d)^2(bc+2ab-3ad).$$

ANSWERS

4. $c^2(x^2+y^2) = a^2(y-mx)^2$.

8. $x-y=0$, or $x+y=0$.

9. $x=0$, or $x-y=0$.

10. $y=0$, or $x=a$.

11. $x-1=0$, or $x-1=0$, or $x+1=0$.

12. $x=0$, or $x-y=0$, or $x+y=0$.

13. $x+y=0$, or $x^2-xy+y^2=0$.

14. $x^2=0$, and $y^2=0$.

15. $x=0$, or $y=0$.

16. $x+1=0$, or $x-1=0$, or $y-2=0$, or $y+2=0$.

17. $x=\pm 1$, and $y=\pm 2$.

18. $y-mx-c=0$, and $y-m'x-c'=0$.

19. $x=\pm a, \pm b$, and $y=\pm a$.

20. $x-a=y$, or $x-a=-y$.

21. $x+y=c$, or $x+y=-c$.

22. $r\cos.(\theta-\alpha)=a$.

36. $bx^2-2hxy+ay^2=(a+b)(hx+ky)$.

SOLUTIONS/HINTS

1. Put $m=-1$ in Ex. 2 of Art. 126.

2. The equation may be written

$$y^2\cos\alpha\,.\,\cos(\alpha-60^\circ)-xy\sin(2\alpha-60^\circ)$$
$$+x^2\sin\alpha\,.\,\sin(\alpha-60^\circ)=0.$$

i.e. $\{y\cos(\alpha-60^\circ)-x\sin(\alpha-60^\circ)\}\{y\cos\alpha-x\sin\alpha\}=0.$

$\therefore$ the two lines are $y=x\tan\alpha$, and $y=x\tan(\alpha-60^\circ)$, or in polar coordinates, $\theta=\alpha$, $\theta=\alpha-60^\circ$. ....(i)

These two lines are inclined to one another at an angle of 60°.

The third line is $x \sin(\alpha + 60°) - y \cos(\alpha + 60°) = \frac{a}{2}$, or in polars, $r \sin(\alpha + 60° - \theta) = \frac{a}{2}$.

Each of the lines (i) cuts this line where $r = \frac{a}{\sqrt{3}}$.

The triangle is equilateral since two equal sides are inclined at 60°, and its area

$$= \frac{1}{2} \cdot \frac{a}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}} \cdot \sin 60° = \frac{a^2}{4\sqrt{3}}.$$

3. On factorizing, the equations to the straight lines are $(A + \sqrt{3}B)x + (B - \sqrt{3}A)y = 0$,

and $(A - \sqrt{3}B)x + (B + \sqrt{3}A)y = 0$.

Put $\tan\alpha = \frac{B}{A}$, divide each by $\sqrt{A^2 + B^2}$, and we obtain

$$y = x \tan(\alpha + 30°), \text{ and } y = x \tan(\alpha - 30°),$$

or in polars, $\theta = \alpha + 30°$; $\theta = \alpha - 30°$. .... (i)

Also the equation $Ax + By + C = 0$ becomes

$$r = -\frac{C}{\sqrt{A^2 + B^2} \cdot \cos(\theta - \alpha)}.$$

Each of the lines (i) cuts this line where

$$r = -\frac{2C}{\sqrt{A^2 + B^2} \cdot \sqrt{3}}.$$

Hence, since the lines (i) are inclined at 60°, the triangle is equilateral, and its area

$$= \frac{1}{2} \cdot \frac{4C^2}{3(A^2 + B^2)} \cdot \sin 60° = \frac{C^2}{\sqrt{3}(A^2 + B^2)}.$$

4. The required equation is, (Art. 122),

$$c^2(x^2 + y^2) = a^2(y - mx)^2.$$

The lines are at right angles if

$$2c^2 = a^2(1 + m^2). \quad \text{(Art. 111)}.$$

5. The required lines are, (Art. 122),

$$4h^2k^2(x^2+y^2)-4hk(hx+yk)(kx+hy)+(h^2+k^2-c^2)(kx+hy)^2=0.$$

These lines are at right angles if

$$8h^2k^2-4h^2k^2-4h^2k^2+(h^2+k^2-c^2)(k^2+h^2)=0, \quad \text{(Art. 111)}$$

i.e. if $h^2+k^2=c^2$.

6. The required lines are, (Art. 122),

$$(x^2+2xy+3y^2)\,.\,2^2+2\,(y-3x)\,(4x+8y)-11\,(y-3x)^2=0,$$

i.e.
$$7x^2-2xy-y^2=0,$$

and the angle between them (Art. 110)

$$\tan^{-1}\frac{2\sqrt{1^2+7}}{7-1}=\tan^{-1}\frac{2\sqrt{2}}{3}.$$

7. On multiplying the first equation by g', the second by g, and subtracting, we have an equation which is satisfied when the two equations are satisfied. But the resulting equation is, on reduction,

$$x^2\,(ag'-a'g)+2xy\,(hg'-h'g)+y^2\,(bg'-b'g)=0,$$

i.e. it gives two straight lines through the origin.

These are at right angles if

$$g\,(a'+b')=g'\,(a+b) \quad \text{(Art. 111)}.$$

8. $x-y=0$, or $x+y=0$. The lines bisecting the angles between the axes.

9. $x=0$, or $x-y=0$. The axis of y and the line $x-y=0$. (See Ex. 8.)

10. $y=0$, or $x=a$. The axis of x and a line parallel to the axis of y at a distance a from it.

11. $x-1=0$, or $x-1=0$, or $x+1=0$. Two coincident lines parallel to the axis of y at a distance 1 from it, and a line also parallel to it, and at a distance -1 from it.

12. $x=0$, or $x-y=0$, or $x+y=0$. The lines of Ex. 8 with the axis of y.

13. $x+y=0$, or $x^2-xy+y^2=0$. The line $x+y=0$ (see Ex. 8) and two imaginary lines through the origin.

14. $x^2 = 0$, and $y^2 = 0$. The origin.

15. $x = 0$, or $y = 0$. The axes.

16. $x + 1 = 0$, or $x - 1 = 0$, or $y - 2 = 0$, or $y + 2 = 0$. Four lines, two parallel to the axis of x, and two parallel to the axis of y.

17. $x = \pm 1$, and $y = \pm 2$. Four points, viz.

$(1, 2)$; $(1, -2)$; $(-1, 2)$; $(-1, -2)$.

18. $y - mx - c = 0$, and $y - m'x - c' = 0$. The point of intersection of these two lines.

19. $x = \pm a, \pm b$, and $y = \pm a$. The 8 points,

(a, a); $(a, -a)$; $(-a, a)$; $(-a, -a)$;

(b, a); $(b, -a)$; $(-b, a)$; $(-b, -a)$.

20. $x - a = y$, or $x - a = -y$. These two straight lines.

21. $x + y = c$, or $x + y = -c$. These two straight lines.

22. $r \cos.(\theta - a) = a$. A straight line. [See Art. 88.]

23. If $b(y - m_1x)(y - m_2x) \equiv ax^2 + 2hxy + by^2$,

then
$$m_1 + m_2 = -\frac{2h}{b}, \text{ and } m_1m_2 = \frac{a}{b}.$$

The lines at right angles to $y - m_1x = 0$ and $y - m_2x = 0$ are $(m_1y + x)(m_2y + x) = 0$,

or
$$m_1 . m_2y^2 + (m_1 + m_2)xy + x^2 = 0,$$

i.e. $\frac{a}{b} . y^2 - \frac{2h}{b} . xy + x^2 = 0$, *i.e.* $ay^2 - 2hxy + bx^2 = 0$.

24. The bisectors of the angles between $x^2 - 2pxy - y^2 = 0$ are, (Art. 112),

$$\frac{x^2 - y^2}{2} = \frac{xy}{-p}, \text{ or } px^2 + 2xy - py^2 = 0.$$

If these are coincident with $x^2 - 2qxy - y^2 = 0$, then

$$\frac{p}{1} = \frac{2}{-2q}; \quad \therefore pq = -1.$$

25. The bisectors of the angles between the first pair are, (Art. 112),

$$\frac{x^2-y^2}{a^2-b^2}=\frac{xy}{h(a+b)}, \quad i.e. \quad \frac{x^2-y^2}{a-b}=\frac{xy}{h},$$

which are also the bisectors between the second pair. Hence etc.

Let either of these bisectors be OQ and let the first pair be OA, OA' and the second pair OB, OB'. We then have $\angle A'OQ=\angle QOA$ and $\angle B'OQ=\angle QOB$. $\therefore$ by subtraction, $\angle B'OA'=\angle AOB$.

26. The angles between the two pairs of lines have the same bisectors. (Art. 112.)

27. Let $ax^2+2hxy+by^2 \equiv b(y-px)(y-mx)$,

and $$a'x^2+2h'xy+b'y^2 \equiv b(y-px)\left(y+\frac{1}{m}x\right).$$

Then $p+m=-\dfrac{2h}{b}$, ...(i) $\quad p-\dfrac{1}{m}=-\dfrac{2h'}{b'}$, ...(iii)

$pm=\dfrac{a}{b}$, ...(ii) $\quad \dfrac{p}{m}=-\dfrac{a'}{b'}$. ......(iv)

(ii) and (iv) give $p^2=-\dfrac{aa'}{bb'}$, so that $p=\dfrac{\sqrt{-aba'b'}}{bb'}$; and (ii) then gives

$$m=\frac{ab'}{\sqrt{-aba'b'}}=-\frac{\sqrt{-aba'b'}}{a'b}.$$

(i) then gives $$-\frac{2h}{b}=\sqrt{-aba'b'}\left(\frac{1}{bb'}-\frac{1}{a'b}\right),$$

so that $$\frac{2ha'b'}{b'-a'}=\sqrt{-aba'b'}.$$

Finally, from (iii),

$$-\frac{2h'}{b'}=\sqrt{-aba'b'}\left[\frac{1}{bb'}-\frac{1}{ab'}\right],$$

i.e. $$\frac{2h'ab}{b-a}=\sqrt{-aba'b'}.$$

28. Let $y - m_1 x = 0$, and $y - m_2 x = 0$ be the two lines, so that $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$. If these be inclined at θ_1 and θ_2 to the axis of x and the bisector at θ, then, as in Art. 112, $2\theta = \theta_1 + \theta_2$ or $\pi + \theta_1 + \theta_2$.

$$\therefore\ \tan 2\theta = \tan(\theta_1 + \theta_2) = \frac{\dfrac{m_1 \sin\omega}{1 + m_1 \cos\omega} + \dfrac{m_2 \sin\omega}{1 + m_2 \cos\omega}}{1 - \dfrac{m_1 \sin\omega}{1 + m_2 \cos\omega} \cdot \dfrac{m_2 \sin\omega}{1 + m_2 \cos\omega}}$$

$$= \frac{\sin\omega \{(m_1 + m_2) + 2m_1 m_2 \cos\omega\}}{1 + (m_1 + m_2)\cos\omega + m_1 m_2 (\cos^2\omega - \sin^2\omega)}$$

$$= \frac{\sin\omega \{-2h + 2a\cos\omega\}}{b - 2h\cos\omega + a(\cos^2\omega - \sin^2\omega)}.$$

If $y = mx$ be the equation of a bisector, then

$$\frac{m \sin\omega}{1 + m\cos\omega} = \tan\theta;$$

$$\therefore\ \tan 2\theta = \frac{\dfrac{2\dfrac{y}{x}\sin\omega}{1 + \dfrac{y}{x}.\cos\omega}}{1 - \dfrac{\dfrac{y^2}{x^2}.\sin^2\omega}{\left(1 + \dfrac{y}{x}\cos\omega\right)^2}} = \frac{2y\sin\omega\,(x + y\cos\omega)}{(x + y\cos\omega)^2 - y^2\sin^2\omega}.$$

Hence the equation becomes

$$\frac{y(x + y\cos\omega)}{(x + y\cos\omega)^2 - y^2\sin^2\omega} = \frac{a\cos\omega - h}{b - 2h\cos\omega + a(\cos^2\omega - \sin^2\omega)},$$

or $$h(x^2 - y^2) + (b - a)xy = (ax^2 - by^2)\cos\omega.$$

29. Let $ax^2 + 2hxy + by^2 \equiv b(y - m_1 x)(y - m_2 x)$, so that

$$m_1 + m_2 = -\frac{2h}{b}, \quad \text{and} \quad m_1 m_2 = \frac{a}{b}.$$

Also $\tan\theta_1 = \dfrac{m_1 \sin\omega}{1 + m_1\cos\omega}$ and $\tan\theta_2 = \dfrac{m_2\sin\omega}{1+m_2\cos\omega}$.

If $\theta_1 + \theta_2 = \pi$, then $\tan\theta_1 + \tan\theta_2 = 0$.

$$\therefore \frac{m_1}{1+m_1\cos\omega} + \frac{m_2}{1+m_2\cos\omega} = 0.$$

$\therefore\ m_1 + m_2 + 2m_1m_2\cos\omega = 0. \qquad \therefore\ h = a\cos\omega.$

30. If

$$x^2 + 2xy\cos\omega + y^2\cos 2\omega \equiv \cos 2\omega\,(y - m_1x)(y - m_2x),$$

then $m_1 + m_2 = -\dfrac{2\cos\omega}{\cos 2\omega}$, and $m_1m_2 = \dfrac{1}{\cos 2\omega}$.

$$\therefore 1 + (m_1+m_2)\cos\omega + m_1m_2 = \frac{\cos 2\omega - 2\cos^2\omega + 1}{\cos 2\omega} = 0.$$

$\therefore$ the lines are perpendicular. (Art. 93.)

31. Transforming to polars, the equation becomes

$r^3\cos 3(\theta+a) + 3ar^2 - 4a^3 = 0. \qquad \therefore\ \cos 3(\theta + a) = 4\dfrac{a^3}{r^3} - 3\dfrac{a}{r}.$

This is satisfied by $\dfrac{a}{r} = \cos(\theta+a)$, $\dfrac{a}{r} = \cos\left(\theta + a - \dfrac{2\pi}{3}\right)$, and $\dfrac{a}{r} = \cos\left(\theta + a - \dfrac{4\pi}{3}\right)$. These equations are

$x\cos(-a) + y\sin(-a) = a$, $x\cos\left(\dfrac{2\pi}{3} - a\right) + y\sin\left(\dfrac{2\pi}{3} - a\right) = a$,

and $\qquad x\cos\left(\dfrac{4\pi}{3} - a\right) + y\sin\left(\dfrac{4\pi}{3} - a\right) = a.$

The equation therefore represents these three straight lines which are inclined at 60° to each other.

By Ex. x. 20,

$$\text{the area of the } \triangle = \frac{a^2}{2}\cdot\frac{(3\sin 120^\circ)^2}{\sin^3 120^\circ} = 3\sqrt{3}\,.\,a^2.$$

32. The terms of the second degree must be a perfect square; $\therefore\ h^2 = ab$. If

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
$$\equiv (\sqrt{a}x + \sqrt{b}y + \lambda_1)(\sqrt{a}x + \sqrt{b}y + \lambda_2),$$

then $\sqrt{a}(\lambda_1 + \lambda_2) = 2g$; $\sqrt{b}(\lambda_1 + \lambda_2) = 2f$. Whence $af^2 = bg^2$.

Also $\lambda_1\lambda_2 = c$. $\therefore (\lambda_1 - \lambda_2)^2 = \dfrac{4(g^2 - ac)}{a}$.

The distance between the two lines

= difference of perpendiculars upon them from the origin

$$= \frac{\lambda_1 \sim \lambda_2}{\sqrt{a+b}} = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}.$$

33. Since the equation represents a pair of straight lines,

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \quad \ldots\ldots\ldots(\text{i})$$

If we put $x = 0$ in the equation of the third pair of lines, we must obtain $by^2 + 2fy + c = 0$, and if we put $y = 0$ in the equation of the third pair of lines, we must obtain

$$ax^2 + 2gx + c = 0,$$

since they cut the axes in the same points as the given lines.

Therefore the equation of the third pair of lines must be of the form

$$ax^2 + by^2 + 2gx + 2fy + c + 2\lambda xy = 0. \quad \ldots\ldots\ldots(\text{ii})$$

Since this is the equation of a pair of lines

$$\therefore abc + 2fg\lambda - af^2 - bg^2 - c\lambda^2 = 0. \quad \ldots\ldots\ldots(\text{iii})$$

Subtracting (iii) from (i) and dividing by $\lambda - h$, we have

$$2fg = c(h + \lambda). \quad \therefore \lambda = \frac{2fg}{c} - h.$$

Substituting for λ in (ii), we have the required equation.

34. By Art. 117, the two lines are

$$ax + \{h - \sqrt{h^2 - ab}\}y + A = 0, \text{ and } ax + \{h + \sqrt{h^2 - ab}\}y + B = 0,$$

where $A + B = 2g$, and $A - B = \dfrac{2fa - 2gh}{\sqrt{h^2 - ab}}$.

By adding, $ax + hy + g = 0$;

By subtracting, $y = \dfrac{fa - gh}{h^2 - ab}$, whence $x = \dfrac{bg - fh}{h^2 - ab}$.

$$x^2+y^2=\frac{a(af^2-2gfh)+b(bg^2-2fgh)+h^2(g^2+f^2)}{(h^2-ab)^2}$$

$$=\frac{a(abc-bg^2-ch^2)+b(abc-af^2-ch^2)+h^2(g^2+f^2)}{(h^2-ab)^2},$$

by using the condition of Art. 116,

$$=\frac{(g^2+f^2)(h^2-ab)-(ac+bc)(h^2-ab)}{(h^2-ab)^2}=\frac{g^2+f^2-ac-bc}{h^2-ab}.$$

35. Let $ax^2+2hxy+by^2\equiv b(y-m_1x)(y-m_2x)$, so that

$$m_1+m_2=-\frac{2h}{b} \text{ and } m_1m_2=\frac{a}{b}. \quad \dots\dots\dots \text{(i)}$$

The line $lx+my=1$ cuts $y-m_1x=0$, where

$$x=\frac{1}{l+mm_1}, \qquad y=\frac{m_1}{l+mm_1}.$$

The equation of the line through this point perpendicular to $y-m_2x=0$, is

$$(l+mm_1)x-1+m_2\{y(l+mm_1)-m_1\}=0. \quad \dots \text{(ii)}$$

The orthocentre lies on this line and also on the line through the origin perpendicular to $lx+my=1$, viz.

$$\frac{x}{l}=\frac{y}{m}=\lambda. \quad \dots\dots\dots\dots\dots\dots\dots \text{(iii)}$$

For the orthocentre we have thus to solve (ii) and (iii).

From (iii) put $x=l\lambda$, $y=m\lambda$ in (ii).

$$\therefore\ \lambda\{l^2+lm(m_1+m_2)+m^2.\,m_1m_2\}=1+m_1m_2.$$

Hence, by (i), $\lambda=\dfrac{a+b}{bl^2-2hlm+am^2}$.

36. If the straight line $lx+my=1$ of the previous article passes through the fixed point (h, k), we have

$$lh+mk=1. \quad \dots\dots\dots\dots\dots\dots\dots \text{(1)}$$

We must eliminate l and m between this and

$$\frac{x'}{l}=\frac{y'}{m}=\frac{a+b}{am^2-2hlm+bl^2}. \quad \dots\dots\dots \text{(2)}$$

The first two of these give

$$\frac{x'}{l} = \frac{y'}{m} = \frac{hx' + ky'}{hl + km} = hx' + ky', \text{ by (1).}$$

Substituting for l, m in (2), we have

$$hx' + ky' = \frac{(a+b)(hx' + ky')^2}{ay'^2 - 2hx'y' + bx'^2},$$ so that the locus required is

$$bx^2 - 2hxy + ay^2 = (a+b)(hx + ky).$$

37. Eliminating x between the two equations, we have

$$y^2(b\cos^2\alpha - 2h\cos\alpha\sin\alpha + a\sin^2\alpha) - 2yp(a\sin\alpha - h\cos\alpha) + ap^2 = 0;$$

$$\therefore\ y_1 + y_2 = \frac{2p(a\sin\alpha - h\cos\alpha)}{(b\cos^2\alpha - 2h\cos\alpha\sin\alpha + a\sin^2\alpha)},$$

and $$y_1y_2 = \frac{ap^2}{b\cos^2\alpha - 2h\sin\alpha\cos\alpha + a\sin^2\alpha}.$$

Whence $$(y_1 - y_2)^2 = \frac{4p^2\cos^2\alpha(h^2 - ab)}{\{b\cos^2\alpha - 2h\cos\alpha\sin\alpha + a\sin^2\alpha\}^2}.$$

Similarly $$(x_1 - x_2)^2 = \frac{4p^2\sin^2\alpha(h^2 - ab)}{(b\cos^2\alpha - 2h\cos\alpha\sin\alpha + a\sin^2\alpha)^2}.$$

$$\therefore\ \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \frac{2p\sqrt{h^2 - ab}}{b\cos^2\alpha - 2h\cos\alpha\sin\alpha + a\sin^2\alpha}.$$

For the area of the triangle, multiply this expression by $\frac{p}{2}$.

38. Let $ax^2 + 2hxy + by^2 \equiv b(y - m_1x)(y - m_2x)$, so that

$$m_1 + m_2 = -\frac{2h}{b}, \quad m_1m_2 = \frac{a}{b}, \text{ and } m_1^2 + m_2^2 = \frac{4h^2 - 2ab}{b^2}.$$

The product of perpendiculars from (x', y')

$$= \frac{(y' - m_1x')(y' - m_2x')}{\sqrt{(1 + m_1^2)(1 + m_2^2)}}$$

$$= \frac{ax'^2 + 2hx'y' + by'^2}{b\sqrt{1 + m_1^2 + m_2^2 + m_1^2m_2^2}} = \frac{ax'^2 + 2hx'y' + by'^2}{\sqrt{(a-b)^2 + 4h^2}}.$$

39. If $\frac{y}{x} = m$ is one root of the equation, $\frac{y}{x} = -\frac{1}{m}$ must be another.

$$\therefore \quad am^4 + bm^3 + cm^2 + dm + e = 0, \ldots\ldots\ldots \text{(i)}$$

and

$$em^4 - dm^3 + cm^2 - bm + a = 0. \ldots\ldots\ldots \text{(ii)}$$

Subtracting, $(a-e)\,m^4 - (a-e) + (b+d)\,m\,(m^2+1) = 0$;

$$\therefore \quad (a-e)\,m^2 + (b+d)\,m - (a-e) = 0. \ldots\ldots \text{(iii)}$$

Multiply (i) by e, (ii) by a, and subtract;

$$\therefore \ (be+ad)\,m^3 + c\,(e-a)\,m^2 + (de+ab)\,m + e^2 - a^2 = 0. \ \text{(iv)}$$

Multiply (iii) by $e+a$ and subtract from (iv).

$$\therefore \ (be+ad)\,m^2 + (e-a)\,(c+a+e)\,m - (ad+be) = 0. \ldots \text{(v)}$$

Eliminating m between (iii) and (v),

$$(b+d)\,(ad+be) + (e-a)^2\,(a+c+e) = 0.$$

Aliter. Since two of the straight lines are at right angles, their equation must be of the form

$$x^2 + \lambda xy - y^2 = 0. \quad \text{(Art. 111.)}$$

The other factor is a quadratic which must be

$$ex^2 + \mu xy - ay^2.$$

Hence the equation given must be equivalent to

$$(x^2 + \lambda xy - y^2)\,(ex^2 + \mu xy - ay^2) = 0.$$

Comparing coefficients, we have

$$d = \lambda e + \mu, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$c = -e - a + \lambda\mu, \quad \ldots\ldots\ldots\ldots\ldots \text{(2)}$$

and

$$b = -\lambda a - \mu. \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

(1) and (3) give $\lambda = -\frac{b+d}{a-e}$ and $\mu = \frac{ad+be}{a-e}$, and then (2) gives $a + c + e = -\frac{(b+d)\,(ad+be)}{(a-e)^2}$.

40. The bisectors of the angles between the lines

$$lx^2 + 2mxy + ny^2 = 0 \text{ are (Art. 112)}$$

$$mx^2 - (l-n)\,xy - my^2 = 0.$$

Hence if $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$,

and $(lx^2 + 2mxy + ny^2)\left[mx^2 - (l-n)xy - my^2\right] = 0$,

represent the same four lines, then

$$\frac{ml}{a} = \frac{2m^2 - l^2 + ln}{b} = \frac{3m(n-l)}{c} = \frac{-2m^2 - ln + n^2}{d} = \frac{-mn}{a}.$$

$$\therefore\ l = -n. \quad \text{Hence}\ \frac{2m^2 - 2l^2}{b} = \frac{-2m^2 + 2l^2}{d};$$

$$\therefore\ b + d = 0,\ \text{and}\ \frac{l}{a} = \frac{-6l}{c}; \quad \therefore\ c + 6a = 0.$$

41. Let the straight lines be

$$(y - m_1x)(y - m_2x)(y - m_3x) = 0,$$

so that $$m_1 + m_2 + m_3 = -\frac{c}{d}, \quad \ldots\ldots\ldots\ldots\ (1)$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{b}{d}, \quad \ldots\ldots\ldots\ldots\ (2)$$

and $$m_1m_2m_3 = -\frac{a}{d}. \quad \ldots\ldots\ldots\ldots\ (3)$$

By the question, $2\tan^{-1} m_2 = \tan^{-1} m_1 + \tan^{-1} m_3$,

$$\therefore\ \frac{2m_2}{1 - m_2^2} = \frac{m_1 + m_3}{1 - m_1m_3} = \frac{-c - dm_2}{d - dm_1m_3},\ \text{from (1)}.$$

$$\therefore\ 2m_2d + 2a = -c + cm_2^2 - dm_2 + dm_2^3,$$

i.e. $$dm_2^3 + cm_2^2 - 3dm_2 - (c + 2a) = 0. \ \ldots\ldots\ (4)$$

But, since $y = m_2x$ satisfies the given equation,

$$\therefore\ dm_2^3 + cm_2^2 + bm_2 + a = 0. \quad \ldots\ldots\ldots\ (5)$$

Subtracting (4) from (5), we have $m_2 = -\dfrac{3a + c}{b + 3d}$.

Substituting this value in (5), we have the given condition.