CHAPTER XXXII.

PROBABILITY.

449. Definition. If an event can happen in a ways and fail in b ways, and each of these ways is equally likely, the **probability**, or the **chance**, of its happening is $\frac{a}{a+b}$, and that of its failing is $\frac{b}{a+b}$.

For instance, if in a lottery there are 7 prizes and 25 blanks, the chance that a person holding 1 ticket will win a prize is $\frac{7}{32}$, and his chance of not winning is $\frac{25}{32}$.

450. The reason for the mathematical definition of probability may be made clear by the following considerations:

If an event can happen in a ways and fail to happen in b ways, and all these ways are equally likely, we can assert that the chance of its happening is to the chance of its failing as a to b. Thus if the chance of its happening is represented by ka, where k is an undetermined constant, then the chance of its failing will be represented by kb.

$$\therefore \text{ chance of happening + chance of failing} = k(a+b).$$

Now the event is certain to happen or to fail; therefore the sum of the chances of happening and failing must represent *certainty*. If therefore we agree to take certainty as our unit, we have

$$1 = k(a+b), \quad \text{or} \quad k = \frac{1}{a+b};$$

$$\therefore \text{ the chance that the event will happen is } \frac{a}{a+b},$$

$$\text{and the chance that the event will not happen is } \frac{b}{a+b}.$$

Cor. If p is the probability of the happening of an event, the probability of its not happening is $1 - p$.

451. Instead of saying that the chance of the happening of an event is $\frac{a}{a+b}$, it is sometimes stated that *the odds are* a *to* b *in favour of the event, or* b *to* a *against the event.*

452. The definition of probability in Art. 449 may be given in a slightly different form which is sometimes useful. If c is the total number of cases, each being equally likely to occur, and of these a are favourable to the event, then the probability that the event will happen is $\frac{a}{c}$, and the probability that it will not happen is $1-\frac{a}{c}$.

Example 1. What is the chance of throwing a number greater than 4 with an ordinary die whose faces are numbered from 1 to 6?

There are 6 possible ways in which the die can fall, and of these two are favourable to the event required;

$$\text{therefore the required chance} = \frac{2}{6} = \frac{1}{3}.$$

Example 2. From a bag containing 4 white and 5 black balls a man draws 3 at random; what are the odds against these being all black?

The total number of ways in which 3 balls can be drawn is ${}^9C_3$, and the number of ways of drawing 3 black balls is ${}^5C_3$; therefore the chance of drawing 3 black balls

$$= \frac{{}^5C_3}{{}^9C_3} = \frac{5 \cdot 4 \cdot 3}{9 \cdot 8 \cdot 7} = \frac{5}{42}.$$

Thus the odds against the event are 37 to 5.

Example 3. Find the chance of throwing at least one ace in a single throw with two dice.

The possible number of cases is 6×6, or 36.

An ace on one die may be associated with any of the 6 numbers on the other die, and the remaining 5 numbers on the first die may each be associated with the ace on the second die; thus the number of favourable cases is 11.

Therefore the required chance is $\frac{11}{36}$.

Or we may reason as follows:

There are 5 ways in which each die can be thrown so as *not* to give an ace; hence 25 throws of the two dice will exclude aces. That is, the chance of *not* throwing one or more aces is $\frac{25}{36}$; so that the chance of throwing one ace at least is $1-\frac{25}{36}$, or $\frac{11}{36}$.

Example 4. Find the chance of throwing more than 15 in one throw with 3 dice.

A throw amounting to 18 must be made up of 6, 6, 6, and this can occur in 1 way; 17 can be made up of 6, 6, 5 which can occur in 3 ways; 16 may be made up of 6, 6, 4 and 6, 5, 5, each of which arrangements can occur in 3 ways.

Therefore the number of favourable cases is

$$1+3+3+3, \quad \text{or} \quad 10.$$

And the total number of cases is 6^3, or 216;

$$\text{therefore the required chance} = \frac{10}{216} = \frac{5}{108}.$$

Example 5. A has 3 shares in a lottery in which there are 3 prizes and 6 blanks; B has 1 share in a lottery in which there is 1 prize and 2 blanks: shew that A's chance of success is to B's as 16 to 7.

A may draw 3 prizes in 1 way;

he may draw 2 prizes and 1 blank in $\frac{3 \cdot 2}{1 \cdot 2} \times 6$ ways;

he may draw 1 prize and 2 blanks in $3 \times \frac{6 \cdot 5}{1 \cdot 2}$ ways;

the sum of these numbers is 64, which is the number of ways in which A can win a prize. Also he can draw 3 tickets in $\frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3}$, or 84 ways;

$$\text{therefore } A\text{'s chance of success} = \frac{64}{84} = \frac{16}{21}.$$

$$B\text{'s chance of success is clearly } \frac{1}{3};$$

$$\text{therefore } A\text{'s chance : } B\text{'s chance} = \frac{16}{21} : \frac{1}{3}$$
$$= 16 : 7.$$

Or we might have reasoned thus: A will get *all blanks* in $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$, or 20 ways; the chance of which is $\frac{20}{84}$, or $\frac{5}{21}$;

$$\text{therefore } A\text{'s chance of success} = 1 - \frac{5}{21} = \frac{16}{21}.$$

453. Suppose that there are a number of events A, B, C,..., of which one must, and only one can, occur; also suppose that a, b, c,... are the numbers of ways respectively in which these events can happen, and that each of these ways is equally likely to occur; it is required to find the chance of each event.

The total number of equally possible ways is $a+b+c+\ldots$, and of these the number favourable to A is a; hence the chance that A will happen is $\dfrac{a}{a+b+c+\dots}$. Similarly the chance that B will happen is $\dfrac{b}{a+b+c+\dots}$; and so on.

454. From the examples we have given it will be seen that the solution of the easier kinds of questions in Probability requires nothing more than a knowledge of the definition of Probability, and the application of the laws of Permutations and Combinations.

EXAMPLES. XXXII. a.

1. In a single throw with two dice find the chances of throwing (1) five, (2) six.

2. From a pack of 52 cards two are drawn at random; find the chance that one is a knave and the other a queen.

3. A bag contains 5 white, 7 black, and 4 red balls: find the chance that three balls drawn at random are all white.

4. If four coins are tossed, find the chance that there should be two heads and two tails.

5. One of two events must happen: given that the chance of the one is two-thirds that of the other, find the odds in favour of the other.

6. If from a pack four cards are drawn, find the chance that they will be the four honours of the same suit.

7. Thirteen persons take their places at a round table, shew that it is five to one against two particular persons sitting together.

8. There are three events A, B, C, one of which must, and only one can, happen; the odds are 8 to 3 against A, 5 to 2 against B: find the odds against C.

9. Compare the chances of throwing 4 with one die, 8 with two dice, and 12 with three dice.

10. In shuffling a pack of cards, four are accidentally dropped; find the chance that the missing cards should be one from each suit.

11. A has 3 shares in a lottery containing 3 prizes and 9 blanks; B has 2 shares in a lottery containing 2 prizes and 6 blanks: compare their chances of success.

12. Shew that the chances of throwing six with 4, 3, or 2 dice respectively are as 1 : 6 : 18.

13. There are three works, one consisting of 3 volumes, one of 4, and the other of 1 volume. They are placed on a shelf at random; prove that the chance that volumes of the same works are all together is $\frac{3}{140}$.

14. A and B throw with two dice; if A throws 9, find B's chance of throwing a higher number.

15. The letters forming the word *Clifton* are placed at random in a row: what is the chance that the two vowels come together?

16. In a hand at whist what is the chance that the 4 kings are held by a specified player?

17. There are 4 shillings and 3 half-crowns placed at random in a line: shew that the chance of the extreme coins being both half-crowns is $\frac{1}{7}$. Generalize this result in the case of m shillings and n half-crowns.

455. We have hitherto considered only those occurrences which in the language of Probability are called *Simple* events. When two or more of these occur in connection with each other, the joint occurrence is called a *Compound* event.

For example, suppose we have a bag containing 5 white and 8 black balls, and two drawings, each of three balls, are made from it successively. If we wish to estimate the chance of drawing first 3 white and then 3 black balls, we should be dealing with a compound event.

In such a case the result of the second drawing might or might not be *dependent* on the result of the first. If the balls are not replaced after being drawn, then if the first drawing gives 3 white balls, the ratio of the black to the white balls remaining is greater than if the first drawing had not given three white; thus the chance of drawing 3 black balls at the second trial is affected by the result of the first. But if the balls are replaced after being drawn, it is clear that the result of the second drawing is not in any way affected by the result of the first.

We are thus led to the following definition:

Events are said to be **dependent** or **independent** according as the occurrence of one does or does not affect the occurrence of the others. Dependent events are sometimes said to be *contingent*.

456. *If there are two independent events the respective probabilities of which are known, to find the probability that both will happen.*

Suppose that the first event may happen in a ways and fail in b ways, all these cases being equally likely; and suppose that the second event may happen in a' ways and fail in b' ways, all these ways being equally likely. Each of the $a+b$ cases may be associated with each of the $a'+b'$ cases, to form $(a+b)(a'+b')$ compound cases all equally likely to occur.

In aa' of these both events happen, in bb' of them both fail, in ab' of them the first happens and the second fails, and in $a'b$ of them the first fails and the second happens. Thus

$\dfrac{aa'}{(a+b)(a'+b')}$ is the chance that both events happen;

$\dfrac{bb'}{(a+b)(a'+b')}$ is the chance that both events fail;

$\dfrac{ab'}{(a+b)(a'+b')}$ is the chance that the first happens and the second fails;

$\dfrac{a'b}{(a+b)(a'+b')}$ is the chance that the first fails and the second happens.

Thus if the respective chances of two independent events are p and p', the chance that both will happen is pp'. Similar reasoning will apply in the case of any number of independent events. Hence it is easy to see that if p_1, p_2, p_3, ... are the respective chances that a number of independent events will separately happen, the chance that they will all happen is $p_1p_2p_3 \dots$; the chance that the two first will happen and the rest fail is $p_1p_2(1-p_3)(1-p_4)\dots$; and similarly for any other particular case.

457. If p is the chance that an event will happen in one trial, the chance that it will happen in any assigned succession of r trials is p^r; this follows from the preceding article by supposing

$$p_1 = p_2 = p_3 = \dots\dots = p.$$

To find the chance that some one at least of the events will happen we proceed thus: the chance that all the events fail is $(1-p_1)(1-p_2)(1-p_3)\dots$, and except in this case some one of the events must happen; hence the required chance is

$$1-(1-p_1)(1-p_2)(1-p_3)\dots.$$

Example 1. Two drawings, each of 3 balls, are made from a bag containing 5 white and 8 black balls, the balls being replaced before the second trial: find the chance that the first drawing will give 3 white, and the second 3 black balls.

The number of ways in which 3 balls may be drawn is ${}^{13}C_3$;

.... 3 white .... ${}^5C_3$;

.... 3 black .... ${}^8C_3$.

Therefore the chance of 3 white at the first trial $= \frac{5 \cdot 4}{1 \cdot 2} \div \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} = \frac{5}{143}$;

and the chance of 3 black at the second trial $= \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \div \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} = \frac{28}{143}$;

therefore the chance of the compound event $= \frac{5}{143} \times \frac{28}{143} = \frac{140}{20449}$.

Example 2. In tossing a coin, find the chance of throwing head and tail alternately in 3 successive trials.

Here the first throw must give either head or tail; the chance that the second gives the opposite to the first is $\frac{1}{2}$, and the chance that the third throw is the same as the first is $\frac{1}{2}$.

Therefore the chance of the compound event $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Example 3. Supposing that it is 9 to 7 against a person A who is now 35 years of age living till he is 65, and 3 to 2 against a person B now 45 living till he is 75; find the chance that one at least of these persons will be alive 30 years hence.

The chance that A will die within 30 years is $\frac{9}{16}$;

the chance that B will die within 30 years is $\frac{3}{5}$;

therefore the chance that both will die is $\frac{9}{16} \times \frac{3}{5}$, or $\frac{27}{80}$;

therefore the chance that both will not be dead, that is that one at least will be alive, is $1 - \frac{27}{80}$, or $\frac{53}{80}$.

458. By a slight modification of the meaning of the symbols in Art. 456, we are enabled to estimate the probability of the concurrence of two *dependent* events. For suppose that *when the first event has happened*, a' denotes the number of ways in which the second event can follow, and b' the number of ways in which it will not follow; then the number of ways in which the two

events can happen together is aa', and the probability of their concurrence is $\dfrac{aa'}{(a+b)(a'+b')}$.

Thus if p is the probability of the first event, and p' the contingent probability that the second will follow, the probability of the concurrence of the two events is pp'.

Example 1. In a hand at whist find the chance that a specified player holds both the king and queen of trumps.

Denote the player by A; then the chance that A has the king is clearly $\frac{13}{52}$; for this particular card can be dealt in 52 different ways, 13 of which fall to A. The chance that, when he has the king, he can also hold the queen is then $\frac{12}{51}$; for the queen can be dealt in 51 ways, 12 of which fall to A.

Therefore the chance required $=\dfrac{13}{52}\times\dfrac{12}{51}=\dfrac{1}{17}$.

Or we might reason as follows:

The number of ways in which the king and the queen can be dealt to A is equal to the number of permutations of 13 things 2 at a time, or $13 \cdot 12$. And similarly the total number of ways in which the king and queen can be dealt is $52 \cdot 51$.

Therefore the chance $=\dfrac{13 \cdot 12}{52 \cdot 51}=\dfrac{1}{17}$, as before.

Example 2. Two drawings, each of 3 balls, are made from a bag containing 5 white and 8 black balls, *the balls not being replaced before the second trial:* find the chance that the first drawing will give 3 white and the second 3 black balls.

At the first trial, 3 balls may be drawn in ${}^{13}C_3$ ways;
and 3 white balls may be drawn in ${}^{5}C_3$ ways;
therefore the chance of 3 white at first trial $=\dfrac{5 \cdot 4}{1 \cdot 2}\div\dfrac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3}=\dfrac{5}{143}$.

When 3 white balls have been drawn and removed, the bag contains 2 white and 8 black balls;

therefore at the second trial 3 balls may be drawn in ${}^{10}C_3$ ways;
and 3 black balls may be drawn in ${}^{8}C_3$ ways;
therefore the chance of 3 black at the second trial

$$=\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3}\div\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3}=\frac{7}{15};$$

therefore the chance of the compound event

$$=\frac{5}{143}\times\frac{7}{15}=\frac{7}{429}.$$

The student should compare this solution with that of Ex. 1, Art. 457.

459. *If an event can happen in two or more different ways* which are mutually exclusive, *the chance that it will happen is the sum of the chances of its happening in these different ways.*

This is sometimes regarded as a self-evident proposition arising immediately out of the definition of probability. It may, however, be proved as follows:

Suppose the event can happen in two ways which cannot concur; and let $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$ be the chances of the happening of the event in these two ways respectively. Then out of $b_1 b_2$ cases there are $a_1 b_2$ in which the event may happen in the first way, and $a_2 b_1$ ways in which the event may happen in the second; *and these ways cannot concur.* Therefore in all, out of $b_1 b_2$ cases there are $a_1 b_2 + a_2 b_1$ cases favourable to the event; hence the chance that the event will happen in one or other of the two ways is

$$\frac{a_1 b_2 + a_2 b_1}{b_1 b_2} = \frac{a_1}{b_1} + \frac{a_2}{b_2}.$$

Similar reasoning will apply whatever be the number of exclusive ways in which the event can happen.

Hence if an event can happen in n ways which are mutually exclusive, and if $p_1, p_2, p_3, \dots p_n$ are the probabilities that the event will happen in these different ways respectively, the probability that it will happen in some one of these ways is

$$p_1 + p_2 + p_3 + \dots\dots + p_n.$$

Example 1. Find the chance of throwing 9 at least in a single throw with two dice.

9 can be made up in 4 ways, and thus the chance of throwing 9 is $\frac{4}{36}$.

10 can be made up in 3 ways, and thus the chance of throwing 10 is $\frac{3}{36}$.

11 can be made up in 2 ways, and thus the chance of throwing 11 is $\frac{2}{36}$.

12 can be made up in 1 way, and thus the chance of throwing 12 is $\frac{1}{36}$.

Now the chance of throwing a number not less than 9 is the sum of these separate chances;

$$\therefore \text{ the required chance} = \frac{4+3+2+1}{36} = \frac{5}{18}.$$

Example 2. One purse contains 1 sovereign and 3 shillings, a second purse contains 2 sovereigns and 4 shillings, and a third contains 3 sovereigns and 1 shilling. If a coin is taken out of one of the purses selected at random, find the chance that it is a sovereign.

Since each purse is equally likely to be taken, the chance of selecting the first is $\frac{1}{3}$; and the chance of then drawing a sovereign is $\frac{1}{4}$; hence the chance of drawing a sovereign so far as it depends upon the first purse is $\frac{1}{3} \times \frac{1}{4}$, or $\frac{1}{12}$. Similarly the chance of drawing a sovereign so far as it depends on the second purse is $\frac{1}{3} \times \frac{2}{6}$, or $\frac{1}{9}$; and from the third purse the chance of drawing a sovereign is $\frac{1}{3} \times \frac{3}{4}$, or $\frac{1}{4}$;

$$\therefore \text{ the required chance} = \frac{1}{12} + \frac{1}{9} + \frac{1}{4} = \frac{4}{9}.$$

460. In the preceding article we have seen that the probability of an event may sometimes be considered as the sum of the probabilities of two or more separate events; but it is very important to notice that the probability of one or other of a series of events is the sum of the probabilities of the separate events *only when the events are mutually exclusive*, that is, when the occurrence of one is incompatible with the occurrence of any of the others.

Example. From 20 tickets marked with the first 20 numerals, one is drawn at random: find the chance that it is a multiple of 3 or of 7.

The chance that the number is a multiple of 3 is $\frac{6}{20}$, and the chance that it is a multiple of 7 is $\frac{2}{20}$; and *these events are mutually exclusive*, hence the required chance is
$$\frac{6}{20} + \frac{2}{20}, \text{ or } \frac{2}{5}.$$

But if the question had been: *find the chance that the number is a multiple of* 3 *or of* 5, it would have been incorrect to reason as follows:

Because the chance that the number is a multiple of 3 is $\frac{6}{20}$, and the chance that the number is a multiple of 5 is $\frac{4}{20}$, therefore the chance that it is a multiple of 3 or 5 is $\frac{6}{20} + \frac{4}{20}$, or $\frac{1}{2}$. For the number on the ticket might be a multiple *both* of 3 and of 5, so that the two events considered are not mutually exclusive.

461. It should be observed that the distinction between simple and compound events is in many cases a purely artificial

one; in fact it often amounts to nothing more than a distinction between two different modes of viewing the same occurrence.

Example. A bag contains 5 white and 7 black balls; if two balls are drawn what is the chance that one is white and the other black?

(i) Regarding the occurrence as a simple event, the chance

$$=(5\times 7)\div {}^{12}C_2=\frac{35}{66}.$$

(ii) The occurrence may be regarded as the happening of one or other of the two following compound events:

(1) drawing a white and then a black ball, the chance of which is

$$\frac{5}{12}\times\frac{7}{11} \text{ or } \frac{35}{132};$$

(2) drawing a black and then a white ball, the chance of which is

$$\frac{7}{12}\times\frac{5}{11}, \text{ or } \frac{35}{132}.$$

And since these events are mutually exclusive, the required chance

$$=\frac{35}{132}+\frac{35}{132}=\frac{35}{66}.$$

It will be noticed that we have here assumed that the chance of drawing two specified balls successively is the same as if they were drawn simultaneously. A little consideration will shew that this must be the case.

EXAMPLES. XXXII. b.

1. What is the chance of throwing an ace in the first only of two successive throws with an ordinary die?

2. Three cards are drawn at random from an ordinary pack: find the chance that they will consist of a knave, a queen, and a king.

3. The odds against a certain event are 5 to 2, and the odds in favour of another event independent of the former are 6 to 5: find the chance that one at least of the events will happen.

4. The odds against A solving a certain problem are 4 to 3, and the odds in favour of B solving the same problem are 7 to 5: what is the chance that the problem will be solved if they both try?

5. What is the chance of drawing a sovereign from a purse one compartment of which contains 3 shillings and 2 sovereigns, and the other 2 sovereigns and 1 shilling?

6. A bag contains 17 counters marked with the numbers 1 to 17. A counter is drawn and replaced; a second drawing is then made: what is the chance that the first number drawn is even and the second odd?

7. Four persons draw each a card from an ordinary pack: find the chance (1) that a card is of each suit, (2) that no two cards are of equal value.

8. Find the chance of throwing six with a single die at least once in five trials.

9. The odds that a book will be favourably reviewed by three independent critics are 5 to 2, 4 to 3, and 3 to 4 respectively; what is the probability that of the three reviews a majority will be favourable?

10. A bag contains 5 white and 3 black balls, and 4 are successively drawn out and not replaced; what is the chance that they are alternately of different colours?

11. In three throws with a pair of dice, find the chance of throwing doublets at least once.

12. If 4 whole numbers taken at random are multiplied together shew that the chance that the last digit in the product is 1, 3, 7, or 9 is $\frac{16}{625}$.

13. In a purse are 10 coins, all shillings except one which is a sovereign; in another are ten coins all shillings. Nine coins are taken from the former purse and put into the latter, and then nine coins are taken from the latter and put into the former: find the chance that the sovereign is still in the first purse.

14. If two coins are tossed 5 times, what is the chance that there will be 5 heads and 5 tails?

15. If 8 coins are tossed, what is the chance that one and only one will turn up head?

16. A, B, C in order cut a pack of cards, replacing them after each cut, on condition that the first who cuts a spade shall win a prize: find their respective chances.

17. A and B draw from a purse containing 3 sovereigns and 4 shillings: find their respective chances of first drawing a sovereign, the coins when drawn not being replaced.

18. A party of n persons sit at a round table, find the odds against two specified individuals sitting next to each other.

19. A is one of 6 horses entered for a race, and is to be ridden by one of two jockeys B and C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win; if C rides A, his chance is trebled: what are the odds against his winning?

20. If on an average 1 vessel in every 10 is wrecked, find the chance that out of 5 vessels expected 4 at least will arrive safely.

462. *The probability of the happening of an event in one trial being known, required the probability of its happening once, twice, three times, ... exactly in* n *trials.*

Let p be the probability of the happening of the event in a single trial, and let $q = 1 - p$; then the probability that the event will happen exactly r times in n trials is the $(r+1)^{\text{th}}$ term in the expansion of $(q+p)^n$.

For if we select any particular set of r trials out of the total number n, the chance that the event will happen in every one of *these* r trials and fail in all the rest is $p^r q^{n-r}$ [Art. 456], and as a set of r trials can be selected in ${}^nC_r$ ways, all of which are equally applicable to the case in point, the required chance is

$${}^nC_r p^r q^{n-r}.$$

If we expand $(p+q)^n$ by the Binomial Theorem, we have

$$p^n + {}^nC_1 p^{n-1} q + {}^nC_2 p^{n-2} q^2 + \dots + {}^nC_{n-r} p^r q^{n-r} + \dots + q^n;$$

thus the terms of this series will represent respectively the probabilities of the happening of the event exactly n times, $n-1$ times, $n-2$ times, ... in n trials.

463. If the event happens n times, or fails only once, twice, ... $(n-r)$ times, it happens r times or more; therefore the chance that it happens *at least* r times in n trials is

$$p^n + {}^nC_1 p^{n-1} q + {}^nC_2 p^{n-2} q^2 + \dots + {}^nC_{n-r} p^r q^{n-r},$$

or the sum of the first $n - r + 1$ terms of the expansion of $(p+q)^n$.

Example 1. In four throws with a pair of dice, what is the chance of throwing doublets twice at least?

In a single throw the chance of doublets is $\frac{6}{36}$, or $\frac{1}{6}$; and the chance of failing to throw doublets is $\frac{5}{6}$. Now the required event follows if doublets are thrown four times, three times, or twice; therefore the required chance is the sum of the first three terms of the expansion of $\left(\frac{1}{6} + \frac{5}{6}\right)^4$.

Thus the chance $= \frac{1}{6^4}(1 + 4 \cdot 5 + 6 \cdot 5^2) = \frac{19}{144}$.

Example 2. A bag contains a certain number of balls, some of which are white; a ball is drawn and replaced, another is then drawn and replaced; and so on: if p is the chance of drawing a white ball in a single trial, find the number of white balls that is most likely to have been drawn in n trials.

The chance of drawing exactly r white balls is ${}^nC_r p^r q^{n-r}$, and we have to find for what value of r this expression is greatest.

Now $${}^nC_r p^r q^{n-r} > {}^nC_{r-1} p^{r-1} q^{n-(r-1)},$$

so long as $$(n-r+1)p > rq,$$

or $$(n+1)p > (p+q)r.$$

But $p+q=1$; hence the required value of r is the greatest integer in $p(n+1)$.

If n is such that pn is an integer, the most likely case is that of pn successes and qn failures.

464. Suppose that there are n tickets in a lottery for a prize of £x; then since each ticket is equally likely to win the prize, and a person who possessed all the tickets *must* win, the money value of each ticket is £$\frac{x}{n}$; in other words this would be a fair sum to pay for each ticket; hence a person who possessed r tickets might reasonably expect £$\frac{rx}{n}$ as the price to be paid for his tickets by any one who wished to buy them; that is, he would estimate £$\frac{r}{n}x$ as the worth of his chance. It is convenient then to introduce the following definition:

If p represents a person's chance of success in any venture and M the sum of money which he will receive in case of success, the sum of money denoted by pM is called his **expectation.**

465. In the same way that *expectation* is used in reference to a person, we may conveniently use the phrase *probable value* applied to things.

Example 1. One purse contains 5 shillings and 1 sovereign: a second purse contains 6 shillings. Two coins are taken from the first and placed in the second; then 2 are taken from the second and placed in the first: find the probable value of the contents of each purse.

The chance that the sovereign is in the first purse is equal to the sum of the chances that it has moved twice and that it has not moved at all;

$$\text{that is, the chance} = \frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot 1 = \frac{3}{4}.$$

$\therefore$ the chance that the sovereign is in the second purse $= \frac{1}{4}$.

Hence the probable value of the first purse

$$= \frac{3}{4} \text{ of } 25s. + \frac{1}{4} \text{ of } 6s. = £1.\ 0s.\ 3d.$$

$\therefore$ the probable value of the second purse

$$= 31s. - 20\tfrac{1}{4}s. = 10s.\ 9d.$$

Or the problem may be solved as follows:

The probable value of the coins removed

$$= \frac{1}{3} \text{ of } 25s. = 8\tfrac{1}{3}s.;$$

the probable value of the coins brought back

$$= \frac{1}{4} \text{ of } (6s. + 8\tfrac{1}{3}s.) = 3\tfrac{7}{12}s.;$$

$\therefore$ the probable value of the first purse

$$= (25 - 8\tfrac{1}{3} + 3\tfrac{7}{12}) \text{ shillings} = £1.\ 0s.\ 3d., \text{ as before.}$$

Example 2. A and B throw with one die for a stake of £11 which is to be won by the player who first throws 6. If A has the first throw, what are their respective expectations?

In his first throw A's chance is $\frac{1}{6}$; in his second it is $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$, because each player must have failed once before A can have a second throw; in his third throw his chance is $\left(\frac{5}{6}\right)^4 \times \frac{1}{6}$ because each player must have failed twice; and so on.

Thus A's chance is the sum of the infinite series

$$\frac{1}{6}\left\{1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots\ldots\right\}.$$

Similarly B's chance is the sum of the infinite series

$$\frac{5}{6} \cdot \frac{1}{6}\left\{1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots\ldots\right\};$$

$\therefore$ A's chance is to B's as 6 is to 5; their respective chances are therefore $\frac{6}{11}$ and $\frac{5}{11}$, and their expectations are £6 and £5 respectively.

466. We shall now give two problems which lead to useful and interesting results.

Example 1. Two players A and B want respectively m and n points of winning a set of games; their chances of winning a single game are p and q respectively, where the sum of p and q is unity; the stake is to belong to the player who first makes up his set: determine the probabilities in favour of each player.

Suppose that A wins in *exactly* $m+r$ games; to do this he must win the last game and $m-1$ out of the preceding $m+r-1$ games. The chance of this is ${}^{m+r-1}C_{m-1}\, p^{m-1} q^r p$, or ${}^{m+r-1}C_{m-1}\, p^m q^r$.

Now the set will necessarily be decided in $m+n-1$ games, and A may win his m games in *exactly* m games, or $m+1$ games, ..., or $m+n-1$ games; therefore we shall obtain the chance that A wins the set by giving to r the values 0, 1, 2, ... $n-1$ in the expression ${}^{m+r-1}C_{m-1}\, p^m q^r$. Thus A's chance is

$$p^m \left\{1 + mq + \frac{m(m+1)}{1 \cdot 2} q^2 + \dots + \frac{\lfloor m+n-2}{\lfloor m-1 \, \lfloor n-1} q^{n-1}\right\};$$

similarly B's chance is

$$q^n \left\{1 + np + \frac{n(n+1)}{1 \cdot 2} p^2 + \dots + \frac{\lfloor m+n-2}{\lfloor m-1 \, \lfloor n-1} p^{m-1}\right\}.$$

This question is known as the "Problem of Points," and has engaged the attention of many of the most eminent mathematicians since the time of Pascal. It was originally proposed to Pascal by the Chevalier de Méré in 1654, and was discussed by Pascal and Fermat, but they confined themselves to the case in which the players were supposed to be of equal skill: their results were also exhibited in a different form. The formulæ we have given are assigned to Montmort, as they appear for the first time in a work of his published in 1714. The same result was afterwards obtained in different ways by Lagrange and Laplace, and by the latter the problem was treated very fully under various modifications.

Example 2. There are n dice with f faces marked from 1 to f; if these are thrown at random, what is the chance that the sum of the numbers exhibited shall be equal to p?

Since any one of the f faces may be exposed on any one of the n dice, the number of ways in which the dice may fall is f^n.

Also the number of ways in which the numbers thrown will have p for their sum is equal to the coefficient of x^p in the expansion of

$$(x^1 + x^2 + x^3 + \dots + x^f)^n;$$

for this coefficient arises out of the different ways in which n of the indices 1, 2, 3, ... f can be taken so as to form p by addition.

Now the above expression $=x^n(1+x+x^2+\ldots+x^{f-1})^n$

$$=x^n\left(\frac{1-x^f}{1-x}\right)^n.$$

We have therefore to find the coefficient of x^{p-n} in the expansion of

$$(1-x^f)^n(1-x)^{-n}.$$

Now $$(1-x^f)^n=1-nx^f+\frac{n(n-1)}{1.2}x^{2f}-\frac{n(n-1)(n-2)}{1.2.3}x^{3f}+\ldots;$$

and $$(1-x)^{-n}=1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\ldots$$

Multiply these series together and pick out the coefficient of x^{p-n} in the product; we thus obtain

$$\frac{n(n+1)\ldots(p-1)}{\lfloor p-n}-n.\frac{n(n+1)\ldots(p-f-1)}{\lfloor p-n-f}+\frac{n(n-1)}{1.2}.\frac{n(n+1)\ldots(p-2f-1)}{\lfloor p-n-2f}-\ldots,$$

where the series is to continue so long as no negative factors appear. The required probability is obtained by dividing this series by f^n.

This problem is due to De Moivre and was published by him in 1730; it illustrates a method of frequent utility.

Laplace afterwards obtained the same formula, but in a much more laborious manner; he applied it in an attempt to demonstrate the existence of a primitive cause which has made the planets to move in orbits close to the ecliptic, and in the same direction as the earth round the sun. On this point the reader may consult Todhunter's *History of Probability*, Art. 987.

EXAMPLES. XXXII. c.

1. In a certain game A's skill is to B's as 3 to 2: find the chance of A winning 3 games at least out of 5.

2. A coin whose faces are marked 2, 3 is thrown 5 times: what is the chance of obtaining a total of 12?

3. In each of a set of games it is 2 to 1 in favour of the winner of the previous game: what is the chance that the player who wins the first game shall win three at least of the next four?

4. There are 9 coins in a bag, 5 of which are sovereigns and the rest are unknown coins of equal value; find what they must be if the probable value of a draw is 12 shillings.

5. A coin is tossed n times, what is the chance that the head will present itself an odd number of times?

6. From a bag containing 2 sovereigns and 3 shillings a person is allowed to draw 2 coins indiscriminately; find the value of his expectation.

7. Six persons throw for a stake, which is to be won by the one who first throws head with a penny; if they throw in succession, find the chance of the fourth person.

8. Counters marked 1, 2, 3 are placed in a bag, and one is withdrawn and replaced. The operation being repeated three times, what is the chance of obtaining a total of 6?

9. A coin whose faces are marked 3 and 5 is tossed 4 times: what are the odds against the sum of the numbers thrown being less than 15?

10. Find the chance of throwing 10 exactly in one throw with 3 dice.

11. Two players of equal skill, A and B, are playing a set of games; they leave off playing when A wants 3 points and B wants 2. If the stake is £16, what share ought each to take?

12. A and B throw with 3 dice: if A throws 8, what is B's chance of throwing a higher number?

13. A had in his pocket a sovereign and four shillings; taking out two coins at random he promises to give them to B and C. What is the worth of C's expectation?

14. In five throws with a single die what is the chance of throwing (1) three aces *exactly*, (2) three aces at least.

15. A makes a bet with B of 5*s*. to 2*s*. that in a single throw with two dice he will throw seven before B throws four. Each has a pair of dice and they throw simultaneously until one of them wins: find B's expectation.

16. A person throws two dice, one the common cube, and the other a regular tetrahedron, the number on the lowest face being taken in the case of the tetrahedron; what is the chance that the sum of the numbers thrown is not less than 5?

17. A bag contains a coin of value M, and a number of other coins whose aggregate value is m. A person draws one at a time till he draws the coin M: find the value of his expectation.

18. If $6n$ tickets numbered 0, 1, 2, ... $6n-1$ are placed in a bag, and three are drawn out, shew that the chance that the sum of the numbers on them is equal to $6n$ is

$$\frac{3n}{(6n-1)(6n-2)}.$$

*INVERSE PROBABILITY.

*467. In all the cases we have hitherto considered it has been supposed that our knowledge of the causes which may produce a certain event is such as to enable us to determine the chance of the happening of the event. We have now to consider problems of a different character. For example, if it is known that an event has happened in consequence of some one of a certain number of causes, it may be required to estimate the probability of each cause being the true one, and thence to deduce the probability of future events occurring under the operation of the same causes.

*468. Before discussing the general case we shall give a numerical illustration.

Suppose there are two purses, one containing 5 sovereigns and 3 shillings, the other containing 3 sovereigns and 1 shilling, and suppose that a sovereign has been drawn: it is required to find the chance that it came from the first or second purse.

Consider a very large number N of trials; then, since before the event each of the purses is equally likely to be taken, we may assume that the first purse would be chosen in $\frac{1}{2}N$ of the trials, and in $\frac{5}{8}$ of these a sovereign would be drawn; thus a sovereign would be drawn $\frac{5}{8} \times \frac{1}{2}N$, or $\frac{5}{16}N$ times from the first purse.

The second purse would be chosen in $\frac{1}{2}N$ of the trials, and in $\frac{3}{4}$ of these a sovereign would be drawn; thus a sovereign would be drawn $\frac{3}{8}N$ times from the second purse.

Now N is very large but is otherwise an arbitrary number; let us put $N = 16n$; thus a sovereign would be drawn $5n$ times from the first purse, and $6n$ times from the second purse; that is, out of the $11n$ times in which a sovereign is drawn it comes from the first purse $5n$ times, and from the second purse $6n$

times. Hence the probability that the sovereign came from the first purse is $\frac{5}{11}$, and the probability that it came from the second is $\frac{6}{11}$.

*469. It is important that the student's attention should be directed to the nature of the assumption that has been made in the preceding article. Thus, to take a particular instance, although in 60 throws with a perfectly symmetrical die it may not happen that ace is thrown exactly 10 times, yet it will doubtless be at once admitted that if the number of throws is continually increased the ratio of the number of aces to the number of throws will tend more and more nearly to the limit $\frac{1}{6}$. There is no reason why one face should appear oftener than another; hence in the long run the number of times that each of the six faces will have appeared will be approximately equal.

The above instance is a particular case of a general theorem which is due to James Bernoulli, and was first given in the *Ars Conjectandi*, published in 1713, eight years after the author's death. Bernoulli's theorem may be enunciated as follows:

If p *is the probability that an event happens in a single trial, then if the number of trials is indefinitely increased, it becomes a certainty that the limit of the ratio of the number of successes to the number of trials is equal to* p*; in other words, if the number of trials is* N, *the number of successes may be taken to be* pN.

See Todhunter's *History of Probability*, Chapter VII. A proof of Bernoulli's theorem is given in the article *Probability* in the *Encyclopædia Britannica*.

*470. *An observed event has happened through some one of a number of mutually exclusive causes: required to find the probability of any assigned cause being the true one.*

Let there be n causes, and *before the event took place* suppose that the probability of the existence of these causes was estimated at $P_1, P_2, P_3, \dots P_n$. Let p_r denote the probability that when the r^{th} cause exists the event will follow: *after the event has occurred* it is required to find the probability that the r^{th} cause was the true one.

Consider a very great number N of trials; then the first cause exists in P_1N of these, and out of this number the event follows in p_1P_1N; similarly there are p_2P_2N trials in which the event follows from the second cause; and so on for each of the other causes. Hence the number of trials in which the event follows is

$$(p_1P_1 + p_2P_2 + \dots + p_nP_n)\,N, \text{ or } N\Sigma(pP);$$

and the number in which the event was due to the r^{th} cause is p_rP_rN; hence *after* the event the probability that the r^{th} cause was the true one is

$$p_rP_rN \div N\Sigma(pP);$$

that is, the probability that the event was produced by the r^{th} cause is

$$\frac{p_rP_r}{\Sigma(pP)}.$$

*471. It is necessary to distinguish clearly between the probability of the existence of the several causes estimated *before* the event, and the probability *after the event has happened* of any assigned cause being the true one. The former are usually called *a priori* probabilities and are represented by $P_1, P_2, P_3, \dots P_n$; the latter are called *a posteriori* probabilities, and if we denote them by $Q_1, Q_2, Q_3, \dots Q_n$, we have proved that

$$Q_r = \frac{p_rP_r}{\Sigma(pP)};$$

where p_r denotes the probability of the event on the hypothesis of the existence of the r^{th} cause.

From this result it appears that $\Sigma(Q) = 1$, which is otherwise evident as the event has happened from one and only one of the causes.

We shall now give another proof of the theorem of the preceding article which does not depend on the principle enunciated in Art. 469.

*472. *An observed event has happened through some one of a number of mutually exclusive causes: required to find the probability of any assigned cause being the true one.*

Let there be n causes, and *before the event took place* suppose that the probability of the existence of these causes was estimated at $P_1, P_2, P_3, \dots P_n$. Let p_r denote the probability that when the r^{th} cause exists the event will follow; then the *antecedent* probability that the event would follow from the r^{th} cause is p_rP_r.

Let Q_r be the *a posteriori* probability that the r^{th} cause was the true one; then the probability that the r^{th} cause was the true one is proportional to the probability that, if in existence, this cause would produce the event;

$$\therefore \quad \frac{Q_1}{p_1P_1} = \frac{Q_2}{p_2P_2} = \dots = \frac{Q_n}{p_nP_n} = \frac{\Sigma(Q)}{\Sigma(pP)} = \frac{1}{\Sigma(pP)};$$

$$\therefore \quad Q_r = \frac{p_rP_r}{\Sigma(pP)}.$$

Hence it appears that in the present class of problems the product P_rp_r, will have to be correctly estimated as a first step; in many cases, however, it will be found that P_1, P_2, P_3, ... are all equal, and the work is thereby much simplified.

Example. There are 3 bags each containing 5 white balls and 2 black balls, and 2 bags each containing 1 white ball and 4 black balls: a black ball having been drawn, find the chance that it came from the first group.

Of the five bags, 3 belong to the first group and 2 to the second; hence

$$P_1 = \frac{3}{5}, \quad P_2 = \frac{2}{5}.$$

If a bag is selected from the first group the chance of drawing a black ball is $\frac{2}{7}$; if from the second group the chance is $\frac{4}{5}$; thus $p_1 = \frac{2}{7}$, $p_2 = \frac{4}{5}$;

$$\therefore p_1P_1 = \frac{6}{35}, \quad p_2P_2 = \frac{8}{25}.$$

Hence the chance that the black ball came from one of the first group is

$$\frac{6}{35} \div \left(\frac{6}{35} + \frac{8}{25}\right) = \frac{15}{43}.$$

*473. When an event has been observed, we are able by the method of Art. 472 to estimate the probability of any particular cause being the true one; we may then estimate the probability of the event happening in a second trial, or we may find the probability of the occurrence of some other event.

For example, p_r is the chance that the event will happen from the r^{th} cause if in existence, and the chance that the r^{th} cause is the true one is Q_r; hence on a second trial the chance that the event will happen from the r^{th} cause is p_rQ_r. Therefore the chance that the event will happen from some one of the causes on a second trial is $\Sigma(pQ)$.

Example. A purse contains 4 coins which are either sovereigns or shillings; 2 coins are drawn and found to be shillings: if these are replaced what is the chance that another drawing will give a sovereign?

This question may be interpreted in two ways, which we shall discuss separately.

I. If we consider that all numbers of shillings are *a priori* equally likely, we shall have three hypotheses; for (i) all the coins may be shillings, (ii) three of them may be shillings, (iii) only two of them may be shillings.

Here $$P_1 = P_2 = P_3;$$

also $$p_1 = 1, \quad p_2 = \frac{1}{2}, \quad p_3 = \frac{1}{6}.$$

Hence probability of first hypothesis $= 1 \div \left(1 + \frac{1}{2} + \frac{1}{6}\right) = \frac{6}{10} = Q_1$,

probability of second hypothesis $= \frac{1}{2} \div \left(1 + \frac{1}{2} + \frac{1}{6}\right) = \frac{3}{10} = Q_2$,

probability of third hypothesis $= \frac{1}{6} \div \left(1 + \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{10} = Q_3$.

Therefore the probability that another drawing will give a sovereign

$$= (Q_1 \times 0) + \left(Q_2 \times \frac{1}{4}\right) + \left(Q_3 \times \frac{2}{4}\right)$$

$$= \frac{1}{4} \cdot \frac{3}{10} + \frac{2}{4} \cdot \frac{1}{10} = \frac{5}{40} = \frac{1}{8}.$$

II. If each coin is equally likely to be a shilling or a sovereign, by taking the terms in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^4$, we see that the chance of four shillings is $\frac{1}{16}$, of three shillings is $\frac{4}{16}$, of two shillings is $\frac{6}{16}$; thus

$$P_1 = \frac{1}{16}, \quad P_2 = \frac{4}{16}, \quad P_3 = \frac{6}{16};$$

also, as before, $$p_1 = 1, \quad p_2 = \frac{1}{2}, \quad p_3 = \frac{1}{6}.$$

Hence $$\frac{Q_1}{6} = \frac{Q_2}{12} = \frac{Q_3}{6} = \frac{Q_1 + Q_2 + Q_3}{24} = \frac{1}{24}.$$

Therefore the probability that another drawing will give a sovereign

$$= (Q_1 \times 0) + \left(Q_2 \times \frac{1}{4}\right) + \left(Q_3 \times \frac{2}{4}\right)$$

$$= \frac{1}{8} + \frac{2}{16} = \frac{1}{4}.$$

*474. We shall now shew how the theory of probability may be applied to estimate the truth of statements attested by witnesses whose credibility is assumed to be known. We shall suppose that each witness states what he believes to be the truth, whether his statement is the result of observation, or deduction, or experiment; so that any mistake or falsehood must be attributed to errors of judgment and not to wilful deceit.

The class of problems we shall discuss furnishes a useful intellectual exercise, and although the results cannot be regarded as of any practical importance, it will be found that they confirm the verdict of common sense.

*475. When it is asserted that the probability that a person speaks the truth is p, it is meant that a large number of statements made by him has been examined, and that p is the ratio of those which are true to the whole number.

*476. Two independent witnesses, A and B, whose probabilities of speaking the truth are p and p' respectively, agree in making a certain statement: what is the probability that the statement is true?

Here the observed event is the fact that A and B make the same statement. Before the event there are four hypotheses; for A and B may both speak truly; or A may speak truly, B falsely; or A may speak falsely, B truly; or A and B may both speak falsely. The probabilities of these four hypotheses are

$$pp', \quad p(1-p'), \quad p'(1-p), \quad (1-p)(1-p') \text{ respectively.}$$

Hence after the observed event, in which A and B make the same statement, the probability that the statement is true is to the probability that it is false as pp' to $(1-p)(1-p')$; that is, the probability that the joint statement is true is

$$\frac{pp'}{pp'+(1-p)(1-p')}.$$

Similarly if a third person, whose probability of speaking the truth is p'', makes the same statement, the probability that the statement is true is

$$\frac{pp'p''}{pp'p''+(1-p)(1-p')(1-p'')};$$

and so on for any number of persons.

*477. In the preceding article it has been supposed that we have no knowledge of the event except the statement made by A and B; if we have information from other sources as to the probability of the truth or falsity of the statement, this must be taken into account in estimating the probability of the various hypotheses.

For instance, if A and B agree in stating a fact, of which the *a priori* probability is P, then we should estimate the probability of the truth and falsity of the statement by

$$Ppp' \text{ and } (1-P)(1-p)(1-p') \text{ respectively.}$$

Example. There is a raffle with 12 tickets and two prizes of £9 and £3. A, B, C, whose probabilities of speaking the truth are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$ respectively, report the result to D, who holds one ticket. A and B assert that he has won the £9 prize, and C asserts that he has won the £3 prize; what is D's expectation?

Three cases are possible; D may have won £9, £3, or *nothing*, for A, B, C may all have spoken falsely.

Now with the notation of Art. 472, we have the *a priori* probabilities

$$P_1=\frac{1}{12}, \quad P_2=\frac{1}{12}, \quad P_3=\frac{10}{12};$$

also $p_1=\frac{1}{2}\times\frac{2}{3}\times\frac{2}{5}=\frac{4}{30}$, $p_2=\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}=\frac{3}{30}$, $p_3=\frac{1}{2}\times\frac{1}{3}\times\frac{2}{5}=\frac{2}{30}$;

$$\therefore \frac{Q_1}{4}=\frac{Q_2}{3}=\frac{Q_3}{20}=\frac{1}{27};$$

hence D's expectation $=\frac{4}{27}$ of £9 $+\frac{3}{27}$ of £3 $=$ £1. 13*s*. 4*d*.

*478. With respect to the results proved in Art. 476, it should be noticed that it was assumed that the statement can be made in two ways only, so that if all the witnesses tell falsehoods they agree in telling the *same* falsehood.

If this is not the case, let us suppose that c is the chance that the two witnesses A and B will agree in telling the same falsehood; then the probability that the statement is true is to the probability that it is false as pp' to $c(1-p)(1-p')$.

As a general rule, it is extremely improbable that two independent witnesses will tell the same falsehood, so that c is usually very small; also it is obvious that the quantity c becomes smaller as the number of witnesses becomes greater. These considerations increase the probability that a statement asserted by two or more independent witnesses is true, even though the credibility of each witness is small.

Example. A speaks truth 3 times out of 4, and B 7 times out of 10; they both assert that a white ball has been drawn from a bag containing 6 balls all of different colours: find the probability of the truth of the assertion.

There are two hypotheses; (i) their coincident testimony is true, (ii) it is false.

Here $$P_1=\frac{1}{6}, \quad P_2=\frac{5}{6};$$

$$p_1=\frac{3}{4}\times\frac{7}{10}, \quad p_2=\frac{1}{25}\times\frac{1}{4}\times\frac{3}{10};$$

for in estimating p_2 we must take into account the chance that A and B will both select the white ball when it has not been drawn; this chance is

$$\frac{1}{5}\times\frac{1}{5} \text{ or } \frac{1}{25}.$$

Now the probabilities of the two hypotheses are as P_1p_1 to P_2p_2, and therefore as 35 to 1; thus the probability that the statement is true is $\frac{35}{36}$.

*479. The cases we have considered relate to the probability of the truth of *concurrent* testimony; the following is a case of *traditionary* testimony.

If A states that a certain event took place, having received an account of its occurrence or non-occurrence from B, what is the probability that the event did take place?

The event happened (1) if they both spoke the truth, (2) if they both spoke falsely; and the event did not happen if only one of them spoke the truth.

Let p, p' denote the probabilities that A and B speak the truth; then the probability that the event did take place is

$$pp'+(1-p)(1-p'),$$

and the probability that it did not take place is

$$p(1-p')+p'(1-p).$$

*480. The solution of the preceding article is that which has usually been given in text-books; but it is open to serious objections, for the assertion that the given event happened if both A and B spoke falsely is not correct except on the supposition that the statement can be made only in two ways. Moreover, although it is expressly stated that A receives his account from B, this cannot generally be taken for granted as it rests on A's testimony.

A full discussion of the different ways of interpreting the question, and of the different solutions to which they lead, will be found in the *Educational Times Reprint*, Vols. XXVII. and XXXII.

*EXAMPLES. XXXII. d.

1. There are four balls in a bag, but it is not known of what colours they are; one ball is drawn and found to be white: find the chance that all the balls are white.

2. In a bag there are six balls of unknown colours; three balls are drawn and found to be black; find the chance that no black ball is left in the bag.

3. A letter is known to have come either from London or Clifton; on the postmark only the two consecutive letters ON are legible; what is the chance that it came from London?

4. Before a race the chances of three runners, A, B, C, were estimated to be proportional to 5, 3, 2; but during the race A meets with an accident which reduces his chance to one-third. What are now the respective chances of B and C?

5. A purse contains n coins of unknown value; a coin drawn at random is found to be a sovereign; what is the chance that it is the only sovereign in the bag?

6. A man has 10 shillings and one of them is known to have two heads. He takes one at random and tosses it 5 times and it always falls head: what is the chance that it is the shilling with two heads?

7. A bag contains 5 balls of unknown colour; a ball is drawn and replaced twice, and in each case is found to be red: if two balls are now drawn simultaneously find the chance that both are red.

8. A purse contains five coins, each of which may be a shilling or a sixpence; two are drawn and found to be shillings: find the probable value of the remaining coins.

9. A die is thrown three times, and the sum of the three numbers thrown is 15: find the chance that the first throw was a four.

10. A speaks the truth 3 out of 4 times, and B 5 out of 6 times: what is the probability that they will contradict each other in stating the same fact?

11. A speaks the truth 2 out of 3 times, and B 4 times out of 5; they agree in the assertion that from a bag containing 6 balls of different colours a red ball has been drawn: find the probability that the statement is true.

12. One of a pack of 52 cards has been lost; from the remainder of the pack two cards are drawn and are found to be spades; find the chance that the missing card is a spade.

13. There is a raffle with 10 tickets and two prizes of value £5 and £1 respectively. A holds one ticket and is informed by B that he has won the £5 prize, while C asserts that he has won the £1 prize: what is A's expectation, if the credibility of B is denoted by $\frac{2}{3}$, and that of C by $\frac{3}{4}$?

14. A purse contains four coins; two coins having been drawn are found to be sovereigns: find the chance (1) that all the coins are sovereigns, (2) that if the coins are replaced another drawing will give a sovereign.

15. P makes a bet with Q of £8 to £120 that three races will be won by the three horses A, B, C, against which the betting is 3 to 2, 4 to 1, and 2 to 1 respectively. The first race having been won by A, and it being known that the second race was won either by B, or by a horse D against which the betting was 2 to 1, find the value of P's expectation.

16. From a bag containing n balls, all either white or black, all numbers of each being equally likely, a ball is drawn which turns out to be white; this is replaced, and another ball is drawn, which also turns out to be white. If this ball is replaced, prove that the chance of the next draw giving a black ball is $\frac{1}{2}(n-1)(2n+1)^{-1}$.

17. If mn coins have been distributed into m purses, n into each, find (1) the chance that two specified coins will be found in the same purse; and (2) what the chance becomes when r purses have been examined and found not to contain either of the specified coins.

18. A, B are two inaccurate arithmeticians whose chance of solving a given question correctly are $\frac{1}{8}$ and $\frac{1}{12}$ respectively; if they obtain the same result, and if it is 1000 to 1 against their making the same mistake, find the chance that the result is correct.

19. Ten witnesses, each of whom makes but one false statement in six, agree in asserting that a certain event took place; shew that the odds are five to one in favour of the truth of their statement, even although the *a priori* probability of the event is as small as $\frac{1}{5^9+1}$.

LOCAL PROBABILITY. GEOMETRICAL METHODS.

*481. The application of Geometry to questions of Probability requires, in general, the aid of the Integral Calculus; there are, however, many easy questions which can be solved by Elementary Geometry.

Example 1. From each of two equal lines of length l a portion is cut off at random, and removed: what is the chance that the sum of the remainders is less than l?

Place the lines parallel to one another, and suppose that after cutting, the right-hand portions are removed. Then the question is equivalent to asking what is the chance that the sum of the right-hand portions is greater than the sum of the left-hand portions. It is clear that the first sum is equally likely to be greater or less than the second; thus the required probability is $\frac{1}{2}$.

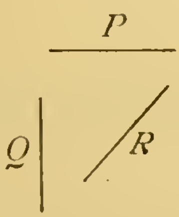
Cor. Each of two lines is known to be of length not exceeding l: the chance that their sum is not greater than l is $\frac{1}{2}$.

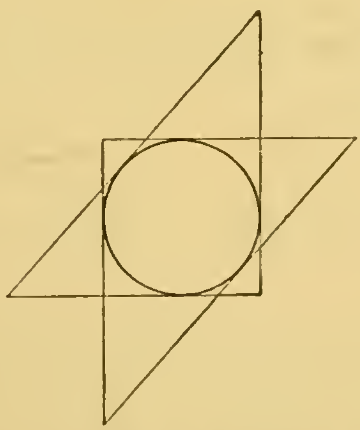
Example 2. If three lines are chosen at random, prove that they are just as likely as not to denote the sides of a possible triangle.

Of three lines one *must* be equal to or greater than each of the other two; denote its length by l. Then all we know of the other two lines is that the length of each lies between 0 and l. But if each of two lines is known to be of random length between 0 and l, it is an even chance that their sum is greater than l. [Ex. 1, Cor.]

Thus the required result follows.

Example 3. Three tangents are drawn at random to a given circle: shew that the odds are 3 to 1 against the circle being *inscribed* in the triangle formed by them.





Draw three random lines P, Q, R, in the same plane as the circle, and draw to the circle the six tangents parallel to these lines.

Then of the 8 triangles so formed it is evident that the circle will be escribed to 6 and inscribed in 2; and as this is true whatever be the original directions of P, Q, R, the required result follows.

*482. Questions in Probability may sometimes be conveniently solved by the aid of co-ordinate Geometry.

Example. On a rod of length $a+b+c$, lengths a, b are measured at random: find the probability that no point of the measured lines will coincide.

Let AB be the line, and suppose $AP=x$ and $PQ=a$; also let a be measured from P towards B, so that x must be less than $b+c$. Again let $AP'=y$, $P'Q'=b$, and suppose $P'Q'$ measured from P' towards B, then y must be less than $a+c$.

Now in favourable cases we must have $AP'>AQ$, or else $AP>AQ'$,

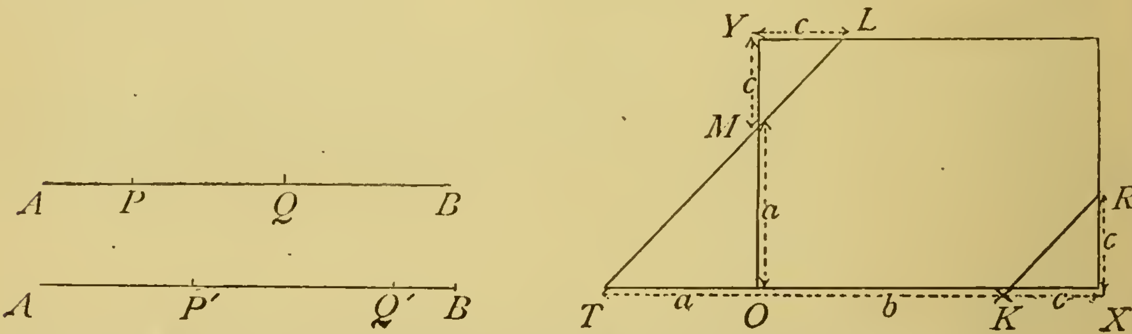
hence $$y>a+x, \text{ or } x>b+y \qquad \text{.......(1)}.$$

Again for all the cases possible, we must have

$$\left.\begin{array}{l} x>0, \text{ and } <b+c \\ y>0, \text{ and } <a+c \end{array}\right\} \qquad \text{.......(2)}.$$

Take a pair of rectangular axes and make OX equal to $b+c$, and OY equal to $a+c$.

Draw the line $y=a+x$, represented by TML in the figure; and the line $x=b+y$ represented by KR.



Then YM, KX are each equal to c, OM, OT are each equal to a.

The conditions (1) are only satisfied by points in the triangles MYL and KXR, while the conditions (2) are satisfied by any points within the rectangle OX, OY;

$$\therefore \text{ the required chance } = \frac{c^2}{(a+c)(b+c)}.$$

*483. We shall close this chapter with some Miscellaneous Examples.

Example 1. A box is divided into m equal compartments into which n balls are thrown at random; find the probability that there will be p compartments each containing a balls, q compartments each containing b balls, r compartments each containing c balls, and so on, where

$$pa+qb+rc+\ldots\ldots=n.$$

Since each of the n balls can fall into any one of the m compartments the total number of cases which can occur is m^n, and these are all equally likely. To determine the number of favourable cases we must find the number of ways in which the n balls can be divided into $p, q, r, \ldots$ parcels containing $a, b, c, \ldots$ balls respectively.

First choose any s of the compartments, where s stands for $p+q+r+\ldots$; the number of ways in which this can be done is $\dfrac{\lfloor m}{\lfloor s \, \lfloor m-s} \ldots\ldots\ldots\ldots (1).$

Next subdivide the s compartments into groups containing $p, q, r, \ldots$ severally; by Art. 147, the number of ways in which this can be done is

$$\frac{\lfloor s}{\lfloor p \, \lfloor q \, \lfloor r \ldots} \ldots\ldots\ldots\ldots (2).$$

Lastly, distribute the n balls into the compartments, putting a into each of the group of p, then b into each of the group of q, c into each of the group of r, and so on. The number of ways in which this can be done is

$$\frac{\lfloor n}{(\lfloor a)^p (\lfloor b)^q (\lfloor c)^r \ldots\ldots} \ldots\ldots\ldots\ldots (3).$$

Hence the number of ways in which the balls can be arranged to satisfy the required conditions is given by the product of the expressions (1), (2), (3). Therefore the required probability is

$$\frac{\lfloor m \, \lfloor n}{m^n (\lfloor a)^p (\lfloor b)^q (\lfloor c)^r \ldots\ldots \lfloor p \, \lfloor q \, \lfloor r \ldots\ldots \lfloor m-p-q-r-\ldots\ldots}.$$

Example 2. A bag contains n balls; k drawings are made in succession, and the ball on each occasion is found to be white: find the chance that the next drawing will give a white ball; (i) when the balls are replaced after each drawing; (ii) when they are not replaced.

(i) Before the observed event there are $n+1$ hypotheses, equally likely; for the bag may contain 0, 1, 2, 3, ... n white balls. Hence following the notation of Art. 471,

$$P_0 = P_1 = P_2 = P_3 = \ldots = P_n;$$

and $$p_0 = 0, \; p_1 = \left(\frac{1}{n}\right)^k, \; p_2 = \left(\frac{2}{n}\right)^k, \; p_3 = \left(\frac{3}{n}\right)^k, \ldots\ldots, \; p_n = \left(\frac{n}{n}\right)^k.$$

Hence after the observed event,

$$Q_r = \frac{r^k}{1^k + 2^k + 3^k + \ldots + n^k}.$$

Now the chance that the next drawing will give a white ball $= \Sigma \frac{r}{n} Q_r$;

thus the required chance $= \dfrac{1}{n} \cdot \dfrac{1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1}}{1^k + 2^k + 3^k + \ldots + n^k}$;

and the value of numerator and denominator may be found by Art. 405.

In the particular case when $k=2$,

$$\text{the required chance } = \frac{1}{n}\left\{\frac{n(n+1)}{2}\right\}^2 \div \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3(n+1)}{2(2n+1)}.$$

If n is indefinitely large, the chance is equal to the limit, when n is infinite, of

$$\frac{1}{n}\cdot\frac{n^{k+2}}{k+2} \div \frac{n^{k+1}}{k+1};$$

and thus the chance is

$$\frac{k+1}{k+2}.$$

(ii) If the balls are not replaced,

$$p_r = \frac{r}{n}\cdot\frac{r-1}{n-1}\cdot\frac{r-2}{n-2}\cdots\cdots\frac{r-k+1}{n-k+1};$$

and

$$Q_r = \frac{p_r}{\Sigma p_r} = \frac{(r-k+1)(r-k+2)\ldots\ldots(r-1)r}{\sum\limits_{r=0}^{r=n}(r-k+1)(r-k+2)\ldots\ldots(r-1)r}$$

$$= (k+1)\,\frac{(r-k+1)(r-k+2)\ldots\ldots(r-1)r}{(n-k+1)(n-k+2)\ldots\ldots(n-1)n(n+1)}. \quad \text{[Art. 394.]}$$

The chance that the next drawing will give a white ball $= \sum\limits_{r=0}^{r=n}\frac{r-k}{n-k}Q_r$

$$= \frac{k+1}{(n-k)(n-k+1)\ldots\ldots n(n+1)}\sum_{r=0}^{r=n}(r-k)(r-k+1)\ldots\ldots(r-1)r$$

$$= \frac{k+1}{(n-k)(n-k+1)\ldots\ldots n(n+1)}\cdot\frac{(n-k)(n-k+1)\ldots\ldots n(n+1)}{k+2}$$

$$= \frac{k+1}{k+2},$$

which is independent of the number of balls in the bag at first.

Example 3. A person writes n letters and addresses n envelopes; if the letters are placed in the envelopes at random, what is the probability that every letter goes wrong?

Let u_n denote the number of ways in which all the letters go wrong, and let $abcd\ldots$ represent that arrangement in which all the letters are in their own envelopes. Now if a in any other arrangement occupies the place of an assigned letter b, this letter must either occupy a's place or some other.

(i) Suppose b occupies a's place. Then the number of ways in which all the remaining $n-2$ letters can be displaced is u_{n-2}, and therefore the numbers of ways in which a may be displaced by interchange with some one of the other $n-1$ letters, and the rest be all displaced is $(n-1)\,u_{n-2}$.

(ii) Suppose a occupies b's place, and b does not occupy a's. Then in arrangements satisfying the required conditions, since a is fixed in b's place, the letters $b, c, d, \ldots$ must be all displaced, which can be done in u_{n-1} ways; therefore the number of ways in which a occupies the place of another letter but not by interchange with that letter is $(n-1)\,u_{n-1}$;

$$\therefore\ u_n = (n-1)\,(u_{n-1} + u_{n-2});$$

from which, by the method of Art. 444, we find $u_n - nu_{n-1} = (-1)^n (u_2 - u_1)$.

Also $u_1 = 0$, $u_2 = 1$; thus we finally obtain

$$u_n = \lfloor n \left\{ \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots + \frac{(-1)^n}{\lfloor n} \right\}.$$

Now the total number of ways in which the n things can be put in n places is $\lfloor n$; therefore the required chance is

$$\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots + \frac{(-1)^n}{\lfloor n}.$$

The problem here involved is of considerable interest, and in some of its many modifications has maintained a permanent place in works on the Theory of Probability. It was first discussed by Montmort, and it was generalised by De Moivre, Euler, and Laplace.

*484. The subject of Probability is so extensive that it is impossible here to give more than a sketch of the principal algebraical methods. An admirable collection of problems, illustrating every algebraical process, will be found in Whitworth's *Choice and Chance;* and the reader who is acquainted with the Integral Calculus may consult Professor Crofton's article *Probability* in the *Encyclopædia Britannica.* A complete account of the origin and development of the subject is given in Todhunter's *History of the Theory of Probability from the time of Pascal to that of Laplace.*

The practical applications of the theory of Probability to commercial transactions are beyond the scope of an elementary treatise; for these we may refer to the articles *Annuities* and *Insurance* in the *Encyclopædia Britannica.*

*EXAMPLES. XXXII. e.

1. What are the odds in favour of throwing at least 7 in a single throw with two dice?

2. In a purse there are 5 sovereigns and 4 shillings. If they are drawn out one by one, what is the chance that they come out sovereigns and shillings alternately, beginning with a sovereign?

3. If on an average 9 ships out of 10 return safe to port, what is the chance that out of 5 ships expected at least 3 will arrive?

4. In a lottery all the tickets are blanks but one; each person draws a ticket, and retains it: shew that each person has an equal chance of drawing the prize.

5. One bag contains 5 white and 3 red balls, and a second bag contains 4 white and 5 red balls. From one of them, chosen at random, two balls are drawn: find the chance that they are of different colours.

6. Five persons A, B, C, D, E throw a die in the order named until one of them throws an ace: find their relative chances of winning, supposing the throws to continue till an ace appears.

7. Three squares of a chess board being chosen at random, what is the chance that two are of one colour and one of another?

8. A person throws two dice, one the common cube, and the other a regular tetrahedron, the number on the lowest face being taken in the case of the tetrahedron; find the average value of the throw, and compare the chances of throwing 5, 6, 7.

9. A's skill is to B's as $1 : 3$; to C's as $3 : 2$; and to D's as $4 : 3$: find the chance that A in three trials, one with each person, will succeed twice at least.

10. A certain stake is to be won by the first person who throws an ace with an octahedral die: if there are 4 persons what is the chance of the last?

11. Two players A, B of equal skill are playing a set of games; A wants 2 games to complete the set, and B wants 3 games: compare their chances of winning.

12. A purse contains 3 sovereigns and two shillings: a person draws one coin in each hand and looks at one of them, which proves to be a sovereign; shew that the other is equally likely to be a sovereign or a shilling.

13. A and B play for a prize; A is to throw a die first, and is to win if he throws 6. If he fails B is to throw, and to win if he throws 6 or 5. If he fails, A is to throw again and to win with 6 or 5 or 4, and so on: find the chance of each player.

14. Seven persons draw lots for the occupancy of the six seats in a first class railway compartment: find the chance (1) that two specified persons obtain opposite seats, (2) that they obtain adjacent seats on the same side.

15. A number consists of 7 digits whose sum is 59; prove that the chance of its being divisible by 11 is $\frac{4}{21}$.

16. Find the chance of throwing 12 in a single throw with 3 dice.

17. A bag contains 7 tickets marked with the numbers 0, 1, 2, ... 6 respectively. A ticket is drawn and replaced; find the chance that after 4 drawings the sum of the numbers drawn is 8.

18. There are 10 tickets, 5 of which are blanks, and the others are marked with the numbers 1, 2, 3, 4, 5: what is the probability of drawing 10 in three trials, (1) when the tickets are replaced at every trial, (2) if the tickets are not replaced?

19. If n integers taken at random are multiplied together, shew that the chance that the last digit of the product is 1, 3, 7, or 9 is $\frac{2^n}{5^n}$; the chance of its being 2, 4, 6, or 8 is $\frac{4^n - 2^n}{5^n}$; of its being 5 is $\frac{5^n - 4^n}{10^n}$; and of its being 0 is $\frac{10^n - 8^n - 5^n + 4^n}{10^n}$.

20. A purse contains two sovereigns, two shillings and a metal dummy of the same form and size; a person is allowed to draw out one at a time till he draws the dummy: find the value of his expectation.

21. A certain sum of money is to be given to the one of three persons A, B, C who first throws 10 with three dice; supposing them to throw in the order named until the event happens, prove that their chances are respectively

$$\left(\frac{8}{13}\right)^2, \quad \frac{56}{13^2}, \quad \text{and} \quad \left(\frac{7}{13}\right)^2.$$

22. Two persons, whose probabilities of speaking the truth are $\frac{2}{3}$ and $\frac{5}{6}$ respectively, assert that a specified ticket has been drawn out of a bag containing 15 tickets: what is the probability of the truth of the assertion?

23. A bag contains $\frac{n(n+1)}{2}$ counters, of which one is marked 1, two are marked 4, three are marked 9, and so on; a person puts in his hand and draws out a counter at random, and is to receive as many shillings as the number marked upon it: find the value of his expectation.

24. If 10 things are distributed among 3 persons, the chance of a particular person having more than 5 of them is $\frac{1507}{19683}$.

25. If a rod is marked at random in n points and divided at those points, the chance that none of the parts shall be greater than $\frac{1}{n}$th of the rod is $\frac{1}{n^n}$.

26. There are two purses, one containing three sovereigns and a shilling, and the other containing three shillings and a sovereign. A coin is taken from one (it is not known which) and dropped into the other; and then on drawing a coin from each purse, they are found to be two shillings. What are the odds against this happening again if two more are drawn, one from each purse?

27. If a triangle is formed by joining three points taken at random in the circumference of a circle, prove that the odds are 3 to 1 against its being acute-angled.

28. Three points are taken at random on the circumference of a circle: what is the chance that the sum of any two of the arcs so determined is greater than the third?

29. A line is divided at random into three parts, what is the chance that they form the sides of a possible triangle?

30. Of two purses one originally contained 25 sovereigns, and the other 10 sovereigns and 15 shillings. One purse is taken by chance and 4 coins drawn out, which prove to be all sovereigns: what is the chance that this purse contains only sovereigns, and what is the probable value of the next draw from it?

31. On a straight line of length a two points are taken at random; find the chance that the distance between them is greater than b.

32. A straight line of length a is divided into three parts by two points taken at random; find the chance that no part is greater than b.

33. If on a straight line of length $a+b$ two lengths a, b are measured at random, the chance that the common part of these lengths shall not exceed c is $\dfrac{c^2}{ab}$, where c is less than a or b; also the chance that the smaller length b lies entirely within the larger a is $\dfrac{a-b}{a}$.

34. If on a straight line of length $a+b+c$ two lengths a, b are measured at random, the chance of their having a common part which shall not exceed d is $\dfrac{(c+d)^2}{(c+a)(c+b)}$, where d is less than either a or b.

35. Four passengers, A, B, C, D, entire strangers to each other, are travelling in a railway train which contains l first-class, m second-class, and n third-class compartments. A and B are gentlemen whose respective *a priori* chances of travelling first, second, or third class are represented in each instance by λ, μ, ν; C and D are ladies whose similar *a priori* chances are each represented by l, m, n. Prove that, for all values of λ, μ, ν (except in the particular case when $\lambda : \mu : \nu = l : m : n$), A and B are more likely to be found both in the company of the same lady than each with a different one.