CHAPTER

8

Matrices

- Definition
- Algebra of Matrices
- Special Matrices
- Adjoint of Square Matrix
- Equivalent Matrices
- System of Simultaneous Linear Equation
- Matrices of Reflection and Rotation
- Characteristic Roots and Characteristic Vector of a Square Matrix

DEFINITION

A rectangular array of symbols (which could be real or complex numbers) along rows and columns is called a matrix.

Thus a system of $m \times n$ symbols arranged in a rectangular formation along m rows and n columns and bonded by the brackets [·] is called an m by n matrix (which is written as $m \times n$ matrix). Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is a matrix of order $m \times n$.

In a compact form, the above matrix is represented by $A = [a_{ij}]$, $1 \le i \le m$, $1 \le j \le n$ or simply $[a_{ij}]_{m \times n}$. The numbers a_{11}, a_{12}, ..., etc., of this rectangular array are called the elements of the matrix. The element a_{ij} belongs to the i^{th} row and j^{th} column and is called the $(i, j)^{th}$ element of the matrix.

Equal Matrices

Two matrices are said to be equal if they have the same order and each element of one is equal to the corresponding element of the other.

Example 8.1 **If a matrix has 8 elements, what are the possible orders it can have?**

Sol. We know that if matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1×8, 8×1, 4×2, 2×4

Example 8.2 **Construct the matrix of order 3 × 2 whose elements are given by $a_{ij} = 2i - j$.**

Sol. In general 3 × 2 matrix is given by $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Now $a_{ij} = 2i - j$

$\Rightarrow$ $a_{11} = 2(1) - 1 = 1$

$a_{12} = 2(1) - 2 = 0$

$a_{21} = 2(2) - 1 = 3$

$a_{22} = 2(2) - 2 = 2$

$a_{31} = 2(3) - 1 = 5$

$a_{32} = 2(3) - 2 = 4$

Hence the required matrix is $\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$

Example 8.3 **If $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$, then find the values of x, y, z, w.**

Sol. We have $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$

Comparing the elements we have

$x + y = 4$, $2x + z = 7$, $x - y = 0$ and $2z + w = 10$

Solving these equations we get $x = 2$ and $y = 2$, $z = 3$, $w = 4$

Example 8.4 **For what values of x and y are the following matrices equal?**

$$A = \begin{bmatrix} 2x+1 & 3y \\ 0 & y^2 - 5y \end{bmatrix}, B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

Sol. We have,

$$\begin{bmatrix} 2x+1 & 3y \\ 0 & y^2 - 5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

$\Rightarrow$ $2x + 1 = x + 3$ (1)

$3y = y^2 + 2$ (2)

$y^2 - 5y = -6$ (3)

$\Rightarrow$ $x = 2$ [From Eq. (1)]

$y = 1$ or 2 [From Eq. (2)]

$y = 2$ or 3 [From Eq. (3)]

$\Rightarrow$ $x = 2$ and $y = 2$

Classification of Matrices

Row Matrix

A matrix having a single row is called a row matrix, e.g., [1 3 5 7].

Column Matrix

A matrix having a single column is called a column matrix, e.g.,

$$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Note: *Matrices consisting of only one column or row are called vectors.*

Square Matrix

An $m \times n$ matrix A is said to be a square matrix if $m = n$, i.e., number of rows = number of columns. For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

is a square matrix of order 3 × 3.

Note: *The diagonal from left-hand side upper corner to right-hand side lower corner is known as leading diagonal or principal diagonal. In the above example, diagonal containing the elements* 1, 3, 5 *is called the leading or principal diagonal.*

Diagonal Matrix

A square matrix all of whose elements, except those in the leading diagonal, are zero is called a diagonal matrix. For a square matrix, $A = [a_{ij}]_{n \times n}$ to be a diagonal matrix, $a_{ij} = 0$, whenever $i \neq j$.

A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots, d_n$ as diagonal elements is denoted by diag $[d_1, d_2, \ldots, d_n]$. For example,

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is a diagonal matrix of order 3 × 3 to be denoted by $A = \text{diag}\,[3\ 5\ -1]$.

Scalar Matrix

A diagonal matrix whose all the leading diagonal elements are equal is called a scalar matrix.

For a square matrix $A = [a_{ij}]_{n \times n}$ to be a scalar matrix,

$$a_{ij} = \begin{cases} 0, & i \neq j \\ m, & i = j \end{cases}$$

where $m \neq 0$. For example,

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a scalar matrix.

Unit Matrix or Identity Matrix

A diagonal matrix of order n which has unity for all its diagonal elements, is called a unit matrix of order n and is denoted by I_n.

Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Triangular Matrix

A square matrix in which all the elements below the diagonal are zero is called upper triangular matrix and a square matrix in which all the elements above diagonal are zero is called lower triangular matrix.

Given a square matrix $A = [a_{ij}]_{n \times n}$, For upper triangular matrix, $a_{ij} = 0$, $i > j$ and for lower triangular matrix, $a_{ij} = 0$, $i < j$. For example,

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

are, respectively, upper and lower triangular matrices.

Note:

- *Diagonal matrix is both upper and lower triangular.*
- *A triangular matrix $A = [a_{ij}]_{n\times n}$ is called strictly triangular if $a_{ii} = 0$ for $1 \leq i \leq n$.*

Null Matrix

If all the elements of a matrix (square or rectangular) are zero, it is called a null or zero matrix. For $A = [a_{ij}]$ to be null matrix, $a_{ij} = 0$, $\forall\ i, j$. For example,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ are null matrices.}$$

Trace of Matrix

The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A, i.e., tr(A). Thus, if $A = [a_{ij}]_{n \times n}$, then

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

Properties of Trace of a Matrix

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar. Then,

(i) $\text{tr}(\lambda A) = \lambda \text{tr}(A)$

(ii) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

(iii) $\text{tr}(AB) = \text{tr}(BA)$

Determinant of Square Matrix

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{\text{th}}$ element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex). If M is the set of square matrices, K is the set of numbers (real or complex) and $f: M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then f(A) is called the determinant of A. It is also denoted by $|A|$ or det(A) or Δ.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = det(A).

Note:

- *If $A_1, A_2, \ldots, A_n$ are square matrices of the same order then $|A_1A_2 \ldots A_n| = |A_1||A_2| \ldots |A_n|$.*
- *If k is scalar, then $|kA| = k^n|A|$, where n is order of the matrix A.*
- *If A and B are square matrices of same order then $|AB| = |BA|$ even though $AB \neq BA$*

Singular and Non-Singular Matrix

A square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$.

Example 8.5 **Find the values of x for which matrix $\begin{bmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{bmatrix}$ singular.**

Sol. Given matrix is singular

$$\Rightarrow \begin{vmatrix} 3 & x-1 & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & x & -x \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{vmatrix} = 0 \quad [R_1 \to R_1 - R_2]$$

$$\Rightarrow \begin{vmatrix} 0 & x & -x \\ -x & 0 & x \\ x+3 & -1 & 2 \end{vmatrix} = 0 \ [R_2 \to R_2 - R_3]$$

$$\Rightarrow \begin{vmatrix} 0 & x & 0 \\ -x & 0 & x \\ x+3 & -1 & 1 \end{vmatrix} = 0 \ [C_3 \to C_3 + C_2]$$

$$\Rightarrow \quad -x[(-x) - x(x+3)] = 0 \Rightarrow x(x^2 + 4x) = 0 \Rightarrow x = 0, -4$$

Hence only one value of x in closed interval $[-4, -1]$ i.e. $x = -4$

ALGEBRA OF MATRICES

Addition and Subtraction of Matrices

Before we give the formal definition of addition and subtraction of matrices, we will discuss an example from a real life situation. Let the marks of the three students S_1, S_2, S_3 in maths, physics and chemistry in two tests are as follows:

$$\begin{array}{c} \text{Test 1} \\ \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix} \end{array} \end{array} \qquad \begin{array}{c} \text{Test 2} \\ \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \end{array} \begin{array}{c} \begin{array}{ccc} P & C & M \end{array} \\ \begin{bmatrix} 55 & 65 & 78 \\ 40 & 65 & 35 \\ 42 & 65 & 70 \end{bmatrix} \end{array} \end{array}$$

Now if we want to find the aggregate marks in both the tests, then we must have

$$\text{Aggregate marks} = \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \overset{\text{Test 1: } P \ C \ M}{\begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix}} + \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \overset{\text{Test 2: } P \ C \ M}{\begin{bmatrix} 55 & 65 & 78 \\ 40 & 65 & 35 \\ 42 & 65 & 70 \end{bmatrix}}$$

$$= \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \overset{P \qquad C \qquad M}{\begin{bmatrix} 40+55 & 40+65 & 60+78 \\ 30+40 & 70+65 & 40+35 \\ 25+42 & 50+65 & 55+70 \end{bmatrix}}$$

$$= \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \overset{\text{Test 1: } P \ C \ M}{\begin{bmatrix} 95 & 105 & 138 \\ 70 & 135 & 75 \\ 67 & 115 & 125 \end{bmatrix}}$$

Thus, any two matrices can be added if they are of the same order and the resulting matrix is of the same order. If two matrices A and B are of the same order, they are said to be conformable for addition.

Let A, B be two matrices, each of order $m \times n$. Then, their sum $A + B$ is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order, their sum $A + B$ is defined to be the matrix of order $m \times n$ such that $(A + B)_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. For example,

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \pm \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 \pm c_1 & b_1 \pm d_1 \\ a_2 \pm c_2 & b_2 \pm d_2 \\ a_3 \pm c_3 & b_3 \pm d_3 \end{bmatrix}$$

Note:

- *Only matrices of the same order can be added or subtracted.*
- *Addition of matrices is commutative* $[A + B = B + A]$ *as well as associative* $[(A + B) + C = A + (B + C)]$.
- *Cancellation laws hold well in case of addition.*
- *The equation* $A + X = O$ *has a unique solution in the set of all* $m \times n$ *matrices (where* O *is null matrix).*

Scalar Multiplication

Before we give the formal definition of scalar multiplication, we will discuss an example from a real life situation. Let the marks of the three students S_1, S_2, S_3 in maths, physics and chemistry are as follows:

$$\begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \overset{P \quad C \quad M}{\begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix}}$$

After the result, the examination body realizes that the test papers were too difficult for the students to perform well. So they decided to give 10% grace marks to each students in each subject. Then the revised result is

$$(1.1) \times \left(\begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \overset{P \quad C \quad M}{\begin{bmatrix} 40 & 40 & 60 \\ 30 & 70 & 40 \\ 25 & 50 & 55 \end{bmatrix}} \right) = \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \overset{P \quad C \quad M}{\begin{bmatrix} 44 & 44 & 66 \\ 33 & 77 & 44 \\ 27.5 & 55 & 60.5 \end{bmatrix}}$$

Thus, the matrix obtained by multiplying every element of a matrix A by a scalar λ is called the scalar multiple of A by λ and is denoted by λA, i.e., if $A = [a_{ij}]$, then $\lambda A = [\lambda a_{ij}]$. For example, if

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 7 & 8 \end{bmatrix}_{2\times3}, \text{ then } 2A = \begin{bmatrix} 4 & 6 & 10 \\ 12 & 14 & 16 \end{bmatrix}_{2\times3}$$

Note: *All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication with scalars.*

Example 8.6 **If** $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$**, find** $3A - 2B$**.**

Sol. $3A - 2B = 3\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 6 & -3 \\ 9 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 8 \\ 14 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6-2 & -3-8 \\ 9-14 & 3-4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -11 \\ -5 & -1 \end{bmatrix}$$

Example 8.7 **If** $A = \text{diag}\,(1\ -1\ 2)$ **and** $B = \text{diag}\,(2\ 3\ -1)$**, then find** $3A + 4B$**.**

Sol. $3A + 4B = 3\ \text{diag}\,(1\ -1\ 2) + 4\ \text{diag}\,(2\ 3\ -1)$

$= \text{diag}(3\ {-3}\ 6) + \text{diag}(8\ 12\ {-4})$
$= \text{diag}(11\ 9\ 2)$

Example 8.8 If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}, \text{ then find } D = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix} \text{ such}$$

that $A + B - D = O$.

Sol. Here $A + B - D = O$.

$\therefore\ D = A + B$

$$\Rightarrow \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 2-2 \\ 3+1 & 4-5 \\ 5+4 & 6+3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$$

$$\Rightarrow \quad D = \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix}$$

Example 8.9 **Let $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ then find $Tr(A) - Tr(B)$.**

Sol. Here to find the value of $Tr(A) - Tr(B)$, we need not to find the matrices A and B

We can find $Tr(A) - Tr(B)$ using the properties of Trace of matrix

That is

$$A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$$

$\Rightarrow\ Tr(A + 2B) = -1$

or $\ T_r(A) + 2T_r(B) = -1$ (i)

$$2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$\Rightarrow\ Tr(2A - B) = 3$

or $\ 2T_r(A) - T_r(B) = 3$ (ii)

Solving (i) and (ii), we get $Tr(A) = 1$ and $Tr(B) = -1$

$\Rightarrow\ Tr(A) - Tr(B) = 2$

Multiplication of Matrices

Before we give the formal definition of how to multiply two matrices, we will discuss an example from a real life situation. Consider a city with two kinds of population: the inner city population and the suburb population. We assume that every year 40% of the inner city population moves to the suburbs, while 30% of the suburb population moves to the inner part of the city. Let I (resp. S) be the initial population of the inner city (resp. the suburban area). So after one year, the population of the inner part is $0.6I + 0.3S$, while the population of the suburbs is $0.4I + 0.7S$.

After two years, the population of the inner city is $0.6(0.6I + 0.3S) + 0.3(0.4I + 0.7S)$ and the suburban population is given by $0.4(0.6I + 0.3S) + 0.7(0.4I + 0.7S)$.

Is there a nice way of representing the two populations after a certain number of years? Let us show how matrices may be helpful to answer this question. Let us represent the two populations in one table (meaning a column object with two entries):

$$\begin{bmatrix} I \\ S \end{bmatrix}$$

So after one year the table which gives the two populations is

$$\begin{bmatrix} 0.6I + 0.3S \\ 0.4I + 0.7S \end{bmatrix}$$

If we consider the following rule (the product of two matrices)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} aI + bS \\ cI + dS \end{bmatrix}$$

then the populations after one year are given by the formula

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} I \\ S \end{bmatrix}$$

After two years, the populations are

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\left(\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} I \\ S \end{bmatrix}\right)$$

Combining this formula with the above result, we get

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6\times0.6+0.3\times0.4 & 0.6\times0.3+0.3\times0.7 \\ 0.4\times0.6+0.7\times0.4 & 0.4\times0.3+0.7\times0.7 \end{bmatrix}$$

In other words, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

In fact, we do not need to have two matrices of the same size to multiply them. Above, we did multiply a (2×2) matrix with a (2×1) matrix [which gave a (2×1) matrix]. In fact, the general rule says that in order to perform the multiplication AB, where A is a $m \times n$ matrix and B is a $k \times l$ matrix, then we must have $n = k$. The result will be a $m \times l$ matrix. For example, we have

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+fz \end{bmatrix}$$

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post-multiplier). Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are two matrices of order $m \times n$ and $n \times p$, respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$= [a_{i1}\ a_{i2} \cdots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= (i^{th} \text{ row of } A)\ (j^{th} \text{ column of } B) \qquad (1)$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$.

Now, we define the product of a row matrix and a column matrix. Let $A = [a_1\ a_2 \ldots a_n]$ be a row matrix and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

be a column matrix. Then,

$$AB = a_1b_1 + a_2b_2 + \cdots + a_nb_n \qquad (2)$$

Thus, from (1), we have $(AB)_{ij}$ is the sum of the product of elements of i^{th} row of A with the corresponding elements of j^{th} column of B. For example,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$$

Here A is a 3×3 matrix and B is a 3×2 matrix. Therefore, A and B are conformable for the product AB and it is of order 3×2 such that

$(AB)_{11}$ = (First row of A) (First column of B)

$$= [2\ 1\ 3] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 2 \times 1 + 1 \times 2 + 3 \times 4 = 16$$

$(AB)_{12}$ = (First row of A) (Second column of B)

$$= [2\ 1\ 3] \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = 2 \times (-2) + 1 \times 1 + 3 \times (-3) = -12$$

$(AB)_{21}$ = (Second row of A) (First column of B)

$$= [3\ -2\ 1] \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 3 \times 1 + (-2) \times 2 + 1 \times 4 = 3$$

Similarly, we have $(AB)_{22} = -11$, $(AB)_{31} = 3$ and $(AB)_{32} = -1$.

$$\therefore \quad AB = \begin{bmatrix} 16 & -12 \\ 3 & -11 \\ 3 & -1 \end{bmatrix}$$

Properties of Matrix Multiplication

- *Commutative law does not necessarily hold for matrices.*
- *If $AB = BA$, then matrices A and B are called commutative matrices.*
- *If $AB = -BA$, then matrices A and B are called anti-commutative matrices.*
- *Matrix multiplication is associative: $A(BC) = (AB)C$.*

Proof:

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ and $C = [c_{ij}]_{p\times q}$. Then, AB is an $m \times p$ matrix and so $(AB)C$ is a $m \times q$ matrix. Also, BC is of order $n \times q$ and so $A(BC)$ is of order $m \times q$. Thus $(AB)C$ and $A(BC)$ are of the same order. Now,

$$((AB)C)_{ij} = \sum_{r=1}^{p} (AB)_{ir}(C)_{rj}$$

$$= \sum_{r=1}^{p}\left(\sum_{s=1}^{n} a_{is}b_{sr}\right)c_{rj} = \sum_{r=1}^{p}\sum_{s=1}^{n}(a_{is}b_{sr})c_{rj}$$

$$= \sum_{r=1}^{p}\sum_{s=1}^{n} a_{is}(b_{sr}c_{rj})$$

[By association law of multiplication of numbers]

$$= \sum_{s=1}^{n} a_{is}\left(\sum_{r=1}^{p}(b_{sr}c_{rj})\right) = \sum_{s=1}^{n} a_{is}(BC)_{sj} = (A(BC))_{ij}$$

for all i, j

Thus, $(AB)C$ and $A(BC)$ are two matrices of the same order such that their corresponding elements are equal. Hence, $(AB)C = A(BC)$.

- *Matrix multiplication is distributive with respect to addition $A(B \pm C) = AB \pm AC$.*
- *If the product $AB = O$, it is not necessary that atleast one of the matrix should be zero matrix. For example, if $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ while neither A nor B is the null matrix.*
- *Cancellation law does not necessarily hold, i.e., if $AB = AC$, then in general $B \neq C$, even if $A \neq O$.*
- *Matrix multiplication $A \times A$ is represented as A^2. Thus $A^n = A\,A \cdots n$ times.*
- *If $A = diag\ (a_1, a_2, a_3, \ldots, a_n)$ and $B = diag\ (b_1, b_2, b_3, \ldots, b_n)$, then $A \times B = diag\ (a_1b_1, a_2b_2, \ldots, a_nb_n)$. Thus $A^n = dign\ (a_1^n, a_2^n, a_3^n, \ldots, a_n^n)$.*
- *If A and B are diagonal matrices of the same order, then $AB = BA$ or diagonal matrices are commutative.*
- *If A and B are commutative, then*

$$(A + B)^2 = (A + B)(A + B)$$
$$= A^2 + AB + BA + B^2$$
$$= A^2 + 2AB + B^2$$

Similarly,

$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$

In general,

$$(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \cdots + {}^nC_nB^n$$

Matrices A and I are always commutative. Hence,

$$(I + A)^n = {}^nC_0 + {}^nC_1A + {}^nC_1A^2 + \cdots + {}^nC_nA^n$$

Transpose of Matrix

The matrix obtained from any given matrix A, by interchanging rows and columns, is called the transpose of A and is denoted by A^T. If $A = [a_{ij}]_{m\times n}$ and $A^T = [b_{ij}]_{n\times m}$, then $b_{ij} = a_{ji}$, $\forall\ i, j$. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}_{3\times 2}, \text{ then } A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}_{2\times 3}$$

Properties of Transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$, A and B being conformable matrices
- $(aA)^T = aA^T$, a being scalar
- $(AB)^T = B^T A^T$ (reversal law), A and B being conformable for multiplication

Proof:
Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices. Then, AB is an $m \times p$ matrix and therefore $(AB)^T$ is a $p \times m$ matrix. Since A^T and B^T are $n \times m$ and $p \times n$ matrices, therefore $B^T A^T$ is a $p \times m$ matrix. Thus, the two matrices $(AB)^T$ and $B^T A^T$ are of the same order such that

$$((AB)^T)_{ij} = (AB)_{ji}$$

$$= \sum_{r=1}^{n} a_{jr} b_{ri}$$

$$= \sum_{r=1}^{n} b_{ri} a_{jr}$$

$$= \sum_{r=1}^{n} (B^T)_{ir} (A^T)_{rj}$$

$$= (B^T A^T)_{ij}$$

Hence, by the definition of equality of two matrices, we have $(AB)^T = B^T A^T$. Transpose of matrix A is also denoted by A'.

- $|A^T| = |A|$

Matrix Polynomial

If matrix A satisfies the polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, then $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$.

Example 8.10 **If $\begin{bmatrix} 1 & -2 & -3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then find the product AB and BA.**

Sol. $AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Example 8.11 **Find the value of x and y that satisfy the equations $\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$.**

Sol. Given $\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3y-2x & 3y-2x \\ 3y & 3y \\ 2y+4x & 2y+4x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

Comparing elements we have

$3y - 2x = 3$ (1)

$2y + 4x + 10$ (2)

Solving (1) and (2) we get we get $x = 3/2$ $y = 2$

Example 8.12 **Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Such that $AB = B$ and $a + d = 2$, then find the value of $(ad - bc)$.**

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p \\ q \end{bmatrix}$

given $AB = B$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} ap+bq \\ cp+dq \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

$\Rightarrow$ $ap + bq = p$ (1)

and $cp + dq = q$ (2)

Eliminating p, q from (1) and (2) we have

$$\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix} = 0$$

$\Rightarrow$ $(a - 1)(d - 1) - bc = 0$

$\Rightarrow$ $ad - (a + d) + 1 - bc = 0$

$\Rightarrow$ $ad - bc = (a + d) - 1 = 2 - 1 = 1$

Example 8.13 **If $A = \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}$, then show that**

$$A^8 = \begin{pmatrix} p^8 & q\left(\dfrac{p^8-1}{p-1}\right) \\ 0 & 1 \end{pmatrix}$$

Sol. $A^2 = \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^2 & pq+q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^2 & q(p+1) \\ 0 & 1 \end{pmatrix}$

$$A^3 = \begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^2 & pq+q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^3 & p^2q+pq+q \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} p^3 & q(p^2+p+1) \\ 0 & 1 \end{pmatrix}$$

Similarly,

$$A^4 = \begin{pmatrix} p^4 & q(p^3+p^2+p+1) \\ 0 & 1 \end{pmatrix} \text{ and so on.}$$

$$\therefore \quad A^8 = \begin{pmatrix} p^8 & q(p^7+p^8+\cdots+1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^8 & q\left(\frac{p^8-1}{p-1}\right) \\ 0 & 1 \end{pmatrix}$$

Example 8.14 **The matrix $R(t)$ is defined by $R(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Show that $R(s)R(t) = R(s+t)$.**

Sol. $R(s)R(t) = \begin{bmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{bmatrix}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

$$= \begin{bmatrix} \cos s\cos t - \sin s\sin t & \cos s\sin t + \sin s\cos t \\ -\sin s\cos t - \cos s\sin t & \cos s\cos t - \sin s\sin t \end{bmatrix}$$

$$= \begin{bmatrix} \cos(s+t) & \sin(s+t) \\ -\sin(s+t) & \cos(s+t) \end{bmatrix}$$

$= R(s+t)$

Example 8.15 **If $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, show that $A^2 = 3A - 2I$. Using this result, show that $A^8 = 255A - 254I$.**

Sol. $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

$$\Rightarrow \quad A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

Now,

$$3A - 2I = 3\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & 0 \\ 3-0 & 6-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

Hence, $A^2 = 3A - 2I$

Now,

$A^4 = (A^2)^2 = (3A - 2I)^2$
$= 9A^2 - 12AI + 4I^2$
$= 9A^2 - 12A + 4I$
$= 9(3A - 2I) - 12A + 4I$
$= 15A - 14I$

$A^8 = (A^4)^2 = (15A - 14I)^2$
$= 225A^2 - 420AI + 196I^2$
$= 225(3A - 2I) - 420A + 196I$
$= 255A - 254I$

Example 8.16 **Matrix A has m rows and $n+5$ columns, matrix B has m rows and $11-n$ columns. If both AB and BA exist, prove that AB and BA are square matrices.**

Sol. If both AB and BA exist, then the number of columns of A is equal to the number of rows of B.

$\therefore \quad n + 5 = m$ (1)

And the number of columns of B is equal to the number of rows of A.

$\therefore \quad 11 - n = m$ (2)

Solving Eqs. (1) and (2), we get $n = 3$ and $m = 8$. Hence, A has order 8×8 and B has order 8×8. Hence, both AB and BA are square matrices.

Example 8.17 **If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $n \in N$, then prove that $A^{4n} = I$.**

Sol. $A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow \quad A^{4n} = I^n = I$

Example 8.18 **Find the value of x for which the matrix product $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ equals an identity matrix.**

Sol. $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$

$$= \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (given)}$$

$\Rightarrow \quad 5x = 1,\ 10x - 2 = 0,\ \therefore x = \frac{1}{5}$

Example 8.19 **Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are two matrices such that they are commutative and $c \neq 3b$. Then find the value of $(a-d)/(3b-c)$.**

Sol. $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

$$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

If $AB = BA$, then

$a + 2c = a + 3b$

$\Rightarrow \quad 2c = 3b \Rightarrow b \neq 0$

$b + 2d = 2a + 4b$

$\Rightarrow \quad 2a - 2d = -3b$

$$\Rightarrow \quad \frac{a-d}{3b-c} = \frac{-\frac{3}{2}b}{3b - \frac{3}{2}b} = -1$$

Example 8.20 **If $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $n \in N$, then prove that $A^n = 2^{n-1} A$.**

Sol. $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$$\Rightarrow \quad A^2=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$$

$$\Rightarrow \quad A^3=A^2A=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}=4A$$

$$\Rightarrow \quad A^n = 2^{n-1}A$$

Example 8.21 **If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, show that $A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$, where k is any positive integer.**

Sol. We have,

$$A^2=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}5 & -8\\ 2 & -3\end{bmatrix}=\begin{bmatrix}1+2\times 2 & -4\times 2\\ 2 & 1-2\times 2\end{bmatrix}$$

and

$$A^3=\begin{bmatrix}5 & -8\\ 2 & -3\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}=\begin{bmatrix}7 & -12\\ 3 & -5\end{bmatrix}=\begin{bmatrix}1+2\times 3 & -4\times 3\\ 3 & 1-2\times 3\end{bmatrix}$$

Thus, it is true for indices 2 and 3. Now, assume

$$A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$$

Then,

$$A^{k+1}=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}3+2k & -4(k+1)\\ k+1 & -1-2k\end{bmatrix}$$

$$=\begin{bmatrix}1+2(k+1) & -4(k+1)\\ k+1 & 1-2(k+1)\end{bmatrix}$$

Thus, if the law is true for A^k, it is also true for A^{k+1}. But it is true for $k = 2, 3$, etc. Hence, by induction, the required result follows.

Example 8.22 **Let $A=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ and X be a matrix such that $A = BX$. Then find X.**

Sol. Let,

$$\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a & b\\ 2c & 2d\end{bmatrix}$$

$\therefore \quad a = 1, b = 2, 2c = 3, 2d = -5$

$$\therefore \quad X=\begin{bmatrix}1 & 2\\ 3/2 & -5/2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2 & 4\\ 3 & -5\end{bmatrix}$$

Concept Application Exercise 8.1

1. Solve the following equations for X and Y:
   $2X-Y=\begin{bmatrix}3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix}$, $2Y+X=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}$
2. If ω is complex cube root of unity and $A=\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}$, then calculate A^{100}.
3. If $A=\begin{bmatrix}1 & 2 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -2 & y\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ and $AB = I_3$, then find the value of $x + y$.
4. If $A=\begin{bmatrix}1 & 0\\ 1/2 & 1\end{bmatrix}$, then find A^{100}.
5. If $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, then find the values of θ satisfying the equation $A^T + A = I_2$.
6. If $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, then find out the values of α, β such that $(\alpha I + \beta A)^2 = A^2$.
7. Consider the matrices
   $A=\begin{bmatrix}4 & 6 & -1\\ 3 & 0 & 2\\ 1 & -2 & 5\end{bmatrix}, B=\begin{bmatrix}2 & 4\\ 0 & 1\\ -1 & 2\end{bmatrix}, C=\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}$
   Out of the given matrix products, which one is not defined.
   a. $(AB)^TC$ **b.** $C^TC(AB)^T$ **c.** C^TAB **d.** A^TABB^TC
8. If $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ p & q & r\end{bmatrix}$, then show that $A^3 = pI + qA + rA^2$.
9. If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}, B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, then find a and b.

Conjugate of Matrix

The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate numbers is called conjugate of A and is denoted by $\bar{A}$. For example, if

$$A=\begin{bmatrix}1+2i & 2-3i & 3+4i\\ 4-5i & 5+6i & 6-7i\\ 8 & 7+8i & 7\end{bmatrix}$$

then,

$$\bar{A}=\begin{bmatrix}1-2i & 2+3i & 3-4i\\ 4+5i & 5-6i & 6+7i\\ 8 & 7-8i & 7\end{bmatrix}$$

Properties of Conjugate

- $\overline{(\bar{A})} = A$
- $\overline{(A+B)} = \bar{A}+\bar{B}$
- $\overline{(\alpha A)} = \bar{\alpha}\bar{A}$, α being any number
- $\overline{(AB)} = \bar{A}\bar{B}$, A and B being conformable for multiplication

Transpose Conjugate of a Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ. The conjugate of the transpose of A is the same as the transpose of the conjugate of A, i.e., $\overline{(A')} = (\overline{A})' = A^\theta$.

If $A = [a_{ij}]_{m\times n}$, then $A^\theta = [b_{ji}]_{n\times m}$ where $b_{ji} = \overline{a}_{ij}$, i.e., the $(j, i)^{\text{th}}$ element of A^θ is equal to the conjugate of $(i, j)^{\text{th}}$ element of A. For example,

$$\text{if } A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}, \text{ then } A^\theta = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$$

Properties of Transpose Conjugate

- $(A^\theta)^\theta = A$
- $(A + B)^\theta = A^\theta + B^\theta$
- $(kA)^\theta = \bar{k}A^\theta$, k being any number
- $(AB)^\theta = B^\theta A^\theta$

SPECIAL MATRICES

Symmetric Matrix

A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all i, j. For example, the matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$$

is symmetric, because $a_{12} = -1 = a_{21}$, $a_{13} = 1 = a_{31}$, $a_{23} = 5 = a_{32}$.

It follows from the definition of a symmetric matrix that A is symmetric $\Leftrightarrow a_{ij} = a_{ji}$ for all i, j $\Leftrightarrow (A)_{ij} = (A^T)_{ij}$ for all $i, j \Leftrightarrow A = A^T$.

Thus, a square matrix A is a symmetric matrix if $A^T = A$. Matrices

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, B = \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$$

are symmetric matrices, because $A^T = A$ and $B^T = B$.

Skew-Symmetric Matrix

A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j. For example, matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} \text{ is skew-symmetric, because}$$

$a_{12} = 2$, $a_{21} = -2 \Rightarrow a_{12} = -a_{21}$
$a_{13} = -3$, $a_{31} = 3 \Rightarrow a_{13} = -a_{31}$
$a_{23} = 5$, $a_{32} = -5 \Rightarrow a_{23} = -a_{32}$

It follows from the definition of a skew-symmetric matrix that A is skew-symmetric. This implies and is implied by

$a_{ij} = -a_{ji}$ for all i, j
$\Leftrightarrow (A)_{ij} = -(A^T)_{ij}$ for all i, j
$\Leftrightarrow A = -A^T$
$\Leftrightarrow A^T = -A$

Thus, a square matrix A is a skew-symmetric matrix if $A^T = -A$. Therefore, matrices

$$A = \begin{bmatrix} 0 & 2i & 3 \\ -2i & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$$

are skew-symmetric matrices, because $A^T = -A$ and $B^T = -B$.

Properties of Symmetric and Skew-Symmetric Matrices

(i) If A is *a* symmetric matrix, then $-A$, kA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices, where $n \in N$, $k \in R$ and B is a square matrix of order that of A

(ii) If A is a skew-symmetric matrix, then
  (*a*) A^{2n} is a symmetric matrix for $n \in N$,
  (*b*) A^{2n+1} is a skew-symmetric matrix for $n \in N$,
  (c) kA is also skew-symmetric matrix, where $k \in R$,
  (d) B^TAB is also skew-symmetric matrix where B is *a* square matrix of order that of A.

(iii) If A, B are two symmetric matrices, then
  (*a*) $A \pm B$, $AB + BA$ are also symmetric matrices,
  (*b*) $AB - BA$ is a skew-symmetric matrix,
  (c) AB is *a* symmetric matrix, when $AB = BA$.

(iv) If A, B are two skew-symmetric matrices, then
  (*a*) $A \pm B$, $AB - BA$ are skew-symmetric matrices,
  (*b*) $AB + BA$ is a symmetric matrix.

(v) If A is a skew-symmetric matrix and C is *a* column matrix, then C^TAC is a zero matrix.

Example 8.23 **Show that the elements on the main diagonal of a skew-symmetric matrix are all zero.**

Sol. Let $A = [a_{ij}]$ be a skew-symmetric matrix. Then, $a_{ij} = -a_{ji}$ for all i, j (by definition). Hence,

$a_{ii} = -a_{ii}$ for all values of i
$\Rightarrow 2a_{ii} = 0$
$\Rightarrow a_{ii} = 0$ for all values of i
$\Rightarrow a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = 0$

Example 8.24 **Let A be a square matrix. Then prove that**

a. $A + A^T$ is a symmetric matrix,

b. $A - A^T$ is a skew-symmetric matrix and

c. AA^T and A^TA are symmetric matrices.

Sol.

a. Let $P = A + A^T$. Then,

$P^T = (A + A^T)^T = A^T + (A^T)^T$ [$\because (A + B)^T = A^T + B^T$]
$\Rightarrow P^T = A^T + A$ [$\because (A^T)^T = A$]
$\Rightarrow P^T = A + A^T = P$

[By commutative law of matrix addition]

Therefore, P is a symmetric matrix.

b. Let $Q = A - A^T$. Then,

$Q^T = (A - A^T)^T = A^T - (A^T)^T$ [$\because (A + B)^T = A^T + B^T$]
$\Rightarrow Q^T = A^T - A$ [$\because (A^T)^T = A$]
$\Rightarrow Q^T = -(A - A^T) = -Q$

Therefore, Q is skew-symmetric.

c. We have,

$(AA^T)^T = ((A^T)^T A^T)$ [By reversal law]

$= AA^T$ [$\because (A^T)^T = A$]

Therefore, AA^T is symmetric. Similarly, it can be proved that A^TA is symmetric.

Example 8.25 **Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.**

Sol. Let A be a square matrix. Then,

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q \text{ (say)}$$

where $P = (1/2)(A + A^T)$ and $Q = (1/2)(A - A^T)$.

Now $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric. Therefore, P is symmetric and Q is skew-symmetric. Hence proved.

Example 8.26 **If matrix** $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = B + C$, **where B is symmetric matrix and C is skew-symmetric matrix. Then find matrix B and C.**

Sol. Here matrix A is expressed as sum of symmetric and skew-symmetric matrix.

Then $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$

Now $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$

$$\Rightarrow B = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 5 & 2 \\ 5 & -6 & 1 \\ 2 & 1 & 8 \end{bmatrix}$$

$$\text{and } C = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 7 \\ 4 & -7 & 0 \end{bmatrix}$$

Example 8.27 $A = \begin{bmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{bmatrix}$ **is symmetric and** $B = \begin{bmatrix} d & 3 & a \\ b-a & e & -2b-c \\ -2 & 6 & -f \end{bmatrix}$ **is skew-symmetric, then find AB.**

Sol. A is symmetric

$\Rightarrow A^T = A$

$$\Rightarrow \begin{bmatrix} 3 & 2 & b \\ a & 5 & 8 \\ -1 & c & 2 \end{bmatrix} = \begin{bmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{bmatrix}$$

$\Rightarrow a = 2, b = -1, c = 8$

B is skew-symmetric

$\Rightarrow B^T = -B$

$$\Rightarrow \begin{bmatrix} d & b-a & -2 \\ 3 & e & 6 \\ a & -2b-c & -f \end{bmatrix} = \begin{bmatrix} -d & -3 & -a \\ a-b & -e & 2b+c \\ 2 & -6 & f \end{bmatrix}$$

$\Rightarrow d = -d, f = -f$ and $e = -e$

$\Rightarrow d = f = 0$

$$\text{So } A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 8 \\ -1 & 8 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 8 \\ -1 & 8 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -6 \\ -31 & 54 & -26 \\ -28 & 9 & -50 \end{bmatrix}$$

Example 8.28 **Let A, B, C, D be (not necessarily square) real matrices such that**

$A^T = BCD$; $B^T = CDA$; $C^T = DAB$ and $D^T = ABC$

for the matrix $S = ABCD$, prove that $S^3 = S$.

Sol. $S = ABCD = A(BCD) = AA^T$ (1)

$S^3 = (ABCD)(ABCD)(ABCD)$

$= (ABC)(DAB)(CDA)(BCD)$

$= D^TC^TB^TA^T$

$= (BCD)^TA^T = AA^T$ (2)

from (1) and (2), $S = S^3$

Example 8.29 **If** $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ **is a matrix satisfying $AA^T = 9I_3$, then find the values of a and b.**

Sol. We have,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$\therefore AA^T = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + 2b + 4 = 0, 2a + 2 - 2b = 0$ and $a^2 + 4 + b^2 = 9$

$\Rightarrow a + 2b + 4 = 0, a - b + 1 = 0$ and $a^2 + b^2 = 5$

Solving $a + 2b + 4 = 0$ and $a - b + 1 = 0$, we get

$a = -2, b = -1$. Clearly, these values satisfy $a^2 + b^2 = 5$.

Hence, $a = -2$ and $b = -1$.

Unitary Matrix

A square matrix is said to be unitary if $\bar{A}'A = I$ since $|\bar{A}'| = |A|$ and $|\bar{A}'A| = |\bar{A}'||A|$, therefore if $\bar{A}'A = I$, we have $|\bar{A}'||A| = 1$.

Thus, the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hermitian and Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $a_{ij} = \bar{a}_{ji}, \forall\, i, j$, i.e., $A = A^{\theta}$. For example,

$$\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}, \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$$

are Hermitian matrices.

A square matrix $A = [a_{ij}]$ is said to be skew-Hermitian if $a_{ij} = \bar{a}_{ji}, \forall\, i, j$, i.e., $A^{\theta} = -A$. For example,

$$\begin{bmatrix} 0 & -2+i \\ 2+i & 0 \end{bmatrix}, \begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$$

are skew-Hermitian matrices.

Note:

- *If A is a Hermitian matrix, then $a_{ii} = \bar{a}_{ii} \Rightarrow a_{ii}$ is real, $\forall\, i$. Thus every diagonal element of a Hermitian matrix must be real.*
- *A Hermitian matrix over the set of real numbers is actually a real symmetric matrix.*
- *If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$, i.e., a_{ii} must be purely imaginary or zero.*
- *A skew-Hermitian matrix over the set of real numbers is actually a real skew-symmetric matrix.*

Orthogonal Matrix

Any square matrix A of order n is said to be orthogonal, if $AA' = A'A = I_n$.

Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$.

Involuntary Matrix

A square matrix such that $A^2 = I$ is called involuntary matrix.

Nilpotent Matrix

A square matrix A is called a nilpotent matrix if there exists a positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A.

Example 8.30 **Show that the matrix** $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ **is idempotent.**

Sol. $A^2 = A \times A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$= A$

Hence, the matrix A is idempotent.

Example 8.31 **Show that** $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ **is a nilpotent matrix of order 3.**

Sol. Let, $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2 \times A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore \quad A^3 = O$

Hence A is nilpotent of order 3.

Example 8.32 **Show that the matrix** $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ **is involutory.**

Sol. $A^2 = A \times A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the given matrix A is involutory.

Example 8.33 **If A and B are n-rowed unitary matrices, then AB and BA are also unitary matrices.**

Sol. Let A and B be two unitary matrices. Then,

$A^{\Theta} A = AA^{\Theta} = I$ and $B^{\Theta}B = BB^{\Theta} = I$

Now,

$$(AB)^{\Theta} (AB) = (B^{\Theta}A^{\Theta}) (AB)$$
$$= B^{\Theta}(A^{\Theta}A)B = B^{\Theta} IB = B^{\Theta}B = I$$

Hence, matrix AB is unitary. Similarly, we can show that $(BA)^{\Theta} (BA) = I \Rightarrow BA$ is also unitary

Example 8.34 **If $abc = p$ and $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$, prove that A is orthogonal if and only if a, b, c are the roots of the equation $x^3 \pm x^2 - p = 0$.**

Sol. Here $AA^T = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$

$$= \begin{bmatrix} a^2+b^2+c^2 & ac+ab+bc & ab+bc+ca \\ ca+ab+bc & a^2+b^2+c^2 & cb+ba+ac \\ ab+cb+ac & bc+ca+ab & a^2+b^2+c^2 \end{bmatrix}$$

Now, $AA^T = I$ if

$a^2 + b^2 + c^2 = 1$ and $ab + bc + ca = 0$

$\Rightarrow$ $(a + b + c)^2 - 2(ab + bc + ca) = 1$ and $ab + bc + ca = 0$

$\Rightarrow$ $a + b + c = \pm 1$ and $ab + bc + ca = 0$

Also, $abc = p$, so that a, b, c are the roots of the equation $x^3 \pm x^2 - p = 0$. Conversely, since a, b, c are the roots of $x^3 \pm x^2 - p = 0$, $a + b + c = \pm 1$ and $ab + bc + ca = 0$, hence $a^2 + b^2 + c^2 = 1$.

Concept Application Exercise 8.2

1. Express the matrix $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.
2. Show that all positive integral powers of a symmetric matrix are symmetric.
3. If A and B be n-rowed orthogonal matrices, then show that AB and BA are also orthogonal matrices.
4. Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $A - \lambda I$ is a singular matrix, then find the values of λ.

ADJOINT OF SQUARE MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then, the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A. Thus,

$$\text{adj } A = [C_{ij}]^T \Rightarrow (\text{adj } A)_{ij} = C_{ji}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where C_{ij} denotes the cofactor of a_{ij} in A. For example,

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Here, $C_{11} = s$, $C_{12} = -r$, $C_{21} = -q$, $C_{22} = p$. Therefore,

$$\text{adj } A = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

Example 8.35 **Find the adjoint of the matrix**

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{1} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{2} & \mathbf{3} \end{bmatrix}$$

Sol. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$$C_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 9$$

$$C_{12} = -\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -3$$

$$C_{13} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$$

$$C_{22} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4$$

$$C_{23} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$$

$$C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

Inverse of Matrix

A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that

$AB = I_n = BA$

In such a case, we say that the inverse of A is B and we write

$A^{-1} = B$

Also from $A(\text{adj } A) = |A|I_n = (\text{adj } A)A$, we can conclude that

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

Properties of Adjoint and Inverse of a Matrix

1. Let A be square matrix of order n. Then $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$.

Proof: Let $A = [a_{ij}]$, and let C_{ij} be cofactor of a_{ij} in A. Then, $(\text{adj } A)_{ij} = C_{ji}$, $\forall\ 1 \le i, j \le n$. Now,

$$(A(\text{adj } A))_{ij} = \sum_{r=1}^{n} (A)_{ir}\, (\text{adj } A)_{rj}$$

$$= \sum_{r=1}^{n} a_{ir} C_{jr} = \begin{cases} |A|, & if \quad i = j \\ 0, & if \quad i \neq j \end{cases}$$

$$\Rightarrow \quad A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \ldots & 0 \\ 0 & |A| & 0 & \ldots & 0 \\ 0 & 0 & |A| & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & |A| \end{bmatrix}$$

$$= |A|\, I_n$$

Similarly,

$$((\text{adj } A)A)_{ij} = \sum_{r=1}^{n} (\text{adj} A)_{ir} (A)_{rj}$$

$$= \sum_{r=1}^{n} C_{ri} a_{rj} = \begin{cases} |A|, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}$$

Hence, $A(\text{adj } A) = |A|I_n = (\text{adj } A)A$

2. Every invertible matrix possesses a unique inverse.

Proof: Let A be an invertible matrix of order $n \times n$. Let B and C be two inverses of A. Then,

$AB = BA = I_n$ (1)

$AC = CA = I_n$ (2)

Now,

$AB = I_n$

$\Rightarrow$ $C(AB) = CI_n$ [pre-multiplying by C]

$\Rightarrow$ $(CA)B = CI_n$ [by associativity]

$\Rightarrow$ $I_n B = CI_n$ [$\because CA = I_n$ from (2)]

$\Rightarrow$ $B = C$ [$\because I_n B = B,\ CI_n = C$]

Hence, an invertible matrix possesses a unique inverse.

3. (*Reversal law*) If A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. In general, if $A, B, C, \ldots$ are invertible matrices, then $(ABC\cdots)^{-1} = \cdots C^{-1}B^{-1}A^{-1}$.

Proof: It is given that A and B are invertible matrices.

$\therefore$ $|A| \neq 0$ and $|B| \neq 0$

$\Rightarrow$ $|A|\,|B| \neq 0$

$\Rightarrow$ $|AB| \neq 0$ [$\because |AB| = |A|\,|B|$]

Hence, AB is an invertible matrix. Now,

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [by associativity]

$= (AI_n)A^{-1}$ [$\because BB^{-1} = I_n$]

$= AA^{-1}$ [$\because AI_n = A$]

$= I_n$ [$\because AA^{-1} = I_n$]

Also,

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$ [by associativity]

$= B^{-1}(I_n B)$ [$\because A^{-1}A = I_n$]

$= B^{-1}B$ [$\because I_n B = B$]

$= I_n$ [$\because B^{-1}B = I_n$]

Thus,

$(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$

Hence,

$(AB)^{-1} = B^{-1}A^{-1}$

4. If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof: A is an invertible matrix.

$\therefore$ $|A| \neq 0$

$\Rightarrow$ $|A^T| \neq 0$ [$\because |A^T| = |A|$]

Hence, A^T is also invertible. Now,

$AA^{-1} = I_n = A^{-1}A$

$\Rightarrow$ $(AA^{-1})^T = (I_n)^T = (A^{-1}A)^T$

$\Rightarrow$ $(A^{-1})^T(A^T) = I_n = A^T(A^{-1})^T$

[by reversal law for transpose]

$\Rightarrow$ $(A^T)^{-1} = (A^{-1})^T$ [by definition of inverse]

5. If A is a non-singular square matrix of order n, then $|\text{adj } A| = |A|^{n-1}$.

Proof: We have,

$A(\text{adj } A) = |A|\, I_n$

$$\Rightarrow \quad A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}_{n \times n}$$

$$\Rightarrow \quad |A(\text{adj } A)| = \begin{vmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & |A| \end{vmatrix} = |A|^n$$

$\Rightarrow$ $|A|\,|\text{adj } A| = |A|^n$ [$\because |AB| = |A|\,|B|$]

$\Rightarrow$ $|\text{adj } A| = |A|^{n-1}$

6. *Reversal law for adjoint*: If A and B are non-singular square matrices of the same order, then $\text{adj }(AB) = (\text{adj } B)(\text{adj } A)$ (using $(AB)^{-1} = B^{-1} . A^{-1}$)

7. If A is an invertible square matrix, then $\text{adj }(A^T) = (\text{adj } A)^T$ (using $(A^T)^{-2} = (A^{-1})^T$)

8. If A is a non-singular square matrix, then $\text{adj }(\text{adj } A) = |A|^{n-2} A$

Proof: We know that $B(\text{adj } B) = |B|I_n$ for every square matrix of order n. Replacing B by adj A, we get

$(\text{adj } A)\,[\text{adj }(\text{adj } A)] = |\text{adj } A|I_n$

$\Rightarrow$ (adj A) [adj (adj A)] $= |A|^{n-1}I_n$ [$\because$ $|\text{adj } A| = |A|^{n-1}$]

$\Rightarrow$ $A\{(\text{adj } A)(\text{adj adj } A)\} = A\{|A|^{n-1}I_n\}$ [pre-multiplying both sides by A]

$\Rightarrow$ $(A \text{ adj } A)(\text{adj adj } A) = |A|^{n-1}(AI_n)$ [by associativity]

$\Rightarrow$ $|A|I_n(\text{adj adj } A) = |A|^{n-1}A$ [$\because$ $AI_n = A$ and A adj $A = |A|I_n$]

$\Rightarrow$ $|A|(I_n(\text{adj adj } A)) = |A|^{n-1}A$

$\Rightarrow$ $|A|(\text{adj adj } A) = |A|^{n-1}A$

$\Rightarrow$ adj adj $A = |A|^{n-2}A$ $\left[\text{multiplying both sides by } \dfrac{1}{|A|}\right]$

9. If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$, i.e., $|A^{-1}| = 1/|A|$.

Proof: Since $|A| \neq 0$, therefore A^{-1} exists such that

$$AA^{-1} = I = A^{-1}A$$

$\Rightarrow$ $|AA^{-1}| = |I|$

$\Rightarrow$ $|A|\,|A^{-1}| = 1$ [$\because$ $|AB| = |A|\,|B|$ and $|I| = 1$]

$\Rightarrow$ $|A^{-1}| = \dfrac{1}{|A|}$ [$\because$ $|A| \neq 0$]

10. Inverse of the k^{th} power of A is the k^{th} power of the inverse of A.

Proof: We have to prove that $(A^{-1})^k = (A^k)^{-1}$.

$$(A^k)^{-1} = (A \times A \times A \cdots \times A)^{-1}$$
$$= (A^{-1}A^{-1}A^{-1} \cdots A^{-1})$$
$$= (A^{-1})^k$$

Example 8.36 **Find the values of K for which matrix $A = \begin{bmatrix} 1 & 0 & -K \\ 2 & 1 & 3 \\ K & 0 & 1 \end{bmatrix}$ is invertible.**

Sol. Matrix A is invertible if $|A| \neq 0$

i.e., $\begin{vmatrix} 1 & 0 & -K \\ 2 & 1 & 3 \\ K & 0 & 1 \end{vmatrix} \neq 0$

$\Rightarrow$ $1(1) - K(-K) \neq 0$

$\Rightarrow$ $|A| = K^2 + 1 \neq 0$ which is true for all real K.

Hence A is invertible for all real values of K

Example 8.37 **Let p be a non-singular matrix, and $I + p + p^2 + \cdots + p^n = O$ then find p^{-1}.**

Sol. We have $I + p + p^2 + \cdots + p^n = O$ (1)

Since p is non-singular matrix, p is invertible

Multiplying (1) both sides by p^{-1},

$\Rightarrow$ $p^{-1} + I + Ip + \cdots + p^{n-1}I = O.p^{-1}$

$\Rightarrow$ $p^{-1} + I(1 + p + \cdots + p^{n-1}) = O$

$\Rightarrow$ $p^{-1} = -I(I + p + p^2 + \cdots + p^{n-1}) = -(-p^n) = p^n$.

Example 8.38 **Given the matrices A and B as $A = \begin{bmatrix} 1 & -1 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. The two matrices X and Y are such that $XA = B$ and $AY = B$ then find the matrix $3(X + Y) = \begin{bmatrix} 4 & -1 \\ 4 & 2 \end{bmatrix}$.**

Sol. Here A is non singular but B is singular hence only A^{-1} exists

Now $XA = B$

$\Rightarrow$ $X = BA^{-1}$ (1)

And $AY = B$

$\Rightarrow$ $Y = A^{-1}B$ (2)

Also $A^{-1} = \dfrac{1}{3}\begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix}$

$$\Rightarrow X = BA^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\Rightarrow Y = A^{-1}B = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow 3(X + Y) = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 2 \end{bmatrix}$$

Example 8.39 **Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of A, then find the value α.**

Sol. Here,

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(1 + 3) + 1(2 + 3) + 1(2 - 1)$$
$$= 4 + 5 + 1 = 10$$

$$\Rightarrow \text{adj } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow B = A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow 10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Hence, $\alpha = 5$.

Example 8.40 **If $A=\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then find the value of $|A|\,|\text{adj}\,A|$.**

Sol. $|A|\,|\text{adj}\,A| = |A \text{ adj } A| = ||A|\,I|$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$= |A|^3 = (a^3)^3 = a^9$$

Example 8.41 **For the matrix $A=\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y so that $A^2 + xI = yA$. Hence, find A^{-1}.**

Sol. We have,

$$A=\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Now,

$$A^2 + xI = yA$$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$\Rightarrow$ $16 + x = 3y$, $y = 8$, $7y = 56$, $5y = 32 + x$

Putting $y = 8$ in $16 + x = 3y$, we get $x = 24 - 16 = 8$. Clearly, $x = 8$ and $y = 8$ also satisfy $7y = 56$ and $5y = 32 + x$. Hence, $x = 8$ and $y = 8$. We have,

$$|A| = \begin{vmatrix} 3 & 1 \\ 7 & 5 \end{vmatrix} = 8 \neq 0$$

So, A is invertible.

Putting $x = 8$, $y = 8$ in $A^2 + xI = yA$, we get

$$A^2 + 8I = 8A$$

$\Rightarrow$ $A^{-1}(A^2 + 8I) = 8A^{-1}A$ [re-multiplying throughout by A^{-1}]

$\Rightarrow$ $A^{-1}A^2 + 8A^{-1}I = 8A^{-1}A$

$\Rightarrow$ $A + 8A^{-1} = 8I$

[$\because A^{-1}A^2 = (A^{-1}A)A = IA = A$, $A^{-1}I = A^{-1}$ and $A^{-1}A = I$]

$\Rightarrow$ $8A^{-1} = 8I - A$

$$\Rightarrow A^{-1} = \frac{1}{8}(8I - A) = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}$$

Example 8.42 **By the method of matrix inversion, solve the system.**

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

Sol. We have,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & -5 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $AX = B$ (1)

Clearly $|A| = -4 \neq 0$. Therefore,

$$\text{adj } A = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj.} A}{|A|} = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8. & 1 & 3 \end{bmatrix}$$

Now,

$$A^{-1}B = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From Eq. (1),

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow$ $x_1 = 1, x_2 = 3, x_3 = 5$ or $y_1 = -1, y_2 = 2, y_3 = 1$

Example 8.43 **If A, B and C are $n \times n$ matrix and det (A) = 2, det (B) = 3 and det (C) = 5, then find the value of the det (A^2BC^{-1}).**

Sol. Given that $|A| = 2$, $|B| = 3$, $|C| = 5$. Now,

$$\det (A^2BC^{-1}) = |A^2BC^{-1}| = \frac{|A|^2|B|}{|C|} = \frac{4\times 3}{5} = \frac{12}{5}$$

Example 8.44 **Matrices A and B satisfy $AB = B^{-1}$, where $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$. Find**

a. without finding B^{-1}, the value of K for which $KA - 2B^{-1} + I = O$.

b. without finding A^{-1}, the matrix X satisfying $A^{-1}XA = B$.

Sol. a. $AB = B^{-1} \Rightarrow AB^2 = I$

Now,

$$KA - 2B^{-1} + I = O$$

$\Rightarrow$ $KAB - 2B^{-1}B + IB = O$

$\Rightarrow$ $KAB - 2I + B = O$

$\Rightarrow$ $KAB^2 - 2B + B^2 = O$

$\Rightarrow KI - 2B + B^2 = O$

$\Rightarrow K\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - 2\begin{bmatrix}2 & -1\\2 & 0\end{bmatrix} + \begin{bmatrix}2 & -1\\2 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\2 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}K & 0\\0 & K\end{bmatrix} - \begin{bmatrix}4 & -2\\4 & 0\end{bmatrix} + \begin{bmatrix}2 & -2\\4 & -2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}K-2 & 0\\0 & K-2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow K = 2$

b. $A^{-1}XA = B$

$\Rightarrow AA^{-1}XA = AB$

$\Rightarrow IXA = AB$

$\Rightarrow XAB = AB^2$

$\Rightarrow XAB = I$

$\Rightarrow XAB^2 = B$

$\Rightarrow XI = B$

$\Rightarrow X = B$

Example 8.45 **If there are three square matrix A, B, C of same order satisfying the equation $A^2 = A^{-1}$ and let $B = A^{2^n}$ & $C = A^{2^{(n-2)}}$ then prove that det.$(B - C) = 0, n \in N$.**

Sol. $B = A^{2^n} = A^{2 \cdot 2^{n-1}}$

$= (A^2)^{2^{n-1}}$

$= (A^{-1})^{2^{n-1}}$

$= (A^{2^{n-1}})^{-1}$

$= \left(A^{2\cdot 2^{n-2}}\right)^{-1}$

$= \left((A^2)^{2^{n-2}}\right)^{-1}$

$= \left((A^{-1})^{-1}\right)^{2^{n-2}} = A^{2^{(n-2)}}$

so $B = C \Rightarrow (B - C) = O \Rightarrow$ det. $(B - C) = 0$

Example 8.46 **If A is a non singular matrix satisfying $AB - BA = A$, then prove that det. $(B + I)$ = det. $(B - I)$.**

Sol. A is non-singular det $A \neq 0$

Given $AB - BA = A$

hence $AB = A + BA = A(I + B)$

$\Rightarrow$ det. $A \cdot$ det. B = det. $A \cdot$ det. (I + B)

$\Rightarrow$ det. B = det. $(I + B)$ (1) (as A is non singular)

again $AB - A = BA$

$\Rightarrow A(B - I) = BA$

$\Rightarrow$ (det. A) $\cdot$ det.$(B - I)$ = det. $B \cdot$ det. A

$\Rightarrow$ det. $(B - I)$ = det. (B) (2)

from (1) and (2), det. $(B - I)$ = det. $(B + I)$

Example 8.47 **If A is a symmetric and B skew symmetric matrix and $A + B$ is non singular and $C = (A + B)^{-1}(A - B)$ then prove that**

(i) $C^T(A + B)C = A + B$
(ii) $C^T(A - B)C = A - B$
(iii) $C^TAC = A$

Sol.

(i) $(A + B)C = (A + B)(A + B)^{-1}(A - B)$

$\Rightarrow (A + B)C = A - B$ (1)

$C^T = (A - B)^T((A + B)^{-1})^T$

$= (A + B)((A + B)^T)^{-1}$

{as $|A + B| \neq 0 \Rightarrow |(A + B)^T| \neq 0 \Rightarrow |A - B| \neq 0$}

$= (A + B)(A - B)^{-1}$ (2)

From (1) and (2),

$C^T(A + B)C = (A + B)(A - B)^{-1}(A - B)$

$= (A + B)$ (3)

(ii) taking transpose in (3)

$C^T(A + B)^T(C^T)^T = (A + B)^T$

$C^T(A - B)C = A - B$ (4)

(iii) adding (3) a1nd (4)

$C^T[A + B + A - B]C = 2A$

$C^TAC = A$

EQUIVALENT MATRICES

If a matrix B is obtained from a matrix A by one or more elementary transformations, then A and B are equivalent matrices and we write $A \sim$ B. Let,

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\2 & 1 & 4 & 3\\3 & 1 & 2 & 4\end{bmatrix}$$

Then

$$A \sim \begin{bmatrix}1 & 2 & 3 & 4\\1 & -1 & 1 & -1\\3 & 1 & 2 & 4\end{bmatrix} \quad [\text{Applying } R_2 \to R_2 + (-1)R_1]$$

$$\sim \begin{bmatrix}1 & 2 & 3 & 1\\1 & -1 & 1 & -2\\3 & 1 & 2 & 2\end{bmatrix} \quad [\text{Applying } C_4 \to C_4 + (-1)C_3]$$

An elementary transformation is called a row transformation or a column transformation accordingly as it is applied to rows or columns.

Theorem 1

Every elementary row (column) transformation of an $m\times n$ matrix (not identity matrix) can be obtained by pre-multiplication (post-multiplication) with the corresponding elementary matrix obtained from the identity matrix $I_m(I_n)$ by subjecting it to the same elementary row (column) transformation.

Theorem 2

Let $C = AB$ be a product of two matrices. Any elementary row (column) transformation of AB can be obtained by subjecting the pre-factor A (post factor B) to the same elementary row (column) transformation.

Method of Finding the Inverse of a Matrix by Elementary Transformations:

Let A be a non singular matrix of order n. Then A can be reduced to the identity matrix I_n by a finite sequence of elementary transformation only. As we have discussed every elementary row transformation of a matrix is equivalent to pre-multiplication by the corresponding elementary matrix. Therefore there exist elementary matrices $E_1, E_2, \ldots E_k$ such that

$$(E_k E_{k-1} \ldots E_2 E_1)A = I_n$$

$$\Rightarrow (E_k E_{k-1} \ldots E_2 E_1)AA^{-1} = I_n A^{-1} \text{ (post multiplying by } A^{-1})$$

$$\Rightarrow (E_k E_{k-1} \ldots E_2 E_1)I_n = A^{-1} \; (\because I_n A^{-1} = A^{-1} \text{and } AA^{-1} = I_n)$$

$$\Rightarrow A^{-1} = (E_k E_{k-1} \ldots E_2 E_1)I_n$$

Algorithm for Finding the Inverse of a Non Singular Matrix by Elementary Row Transformations

Let A be non-singular matrix of order n

Step I : Write $A = I_n A$

Step II : Perform a sequence of elementary row operations successively on A on the LHS and the pre factor I_n on the RHS till we obtain the result $I_n = BA$

Step III : Write $A^{-1} = B$

The following steps will be helpful to find the inverse of a square matrix of order 3 by using elementary row transformations.

Step I : Introduce unity at the intersection of first row and first column either by interchanging two rows or by adding a constant multiple of elements of some other row to first row.

Step II : After introducing unity at (1,1) place introduce zeros at all other places in first column.

Step III : Introduce unity at the intersection of 2nd row and 2nd column with the help of 2nd and 3rd row.

Step IV : Introduce zeros at all other places in the second column except at the intersection of 2nd row and 2nd column.

Step V : Introduce unity at the intersection of 3rd row and third column.

Step VI : Finally introduce zeros at all other places in the third column except at the intersection of third row and third column.

Example 8.48 **Using elementary transformation, find the inverse of the matrix** $A = \begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{-a}\right) \end{bmatrix}$.

Sol. $A = \begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix}$

We write,

$$\begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & \dfrac{b}{a} \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \frac{R_1}{a}\right)$$

$$\text{or} \quad \begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & \dfrac{1}{a} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ \dfrac{-c}{a} & 1 \end{bmatrix} A \qquad (R_2 \to R_2 - cR_1)$$

$$\text{or} \quad \begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ -c & a \end{bmatrix} A \qquad (R_2 \to aR_2)$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix} A \qquad \left(R_1 \to R_1 - \frac{b}{a}R_2\right)$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

Example 8.49 **Obtain the inverse of the following matrix using elementary operations** $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol. We have, $A = IA$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \text{ (Applying } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (Applying } R_3 \to R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (Applying } R_1 \to R_1 - 2R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \text{ (Applying } R_3 \to R_3 + 5R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \dfrac{5}{2} & \dfrac{-3}{2} & \dfrac{1}{2} \end{bmatrix} A \text{ (Applying } R_3 \to \frac{1}{2}R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \,(R_1 \to R_1 + R_3 \text{ and } R_2 \to R_2 - 2R_3)$$

Hence,

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Concept Application Exercise 8.3

1. For any 2×2 matrix A, if $A\,(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then find $|A|$, i.e., det A.
2. Find the multiplicative inverse of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
3. Find the value of x for which the matrix $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ is inverse of $B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$.
4. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $B = \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}$ where $0 < \beta < \pi/2$, then prove that $BAB = A^{-1}$. Also find the least positive value of α for which $BA^4B = A^{-1}$.
5. By using elementary operations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.
6. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and $CB = D$.

   Solve the equation $AX = B$.

SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

Consider the following system of n linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$$

or $AX = B$

Then $n \times n$ matrix A is called the coefficient matrix of the system of linear equations.

Homogeneous and Non-Homogeneous System of Linear Equations

A system of equations $AX = B$ is called a homogeneous system if $B = O$. Otherwise, it is called a non-homogeneous system of equations.

Solution of a System of Equations

Consider the system of equations $AX = B$. A set of values of the variables $x_1, x_2, \ldots, x_n$ which simultaneously satisfy all the equations is called a solution of the system of equations.

Consistent System

If the system of the equations has one or more solutions, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations.

Solution of a Non-Homogeneous System of Linear Equations

There are two methods of solving a non-homogeneous system of simultaneous linear equations:

a. Cramer's rule: Discussed in the chapter on determinants.

b. Matrix method:

Consider the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad (1)$$

If

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then, Eq. (1) is equivalent to the matrix equation

$$AX = D \qquad (2)$$

Multiplying both sides of Eq. (2) by the inverse matrix A^{-1}, we get

$$A^{-1}(AX) = A^{-1}D \Rightarrow IX = A^{-1}D \qquad [\because A^{-1}A = I]$$

$$\Rightarrow X = A^{-1}D \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad (3)$$

where A_1, B_1, etc., are the cofactors of a_1, b_1, etc., in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ (\Delta \neq 0)$$

(i) If A is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$.

(ii) If A is singular matrix, and $(\text{adj } A)\,D = O$, then the system of the equations given by $AX = D$ is consistent with infinitely many solutions.

(iii) If A is a singular matrix and (adj A) $D \neq O$, then the system of equations given by $AX = D$ is inconsistent and has no solution.

Solution of Homogeneous System of Linear Equations

Let $AX = O$ be a homogeneous system of n linear equations with n unknowns. Now if A is non-singular, then the system of equations will have a unique solution, i.e., trivial solution and if A is a singular, then the system of equations will have infinitely many solutions.

Example 8.50 **Solve the following system of equations, using matrix method: $x + 2y + z = 7, x + 3z = 11, 2x - 3y = 1$.**

Sol. The given system of equations is

$$x + 2y + z = 7$$
$$x + 0y + 3z = 11$$
$$2x - 3y + 0z = 1$$

or $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

or $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 17 \\ 1 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 18$$

So, the given system of equations has a unique solution given by $X = A^{-1}B$.

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

Now,

$$X = A^{-1} B$$

$$\Rightarrow \quad X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 - 33 + 6 \\ 42 - 22 - 2 \\ -21 + 77 - 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \Rightarrow x = 2, y = 1, z = 3$$

Example 8.51 **Show that the following system of equations is inconsistent.**

$$\mathbf{2x + 4y + 6z = 8}$$
$$\mathbf{x + 2y + 3z = 5}$$
$$\mathbf{x + y + 3z = 4}$$

Sol. The given system is

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$$

or

$$AX = B$$

where

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$$

Now,

$$\text{adj } A = \begin{bmatrix} 3 & 0 & -1 \\ -6 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{adj }(A)\, B = \begin{bmatrix} 3 & 0 & -1 \\ -6 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ -40 \\ 0 \end{bmatrix}$$

Thus, the system of equations is inconsistent.

MATRICES OF REFLECTION AND ROTATION

Reflection Matrix

Reflection in the x-axis

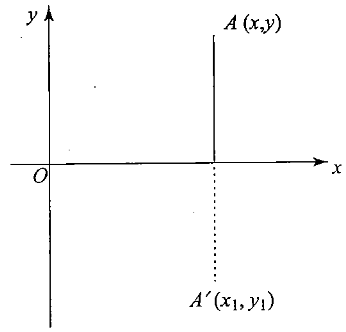

Fig. 8.1

Let A be any point and A' be its image after reflection in the x-axis.

If the coordinates of A and A' be (x, y) and (x_1, y_1), respectively, then $x_1 = x$ and $y_1 = -y$. These may be written as

$$\left.\begin{aligned} x_1 &= 1 \times x + 0 \times y \\ y_1 &= 0 \times x + (-1)y \end{aligned}\right\}$$

Thus, system of equation in the matrix form will be

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point A (x, y) in the x-axis Similarly, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ will describe the reflection of a point (x, y) in the y-axis.

Reflection through the origin

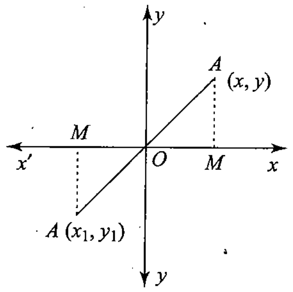

Fig. 8.2

If $A'(x_1, y_1)$ be the image of $A(x, y)$ after reflection through the origin, then

$$\left.\begin{aligned} x_1 &= -x \\ y_1 &= -y \end{aligned}\right\}$$

$\Rightarrow \quad x_1 = (-1)x + 0 \times y$ and $y_1 = 0 \times x + (-1)y$

Thus, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point $A(x, y)$ through the origin.

Reflection in the line y = x

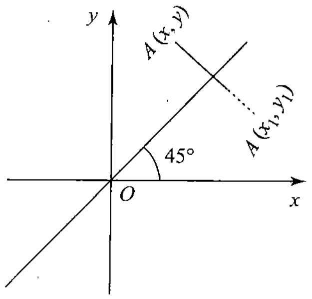

Fig. 8.3

In this case $x_1 = 0 \times x + 1 \times y$

$y_1 = 1 \times x + 0 \times y$

And the reflection matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Reflection in line y = x tan θ

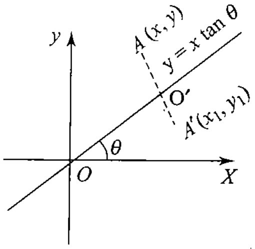

Fig. 8.4

Considering the line $y = x \tan \theta$ shown in Fig. 8.4, we have

$x_1 = x \cos 2\theta + y \sin 2\theta \quad (\because O'$ is the mid-point of $AA')$

$y_1 = x \sin 2\theta - y \cos 2\theta$

In matrix form, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

describes the reflection of a point (x, y) in the line $y = \tan \theta$.

Note: *By putting $\theta = 0, \pi/2, \pi/4$ we can get the reflection matrices in x-axis, y-axis and the line $y = x$, respectively.*

Rotation Through an Angle θ

Let $A\ (x, y)$ be any point such that $OA = r$ and $\angle AOX = \phi$.
Let OA rotate through an angle θ in the anti-clockwise direction such that $A'(x_1, y_1)$ is the new position. Then

$OA' = r$

$x_1 = x \cos \theta - y \sin \theta$

$y_1 = x \sin \theta + y \cos \theta$

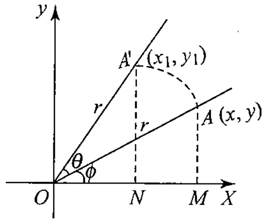

Fig. 8.5

In matrix form, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 8.52 **Find the image of the point (2, –4) under the transformations $(x, y) \to (x + 3y, y - x)$.**

Sol. Let (x_1, y_1) be the image of the point (x, y). Then by the given transformation

$x_1 = 1 \times x + 3 \times y$

$y_1 = (-1)\, x + 1 \times y$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$$

Therefore, the image is (–10, –6).

Example 8.53

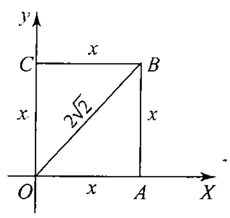

Fig. 8.6

Write down the 2 × 2 matrix A which corresponds to a counterclockwise rotation of 60° about the origin. In Fig. 8.6,

the square $OABC$ has it diagonal OB of $2\sqrt{2}$ units in length. The square is rotated counterclockwise about O through 60°. Find the coordinates of the vertices of the square after rotating.

Sol. The matrix of rotation will be

$$\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

Since each side of the square is x,

$\therefore\ x^2 + x^2 = (2\sqrt{2})^2$

$\Rightarrow x = 2$ units

Therefore the coordinates of the vertices O, A, B, C are (0, 0), (2, 0), (2, 2), (0, 2), respectively.

Let after rotation A, B, C map into A', B', C', respectively, while O maps into itself. Then,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

$\therefore\ A\,(2, 0) \rightarrow A'\,(1, \sqrt{3})$

Similarly,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3} \\ \sqrt{3}+1 \end{bmatrix}$$

$\therefore\ B\,(2, 2) \rightarrow B'\,(1-\sqrt{3}, \sqrt{3}+1)$

and

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

$\therefore\ C\,(0, 2) \rightarrow C'\,(-\sqrt{3}, 1)$

CHARACTERISTIC ROOTS AND CHARACTERISTIC VECTOR OF A SQUARE MATRIX

Definition

Any non-zero vector, X, is said to be a characteristic vector of a matrix A, if there exists a number λ such that $AX = \lambda X$. And then λ is said to be a characteristic root of the matrix A corresponding to the characteristic vector X and vice versa. Characteristic roots (vectors) are also often called proper values, latent values or eigenvalues (vectors).

Note:

It will be useful to remember that

(*i*) *a characteristic vector of a matrix cannot correspond two different characteristic roots, but*

(*ii*) *a characteristic root of a matrix can correspond to different characteristic vectors. Thus, if*

$$AX = \lambda_1 X,\ AX = \lambda_2 X,\ \lambda_2 \neq \lambda_1$$
$$\lambda_1 X = \lambda_2 X \Rightarrow (\lambda_1 - \lambda_2)X = O$$

But $X \neq O$ and $(\lambda_1 - \lambda_2) \neq 0$. And therefore $(\lambda_1 - \lambda_2)$; $X \neq O$. Thus we have a contradiction and as such we see the truth of statement (i).

But if $AX = \lambda X$, then also $A(kX) = \lambda(kX)$, so that kX is also a characteristic vector of A corresponding to the same characteristic root λ. Thus we have the truth of statement (ii).

Determinant of Characteristic Roots and Vectors

If λ be a characteristic root and X, a corresponding characteristic vector of a matrix A, then we have

$$AX = \lambda X = \lambda IX \Rightarrow (A - \lambda I)X = O$$

Since $X \neq O$, we deduce that the matrix $(A - \lambda I)$ is singular so that its determinant

$$|A - \lambda I| = 0$$

Thus, every characteristic root λ of a matrix A is a root of its characteristic equation

$$|A - \lambda I| = 0 \qquad (1)$$

Conversely, if λ be any root of the characteristic equation [Eq. (1)], then the matrix equation $(A - \lambda I)X = O$ necessarily possesses a non-zero solution X so that there exists a vector $X \neq O$ such that $AX = \lambda IX = \lambda X$.

Thus, every root of the characteristic equation of a matrix is a characteristic root of the matrix.

If A be n-rowed, then the characteristic equation $|A - \lambda I| = 0$ is of n^{th} degree so that every n-rowed square matrix possesses n characteristic roots, which, of course, may not all be distinct.

Example 8.54 **Show that the two matrices A, $P^{-1}AP$ have the same characteristic roots.**

Sol. Let,

$$\begin{aligned} P^{-1}AP &= B \\ \therefore\ B - \lambda I &= P^{-1}AP - \lambda I \\ &= P^{-1}AP - P^{-1}\lambda IP \\ &= P^{-1}(A - \lambda I)P \\ \Rightarrow |B - \lambda I| &= |P^{-1}|\,|A - \lambda I|\,|P| \\ &= |A - \lambda I|\,|P^{-1}|\,|P| \\ &= |A - \lambda I|\,|P^{-1}P| \\ &= |A - \lambda I|\,|I| = |A - \lambda I| \end{aligned}$$

Thus, the two matrices A and B have the same characteristic determinants and hence the same characteristic equations and the same characteristic roots. The same thing may also be seen in another way. Now,

$$AX = \lambda X$$
$$\Rightarrow P^{-1}AX = \lambda P^{-1}X$$
$$\Rightarrow (P^{-1}AP)(P^{-1}X) = \lambda(P^{-1}X)$$

Example 8.55 **Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n eigen values of a square matrix A of order n, then the eigen values of the matrix A^2 be $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$.**

Sol. $AX = \lambda X$

$$\Rightarrow A(AX) = A(\lambda X)$$
$$\Rightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$$

i.e.,

$$A^2X = \lambda^2 X$$

Hence, eigenvalue of A^2 is λ^2. Thus if $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A, then $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$ are eigenvalues of A^2.

Example 8.56 **Show that the characteristic roots of an idempotent matrix are either zero or unity.**

Sol. Since A is an idempotent matrix, hence,

$$A^2 = A$$

Let X be a latent vector of the matrix A corresponding to the latent root λ so that

$$AX = \lambda X \qquad (1)$$

or

$$(A - \lambda I)\, X = O$$

such that

$$X \neq O$$

On pre-multiplying Eq. (1) by A, we get

$$A(AX) = A(\lambda X) = \lambda(AX)$$

i.e.,

$$(AA)X = \lambda(AX)$$

$\Rightarrow\ AX = \lambda^2 X$

$\Rightarrow\ \lambda X = \lambda^2 X \quad (\because A^2 = A)$

$\Rightarrow\ (\lambda^2 - \lambda)\, X = O \quad (\because AX = \lambda X)$

$\Rightarrow\ \lambda^2 - \lambda = 0$

$\Rightarrow\ \lambda(\lambda - 1) = 0 \quad (\because X \neq 0)$

$\Rightarrow\ \lambda = 0, \lambda = 1$

Example 8.57 **Find the characteristic roots of the two-rowed orthogonal matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that they are of unit modulus.**

Sol. We have,

$$|A - \lambda I| = \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$$

$$= (\cos\theta - \lambda)^2 + \sin^2\theta$$

Therefore characteristic equation of A is

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$\Rightarrow\ \cos\theta - \lambda = \pm i \sin\theta$

$\Rightarrow\ \lambda = \cos\theta \pm i \sin\theta$

which are of unit modulus.

Example 8.58 **Prove that the product of the characteristic roots of a square matrix of order n is equal to the determinant of the matrix.**

Sol: Let $A = [a_{ij}]$ be a given square matrix. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the characteristic roots of A. If $\phi(\lambda)$ is the characteristic function, then

$$\phi(\lambda) = |A - \lambda I|$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n [\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n] \qquad (1)$$

$$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \qquad (2)$$

On putting $\lambda = 0$, we have

$$\phi(0) = |A| = \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \times \lambda_n = (-\lambda)^n p_n \qquad (3)$$

Hence, the product of characteristic roots of a square matrix is equal to the determinant of the matrix.

Example 8.59 **If A is non-singular, prove that the eigen values of A^{-1} are the reciprocals of the eigen value of A.**

Sol. Let A be an eigen value of A and X be a corresponding eigenvector. Then,

$$AX = \lambda X$$

$\Rightarrow\ X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$

$\Rightarrow\ \dfrac{1}{\lambda} X = A^{-1}X \quad [\because A \text{ is non-singular} \Rightarrow \lambda \neq 0]$

$\Rightarrow\ A^{-1}X = \dfrac{1}{\lambda} X$

Therefore, $1/\lambda$ is an eigenvalue of A^{-1} and X is a corresponding eigenvector.

Example 8.60 **If α is a characteristic root of a non-singular matrix, then prove that $|A/\alpha|$ is a characteristic root of adj A.**

Sol. Since α is a characteristic root of a non-singular matrix, therefore $\alpha \neq 0$. Also α is a characteristic root of A implies that there exists a non-zero vector X such that

$$AX = \alpha X$$

$\Rightarrow\ (\text{adj } A)(AX) = (\text{adj } A)(\alpha X)$

$\Rightarrow\ [(\text{adj } A)A]\, X = \alpha\, (\text{adj } A)X$

$\Rightarrow\ |A|\, IX = \alpha(\text{adj } A)X \quad [\because (\text{adj } A)A = |A|\, I]$

$\Rightarrow\ |A|\, X = \alpha(\text{adj } A)X$

$\Rightarrow\ \dfrac{|A|}{\alpha} X = (\text{adj } A)X$

$\Rightarrow\ (\text{adj } A)X = \dfrac{|A|}{\alpha} X$

Since X is a non-zero vector, therefore $|A/\alpha|$ is a characteristic root of the matrix adj A.

EXERCISES

Subjective Type

Solutions on page 8.36

1. If $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then prove that $(pI + qX)^m = p^m I + mp^{m-1} qX$, $\forall$ $p, q \in R$, where I is a two-rowed unit matrix and $m \in N$.

2. If A is an upper triangular matrix of order $n \times n$ and B is a lower triangular matrix of order $n \times n$, then prove that $(A' + B) \times (A + B')$ will be a diagonal matrix of order $n \times n$ [assume all elements of A and B to be non-negative and an elements of $(A' + B) \times (A + B')$ as C_{ij}].

3. If B, C are square matrices of order n and if $A = B + C$, $BC = CB$, $C^2 = O$, then without using mathematical induction, show that for any positive integer p, $A^{p+1} = B^p[B + (p + 1)C]$.

4. If $D = \text{diag}\ [d_1, d_2, \ldots, d_n]$, then prove that $f(D) = \text{diag}\ [f(d_1), f(d_2), \ldots, f(d_n)]$, where $f(x)$ is a polynomial with scalar coefficient.

5. Show that the solutions of the equation

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}^2 = O \text{ are } \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\alpha\beta} & -\beta \\ \alpha & \mp\sqrt{\alpha\beta} \end{bmatrix}$$

where α, β are arbitrary.

6. If $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$, then prove that $A^2 + 3A + 2I = O$, hence find B and C matrices of order 2 with integer elements, if $A = B^3 + C^3$.

7. Find the possible square roots of the two-rowed unit matrix I.

8. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then show that $A^2 - 4A - 5I = O$, where I and 0 are the unit matrix and the null matrix of order 3, respectively. Use this result to find A^{-1}.

9. If S is a real skew-symmetric matrix, then prove that $I - S$ is non-singular and the matrix $A = (I + S)(I - S)^{-1}$ is orthogonal.

10. If B and C are non-singular matrices and O is null matrix, then show that $\begin{bmatrix} A & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} O & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$.

11. Show that every square matrix A can be uniquely expressed as $P + iQ$, where P and Q are Hermitian matrices.

12. Express A as the sum of a Hermitian and a skew-Hermitian matrix, where $A = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix}$.

Objective Type

Solutions on page 8.38

Each question has four choices a, b, c and d, out of which *only one* is correct. Find the correct answer.

1. The inverse of a skew-symmetric matrix of odd order is
   a. a symmetric matrix **b.** a skew symmetric
   c. diagonal matrix **d.** does not exist

2. Let A and B be two 2×2 matrices. Consider the statements
   (i) $AB = O \Rightarrow A = O$ or $B = O$
   (ii) $AB = I_2 \Rightarrow A = B^{-1}$
   (iii) $(A + B)^2 = A^2 + 2AB + B^2$
   Then
   a. (i) and (ii) are false, (iii) is true
   b. (ii) and (iii) are false, (i) is true
   c. (i) is false, (ii) and (iii) are true
   d. (i) and (iii) are false, (ii) is true

3. The equation $\begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$ has

   (i) for $y = 0$ **(p)** rational roots
   (ii) for $y = -1$ **(q)** irrational roots
   (r) integral roots

   Then

	(i)	(ii)
a.	(p)	(r)
b.	(q)	(p)
c.	(p)	(q)
d.	(r)	(p)

4. The number of diagonal matrix A of order n for which $A^3 = A$ is
   a. 1 **b.** 0
   c. 2^n **d.** 3^n

5. If $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is n^{th} root of I_2, then choose the correct statements:
   (i) if n is odd, $a = 1, b = 0$
   (ii) if n is odd, $a = -1, b = 0$
   (iii) if n is even, $a = 1, b = 0$
   (iv) if n is even, $a = -1, b = 0$
   a. i, ii, iii **b.** ii, iii, iv
   c. i, ii, iii, iv **d.** i, iii, iv

6. A is a 2×2 matrix such that $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $A^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
   The sum of the elements of A is
   a. -1 **b.** 0
   c. 2 **d.** 5

7. The product of matrices $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a null matrix if $\theta - \phi =$

a. $2n\pi, n \in Z$ b. $n\frac{\pi}{2}, n \in Z$
c. $(2n+1)\frac{\pi}{2}, n \in Z$ d. $n\pi, n \in Z$

8. Let A, B be two matrices such that they commute, then for any positive integer n,
(i) $AB^n = B^n A$ (ii) $(AB)^n = A^n B^n$
a. only (i) is correct
b. Both (i) and (ii) are correct
c. only (ii) is correct
d. none of (i) and (ii) is correct

9. If $A = [a_{ij}]_{4\times4}$, such that $a_{ij} = \begin{cases} 2, & \text{when } i = j \\ 0, & \text{when } i \neq j \end{cases}$, then $\left\{\frac{\det(\text{adj}(\text{adj } A))}{7}\right\}$ is (where $\{\cdot\}$ represents fractional part function)
a. 1/7 **b.** 2/7
c. 3/7 **d.** none of these

10. If $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is to be the square root of two-rowed unit matrix, then α, β and γ should satisfy the relation
a. $1 - \alpha^2 + \beta\gamma = 0$ **b.** $\alpha^2 + \beta\gamma - 1 = 0$
c. $1 + \alpha^2 + \beta\gamma = 0$ **d.** $1 - \alpha^2 - \beta\gamma = 0$

11. If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is
a. $3I$ **b.** O
c. I **d.** $2I$

12. If A and B are square matrices of order n, then $A - \lambda I$ and $B - \lambda I$ commute for every scalar λ, only if
a. $AB = BA$ **b.** $AB + BA = O$
c. $A = -B$ **d.** none of these

13. Matrix A such that $A^2 = 2A - I$, where I is the identity matrix, then for $n \geq 2$, A^n is equal to
a. $2^{n-1}A - (n-1)I$ **b.** $2^{n-1}A - I$
c. $nA - (n-1)I$ **d.** $nA - I$

14. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A+1)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the value of $a + b + c + d$ is
a. 2 **b.** 1
c. 4 **d.** none of these

15. If $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then A^8 equals
a. $4B$ **b.** $128B$
c. $-128B$ **d.** $-64B$

16. If $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, then sum of all the elements of matrix A is
a. 0 **b.** 1
c. 2 **d.** -3

17. If $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$, then $A(\bar{A}^T)$ equals
a. O **b.** I
c. $-I$ **d.** $2I$

18. Identify the incorrect statement in respect of two square matrices A and B conformable for sum and product:
a. $t_r(A + B) = t_r(A) + t_r(B)$ **b.** $t_r(\alpha A) = \alpha t_r(A), \alpha \in R$
c. $t_r(A^T) = t_r(A)$ **d.** none of these

19. A is an involuntary matrix given by
$A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$, then the inverse of $A/2$ will be
a. $2A$ **b.** $\frac{A^{-1}}{2}$
c. $\frac{A}{2}$ **d.** A^2

20. If A is a non-singular matrix such that $AA^T = A^TA$ and $B = A^{-1}A^T$, then matrix B is
a. involuntary **b.** orthogonal
c. idempotent **d.** none of these

21. If P is an orthogonal matrix and $Q = PAP^T$ and $x = P^T Q^{1000} P$, then x^{-1} is, where A is involutary matrix
a. A **b.** I
c. A^{1000} **d.** none of these

22. If n^{th}-order square matrix A is a orthogonal, then, $|\text{adj}(\text{adj } A)|$ is
a. always -1 if n is even **b.** always 1 if n is odd
c. always 1 **d.** none of these

23. If both $A - \frac{1}{2}I$ and $A + \frac{1}{2}$ are orthogonal matrices, then
a. A is orthogonal
b. A is skew-symmetric matrix of even order
c. $A^2 = \frac{3}{4}I$
d. none of these

24. In which of the following type of matrix inverse does not exist always
a. idempotent **b.** orthogonal
c. involuntary **d.** none of these

25. If A is an orthogonal matrix, then A^{-1} equals
a. A^T **b.** A
c. A^2 **d.** none of these

26. If Z is an idempotent matrix, then $(I + Z)^n$
a. $I + 2^nZ$ **b.** $I + (2^n - 1)Z$
c. $I - (2^n - 1)Z$ **d.** none of these

27. If A and B are two matrices such that $AB = B$ and $BA = A$, then

a. $(A^5 - B^5)^3 = A - B$
b. $(A^5 - B^5)^3 = A^3 - B^3$
c. $A - B$ is idempotent
d. $A - B$ is nilpotent

28. If A is a nilpotent matrix of index 2, then for any positive integer n, $A(I + A)^n$ is equal to

a. A^{-1}
b. A
c. A^n
d. I_n

29. Let A be an n^{th}-order square matrix and B be its adjoint, then $|AB + KI_n|$ is (where K is a scalar quantity)

a. $(|A| + K)^{n-2}$
b. $(|A| + K)^n$
c. $(|A| + K)^{n-1}$
d. none of these

30. If $A^2 = I$, then the value of $\det(A - I)$ is (where A has order 3)

a. 1
b. −1
c. 0
d. cannot say anything

31. If $A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}, B = \begin{bmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{bmatrix}$ and if A is invertible, then which of the following is not true?

a. $|A| = |B|$
b. $|A| = -|B|$
c. $|\text{adj } A| = |\text{adj } B|$
d. A is invertible if and only if B is invertible

32. If A and B are two non-singular matrices such that $AB = C$, then $|B|$ is equal to

a. $\dfrac{|C|}{|A|}$
b. $\dfrac{|A|}{|C|}$
c. $|C|$
d. none of these

33. If A and B are squares matrices such that $A^{2006} = O$ and $AB = A + B$, then $\det(B)$ equals

a. 0
b. 1
c. −1
d. none of these

34. If matrix A is given by $A = \begin{bmatrix} 6 & 11 \\ 2 & 4 \end{bmatrix}$, then the determinant of $A^{2005} - 6A^{2004}$ is

a. 2^{2006}
b. $(-11)\, 2^{2005}$
c. $-2^{2005} \cdot 7$
d. $(-9)\, 2^{2004}$

35. If A is a non-diagonal involutory matrix, then

a. $A - I = O$
b. $A + I = O$
c. $A - I$ is non-zero singular
d. none of these

36. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then the values of a and c are equal to

a. 1, 1
b. 1, − 1
c. 1, 2
d. −1, 1

37. If A and B are two non-singular matrices of the same order such that $B^r = I$, for some positive integer $r > 1$. Then $A^{-1} B^{r-1} A - A^{-1} B^{-1} A =$

a. I
b. $2I$
c. O
d. $-I$

38. For two unimodular complex numbers z_1 and z_2, $\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$ is equal to

a. $\begin{bmatrix} z_1 & z_2 \\ \bar{z}_1 & \bar{z}_2 \end{bmatrix}$
b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
c. $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$
d. none of these

39. If $A(\alpha, \beta) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^\beta \end{bmatrix}$, then $A(\alpha, \beta)^{-1}$ is equal to

a. $A(-\alpha, -\beta)$
b. $A(-\alpha, \beta)$
c. $A(\alpha, -\beta)$
d. $A(\alpha, \beta)$

40. If $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ and $a^2 + b^2 + c^2 + d^2 = 1$, then A^{-1} is equal to

a. $\begin{bmatrix} a+ib & -c-id \\ -c+id & a-ib \end{bmatrix}$
b. $\begin{bmatrix} a+ib & -c+id \\ -c+id & a-ib \end{bmatrix}$
c. $\begin{bmatrix} a-ib & -c-id \\ -c-id & a+ib \end{bmatrix}$
d. none of these

41. If $A^3 = O$, then $I + A + A^2$ equals

a. $I - A$
b. $(I + A^1)^{-1}$
c. $(I - A)^{-1}$
d. none of these

42. If A is order 3 square matrix such that $|A| = 2$, then $|\text{adj}(\text{adj}(\text{adj } A))|$ is

a. 512
b. 256
c. 64
d. none of these

43. $(-A)^{-1}$ is always equal to (where A is n^{th}-order square matrix)

a. $(-1)^n A^{-1}$
b. $-A^{-1}$
c. $(-1)^{n-1} A^{-1}$
d. none of these

44. For each real x, $-1 < x < 1$. Let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z = \dfrac{x+y}{1+xy}$. Then

a. $A(z) = A(x)\, A(y)$
b. $A(z) = A(x) - A(y)$
c. $A(z) = A(x) + A(y)$
d. $A(z) = A(x)\, [A(y)]^{-1}$

45. If $\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-2}$, then the value of x is

a. $a/125$
b. $2a/125$
c. $2a/25$
d. none of these

46. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $f(x) = \dfrac{1+x}{1-x}$, then $f(A)$ is

a. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$
c. $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ d. none of these

47. If A is a square matrix of order n such that $|\text{adj}(\text{adj } A)| = |A|^9$, then the value of n can be
a. 4 b. 2
c. either 4 or 2 d. none of these

48. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, then $A^T A^{-1}$ is
a. $\begin{bmatrix} -\cos 2x & \sin 2x \\ -\sin 2x & \cos 2x \end{bmatrix}$ b. $\begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$
c. $\begin{bmatrix} \cos 2x & \cos 2x \\ \cos 2x & \sin 2x \end{bmatrix}$ d. none of these

49. If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$ and I is a 2 × 2 unit matrix, then $(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \sin\alpha \end{bmatrix}$ is
a. $-I + A$ b. $I - A$
c. $-I - A$ d. none of these

50. The matrix X for which $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$ is
a. $\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$ b. $\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}$
c. $\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 6 & 2 \\ \frac{11}{2} & 2 \end{bmatrix}$

51. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1} BA)^n$ is equal to
a. $A^{-n} B^n A^n$ b. $A^n B^n A^{-n}$
c. $A^{-1} B^n A$ d. $n(A^{-1} BA)$

52. If A is singular matrix, then adj A is
a. singular b. non-singular
c. symmetric d. not defined

53. The inverse of a diagonal matrix is
a. a diagonal matrix b. a skew-symmetric matrix
c. a symmetric matrix d. none of these

54. If P is non-singular matrix, then value of adj(P^{-1}) in terms of P is
a. $P/|P|$ b. $P|P|$
c. P d. none of these

55. If adj $B = A$, $|P| = |Q| = 1$, then adj$(Q^{-1} BP^{-1})$ is
a. PQ b. QAP
c. PAQ d. $PA^{-1}Q$

56. If A is non-singular and $(A - 2I)(A - 4I) = O$, then $\frac{1}{6}A + \frac{4}{3}A^{-1}$ is equal to
a. O b. I
c. $2I$ d. $6I$

57. If $A(\alpha, \beta) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^\beta \end{bmatrix}$, then $A(\alpha, \beta)^{-1}$ in terms of function of A is
a. $A(\alpha, -\beta)$ b. $A(-\alpha, -\beta)$
c. $A(-\alpha, \beta)$ d. none of these

58. If A and B are two square matrices such that $B = -A^{-1} BA$, then $(A + B)^2$ is equal to
a. $A^2 + B^2$ b. O
c. $A^2 + 2AB + B^2$ d. $A + B$

59. Let a and b be two real numbers such that $a > 1, b > 1$.
If $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, then $\lim_{n\to\infty} A^{-n}$ is
a. unit matrix b. null matrix
c. $2I$ d. none of these

60. Let $f(x) = \frac{1+x}{1-x}$. If A is matrix for which $A^3 = O$, then $f(A)$ is
a. $I + A + A^2$ b. $I + 2A + 2A^2$
c. $I - A - A^2$ d. none of these

61. If A and B are two non-zero square matrices of the same order such that the product $AB = O$, then
a. both A and B must be singular
b. exactly one of them must be singular
c. both of them are non-singular
d. none of these

62. If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A =$
a. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
c. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ d. $-\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

63. If $A^2 - A + I = 0$, then the inverse of A is
a. A^{-2} b. $A + I$
c. $I - A$ d. $A - I$

64. The number of solutions of the matrix equation $X^2 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ is
a. more than 2 b. 2
c. 0 d. 1

65. If A and B are symmetric matrices of the same order and $X = AB + BA$ and $Y = AB - BA$, then $(XY)^T$ is equal to
a. XY b. YX
c. $-YX$ d. none of these

66. If A is a 3×3 skew-symmetric matrix, then trace of A is equal to

a. -1 **b.** 1
c. $|A|$ **d.** none of these

67. Elements of a matrix A of order 10×10 are defined as $a_{ij} = w^{i+j}$ (where w is cube root of unity), then trace (A) of the matrix is

a. 0 **b.** 1
c. 3 **d.** none of these

68. Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where $\alpha \in R$. Then $(F(\alpha))^{-1}$ is equal to

a. $F(\alpha^{-1})$ **b.** $F(-\alpha)$
c. $F(2\alpha)$ **d.** none of these

69. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $G(y) = \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix}$, then $[F(x)\, G(y)]^{-1}$ is equal to

a. $F(-x)\, G(-y)$ **b.** $G(-y)\, F(-x)$
c. $F(x^{-1})\, G(y^{-1})$ **d.** $G(y^{-1})\, F(x^{-1})$

70. If A is a skew-symmetric matrix and n is odd positive integer, then A^n is

a. a skew-symmetric matrix **b.** a symmetric matrix
c. a diagonal matrix **d.** none of these

71. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where $bc \neq 0$) satisfies the equations $x^2 + k = 0$, then

a. $a + d = 0$ **b.** $k = -|A|$
c. $k = |A|$ **d.** none of these

72. If $A, B, A + I, A + B$ are idempotent matrices, then AB is equal to

a. BA **b.** $-BA$
c. I **d.** O

73 The matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is

a. idempotent matrix **b.** involutory matrix
c. nilpotent matrix **d.** none of these

74. If A is symmetric as well as skew-symmetric matrix, then A is

a. diagonal matrix **b.** null matrix
c. triangular matrix **d.** none of these

75. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1}BA)^n$ is equal to

a. $A^{-n}B^nA^n$ **b.** $A^nB^nA^{-n}$
c. $A^{-1}B^nA$ **d.** $n(A^{-1}BA)$

76. Which of the following is an orthogonal matrix?

a. $\begin{bmatrix} 6/7 & 2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$ **b.** $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$

c. $\begin{bmatrix} -6/7 & -2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ -3/7 & 6/7 & 2/7 \end{bmatrix}$ **d.** $\begin{bmatrix} 6/7 & -2/7 & 3/7 \\ 2/7 & 2/7 & -3/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$

77. If $k \in R_0$, then $\det\{\operatorname{adj}(kI_n)\}$ is equal to

a. k^{n-1} **b.** $k^{n(n-1)}$
c. k^n **d.** k

78. Given that matrix $A = \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$. If $xyz = 60$ and $8x + 4y + 3z = 20$, then $A(\text{adj } A)$ is equal to

a. $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ **b.** $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$

c. $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ **d.** $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$

79. Let $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$. Then $\text{tr}(A) - \text{tr}(B)$ has the value equal to

a. 0 **b.** 1
c. 2 **d.** none

80. If $A_1, A_3, \ldots, A_{2n-1}$ are n skew symmetric matrices of same order, then $B = \sum_{r=1}^{n} (2r - 1)(A_{2r-1})^{2r-1}$ will be

a. symmetric
b. skew-symmetric
c. neither symmetric nor skew-symmetric
d. data not adequate

81. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$. Which of the following is true?

a. $AX = B$ has a unique solution
b. $AX = B$ has exactly three solutions
c. $AX = B$ has infinitely many solutions
d. $AX = B$ is inconsistent

82. Consider three matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$. Then the value of the sum $\text{tr}(A) + \text{tr}\left(\frac{ABC}{2}\right) + \text{tr}\left(\frac{A(BC)^2}{4}\right) + \text{tr}\left(\frac{A(BC)^3}{8}\right) + \cdots + \infty$ is

a. 6 b. 9
c. 12 d. none

Multiple Correct Answers Type

Solutions on page 8.46

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If A is unimodular, then which of the following is unimodular?
   a. $-A$
   b. A^{-1}
   c. adj A
   d. ωA, where ω is cube root of unity

2. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2 + 2AB$, then
   a. $a = -1$ b. $a = 1$
   c. $b = 2$ d. $b = -2$

3. If $AB = A$ and $BA = B$, then which of the following is/are true?
   a. A is idempotent b. B is idempotent
   c. A^T is idempotent d. none of these

4. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is an orthogonal matrix, then
   a. $a = -2$ b. $a = 2, b = 1$
   c. $b = -1$ d. $b = 1$

5. Let A and B are two non-singular square matrices, A^T and B^T are the transpose matrices of A and B, respectively, then which of the following are correct?
   a. B^TAB is symmetric matrix if A is symmetric
   b. B^TAB is symmetric matrix if B is symmetric
   c. B^TAB is skew-symmetric matrix for every matrix A
   d. B^TAB is skew-symmetric matrix if A is skew-symmetric

6. If $A(\theta) = \begin{bmatrix} \sin\theta & i\cos\theta \\ i\cos\theta & \sin\theta \end{bmatrix}$, then which of the following is not true?
   a. $A(\theta)^{-1} = A(\pi - \theta)$
   b. $A(\theta) + A(\pi + \theta)$ is a null matrix
   c. $A(\theta)$ is invertible for all $\theta \in R$
   d. $A(\theta)^{-1} = A(-\theta)$

7. If A is a matrix such that $A^2 + A + 2I = O$, then which of the following is/are true?
   a. A is non-singular b. A is symmetric
   c. A cannot be skew-symmetric d. $A^{-1} = -\frac{1}{2}(A + I)$

8. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then
   a. adj(adj A) = A b. |adj(adj A)| = 1
   c. |adj A| = 1 d. none of these

9. If $\begin{pmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{pmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then
   a. $a = \cos 2\theta$ b. $a = 1$
   c. $b = \sin 2\theta$ d. $b = -1$

10. If $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -1/3 \end{bmatrix}$, then
   a. $|A| = -1$ b. adj $A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -3 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}$
   c. $A = \begin{bmatrix} 1 & 1/3 & 7 \\ 0 & 1/3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ d. $A = \begin{bmatrix} 1 & -1/3 & -7 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11. If B is an idempotent matrix, and $A = I - B$, then
   a. $A^2 = A$ b. $A^2 = I$
   c. $AB = O$ d. $BA = O$

12. Which of the following statements is/are true about square matrix A of order n?
   a. $(-A)^{-1}$ is equal to $-A^{-1}$ when n is odd only.
   b. If $A^n = O$, then $I + A + A^2 + \cdots + A^{n-1} = (I - A)^{-1}$.
   c. If A is skew-symmetric matrix of odd order, then its inverse does not exist.
   d. $(A^T)^{-1} = (A^{-1})^T$ holds always.

13. If $A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$, then $A_iA_k + A_kA_i$ is equal to
   a. $2I$ if $i = k$ b. O if $i \neq k$
   c. $2I$ if $i \neq k$ d. O always

14. If $A = (a_{ij})_{n \times n}$ and f is a function, we define $f(A) = (f(a_{ij}))_{n \times n}$. Let $A = \begin{pmatrix} \pi/2 - \theta & \theta \\ -\theta & \pi/2 - \theta \end{pmatrix}$. Then
   a. sin A is invertible b. sin A = cos A
   c. sin A is orthogonal d. sin (2A) = 2 sin A cos A

15. If $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal, then

a. $a = \pm \frac{1}{\sqrt{2}}$ b. $b = \pm \frac{1}{\sqrt{12}}$

c. $c = \pm \frac{1}{\sqrt{3}}$ d. none of these

16. If A and B are two invertible matrices of the same order, then adj(AB) is equal to

a. adj(B) adj(A) b. $|B|\,|A|\,B^{-1}A^{-1}$

c. $|B|\,|A|\,A^{-1}B^{-1}$ d. $|A|\,|B|\,(AB)^{-1}$

17. If A, B and C are three square matrices of the same order, then $AB = AC \Rightarrow B = C$. Then

a. $|A| \neq 0$ b. A is invertible

c. A may be orthogonal d. A is symmetric

18. Suppose $a_1, a_2, \ldots$ are real numbers, with $a_1 \neq 0$. If $a_1, a_2, a_3, \ldots$ are in A.P., then

a. $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{bmatrix}$ is singular (where $i = \sqrt{-1}$)

b. the system of equations $a_1x + a_2y + a_3z = 0$, $a_4x + a_5y + a_6z = 0$, $a_7x + a_8y + a_9z = 0$ has infinite number of solutions

c. $B = \begin{bmatrix} a_1 & ia_2 \\ ia_2 & a_1 \end{bmatrix}$ is non-singular

d. none of these

19. If α, β, γ are three real numbers and $A = \begin{bmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{bmatrix}$, then which of following is/are true?

a. A is singular b. A is symmetric

c. A is orthogonal d. A is not invertible

20. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then which of following is not true?

a. $\lim_{n\to\infty} \frac{1}{n^2} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ b. $\lim_{n\to\infty} \frac{1}{n} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

c. $A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \forall\, n \neq N$ d. none of these

21. If C is skew-symmetric matrix of order n and X is $n \times 1$ column matrix, then X^TCX is

a. singular b. non-singular

c. invertible d. non-invertible

22. If $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} b+c & c+a & b-c \\ c-b & c+b & a-b \\ b-c & a-c & a+b \end{bmatrix}$ $(a, b, c \neq 0)$, then SAS^{-1} is

a. symmetric matrix b. diagonal matrix

c. invertible matrix d. singular matrix

23. If D_1 and D_2 are two 3×3 diagonal matrices, then which of the following is/are true?

a. D_1D_2 is diagonal matrix b. $D_1D_2 = D_2D_1$

c. $D_1^2 + D_2^2$ is a diagonal matrix d. none of these

24. If A and B are symmetric and commute, then which of the following is/are symmetric?

a. $A^{-1}B$ b. AB^{-1}

c. $A^{-1}B^{-1}$ d. None of these

25. A skew-symmetric matrix A satisfies the relation $A^2 + I = O$, where I is a unit matrix then A is

a. idempotent b. orthogonal

c. of even order d. odd order

26. If $AB = A$ and $BA = B$, then

a. $A^2B = A^2$ b. $B^2A = B^2$

c. $ABA = A$ d. $BAB = B$

27. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Then

a. $A^2 - 4A - 5I_3 = O$ b. $A^{-1} = \frac{1}{5}(A - 4I_3)$

c. A^3 is not invertible d. A^2 is invertible

Reasoning Type

Solutions on page 8.49

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** $|\text{adj}(\text{adj}(\text{adj } A))| = |A|^{(n-1)^3}$, where n is order of matrix A.

Statement 2: $|\text{adj } A| = |A|^n$.

2. **Statement 1:** If $D = \text{diag}\,[d_1, d_2, \ldots, d_n]$, then $D^{-1} = \text{diag}\,[d_1^{-1}, d_2^{-1} \ldots, d_n^{-1}]$.

Statement 2: If $D = \text{diag}\,[d_1, d_2, \ldots, d_n]$, then $D^n = \text{diag}\,[d_1^n, d_2^n \ldots, d_n^n]$.

3. **Statement 1:** Matrix 3×3, $a_{ij} = \frac{i-j}{i+2j}$ cannot be expressed as a sum of symmetric and skew-symmetric matrix.

Statement 2: Matrix 3×3, $a_{ij} = \frac{i-j}{i+2j}$ is neither symmetric nor skew-symmetric.

4. **Statement 1:** If a, b, c, d are real numbers and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^3 = O$, then $A^2 = O$.
Statement 2: For matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have
$A^2 - (a + d)A + (ad - bc)I = O$.

5. **Statement 1:** If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then
$[F(\alpha)]^{-1} = F(-\alpha)$.
Statement 2: For matrix $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$,
we have $[G(\beta)]^{-1} = G(-\beta)$.

6. **Statement 1:** $A = \begin{bmatrix} 4 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Then $(AB)^{-1}$ does not exist.
Statement 2: Since $|A| = 0$, $(AB)^{-1} = B^{-1}A^{-1}$ is meaningless.

7. **Statement 1:** The determinant of a matrix $A = [a_{ij}]_{5 \times 5}$ where $a_{ij} + a_{ji} = 0$ for all i and j is zero.
Statement 2: The determinant of a skew-symmetric matrix of odd order is zero.

8. **Statement 1:** The inverse of the matrix $A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0$, $i \geq j$ is $B = [a_{ij}^{-1}]_{n \times n}$.
Statement 2: The inverse of singular matrix does not exist.

9. **Statement 1:** For a singular square matrix A, $AB = AC \Rightarrow B = C$.
Statement 2: If $|A| = 0$, then A^{-1} does not exist.

10. **Statement 1:** If A, B, C are matrices such that $|A_{3\times 3}| = 3$, $|B_{3 \times 3}| = -1$ and $|C_{2 \times 2}| = +2$, then $|2ABC| = -12$.
Statement 2: For matrices A, B, C of the same order, $|ABC| = |A|\,|B|\,|C|$.

11. **Statement 1:** If the matrices A, B, $(A + B)$ are non-singular, then $[A(A + B)^{-1} B]^{-1} = B^{-1} + A^{-1}$.
Statement 2: $[A(A + B)^{-1} B]^{-1} = [A(A^{-1} + B^{-1})B]^{-1}$
$= [(I + AB^{-1})B]^{-1}$
$= [(B + AB^{-1}B)]^{-1}$
$= [(B + AI)]^{-1}$
$= [(B + A)]^{-1}$
$= B^{-1} + A^{-1}$.

12. **Statement 1:** Let A, B be two square matrices of the same order such that $AB = BA$, $A^m = O$ and $B^n = O$ for some positive integers m, n, then there exists a positive integer r such that $(A + B)^r = O$.
Statement 2: If $AB = BA$ then $(A + B)^r$ can be expanded as binomial expansion.

13. **Statement 1:** If $A = [a_{ij}]_{n \times n}$ is such that $a_{ij} = \bar{a}_{ji}, \forall\ i, j$ and $A^2 = O$, then matrix A null matrix.
Statement 2: $|A| = 0$.

14. **Statement 1:** If A is orthogonal matrix of order 2, then $|A| = \pm 1$.
Statement 2: Every two-rowed real orthogonal matrix is of any one of the forms $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

Linked Comprehension Type

Solutions on page 8.50

Based upon each paragraph, some multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Let A is matrix of order 2×2 such that $A^2 = O$.

1. $A^2 - (a + d)A + (ad - bc)I$ is equal to
a. I **b.** O
c. $-I$ **d.** none of these

2. tr(A) is equal to
a. 1 **b.** 0
c. -1 **d.** none of these

3. $(I + A)^{100} =$
a. $100A$ **b.** $100(I + A)$
c. $100I + A$ **d.** $I + 100A$

For Problems 4–6

If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$, then

4. Which of the following is true?
a. If matrix A is singular then matrix B is non-singular.
b. If matrix A is non-singular then matrix B is singular.
c. If matrix A is singular then matrix B is also singular.
d. Cannot say anything.

5. $(A + B)^7$ is equal to
a. $7(A + B)$ **b.** $7 \cdot I_{3\times 3}$
c. $64\,(A + B)$ **d.** $128\,I$

6. $(A + I)^5$ is equal to (where I is identity matrix)
a. $I + 60I$ **b.** $I + 16A$
c. $I + 31A$ **d.** none of these

For Problems 7–8

Consider an arbitrary 3×3 matrix $A = [a_{ij}]$, a matrix $B = [b_{ij}]$ is formed such that b_{ij} is the sum of all the elements except a_{ij} in the i^{th} row of A. Answer the following questions.

7. If there exists a matrix X with constant elements such that $AX = B$, then X is
a. skew-symmetric **b.** null matrix
c. diagonal matrix **d.** none of these

8. The value of $|B|$ is equal to
a. $|A|$ **b.** $|A|/2$
c. $2|A|$ **d.** none of these

For Problems 9–11

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ satisfies $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. And trace of a square matrix X is equal to the Sum of elements in its principal diagonal.

Further consider a matrix $\underset{3\times3}{U}$ with its column as U_1, U_2, U_3 such that

$$A^{50}U_1 = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix}, A^{50}U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A^{50}U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then answer the following questions.

9. The values of $|A^{50}|$ equals

a. 0 **b.** 1
c. –1 **d.** 25

10. Trace of A^{50} equals

a. 0 **b.** 1
c. 2 **d.** 3

11. The value of $|U|$equals

a. 0 **b.** 1
c. 2 **d.** –1

For Problems 12–14

Let A be a square matrix of order 2 or 3 and I be the identity matrix of the same order. Then the matrix $A - \lambda I$ is called characteristic matrix of the matrix A, where λ is some complex number. The determinant of the characteristic matrix is called characteristic determinant of the matrix A which will of course be a polynomial of degree 3 in λ. The equation $\det(A - \lambda I) = 0$ is called characteristic equation of the matrix A and its roots (the values of λ) are called characteristic roots or eigenvalues. It is also known that every square matrix has its characteristic equation.

12. The eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ are

a. 2, 1, 1 **b.** 2, 3, –2
c. –1, 1, 3 **d.** none of these

13. Which of the following matrices do not have eigenvalues as 1 and –1?

a. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ (where $i = \sqrt{-1}$)

c. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

14. If one of the eigenvalues of a square matrix A of order 3×3 is zero, then

a. det A must be non-zero **b.** det A must be zero
c. adj A must be a zero matrix **d.** none of these

For Problems 15–17

Let A be a $m \times n$ matrix. If there exists a matrix L of type $n \times m$ such that $LA = I_n$, then L is called left inverse of A. Similarly, if there exists a matrix R of type $n \times m$ such that $AR = I_m$, then R is called right inverse of A. For example, to find right inverse of matrix

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$, we take $R = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$

and solve $AR = I_3$, i.e.,

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow x - u = 1$ $\quad y - v = 0$ $\quad z - w = 0$

$x + u = 0$ $\quad y + v = 1$ $\quad z + w = 0$

$2x + 3u = 0$ $\quad 2y + 3v = 0$ $\quad 2z + 3w = 1$

As this system of equations is inconsistent, we say there is no right inverse for matrix A.

15. Which of the following matrices is NOT left inverse of matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$?

a. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & -7 & 3 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

c. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 3 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

16. The number of right inverses for the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ is

a. 0 **b.** 1
c. 2 **d.** infinite

17. For which of the following matrices, the number of left inverses is greater than the number of right inverses?

a. $\begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix}$ **d.** $\begin{bmatrix} 3 & 3 \\ 1 & 1 \\ 4 & 4 \end{bmatrix}$

For Problems 18–20

If e^A is defined as $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \frac{1}{2}\begin{bmatrix} f(x) & g(x) \\ g(x) & f(x) \end{bmatrix}$

where $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $0 < x < 1$, then I is an identity matrix.

18. $\int \frac{g(x)}{f(x)} dx$ is equal to

a. $\log(e^x + e^{-x}) + c$ **b.** $\log(e^x - e^{-x}) + c$
c. $\log(e^{2x} - 1) + c$ **d.** none of these

19. $\int (g(x)+1)\sin x\,dx$ is equal to

a. $\dfrac{e^x}{2}(\sin x - \cos x) + c$ **b.** $\dfrac{e^{2x}}{5}(2\sin x - \cos x) + c$

c. $\dfrac{e^x}{5}(\sin 2x - \cos 2x) + c$ **d.** none of these

20. $\int \dfrac{f(x)}{\sqrt{g(x)}}dx$ is equal to

a. $\dfrac{1}{2\sqrt{e^x-1}} - \operatorname{cosec}^{-1}(e^x) + c$

b. $\dfrac{2}{\sqrt{e^x - e^{-x}}} - \sec^{-1}(e^x) + c$

c. $\dfrac{1}{2\sqrt{e^{2x}-1}} + \sec^{-1}(e^x) + c$

d. none of these

Matrix-Match Type

Solutions on page 8.52

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, b→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1.

Column I	Column II
a. $(I-A)^n$ is if A is idempotent	**p.** $2^{n-1}(I-A)$
b. $(I-A)^n$ is if A is involuntary	**q.** $I - nA$
c. $(I-A)^n$ is if A is nilpotent of index 2	**r.** A
d. If A is orthogonal, then $(A^T)^{-1}$	**s.** $I - A$

2.

Column I	Column II
a. If A is an idempotent matrix and I is an identity matrix of the same order, then the value of n, such that $(A+I)^n = I + 127$ is	**p.** 9
b. If $(I-A)^{-1} = I + A + A^2 + \cdots + A^7$, then $A^n = O$ where n is	**q.** 10
c. If A is matrix such that $a_{ij} = (i+j)(i-j)$, then A is singular if order of matrix is	**r.** 7
d. If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric, then order of A can be	**s.** 8

3.

Column I (A, B, C are matrices)	Column II
a. If $\|A\| = 2$, then $\|2A^{-1}\| =$ (where A is of order 3)	**p.** 1
b. If $\|A\| = 1/8$, then $\|\text{adj}(\text{adj}(2A))\| =$ (where A is of order 3)	**q.** 4
c. If $(A+B)^2 = A^2 + B^2$, and $\|A\| = 2$, then $\|B\| =$ (where A and B are of odd order)	**r.** 24
d. $\|A_{2\times 2}\| = 2$, $\|B_{3\times 3}\| = 3$ and $\|C_{4\times 4}\| = 4$, then $\|ABC\|$ is equal to	**s.** 0
	t. does not exist

Integer Type

Solutions on page 8.53

1. $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and $(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$ (where I is the 2×2 identity matrix), then the product of all elements of matrix V is.

2. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix and $f(x) = x - x^2$ and $bc = 1/4$ then the value of $1/f(a)$ is.

3. Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$ then the minimum value of $\sum(\cos^x\theta + \sin^x\theta)$, $\theta \in R$.

4. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ and $f(x)$ is defined as $f(x) = \det.(A^TA^{-1})$ then the value of $\underbrace{f(f(f(f\ldots\ldots\ldots f(x))))}_{n\text{-times}}$ is $(n \ge 2)$.

5. The equation $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a solution for (x, y, z) besides $(0, 0, 0)$. Then the value of k is.

6. If A is an idempotent matrix satisfying, $(I - 0.4A)^{-1} = I - \alpha A$ where I is the unit matrix of the same order as that of A then the value of $|9\alpha|$ is equal to.

7. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B = [a\, b\, c]$ and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+2)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$ be three given matrices, where a, b, c and $x \in R$, Given that $\text{tr}\cdot(AB) = \text{tr}\cdot(C)$ $x \in R$, where $\text{tr}\cdot(A)$ denotes trace of A. If $f(x) = ax^2 + bx + c$ then the value of $f(1)$ is.

8. Let A be the set of all 3×3 skew symmetric matrices whose entries are either -1, 0 or 1. If there are exactly three 0's, three 1's and three (-1)'s, then the number of such matrices, is.

9. Let $A = [a_{ij}]_{3\times3}$ be a matrix such that $AA^T = 4I$ and $a_{ij} + 2c_{ij} = 0$ where c_{ij} is the cofactor of a_{ij} and I is the unit matrix of order 3.

$$\begin{vmatrix} a_{11}+4 & a_{12} & a_{13} \\ a_{21} & a_{22}+4 & a_{23} \\ a_{31} & a_{32} & a_{33}+4 \end{vmatrix} + 5\lambda \begin{vmatrix} a_{11}+1 & a_{12} & a_{13} \\ a_{21} & a_{22}+1 & a_{23} \\ a_{31} & a_{32} & a_{33}+1 \end{vmatrix} = 0$$

then the value of 10λ is.

10. Let S be the set which contains all possible values of l, m, n, p, q, r for which

$$A = \begin{bmatrix} l^2-3 & p & 0 \\ 0 & m^2-8 & q \\ r & 0 & n^2-15 \end{bmatrix}$$ be a non-singular idempotent matrix. Then the sum of all the elements of the set S is.

11. If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and tr(A) = 12, then the value of $|A|^{1/2}$ is.

12. If A is a square matrix of order3 such that $|A| = 2$ then $|(\text{adj } A^{-1})^{-1}|$ is.

Archives

Solutions on page 8.54

Subjective Type

1. Given a matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^TA = I$, then find the value of $a^3 + b^3 + c^3$. **(IIT-JEE, 2003)**

2. If M is a 3×3 matrix, where det $M = 1$ and $MM^T = I$, where 'I' is an identity matrix, prove that det $(M - I) = 0$. **(IIT-JEE, 2004)**

3. If $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $AX = U$ has infinitely many solutions, prove that $BX = V$ has no unique solution. **(IIT-JEE, 2004)**

4. Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)' = (-1)^k AB$, where $(AB)'$ is the transpose of the matrix AB, then find the possible values of k. **(IIT-JEE, 2008)**

Objective Type

Multiple choice questions with one correct answer

1. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then value of α for which $A^2 = B$ is

   a. 1 **b.** -1
   c. 4 **d.** no real values

   (IIT-JEE, 2003)

2. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then the value of α is

   a. ± 1 **b.** ± 2 **c.** ± 3 **d.** ± 5

   (IIT-JEE, 2004)

3. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$. The values of c and d are

   a. $(-6, -11)$ **b.** $(6, 11)$
   c. $(-6, 11)$ **d.** $(6, -11)$

   (IIT-JEE, 2005)

4. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $x = P^TQ^{2005}P$, then X is equal to

   a. $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

   b. $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

   c. $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

   d. $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

   (IIT-JEE, 2005)

5. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

   a. 0 **b.** $2^9 - 1$ **c.** 168 **d.** 2

   (IIT-JEE, 2010)

6. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \theta & 1 \end{bmatrix}$, where each of $a, b,$ and c is either ω or ω^2. Then the number of distinct matrices in the set S is

   a. 2 **b.** 6 **c.** 4 **d.** 8

   (IIT-JEE, 2011)

Multiple choice questions with one or more correct answer

1. Let M and N be two 3×3 non-singular skew symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2 (M^TN)^{-1} (MN^{-1})^T$ is equal to

   a. M^2 **b.** $-N^2$ **c.** $-M^2$ **d.** MN

   (IIT-JEE, 2011)

Comprehension

Read the passages given below and answer the questions that follow.

For Problems 1–3

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, and U_1, U_2 and U_3 are columns of a 3 × 3 matrix U. If column matrices U_1, U_2 and U_3 satisfy

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

then answer the following questions. **(IIT-JEE, 2006)**

1. The value $|U|$ is
   a. 3 **b.** −3 **c.** 3/2 **d.** 2
2. The sum of the elements of the matrix U^{-1} is
   a. −1 **b.** 0 **c.** 1 **d.** 3
3. The value of $[3\ \ 2\ \ 0]\ U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is
   a. 5 **b.** 5/2 **c.** 4 **d.** 3/2

For Problems 4–6

Let A be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0. **(IIT-JEE, 2009)**

4. The number of matrices in A is
   a. 12 **b.** 6 **c.** 9 **d.** 3
5. The number of matrices A in A for which the system of linear equations
   $$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
   has a unique solution is
   a. less than 4 **b.** at least 4 but less than 7
   c. at least 7 but less than 10 **d.** at least 10
6. The number of matrices A in A for which the system of linear equations
   $$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
   is inconsistent is
   a. 0 **b.** more than 2
   c. 2 **d.** 1

For Problems 7–9

Let P be an odd prime number and T_p be the following set of 2 × 2 matrices:

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, \ldots, p-1\} \right\}$$ **(IIT-JEE, 2010)**

7. The number of A in T_p such that A is either symmetric or skew – symmetric or both, and det(A) divisible by p is
   a. $(p-1)^2$ **b.** $2(p-1)$
   c. $(p-1)^2 + 1$ **d.** $2p - 1$
8. The number of A in T_p such that the trace of A is not divisible by p but det(A) divisible by p is [Note : The trace of matrix is the sum of its diagonal entries].
   a. $(p-1)(p^2 - p + 1)$ **b.** $p^3 - (p-1)^2$
   c. $(p-1)^2$ **d.** $(p-1)(p^2-2)$
9. The number of A in T_p such that det(A) is not divisible by p is
   a. $2p^2$ **b.** $p^3 - 5p$
   c. $p^3 - 3p$ **d.** $p^3 - p^2$

For Problems 10–12

Let a, b and c be three real numbers satisfying $[a\ b\ c] \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0]$ **(IIT-JEE, 2011)**

10. If the point $P(a, b, c)$ with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is
    a. 0 **b.** 12 **c.** 7 **d.** 6
11. Let ω be a solution of $x^3 - 1 = 0$ with Im(ω) > 0. If $a = 2$ with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to
    a. −2 **b.** 2 **c.** 3 **d.** −3
12. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} + \frac{1}{\beta} \right)^n$ is
    a. 6 **b.** 7 **c.** $\frac{6}{7}$ **d.** ∞

Integer type

1. Let K be a positive real number and

   $$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2 \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

   If det (adj A) + det (adj B) = 10^6, then $[k]$ is equal to.

   [Note: adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k]. **(IIT-JEE, 2010)**

2. Let M be a 3 × 3 matrix satisfying

   $$M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \text{ and } M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

   Then the sum of the diagonal entries of M is. **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. We have

$$X^2 = X \times X = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = O$$

$\Rightarrow \quad X^3 = X^2 \times X = O \times X = O$

Similarly, $X^4 = X^5 = X^6 = \cdots = O$

Now, by binomial theorem, we have

$$(pI + qX)^m = (pI)^m + {}^mC_1 (pI)^{m-1}(qX) + {}^mC_2 (pI)^{m-2}(qX)^2 + \cdots + {}^mC_m (qX)^m$$

$$= p^m I + mp^{m-1} qX + \frac{m(m-1)}{2!} p^{m-2}(qX)^2 + \cdots + q^m X^m$$

$$= p^m I + mp^{m-1} qX + O + O + \cdots + O \; (\because X^2 = X^3 = \cdots = X^m = O)$$

$\Rightarrow \quad (pI + qX)^m = p^m + mp^{m-1}qX, \quad \forall\, p, q \in R$

2. A is an upper triangular matrix. So,

$$A = \begin{cases} a_{ij} = 0, & i > j \\ a_{ij} \neq 0, & i \le j \end{cases} \text{ and } A' = \begin{cases} a_{ij} = 0, & i < j \\ a_{ij} \neq 0, & i \ge j \end{cases}$$

B is lower triangular matrix. So,

$$B = \begin{cases} b_{ij} = 0, & i < j \\ b_{ij} \neq 0, & i \ge j \end{cases}$$

and $B' \begin{cases} b_{ij} = 0, & i > j \\ b_{ij} \neq 0, & i \le j \end{cases}$

Let, $C' = A' + B = \begin{cases} c'_{ij} = 0, & i < j \\ c'_{ij} \neq 0, & i \ge j \end{cases}$

$$C'' = (A + B') = \begin{cases} c''_{ij} = 0, & i > j \\ c''_{ij} \neq 0, & i \le j \end{cases}$$

$\therefore \quad (A' + B) \times (A + B') = C$ (let) for C, $c_{ij} \neq 0$ (for all i and j)

Therefore, $(A' + B) \times (A + B')$ is a matrix of order $n \times n$, $\forall\, c_{ij} \neq 0$ in any case.

3. $A^{p+1} = (B + C)^{p+1}$

We can expand $(B + C)^{p+1}$ like binomial expansion as $BC = CB$.

$\therefore \quad (B + C)^{p+1} = {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + {}^{p+1}C_2 B^{p-1} C^2 + \cdots + {}^{p+1}C_{p+1} C^{p+1}$

$$= {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + O + O + \cdots + O \; (\because C^2 = O \Rightarrow C^3 = C^4 = \cdots = O)$$

$$= B^{p+1} + (p+1)B^p C$$

$$= B^p[B + (p+1)C]$$

4. Let, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$\Rightarrow \quad f(D) = a_0 I + a_1 D + a_2 D^2 + \cdots + a_n D^n$

$= a_0 \times \text{diag}\,(1, 1, \ldots, 1)$

$+ a_1 \times \text{diag}\,(d_1, d_2, \ldots, d_n)$

$+ a_2 \times \text{diag}\,(d_1^2, d_2^2, \ldots, d_n^2)$

$+ \cdots$

$\vdots$

$+ a_n \times \text{diag}\,(d_1^n, d_2^n, \ldots, d_n^n)$

$= \text{diag}\,(a_0 + a_1 d_1 + a_2 d_1^2 + \cdots + a_n d_1^n,$

$a_0 + a_1 d_2 + a_2 d_2^2 + \cdots + a_n d_2^n,$

$a_0 + a_1 d_3 + a_2 d_3^2 + \cdots + a_n d_3^n$

$\vdots$

$a_0 + a_1 d_n + a_2 d_n^2 + \cdots + a_n d_n^n)$

$= \text{diag}\,(f(d_1), f(d_2), \ldots, f(d_n))$

5. Given equation is

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x^2 + yz & xy + yt \\ zx + tz & zy + t^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x^2 + yz = 0 \quad (1)$

$y(x + t) = 0 \quad (2)$

$z(x + t) = 0 \quad (3)$

$yz + t^2 = 0 \quad (4)$

From Eqs. (1) and (4), we have $x^2 = t^2$ or $x = \pm t$.

Case I:

If $x = t$, from Eqs. (2) and (3), $y = 0$, $z = 0$. Then from Eq. (1), $x = 0 = t$.

Case II:

If $x = -t$, then Eqs. (2) and (3) are satisfied for all values of y and z. If we take $y = -\beta$, $z = \alpha$, then from Eq. (1),

$$x = \pm\sqrt{\alpha\beta} = -t$$

Obviously, Case I is included in Case II ($\alpha = 0 = \beta$). Hence, the general solution of the given equation is

$$x = -t = \pm\sqrt{\alpha\beta},\ y = -\beta,\ z = \alpha$$

$$\Rightarrow \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\alpha\beta} & -\beta \\ \alpha & \mp\sqrt{\alpha\beta} \end{bmatrix}$$

where α, β are arbitrary.

6. $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$$

$\Rightarrow \quad A^2 + 3A + 2I$

$$= \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} + 3\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad A^2 + 3A + 2I = O \quad (1)$

From Eq. (1),

$A^3 + 3A^2 + 2A = O$

$\Rightarrow \quad (A + I)^3 - A = I^3$

$\Rightarrow \quad A = (A + I)^3 - I^3 = (A + I)^3 + (-I)^3$

$\Rightarrow \quad B = A + I$ and $C = -I$

$$\therefore \quad B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

and $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

7. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square root of the matrix.

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $A^2 = I$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2+bc = 1 \quad (1)$$

$$ab+bd = 0 \quad (2)$$

$$ac+cd = 0 \quad (3)$$

$$cb+d^2 = 1 \quad (4)$$

If $a + d = 0$, the above four equations hold simultaneously if $d = -a$ and $a^2 + bc = 1$. Hence, one possible square root of I is

$$A=\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$$

where α, β, γ are the three numbers related by the condition $\alpha^2 + \beta\gamma = 1$.

If $a + d \neq 0$, then above four equations hold simultaneously if $b = 0, c = 0, a = 1, d = 1$ or if $b = 0, c = 0, a = -1, d = -1$.

Hence, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

i.e., $\pm I$ are other possible square roots of I.

8. Given $A=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

$$\therefore \quad A^2 = A\times A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\times\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix}$$

$$=\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

$$\therefore \quad A^2-4A-5I=\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}-4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}-5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}-\begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix}-\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$=\begin{pmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \quad A^2-4A-5I=O$$

or

$$5I = A^2 - 4A$$

Multiplying by A^{-1}, we get

$$5A^{-1} = A - 4I$$

$$=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}-4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}-\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 1-4 & 2-0 & 2-0 \\ 2-0 & 1-4 & 2-0 \\ 2-0 & 2-0 & 1-4 \end{pmatrix}$$

$$=\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\therefore \quad A^{-1}=\frac{1}{5}\begin{pmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{pmatrix}=\begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}$$

9. First, we will show that $I - S$ is non-singular. The equality $|I - S| = 0$ implies that I is a characteristic root of the matrix S, but this is not possible, for a real skew-symmetric matrix can have zero or purely imaginary numbers as its characteristic roots. Thus $|I - S| \neq 0$, i.e., $I - S$ is non-singular. We have,

$$A^T = [(I-S)^{-1}]^T (I+S)^T$$

$$= [(I-S)^T]^{-1} (I+S)^T$$

But

$$(I-S)^T = I^T - S^T = I + S$$

and

$$(I+S)^T = I^T + S^T = I - S$$

$$\therefore \quad A^T = (I+S)^{-1}(I-S)$$

$$\therefore \quad A^TA = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$$

$$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$$

$$= I$$

Thus, A is orthogonal.

10. Let $P=\begin{bmatrix} A & B \\ C & O \end{bmatrix}$

Consider the matrix equation $PX = Q$

or

$$\begin{bmatrix} A & B \\ C & O \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

which is equivalent to the following system of equations

$$Ax_1 + Bx_2 = q_1 \quad (1)$$

$$Cx_1 + O = q_2 \quad (2)$$

From Eq. (2), we have

$$Cx_1 = q_2$$

or

$$C^{-1}Cx_1 = C^{-1}q_2$$

$$\Rightarrow x_1 = q_2C^{-1} \quad (3)$$

Putting the value of x_1 in Eq. (1), we get

$$q_2AC^{-1} + Bx_2 = q_1$$

or

$$q_2B^{-1}AC^{-1} + B^{-1}Bx_2 = B^{-1}q_1$$

$$\Rightarrow x_2 = B^{-1}q_1 - q_2B^{-1}AC^{-1} \quad (4)$$

Both Eqs. (3) and (4) are equivalent to the matrix equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix} Q$$

$$\Rightarrow \quad P^{-1} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$$

11. Let $P = 1/2(A + A^\theta)$

and $Q = 1/2i(A - A^\theta)$. Then,

$$A = P + iQ \qquad (1)$$

Now,

$$P^\theta = \left\{\frac{1}{2}(A + A^\theta)\right\}^\theta$$

$$= \frac{1}{2}(A^\theta + (A^\theta)^\theta)$$

$$= \frac{1}{2}(A^\theta + A) = \frac{1}{2}(A + A^\theta) = P$$

Therefore p is a Hermitian matrix. Also,

$$Q^\theta = \left\{\frac{1}{2}(A - A^\theta)\right\}^\theta$$

$$= \left(\frac{\bar{1}}{2i}\right)(A - A^\theta)^\theta$$

$$= -\frac{1}{2i}\{A^\theta - (A^\theta)^\theta\}$$

$$= -\frac{1}{2i}(A^\theta - A)$$

$$= \frac{1}{2i}(A - A^\theta)$$

$$= Q$$

Therefore, Q is also Hermitian matrix.

Thus A can be expressed in the form (1). Since A is unique, let $A = R + iS$ where R and S are both Hermitian matrices. We have,

$$A^\theta = (R + iS)^\theta$$

$$= R^\theta + (IS)^\theta$$

$$= R^\theta - iS^\theta$$

$$= R - iS \qquad \text{(since } R \text{ and } S \text{ are both Hermitian)}$$

$$\therefore \quad A + A^\theta = (R + iS) + (R - iS) = 2R$$

$$\Rightarrow \quad R = \frac{1}{2}(A + A^\theta) = P$$

Also,

$$A - A^\theta = (R + iS) - (R - iS) = 2iS$$

$$\Rightarrow \quad S = \frac{1}{2i}(A - A^\theta) = Q$$

Hence expression (1) for A is unique.

12. We have,

$$A = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix}$$

$$\Rightarrow \quad A^T = \begin{bmatrix} 2+3i & -3-i & 3-2i \\ 2 & 7 & i \\ 5 & 3-i & 2+i \end{bmatrix}$$

$$\Rightarrow \quad \bar{A}^T = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

or

$$A^\Theta = \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$\therefore \quad A + A^\Theta = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} + \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & 3+2i & 4 \end{bmatrix} \qquad (1)$$

and

$$A - A^\Theta = \begin{bmatrix} 2+3i & 2 & 5 \\ -3-i & 7 & 3-i \\ 3-2i & i & 2+i \end{bmatrix} - \begin{bmatrix} 2-3i & -3+i & 3+2i \\ 2 & 7 & -i \\ 5 & 3+i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix} \qquad (2)$$

Adding Eqs. (1) and (2), we get

$$2A = \begin{bmatrix} 4 & -1+i & 8+2i \\ -1-i & 14 & 3-2i \\ 8-2i & 3+2i & 4 \end{bmatrix} + \begin{bmatrix} 6i & 5-i & 2-2i \\ -5-i & 0 & 3 \\ -2-2i & -3 & 2i \end{bmatrix}$$

$$\text{Hence, } A = \begin{bmatrix} 2 & -\frac{1}{2}+\frac{i}{2} & 4+i \\ -\frac{1}{2}-\frac{i}{2} & 7 & \frac{3}{2}-i \\ 4-i & \frac{3}{2}+i & 2 \end{bmatrix} + \begin{bmatrix} 3i & \frac{5}{2}-\frac{i}{2} & 1-i \\ -\frac{5}{2}-\frac{i}{2} & 0 & \frac{3}{2} \\ -1-i & -\frac{3}{2} & i \end{bmatrix}$$

Objective Type

1. d. Let A be a skew-symmetric matrix of order n. By definition, $A' = -A$

$\Rightarrow \quad |A'| = |-A|$

$\Rightarrow \quad |A| = (-1)^n |A|$

$\Rightarrow \quad |A| = -|A| \qquad [\because n \text{ is odd}]$

$\Rightarrow \quad 2|A| = 0$

$\Rightarrow \quad |A| = 0$

Hence, A^{-1} does not exist.

2. d. (i) is false.

If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

(ii) is true as the product AB is an identity matrix, if and only if B is inverse of the matrix A.

(iii) is false since matrix multiplication in not commutative.

3. c. $[1 \quad x \quad y]\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = 10$

$\Rightarrow \quad [1 \quad 3+2x \quad 1-x+y]\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = [0]$

$\Rightarrow \quad [1 + 3x + 2x^2 + y - xy + y^2] = [0]$

$\Rightarrow \quad 2x^2 + y^2 + y + 3x - xy + 1 = 0$

If $y = 0$, $2x^2 + 3x + 1 = 0$

$\Rightarrow \quad (2x-1)(x+1) = 0$

$\Rightarrow \quad x = -1/2, -1$ (rational roots)

If $\quad y = -1, 2x^2 + 4x + 1 = 0$

$\Rightarrow \quad x = \frac{-4 \pm \sqrt{12}}{4} = \frac{-2 \pm \sqrt{3}}{2}$ (irrational roots)

4. d. $A = \text{diag}(d_1, d_2, \ldots, d_n)$

Given, $A^3 = A$

$\Rightarrow \quad \text{diag}(d_1^3, d_2^3, \ldots, d_n^3) = \text{diag}(d_1, d_2, \ldots, d_n)$

$\Rightarrow \quad d_1^3 = d_1, d_2^3 = d_2, \ldots, d_n^3 = d_n$

Hence, all $d_1, d_2, d_3, \ldots, d_n$ have three possible values ± 1, 0. Each diagonal element can be selected in three ways. Hence, the number of different matrices is 3^n.

5. d. If A is n^{th} root of I_2, then $A^n = I_2$. Now,

$$A^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

Thus,

$$A^n = \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix}$$

Now,

$$A^n = I \Rightarrow \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a^n = 1, b = 0$

6. d. $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (1)

$A^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (2)

Let A be given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The first equation gives

$a - b = -1$ (3)

$c - d = 2$ (4)

For second equation gives

$$A^2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = A\left(A\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = A\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This gives

$-a + 2b = 1$ (5)

$-c + 2d = 0$ (6)

Eqs. (3) + (5) $\Rightarrow b = 0$ and $a = -1$

Eqs. (4) + (6) $\Rightarrow d = 2$ and $c = 4$

So the sum $a + b + c + d = 5$.

7. c. $AB = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos\theta\cos\phi\sin\theta\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi(\cos(\theta-\phi)) & \cos\theta\sin\phi(\cos(\theta-\phi)) \\ \sin\theta\cos\phi(\cos(\theta-\phi)) & \sin\theta\sin\phi(\cos(\theta-\phi)) \end{bmatrix}$$

$$= (\cos(\theta-\phi))\begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{bmatrix}$$

Now, $AB = O \Rightarrow \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = (2n+1)\pi/2, n \in Z$.

8. b. $AB^n = ABBBBB\cdots B$

$= (AB)BBB\cdots B$

$= B(AB)BBB\cdots B$

$= BB(AB)BB\cdots B$

$\vdots$

$= B^nA$

$(AB)^n = (AB)(AB)(AB)\cdots(AB)$

$= A(BA)(BA)(BA)\cdots(BA)B$

$= A(AB)(AB)(AB)\cdots(AB)B$

$= A^2(BA)(BA)(BA)\cdots(BA)B^2$

$= A^2(AB)(AB)(AB)\cdots(AB)B^2$

$= A^3(BA)(BA)(BA)\cdots(BA)B^3$

$\vdots$

$= A^nB^n$

9. a. From given data $|A| = 2^4$

$\Rightarrow \quad |\text{adj}(\text{adj } A)| = (2^4)^9 = 2^{36}$

$\Rightarrow \quad \left\{\frac{\det(\text{adj}(\text{adj } A))}{7}\right\} = \left\{\frac{2^{36}}{7}\right\} = \left\{\frac{(7+1)^{12}}{7}\right\} = \frac{1}{7}$

10. b. We have,

$$\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad \alpha^2 + \beta\gamma - 1 = 0$

11. c. Given $A^2 = A$. Now,

$(I + A)^3 - 7A = I^3 + 3I^2A + 3IA^2 + A^3 - 7A$

$= I + 3A + 3A + A - 7A$

$= I + O$

$= I$

12. a. $(A - \lambda I)(B - \lambda I) = (B - \lambda I)(A - \lambda I)$

$\Rightarrow \quad AB - \lambda(A + B)I + \lambda^2 I^2 = BA - \lambda(B + A)I + \lambda^2 I^2$

$\Rightarrow \quad AB = BA$

13. c. Given, $A^2 = 2A - I$

Now, $A^3 = A(A^2)$
$= A(2A - I)$
$= 2A^2 - A$
$= 2(2A - I) - A$
$= 3A - 2I$

$A^4 = A(A^3)$
$= A(3A - 2I)$
$= 3A^2 - 2A$
$= 3(2A - I) - 2A$
$= 4A - 3I$

Following this, we can say $A^n = nA - (n-1)I$.

14. a. We have, $A^2 = O, A^k = O, \forall k \geq 2$

Thus,

$(A + I)^{50} = I + 50A$

$\Rightarrow (A + I)^{50} = I + 50A$

$\Rightarrow a = 1, b = 0, c = 0, d = 1$

15. b. We have,

$A = iB$

$$\Rightarrow A^2 = (iB)^2 = i^2B^2 = -B^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = -2B$$

$\Rightarrow A^4 = (-2B)^2 = 4B^2 = 4(2B) = 8B$

$\Rightarrow (A^4)^2 = (8B)^2$

$\Rightarrow A^8 = 64B^2 = 128B$

16. b. Since the product matrix is 3×3 matrix and the pre-multiplier of A is a 3×2 matrix, therefore A is 2×3 matrix. Let,

$A = \begin{bmatrix} l & m & n \\ x & y & z \end{bmatrix}$. Then the given equation becomes

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} l & m & n \\ x & y & z \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2l - x & 2m - y & 2n - z \\ l & m & x \\ -3l + 4x & -3m + 4y & -3n + 4z \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$\Rightarrow 2l - x = -1, 2m - y = -8, 2n - z = -10, l = 1, m = -2, n = -5$

$\Rightarrow x = 3, y = 4, z = 0, l = 1, m = -2, n = -5$

$$\Rightarrow A = \begin{bmatrix} l & m & n \\ x & y & z \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$$

17. b. Let, $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\therefore A^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1-i & -1 \end{bmatrix}$$

$$\Rightarrow (\bar{A}^T) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore A(\bar{A}^T) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

18. d. See theory.

19. a. A is involuntary. Hence,

$A^2 = I \Rightarrow A = A^{-1}$

Also,

$(kA)^{-1} = \frac{1}{k}(A)^{-1}$

$\Rightarrow \left(\frac{1}{2}A\right)^{-1} = 2(A)^{-1} \Rightarrow 2A$

20. b. Given,

$B = A^{-1}A^T$

$\Rightarrow B^T = (A^{-1}A^T)^T = A \times (A^{-1})^T$

$\Rightarrow B \times B^T = A^{-1}A^T \times A \times (A^{-1})^T = A^{-1} \times (A^T \times A)(A^{-1})^T$

$= A^{-1}(A \times A^T)(A^{-1})^T$

$= (A^{-1}A) \times (A^{-1}A)^T = I \times I^T = I$

21. b. $P^TP = I$

$Q = PAP^T$

$\therefore x = P^TQ^{1000}P = P^T(PAP^T)^{1000}P$

$= P^TPAP^T(PAP^T)^{999}P$

$= IAP^T \cdot PAP^T(PAP^T)^{998}P$

$= AIAP^T(PAP^T)^{998}P$

$= A^2P^TPAP^T(PAP^T)^{997}P$

$= A^3P^T((PAP^T)^{997}P$

$\vdots$

$= A^{1000} = I$ ($\because$ A is involuntary)

Hence, $x^{-1} = I$.

22. b. Since A is orthogonal, hence

$AA^T = I$

$\Rightarrow |AA^T| = 1$

$\Rightarrow |A^2| = 1$

$\Rightarrow |A| = \pm 1$

Now, $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

23. b. $\left(A' - \frac{1}{2}I\right)\left(A - \frac{1}{2}I\right) = I$ and $\left(A' + \frac{1}{2}I\right)\left(A + \frac{1}{2}I\right) = I$

$\Rightarrow A + A' = 0$ (subtracting the two results)

$\Rightarrow A' = -A$

$\Rightarrow A^2 = -\frac{3}{4}I$

$\Rightarrow \left(\frac{-3}{4}\right)^n = (\det(A))^2$

$\Rightarrow n$ is even

24. a. For involuntary matrix,

$A^2 = I$

$\Rightarrow |A^2| = |I| \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

For idempotent matrix,

$A^2 = A$

$\Rightarrow |A^2| = |A| \Rightarrow |A|^2 = |A| \Rightarrow |A| = 0$ or 1

For orthogonal matrix,

$AA^T = I$

$\Rightarrow |AA^T| = |I| \Rightarrow |A||A^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$

Thus if matrix A is idempotent it may not be invertible.

25. a. $A \times A^T = I$

$\Rightarrow |A \times A^T| = |I|$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1$

$\Rightarrow A^{-1}$ exists

$\Rightarrow \quad A^{-1} \times A \times A^T = A^{-1} \times I$

$\Rightarrow \quad A^{-1} = A^T$

26. b. Z is idempotent, then

$Z^2 = Z \Rightarrow Z^3, Z^4, \ldots, Z^n = Z$

$\therefore \quad (I + Z)^n = {}^nC_0 I^n + {}^nC_1 I^{n-1}Z + {}^nC_2 I^{n-2}Z^2 + \cdots + {}^nC_n Z^n$

$= {}^nC_0 I + {}^nC_1 Z + {}^nC_2 Z + {}^nC_3 Z + \cdots + {}^nC_n Z$

$= I + ({}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n)Z$

$= I + (2^n - 1)Z$

27. d. Since $AB = B$ and $BA = A$, so

$BAB = B^2$

$\Rightarrow \quad (BA)B = B^2$

$\Rightarrow \quad AB = B^2$

$\Rightarrow \quad B = B^2$

Hence, B is idempotent and similarly A.

$(A - B)^2 = A^2 - AB - BA + B^2 = A - B - A + B = O$

Therefore, $A - B$ is nilpotent.

28. b. $A^2 = O, A^3 = A^4 = \cdots = A^n = O$

Then, $A(I + A)^n = A(I + nA) = A + nA^2 = A$

29. b. We have, $AB = A(\text{adj } A) = |A|\, I_n$

$\therefore \quad AB + KI_n = |A|I_n + KI_n$

$\Rightarrow \quad AB + KI_n = (|A| + k)I_n$

$\Rightarrow \quad |AB + KI_n| = |(|A| + k)I_n| \quad (\because |aI_n| = a^n)$

$= (|A| + K)^n$

30. d. $\det(A - I) = \det(A - A^2)$

$= \det A(I - A)$

$= \det A \times \det(I - A)$

$= -\det A \times \det(A - I)$

Now,

$A^2 = I$

$\Rightarrow \quad \det(A^2) = \det(I)$

$\Rightarrow \quad (\det A)^2 = 1$

$\Rightarrow \quad \det(A) = \pm 1$

Thus, det (A) can be 1 or -1, from which we cannot say anything about $\det(A - I)$.

31. a. $|B| = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$ (Multiplying R_2 by -1)

$= -\begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix}$ (Multiplying C_2 by -1)

$= \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix}$ (Changing R_1 with R_2)

$= -\begin{vmatrix} p & a & x \\ q & b & y \\ r & c & z \end{vmatrix}$

$= -\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$

Hence $|A| = -|B|$, obviously when $|A| \neq 0$, $|B| \neq 0$. Also, $|\text{adj } B| = |B|^2 = (-|A|)^2 = |A|^2$.

32 a. $AB = C$

$\Rightarrow \quad |AB| = |C|$

$\Rightarrow \quad |A||B| = |C|$

$\Rightarrow \quad |B| = \dfrac{|C|}{|A|}$

33 a. $AB = A + B$

$\Rightarrow \quad B = AB - A = A(B - I)$

$\Rightarrow \quad \det(B) = \det(A)\det(B - I) = 0$

$\Rightarrow \quad \det(B) = 0$

34. b. $|A^{2005} - 6A^{2004}| = |A|^{2004}|A - 6I|$

$= 2^{2004}\begin{vmatrix} 0 & 11 \\ 2 & -2 \end{vmatrix} = (-22)\, 2^{2004} = (-11)\,(2)^{2005}$

35. c. $A^2 = I$

$\Rightarrow \quad A^2 - I = O$

$\Rightarrow \quad (A + I)(A - I) = 0$

Therefore, either $|A + I| = 0$ or $|A - I| = 0$. If $|A - I| \neq 0$, then $(A + I)(A - I) = O \Rightarrow A - I = O$ which is not so.

$\therefore \quad |A - I| = 0$ and $A - I \neq O$

36. b. We have,

$I = AA^{-1}$

$= \dfrac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -08 & 6 & 2c \\ 5 & -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & c+1 \\ 0 & 1 & 2\ (c+1) \\ 4(1-a) & 3\ (a-1) & 2+ac \end{bmatrix}$

Comparing the elements of AA^{-1} with those of I, we have

$c + 1 = 0 \Rightarrow c = -1$

$\Rightarrow \quad c = -1$ and $a - 1 = 0 \Rightarrow a = 1$

37. c. Given $B^r = I \Rightarrow B^rB^{-1} = IB^{-1} \Rightarrow B^{r-1} = B^{-1}$

$\Rightarrow \quad A^{-1}B^{r-1}A - A^{-1}B^{-1}A = A^{-1}B^{-1}A - A^{-1}B^{-1}A = O$

38. c. $\begin{vmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_1 & z_1 \end{vmatrix}^{-1}\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$

$= \left(\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}\right)^{-1}$

$= \begin{bmatrix} z_1\bar{z}_1 + z_2\bar{z}_2 & 0 \\ 0 & z_2\bar{z}_2 + z_1\bar{z}_1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} |z_1|^2 + |z_2|^2 & 0 \\ 0 & |z_1|^2 + |z_2|^2 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

39. a. We have,

$A(\alpha, \beta)^{-1} = \dfrac{1}{e^\beta}\begin{bmatrix} e^\beta \cos\alpha & -e^\beta \sin\alpha & 0 \\ e^\beta \sin\ \alpha & e^\beta \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= A(-\alpha, -\beta)$

40. a. We have,

$|A| = (a+ib)(a-ib) - (-c+id)(c+id)$

$= a^2 + b^2 + c^2 + d^2 = 1$

and $\text{adj}(A) = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

Then $A^{-1} = \begin{bmatrix} a-ib & -c-id \\ -c+id & a-ib \end{bmatrix}$

41. c. Given

$A^3 = O$

Now,

$(I-A)(I+A+A^2)$

$= I^2 + IA + IA^2 - AI - A^2 - A^3$

$= I - A^3$

$= I$

$\Rightarrow \quad (I-A)^{-1} = I + A + A^2$

42. b. We know that $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

$\Rightarrow \quad |\text{adj}(\text{adj}(\text{adj } A))| = |\text{adj } A|^{(n-1)^2}$

$= |A|^{(n-1)^3}$

$= 2^8 = 256$

43. b. $(-A)^{-1} = \dfrac{\text{adj}(-A)}{|-A|} = \dfrac{(-1)^{n-1}\text{adj}(A)}{(-1)^n |A|} = \dfrac{\text{adj}(A)}{-|A|} = -A^{-1}$

44. a. $A(x)A(y) = (1-x)^{-1}(1-y)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}\begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix}$

$= (1+xy-(x+y))^{-1}\begin{bmatrix} 1+xy & -(x+y) \\ -(x+y) & 1+xy \end{bmatrix}$

$= \left(1 - \dfrac{x+y}{1+xy}\right)^{-1}\begin{bmatrix} 1 & -\dfrac{x+y}{1+xy} \\ -\dfrac{x+y}{1+xy} & 1 \end{bmatrix} = A(z)$

45. b. Let, $A = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}$

$\Rightarrow \quad \text{adj}(A) = \begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$

$\Rightarrow \quad A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix} = \dfrac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$

$\Rightarrow \quad A^{-2} = (A^{-1})^2 = \dfrac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}\dfrac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$

$= \dfrac{1}{625}\begin{bmatrix} 25 & 0 \\ 10a & 25 \end{bmatrix}$

$= \begin{bmatrix} \dfrac{1}{25} & 0 \\ \dfrac{2a}{125} & \dfrac{1}{25} \end{bmatrix}$

Now,

$\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{25} & 0 \\ \dfrac{2a}{125} & \dfrac{1}{25} \end{bmatrix}$

$\Rightarrow \quad x = 2a/125$

46. c. $f(x) = \dfrac{1+x}{1-x}$

$\Rightarrow \quad (1-x)f(x) = 1+x$

$\Rightarrow \quad (I-A)f(A) = (I+A)$

$\Rightarrow \quad f(A) = (I-A)^{-1}(I+A)$

$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)$

$\Rightarrow \quad f(A) = \left(\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right)$

$= \dfrac{\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}{-4}$

$= \dfrac{\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}}{-4}$

$= \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

47. a. We know that in a square matrix of order n,

$|\text{adj } A| = |A|^{n-1}$

$\Rightarrow \quad |\text{adj}(\text{adj } A)| = |\text{adj } A|^{n-1} = |A|^{(n-1)^2}$

$\Rightarrow \quad n^2 - 2n - 8 = 0$

$\Rightarrow \quad n = 4$ as $n = -2$ is not possible

48. b. $|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + \tan^2 x \neq 0$

So A is invertible. Also,

$\text{adj } A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

Now,

$A^{-1} = \dfrac{1}{|A|}\text{adj } A$

$\Rightarrow \quad A^{-1} = \dfrac{1}{(1+\tan^2 x)}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$= \begin{bmatrix} \dfrac{1}{1+\tan^2 x} & \dfrac{-\tan x}{1+\tan^2 x} \\ \dfrac{\tan x}{1+\tan^2} & \dfrac{1}{1+\tan^2 x} \end{bmatrix}$

$\therefore \quad A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{1+\tan^2 x} & \dfrac{-\tan x}{1+\tan^2 x} \\ \dfrac{\tan x}{1+\tan^2} & \dfrac{1}{1+\tan^2 x} \end{bmatrix}$

$= \begin{bmatrix} \dfrac{1-\tan^2 x}{1+\tan^2 x} & \dfrac{-2\tan x}{1+\tan^2 x} \\ \dfrac{2\tan x}{1+\tan^2 x} & \dfrac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$

$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

49. b. Since $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and given $A=\begin{bmatrix}0 & \tan\alpha/2\\-\tan\alpha/2 & 0\end{bmatrix}$

$\therefore\quad I-A=\begin{bmatrix}1 & -\tan\alpha/2\\ \tan\alpha/2 & 1\end{bmatrix}$ (1)

Now, $(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$$=\begin{bmatrix}1 & \tan\alpha/2\\-\tan\alpha/2 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}1 & \tan\alpha/2\\-\tan\alpha/2 & 1\end{bmatrix}\begin{bmatrix}\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} & -\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2}\\ \dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2} & \dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}+\dfrac{2\tan^2\alpha/2}{1+\tan^2\alpha/2} & -\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2}+\dfrac{\tan\alpha/2\,(1-\tan^2\alpha/2)}{1+\tan^2\alpha/2}\\ \dfrac{-\tan\alpha/2\,(1-\tan^2\alpha/2)}{1+\tan^2\alpha/2}+\dfrac{2\tan\alpha/2}{1+\tan^2\alpha/2} & \dfrac{2\tan^2\alpha/2}{1+\tan^2\alpha/2}+\dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)} & -\dfrac{\tan\alpha/2\,(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}\\ \dfrac{\tan\alpha/2\,(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)} & \dfrac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}\end{bmatrix}$$

$$=\begin{bmatrix}1 & -\tan\alpha/2\\ \tan\alpha/2 & 1\end{bmatrix}$$

$=I-A$ [Using (1)]

50. d. Let, $A=\begin{bmatrix}1 & -4\\3 & -2\end{bmatrix}$ and $B=\begin{bmatrix}-16 & -6\\7 & 2\end{bmatrix}$

Then the matrix equation is $AX=B$.

$\therefore\quad |A|=\begin{vmatrix}1 & -4\\3 & -2\end{vmatrix}=-2+12\neq 0$

So A is an invertible matrix. Also,

$$\text{adj } A=\begin{bmatrix}-2 & -3\\4 & 1\end{bmatrix}^T=\begin{bmatrix}-2 & 4\\-3 & 1\end{bmatrix}$$

So,

$$A^{-1}=\frac{1}{|A|}\text{adj} A=\frac{1}{10}\begin{bmatrix}-2 & 4\\-3 & 1\end{bmatrix}$$

Now,

$AX=B$

$\Rightarrow\ A^{-1}(AX)=A^{-1}B$

$\Rightarrow\ (A^{-1}A)X=A^{-1}B$

$\Rightarrow\ IX=A^{-1}B$

$\Rightarrow\ X=A^{1}B$

$$\Rightarrow\ X=\frac{1}{10}\begin{bmatrix}-2 & 4\\-3 & 1\end{bmatrix}\begin{bmatrix}-16 & -6\\7 & 2\end{bmatrix}=\begin{bmatrix}6 & 2\\11/2 & 2\end{bmatrix}$$

51. c. $(A^{-1}BA)^2=(A^{-1}BA)(A^{-1}BA)$

$=A^{-1}B(AA^{-1})BA$

$=A^{-1}BIBA=A^{-1}B^2A$

$\Rightarrow\ (A^{-1}BA)^3=(A^{-1}B^2A)(A^{-1}BA)$

$=A^{-1}B^2(AA^{-1})BA$

$=A^{-1}B^2IBA$

$=A^{-1}B^3A$ and so on

$\Rightarrow\ (A^{-1}BA)^n=A^{-1}B^nA$

52. a. $A \text{ adj } A=|A|\,I$

$\Rightarrow\ |A \text{ adj } A|=|A|^n$ [If A is of order $n\times n$]

$\Rightarrow\ |A|\,|\text{adj } A|=|A|^n$

$\Rightarrow\ |\text{adj } A|=|A|^{n-1}$

Now, A is singular,

$\therefore\ |A|=0$

$\Rightarrow\ |\text{adj } A|=0$

Hence adj A is singular.

53. a. $A=\text{diag}(d_1, d_2, d_3, ..., d_n)$

$\Rightarrow\ |A|=(d_1\times d_2\times d_3\times d_4\cdots d_n)$

Now,

Cofactor of d_1 is $d_2d_3\cdots d_n$

Cofactor of d_2 is $d_1\times d_3\times d_4\cdots d_n$

Cofactor of d_3 is $d_1\times d_2\times d_4\cdots d_n$

$\vdots$

Cofactor of d_n is $d_1\times d_2\times d_3\cdots d_{n-1}$

$\Rightarrow\ A^{-1}=\dfrac{1}{|A|}\text{adj}(A)=\text{diag}(d_1^{-1}, d_2^{-1}, d_3^{-1}, ..., d_n^{-1})$

Hence, A^{-1} is also a diagonal matrix.

54. a. We know that for any non-singular matrix A,

$$A^{-1}=\frac{1}{|A|}\text{adj}(A)$$

Now put $A=P^{-1}$. Then we have

$$(P^{-1})^{-1}=\frac{1}{|P^{-1}|}\text{adj}(P^{-1})$$

$\Rightarrow\ P=|P|\text{adj}(P^{-1})$

$\Rightarrow\ \text{adj}(P^{-1})=\dfrac{P}{|P|}$

55. c. $\text{adj}(Q^{-1}BP^{-1})=\text{adj}(P^{-1})\text{adj}(B)\text{adj}(Q^{-1})$

$=\dfrac{P}{|P|}A\dfrac{Q}{|Q|}$

$=PAQ$

56. b. We have,

$(A-2I)(A-4I)=O$

$\Rightarrow\ A^2-2A-4A+8I=O$

$\Rightarrow\ A^2-6A+8I=O$

$\Rightarrow\ A^{-1}(A^2-6A+8I)=A^{-1}O$

$\Rightarrow\ A-6I+8A^{-1}=O$

$\Rightarrow\ A+8A^{-1}=6I$

$\Rightarrow\ \dfrac{1}{6}A+\dfrac{4}{3}A^{-1}=1$

57. b. $|A(\alpha,\beta)|=\cos^2\alpha e^\beta+\sin^2\alpha e^\beta=e^\beta$

Now,

$$A(\alpha,\beta)^{-1}=\frac{1}{e^\beta}\text{adj}(A(\alpha,\beta))$$

$$= \frac{1}{e^{\beta}}\begin{bmatrix} e^{\beta}\cos\alpha & -\sin\alpha e^{\beta} & 0 \\ e^{\beta}\sin\alpha & \cos\alpha e^{\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{-\beta} \end{bmatrix}$$

$= A(-\alpha, -\beta)$

58. a. As $B = -A^{1}BA$, we get

$AB = -BA$ or $AB + BA = O$

Now,

$(A + B)^2 = (A + B)(A + B)$

$= A^2 + BA + BA + B^2$

$= A^2 + O + B^2$

$= A^2 + B^2$

59. b. $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$\Rightarrow A^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$

$\Rightarrow A^3 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}$

$\Rightarrow A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$\Rightarrow (A^n)^{-1} = \frac{1}{a^n b^n}\begin{pmatrix} b^n & 0 \\ 0 & a^n \end{pmatrix} = \begin{pmatrix} a^{-n} & 0 \\ 0 & b^{-n} \end{pmatrix}$

$\Rightarrow \lim_{n\to\infty}(A^n)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ as $a > 1$ and $b > 1$

60. b. $(I - A)f(A) = I + A$

$\Rightarrow f(A) = (I + A)(I - A)^{-1}$

$= (I + A)(I + A + A^2)$

$= I + A + A^2 + A + A^2 + A^3$

$= I + 2A + 2A^2$

61. d. If possible assume that A is non-singular, then A^{-1} exists. Thus,

$AB = O \Rightarrow A^{-1}(AB) = (A^{-1}A)B = O$

$\Rightarrow IB = O$ or $B = O \times$ (a contradiction)

Hence, both A and B must be singular.

62. a. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1}$

$\Rightarrow A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

63. c. $A^2 - A + I = 0$

$\Rightarrow I = A - A^2$

$IA^{-1} = AA^{-1} - A^2A^{-1}$

$\Rightarrow A^{-1} = I - A$

64. a. Let, $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Rightarrow X^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$

$\Rightarrow a^2 + bc = 1$ and $ab + bd = 1 \Rightarrow b(a + d) = 1$

$ac + cd = 2 \Rightarrow c(a + d) = 2 \Rightarrow 2b = c$

Also,

$bc + d^2 = 3 \Rightarrow d^2 - a^2 = 2$

$\Rightarrow (d - a)(a + d) = 2 \Rightarrow d - a = 2b$ (using $bc = 1 - a^2$)

$a + d = 1/b$

$\Rightarrow 2d = 2b + 1/b, \quad 2a = 1/b - 2b$

$d = b + 1/2b, \quad a = 1/(2b) - b$

$c = 2b$

$\Rightarrow \left(b^2 + \frac{1}{4b^2} + 1\right) + 2b^2 = 3$

$\Rightarrow 3b^2 + \frac{1}{4b^2} = 2$

$\Rightarrow 3x + \frac{1}{4x} = 2$

$\Rightarrow b = \pm\frac{1}{\sqrt{6}}$ or $b = \pm\frac{1}{\sqrt{2}}$

Therefore, matrices are

$$\begin{pmatrix} 0 & 1/\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 & -1/\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} \\ 2/\sqrt{6} & 4\sqrt{6} \end{pmatrix}$$

65. c. Given that

$X = AB + BA \Rightarrow X = X^T$

and

$Y = AB - BA$

$\Rightarrow Y = -Y^T$

Now, $(XY)^T = Y^TX^T = -YX.$

66. c. As A is a skew-symmetric matrix,

$A^T = -A$

$\Rightarrow a_{ii} = 0, \forall i$

$\Rightarrow \text{tr}(A) = 0$

Also,

$|A| = |A^T| = |-A| = (-1)^3|A|$

$\Rightarrow 2|A| = 0$

$\Rightarrow |A| = 0$

67. d. $\text{tr}(A) = \sum_{i=j} a_{ij}$

$= (a_{11} + a_{22} + a_{33} + \cdots + a_{10\times 10})$

$= (w^2 + w^4 + w^6 + \cdots + w^{20})$

$= w^2(1 + w^2 + w^4 + \cdots + w^{18})$

$= w^2[(1 + w + w^2) + \cdots + (1 + w + w^2) + 1]$

$= w^2 \times 1$

$\Rightarrow \text{tr}(A) = w^2$

68. b. We have,

$$F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{adj}(F(\alpha)) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also,

$\det(F(\alpha)) = 1$

$$\Rightarrow \quad [F(\alpha)]^{-1} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(-\alpha)$$

69. b. We have,

$$[F(x)G(y)]^{-1} = [G(y)]^{-1}[F(x)]^{-1}$$
$$= G(-y)F(-x)$$

70. a. Given A is skew-symmetric. Hence,

$$A^T = -A$$
$$\Rightarrow \quad A^n = (-A^T)^n = -(A^T)^n = -(A^n)^T \text{ (given } n \text{ is odd)}$$

Hence, A^n is skew-symmetric.

71. c. We have,

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + db \\ ac + cd & bc + d^2 \end{bmatrix}$$

As A satisfies $x^2 + k = 0$, therefore

$$A^2 + kI = O$$
$$\Rightarrow \quad \begin{bmatrix} a^2 + bc + k & (a+d)b \\ (a+d)c & bc + d^2 + k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Rightarrow \quad a^2 + bc + k = 0,\ bc + d^2 + k = 0$$

and $(a + d)b = (a + d)c = 0$

As $bc \neq 0, b \neq 0, c \neq 0$, so

$$a + d = 0$$
$$\Rightarrow \quad a = -d$$

Also,

$$k = -(a^2 + bc)$$
$$= -(d^2 + bc)$$
$$= -((-ad) + bc)$$
$$= |A|$$

72. b. Given $A, B, A + I, A + B$ are idempotent. Hence,

$$A^2 = A,\ B^2 = B,\ (A + I)^2 = A + I \text{ and } (A + B)^2 = A + B$$
$$\Rightarrow \quad A^2 + B^2 + AB + BA = A + B$$
$$\Rightarrow \quad A + B + AB + BA = A + B$$
$$\Rightarrow \quad AB + BA = O$$

73. b.
$$A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 25 - 24 + 0 & 40 - 40 + 0 & 0 + 0 + 0 \\ -15 + 15 + 0 & -24 + 25 + 0 & 0 + 0 + 0 \\ -5 + 6 - 1 & -8 + 10 - 2 & 0 + 0 + 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, the matrix A is involutory.

74. b. Let $A = [a_{ij}]$. Since A is skew-symmetric, therefore

$$a_{ii} = 0 \text{ and } a_{ij} = -a_{ij}\ (i \neq j)$$

A is symmetric as well, so $a_{ij} = a_{ji}$ for all i and j.

$$\therefore \quad a_{ij} = 0 \text{ for all } i \neq j$$

Hence, $a_{ij} = 0$ for all i and j, i.e., A is a null matrix.

75. c. $(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA)$

$$= A^{-1}B(AA^{-1})BA$$
$$= A^{-1}BIBA = A^{-1}B^2A$$
$$(A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA)$$
$$= A^{-1}B^2(AA^{-1})BA$$
$$= A^{-1}B^3A \text{ and so on}$$
$$\therefore \quad (A^{-1}BA)^n = A^{-1}B^nA$$

76. a. Matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is orthogonal if

$$\sum a_i^2 = \sum b_i^2 = \sum c_i^2 = 1; \sum a_ib_i = \sum b_ic_i = \sum c_ia_i = 0$$

77. b. $(kI_n)\text{ adj }(kI_n) = |kI_n|I_n$ [using $A(\text{adj } A) = |A|I$]

$$\text{adj }(kI_n) = k^{n-1}I_n$$
$$|\text{adj }(kI_n)| = k^{n(n-1)}$$

78. c. $A \text{ adj } A = |A|\, I$

$$|A| = xyz - 8x - 3(z - 8) + 2(2 - 2y)$$
$$|A| = xyz - (8x + 3z + 4y) + 28$$
$$= 60 - 20 + 28$$
$$= 68$$
$$\Rightarrow \quad A\,(\text{adj } A) = 68\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$$

79. c. $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

$$\Rightarrow \quad \text{tr}(A) + 2\text{tr}(B) = -1$$

and $2\,\text{tr}(A) - \text{tr}(B) = 3$

Let $\text{tr}(A) = x$ and $\text{tr}(B) = y$. Then,

$$x + 2y = -1 \text{ and } 2x - y = 3$$

Solving, $x = 1$ and $y = -1$. Hence,

$$\text{tr}(A) - \text{tr}(B) = x - y = 2$$

80. b. $B = A_1 + 3A_3^3 + \cdots + (2n - 1)(A_{2n-1})^{2n-1}$

$$B^T = -[A_1 + 3A_3^3 + \cdots + (2n - 1)(A_{2r-1})^{2r-1}]$$
$$= -B$$

Hence, B is skew-symmetric.

81. a. $|A| = 1(0 - 10) - 2(2 - 6) + 3(4 - 0)$

$$= -10 + 8 + 12 = 10$$
$$\Rightarrow \quad |A| \neq 0$$
$$\Rightarrow \quad \text{Unique solution}$$

82. a. $BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \quad \text{tr}(A) + \text{tr}\left(\frac{ABC}{2}\right) + \text{tr}\left(\frac{A(BC)^2}{4}\right) + \text{tr}\left(\frac{A(BC)^3}{8}\right) + \cdots + \infty$$
$$= \text{tr}(A) + \text{tr}\left(\frac{A}{2}\right) + \text{tr}\left(\frac{A}{2^2}\right) + \cdots$$
$$= \text{tr}(A) + \frac{1}{2}\,\text{tr}(A) + \frac{1}{2^2}\,\text{tr}(A) \cdots$$
$$= \frac{\text{tr}(A)}{1 - (1/2)}$$
$$= 2\text{tr}(A) = 2(2 + 1) = 6$$

Multiple Correct Answers Type

1. b, c.

$\det(-A) = (-1)^n \det(A)$

$\det(A^{-1}) = \dfrac{1}{\det(A)} = 1$

$\det(\text{adj } A) = |A|^{n-1} = 1$

Hence, $|\omega A| = \omega^n|A| = 1$ only when $n = 3k, k \in Z$.

2. a, d.

Given, $(A + B)^2 = A^2 + B^2 + 2AB$

$\Rightarrow \quad (A + B)(A + B) = A^2 + B^2 + 2AB$

$\Rightarrow \quad A^2 + AB + BA + B^2 = A^2 + B^2 + 2AB \Rightarrow BA = AB$

$$\Rightarrow \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+2 & -a+1 \\ b-2 & -b-1 \end{bmatrix} = \begin{bmatrix} a-b & 1+1 \\ 2a+b & 2-1 \end{bmatrix}$$

The corresponding elements of equal matrices are equal.

$a + 2 = a - b, -a + 1 = 2 \Rightarrow a = -1$

$b - 2 = 2a + b, -b - 1 = 1 \Rightarrow b = -2$

$\Rightarrow \quad a = -1, b = -2$

3. a, b, c.

Given, $AB = A, BA = B$

$\Rightarrow \quad B \times AB = B \times A$

$\Rightarrow \quad (BA)B = B$

$\Rightarrow \quad B^2 = B$

Also,

$A \times B \times A = AB$

$\Rightarrow \quad (AB)A = A$

$\Rightarrow \quad A^2 = A$

Now $(A^T)^2 = (A^T \times A^T) = (A \times A)^T = (A^2)^T = A^T$

Similarly, $(B^T)^2 = B^T$

$\Rightarrow \quad A^T$ and B^T are idempotent

4. a, c.

A is an orthogonal matrix.

$\therefore \quad AA^T = I$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & - & 2 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow \quad a + 4 + 2b = 0, 2a + 2 - 2b = 0$ and $a^2 + 4 + b^2 = 9$

$\Rightarrow \quad a + 2b + 4 = 0, a - b + 1 = 0$ and $a^2 + b^2 = 5$

$\Rightarrow \quad a = -2, b = -1$

5. a, d.

$(B^TAB)^T = B^TA^T(B^T)^T = B^TA^TB = B^TAB$ if A is symmetric.

Therefore, B^TAB is symmetric if A is symmetric.

Also, $(B^TAB)^T = B^TA^TB = B^T(-A)B = -(B^TA^TB)$

Therefore, B^TAB if A is skew-symmetric if A is skew-symmetric.

6. a, b, c.

We have, $|A(\theta)| = 1$

Hence, A is invertible.

$$A(\pi + \theta) = \begin{bmatrix} \sin(\pi+\theta) & i\cos(\pi+\theta) \\ i\cos(\pi+\theta) & \sin(\pi+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta & -i\cos\theta \\ -i\cos\theta & -\sin\theta \end{bmatrix} = -A(\theta)$$

$$\text{adj }(A(\theta)) = \begin{bmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{bmatrix}$$

$$\Rightarrow \quad A(\theta)^{-1} = \begin{bmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{bmatrix} = A(\pi - \theta)$$

7. a, c, d.

Given, $A^2 + A + 2I = O$

$\Rightarrow \quad A^2 + A = -2I$

$\Rightarrow \quad |A^2 + A| = |-2I|$

$\Rightarrow \quad |A||A + I| = (-2)^n$

$\Rightarrow \quad |A| \neq 0$

Therefore, A is non-singular, hence its inverse exists. Also, multiplying the given equation both sides with A^{-1}, we get

$A^{-1} = -\dfrac{1}{2}(A + I)$

8. a, b, c.

$$\because \quad |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) + 3(2-0) + 4(-2-0) = 1$$

$\therefore \quad \text{adj(adj } A) = |A|^{3-2}A = A$ and $|\text{adj(adj } A)| = |A| = 1$

Also,

$|\text{adj } A| = |A|^{3-1} = |A|^2 = 1^2 = 1$

9. a, c.

We have,

$$\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \frac{1}{1+\tan^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \cos^2\theta\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$= \cos^2\theta\begin{bmatrix} 1-\tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1-\tan^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$\therefore \quad a = \cos 2\theta, b = \sin 2\theta$

10. a, b, c.

$|A^{-1}| = -1 \Rightarrow |A| = -1$

Now, use $\text{adj } A = |A|A^{-1}$ and $A = (A^{-1})^{-1}$

11. a, c, d.

B is an idempotent matrix

$\therefore \quad B^2 = B$

Now,

$$\begin{aligned}A^2 &= (I-B)^2\\ &= (I-B)(I-B)\\ &= I - IB - IB + B^2\\ &= I - B - B + B^2\\ &= I - 2B + B^2\\ &= I - 2B + B\\ &= I - B\\ &= A\end{aligned}$$

Therefore, A is idempotent. Again,

$$AB = (I-B)B = IB - B^2 = B - B^2 = B^2 - B^2 = O$$

Similarly, $BA = B(I-B) = BI - B^2 = B - B = O$.

12. b, c.

$$(-A)^{-1} = \frac{\text{adj}(-A)}{|-A|} = \frac{(-1)^{n-1}\text{adj}(A)}{(-1)^n\,|A|} = \frac{\text{adj}(A)}{-|A|} = -A^{-1}$$ (for any value of n)

Given, $A^n = O$

Now,

$$(I-A)(I + A + A^2 + \cdots + A^{n-1}) = I - A^n = I$$

$$\Rightarrow\quad (I-A)^{-1} = I + A + A^2 + \cdots + A^{n-1}$$

13. a, b.

Let $I = k = 1$ (say). Then,

$$A_iA_k = A_kA_i = A_1A_1$$

$$A_iA_k = A_1A_1 = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I$$

$$A_2A_2 = \begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = I$$

$$\therefore\quad A_iA_k + A_kA_i = I + I = 2I$$

If $i \neq k$ let $i = 3$ and $k = 2$, then

$$A_iA_k = A_1A_2 = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix} = \begin{bmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{bmatrix}$$

$$A_2A_1 = \begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} = \begin{bmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&-i\end{bmatrix}$$

$$\Rightarrow\quad A_1A_2 + A_2A_1 = O$$

14. a, c.

$$\sin A = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \text{ and } \cos A = \begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$\therefore\quad |\sin A| = \cos^2\theta + \sin^2\theta = 1.$

Hence $\sin A$ is invertible.

Also, $$(\sin A)\times(\sin A)^T = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta+\sin^2\theta & 0\\ 0 & \cos^2\theta+\sin^2\theta\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I$$

Hence, $\sin A$ is orthogonal. Also,

$$2\sin A\cos A = 2\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$$= 2\begin{bmatrix}2\sin\theta\cos\theta & \cos^2\theta+\sin^2\theta\\ \cos^2\theta-\sin^2\theta & 0\end{bmatrix} = 2\begin{bmatrix}\sin 2\theta & 1\\ \cos 2\theta & 0\end{bmatrix} \neq \sin 2A$$

15. a. c. Let,

$$A = \begin{bmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{bmatrix}$$

Now,

$$A^T = \begin{bmatrix}0 & a & a\\ 2b & b & -b\\ c & -c & c\end{bmatrix}$$

Hence, A is orthogonal. Therefore,

$$AA^T = I \Rightarrow \begin{bmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{bmatrix}\begin{bmatrix}0 & a & a\\ 2b & b & -b\\ c & -c & c\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Equating the corresponding elements, we get

$$4b^2 + c^2 = 1 \quad (1)$$

$$2b^2 - c^2 = 0 \quad (2)$$

$$a^2 + b^2 + c^2 = 1 \quad (3)$$

Solving Eqs. (1), (2) and (3), we get

$$a = \pm\frac{1}{\sqrt{2}},\ b = \pm\frac{1}{\sqrt{6}},\ c = \pm\frac{1}{\sqrt{3}}$$

16. a, b, d. See theory.

17. a, b, c.

If $|A| \neq 0$, then

$$AB = AC$$

$$\Rightarrow\quad A^{-1}AB = A^{-1}AC$$

$$\Rightarrow\quad B = C$$

Also if A is orthogonal matrix, then, $AA^T = I$

$\Rightarrow\quad |AA^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow A$ is invertible

18. a, b, c.

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$|a| = 3\begin{vmatrix}a_1 & a_2 & a_3\\ d & d & d\\ d & d & d\end{vmatrix} = 0$$

where d is the common difference of the A.P.

Therefore, the given system of equations has infinite number of solutions. Also,

$$|B| = a_1^2 + a_2^2 \neq 0$$

19. a, b, d.

$$A=\begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ \cos\beta & \sin\beta & 0\\ \cos\gamma & \sin\gamma & 0\end{bmatrix}\begin{bmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \sin\alpha & \sin\beta & \sin\gamma\\ 0 & 0 & 0\end{bmatrix}$$

Thus, A is symmetric and $|A| = 0$, hence singular and not invertible. Also,

$$AA^T \neq I$$

20. b, c.

$$A^{-1}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$$

$$\Rightarrow\quad (A^{-1})^2=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}$$

Similarly,

$$(A^{-1})^3=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$$

and $(A^{-1})^n=\begin{bmatrix}1 & 0\\ -n & 1\end{bmatrix}\Rightarrow \lim\limits_{n\to\infty}\frac{1}{n}A^{-n}=\lim\limits_{n\to\infty}\begin{bmatrix}1/n & 0\\ -1 & 1/n\end{bmatrix}=\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}$

$$\lim_{n\to\infty}\frac{1}{n^2}A^{-n}=\lim_{n\to\infty}\begin{bmatrix}1/n^2 & 0\\ -1/n & 1/n^2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

21. a, d.

Here X is a $n\times 1$ matrix, C is a $n\times n$ matrix and X^T is a $1\times n$ matrix. Hence X^TCX is a 1×1 matrix. Let $X^TCX = k$. Then,

$(X^TCX)^T = X^TC^T(X^T)^T = X^T(-C)X = -X^TCX = -k$

$\Rightarrow\quad k = -k$

$\Rightarrow\quad k = 0$

$\Rightarrow\quad X^TCX$ is null matrix

22. a, b, c.

$$S=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\Rightarrow S^{-1}=\frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$$

We have,

$$SA=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}b+c & c+a & b-c\\ c-b & c+a & a-b\\ b-c & a-c & a+b\end{bmatrix}$$

$$=\begin{bmatrix}0 & 2a & 2a\\ 2b & 0 & 2b\\ 2c & 2c & 0\end{bmatrix}$$

$$\therefore\quad SAS^{-1}=\begin{bmatrix}0 & 2a & 2a\\ 2b & 0 & 2b\\ 2c & 2c & 0\end{bmatrix}\frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}0 & a & a\\ b & 0 & b\\ c & c & 0\end{bmatrix}\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}2a & 0 & 0\\ 0 & 2b & 0\\ 0 & 0 & 2c\end{bmatrix}$$

$= \text{diag}(2a, 2b, 2c)$

23. a, b, c.

All are properties of diagonal matrix.

24. a, b, c.

Given that A and B commute, we have

$AB = BA\quad (\because A$ and B are symmetric$)$ (1)

Also,

$A^T = A,\ B^T = B$ (2)

$(A^{-1}B)^T = B^T(A^{-1})^T = BA^{-1}$

$(\because$ if A is symmetric, A^{-1} is also symmetric$)$

Also from Eq. (1),

$ABA^{-1} = B$ (3)

$\Rightarrow\quad A^{-1}ABA^{-1} = A^{-1}B$

$\Rightarrow\quad IBA^{-1} = A^{-1}B$

$\Rightarrow\quad BA^{-1} = A^{-1}B$

Hence, from Eq. (2),

$(A^{-1}B)^T = A^{-1}B$

Thus, $A^{-1}B$ is symmetric. Similarly, AB^{-1} is also symmetric. Also,

$BA = AB$

$\Rightarrow\quad (BA)^{-1} = (AB)^{-1}$

$\Rightarrow\quad A^{-1}B^{-1} = B^{-1}A^{-1}$

$\Rightarrow\quad (A^{-1}B^{-1})^T = (B^{-1}A^{-1})^T$

$= (A^{-1})^T(B^{-1})^T$

$= A^{-1}B^{-1}$

Hence, $A^{-1}B^{-1}$ is symmetric.

25. b, c.

Since A is skew-symmetric, $A^T = -A$. We have,

$A^2 + I = O$

$\Rightarrow\quad A^2 = -I$ or $AA = -I$

$\Rightarrow\quad A(-A) = I$

$\Rightarrow\quad AA^T = I$

Again, we know that

$|A| = |A^T|$ and $|kA| = k^n|A|$

where n is the order of A. Now,

$A^T = (-1)^n \times A$

$\Rightarrow\quad |A^T| = (1)^n\,|A|$

ΦΙ $\quad [1 - (-1)^n]\,|A| = 0$

Hence either $|A| = 0$ or $1 - (-1)^n = 0$, i.e., n is even. But

$A^2 = O - I = -I$

$\Rightarrow\quad |A|^2 = (-1)^n\,|I| = (-1)^n \neq 0$

Hence, the only possibility is that A is of even order.

26. a, b, c, d.

We have, $A^2B = A(AB) = AA = A^2$, $B^2A = B(BA) = BB = B^2$,

$ABA = A(BA) = AB = A$ and $BAB = B(AB) = BA = B$.

27. a, b, d.

$$A^2=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}=\begin{bmatrix}9 & 8 & 8\\ 8 & 9 & 8\\ 8 & 8 & 9\end{bmatrix}$$

We have,

$$A^2-4A-5I_3=\begin{bmatrix}9 & 8 & 8\\ 8 & 9 & 8\\ 8 & 8 & 9\end{bmatrix}-4\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=O$$

$\Rightarrow\quad 5I_3 = A^2 - 4A = A(A - 4I_3)$

$\Rightarrow \quad I_3 = \frac{1}{5}(A - 4I_3) \Rightarrow A^{-1} = \frac{1}{5}(A - 4I_3)$

Note that $|A| = 5$. Since $|A^3| = |A|^3 = 5^3 \neq 0$, A^3 is invertible. Similarly, A^2 is invertible.

Reasoning Type

1. c. We know that $|\text{adj } A| = |A|^{n-1}$. Hence, statement 2 is false.

Now,

$|\text{adj}(\text{adj } A)| = |\text{adj } A|^{n-1} = ||A|^{n-1}|^{n-1} = |A|^{(n-1)^2}$

Then,

$|\text{adj}(\text{adj}(\text{adj } A))| = |\text{adj}(\text{adj } A))|^{n-1}$

$= ||A|^{(n-1)^2}|^{n-1}$

$= |A|^{(n-1)^3}$

Hence, statement 1 is true.

2. b. Both the statements are true as both are standard properties of diagonal matrix. But statement 2 does not explain statement 1.

3. d. Matrix $a_{ij} = \frac{i-j}{i+2j}$ is $A = \begin{bmatrix} 0 & -\frac{1}{5} & -\frac{2}{7} \\ \frac{1}{4} & 0 & -\frac{1}{8} \\ \frac{2}{5} & \frac{1}{7} & 0 \end{bmatrix}$, which is neither symmetric nor skew-symmetric. But this is not the reason for which A cannot be expressed as sum of symmetric and skew-symmetric matrix. In fact any matrix can be expressed as a sum of symmetric and skew-symmetric matrix. Hence, statement 1 is false but statement 2 is true.

4. a. Given,

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \quad A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

Hence,

$A^2 - (a + d)A + (ad - bc)I$

$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a^2+bc-(a^2+ad)+(ad-bc) & ab+bd-(ab+bd) \\ ac+cd-(ac+cd) & bc+d^2-(ad+d^2)+(ad-bc) \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= O$

Given,

$A^3 = O$

$\Rightarrow \quad |A| = 0$ or $ad - bc = 0$

$\Rightarrow \quad A^2 - (a + d)A = O$ or $A^2 = (a + d)A \qquad (1)$

Case (i)

$a + d = 0$

From Eq. (1),

$A^2 = O$

Case (ii)

$a + d \neq 0$

Given,

$A^3 = O$

$\Rightarrow \quad A^2A = O$

$\Rightarrow \quad (A + d)A \cdot A = O$

$\Rightarrow \quad A^2 = O$

5. b. $\text{adj}(F(\alpha)) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Also,

$|F(\alpha)| = 1$

Then,

$[F(\alpha)]^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= F(-\alpha)$

Similarly, we can prove that $[G(\beta)]^{-1} = G(-\beta)$.

But again given matrices $F(\alpha)$ and $G(\beta)$ are special matrices for which this type of result holds.

In general, such result is not true. You can verify with any other matrix. Hence, both statements are true but statement 2 is not correct explanation of statement 1.

6. a. Statement 1 is true as $|A| = 0$. Since $|B| \neq 0$, statement 2 is also true and correct explanation of statement 1.

7. a. $A = -A^T \Rightarrow |A| = -|A^T| = -|A|$

$\Rightarrow \quad 2|A| = 0$

$\Rightarrow \quad |A| = 0$

8. d. Statement 1 is false.

$\because \quad A = [A_{ij}]_{n\times n}$ where $a_{ij} = 0, i \geq j$

Therefore, $|A| = 0$ and hence A is singular. So, inverse of A is not defined.

In statement 2, $|A| = 0$. Therefore, inverse of A is not defined.

9. d. A^{-1} exists only for non-singular matrix.

$\therefore \quad AB = AC \Rightarrow B = C$ if A^{-1} exists

10. d. ABC is not defined, as order of A, B and C are such that they are not conformable for multiplication.

11. c. $[A(A + B)^{-1}B]^{-1} = B^{-1}((A + B)^{-1})^{-1})A^{-1}$

$= B^{-1}(A + B)A^{-1} = (B^{-1}A + I)A^{-1} = B^{-1}I + IA^{-1} = B^{-1} + A^{-1}$

Hence, statement 1 is true. Statement 2 is false as $(A + B)^{-1} = A^{-1} + B^{-1}$ is not true.

12. b. Since $AB = BA$, we have

$(A + B)^r = {}^rC_0A^r + {}^rC_1A^{r-1}B + {}^rC_2A^{r-2}B^2 + \cdots + {}^rC_sA^{r-s}B^s + \cdots + {}^rC_{r-1}AB^{r-1} + {}^rC_rB^r$

If $r = m + n$, then

$A^{r-p}B^p = A^mB^{r-p-m}B^p = O$ if $p \leq n$

and $A^{r-p}B^p = A^{r-p}B^nB^{p-n} = O$ if $p > n$

Then, $(A + B)^r = O$, for $r = m + n$

Thus, both the statements are correct but statement 2 is not correctly explaining statement 1.

13. b. Let, $A=\begin{bmatrix} d_1 & z_1 & z_2 \\ \bar{z}_1 & d_2 & z_3 \\ \bar{z}_2 & \bar{z}_3 & d_3 \end{bmatrix}$.

$A^2 = O$

$$\Rightarrow \begin{bmatrix} d_1 & z_1 & z_2 \\ \bar{z}_1 & d_2 & z_3 \\ \bar{z}_2 & \bar{z}_3 & d_3 \end{bmatrix}\begin{bmatrix} d_1 & z_1 & z_2 \\ \bar{z}_1 & d_2 & z_3 \\ \bar{z}_2 & \bar{z}_3 & d_3 \end{bmatrix}$$

$$=\begin{bmatrix} d_1^2+|z_1|^2+|z_2|^2 & d_1z_1+d_2z_1+z_2\bar{z}_3 & d_1z_2+z_1z_3+z_2d_3 \\ d_1\bar{z}_1+d_2\bar{z}_1+z_3\bar{z}_2 & d_2^2+|z_1|^2+|z_3|^2 & \bar{z}_1z_2+d_2z_3+z_3d_3 \\ d_1\bar{z}_2+\bar{z}_3\bar{z}_1+d_3\bar{z}_2 & z_1\bar{z}_2+d_2\bar{z}_3+d_3\bar{z}_3 & d_3^2+|z_1|^2+|z_2|^2 \end{bmatrix}=O$$

$\Rightarrow$ Diagonal elements $d_1 = d_2 = d_3 = 0$ and $|z_1| = |z_2| = |z_3| = 0$

$\Rightarrow$ $z_1 = z_2 = z_3 = 0$

$\Rightarrow$ A = Null matrix

Thus, statement 1 is true. Also,

$A^2 = O \Rightarrow |A|^2 = 0$ or $|A| = 0$

Thus, statement 2 is true but it does not explain statement 1.

14. a. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ $a^2 + b^2 = 1$ (1)

$c^2 + d^2 = 1$ (2)

$ac + bd = 0$ (3)

$\Rightarrow$ $\frac{a}{d} = \frac{-b}{c} = k$ (let)

$\Rightarrow$ $c^2 + d^2 = 1/k^2$ or $k^2 = 1$ or $k = \pm 1$

$\Rightarrow$ $\frac{a}{d} = \frac{-b}{c} = \pm 1$

Also, we must have $a, b, c, d \in [-1, 1]$ for Eqs. (1) and (2) to get defined. Hence, without loss of generality, we can assume $a = \cos\theta$ and $b = \sin\theta$.

So for $\frac{a}{d} = \frac{-b}{c} = 1$, we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and

for $\frac{a}{d} = \frac{-b}{c} = -1$, we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Linked Comprehension Type

For Problems 1–3

1. b, 2. b, 3. d.

Sol.

1. Let,

$a=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow$ $A^2 - (a + d)A + (ad - bc)I$

$$=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}-(a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}+(ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}-\begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}+\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$= O$

2. If $A = O$, tr$(A) = 0$. Suppose $A \neq O$ and $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$|A| = 0$ and $A^2 - (A + d)A + (ad - bc)I = 0$

$\Rightarrow$ $a + d = 0$

3. $(I + A)^{100} = {}^{100}C_0I^{100} + {}^{100}C_1I^{99}A + {}^{100}C_2I^{98}A^2 + \cdots + {}^{100}C_{100}A^{100}$

$= I + 100A + O + O + \cdots + O$

$= I + 100A$

For Problems 4–6

4. c, 5. c, 6. c.

Sol.

4. $AB = A \Rightarrow |AB| = |A|$ (1)

$\Rightarrow$ $|A| = 0$ or $|B| = 1$

$BA = B \Rightarrow |BA| = |B|$ (2)

$\Rightarrow$ $|A| = 1$ or $|B| = 0$

If $|A| = 0$, then from Eq. (2), $|B| = 0$

If $|B| = 0$, then from Eq. (1), $|A| = 0$

5. $AB = A, BA = B$

$ABA = A^2 \Rightarrow A(BA) = A^2 \Rightarrow AB = A^2 \Rightarrow A = A^2$

Similarly, $B^2 = B$

$(A + B)^2 = A^2 + B^2 + AB + BA$

$= A + B + A + B = 2(A + B)$

$(A + B)^3 = (A + B)^2(A + B) = 2(A + B)^2 = 2^2(A + B)$

$\Rightarrow$ $(A + B)^7 = 2^6(A + B) = 64\,(A + B)$

6. $(A + I)^5 = I + 5A + 10A^2 + 10A^3 + 5A^4 + A^5$

$= I + 5A + 10A + 10A + 5A + A$

$(\because A^2 = A \Rightarrow A^3 = A^4 = A^5 = \cdots = A)$

$= I + 31A$

For Problems 7–8

7. d, 8. c.

Sol.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow B=\begin{bmatrix} a_{12}+a_{13} & a_{11}+a_{13} & a_{11}+a_{12} \\ a_{22}+a_{23} & a_{21}+a_{23} & a_{21}+a_{22} \\ a_{32}+a_{33} & a_{31}+a_{33} & a_{31}+a_{32} \end{bmatrix}$$

$\Rightarrow$ $X = A^{-1}B$

$$=\frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{bmatrix} a_{12}+a_{13} & a_{11}+a_{13} & a_{11}+a_{12} \\ a_{22}+a_{23} & a_{21}+a_{23} & a_{21}+a_{22} \\ a_{32}+a_{33} & a_{31}+a_{33} & a_{31}+a_{32} \end{bmatrix}$$

$$=\frac{1}{|A|}\begin{bmatrix} 0 & |A| & |A| \\ |A| & 0 & |A| \\ |A| & |A| & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$ $|A^{-1}B| = 2$

$\Rightarrow$ $|A^{-1}||B| = 2$

$\Rightarrow$ $|B| = 2|A|$

For Problems 9–11

9. b, 10. d, 11. b.

Sol. $A^n - A^{n-2} = A^2 - I \Rightarrow A^{50} = A^{48} + A^2 - I$

Further,

$$A^{48} = A^{46} + A^2 - I$$
$$A^{46} = A^{44} + A^2 - I$$
$$\vdots \quad \vdots \quad \vdots \; \vdots$$
$$A^4 = A^2 + A^2 - I$$

$$A^{50} = 25A^2 - 24I$$

Here,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

$\therefore \quad |A^{50}| = 1$

Also, $\text{tr}(A^{50}) = 1 + 1 + 1 = 3$. Further,

$$\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cup_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$\cup_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \cup_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \cup \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ i.e., } |\cup| = 1$$

For Problems 12–14

12. c, 13. d, 14. b.

Sol.

12. $$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \quad A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 4 \\ -1 & -1 & -2-\lambda \end{bmatrix}$$

$\Rightarrow \quad \det(A - \lambda I) = -(\lambda - 1)(\lambda + 1)(\lambda - 3)$

Thus, the characteristic roots are –1, 1 and 3.

13. Option (a) is not correct since its characteristic determinant is

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

The characteristic equation is $\lambda^2 - 1 = 0$. Therefore, $\lambda = 1, -1$

Hence, eigenvalues are 1 and –1.

We similarly note that matrices given in options (b) and (c) have eigenvalues 1 and –1. Hence, they are not correct.

Option (d) has characteristic equation $(1 - \lambda)^2 = 0$. Hence, eigenvalues are not 1 and –1.

14. b. Let, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\Rightarrow \quad A - \lambda I = \begin{bmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{bmatrix}$$

$$\Rightarrow \quad \det(A - \lambda I) = (a_1 - \lambda)[(b_2 - \lambda)(c_3 - \lambda) - b_3c_2] - b_1[a_2(c_3 - \lambda) - a_3c_2] + c_1[a_2 b_3 - a_3(b_2 - \lambda)]$$

Now one of the eigen values is zero, so one root of equation should be zero. Therefore, constant term in the above polynomial is zero.

$\therefore \quad a_1b_2c_3 - a_1b_3c_2 - b_1a_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1a_3b_2 = 0$ (collecting constant terms)

But this value is value of determinant of A.

$\therefore \quad \det A = 0$

For Problems 15–17

15. c, 16. d, 17. c.

Sol.

15. As second row of all the options is same, we are to look at the elements of the first row. Let the left inverse be $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad a + b + 2c = 1$

$-a + b + 3c = 1$, i.e., $b = \dfrac{1-5c}{2}, a = \dfrac{1+c}{2}$

Thus, matrices in the options (a), (b) and (d) are the inverses and matrix in option (c) is not the left inverse.

16. Let right inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, $a - c + 2e = 1$

$b - d + 2f = 0$

$2a - c + e = 0$

$2b - d + f = 1$

This system of equations has infinite solutions.

17. By observation there cannot be any left inverse for options (b) and (d). So we will check for (a) and (c) only.

For option (a), let the left inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $a-3b=1, 2a+2b=0$ and $4a+b=0$ which is not possible.

For option (c),

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $a+2b+5c=1, 4a-3b+4c=0, d+2e+5f=0, 4d-3e+4f=1$

Therefore, there are infinite number of left inverses.

$$\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $a+4d=1, 2a-3d=0$ and $5a+4d=0$

which is not possible. Therefore, there is no right inverse.

For Problems 18–20

18. a, 19. b, 20. c.

Sol. $A=\begin{bmatrix} x & x \\ x & x \end{bmatrix} \Rightarrow A^2=\begin{bmatrix} 2x^2 & 2x^2 \\ 2x^2 & 2x^2 \end{bmatrix}, A^3=\begin{bmatrix} 2^2x^2 & 2^2x^2 \\ 2^2x^2 & 2^2x^2 \end{bmatrix}$

and so on. Then

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$=\begin{bmatrix} 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots & x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots \\ x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots & 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+\cdots \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\cdots\right)+\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\cdots\right)-\frac{1}{2} \\ \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\cdots\right)-\frac{1}{2} & \frac{1}{2}\left(1+2x+\frac{2^2x^2}{2!}+\cdots\right)+\frac{1}{2} \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} e^{2x}+1 & e^{2x}-1 \\ e^{2x}-1 & e^{2x}+1 \end{bmatrix}$$

$\Rightarrow$ $f(x)=e^{2x}+1$ and $g(x)=e^{2x}-1$

18. $\int \frac{e^{2x}-1}{e^{2x}+1}dx=\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx=\log_e(e^x+e^{-x})+c$

19. $\int (g(x)+1)\sin x dx=\int e^{2x}\sin x dx=\frac{e^{2x}}{5}(2\sin x-\cos x)+c$

20. $\int \frac{e^{2x}+1}{\sqrt{e^{2x}-1}}dx$

$$=\int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx+\int \frac{1}{\sqrt{e^{2x}-1}}dx$$

$$=\int \frac{e^{2x}}{\sqrt{e^{2x}-1}}dx+\int \frac{e^x}{e^x\sqrt{e^{2x}-1}}dx$$

$$=\frac{1}{2\sqrt{e^{2x}-1}}+\sec^{-1}(e^x)+c$$

Matrix-Match Type

1. a → s; b → p; c → q; d → r.

a. Since A is idempotent, hence,

$A^2=A$

$\Rightarrow$ $A^3=AA^2=AA=A^2=A, A^4=AA^3=AA=A^2=A$

$\Rightarrow$ $A^n=A$

$\Rightarrow$ $(I-A)^n={}^nC_0I-{}^nC_1A+{}^nC_2A^2-{}^nC_3A^3+\cdots$

$=I+(-{}^nC_1+{}^nC_2-{}^nC_3+\cdots)A$

$=I+[({}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+\cdots)-{}^nC_0]A=I-A$

b. A is involuntary. Hence,

$A^2=I$

$\Rightarrow$ $A^3=A^5=\cdots=A$ and $A^2=A^4=A^6=\cdots=I$

$\Rightarrow$ $(I-A)^n={}^nC_0I-{}^nC_1A+{}^nC_2A^2-{}^nC_3A^3+\cdots$

$={}^nC_0I-{}^nC_1A+{}^nC_2I-{}^nC_3A+{}^nC_4I-\cdots$

$=({}^nC_0+{}^nC_2+{}^nC_4+\cdots)I-({}^nC_1A+{}^nC_3+{}^nC_5+\cdots)$

$A=2^{n-1}(I-A)$

$\Rightarrow$ $[(I-A)^n]A^{-1}=2^{n-1}(I-a)A^{-1}=2^{n-1}(A^{-1}-I)$

c. If A is nilpotent of index 2, then

$A^2=A^3=A^4\cdots=A^n=O$

$\Rightarrow$ $(I-A)^n={}^nC_0I-{}^nC_1A+{}^nC_2A^2-{}^nC_3A^3+\cdots$

$=I-nA+O+O+\cdots$

$=I-nA$

d. A is orthogonal. Hence,

$AA^T=I$

$\Rightarrow$ $(A^T)^{-1}=A$

2. a → r; b → s; c → p, r; d → p, q, r, s.

a. Since A is idempotent, $A^2=A^3=A^4=\cdots=A$. Now,

$(A+I)^n=I+{}^nC_1A+{}^nC_2A^2+\cdots+{}^nC_nA^n$

$=I+{}^nC_1A+{}^nC_2A+\cdots+{}^nC_nA$

$=I+{}^nC_1A+{}^nC_2A+\cdots+{}^nC_nA$

$=I+({}^nC_1+{}^nC_2+\cdots+{}^nC_n)A$

$=I+(2^n-1)A$

$\Rightarrow$ $2^n-1=127$

$\Rightarrow$ $n=7$

b. We have,

$(I-A)(I+A+A^2+\cdots+A^7)$

$=I+A+A^2+\cdots+A^7+(-A-A^2-A^3-A^4\cdots-A^8)$

$=I-A^8$

$=I$ (if $A^8=O$)

c. Here matrix A is skew-symmetric and since $|A|=|A^T|=(-1)^n|A|$, so $|A|(1-(-1)^n)=0$. As n is odd, hence $|A|=0$. Hence A is singular.

d. If A is symmetric, A^{-1} is also symmetric for matrix of any order.

3. a → q; b → p; c → s; d → r.

a. $|A| = 2 \Rightarrow |2A^{-1}| = 2^3/|A| = 4$

b. $|\text{adj}(\text{adj}(2A))| = |2A|^4 = 2^{12}|A|^4 = 2^{12}/2^{12} = 1$

c. $(A + B)^2 = A^2 + B^2$

$\Rightarrow AB + BA = O$

$\Rightarrow |AB| = |-BA| = -|BA| = -|AB|$

$\Rightarrow |AB| = 0$

$\Rightarrow |B| = 0$

d. Product ABC is not defined.

Integer Type

1.(0) $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$\Rightarrow A^2 = A.A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow A^4 = A^2.A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

$\Rightarrow A^8 = \begin{bmatrix} 3^4 & 0 \\ 0 & 3^4 \end{bmatrix}$

and $A^6 = A^4.A^2 = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & 3^3 \end{bmatrix}$

Let $V = \begin{bmatrix} x \\ y \end{bmatrix}$

$A^8 + A^6 + A^4 + A^2 + I$

$\begin{bmatrix} 81 & 0 \\ 0 & 81 \end{bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}$

$(A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 121x \\ 121y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

$\Rightarrow x = 0$ and $y = 1/11$

$\Rightarrow V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1/11 \end{bmatrix}$

2.(4) $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix.

$\Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+bc & ab+b-ab \\ ac+c-ac & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+bc & b \\ c & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

$\Rightarrow a^2 + bc = a$

$a - a^2 = bc = 1/4$ (given)

$f(a) = 1/4$

3.(2) $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$\Rightarrow A^2 = A.A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A^2 = I \Rightarrow A^4 = A^6 = A^8 = \cdots = I$

Now $A^x = I$

$\Rightarrow x = 2, 4, 6, 8, \ldots$

$\therefore \Sigma(\cos^x\theta + \sin^x\theta)$

$= (\cos^2\theta + \sin^2\theta) + (\cos^4\theta + \sin^4\theta) + (\cos^6\theta + \sin^6\theta) + \cdots$

$= (\cos^2\theta + \cos^4\theta + \cos^6\theta + \cdots) + (\sin^4\theta + \sin^4\theta + \sin^6\theta + \cdots)$

$= \dfrac{\cos^2\theta}{1-\cos^2\theta} + \dfrac{\sin^2\theta}{1-\sin^2\theta}$

$= \cot^2\theta + \tan^2\theta$

which has minimum value 2

4.(1) $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

Hence, det. $A = \sec^2 x$

$\therefore$ det $A^T = \sec^2 x$

Now $f(x) = \text{det.}\,(A^T A^{-1})$

$= (\text{det. } A^T)\,(\text{det. } A^{-1})$

$= (\text{det. } A^T)\,(\text{det. } A)^{-1}$

$= \dfrac{\text{det.}(A^T)}{\text{det.}(A)} = 1$

Hence, $f(x) = 1$.

5.(2) $\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{vmatrix} = 0$

$\Rightarrow 1(3k - 16) - 2(k - 12) + 2(4 - 9) = 0$

$\Rightarrow 3k - 16 - 2k + 24 - 10 = 0$

$\Rightarrow k = 2$

6.(9) Given $A^2 = A$

$\Rightarrow I = (I - 0.4A)(I - \alpha A)$

$= I - I\alpha A - 0.4AI + 0.4\alpha A^2$

$= I - A\alpha - 0.4A + 0.4\alpha A$

$= I - A(0.4 + \alpha) + 0.4\alpha A$

$\Rightarrow 0.4\alpha = 0.4 + \alpha$

$\Rightarrow \alpha = -2/3$

$\Rightarrow |9\alpha| = 6$

7.(7) We have $AB = \begin{bmatrix} 3ax^2 & 3bx^2 & 3cx^2 \\ a & b & c \\ 6ax & 6bx & 6cx \end{bmatrix}$

Now tr·(AB) = tr·(C)

$\Rightarrow \quad 3ax^2 + b + 6cx = (x+2)^2 + 2x + 5x^2 \quad \forall x \in R$ (Identity)

$\Rightarrow \quad 3ax^2 + 6cx + b = 6x^2 + 6x + 4$

$\Rightarrow \quad a = 2, c = 1, b = 4$

8.(8) In a skew symmetric matrix, diagonal elements are zero.

Also $a_{ij} + a_{ji} = 0$

Hence, number of matrices = 2 × 2 × 2 = 8

9.(4) Given that $AA^T = 4I$

$\Rightarrow \quad |A|^2 = 4$

$\Rightarrow \quad |A| = \pm 2,$

so $A^T = 4A^{-1} = 4\dfrac{adj\, A}{|A|}$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \frac{4}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Now $a_{ij} = \dfrac{4}{|A|} c_{ij}$

$\Rightarrow \quad -2c_{ij} = \dfrac{4}{|A|} c_{ij}$(as $a_{ij} + 2c_{ij} = 0$)

$\Rightarrow \quad |A| = -2$

Now $|A + 4I| = |A + AA^T|$

$= |A|\,|I + A^T|$

$= -2\,|(I + A)^T|$

$= -2\,|I + A|$

$\Rightarrow \quad |A + 4I| + 2\,|A + I| = 0,$

so on comparing, we get $5\lambda = 2 \Rightarrow \lambda = \dfrac{2}{5}$

Hence, $10\lambda = 4$

10.(0) For idempotent matrix, $A^2 = A$

$\Rightarrow \quad A^{-1}A^2 = A^{-1}A \qquad (\because \; A$ is non-singular)

$\Rightarrow \quad A = I$

Thus non-singular idempotent matrix is always a unit matrix.

$\therefore \quad l^2 - 3 = 1 \quad \Rightarrow \quad l = \pm 2$

$m^2 - 8 = 1 \quad \Rightarrow \quad m = \pm 3$

$n^2 - 15 = 1 \quad \Rightarrow \quad n = \pm 4$

and $p = q = r = 0$

$\Rightarrow$ required sum is 0.

11.(8) A diagonal matrix is commutative with every square matrix if it is scalar matrix so every diagonal element is 4.

$\therefore \quad |A| = 64$

12.(4) $|\text{adj } A^{-1}| = |A^{-1}|^2 = \dfrac{1}{|A|^2}$

$\Rightarrow \quad |(\text{adj } A^{-1})^{-1}| = \dfrac{1}{|adj\ A^{-1}|}$

$= |A|^2 = 2^2 = 4$

Archives

Subjective Type

1. $A^TA = I$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a^2 + b^2 + c^2 = 1$ (1)

and

$ab + bc + ca = 0$ (2)

Now,

$a^3 + b^3 + c^3 = (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$
$+ 3abc = (a + b + c) + 3$ (3)

Now,

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,(ab + bc + ca) = 1 + 2 \times 0 = 1$

$\Rightarrow \quad a + b + c = 1$ (since a, b, c are real positive numbers)

Now from Eq. (3), $a^3 + b^3 + c^3 = 1 + 3 = 4$

Alternative solution:

$A^TA = I$

$\Rightarrow \quad |A^TA| = |I| \Rightarrow |A|^2 = 1$

$\Rightarrow \quad (a^3 + b^3 + c^3 - 3abc)^2 = 1$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = 1$

(since a, b, c are positive real numbers)

$\Rightarrow \quad a^3 + b^3 + c^3 \geq 3abc \quad (\because$ A.M. ≥ G.M.)

$\Rightarrow \quad a^3 + b^3 + c^3 = 4$

2. We are given that $MM^T = I$, where M is a square matrix of order 3 and det $M = 1$. Now,

det $(M - I)$ = det $(M - MM^T)$ $\quad [\because$ Given $MM^T = I]$

= det $[M(I - M^T)]$

= (det M) [det $(I - M^T)$] $\quad [\because |AB| = |A|\,|B|]$

= –(det M) [det $(M^T - 1)$]

= –1[det $(M^T - 1)$] $\quad [\because$ det $(M) = 1]$

= –det $[(M - I)]^T$

= –det $(M - I)$]

$\Rightarrow \quad 2$ det $(M - I) = 0$

$\Rightarrow \quad$ det $(M - I) = 0$

3. Given that $A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$

and $AX = U$ has infinitely many solutions. Hence,

$|A| = 0$ and $|A_1| = |A_2| = |A_3| = 0$

Now, $|A| = 0 \Rightarrow \begin{vmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{vmatrix} = a(bc - bd) - 1(c - d) = 0$

$\Rightarrow \quad (ab - 1)\,(c - d) = 0$

$\Rightarrow \quad ab = 1$ or $c = d$ (1)

Also,

$$|A_1| = \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix} = 0$$

$\Rightarrow f(bc - bd) - 1(gc - hd) = 0$

$\Rightarrow fb(c - d) = gc - hd$ (2)

$$|A_2| = \begin{vmatrix} a & f & 0 \\ 1 & g & d \\ 1 & h & c \end{vmatrix} = 0$$

$\Rightarrow a(gc - hd) - f(c - d) = 0$

$\Rightarrow a(gc - hd) = f(c - d)$ (3)

$$|A_3| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$$

$\Rightarrow a(bh - bg) - 1(h - g) + f(b - b) = 0$

$\Rightarrow ab(h - g) - (h - g) = 0$

$\Rightarrow ab = 1$ or $h = g$ (4)

For $AX = U$ to have infinitely many solutions,

$c = d$ and $h = g$.

Now taking $BX = V$ where $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$, we have

$$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$$

(since in view of $c = d$ and $g = h$, C_2 and C_3 are identical)

$\Rightarrow BX = V$ has no unique solution

Also,

$$|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0 \quad (\because c = d, g = h)$$

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df \quad (\because c = d)$$

$$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} = -a^2df$$

Therefore, if $adf \neq 0$, then $|B_2| = |B_3| \neq 0$. Hence, no solution exists.

4. $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow AB = BA$

As A is symmetric and B is skew-symmetric,

$(AB)' = -AB$

$\Rightarrow k$ is an odd integer

Objective Type

Multiple choice questions with one correct answer

1. d. Given that $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ and $A^2 = B$. Hence,

$$\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\Rightarrow \alpha^2 = 1, \alpha + 1 = 5$

$\Rightarrow \alpha = \pm 1, \alpha = 4$

Hence, there is no real value.

2. c. $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$

Now,

$|A| = \alpha^2 - 4$

$\Rightarrow (\alpha^2 - 4)^3 = 125 = 5^3$

$\Rightarrow \alpha^2 - 4 = 5$

$\Rightarrow \alpha = \pm 3$

3. c. We have,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}(A^2 + cA + dI).$$

$\Rightarrow 6AA^{-1} = A^3 + cA^2 + dAI$

$\Rightarrow A^3 + cA^2 + dA - 6I = O$

We find that

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix}$$

$\therefore A^3 + cA^2 + dA - 6I = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix} + c\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} + d\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 1 + c + d - 6 & 0 & 0 \\ 0 & -11 - c + d - 6 & 19 + 5c + d \\ 0 & -38 - 10c - 2d & 46 + 14c + 4d - 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow 1 + c + d - 6 = 0$

$-11 - c + d - 6 = 0$

$\Rightarrow c + d = 5$ and $-c + d = 17$

On solving, we get $c = -6$, $d = 11$. They also satisfy the equations

$-38 - 10c - 2d = 0$

$46 + 14c + 4d - 6 = 0$
$19 + 5c + d = 0$

4.a. Given that

$$P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$Q = PAP^T$ and
$X = P^T Q^{2005} P$

We have,

$$P^T P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We observe that

$Q = PAP^T$
$\Rightarrow Q^2 = (PAP^T)(PAP^T)$
$= PA(P^TP) AP^T$
$= PA(IA) P^T$
$= PA^2 P^T$

Proceeding in the same way, we get
$Q^{2005} = PA^{2005} P^T$

Also,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

And proceeding in the same way,

$$A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

Now,

$X = P^T Q^{2005} P$
$= P^T(PA^{2005}P^T)P$
$= (P^TP)A^{2005}(P^TP)$
$= IA^{2005}I$
$= A^{2005}$

$$= \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

5. a. Three planes cannot intersect at two distinct points.

6. a. For being non–singular

$$\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$$

$\Rightarrow ac\omega^2 - (a + c)\,\omega + 1 \neq 0$

Hence number of possible triplets of (a, b, c) is 2.
i.e. $(\omega, \omega^2, \omega)$ and (ω, ω, ω).

Multiple choice questions with one or more correct answer

1. c. Given $MN = NM$

$M^2N^2 (M^TN)^{-1} (MN^{-1})^T$
$= M^2N^2 N^{-1} (M^T)^{-1} (N^{-1})^T . M^T$
$= M^2N. (M^T)^{-1}(N^{-1})^T M^T$
$= -M^2.N(M)^{-1} (N^T)^{-1} M^T$
$= + M^2NM^{-1}N^{-1} M^T$
$= -M.NMM^{-1} N^{-1} M$ $\quad (\because MN = NM)$
$= -MNN^{-1}M = -M^2.$

Comprehension

For Problems 1–3

1. a, 2. b, 3. a.

Sol.

1. Let $U_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ 2a + b \\ 3a + 2b + c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a = 1, b = -2, c = 1$

$$\therefore U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Similarly,

$$U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}, U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \Rightarrow |U| = 3$$

2. $$U^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$$

Hence, sum of elements of U^{-1} is $\frac{1}{3}(0) = 0$.

3. $$\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -8 \\ -5 \end{bmatrix} = 5$$

For Problems 4–6

4. a, 5. b, 6. b.

Sol.

4. Let the matrix be

$$\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

We have five entries as 1 and remaining four entries as 0.

Since matrix is symmetric, we must have even number of zeros for $i \neq j$. We have two cases.

(i) Two entries in diagonal are zero. We can select two places from three (in diagonal) in ${}^3C_2$ ways. Now we have to select elements for upper triangle. For upper triangle, we have three places of which one entry is '0' and two are '1'. One place from three can be selected in ${}^3C_1$ ways. Hence, the number of matrices is ${}^3C_2 \times {}^3C_1 = 9$.

(ii) If all the entries in the principal diagonal are 1, we have two '0' and one '1' in upper triangle. Hence, the number of matrices is 3. Therefore, total number of matrices is 12.

5. $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 1 \end{bmatrix}$

Either $b = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$$A = \begin{bmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{bmatrix}$$

Either $a = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$$A = \begin{bmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$$

Either $a = 0$ or $b = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$$A = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$$

$a = b = 0 \Rightarrow |A| = 0$

$a = c = 0 \Rightarrow |A| = 0$

$b = c = 0 \Rightarrow |A| = 0$

Therefore, there will be only six matrices.

6. The six matrices A for which $|A| = 0$ are:

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (infinite solutions)

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (inconsistent)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (inconsistent)

For Problems 7–9

7. d, 8. c, 9. d.

Sol.

7. d We must have $a^2 - b^2 = kp$

$\Rightarrow$ $(a + b)(a - b) = kp$

$\Rightarrow$ either $a - b = 0$ or $a + b$ is multiple of p

when $a = b$; number of matrices is p

and when $a + b =$ multiple of $p \Rightarrow a, b$ has $p - 1$

$\therefore$ Total number of matrices $= p + p - 1$
$= 2p - 1$.

8. c.

9. d.

For Problems 10–12

10. d, 11. a, 12. b.

Sol.

10. $a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

$a + b + c = 0$

Solving these, we get

$b = 6a \Rightarrow c = -7a$

now $2x + y + z = 0$

$\Rightarrow 2a + 6a + (-7a) = 1 \Rightarrow a = 1, b = 6, c = -7.$

11. $a = 2$, b and c satisfies (E)

$b = 12, c = -14$

$$\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$$

$$= \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}}$$

$= 3\omega + 1 + 3\omega^2$

$= -2.$

12. $ax^2 + bx + c = 0$

$\Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow \alpha = 1, \beta = -7$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{1} - \frac{1}{7}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n$$

$$= \frac{1}{1 - \frac{6}{7}} = 7$$

Integer type

1.(4) $|A| = (2k+1)^3$, $|B| = 0$ (since B is a skew-symmetric matrix of order 3)

$\Rightarrow$ $\det(\text{adj } A) = |A|^{n-1} = ((2k+1)^3)^2 = 10^6$

$\Rightarrow$ $2k+1 = 10 \Rightarrow 2k = 9$

$\Rightarrow$ $[K] = 4$.

2.(9) let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow b = -1, e = 2, h = 3$

$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow a = 0, d = 3, g = 2$

$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} = g + h + i = 12 \Rightarrow i = 7$

Therefore, sum of diagonal elements = 9.