CHAPTER V.

GEOMETRICAL PROGRESSION.

51. Definition. Quantities are said to be in **Geometrical Progression** when they increase or decrease by a *constant factor*.

Thus each of the following series forms a Geometrical Progression:

$$3,\ 6,\ 12,\ 24,\ \ldots\ldots\ldots\ldots$$

$$1,\ -\frac{1}{3},\ \frac{1}{9},\ -\frac{1}{27},\ldots\ldots\ldots$$

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots\ldots\ldots$$

The constant factor is also called the *common ratio*, and it is found by dividing *any* term by that which immediately *precedes* it. In the first of the above examples the common ratio is 2; in the second it is $-\frac{1}{3}$; in the third it is r.

52. If we examine the series

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots\ldots$$

we notice that *in any term the index of* r *is always less by one than the number of the term in the series.*

Thus the 3^{rd} term is ar^2;

the 6^{th} term is ar^5;

the 20^{th} term is ar^{19};

and, generally, the p^{th} term is ar^{p-1}.

If n be the number of terms, and if l denote the last, or n^{th} term, we have $l = ar^{n-1}$.

53. Definition. When three quantities are in Geometrical Progression the middle one is called the **geometric mean** between the other two.

To find the geometric mean between two given quantities.

Let a and b be the two quantities; G the geometric mean. Then since a, G, b are in G. P.,

$$\frac{b}{G} = \frac{G}{a},$$

each being equal to the common ratio;

$$\therefore G^2 = ab;$$

whence $$G = \sqrt{ab}.$$

54. *To insert a given number of geometric means between two given quantities.*

Let a and b be the given quantities, n the number of means.

In all there will be $n+2$ terms; so that we have to find a series of $n+2$ terms in G. P., of which a is the first and b the last.

Let r be the common ratio;

then $$b = \text{the } (n+2)^{\text{th}} \text{ term}$$
$$= ar^{n+1};$$
$$\therefore r^{n+1} = \frac{b}{a};$$
$$\therefore r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence the required means are ar, ar^2,... ar^n, where r has the value found in (1).

Example. Insert 4 geometric means between 160 and 5.

We have to find 6 terms in G. P. of which 160 is the first, and 5 the sixth.

Let r be the common ratio;

$$\text{then } 5 = \text{the sixth term}$$
$$= 160r^5;$$
$$\therefore r^5 = \frac{1}{32};$$

whence $$r = \frac{1}{2};$$

and the means are 80, 40, 20, 10.

55. *To find the sum of a number of terms in Geometrical Progression.*

Let a be the first term, r the common ratio, n the number of terms, and s the sum required. Then

$$s = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1};$$

multiplying every term by r, we have

$$rs = ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n.$$

Hence by subtraction,

$$rs - s = ar^n - a;$$

$$\therefore (r-1)\,s = a\,(r^n - 1);$$

$$\therefore s = \frac{a(r^n - 1)}{r - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Changing the signs in numerator and denominator,

$$s = \frac{a(1 - r^n)}{1 - r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

NOTE. It will be found convenient to remember both forms given above for s, using (2) in all cases except when r is *positive and greater than* 1.

Since $ar^{n-1} = l$, the formula (1) may be written

$$s = \frac{rl - a}{r - 1};$$

a form which is sometimes useful.

Example. Sum the series $\frac{2}{3}$, -1, $\frac{3}{2}$, ... to 7 terms.

The common ratio $= -\frac{3}{2}$; hence by formula (2)

the sum $$= \frac{\frac{2}{3}\left\{1 - \left(-\frac{3}{2}\right)^7\right\}}{1 + \frac{3}{2}}$$

$$= \frac{\frac{2}{3}\left\{1 + \frac{2187}{128}\right\}}{\frac{5}{2}}$$

$$= \frac{2}{3} \times \frac{2315}{128} \times \frac{2}{5}$$

$$= \frac{463}{96}.$$

56. Consider the series $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\ldots$

$$\text{The sum to } n \text{ terms} = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$$

$$= 2\left(1-\frac{1}{2^n}\right)$$

$$= 2-\frac{1}{2^{n-1}}.$$

From this result it appears that however many terms be taken the sum of the above series is always less than 2. Also we see that, by making n sufficiently large, we can make the fraction $\frac{1}{2^{n-1}}$ as small as we please. Thus by taking a sufficient number of terms the sum can be made to differ by as little as we please from 2.

In the next article a more general case is discussed.

57. From Art. 55 we have $s = \frac{a(1-r^n)}{1-r}$

$$= \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Suppose r is a proper fraction; then the greater the value of n the smaller is the value of r^n, and consequently of $\frac{ar^n}{1-r}$; and therefore by making n sufficiently large, we can make the sum of n terms of the series differ from $\frac{a}{1-r}$ by as small a quantity as we please.

This result is usually stated thus: *the sum of an infinite number of terms of a decreasing Geometrical Progression is* $\frac{\mathrm{a}}{1-\mathrm{r}}$; or more briefly, *the sum to infinity is* $\frac{\mathrm{a}}{1-\mathrm{r}}$.

Example 1. Find three numbers in G. P. whose sum is 19, and whose product is 216.

Denote the numbers by $\frac{a}{r}$, a, ar; then $\frac{a}{r} \times a \times ar = 216$; hence $a=6$, and the numbers are $\frac{6}{r}$, 6, $6r$.

$$\therefore \frac{6}{r}+6+6r=19;$$

$$\therefore 6-13r+6r^2=0;$$

whence $$r=\frac{3}{2} \text{ or } \frac{2}{3}.$$

Thus the numbers are 4, 6, 9.

Example 2. The sum of an infinite number of terms in G. P. is 15, and the sum of their squares is 45; find the series.

Let a denote the first term, r the common ratio; then the sum of the terms is $\frac{a}{1-r}$; and the sum of their squares is $\frac{a^2}{1-r^2}$.

Hence $$\frac{a}{1-r}=15 \qquad \text{.....(1)},$$

$$\frac{a^2}{1-r^2}=45 \qquad \text{.....(2)}.$$

Dividing (2) by (1) $$\frac{a}{1+r}=3 \qquad \text{.....(3)},$$

and from (1) and (3) $$\frac{1+r}{1-r}=5;$$

whence $r=\frac{2}{3}$, and therefore $a=5$.

Thus the series is 5, $\frac{10}{3}$, $\frac{20}{9}$,......

EXAMPLES. V. a.

1. Sum $\frac{1}{2}, \frac{1}{3}, \frac{2}{9},\ldots$ to 7 terms.

2. Sum $-2,\ 2\frac{1}{2},\ -3\frac{1}{8},\ldots$ to 6 terms.

3. Sum $\frac{3}{4},\ 1\frac{1}{2},\ 3,\ldots$ to 8 terms.

4. Sum 2, -4, 8,... to 10 terms.

5. Sum 16·2, 5·4, 1·8,... to 7 terms.

6. Sum 1, 5, 25,... to p terms.

7. Sum 3, -4, $\frac{16}{3}$,... to $2n$ terms.

8. Sum 1, $\sqrt{3}$, 3,... to 12 terms.

9. Sum $\frac{1}{\sqrt{2}},\ -2,\ \frac{8}{\sqrt{2}},\ldots$ to 7 terms.

10. Sum $-\frac{1}{3}, \frac{1}{2}, -\frac{3}{4}, \ldots$ to 7 terms.

11. Insert 3 geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$.

12. Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

13. Insert 6 geometric means between 14 and $-\frac{7}{64}$.

Sum the following series to infinity:

14. $\frac{8}{5}, -1, \frac{5}{8}, \ldots$

15. $\cdot 45, \cdot 015, \cdot 0005, \ldots$

16. $1\cdot 665, -1\cdot 11, \cdot 74, \ldots$

17. $3^{-1}, 3^{-2}, 3^{-3}, \ldots$

18. $3, \sqrt{3}, 1, \ldots$

19. $7, \sqrt{42}, 6, \ldots$

20. The sum of the first 6 terms of a G. P. is 9 times the sum of the first 3 terms; find the common ratio.

21. The fifth term of a G. P. is 81, and the second term is 24; find the series.

22. The sum of a G. P. whose common ratio is 3 is 728, and the last term is 486; find the first term.

23. In a G. P. the first term is 7, the last term 448, and the sum 889; find the common ratio.

24. The sum of three numbers in G. P. is 38, and their product is 1728; find them.

25. The continued product of three numbers in G. P. is 216, and the sum of the product of them in pairs is 156; find the numbers.

26. If S_p denote the sum of the series $1 + r^p + r^{2p} + \ldots$ *ad inf.*, and s_p the sum of the series $1 - r^p + r^{2p} - \ldots$ *ad inf.*, prove that

$$S_p + s_p = 2S_{2p}.$$

27. If the p^{th}, q^{th}, r^{th} terms of a G. P. be a, b, c respectively, prove that

$$a^{q-r}\, b^{r-p}\, c^{p-q} = 1.$$

28. The sum of an infinite number of terms of a G. P. is 4, and the sum of their cubes is 192; find the series.

58. Recurring decimals furnish a good illustration of infinite Geometrical Progressions.

Example. Find the value of $\cdot 4\dot{2}\dot{3}$.

$$\begin{aligned} \cdot 4\dot{2}\dot{3} &= \cdot 4232323\ldots\ldots \\ &= \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \ldots\ldots \\ &= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots\ldots; \end{aligned}$$

that is, $$\cdot 4\dot{2}\dot{3} = \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots\dots\right)$$

$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}}$$

$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{100}{99}$$

$$= \frac{4}{10} + \frac{23}{990}$$

$$= \frac{419}{990},$$

which agrees with the value found by the usual arithmetical rule.

59. The general rule for reducing any recurring decimal to a vulgar fraction may be proved by the method employed in the last example; but it is easier to proceed as follows.

To find the value of a recurring decimal.

Let P denote the figures which do not recur, and suppose them p in number; let Q denote the recurring period consisting of q figures; let D denote the value of the recurring decimal; then

$$D = \cdot PQQQ\dots\dots\dots\dots;$$

$$\therefore\ 10^p \times D = P\cdot QQQ\dots\dots\dots\dots;$$

and $$10^{p+q} \times D = PQ\cdot QQQ\ \dots\dots\dots;$$

therefore, by subtraction, $(10^{p+q} - 10^p)\,D = PQ - P$;

that is, $$10^p\,(10^q - 1)\,D = PQ - P;$$

$$\therefore\ D = \frac{PQ - P}{(10^q - 1)\,10^p}.$$

Now $10^q - 1$ is a number consisting of q nines; therefore the denominator consists of q nines followed by p ciphers. Hence we have the following rule for reducing a recurring decimal to a vulgar fraction:

For the numerator subtract the integral number consisting of the non-recurring figures from the integral number consisting of the non-recurring and recurring figures; for the denominator take a number consisting of as many nines as there are recurring figures followed by as many ciphers as there are non-recurring figures.

60. *To find the sum of* n *terms of the series*

$$a, \ (a+d)r, \ (a+2d)r^2, \ (a+3d)r^3, \ldots\ldots\ldots$$

in which each term is the product of corresponding terms in an arithmetic and geometric series.

Denote the sum by S; then

$$S = a + (a+d)r + (a+2d)r^2 + \ldots + (a+\overline{n-1}d)r^{n-1};$$

$$\therefore \ rS = \qquad ar + (a+d)r^2 + \ldots + (a+\overline{n-2}d)r^{n-1} + (a+\overline{n-1}d)r^n.$$

By subtraction,

$$S(1-r) = a + (dr + dr^2 + \ldots + dr^{n-1}) - (a+\overline{n-1}d)r^n$$

$$= a + \frac{dr(1-r^{n-1})}{1-r} - (a+\overline{n-1}d)r^n;$$

$$\therefore \ S = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+\overline{n-1}d)r^n}{1-r}.$$

Cor. Write S in the form

$$\frac{a}{1-r} + \frac{dr}{(1-r)^2} - \frac{dr^n}{(1-r)^2} - \frac{(a+\overline{n-1}d)r^n}{1-r};$$

then if $r<1$, we can make r^n as small as we please by taking n sufficiently great. In this case, *assuming that all the terms which involve* r^n *can be made so small that they may be neglected,* we obtain $\dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$ for the sum to infinity. We shall refer to this point again in Chap. XXI.

In summing to infinity series of this class it is usually best to proceed as in the following example.

Example 1. If $x<1$, sum the series

$$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \text{ to infinity.}$$

Let

$$S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots;$$

$$\therefore \ xS = \qquad x + 2x^2 + 3x^3 + \ldots\ldots;$$

$$\therefore \ S(1-x) = 1 + x + x^2 + x^3 + \ldots\ldots$$

$$= \frac{1}{1-x};$$

$$\therefore \ S = \frac{1}{(1-x)^2}.$$

Example 2. Sum the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots$ to n terms.

Let
$$S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots\ldots+\frac{3n-2}{5^{n-1}};$$

$$\therefore\ \frac{1}{5}S=\quad\frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\ldots\ldots+\frac{3n-5}{5^{n-1}}+\frac{3n-2}{5^n};$$

$$\therefore\ \frac{4}{5}S=1+\left(\frac{3}{5}+\frac{3}{5^2}+\frac{3}{5^3}+\ldots\ldots+\frac{3}{5^{n-1}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+\ldots\ldots+\frac{1}{5^{n-2}}\right)-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{5}\cdot\frac{1-\frac{1}{5^{n-1}}}{1-\frac{1}{5}}-\frac{3n-2}{5^n}$$

$$=1+\frac{3}{4}\left(1-\frac{1}{5^{n-1}}\right)-\frac{3n-2}{5^n}$$

$$=\frac{7}{4}-\frac{12n+7}{4\,.\,5^n};$$

$$\therefore\ S=\frac{35}{16}-\frac{12n+7}{16\,.\,5^{n-1}}.$$

EXAMPLES. V. b.

1. Sum $1+2a+3a^2+4a^3+\ldots$ to n terms.

2. Sum $1+\frac{3}{4}+\frac{7}{16}+\frac{15}{64}+\frac{31}{256}+\ldots$ to infinity.

3. Sum $1+3x+5x^2+7x^3+9x^4+\ldots$ to infinity.

4. Sum $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\ldots$ to n terms.

5. Sum $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$ to infinity.

6. Sum $1+3x+6x^2+10x^3+\ldots$ to infinity.

7. Prove that the $(n+1)^{\text{th}}$ term of a G. P., of which the first term is a and the third term b, is equal to the $(2n+1)^{\text{th}}$ term of a G. P. of which the first term is a and the fifth term b.

8. The sum of $2n$ terms of a G. P. whose first term is a and common ratio r is equal to the sum of n of a G. P. whose first term is b and common ratio r^2. Prove that b is equal to the sum of the first two terms of the first series.

9. Find the sum of the infinite series

$$1+(1+b)r+(1+b+b^2)r^2+(1+b+b^2+b^3)r^3+\ldots,$$

r and b being proper fractions.

10. The sum of three numbers in G. P. is 70; if the two extremes be multiplied each by 4, and the mean by 5, the products are in A. P.; find the numbers.

11. The first two terms of an infinite G. P. are together equal to 5, and every term is 3 times the sum of all the terms that follow it; find the series.

Sum the following series:

12. $x+a,\ x^2+2a,\ x^3+3a\ldots$ to n terms.

13. $x(x+y)+x^2(x^2+y^2)+x^3(x^3+y^3)+\ldots$ to n terms.

14. $a+\frac{1}{3},\ 3a-\frac{1}{6},\ 5a+\frac{1}{12}+\ldots$ to $2p$ terms.

15. $\frac{2}{3}+\frac{3}{3^2}+\frac{2}{3^3}+\frac{3}{3^4}+\frac{2}{3^5}+\frac{3}{3^6}+\ldots$ to infinity.

16. $\frac{4}{7}-\frac{5}{7^2}+\frac{4}{7^3}-\frac{5}{7^4}+\frac{4}{7^5}-\frac{5}{7^6}+\ldots$ to infinity.

17. If a, b, c, d be in G. P., prove that

$$(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2.$$

18. If the arithmetic mean between a and b is twice as great as the geometric mean, shew that $a : b=2+\sqrt{3} : 2-\sqrt{3}$.

19. Find the sum of n terms of the series the r^{th} term of which is

$$(2r+1)2^r.$$

20. Find the sum of $2n$ terms of a series of which every even term is a times the term before it, and every odd term c times the term before it, the first term being unity.

21. If S_n denote the sum of n terms of a G. P. whose first term is a, and common ratio r, find the sum of $S_1, S_3, S_5, \ldots S_{2n-1}$.

22. If $S_1, S_2, S_3, \ldots S_p$ are the sums of infinite geometric series, whose first terms are $1, 2, 3, \ldots p$, and whose common ratios are

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{p+1} \text{ respectively,}$$

prove that
$$S_1+S_2+S_3+\ldots+S_p=\frac{p}{2}(p+3).$$

23. If $r<1$ and positive, and m is a positive integer, shew that

$$(2m+1)r^m(1-r)<1-r^{2m+1}.$$

Hence shew that nr^n is indefinitely small when n is indefinitely great.