Engineering Mathematics Test 3

(Linear Algebra, Partial Differential Equations)

Number of Questions: 25 **Time: 60 min.**

Directions for questions 1 to 25: Select the correct alternative from the given choices.

1. If A is a square matrix of order 5 with $A^{-1} = A^T$ and non-negative determinant, then the determinant of A is ______.

(A) 0 (B) 1
(C) 2 (D) 5

2. For two matrices A and B, if $AB = A$ and $BA = B$, then which of the following statements is/are correct?

I. A is an idempotent matrix.
II. B is an idempotent matrix.

(A) I only
(B) II only
(C) Both I and II
(D) Neither I nor II

3. Consider the matrix $A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$. Which of the following is NOT equal to the determinant of A?

(A) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (B) $\begin{vmatrix} a+bc & a & 1+a \\ b+ca & b & 1+b \\ c+ab & c & 1+c \end{vmatrix}$

(C) $\begin{vmatrix} 0 & a-b & bc-ac \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix}$ (D) $\begin{vmatrix} 1 & a+1 & a^2+a \\ 1 & b+1 & b^2+b \\ 1 & c+1 & c^2+c \end{vmatrix}$

4. For a non-singular square matrix A, if $A^3 = A$, then A must be ______.

(A) a nilpotent matrix
(B) an idempotent matrix
(C) an involutory matrix
(D) None of these

5. If A is a matrix of order 6×9 with rank 5, then which of the following is true?

(A) All the rows of A are linearly independent.
(B) 5 columns of A are linearly independent.
(C) AA^T is invertible.
(D) A^TA is invertible.

6. The rank of the matrix $P = \begin{bmatrix} 1 & 2 & 4 & -3 \\ 2 & -3 & 5 & -4 \\ 4 & 1 & 13 & -10 \\ 3 & -8 & 6 & -5 \end{bmatrix}$ is ____.

(A) 1 (B) 2
(C) 3 (D) 4

7. The value of x_3 in the solution of the system of linear equations $x_1 + 2x_2 + 2x_3 = 4$, $2x_1 - 2x_2 - x_3 = -3$, $4x_1 + x_2 + 2x_3 = 3$ is —

(A) 1 (B) −1
(C) 2 (D) −2

8. For a homogeneous system of linear equations $AX = O$ with four equations in four unknowns, if the number of linearly independent solutions is one, then the rank of A is ______.

(A) 1 (B) 2
(C) 3 (D) 4

9. If $A = \begin{bmatrix} 1 & 10 & 16 & -20 \\ 0 & -1 & 159 & 237 \\ 0 & 0 & 1 & -431 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; then the determinant of $A^9 - 7A^5 + 4A$ is _____.

(A) 4 (B) −4
(C) 16 (D) −16

10. If the characteristic equation of a 2×2 matrix A is $\lambda^2 - 4\lambda + 1 = 0$, then the trace and determinant of A respectively are ______.

(A) −1 and 4 (B) 1 and −4
(C) 4 and 1 (D) 4 and −1

11. If λ_1 and λ_2 are the eigenvalues of a 2×2 non-singular matrix A, then the eigenvalues of ad joint of A are ______.

(A) λ_1 and λ_2 (B) λ_1^2 and λ_2^2
(C) $\lambda_1 + \lambda_2$ and $\lambda_1 - \lambda_2$ (D) $\lambda_1 \times \lambda_2$ and $\frac{\lambda_1}{\lambda_2}$

12. If A is a 3×3 matrix with the characteristic equation $\lambda^3 - 5\lambda^2 + 2\lambda - 3 = 0$, then $3A^9 - 15A^8 + 6A^7 - 11A^6 + 10A^5 - 4A^4 + 10A^3 - 20A^2 + 8A - 9I$ is equal to _______.

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

13. Which of the following is NOT an eigenvector of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & -1 & 4 \end{bmatrix}$?

(A) $\begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix}$ B) $\begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}$

(C) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

14. If the system of linear equations
$2x + 3y + 4z = 1$
$5x - y + z = 4$
$3x + ay - 3z = 3$
has a unique solution, then the value of $a + 4$ _______
(A) must be equal to 0,
(B) should not be equal to 0
(C) can be any real number,
(D) can be any rational number

15. If $a_1, b_1, c_1, d_1, a_2, b_2, c_2$ and d_2 are any non zero real numbers, then which of the following types of solution is NOT possible for the system of linear equations.
$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
(A) Unique solution
(B) No solution
(C) Infinitely many solution
(D) None of these

16. The partial differential equation of
$z = f(x + at) - g(x - at)$ is ______.
(A) $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$ (B) $\frac{\partial z}{\partial t} = a\frac{\partial z}{\partial x}$
(C) $\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$ (D) $\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2}$

17. The first order partial differential equation by eliminating the arbitary function from $z = f(x^3 - y^3)$ is
(A) $p + q = 0$ (B) $yp + xq = 0$
(C) $y^2p + x^2q = 0$ (D) $2y^2p + 3x^2q = 0$

18. The general solution of the partial differential equation $x^3(y - z)p + y^3(z - x)q = z^3(x - y)$ is
(A) $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$
(B) $\phi\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$
(C) $\phi\left(\frac{1}{x} - \frac{1}{y} - \frac{1}{z}, \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{z^2}\right) = 0$
(D) $\phi\left(\frac{1}{x^2} - \frac{1}{y^2} - \frac{1}{z^2}, \frac{1}{x} - \frac{1}{y} + \frac{1}{z}\right) = 0$

19. The solution of the partial differential equation $xy^2z^2p + x^2yz^2q = x^2y^2z$ is
(A) $x^2 + y^2 = \phi(x^2 - y^2)$
(B) $x^2 + z^2 = \phi(x^2 - z^2)$
(C) $x^2 - y^2 = \phi(y^2 - z^2)$
(D) $y^2 - z^2 = \phi(x^2 - y^2 - z^2)$

20. The solution of $(p - q)(z - xp - yq) = 1$ is __________
(A) $z = ax - by + \frac{1}{a+b}$ (B) $z = ax - by$
(C) $z = ax + by$ (D) $z = ax + by + \frac{1}{a-b}$

21. If $u(x, y) = X(x).Y(y)$ be the solution of the partial differential equation $4\frac{\partial u}{\partial x} + 5\frac{\partial u}{\partial y} = 0$, which is obtained by solving it by the method of separation of variables, then $X(x)$ (the function of x only in $u(x, y)$) is _______
[Note: Here c and k are arbitrary constants]
(A) $X(x) = ce^{(kx)}x^2$ (B) $X(x) = ce^{\left(\frac{4k}{x}\right)}$
(C) $X(x) = ce^{\left(\frac{k}{4}\right)x}$ (D) $X(x) = ce^{(-5k)}x^2$

22. Which of the following second order partial differential equations is an elliptic equation?
(A) $3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - 5\frac{\partial^2 u}{\partial y^2} + 7\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 7x^2$
(B) $3\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 6x^2y$
(C) $-3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} + 4x\frac{\partial u}{\partial x} - 7y\frac{\partial u}{\partial y} = 0$
(D) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x} + 5\frac{\partial u}{\partial y} = 6xy^2$

23. The Fourier cosine series of the function $f(x) = \frac{1}{2}$, $0 \le x \le 1$ is
(A) 1 (B)
(C) 0 (D)

24. The Fourier series of $f(x) = e^{2x}$ in the $\frac{1}{4}\frac{a_0}{2}$ interval $(0, 2\pi)$ is
$f(x) = + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, then the value of $\frac{a_0}{2}$ is
(A) $\frac{e^{4\pi} - 1}{4\pi}$ (B) $\frac{e^{2\pi} - 1}{2\pi}$
(C) $\frac{e^{4\pi} - 1}{2\pi}$ (D) $\frac{e^{2\pi} - 1}{4\pi}$

25. The Fourier series of the function

$$f(x) = \begin{cases} -2 \text{ for} -\pi < x < \frac{-\pi}{2} \\ 0 \text{ for} \frac{-\pi}{2} < x < \frac{\pi}{2} \\ 2 \text{ for} \frac{\pi}{2} < x < \pi \end{cases}$$

is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, then $b_n =$

(A) $\frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \cos n\pi\right]$

(B) $\frac{4}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$

(C) $\frac{1}{4n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$

(D) $\frac{4}{n\pi}\left[\sin\frac{n\pi}{2} - \sin n\pi\right]$

Answer Keys

1. B	**2.** C	**3.** B	**4.** C	**5.** B	**6.** B	**7.** A	**8.** C	**9.** D	**10.** C
11. A	**12.** C	**13.** D	**14.** B	**15.** A	**16.** A	**17.** C	**18.** B	**19.** C	**20.** D
21. C	**22.** B	**23.** B	**24.** A	**25.** B					

Hints and Explanations

1. Given $A^{-1} = A^T$ and det of A is non-negative

$\therefore$ $A.A^T = A^T.A = I_5$

$\Rightarrow$ Det of $(A.A^T)$ = Det of I_5

$\Rightarrow$ $|A.A^T| = |I_5|$

$\Rightarrow$ $|A|\,|A^T| = 1$

$\Rightarrow$ $|A|\,.\,|A| = 1$ ($\because |A| = |A^T|$)

$\Rightarrow$ $|A|^2 = 1 \Rightarrow |A| = \pm 1$

$\therefore$ The determinant of $A = 1$.

($\because$ $|A|$ is non-negative) Choice (B)

2. Given $AB = A$ and $BA = B$

Consider $BA = B$

$\Rightarrow$ $A(BA) = AB$

$\Rightarrow$ $(AB)A = A$ ($\because AB = A$)

$AA = A \Rightarrow A^2 = A$

$\Rightarrow$ A is an idempotent matrix $\rightarrow (1)$

Consider $AB = A$

$\Rightarrow$ $B(AB) = BA$

$\Rightarrow$ $(BA)B = B$ ($\because BA = B$)

$\Rightarrow$ $BB = B \Rightarrow B^2 = B$

$\Rightarrow$ B is an idempotent matrix $\rightarrow (2)$

$\therefore$ From (1) and (2), both I and II are correct.

Choice (C)

3. Given $A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$

Consider the determinant given in option (B)

$$\begin{vmatrix} a+bc & a & 1+a \\ b+ca & b & 1+b \\ c+ab & c & 1+c \end{vmatrix} = \begin{vmatrix} a & a & 1+a \\ b & b & 1+b \\ c & c & 1+c \end{vmatrix} + \begin{vmatrix} bc & a & 1+a \\ ca & b & 1+b \\ ab & c & 1+c \end{vmatrix}$$

$$= 0 + \begin{vmatrix} bc & a & 1 \\ ca & b & 1 \\ ab & c & 1 \end{vmatrix} + \begin{vmatrix} bc & a & a \\ ca & b & b \\ ab & c & c \end{vmatrix}$$

$$= \begin{vmatrix} bc & a & 1 \\ ca & b & 1 \\ ab & c & 1 \end{vmatrix} + 0$$

$$= -1 \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

= –Det of $A \neq$ Det of A

$\therefore$ The determinant given in option B is NOT equal to det of A. Choice (B)

4. Given A is non-singular and $A^3 = A$

$\Rightarrow$ $AA^2 = A$

$\Rightarrow$ $A^{-1}(AA^2) = A^{-1}A$

$\Rightarrow$ $(A^{-1}A)A^2 = A^{-1}A$

$\Rightarrow$ $A^2 = I$

$\Rightarrow$ A must be an involutory matrix. Choice (C)

5. Given A is a matrix of order 6×9.

Rank of $A = 5$

$\therefore$ Maximum number rows/columns of A that are linearly independent = 5.

$\therefore$ Option (B) is TRUE. Choice (B)

6. Given matrix is $P = \begin{bmatrix} 1 & 2 & 4 & -3 \\ 2 & -3 & 5 & -4 \\ 4 & 1 & 13 & -10 \\ 3 & -8 & 6 & -5 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 - 4R_1$ and $R_4 \rightarrow R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 4 & -3 \\ 0 & -7 & -3 & 2 \\ 0 & -7 & -3 & 2 \\ 0 & -14 & -6 & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ and $R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 4 & -3 \\ 0 & -7 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to \frac{-1}{7} R_2$$

$$\therefore \quad P \sim \begin{bmatrix} 1 & 2 & 4 & -3 \\ 0 & 1 & 3/7 & -2/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in Row Echelon form.

$\therefore$ The rank of P = the number of non-zero rows in its Row Echelon form = 2. Choice (B)

7. Given system of equations is

$x_1 + 2x_2 + 2x_3 = 4$

$2x_1 - 2x_2 - x_3 = -3 \to (1)$

$4x_1 + x_2 + 2x_3 = 3$

It can be written in matrix form as

$AX = B \to (2)$

Where $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & -1 \\ 4 & 1 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix}$

Consider the augmented matrix

$$[A/B] = \begin{vmatrix} 1 & 2 & 2 & 4 \\ 2 & -2 & -1 & -3 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -6 & -5 & -11 \\ 0 & -7 & -6 & -13 \end{bmatrix}$$

$R_3 \to 6R_3 - 7R_2$

$$\therefore [A/B] \sim \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -6 & -5 & -11 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Hence the system of equations that has same solution as that of $AX = B$ is

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -6 & -5 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -1 \end{bmatrix}$$

$\Rightarrow \quad x_1 + 2x_2 + 2x_3 = 4$

$-6x_2 - 5x_3 = -11$

$-x_3 = -1 \Rightarrow x_3 = 1.$ Choice (A)

8. Given the system of equations $AX = O$ has

Number of unknowns $= n = 4$

Number of equations = 4

$\therefore$ A is a 4×4 matrix

Also given the number of linearly independent solutions = 1. We know that the number of linearly independent solutions of a system of homogeneous linear equations.

$AX = O$ is $= n - r$

Where n = the number of unknowns and r = the rank of A.

$\therefore \quad n - r = 1$

$\Rightarrow \quad 4 - r = 1$

$\Rightarrow \quad r = 4 - 1 = 3$

$\therefore$ The rank of $A = 3$ Choice (C)

9. Given A = $\begin{bmatrix} 1 & 10 & 16 & -20 \\ 0 & -1 & 159 & 237 \\ 0 & 0 & 1 & -431 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The eigenvalues of A are 1, –1, 1 and 1.

If $\lambda = -1$ is an eigenvalue of A, then $(-1)^9 - 7(-1)^5 + 4(-1) = 2$ is an eigenvalue of $A^9 - 7A^5 + 4A$.

Also if $\lambda = 1$ is an eigenvalue of A then

$(1)^9 - 7(1)^5 + 4(1) = -2$ is an eigenvalue of $A^9 - 7A + 4A$

$\therefore$ The eigenvalues of $A^9 - 7A^5 + 4A$ are –2, 2, –2 and –2.

$\therefore$ The determinant of $A^9 - 7A^5 + 4A$ = Product of the eigenvalues of $A^9 - 7A^5 + 4A = (-2)(2)(-2)(-2) = -16.$ Choice (D)

10. Given the characteristic equation of a 2×2 matrix A is $\lambda^2 - 4\lambda + 1 = 0$

Let λ_1 and λ_2 be the eigenvalues of A.

$\therefore$ Trace of A = sum of the eigenvalues of A

$= \lambda_1 + \lambda_2 = (-(-4)) = 4$

Determinant of A = Product of the eigenvalues of $A = \lambda_1 . \lambda_2 = 1.$ Choice (C)

11. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix with λ_1 and λ_2 as its eigenvalues.

The characteristic equation of A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (a - \lambda)(d - \lambda) - bc = 0$ ---- (1)

The adjoint of A is $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\therefore$ The characteristic equation of adj(A) is

$$\begin{vmatrix} d-\lambda & -b \\ -c & a-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (d - \lambda)(a - \lambda) - bc = 0$ ---- (2)

As (1) and (2) are one and the same and λ_1 and λ_2 being the roots of (1), λ_1 and λ_2 will be the roots of (2).

$\therefore$ The eigenvalues of adj(A) are λ_1 and λ_2.

Choice (A)

12. The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 2\lambda - 3 = 0$

$\therefore$ By Cayley Hamilton theorem, we have

$A^3 - 5A^2 + 2A - 3I = 0$ ----- (1)

Consider

$3A^9 - 15A^8 + 6A^7 - 11A^6 + 10A^5 - 4A^4 + 10A^3 - 20A^2 + 8A - 9I$

$= 3A^6(A^3 - 5A^2 + 2A - 3I) - 2A^6 + 10A^5 - 4A^4 + 10A^3 - 20A^2 + 8A - 9I$

$= 3A^6 \times 0 - 2A^3(A^3 - 5A^2 + 2A - 3I) + 4A^3 - 20A^2 + 8A - 9I$

(From (1)) $= 0 - 2A^3 \times 0 + 4(A^3 - 5A^2 + 2A - 3I) + 3I$

(From (1)) $= 4 \times 0 + 3I$ (From (1))

$$= 3I = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Choice (C)

13. Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & -1 & 4 \end{bmatrix}$

The eigenvalues of A are 1, 3 and 4

If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A, then x should satisfy any one of the three conditions. $AX = x$, $AX = 3X$ and $AX = 4X$

From the options given, it can be easily observed that the vectors given in options (A), (B) and (C), will satisfy one of these three conditions.

Consider the vector given in option (D),

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ -2 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

For, $\lambda = 1, 2$ or 3

$\therefore$ Its not an eigenvector of A

Choice (D)

14. Given system of linear equations is

$2x + 3y + 4z = 1$

$5x - y + z = 4$

$3x + ay - 3z = 3$ --------- (1)

It can be written in matrix form as $AX = B$

Where $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & -1 & 1 \\ 3 & a & -3 \end{bmatrix}$;

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ consider the augmented matrix

$$[A|B] = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 5 & -1 & 1 & 4 \\ 3 & a & -3 & 3 \end{bmatrix}$$

$R_2 \to 2R_2 - 5R_1$, $R_3 \to 2R_3 - 3R_1$

$$\sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & -17 & -18 & 3 \\ 0 & 2a-9 & -18 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A|B] \sim \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & -17 & -18 & 3 \\ 0 & 2a+8 & 0 & 0 \end{bmatrix}$$

The given system of equations has a unique solution, if $P(A) = p([A/B]) = 3$ (= The no. of unknowns)

This is possible only if $2a + 8 \neq 0$

$\Rightarrow \quad a + 4 \neq 0$.

Choice (B)

15. Given system of equations is

$a_1x + b_1y + c_1z = d_1$ ______(1)

$a_2x + b_2y + c_2z = d_2$

It can be written in matrix form as

$AX = B$

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$; $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Here two possibilities arise

(i) $P(A) \neq P(A/B])$

In this case, (1) has no solution

(ii) $P(A) = P([A/B]) < 3$(= The no. of unknowns)

In this case, (1) has infinitely many solutions.

So, the given system (1) do not have a unique solution.

Choice (A)

16. $z = f(x + at) - g(x - at)$

$$\frac{\partial z}{\partial x} = f^1(x+at) - g^1(x-at)$$

$$\frac{\partial^2 z}{\partial x^2} = f^{11}(x+at) - g^{11}(x-at)$$

$$\frac{\partial z}{\partial t} = af^1(x+at) + ag^1(x-at)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2f^{11}(x+at) - a^2g^{11}(x-at)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Choice (A)

17. Given $z = f(x^3 - y^3)$

Let $x^3 - y^3 = u$

$z = f(u)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} = f\ (u).3x^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} = -f\ (u)3y^2$$

$$y^2\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} = f\ (u)3x^2y^2\ - f^1(u)3x^2y^2 = 0$$

∴ The first order partial dE is $y^2p + x^2q = 0$

Choice (C)

18. $x^3(y - z)\,p + y^3\,(z - x)q = z^3(x - y)$

The subsidiary equation of the given differential equation is

$$\frac{dx}{x^3(y-z)} = \frac{dy}{y^3(z-x)} = \frac{dz}{z^3(x-y)} \quad \text{............ (1)}$$

using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$ each fraction of

(1) is equal to $\dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$

$$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

On integrating the above, we get

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = C \text{ or } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1 \quad \text{........... (2)}$$

using the multipliers $\frac{1}{x^3}, \frac{1}{y^3}$ and $\frac{1}{z^3}$ each of the fraction (1) equal to $\dfrac{\frac{1}{x^3}dx + \frac{1}{y^3}dy + \frac{1}{z^3}dz}{0}$

On integrating both the sides, we get

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = C_2 \quad \text{............ (3)}$$

From (2) and (3) the general solutions is

$$\varphi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 0.$$

Choice (B)

19. $xy^2z^2p + x^2yz^2q = x^2\,y^2\,z$ .... (1)

The subsidiary equations of (1) are

$$\frac{dx}{xy^2z^2} = \frac{dy}{x^2yz^2} = \frac{dz}{x^2y^2z} \quad \text{............ (2)}$$

Considering first two fractions of (2) we have $x\,dx = y\,dy$.

On integrating we get $x^2 = y^2$ or $x^2 - y^2 = C_1$ .... (3)

Considering the last two fractions of (2), we have $y\,dy = z\,dz$. On integrating, we get

$y^2 - z^2 = C_2$ .... (4)

From (3) and (4) the general solution of (1) is

$x^2 - y^2 = \phi\,(y^2 - z^2)$. Choice (C)

20. Given $(p - q)\,(z - x\,p - yq) = 1$

$$z = x\,p + yq + \frac{1}{p-q}$$

This is a clairaut equation and its solution is

$$z = ax + by + \frac{1}{a-b}$$

Choice (D)

21. Given $u = x(x), y(y)$ ------ (1) is the solution of the PDE

$$4\frac{\partial u}{\partial x} + 5\frac{\partial u}{\partial y} = 0 \text{ ----------- (2)}$$

Obtained by solving (2) by the method of separation of variables

$$\therefore \frac{\partial u}{\partial x} = x^1y \text{ and } \frac{\partial u}{\partial y} = xy^1$$

where $x^1 = \dfrac{dx}{dx}$ and $y^1 = \dfrac{dY}{dy}$

∴ (2) becomes

$4x^1y + 5xy^1 = 0$

$$\Rightarrow \frac{4x^1}{x} + 5\frac{y^1}{y} = 0$$

$$\Rightarrow \frac{4x^1}{x} = -5\frac{y^1}{y} = k \text{ (say) where } k \text{ is a constant}$$

$$\Rightarrow \frac{4x^1}{x} = k \text{ and } \frac{-5y^1}{y} = k$$

$$\Rightarrow x^1 = \frac{kx}{4} \qquad y^1 = \frac{-ky}{5}$$

$$\Rightarrow x^1 - \frac{kx}{4} = 0 \Rightarrow y^1 + \frac{ky}{5} = 0$$

$$\Rightarrow \frac{dX}{dx} = \frac{kx}{4} \Rightarrow \frac{dx}{x} = \frac{k}{4}\ dx$$

$$\Rightarrow \int\frac{dx}{x} = \frac{k}{4}\int dx \Rightarrow = \frac{k}{4}x + c^1$$

$$\Rightarrow x = e^{\frac{kx}{4}} + c^1 = e^{\frac{kx}{4}}c^1$$

$$\Rightarrow x = ce^{\left(\frac{kx}{4}\right)}; \text{ where } c = e^{c^1}$$

$$\therefore x\,(x) = ce^{\left(\frac{kx}{4}\right)}.$$

Choice (C)

22. A PDE is of the form

$Au_{xx} + Bu_{xy} + Cu_{yy} + F(x, y, u, u_x, u_y) = 0$ ----- (1)

is elliptic, if $B^2 - 4AC < 0$

From the PDE in the options, consider the PDE in option (B)

Comparing it with (1), we have

$A = 3, B = -4$ and $C = 5$

$\therefore\ B^2 - 4AC = (-4)^2 - 4 \times 3 \times 5 = -44 < 0$

$\Rightarrow\ B^2 - 4AC < 0$

Hence the PDE given in option (B) is elliptic

Also, it can be easily observed that the PDE given in options (A), (C) and (D) do not satisfy the property, $B^2 - 4AC < 0$. Choice (B)

23. The coefficients of Fourier cosine series are given by

$$a_0 = \frac{2}{1}\int_0^1 f(x)dx \text{ and}$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$\therefore\ a_0 = \frac{2}{1}\int_0^1 \frac{1}{2}dx = x\Big]_0^1 = 1$$

$$a_n = \frac{2}{1}\int_0^1 \frac{1}{2}\cos(n\pi x)dx = \int_0^1 \cos(n\pi x)dx$$

$$= \frac{\sin(n\pi x)}{n\pi}\Big]_0^1 = 0$$

$\therefore$ The required series is $f(x) = \frac{1}{2}$. Choice (B)

24. In the Fourier series of e^{2x}; a_0 is given by $\frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{2x}dx = \frac{1}{2\pi}\left[e^{2x}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[e^{4\pi} - 1\right]\ \frac{a_0}{2} = \frac{e^{4\pi}-1}{2\times 2\pi} = \frac{e^{4\pi}-1}{4\pi}.$$ Choice (A)

25. In the Fourier series of $f(x)$, b_n is given by $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\frac{-\pi}{2}}(-2)\sin nx dx + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 0.\sin nx dx + \int_{\frac{\pi}{2}}^{\pi} 2\sin nx dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{\frac{-\pi}{2}} + \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\frac{\pi}{2}}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{\cos n\frac{\pi}{2}}{n}\right]$$

$$= \frac{4}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right].$$ Choice (B)