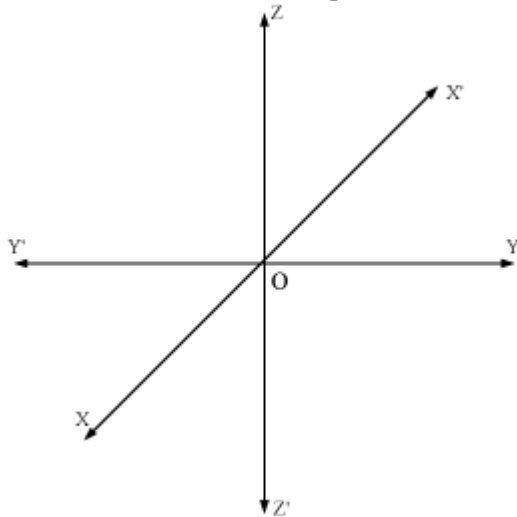


Introduction to Three Dimensional Geometry

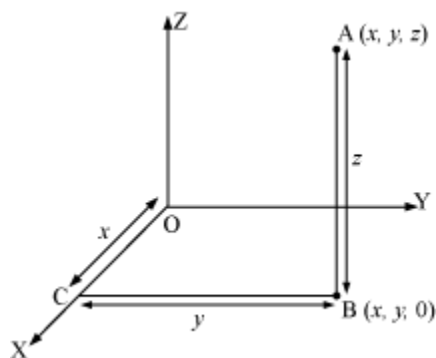
Rectangular Coordinate System

- If we draw three planes intersecting at O such that they are mutually perpendicular to each other, then these will intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$. These lines constitute the **rectangular coordinate system** and are respectively known as the x , y , and z -axes.
- Point O is called the **origin** of the coordinate system.



- The distances measured from XY -plane upwards in the direction of OZ are taken as positive and those measured downward in the direction of OZ' are taken as negative.
- The distances measured to the right of ZX -plane along OY are taken as positive and those measured to the left of ZX -plane along OY' are taken as negative.
- The distances measured in front of YZ -plane along OX are taken as positive and those measured at the back of YZ -plane along OX' are taken as negative.
- The planes XOY , YOZ , and ZOX are known as the three **coordinate planes** and are respectively called the XY -plane, the YZ -plane, and the ZX -plane.
- The three coordinate planes divide the space into eight parts known as **octants**. These octants are named as $XOYZ$, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$, and $XOY'Z'$ and are denoted by I, II, III, IV, V, VI, VII, and VIII respectively.
- If a point A lies in the first octant of a coordinate space, then the lengths of the perpendiculars drawn from point A to the planes XY , YZ and ZX are represented by x , y , and z respectively and are called the **coordinates** of point A . This means that the

coordinates of point A are (x, y, z) . However, if point A would have been in any other quadrant, then the signs of x, y , and z would change accordingly.



- The coordinates of the origin are $(0, 0, 0)$.
- The sign of the coordinates of a point determines the octant in which the point lies. The following table shows the signs of the coordinates in the eight octants.

Octants →	I	II	III	IV	V	VI	VII	VIII
Coordinates ↓								
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

- The coordinates of a point lying on different axes are as follows:
- The coordinates of a point lying on the x -axis will be of the form $(x, 0, 0)$.
- The coordinates of a point lying on the y -axis will be of the form $(0, y, 0)$.
- The coordinates of a point lying on the z -axis will be of the form $(0, 0, z)$.
- The coordinates of a point lying on different planes are as follows:
- The coordinates of a point lying in the XY -plane will be of the form $(x, y, 0)$.
- The coordinates of a point lying in the YZ -plane will be of the form $(0, y, z)$.

- The coordinates of a point lying in the ZX-plane will be of the form $(x, 0, z)$.
- Let's now try and solve the following puzzle to check whether we have understood the basic concepts that we just studied.

Solved Examples

Example 1 State whether the following statements are true or false.

1. The point $(-5, 0, 1)$ lies on the y -axis.
2. The point $(1, 7, -1)$ lies in octant V, whereas the point $(-10, -8, 6)$ lies in octant VIII.
3. The x, y , and z coordinates of the point $(5, 7, 18)$ are 5, 7 and 18 respectively.
4. The point $(0, 0, 19)$ lies in the ZX-plane.
5. The point $(-2, 11, 8)$ lies in octant II.

Solution:

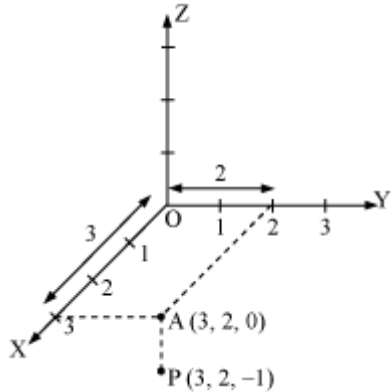
1. False. The point $(-5, 0, 1)$ does not lie on the y -axis since the point that lies on the y -axis is of the form $(0, y, 0)$.
2. False. Point $(-10, -8, 6)$ lies in octant III.
3. True.
4. False. A point lying in the ZX-plane is of the form $(x, 0, z)$.
5. True.

Example 2 Locate point $(3, 2, -1)$ in a three-dimensional space.

Solution: We have to locate point $(3, 2, -1)$ in a three-dimensional space. In order to do this, we will first draw the three axes.

Then, starting from the origin, when we move 2 units in the positive y -direction and then 3 units in the positive x -direction, we will reach at the point $A(3, 2, 0)$.

From point A(3, 2, 0), we move 1 unit in the negative z-direction to reach the required point i.e., P(3, 2, -1).



Distance between Two Points in Three-Dimensional Space

- The distance formula that is used for finding the distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ lying in three-dimensional space is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Example: The distance between points A(3, 7, 8) and B(5, 0, 1) is given by

$$AB = \sqrt{(5-3)^2 + (0-7)^2 + (1-8)^2} = \sqrt{102}$$

Solved Examples

Example 1: Prove that points $(-4, 8, 1)$, $(2, 4, -1)$ and $(-2, 2, 5)$ are the vertices of an equilateral triangle.

Solution: Let the given points be A(-4, 8, 1), B(2, 4, -1) and C(-2, 2, 5) respectively.

We know that in an equilateral triangle, the lengths of all the sides are the same.

On using the distance formula, we obtain

$$AB = \sqrt{[2 - (-4)]^2 + (4 - 8)^2 + (-1 - 1)^2} = \sqrt{(6)^2 + (-4)^2 + (-2)^2} = \sqrt{36 + 16 + 4} = \sqrt{56}$$

$$BC = \sqrt{(-2 - 2)^2 + (2 - 4)^2 + [5 - (-1)]^2} = \sqrt{(-4)^2 + (-2)^2 + (6)^2} = \sqrt{16 + 4 + 36} = \sqrt{56}$$

$$CA = \sqrt{[-4 - (-2)]^2 + (8 - 2)^2 + (1 - 5)^2} = \sqrt{(-2)^2 + (6)^2 + (-4)^2} = \sqrt{4 + 36 + 16} = \sqrt{56}$$

Thus,

$$AB = BC = CA = \sqrt{56} \text{ units}$$

Thus, the given vertices are the vertices of an equilateral triangle.

Example 2: Find a point lying in the XY-plane such that the sum of the squares of its x and y coordinates is 5. Also, its distance from point $(3, 5, 0)$ is 5 units, while its distance from point $(5, -1, 2)$ is 7 units.

Solution: The two given points are $A(3, 5, 0)$ and $B(5, -1, 2)$.

We know that any point lying in the XY-plane is of the form $(x, y, 0)$. Hence, let the required point be $C(x, y, 0)$.

It is given that the distance between points A and C is 5 units.

On applying the distance formula, we obtain

$$\begin{aligned} AC &= \sqrt{(x-3)^2 + (y-5)^2 + (0-0)^2} \\ &= \sqrt{x^2 + 9 - 6x + y^2 + 25 - 10y} = \sqrt{x^2 + y^2 - 6x - 10y + 34} \end{aligned}$$

Thus,

$$\begin{aligned} \sqrt{x^2 + y^2 - 6x - 10y + 34} &= 5 \\ \Rightarrow x^2 + y^2 - 6x - 10y + 34 &= 25 \\ \Rightarrow x^2 + y^2 - 6x - 10y + 9 &= 0 \end{aligned}$$

It is given that $x^2 + y^2 = 5$. Hence,

$$5 - 6x - 10y + 9 = 0$$

$$\Rightarrow -6x - 10y + 14 = 0$$

$$\Rightarrow 3x + 5y - 7 = 0 \dots (1)$$

It is also given that the distance between points B and C is 7 units.

On applying the distance formula, we obtain

$$\begin{aligned} BC &= \sqrt{(x-5)^2 + (y+1)^2 + (0-2)^2} \\ &= \sqrt{x^2 + 25 - 10x + y^2 + 1 + 2y + 4} = \sqrt{x^2 + y^2 - 10x + 2y + 30} \end{aligned}$$

Thus,

$$\begin{aligned} \sqrt{x^2 + y^2 - 10x + 2y + 30} &= 7 \\ \Rightarrow x^2 + y^2 - 10x + 2y + 30 &= 49 \\ \Rightarrow x^2 + y^2 - 10x + 2y - 19 &= 0 \end{aligned}$$

It is given that $x^2 + y^2 = 5$. Hence,

$$5 - 10x + 2y - 19 = 0$$

$$\Rightarrow -10x + 2y - 14 = 0$$

$$\Rightarrow 5x - y + 7 = 0 \dots (2)$$

On solving equations (1) and (2), we obtain

$$x = -1 \text{ and } y = 2$$

Thus, the required point is $(-1, 2, 0)$.

Example 3: Show that points $(4, 3, -3)$, $(3, 2, -1)$, and $(2, 1, 1)$ are collinear.

Solution: Let the given points be $P(4, 3, -3)$, $Q(3, 2, -1)$, and $R(2, 1, 1)$.

We know that points are said to be collinear if they lie on a line.

On applying the distance formula, we obtain

$$PQ = \sqrt{(3-4)^2 + (2-3)^2 + [-1-(-3)]^2} = \sqrt{(-1)^2 + (-1)^2 + (2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$QR = \sqrt{(2-3)^2 + (1-2)^2 + [1-(-1)]^2} = \sqrt{(-1)^2 + (-1)^2 + (2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$RP = \sqrt{(4-2)^2 + (3-1)^2 + (-3-1)^2} = \sqrt{(2)^2 + (2)^2 + (-4)^2} = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$$

Thus, $PQ + QR = RP$.

Hence, the given points i.e., P, Q, and R are collinear.

Section Formula

- The coordinates of the point that divides the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m: n$ are given by

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right).$$

This formula is known as the section formula.

- The coordinates of a point that divides the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $m: n$ are given by

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right).$$

- If X is the mid-point of the line segment joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then X divides AB in the ratio 1:1. Hence, by using the section formula, the coordinates of point X will be given by

$$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2} \right).$$

- If a point R divides the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $k: 1$, then, by using the section formula, the coordinates of point R will be given by

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1} \right).$$

- The coordinates of the centroid of a triangle whose vertices are (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) are given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right).$$

Solved Examples

Example 1: The coordinates of the vertices of ΔWXY are $W(-12, 5, 6)$, $X(-2, 1, -8)$ and $Y(-1, -6, -7)$. Does the centroid of ΔWXY lie on the XZ -plane?

Solution: It is given that the coordinates of the vertices of ΔWXY are $W(-12, 5, 6)$, $X(-2, 1, -8)$ and $Y(-1, -6, -7)$.

We know that the coordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) are given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

Thus, the coordinates of the centroid of ΔWXY are given by

$$\begin{aligned} & \left(\frac{(-12) + (-2) + (-1)}{3}, \frac{5 + 1 + (-6)}{3}, \frac{6 + (-8) + (-7)}{3} \right) \\ &= \left(\frac{-12 - 2 - 1}{3}, \frac{5 + 1 - 6}{3}, \frac{6 - 8 - 7}{3} \right) \\ &= \left(\frac{-15}{3}, 0, \frac{-9}{3} \right) \\ &= (-5, 0, -3) \end{aligned}$$

We also know that any point whose coordinates are of the form $(x, 0, z)$ lies on the XZ -plane. The coordinates of the centroid of ΔWXY are $(-5, 0, -3)$.

Hence, the centroid of ΔWXY lies on the XZ -plane.

Example 2: Find the ratio in which the line segment joining the points $(23, 16, -19)$ and $(-15, 20, 21)$ is divided externally by the ZX -plane.

Solution: Let the given points be $A(23, 16, -19)$ and $B(-15, 20, 21)$.

Let point $C(x, y, z)$ divide the line segment AB externally in the ratio $k: 1$.

Accordingly, the coordinates of point C are

$$\begin{aligned} & \left(\frac{k(-15) - 1(23)}{k - 1}, \frac{k(20) - 1(16)}{k - 1}, \frac{k(21) - 1(-19)}{k - 1} \right) \\ &= \left(\frac{-15k - 23}{k - 1}, \frac{20k - 16}{k - 1}, \frac{21k + 19}{k - 1} \right) \end{aligned}$$

Since point C lies on the ZX -plane, its y -coordinate is zero.

Therefore,

$$\begin{aligned}\frac{20k-16}{k-1} &= 0 \\ \Rightarrow 20k-16 &= 0 \\ \Rightarrow k &= \frac{16}{20} = \frac{4}{5}\end{aligned}$$

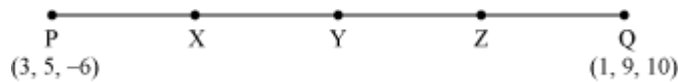
Thus, the ZX-plane divides AB externally in the ratio 4: 5.

Example 3: Find the coordinates of the points that divide the line segment joining the points P(3, 5, -6) and Q(1, 9, 10) into four equal parts.

Solution: The given points are P(3, 5, -6) and Q(1, 9, 10).

Let points X, Y, and Z divide the line segment PQ. Hence,

$$PX = XY = YZ = ZQ.$$



From the figure, we can clearly see that point Y is the mid-point of line segment PQ.

Hence, the coordinates of point Y are given by

$$\left(\frac{3+1}{2}, \frac{5+9}{2}, \frac{-6+10}{2}\right) = \left(\frac{4}{2}, \frac{14}{2}, \frac{4}{2}\right) = (2, 7, 2)$$

Again, X is the mid-point of line segment PY. Hence, the coordinates of point X are given by

$$\left(\frac{3+2}{2}, \frac{5+7}{2}, \frac{-6+2}{2}\right) = \left(\frac{5}{2}, \frac{12}{2}, \frac{-4}{2}\right) = \left(\frac{5}{2}, 6, -2\right)$$

Again, Z is the mid-point of line segment YQ. Hence, the coordinates of point Z are given by

$$\left(\frac{2+1}{2}, \frac{7+9}{2}, \frac{2+10}{2}\right) = \left(\frac{3}{2}, \frac{16}{2}, \frac{12}{2}\right) = \left(\frac{3}{2}, 8, 6\right)$$

Thus, the coordinates of the points that divide the line segment joining the given points are

$$(2, 7, 2), \left(\frac{5}{2}, 6, -2\right), \text{ and } \left(\frac{3}{2}, 8, 6\right).$$

