Determinants

- Determinant of a square matrix A is denoted by $|A|$ or det (A).

- Determinant of a matrix $A = [a]_{1\times 1}$ is $|A| = |a| = a$

- Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by, $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

- Determinant of a matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is given by (expanding along R_1):

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + (-1)^{1+2} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (-1)^{1+3} a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Similarly, we can find the determinant of A by expanding along any other row or along any column.

- The various properties of determinants are as follows:

  - If the rows and the columns of a square matrix are interchanged, then the value of the determinant remains unchanged.

    Example:

    $$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

    This property is same as saying, if A is a square matrix, then $|A| = |A'|$

  - If we interchange any two rows (or columns), then sign of determinant changes.

    Example:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}, \text{ by applying } C_1 \leftrightarrow C_2$$

$$= \begin{vmatrix} b_3 & a_3 & c_3 \\ b_2 & a_2 & c_2 \\ b_1 & a_1 & c_1 \end{vmatrix}, \text{ by applying } R_1 \leftrightarrow C_3$$

- If any two rows or any two columns of a determinant are identical or proportional, then the value of the determinant is zero.

  Example:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix} = 0, \text{ where } k \text{ is a constant}$$

- If each element of a row or a column of determinant is multiplied by a constant *a*, then its determinant value gets multiplied by *a*.

  Example: Area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

  Since area is always positive, we take the absolute value of the above determinant.

- If A is a square matrix, then $A\,(adj\,A) = (adj\,A)\,A = |A|\,I$
- A square matrix A is said to be singular, if A=0
- A square matrix A is said to be non-singular, if $|A| \neq 0$
- If A and B are square matrices of same order, then $|AB| = |A||B|$

  Therefore, if A and B are non-singular matrices of same order, then AB and BA are also non-singular matrices of same order.

- If A is a non-singular matrix of order n, then $|adj\,A| = |A|^{n-1}$

- A square matrix A is invertible, if and only if A is non-singular and inverse of A is given by the formula:

$$A^{-1} = \frac{1}{|A|}(adj\,A)$$

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- If A is a non-singular matrix of order n, then $\left(Adj A\right|(adj A)\big| = |A|^{n-1}$
- A square matrix A is invertible, if and only if A is non-singular and inverse of A is given by the formula:

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- The system of following linear equations $\begin{matrix} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{matrix}$ can be written as $AX = B$, where

  $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- A system of linear equations is said to be consistent, if its solution (one or more) exists.

- A system of linear equations is said to be inconsistent, if its solution does not exist.

- Unique solution of the equation $AX = B$ is given by $X = A^{-1} B$, where $|A| \neq 0$

- For a square matrix A in equation $AX = B$, if

  - $|A| \neq 0$, then there exists a unique solution
  - $|A| \neq 0$ and $(adjA)\,B \neq O$, then no solution exists
  - $|A| \neq 0$ and $(adjA)\,B \neq O$, then the system may or may not be consistent

Example 2: Solve the following system of linear equations:

$$x - 3y + 4z = 12$$
$$2x + 2y - 3z = -7$$
$$6x - y + 2z = 13$$

Solution: The given system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 2 & -3 \\ 6 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and} B = \begin{bmatrix} 12 \\ -7 \\ 13 \end{bmatrix}$$

Now, $|A| = 1[2\times2 - (-1)(-3)] + 3[2\times2 - 6(-3)] + 4[2\times(-1) - 6\times2] = 11 \neq 0$

Therefore, A is a non-singular matrix and hence, the given system of linear equations has only one solution.

Now,
$A_{11} = [2\times2 - (-1)(-3)] = 1$
$A_{12} = -[2\times2 - 6(-3)] = -22$
$A_{13} = [2(-1) - 6\times2] = -14$
$A_{21} = -[(-3)\times2 - (-1)\times4] = 2$
$A_{22} = [1\times2 - 6\times4] = -22$
$A_{23} = -[1(-1) - 6(-3)] = -17$
$A_{31} = [(-3)(-3) - 4\times2] = 1$
$A_{32} = -[1(-3) - 2\times4] = 11$
$A_{33} = [1\times2 - 2(-3)] = 8$

$$\therefore A^{-1} = \frac{1}{|A|}(adj A) = \frac{1}{11}\begin{bmatrix} 1 & 2 & 1 \\ -22 & -22 & 11 \\ -14 & -17 & 8 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 2 & 1 \\ -22 & -22 & 11 \\ -14 & -17 & 8 \end{bmatrix}\begin{bmatrix} 12 \\ -7 \\ 13 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 33 \\ 55 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$x = 1, y = 3$, and $z = 5$