1.4 UNIVERSAL GRAVITATION

1.200 We have

$$\frac{Mv^2}{r} = \frac{\gamma M m_s}{r^2} \text{ or } r = \frac{\gamma m_s}{v^2}$$

Thus
$$\omega = \frac{v}{r} = \frac{v}{\gamma m_s / v^2} = \frac{v^3}{\gamma m_s}$$

(Here m_s is the mass of the Sun.)

So $$T = \frac{2\pi \gamma m_s}{v^3} = \frac{2\pi \times 6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(34{\cdot}9 \times 10^3)^3} = 1{\cdot}94 \times 10^7 \text{ sec} = 225 \text{ days.}$$

(The answer is incorrectly written in terms of the planetary mass M)

1.201 For any planet

$$M R \omega^2 = \frac{\gamma M m_s}{R^2} \text{ or } \omega = \sqrt{\frac{\gamma m_s}{R^3}}$$

So,
$$T = \frac{2\pi}{\omega} = 2\pi R^{3/2} / \sqrt{\gamma m_s}$$

(a) Thus
$$\frac{T_J}{T_E} = \left(\frac{R_J}{R_E}\right)^{3/2}$$

So
$$\frac{R_J}{R_E} = (T_J / T_E)^{2/3} = (12)^{2/3} = 5 \cdot 24.$$

(b)
$$V_J^2 = \frac{\gamma m_s}{R_J}, \text{ and } R_J = \left(T \frac{\sqrt{\gamma m_s}}{2\pi}\right)^{2/3}$$

So
$$V_J^2 = \frac{(\gamma m_s)^{2/3} (2\pi)^{2/3}}{T^{2/3}} \quad \text{or,} \quad V_J = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3}$$

where $T = 12$ years. $m_s =$ mass of ths Sun.

Putting the values we get $V_J = 12{\cdot}97$ km/s

$$\text{Acceleration} = \frac{v_J^2}{R_J} = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3} \times \left(\frac{2\pi}{T\sqrt{\gamma m_s}}\right)^{2/3}$$

$$= \left(\frac{2\pi}{T}\right)^{4/3} (\gamma m_s)^{1/3}$$

$$= 2{\cdot}15 \times 10^{-4} \text{ km/s}^2$$

1.202 Semi-major axis= $(r+R)/2$

It is sufficient to consider the motion be along a circle of semi-major axis $\frac{r+R}{2}$ for T does not depend on eccentricity.

Hence $$T = \frac{2\pi\left(\frac{r+R}{2}\right)^{3/2}}{\sqrt{\gamma m_s}} = \pi\sqrt{(r+R)^3/2\gamma m_s}$$

(again m_s is the mass of the Sun)

1.203 We can think of the body as moving in a very elongated orbit of maximum distance R and minimum distance 0 so semi major axis = $R/2$. Hence if τ is the time of fall then

$$\left(\frac{2\tau}{T}\right)^2 = \left(\frac{R/2}{R}\right)^3 \quad \text{or} \quad \tau^2 = T^2/32$$

or $$\tau = T/4\sqrt{2} = 365/4\sqrt{2} = 64{\cdot}5 \text{ days.}$$

1.204 $T = 2\pi R^{3/2}/\sqrt{\gamma m_s}$

If the distances are scaled down, $R^{3/2}$ decreases by a factor $\eta^{3/2}$ and so does m_s . Hence T does not change.

1.205 The double star can be replaced by a single star of mass $\frac{m_1 m_2}{m_1+m_2}$ moving about the centre of mass subjected to the force $\gamma m_1 m_2/r^2$. Then

$$T = \frac{2\pi r^{3/2}}{\sqrt{\gamma m_1 m_2 / \frac{m_1 m_2}{m_1+m_2}}} = \frac{2\pi r^{3/2}}{\sqrt{\gamma M}}$$

So $$r^{3/2} = \frac{T}{2\pi}\sqrt{\gamma M}$$

or, $$r = \left(\frac{T}{2\pi}\right)^{2/3}(\gamma M)^{1/3} = \sqrt[3]{\gamma M (T/2\pi)^2}$$

1.206 (a) The gravitational potential due to m_1 at the point of location of m_2 :

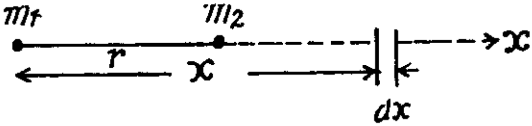
$$V_2 = \int_r^\infty \vec{G}\cdot d\vec{r} = \int_r^\infty -\frac{\gamma m_1}{x^2}\,dx = -\frac{\gamma m_1}{r}$$

So, $$U_{21} = m_2 V_2 = -\frac{\gamma m_1 m_2}{r}$$

Similarly $$U_{12} = -\frac{\gamma m_1 m_2}{r}$$

Hence

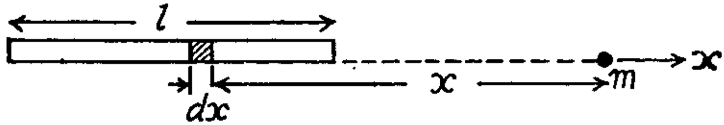
$$U_{12} = U_{21} = U = -\frac{\gamma m_1 m_2}{r}$$



(b) Choose the location of the point mass as the origin. Then the potential energy dU of an element of mass $dM = \frac{M}{l}dx$ of the rod in the field of the point mass is

$$dU = -\gamma m \frac{M}{l} dx \frac{1}{x}$$

where x is the distance between the element and the point. (Note that the rod and the point mass are on a straight line.) If then a is the distance of the nearer end of the rod from the point mass.



$$U = -\gamma \frac{mM}{l} \int_a^{a+l} \frac{dx}{x} = -\gamma m \frac{M}{l} \ln\left(1 + \frac{l}{a}\right)$$

The force of interaction is

$$F = -\frac{\partial U}{\partial a}$$

$$= \gamma \frac{mM}{l} \times \frac{1}{1 + \frac{l}{a}} \left(-\frac{l}{a^2}\right) = -\frac{\gamma mM}{a(a+l)}$$

Minus sign means attraction.

1.207 As the planet is under central force (gravitational interaction), its angular momentum is conserved about the Sun (which is situated at one of the focii of the ellipse)

So, $$m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1^2 = \frac{v_2^2 r_2^2}{r_1^2} \qquad (1)$$

From the conservation of mechanical energy of the system (Sun + planet),

$$-\frac{\gamma m_s m}{r_1} + \frac{1}{2} m v_1^2 = -\frac{\gamma m_s m}{r_2} + \frac{1}{2} m v_2^2$$

or, $$-\frac{\gamma m_s}{r_1} + \frac{1}{2} v_2^2 \frac{r_2^2}{r_1^2} = -\left(\frac{\gamma m_s}{r_2}\right) + \frac{1}{2} v_2^2 \quad \text{[Using (1)]}$$

Thus, $$v_2 = \sqrt{2 \gamma m_s r_1 / r_2 (r_1 + r_2)} \qquad (2)$$

Hence $$M = m v_2 r_2 = m \sqrt{2 \gamma m_s r_1 r_2 / (r_1 + r_2)}$$

1.208 From the previous problem, if r_1 , r_2 are the maximum and minimum distances from the sun to the planet and v_1 , v_2 are the corresponding velocities, then, say,

$$E = \frac{1}{2} m v_2^2 - \frac{\gamma m m_s}{r_2}$$

$$= \frac{\gamma m m_s}{r_1 + r_2} \cdot \frac{r_1}{r_2} - \frac{\gamma m m_s}{r_2} = -\frac{\gamma m m_s}{r_1 + r_2} = -\frac{\gamma m m_s}{2a} \text{ [Using Eq. (2) of 1.207]}$$

where $2a$ = major axis $= r_1 + r_2$. The same result can also be obtained directly by writing an equation analogous to Eq (1) of problem 1.191.

$$E = \frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} - \frac{\gamma m m_s}{r}$$

(Here M is angular momentum of the planet and m is its mass). For extreme position $\dot{r} = 0$ and we get the quadratic

$$Er^2 + \gamma m m_s r - \frac{M^2}{2m} = 0$$

The sum of the two roots of this equation are

$$r_1 + r_2 = -\frac{\gamma m m_s}{E} = 2a$$

Thus $$E = -\frac{\gamma m m_s}{2a} = \text{ constant}$$

1.209 From the conservtion of angular momentum about the Sun.

$$m v_0 r_0 \sin\alpha = m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1 r_1 = v_2 r_2 = v_0 r_2 \sin\alpha \qquad (1)$$

From conservation of mechanical energy,

$$\frac{1}{2} m v_0^2 - \frac{\gamma m_s m}{r_0} = \frac{1}{2} m v_1^2 - \frac{\gamma m_s m}{r_1}$$

or, $$\frac{v_0^2}{2} - \frac{\gamma m_s}{r_0} = \frac{v_0^2 r_0^2 \sin^2\alpha}{2 r_1^2} - \frac{\gamma m_s}{r_1} \text{ (Using 1)}$$

or, $$\left(v_0^2 - \frac{2\gamma m_s}{r_0}\right) r_1^2 + 2\gamma m_s r_1 - v_0^2 r_0^2 \sin\alpha = 0$$

So, $$r_1 = \frac{-2\gamma m_s \pm \sqrt{4\gamma^2 m_s^2 + 4\left(v_0^2 r_0^2 \sin^2\alpha\right)\left(v_0^2 - \frac{2\gamma m_s}{r_0}\right)}}{2\left(v_0^2 - \frac{2\gamma m_s}{r_0}\right)}$$

$$= \frac{1 \pm \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{\gamma m_s}\left(\frac{2}{r_0} - \frac{v_0^2}{r m_s}\right)}}{\left(\frac{2}{r_0} - \frac{v_0}{\gamma m_s}\right)} = \frac{r_0\left[1 \pm \sqrt{1 - (2-\eta)\eta \sin^2\alpha}\right]}{(2-\eta)}$$

where $\eta = v_0^2 r_0 / \gamma m_s$, ($m_s$ is the mass of the Sun).

1.210 At the minimum separation with the Sun, the cosmic body's velocity is perpendicular to its position vector relative to the Sun. If r_{min} be the sought minimum distance, from conservation of angular momentum about the Sun (C).

$$m v_0 \, l = m v r_{min} \text{ or, } v = \frac{v_0 \, l}{r_{min}} \tag{1}$$

From conservation of mechanical energy of the system (sun + cosmic body),

$$\frac{1}{2} m v_0^2 = -\frac{\gamma \, m_s \, m}{r_{min}} + \frac{1}{2} m v^2$$

So, $$\frac{v_0^2}{2} = -\frac{\gamma \, m_s}{r_{min}} + \frac{v_0^2}{2 r_{min}^2} \quad \text{(using 1)}$$

or, $$v_0^2 \, r_{min}^2 + 2\gamma \, m_s \, r_{min} - v_0^2 \, l^2 = 0$$

So, $$r_{min} = \frac{-2\gamma \, m_s \pm \sqrt{4\gamma^2 m^2 + 4 v_0^2 v_0^2 l^2}}{2 v_0^2} = \frac{-\gamma \, m_s \pm \sqrt{\gamma^2 m_s^2 + v_0^4 l^2}}{v_0^2}$$

Hence, taking positive root

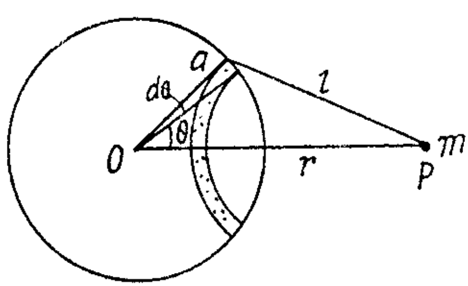
$$r_{min} = (\gamma \, m_s / v_0^2) \left[\sqrt{1 + (l \, v_0^2 / \gamma \, m_s)^2} - 1 \right]$$

1.211 Suppose that the sphere has a radius equal to a. We may imagine that the sphere is made up of concentric thin spherical shells (layers) with radii ranging from 0 to a, and each spherical layer is made up of elementry bands (rings). Let us first calculate potential due to an elementry band of a spherical layer at the point of location of the point mass m (say point P) (Fig.). As all the points of the band are located at the distance l from the point P, so,

$$\partial \varphi = -\frac{\gamma \, \partial M}{l} \text{ (where mass of the band)} \tag{1}$$

$$\partial M = \left(\frac{dM}{4\pi a^2} \right) (2\pi a \sin\theta)(a \, d\theta)$$

$$= \left(\frac{dM}{2} \right) \sin\theta \, d\theta \tag{2}$$



And $l^2 = a^2 + r^2 - 2 \, ar\cos\theta$ (3)

Differentiating Eq. (3), we get

$l dl = ar \sin\theta \, d\theta$ (4)

Hence using above equations

$$\partial \varphi = -\left(\frac{\gamma \, dM}{2ar} \right) dl \tag{5}$$

Now integrating this Eq. over the whole spherical layer

$$d\varphi = \int \partial \varphi = -\frac{\gamma\, dM}{2ar} \int_{r-a}^{r+a}$$

So $$d\varphi = -\frac{\gamma\, dM}{r} \tag{6}$$

Equation (6) demonstrates that the potential produced by a thin uniform spherical layer outside the layer is such as if the whole mass of the layer were concentrated at it's centre;
Hence the potential due to the sphere at point P;

$$\varphi = \int d\varphi = -\frac{\gamma}{r}\int dM = -\frac{\gamma M}{r} \tag{7}$$

This expression is similar to that of Eq. (6)
Hence thte sought potential energy of gravitational interaction of the particle m and the sphere,

$$U = m\varphi = -\frac{\gamma M m}{r}$$

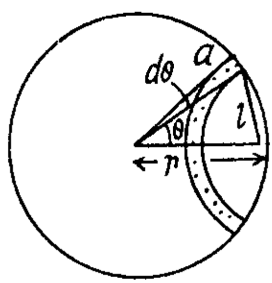
(b) Using the Eq., $$G_r = -\frac{\partial \varphi}{\partial r}$$

$$G_r = -\frac{\gamma M}{r^2} \qquad \text{(using Eq. 7)}$$

So $$\vec{G} = -\frac{\gamma M}{r^3}\vec{r} \text{ and } \vec{F} = m\vec{G} = -\frac{\gamma m M}{r^3}\vec{r} \tag{8}$$

1.212 (The problem has already a clear hint in the answer sheet of the problem book). Here we adopt a different method.

Let m be the mass of the spherical layer, wich is imagined to be made up of rings. At a point inside the spherical layer at distance r from the centre, the gravitational potential due to a ring element of radius a equals,



$$d\varphi = -\frac{\gamma m}{2ar}\, dl \quad \text{(see Eq. (5) of solution of 1.211)}$$

So, $$\varphi = \int d\varphi = -\frac{\gamma m}{2ar}\int_{a-r}^{a+r} dl = -\frac{\gamma m}{a} \tag{1}$$

Hence $$G_r = -\frac{\partial \varphi}{\partial r} = 0\,.$$

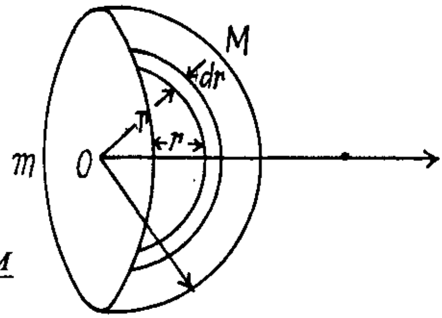
Hence gravitational field strength as well as field force becomes zero, inside a thin sphereical layer.

1.213 One can imagine that the uniform hemisphere is made up of thin hemispherical layers of radii ranging from 0 to R. Let us consider such a layer (Fig.). Potential at point O, due to this layer is,

$$d\varphi = -\frac{\gamma dm}{r} = -\frac{3\gamma M}{R^3} r dr, \text{ where } dm = \frac{M}{(2/3)\pi R^3}\left(\frac{4\pi r^2}{2}\right) dr$$

(This is because all points of each hemispherical shell are equidistant from O.)

Hence, $\varphi = \int d\varphi = -\frac{3\gamma M}{R^3}\int_0^R r dr = -\frac{3\gamma M}{2R}$



Hence, the work done by the gravitational field force on the particle of mass m, to remove it to infinity is given by the formula

$A = m\varphi$, since $\varphi = 0$ at infinity.

Hence the sought work,

$$A_{0\to\infty} = -\frac{3\gamma mM}{2R}$$

(The work done by the external agent is $-A$.)

1.214 In the solution of problem 1.211, we have obtained φ and G due to a uniform shpere, at a distance r from it's centre outside it. We have from Eqs. (7) and (8) of 1.211,

$$\varphi = -\frac{\gamma M}{r} \text{ and } \vec{G} = -\frac{\gamma M}{r^3}\vec{r} \qquad \text{(A)}$$

Accordance with the Eq. (1) of the solution of 1.212, potential due to a spherical shell of radius a, at any point, inside it becomes

$$\varphi = \frac{\gamma M}{a} = \text{Const. and } G_r = -\frac{\partial\varphi}{\partial r} = 0 \qquad \text{(B)}$$

For a point (say P) which lies inside the uniform solid sphere, the potential φ at that point may be represented as a sum.

$$\varphi_{inside} = \varphi_1 + \varphi_2$$

where φ_1 is the potential of a solid sphere having radius r and φ_2 is the potential of the layer of radii r and R. In accordance with equation (A)

$$\varphi_1 = -\frac{\gamma}{r}\left(\frac{M}{(4/3)\pi R^3}\frac{4}{3}\pi r^3\right) = -\frac{\gamma M}{R^3} r^2$$

The potential φ_2 produced by the layer (thick shell) is the same at all points inside it. The potential φ_2 is easiest to calculate, for the point positioned at the layer's centre. Using Eq. (B)

$$\varphi_2 = -\gamma\int_r^R \frac{dM}{r} = -\frac{3}{2}\frac{\gamma M}{R^3}(R^2 - r^2)$$

where $dM = \frac{M}{(4/3)\pi R^3} 4\pi r^2 dr = \left(\frac{3M}{R^3}\right) r^2 dr$

is the mass of a thin layer between the radii r and $r + dr$.

Thus $$\varphi_{inside} = \varphi_1 + \varphi_2 = \left(\frac{\gamma M}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) \qquad \text{(C)}$$

From the Eq. $$G_r = \frac{-\partial \varphi}{\partial r}$$

$$G_r = \frac{\gamma M r}{R^3}$$

or $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma \frac{4}{3}\pi \rho \vec{r}$$

(where $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$, is the density of the sphere) (D)

The plots $\varphi(r)$ and $G(r)$ for a uniform sphere of radius R are shown in figure of answersheet.

Alternate : Like Gauss's theorem of electrostatics, one can derive Gauss's theorem for gravitation in the form $\oint \vec{G} \cdot d\vec{S} = -4\pi\gamma\, m_{inclosed}$. For calculation of $\vec{G}$ at a point inside the sphere at a distance r from its centre, let us consider a Gaussian surface of radius r, Then,

$$G_r 4\pi r^2 = -4\pi\gamma\left(\frac{M}{R^3}\right)r^3 \quad \text{or,} \quad G_r = -\frac{\gamma M}{R^3} r$$

Hence, $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma \frac{4}{3}\pi \rho \vec{r} \left(\text{as } \rho = \frac{M}{(4/3)\pi R^3}\right)$$

So, $$\varphi = \int_r^\infty G_r\, dr = \int_r^R -\frac{\gamma M}{R^3} r\, dr + \int_R^\infty -\frac{\gamma M}{r^2} dr$$

Integrating and summing up, we get,

$$\varphi = -\frac{\gamma M}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

And from Gauss's theorem for outside it :

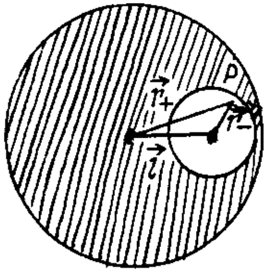
$$G_r 4\pi r^2 = -4\pi\gamma M \quad \text{or} \quad G_r = -\frac{\gamma M}{r^2}$$

Thus $$\varphi(r) = \int_r^\infty G_r\, dr = -\frac{\gamma M}{r}$$

1.215 Treating the cavity as negative mass of density $-\rho$ in a uniform sphere density $+\rho$ and using the superposition principle, the sought field strength is :

$$\vec{G} = \vec{G}_1 + \vec{G}_2$$

or $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\,\vec{r}_+ + -\frac{4}{3}\gamma\pi(-\rho)\,\vec{r}_-$$



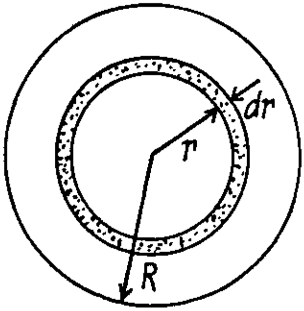
(where $\vec{r}_+$ and $\vec{r}_-$ are the position vectors of an orbitrary point P inside the cavity with respect to centre of sphere and cavity respectively.)

Thus $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\left(\vec{r}_+ - \vec{r}_-\right) = -\frac{4}{3}\pi\gamma\rho\,\vec{l}$$

1.216 We partition the solid sphere into thin spherical layers and consider a layer of thickness dr lying at a distance r from the centre of the ball. Each spherical layer presses on the layers within it. The considered layer is attracted to the part of the sphere lying within it (the outer part does not act on the layer). Hence for the considered layer

$$dp\,4\pi r^2 = dF$$

or, $$dP\,4\pi r^2 = \frac{\gamma\left(\frac{4}{3}\pi r^3 \rho\right)(4\pi r^2\, dr\, \rho)}{r^2}$$



(where ρ is the mean density of sphere)

or, $$dp = \frac{4}{3}\pi\gamma\rho^2 r\, dr$$

Thus $$p = \int_r^R dp = \frac{2\pi}{3}\gamma\rho^2(R^2 - r^2)$$

(The pressure must vanish at $r = R$.)

or, $$p = \frac{3}{8}\left(1 - (r^2/R^2)\right)\gamma M^2/\pi R^4, \text{ Putting } \rho = M/(4/3)\pi R^3$$

Putting $r = 0$, we have the pressure at sphere's centre, and treating it as the Earth where mean density is equal to $\rho = 5{\cdot}5 \times 10^3\ \text{kg/m}^3$ and $R = 64 \times 10^2$ km

we have, $$p = 1{\cdot}73 \times 10^{11}\ \text{Pa} \quad \text{or} \quad 1{\cdot}72 \times 10^6\ \text{atms.}$$

1.217 (a) Since the potential at each point of a spherical surface (shell) is constant and is equal to $\varphi = -\frac{\gamma m}{R}$, [as we have in Eq. (1) of solution of problem 1.212]

We obtain in accordance with the equation

$$U = \frac{1}{2}\int dm\, \varphi = \frac{1}{2}\varphi\int dm$$

$$= \frac{1}{2}\left(-\frac{\gamma m}{R}\right)m = -\frac{\gamma m^2}{2R}$$

(The factor $\frac{1}{2}$ is needed otherwise contribution of different mass elements is counted twice.)

(b) In this case the potential inside the sphere depends only on r (see Eq. (C) of the solution of problem 1.214)

$$\varphi = -\frac{3\gamma m}{2R}\left(1 - \frac{r^2}{3R^2}\right)$$

Here dm is the mass of an elementry spherical layer confined between the radii r and $r + dr$:

$$dm = (4\pi r^2 dr\, \rho) = \left(\frac{3m}{R^3}\right) r^2\, dr$$

$$U = \frac{1}{2}\int dm\, \varphi$$

$$= \frac{1}{2}\int_0^R \left(\frac{3\,m}{R^3}\right) r^2\, dr \left\{-\frac{3\,\gamma\, m}{2\,R}\left(1 - \frac{r^2}{3\,R^2}\right)\right\}$$

After integrating, we get

$$U = -\frac{3}{5}\frac{\gamma\, m^2}{R}$$

1.218 Let $\omega = \sqrt{\frac{\gamma M_E}{r^3}}$ = circular frequency of the satellite in the outer orbit,

$\omega_0 = \sqrt{\frac{\gamma M_E}{(r - \Delta r)^3}}$ = circular frequency of the satellite in the inner orbit.

So, relative angular velocity = $\omega_0 \pm \omega$ where – sign is to be taken when the satellites are moving in the same sense and + sign if they are moving in opposite sense.
Hence, time between closest approaches

$$= \frac{2\,\pi}{\omega_0 \pm\, \omega} = \frac{2\,\pi}{\sqrt{\gamma\, M_E}\,/\,r^{3/2}}\frac{1}{\frac{3\,\Delta\, r}{2\,r} + \delta} = \begin{cases} 4{\cdot}5 \text{ days } (\delta = 0) \\ 0{\cdot}80 \text{ hour } (\delta = 2) \end{cases}$$

where δ is 0 in the first case and 2 in the second case.

1.219 $\omega_1 = \frac{\gamma M}{R^2} = \frac{6{\cdot}67 \times 10^{-11} \times 5{\cdot}96 \times 10^{24}}{(6{\cdot}37 \times 10^6)^2} = 9{\cdot}8 \text{ m/s}^2$

$$\omega_2 = \omega^2 R = \left(\frac{2\,\pi}{T}\right)^2 R = \left(\frac{2 \times 22}{24 \times 3600 \times 7}\right)^2 6{\cdot}37 \times 10^6 = 0{\cdot}034 \text{ m/s}^2$$

and $\quad \omega_3 = \frac{\gamma\, M_S}{R^2_{mean}} = \frac{6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(149{\cdot}50 \times 10^6 \times 10^3)^2} = 5{\cdot}9 \times 10^{-3} \text{ m/s}^2$

Then $\qquad \omega_1 : \omega_2 : \omega_3 = 1 : 0{\cdot}0034 : 0{\cdot}0006$

1.220 Let h be the sought height in the first case. so

$$\frac{99}{100}\, g = \frac{\gamma M}{(R + h)^2}$$

$$= \frac{\gamma M}{R^2\left(1 + \frac{h}{R}\right)^2} = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$$

or $$\frac{99}{100} = \left(1+\frac{h}{R}\right)^{-2}$$

From the statement of the problem, it is obvious that in this case $h << R$

Thus $$\frac{99}{100} = \left(1-\frac{2h}{R}\right) \text{ or } h = \frac{R}{200} = \left(\frac{6400}{200}\right)\text{km} = 32\text{ km}$$

In the other case if h' be the sought height, than

$$\frac{g}{2} = g\left(1+\frac{h'}{R}\right)^{-2} \text{ or } \frac{1}{2} = \left(1+\frac{h'}{R}\right)^{-2}$$

From the language of the problem, in this case h' is not very small in comparision with R. Therefore in this case we cannot use the approximation adopted in the previous case.

Here, $\left(1+\frac{h'}{R}\right)^2 = 2$ So, $\frac{h'}{R} = \pm\sqrt{2}-1$

As $-ve$ sign is not acceptable

$$h' = (\sqrt{2}-1)R = (\sqrt{2}-1)\,6400\text{ km} = 2650\text{ km}$$

1.221 Let the mass of the body be m and let it go upto a height h.
From conservation of mechanical energy of the system

$$-\frac{\gamma M m}{R} + \frac{1}{2} m v_0^2 = \frac{-\gamma M m}{(R+h)} + 0$$

Using $\frac{\gamma M}{R^2} = g$, in above equation and on solving we get,

$$h = \frac{R v_0^2}{2gR - v_0^2}$$

1.222 Gravitational pull provides the required centripetal acceleration to the satelite. Thus if h be the sought distance, we have

so, $$\frac{m v^2}{(R+h)} = \frac{\gamma m M}{(R+h)^2} \text{ or, } (R+h)v^2 = \gamma M$$

or, $$R v^2 + h v^2 = g R^2, \text{ as } g = \frac{\gamma M}{R^2}$$

Hence $$h = \frac{gR^2 - R v^2}{v^2} = R\left[\frac{gR}{v^2} - 1\right]$$

1.223 A satellite that hovers above the earth's equator and corotates with it moving from the west to east with the diurnal angular velocity of the earth appears stationary to an observer on the earth. It is called geostationary. For this calculation we may neglect the annual motion of the earth as well as all other influences. Then, by Newton's law,

$$\frac{\gamma M m}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r$$

where M = mass of the earth, T = 86400 seconds = period of daily rotation of the earth and r = distance of the satellite from the centre of the earth. Then

$$r = \sqrt[3]{\gamma M \left(\frac{T}{2\pi} \right)^2}$$

Substitution of $M = 5{\cdot}96 \times 10^{24}$ kg gives

$$r = 4{\cdot}220 \times 10^4 \text{ km}$$

The instantaneous velocity with respect to an inertial frame fixed to the centre of the earth at that moment will be

$$\left(\frac{2\pi}{T} \right) r = 3{\cdot}07 \text{ km/s}$$

and the acceleration will be the centripetal acceleration.

$$\left(\frac{2\pi}{T} \right)^2 r = 0{\cdot}223 \text{ m/s}^2$$

1.224 We know from the previous problem that a satellite moving west to east at a distance $R = 2{\cdot}00 \times 10^4$ km from the centre of the earth will be revolving round the earth with an angular velocity faster than the earth's diurnal angualr velocity. Let

ω = angular velocity of the satellite

$\omega_0 = \frac{2\pi}{T}$ = anuglar velocity of the earth. Then

$$\omega - \omega_0 = \frac{2\pi}{\tau}$$

as the relative angular velocity with respect to earth. Now by Newton's law

$$\frac{\gamma M}{R^2} = \omega^2 R$$

So,

$$M = \frac{R^3}{\gamma} \left(\frac{2\pi}{\tau} + \frac{2\pi}{T} \right)^2$$

$$= \frac{4\pi^2 R^3}{\gamma T^2} \left(1 + \frac{T}{\tau} \right)^2$$

Substitution gives

$$M = 6{\cdot}27 \times 10^{24} \text{ kg}$$

1.225 The velocity of the satellite in the inertial space fixed frame is $\sqrt{\frac{\gamma M}{R}}$ east to west. With respect to the Earth fixed frame, from the $\vec{v}_1' = \vec{v} - (\vec{w} \times \vec{r})$ the velocity is

$$v' = \frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}} = 7{\cdot}03 \text{ km/s}$$

Here M is the mass of the earth and T is its period of rotation about its own axis.

It would be $-\frac{2\pi R}{T}+\sqrt{\frac{\gamma M}{R}}$, if the satellite were moving from west to east.

To find the acceleration we note the formula

$$m\vec{w}' = \vec{F} + 2m(\vec{v}' \times \vec{\omega}) + m\omega^2 \vec{R}$$

Here $\vec{F} = -\frac{\gamma M m}{R^3}\vec{R}$ and $\vec{v}' \perp \vec{\omega}$ and $\vec{v}' \times \vec{\omega}$ is directed towards the centre of the Earth.

Thus $$w' = \frac{\gamma M}{R^2} + 2\left(\frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}}\right)\frac{2\pi}{T} - \left(\frac{2\pi}{T}\right)^2 R$$

toward the earth's rotation axis

$$= \frac{\gamma M}{R^2} + \frac{2\pi}{T}\left[\frac{2\pi R}{T} + 2\sqrt{\frac{\gamma M}{R}}\right] = 4{\cdot}94 \text{ m/s}^2 \text{ on substitution.}$$

1.226 From the well known relationship between the velocities of a particle w.r.t a space fixed frame (K) rotating frame (K') $\vec{v} = \vec{v}' + (\vec{w} \times \vec{r})$

$$v'_1 = v - \left(\frac{2\pi}{T}\right)R$$

Thus kinetic energy of the satellite in the earth's frame

$$T_1' = \frac{1}{2}m v'^2_1 = \frac{1}{2}m\left(v - \frac{2\pi R}{T}\right)^2$$

Obviously when the satellite moves in opposite sense comared to the rotation of the Earth its velocity relative to the same frame would be

$$v_2' = v + \left(\frac{2\pi}{T}\right)R$$

And kinetic energy

$$T_2' = \frac{1}{2}m v'^2_2 = \frac{1}{2}m\left(v + \frac{2\pi R}{T}\right)^2 \qquad (2)$$

From (1) and (2)

$$T' = \frac{\left(v + \frac{2\pi R}{T}\right)^2}{\left(v - \frac{2\pi R}{T}\right)^2} \qquad (3)$$

Now from Newton's second law

$$\frac{\gamma M m}{R^2} = \frac{m v^2}{R} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{R}} = \sqrt{gR} \qquad (4)$$

Using (4) and (3)

$$\frac{T_2'}{T_1'} = \frac{\left(\sqrt{gR} + \frac{2\pi R}{T}\right)^2}{\left(\sqrt{gR} - \frac{2\pi R}{T}\right)^2} = 1{\cdot}27 \quad \text{nearly (Using Appendices)}$$

1.227 For a satellite in a circular orbit about any massive body, the following relation holds between kinetic, potential & total energy :

$$T = -E, \; U = 2E \qquad (1)$$

Thus since total mechanical energy must decrease due to resistance of the cosmic dust, the kintetic energy will increase and the satellite will 'fall', We see then, by work energy theorm

$$dT = -dE = -dA_{fr}$$

So,
$$mv\,dv = \alpha v^2 v\,dt \quad \text{or,} \quad \frac{\alpha\,dt}{m} = \frac{dv}{v^2}$$

Now from Netow's law at an arbitray radius r from the moon's centre.

$$\frac{v^2}{r} = \frac{\gamma M}{r^2} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{r}}$$

(M is the mass of the moon.) Then

$$v_i = \sqrt{\frac{\gamma M}{\eta R}}, \; v_f = \sqrt{\frac{\gamma M}{R}}$$

where R= moon's radius. So

$$\int_{v_1}^{v_f} \frac{dv}{v^2} = \frac{\alpha}{m}\int_0^{\tau} dt = \frac{\alpha\tau}{m}$$

or,
$$\tau = \frac{m}{\alpha}\left(\frac{1}{v_i} - \frac{1}{v_f}\right) = \frac{m}{\alpha\sqrt{\gamma\frac{M}{R}}}(\sqrt{\eta} - 1) = \frac{m}{\alpha\sqrt{gR}}(\sqrt{\eta} - 1)$$

where g is moon's gravity. The averaging implied by Eq. (1) (for noncircular orbits) makes the result approximate.

1.228 From Newton's second law

$$\frac{\gamma Mm}{R^2} = \frac{mv_0^2}{R} \text{ or } v_0 = \sqrt{\frac{\gamma M}{R}} = 1{\cdot}67 \text{ km/s} \qquad (1)$$

From conservation of mechanical energy

$$\frac{1}{2}mv_e^2 - \frac{\gamma Mm}{R} = 0 \text{ or, } v_e = \sqrt{\frac{2\gamma M}{R}} = 2{\cdot}37 \text{ km/s}^2 \qquad (2)$$

In Eq. (1) and (2), M and R are the mass of the moon and its radius. In Eq. (1) if M and R represent the mass of the earth and its radius, then, using appendices, we can easily get

$$v_0 = 7{\cdot}9 \text{ km/s} \quad \text{and} \quad v_c = 11{\cdot}2 \text{ km/s}.$$

1.229 In a parabolic orbit, $E = 0$

So $$\frac{1}{2} m v_i^2 - \frac{\gamma M m}{R} = 0 \text{ or, } v_i = \sqrt{2}\sqrt{\frac{\gamma M}{R}}$$

where M = mass of the Moon, R = its radius. (This is just the escape velocity.)
On the other hand in orbit

$$m v_f^2 R = \frac{\gamma M m}{R^2} \text{ or } v_f = \sqrt{\frac{\gamma M}{R}}$$

Thus $$\Delta v = (1 - \sqrt{2})\sqrt{\frac{\gamma M}{R}} = -0{\cdot}70 \text{ km/s}.$$

1.230 From 1.228 for the Earth surface

$$v_0 = \sqrt{\frac{\gamma M}{R}} \text{ and } v_e = \sqrt{\frac{2\gamma M}{R}}$$

Thus the sought additional velocity

$$\Delta v = v_e - v_0 = \sqrt{\frac{\gamma M}{R}}(\sqrt{2} - 1) = gR(\sqrt{2} - 1)$$

This 'kick' in velocity must be given along the direction of motion of the satellite in its orbit.

1.231 Let r be the sought distance, then

$$\frac{\gamma \eta M}{(nR - r)^2} = \frac{\gamma M}{r^2} \text{ or } \eta r^2 = (nR - r)^2$$

or $$\sqrt{\eta}\, r = (nR - r) \text{ or } r = \frac{nR}{\sqrt{\eta} + 1} = 3{\cdot}8 \times 10^4 \text{ km}.$$

1.232 Between the earth and the moon, the potential energy of the spaceship will have a maximum at the point where the attractions of the earth and the moon balance each other. This maximum P.E. is approximately zero. We can also neglect the contribution of either body to the p.E. of the spaceship sufficiently near the other body. Then the minimum energy that must be imparted to the spaceship to cross the maximum of the P.E. is clearly (using E to denote the earth)

$$\frac{\gamma M_E m}{R_E}$$

With this energy the spaceship will cross over the hump in the P.E. and coast down the hill of p.E. towards the moon and crashland on it. What the problem seeks is the minimum energy reguired for softlanding. That reguies the use of rockets to loving about the braking of the spaceship and since the kinetic energy of the gases ejected from the rocket will always be positive, the total energy required for softlanding is greater than that required for crashlanding. To calculate this energy we assume that the rockets are used fairly close to the moon when the spaceship has nealy attained its terminal velocity on the moon $\sqrt{\frac{2\gamma M_O}{R_0}}$ where M_0 is the mass of the moon and R_0 is its radius. In general $dE = vdp$ and since the speed of the ejected gases is not less than the speed of the rocket, and momentum transfered to the ejected gases must equal the momentum of the spaceship the energy E of the gass ejected is not less than the kinetic energy of spaceship

$$\frac{\gamma M_0 m}{R_0}$$

Addding the two we get the minimum work done on the ejected gases to bring about the softlanding.

$$A_{\min} \approx \gamma m \left(\frac{M_E}{R_E} + \frac{M_0}{R_0}\right)$$

On substitution we get $1{\cdot}3 \times 10^8$ kJ.

1.233 Assume first that the attraction of the earth can be neglected. Then the minimum velocity, that must be imparted to the body to escape from the Sun's pull, is, as in 1·230, equal to

$$(\sqrt{2} - 1)\, v_1$$

where $v_1^2 = \gamma M_s / r$, r = radius of the earth's orbit, M_s = mass of the Sun.

In the actual case near the earth, the pull of the Sun is small and does not change much over distances, which are several times the radius of the Earth. The velocity v_3 in question is that which overcomes the earth's pull with sufficient velocity to escape the Sun's pull. Thus

$$\frac{1}{2} m v_3^2 - \frac{\gamma M_E}{R} = \frac{1}{2} m (\sqrt{2} - 1)^2 v_1^2$$

where R = radius of the earth, M_E = mass of the earth.

Writing $v_1^2 = \gamma M_E / R$, we get

$$v_3 = \sqrt{2 v_2^2 + (\sqrt{2} - 1)^2 v_1^2} = 16{\cdot}6 \text{ km/s}$$