Chapter 10. Three Dimensional Geometry

1 Mark Questions

1. Write the distance of a point $P(a, b, c)$ from X-axis. **Delhi 2014C**

> Let any point on X-axis be $Q(x, 0, 0)$. Then, use the formula for distance of point $R(x_1, y_1, z_1)$ from $S(x_2, y_2, z_2)$
> $$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Given point is $P(a, b, c)$.

Let the coordinates of the point on X-axis be (a,0,0). **(1/2)**

[$\because$ x-coordinate of both points will be same]

$\therefore$ Required distance

$$= \sqrt{(a-a)^2 + (0-b)^2 + (0-c)^2}$$

$$= \sqrt{0 + b^2 + c^2}$$

$$= \sqrt{b^2 + c^2}$$ **(1/2)**

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equation for the line. **All India 2014**

Given cartesian equation of a line is

$$\frac{3-x}{5}=\frac{y+4}{7}=\frac{2z-6}{4}$$

On rewriting the given equation in standard form, we get

$$\frac{x-3}{-5}=\frac{y+4}{7}=\frac{z-3}{2}=\lambda \qquad \text{[let]}$$

$\Rightarrow \qquad x=-5\lambda+3, \quad y=7\lambda-4$

and $\qquad z=2\lambda+3$ **(1/2)**

Now, $x\hat{i}+y\hat{j}+z\hat{k}$

$$=(-5\lambda+3)\hat{i}+(7\lambda-4)\hat{j}+(2\lambda+3)\hat{k}$$

$$=3\hat{i}-4\hat{j}+3\hat{k}+\lambda(-5\hat{i}+7\hat{j}+3\hat{k})$$

$$\Rightarrow \vec{r}=(3\hat{i}-4\hat{j}+3\hat{k})+\lambda(-5\hat{i}+7\hat{j}+3\hat{k})$$

which is the required equation of line in vector form. **(1/2)**

3. Write the equation of the straight line through the point (α, β, γ) and parallel to Z-axis.

The vector equation of a line parallel to Z-axis is $\vec{m}=0\hat{i}+0\hat{j}+\hat{k}$. Then, the required line passes through the point $A(\alpha, \beta, \gamma)$ whose position vector is $\vec{r_1}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{j}$ and is parallel to the vector $\vec{m}=(0\hat{i}+0\hat{j}+\hat{k})$. **(1/2)**

$\therefore$ The equation is $\vec{r}=\vec{r_1}+\lambda\vec{m}$

$$=(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})+\lambda(0\hat{i}+0\hat{j}+\hat{k})$$

$$=(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})+\lambda(\hat{k})$$ **(1/2)**

4. Find the direction cosines of the line

$$\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}.$$

Delhi 2013C

Given, equation of line is $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$

It can be rewritten as $\frac{x-4}{-2}=\frac{y}{6}=\frac{z-1}{-3}$

Here, DR's of the line are −2, 6, − 3.

Now, $\sqrt{(-2)^2+6^2+(-3)^2}$

$$=\sqrt{4+36+9}$$

$$=\sqrt{49}=7 \text{ units}$$

$\therefore$ DC's of a line are $-\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}$. **(1)**

5. If a unit vector $\hat{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ. **Delhi 2013**

Given unit vector $\vec{a}$ makes angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with j and θ $\hat{k}$. So, $\alpha=\frac{\pi}{3}$ with $\hat{i}$, $\beta=\frac{\pi}{4}$ with $\hat{j}$ and $\gamma=\theta$ with $\hat{k}$.

Now, $\cos^2\frac{\pi}{3}+\cos^2\frac{\pi}{4}+\cos^2\theta=1$

$[\because \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1]$

$$\Rightarrow \quad \frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$$

$$\Rightarrow \quad \cos^2\theta=1-\frac{3}{4}=\frac{1}{4} \quad \textbf{(1/2)}$$

$$\Rightarrow \quad \cos\theta=\pm\frac{1}{2}$$

$\Rightarrow \cos\theta=\frac{1}{2}$, as θ is an acute angle.

$$\therefore \quad \theta=\frac{\pi}{3} \quad \textbf{(1/2)}$$

6. Find the cartesian equation of the line which passes through the point (−2,4,− 5) and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$.

HOTS; Delhi 2013

If two lines are parallel, then they both have same direction ratios. Use this result and simplify it.

Given, the required line is parallel to the line

$$\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6} \text{ or } \frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$$

$\therefore$ DR's of both lines are proportional to each other. **(1/2)**

The required equation of the line passing through $(-2, 4, -5)$ having DR's $(3, -5, 6)$ is

$$\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}.$$ **(1/2)**

7. If a line has direction ratios $2, -1, -2$, then what are its direction cosines? **Delhi 2012**

Given, DR's of the line are $2, -1, -2$.

$\therefore$ Direction cosines of the line are

$$=\frac{2}{\sqrt{(2)^2+(-1)^2+(-2)^2}}, \frac{-1}{\sqrt{(2)^2+(-1)^2+(-2)^2}}, \frac{-2}{\sqrt{(2)^2+(-1)^2+(-2)^2}}$$

$$\left[\because l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}\right]$$

$$=\frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{-2}{\sqrt{4+1+4}}$$

$$=\frac{2}{\sqrt{9}}, \frac{-1}{\sqrt{9}}, \frac{-2}{\sqrt{9}}, \text{ i.e. } \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$$ **(1)**

8. What are the direction cosines of a line which makes equal angles with the coordinate axes? **Foreign 2011; All India 2009, 2008C**

Given, line makes equal angles with coordinate axes. Let α, β and γ be the angle made by the line with coordinate axes.

Than, $\alpha = \beta = \gamma \Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow \quad l = m = n \quad$ ...(i)

$[\because l = \cos\alpha, m = \cos\beta, n = \cos\gamma]$

We know that, $l^2 + m^2 + n^2 = 1$

$\therefore \quad l^2 + l^2 + l^2 = 1 \quad$ [from Eq.(i)]

$\Rightarrow \quad 3l^2 = 1 \Rightarrow l^2 = \frac{1}{3} \Rightarrow l = \pm\frac{1}{\sqrt{3}}$

From Eq. (i), direction cosines of a line are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$. **(1)**

9. Write the vector equation of the line given by $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. **Delhi 2011,2010**

Given equation of line in cartesian form is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}.$$

The point on the line is (5, – 4, 6) and DR's are (3, 7, 2).

We know that, vector equation of a line , if point is $\vec{a}$ and direction of a line is $\vec{b}$, is $\vec{r} = \vec{a} + \lambda\vec{b}$.

Here, $\vec{a} = (5, -4, 6)$ and $\vec{b} = (3, 7, 2)$.

So, equation of line in vector form is

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k}) \quad \textbf{(1)}$$

10. Equation of line is $\frac{4-x}{2} = \frac{y+3}{2} = \frac{z+2}{1}$.

Find the direction cosines of a line parallel to above line. **HOTS; All India 2011C**

Given equation of line can be written as

$$\frac{x-4}{-2}=\frac{y+3}{2}=\frac{z+2}{1}$$

Here, DR's of a line are −2, 2, 1.

∴ DC's of line parallel to above line are given by $\frac{-2}{\sqrt{(-2)^2+(2)^2+(1)^2}}$,

$$\frac{2}{\sqrt{(-2)^2+(2)^2+(1)^2}}, \frac{1}{\sqrt{(-2)^2+(2)^2+(1)^2}}$$

or $$\frac{-2}{\sqrt{4+4+1}}, \frac{2}{\sqrt{4+4+1}}, \frac{1}{\sqrt{4+4+1}}$$

or $$\frac{-2}{\sqrt{9}}, \frac{2}{\sqrt{9}}, \frac{1}{\sqrt{9}} \text{ or } -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}.$$

Hence, required DC's of a line parallel to the given line are $\left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$. **(1)**

NOTE Before we can use the DR's of a line, first we ensure that coefficients of x, y and z are unity with positive sign.

11. If the equations of line AB is $\frac{3-x}{1}=\frac{y+2}{-2}=\frac{z-5}{4}$, then write the direction ratios of the line parallel to above line AB. **Delhi 2011C**

Given equation of line can be written as

$$\frac{x-3}{-1}=\frac{y+2}{-2}=\frac{z-5}{4}$$

∴ DR's of the line parallel to above line are −1, −2, 4.

[∵ parallel lines have same DR's] **(1)**

12. Find the distance of point (2, 3, 4) from X-axis. **Delhi 2010C**

Do same as Que. 1. **[Ans. 5]**

13. Write the equation of line parallel to the line $\frac{x-2}{-3}=\frac{y+3}{2}=\frac{z+5}{6}$ and passing through point (1, 2, 3). **All India 2009C**

Do same as Que. 6. $\left[\textbf{Ans. } \frac{x-1}{-3}=\frac{y-2}{2}=\frac{z-3}{6}\right]$

14. Write the direction cosines of a line parallel to the line $\frac{3-x}{3}=\frac{y+2}{-2}=\frac{z+2}{6}$. **Delhi 2009C**

Do same as Que. 10. $\left[\textbf{Ans. } \frac{-3}{7},\frac{-2}{7},\frac{6}{7}\right]$

15. The equation of line is

$$\frac{2x-5}{4}=\frac{y+4}{3}=\frac{6-z}{6}.$$

Find the direction cosines of the line parallel to this line. **All India 2008**

Do same as Que. 10. $\left[\textbf{Ans. } \frac{2}{7},\frac{3}{7},\frac{-6}{7}\right]$

16. The equation of line is given by $\frac{4-x}{2}=\frac{y+3}{5}=\frac{z+2}{6}$. Write the direction cosines of the line parallel to above line.

Delhi 2008C

Do same as Que. 10. $\left[\textbf{Ans. } \frac{-2}{\sqrt{65}},\frac{5}{\sqrt{65}},\frac{6}{\sqrt{65}}\right]$

17. If $P=(1,5,4)$ and $Q=(4,1,-2)$, then find the direction ratios of PQ. **All India 2008**

Given points are $P(1, 5, 4)$ and $Q(4, 1, -2)$.

$\therefore$ Direction ratios of $PQ = 4-1, 1-5, -2-4$

$= 3, -4, -6$ **(1)**

$\left[\because \text{DR's of line joining points } P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2) \text{ are } x_2-x_1, y_2-y_1, z_2-z_1.\right]$

4 Marks Questions

18. Show that the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersect. Also, find their point of intersection. **Delhi 2014**

Given lines can be rewritten as

$$\vec{r} = (3\lambda + 1)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} \quad \text{...(i)}$$

and $$\vec{r} = (4 + 2\mu)\hat{i} + 0\hat{j} + (3\mu - 1)\hat{k} \quad \text{...(ii)} \ \textbf{(1)}$$

Both lines intersect at a point, when their respective components along $\hat{i}, \hat{j}$ and $\hat{k}$ are equal.

$\therefore \quad 3\lambda + 1 = 4 + 2\mu$

$\Rightarrow \quad 3\lambda - 2\mu = 3 \quad \text{...(iii)}$

$1 - \lambda = 0 \quad \text{...(iv)}$

and $\quad 3\mu - 1 = -1 \quad \text{...(v)}$ **(1)**

From Eq. (iv), we get $\lambda = 1$ and put the value of λ in Eq. (iii), we get

$3(1) - 2\mu = 3$

$\Rightarrow \quad -2\mu = 3 - 3$

$\Rightarrow \quad \mu = 0$

On putting the value of μ in Eq. (v), we get

$3(0) - 1 = -1 \Rightarrow 0 - 1 = -1$

$\Rightarrow \quad -1 = -1$, which is true

So, both lines intersect each other. **(1)**

Point of intersection of both lines can be obtained by putting $\lambda = 1$ in Eq. (i), then we get

$\vec{r} = 4\hat{i} + 0\hat{j} - \hat{k}$, which is the position vector of the point of intersection $(4, 0, -1)$. **(1)**

19. Find the direction cosines of the line $\frac{x+2}{2} = \frac{2y-7}{6} = \frac{5-z}{6}$. Also, find the vector equation of the line through the point $A(-1, 2, 3)$ and parallel to the given line.

Delhi 2014C

Given equation of line is

$$\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$$

This equation can be written as

$$\frac{x+2}{2}=\frac{y-7/2}{3}=\frac{z-5}{-6}$$

So, direction ratio's of line are 2, 3, – 6. **(1)**

Now, direction cosines of a line are

$$l=\frac{2}{\sqrt{2^2+3^2+(-6)^2}}, \quad m=\frac{3}{\sqrt{2^2+3^2+(-6)^2}},$$

$$n=\frac{-6}{\sqrt{2^2+3^2+(-6)^2}}$$

$$\left[\because l=\frac{\pm a}{\sqrt{a^2+b^2+c^2}}, m=\frac{\pm b}{\sqrt{a^2+b^2+c^2}}, n=\frac{\pm c}{\sqrt{a^2+b^2+c^2}}\right]$$

(1)

$$\Rightarrow l=\frac{2}{\sqrt{49}}, m=\frac{3}{\sqrt{49}}, h=\frac{-6}{\sqrt{49}}$$

So, direction cosines of given line are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$.

Now, DR's of a line parallel to given line are 2, 3, –6 and it passes through the point A (–1, 2, 3). So, required equation of line parallel to given line is

$$\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{-6}$$ **(1)**

20. Find the angle between the lines

$$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$$

and $\vec{r}=7\hat{i}-6\hat{j}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$.

Foreign 2014; All India 2008 C

If vector form of lines are $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$, then angle between them is

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$$

The given equations of lines are

$$\vec{r} = (2\hat{i} - 5\hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k}) \quad \ldots(i)$$

and $\vec{r} = (7\hat{i} - 6\hat{j} - 6\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k}) \ldots(ii)$ **(1)**

On comparing Eqs. (i) and (ii) with vector form of equation of line, i.e.

$\vec{r} = \vec{a} + \lambda \vec{b}$, we get

$$\vec{a_1} = 2\hat{i} - 5\hat{j} + \hat{k}, \vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

and $\vec{a_2} = 7\hat{i} - 6\hat{j} - 6\hat{k}, \vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$ **(1)**

We know that, angle between two lines is given by

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$$

$$\therefore \cos\theta = \left|\frac{(3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{(3)^2 + (2)^2 + (6)^2} \cdot \sqrt{(1)^2 + (2)^2 + (2)^2}}\right|$$

(1)

$$\Rightarrow \quad \cos\theta = \left|\frac{3 + 4 + 12}{\sqrt{49} \times \sqrt{9}}\right|$$

$$\Rightarrow \quad \cos\theta = \left|\frac{19}{7 \times 3}\right| \Rightarrow \cos\theta = \frac{19}{21}$$

Hence, angle between given two lines is

$\theta = \cos^{-1}\left(\frac{19}{21}\right)$. **(1)**

21. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also, find their point of intersection. **Delhi 2014**

The given lines are

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=\lambda \qquad \text{[let] ...(i)}$$

and $$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=\mu \qquad \text{[let] ...(ii)}$$

Then, any point on line (i) is

$$P(3\lambda-1, 5\lambda-3, 7\lambda-5) \qquad \text{...(iii)}$$

and any point on line (ii) is

$$Q(\mu+2, 3\mu+4, 5\mu+6) \qquad \text{...(iv)}$$

If lines (i) and (ii) intersect, then these points must coincide.

$$\therefore \qquad 3\lambda-1=\mu+2$$
$$5\lambda-3=3\mu+4$$
$$7\lambda-5=5\mu+6$$
$$\Rightarrow \qquad 3\lambda-\mu=3 \qquad \text{...(v)}$$
$$5\lambda-3\mu=7 \qquad \text{...(vi)}$$
$$7\lambda-5\mu=11 \qquad \text{...(vii)} \ \mathbf{(1)}$$

On multiplying Eq. (v) by 3 and then subtracting Eq. (vi) from it, we get

$$9\lambda-3\mu-5\lambda+3\mu=9-7$$
$$\Rightarrow \qquad 4\lambda=2 \Rightarrow \lambda=\frac{1}{2}$$

On putting the value of λ in Eq. (v), we get

$$3\times\frac{1}{2}-\mu=3$$
$$\Rightarrow \qquad \frac{3}{2}-\mu=3 \Rightarrow \mu=-\frac{3}{2} \qquad \mathbf{(1)}$$

On putting the values of λ and μ in Eq. (vii), we get

$$7\times\frac{1}{2}-5\left(-\frac{3}{2}\right)=11$$
$$\Rightarrow \qquad \frac{7}{2}+\frac{15}{2}=11 \Rightarrow \frac{22}{2}=11$$
$$\Rightarrow \qquad 11=11, \text{ which is true.} \qquad \mathbf{(1)}$$

[illegible]

Hence, lines (i) and (ii) intersect and their point of intersection is

$$P\left(3\times\frac{1}{2}-1, 5\times\frac{1}{2}-3, 7\times\frac{1}{2}-5\right)$$

[put $\lambda = \frac{1}{2}$ in Eq. (iii)]

i.e. $$P\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$$ **(1)**

22. Find the value of p, so that the lines

$$l_1: \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$$

and $$l_2: \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are perpendicular to each other. Also, find the equation of a line passing through a point $(3, 2, -4)$ and parallel to line l_1.

All India 2014

Equation of the given lines can be written in standard form as

$$l_1 : \frac{x-1}{-3} = \frac{y-2}{\frac{p}{7}} = \frac{z-3}{2}$$

and $$l_2 : \frac{x-1}{-\frac{3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$ **(1)**

Direction ratios of these lines are $-3, \frac{p}{7}, 2$ and $-\frac{3p}{7}, 1, -5$, respectively. **(1)**

We know that, two lines of direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular to each other, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\therefore \quad (-3)\left(\frac{-3p}{7}\right) + \left(\frac{p}{7}\right)(1) + (2)(-5) = 0$$

$$\Rightarrow \quad \frac{9p}{7} + \frac{p}{7} - 10 = 0$$

$$\Rightarrow \quad \frac{10p}{7} = 10 \Rightarrow p = 7$$ **(1)**

Thus, the value of p is 7.

Also, we know that, the equation of a line which passes through the point (x_1, y_1, z_1) with direction ratios a, b, c is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Hence, required line is parallel to line l_1.

So, $a = -3$, $b = \frac{7}{7} = 1$ and $c = 2$

Now, equation of line passing through the point $(3, 2, -4)$ and having direction ratios $(-3, 1, 2)$ is

$$\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2}$$

$$\Rightarrow \quad \frac{3-x}{3} = \frac{y-2}{1} = \frac{z+4}{2}$$ **(1)**

23. A line passes through the point $(2, -1, 3)$ and is perpendicular to the lines

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda\,(2\hat{i} - 2\hat{j} + \hat{k})$

and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu\,(\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation in vector and cartesian forms.

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Given lines are $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$

and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$.

On comparing with vector form of equation of line $\vec{r} = a + \lambda b$, we get

$b_1 = 2\hat{i} - 2\hat{j} + \hat{k}$ and $b_2 = \hat{i} + 2\hat{j} + 2\hat{k}$. The required line is perpendicular to the given lines. **(1)**

So, it is parallel to the vector

$$\vec{b} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= (-4-2)\,\hat{i} - (4-1)\,\hat{j} + (4+2)\,\hat{k}$$

$$= -6\hat{i} - 3\hat{j} + 6\hat{k}$$ **(1)**

Thus, the required line passes through the point $(2, -1, 3)$ and parallel to the vector $-6\hat{i} - 3\hat{j} + 6\hat{k}$.

So, its vector equation is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k})$$

The equation can be rewritten as

$$x\hat{i} + y\hat{j} + z\hat{k} = (2-6\lambda)\,\hat{i} + (-1-3\lambda)\,\hat{j} + (3+6\lambda)\,\hat{k}$$

(1)

On comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ from both sides, we get

$$x = 2 - 6\lambda, \quad y = -1 - 3\lambda, \quad z = 3 + 6\lambda$$

$$\Rightarrow \frac{x-2}{-6} = \lambda, \quad \frac{y+1}{-3} = \lambda, \quad \frac{z-3}{6} = \lambda$$

$$\Rightarrow \frac{2-x}{6} = \frac{-y-1}{3} = \frac{z-3}{6}$$

which is the required cartesian form of the line. **(1)**

24. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$$

and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Foreign 2014; Delhi 2008

Given equations of lines are

$$\vec{r} = (\hat{i} + \hat{j}) + \lambda (2\hat{i} - \hat{j} + \hat{k}) \quad ...(i)$$

and $$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu (3\hat{i} - 5\hat{j} + 2\hat{k}) \quad ...(ii)$$

On comparing above equations with vector equation

$$\vec{r} = \vec{a} + \lambda \vec{b}, \text{ we get}$$

$$\vec{a_1} = \hat{i} + \hat{j}, \vec{b_1} = 2\hat{i} - \hat{j} + \hat{k}$$

and $$\vec{a_2} = 2\hat{i} + \hat{j} - \hat{k}, \vec{b_2} = 3\hat{i} - 5\hat{j} + 2\hat{k} \quad \textbf{(1)}$$

Now, we know that, the shortest distance between two lines is given by

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| \quad ...(iii)$$

$$\therefore \quad \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= \hat{i}(-2+5) - \hat{j}(4-3) + \hat{k}(-10+3)$$

$$\Rightarrow \vec{b_1} \times \vec{b_2} = 3\hat{i} - \hat{j} - 7\hat{k} \qquad \text{...(iv)} \ \textbf{(1)}$$

and $|\vec{b_1} \times \vec{b_2}| = \sqrt{(3)^2 + (-1)^2 + (-7)^2}$

$$= \sqrt{9 + 1 + 49} = \sqrt{59} \qquad \text{...(v)}$$

Also, $\vec{a_2} - \vec{a_1} = (2\hat{i} + \hat{j} - \hat{k}) - (\hat{i} + \hat{j})$

$$= \hat{i} - \hat{k} \qquad \text{...(vi)} \ \textbf{(1)}$$

From Eqs. (iii), (iv), (v) and (vi), we get

$$d = \left| \frac{(3\hat{i} - \hat{j} - 7\hat{k}) \cdot (\hat{i} - \hat{k})}{\sqrt{59}} \right|$$

$$\Rightarrow \qquad d = \left| \frac{3 - 0 + 7}{\sqrt{59}} \right| = \frac{10}{\sqrt{59}}$$

Hence, required shortest distance is $\frac{10}{\sqrt{59}}$ units. **(1)**

25. Find the shortest distance between the two lines whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and

$\vec{r} = (4\hat{i} + 5\hat{j} + \hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$ **Delhi 2014C**

Given equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \quad \ldots(i)$$

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k}) \quad \ldots(ii)$$

On comparing with $\vec{r} = \vec{a} + \lambda \vec{b}$, we get

$$\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$$

and $\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}, \vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

Now, $\vec{a_2} - \vec{a_1} = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

and $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$ **(1)**

$$= \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6) = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-9)^2 + 3^2 + 9^2}$

$\Rightarrow \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{81 + 9 + 81} = \sqrt{171}$ **(1)**

Now, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} - \vec{b_2})$

$$= (3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot (-9\hat{i} + 3\hat{j} + 9\hat{k})$$

$$= -27 + 9 + 27 = 9$$ **(1)**

$\therefore$Shortest distance between two lines is

$\text{SD} = \left| \dfrac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \dfrac{9}{\sqrt{171}} \right|$ units **(1)**

26. Find the shortest distance between the following lines:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}, \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

Foreign 2014; Delhi 2008

Given equations of lines are

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \qquad \text{...(i)}$$

and $$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \qquad \text{...(ii)}$$

On comparing above equations with one point form of equation of line, i.e.

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}, \text{ we get}$$

$$a_1=1, b_1=-2, c_1=1, x_1=3,$$
$$y_1=5, z_1=7$$

and $$a_2=7, b_2=-6,$$
$$c_2=1, x_2=-1,$$
$$y_2=-1, z_2=-1 \qquad \textbf{(1)}$$

We know that, the shortest distance between two lines is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

$$\therefore \quad d=\frac{\begin{vmatrix} -4 & -6 & -8 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}}{\sqrt{(-2+6)^2+(7-1)^2+(-6+14)^2}} \qquad \textbf{(1)}$$

$$[\because x_2-x_1=-1-3=-4, y_2-y_1=-1-5=-6,$$
$$z_2-z_1=-1-7=-8]$$

$$=\left|\frac{-4(-2+6)+6(1-7)-8(-6+14)}{\sqrt{(4)^2+(6)^2+(8)^2}}\right|$$

$$=\left|\frac{-4(4)+6(-6)-8(8)}{\sqrt{16+36+64}}\right|=\left|\frac{-16-36-64}{\sqrt{116}}\right|$$

(1)

$$= \left| \frac{-116}{\sqrt{116}} \right| = \frac{116}{\sqrt{116}} = \frac{(\sqrt{116})^2}{\sqrt{116}} = \sqrt{116}$$

Hence, the required shortest distance is $\sqrt{116}$ units. **(1)**

27. Find the distance between the lines l_1 and l_2 given by $l_1: \vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$,
$l_2: \vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$. **Foreign 2014**

Given equation of lines are

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$$

On comparing with $\vec{r} = \vec{a} + \lambda \vec{b}$, we get

$$a_1 = \hat{i} + 2\hat{j} - 4\hat{k}, \vec{b}_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

and $\vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}, \vec{b}_2 = 4\hat{i} + 6\hat{j} + 12\hat{k}$.

(1)

Now, $\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 3\hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - 4\hat{k})$

$$= 2\hat{i} + \hat{j} - \hat{k}$$

and $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 4 & 6 & 12 \end{vmatrix}$

$= \hat{i}(36 - 36) - \hat{j}(24 - 24) + \hat{k}(12 - 12) = 0$ **(1)**

So, both given lines are parallel and

$$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Then, $\vec{b} \times (\overline{a_2} - \overline{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$

$= \hat{i}(-3 - 6) - \hat{j}(-2 - 12) + \hat{k}(2 - 6)$

$= -9\hat{i} + 14\hat{j} - 4\hat{k}$ **(1)**

Now, required distance between given lines is given by

$$d = \left| \frac{\vec{b} \times (\overline{a_2} - \overline{a_1})}{|\vec{b}|} \right| = \left| \frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{\sqrt{(2)^2 + (3)^2 + (6)^2}} \right|$$

$$= \frac{\sqrt{81 + 196 - 16}}{\sqrt{4 + 9 + 36}} = \frac{\sqrt{261}}{\sqrt{49}}$$

$= \frac{\sqrt{261}}{7}$ units **(1)**

28. Find the vector and cartesian equations of the line passing through the point (2, 1, 3) and perpendicular to the lines **All India 2014**

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3} \text{ and } \frac{x}{-3}=\frac{y}{2}=\frac{z}{5}.$$

Any line through the point (2, 1, 3) can be written as

$$\frac{x-2}{a}=\frac{y-1}{b}=\frac{z-3}{c} \quad \text{...(i)}$$

where, a, b and c are the direction ratios of line (i).

Now, the line (i) is perpendicular to the lines

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$$

and $$\frac{x-0}{-3}=\frac{y-0}{2}=\frac{z-0}{5}.$$

Direction ratios of these two lines are (1, 2, 3) and (−3, 2, 5), respectively. **(1)**

$$\therefore \quad a+2b+3c=0 \quad \text{...(ii)}$$

[$\because$ if two lines are perpendicular, then $a_1a_2+b_1b_2+c_1c_2=0$]

and $$-3a+2b+5c=0 \quad \text{...(iii)}$$

In Eqs. (ii) and (iii), by cross-multiplication, we get

$$\frac{a}{10-6}=\frac{-b}{-9-5}=\frac{c}{2+6}$$

$$\Rightarrow \quad \frac{a}{4}=\frac{b}{14}=\frac{c}{8}$$

$$\Rightarrow \quad \frac{a}{2}=\frac{b}{7}=\frac{c}{4}=\lambda \quad \text{[say]}$$

$$\therefore \quad a=2\lambda, b=7\lambda \text{ and } c=6\lambda \quad \textbf{(1)}$$

On substituting the values of a, b and c in Eq. (i), we get

$$\frac{x-2}{2\lambda}=\frac{y-1}{7\lambda}=\frac{z-3}{6\lambda}$$

$$\Rightarrow \quad \frac{x-2}{}=\frac{y-1}{}=\frac{z-3}{} \quad \textbf{(1)}$$

$\rightarrow \quad \overline{\quad 2 \quad} = \overline{\quad 7 \quad} = \overline{\quad 6 \quad}$

which is the required cartesian equation of the line.

The vector equation of line which passes through (2, 1, 3) and parallel to the vector $2\hat{i} + 7\hat{j} + 6\hat{k}$ is

$$\vec{r} = 2\hat{i} + \hat{j} + 3\hat{k} + \lambda(2\hat{i} + 7\hat{j} + 6\hat{k})$$

which is the required vector equation of the line. **(1)**

29. The cartesian equation of a line is $6x - 2 = 3y + 1 = 2z - 2$. Find the direction cosines of the line. Write down the cartesian and vector equations of a line passing through $(2, -1, -1)$ which are parallel to the given line. **Delhi 2013C**

Given equation of line is

$$6x - 2 = 3y + 1 = 2z - 2$$

or $$\frac{x - 2/6}{1/6} = \frac{y + 1/3}{1/3} = \frac{z - 2/2}{1/2}$$

$$\Rightarrow \quad \frac{x-1/3}{1/6}=\frac{y+1/3}{1/3}=\frac{z-1}{1/2}$$

$\therefore$ DC's of a line are $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. **(1)**

The equation of a line passing through $(2, -1, -1)$ and parallel to the given line is

$$\frac{x-2}{1/6}=\frac{y+1}{1/3}=\frac{z+1}{1/2}=\lambda \qquad \text{[say]}\,\mathbf{(1)}$$

$$\left[\because \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}\right]$$

$$\Rightarrow \quad x=2+\frac{\lambda}{6}, y=-1+\frac{\lambda}{3} \text{ and } z=-1+\frac{\lambda}{2}$$

Now, $x\hat{i}+y\hat{j}+z\hat{k}=\left(2+\frac{\lambda}{6}\right)\hat{i}+\left(-1+\frac{\lambda}{3}\right)\hat{j}+\left(-1+\frac{\lambda}{2}\right)\hat{k}$ **(1)**

$$\Rightarrow \quad \vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\lambda\left(\frac{1}{6}\hat{i}+\frac{1}{3}\hat{j}+\frac{1}{2}\hat{k}\right)$$

which is the required equation of line in vector form. **(1)**

30. Find the shortest distance between the two lines whose vector equations are

$$\vec{r}=(6\hat{i}+2\hat{j}+2\hat{k})+\lambda(\hat{i}-2\hat{j}+2\hat{k})$$

and $\vec{r}=(-4\hat{i}-\hat{k})+\mu(3\hat{i}-2\hat{j}-2\hat{k})$.

Delhi 2013C; Foreign 2011

Given equations of lines are

$$\vec{r} = (6\hat{i} + 2\hat{j} + 2\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$$

and $\vec{r} = (-4\hat{i} - \hat{k}) + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

which are of the form $\vec{r} = \vec{a} + \lambda\vec{b}$.

Here, $\vec{a_1} = 6\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b_1} = \hat{i} - 2\hat{j} + 2\hat{k}$;

$\vec{a_2} = -4\hat{i} - \hat{k}$ and $\vec{b_2} = 3\hat{i} - 2\hat{j} - 2\hat{k}$

Then, $\vec{a_2} - \vec{a_1} = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k})$

$= -10\hat{i} - 2\hat{j} - 3\hat{k}$ **(1)**

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix}$$

$= \hat{i}(4 + 4) - \hat{j}(-2 - 6) + \hat{k}(-2 + 6)$

$= 8\hat{i} + 8\hat{j} + 4\hat{k}$ **(1)**

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(8)^2 + (8)^2 + (4)^2}$

$= \sqrt{64 + 64 + 16} = \sqrt{144} = 12$ **(1)**

Now, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})$

$= (-10\hat{i} - 2\hat{j} - 3\hat{k}) \cdot (8\hat{i} + 8\hat{j} + 4\hat{k})$

$= -80 - 16 - 12 = -108$

$\therefore \quad \text{Required SD} = \left| \dfrac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$

$= \left| \dfrac{-108}{12} \right| = 9$ units **(1)**

31. Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k});$$
$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

are intersecting. Hence, find their point of intersection. **All India 2013**

Given vector lines are

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

Their cartesian form are

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2} = r \quad \text{[say] ...(i)}$$

and $\frac{x-5}{3} = \frac{y+2}{2} = \frac{z-0}{6} = p$ [say] ...(ii)**(1)**

Let $(r+3, 2r+2, 2r-4)$ and $(3p+5, 2p-2, 6p)$ be two points on the lines (i) and (ii), respectively.

If these lines intersect each other, then

$$r + 3 = 3p + 5$$

$\Rightarrow$ $r - 3p = 2$ ...(iii)

$$2r + 2 = 2p - 2$$

$\Rightarrow$ $r - p = -2$ ...(iv)

and $2r - 4 = 6p \Rightarrow 2r - 6p = 4$

$\Rightarrow$ $r - 3p = 2$ ...(v) **(1)**

Now, subtracting Eq. (v) from Eq. (iv), we get

$$2p = -4 \Rightarrow p = -2$$

On putting $p = -2$ in Eq. (iv), we get

$$r - (-2) = -2 \Rightarrow r = -4$$

$\therefore$ Any point on line (i) is

$(-4+3, -8+2, -8-4) = (-1, -6, -12)$ **(1)**

and any point on line (ii) is

$(-6+5, -4-2, -12) = (-1, -6, -12)$

Since, both points are same, therefore both lines intersect each other at point $(-1, -6, -12)$. **(1)**

32. Find the vector and cartesian equations of line passing through point $(1, 2, -4)$ and perpendicular to two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$$

and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **Delhi 2012**

Let the required equation of line passing through (1,2,–4) be

$$\frac{x-1}{a}=\frac{y-2}{b}=\frac{z+4}{c} \quad \ldots(i)$$

Given that line (i) is perpendicular to lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \quad \ldots(ii)$$

and $$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \quad \ldots(iii)$$ **(1)**

We know that, when two lines are perpendicular, then we have

$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$, where a_1, b_1, c_1 and a_2, b_2, c_2 are the DR's of two lines.

Using this property, first in Eqs. (i) and (ii) and then in Eqs. (i) and (iii), we get

$$3a - 16b + 7c = 0 \quad \ldots(iv)$$

and $$3a + 8b - 5c = 0 \quad \ldots(v)$$ **(1)**

On subtracting Eq. (v) from Eq. (iv), we get

$$3a - 16b = -7c$$
$$\underline{-3a - 8b = -5c}$$
$$-24b = -12c$$

$$\Rightarrow \quad b = \frac{c}{2}$$

On putting $b = \frac{c}{2}$ in Eq. (iv), we get

$$3a - 16\left(\frac{c}{2}\right) + 7c = 0$$

$$\Rightarrow \quad 3a - 8c + 7c = 0$$

$$\Rightarrow \quad 3a - c = 0$$

$\Rightarrow \qquad a = \frac{c}{3}$ **(1)**

On putting $a = \frac{c}{3}$ and $b = \frac{c}{2}$ in Eq. (i), we get the required equation of line in cartesian form as

$$\frac{x-1}{\left(\frac{c}{3}\right)} = \frac{y-2}{\left(\frac{c}{2}\right)} = \frac{z+4}{c}$$

[on multiplying denominator by 6]

$$\Rightarrow \quad \frac{x-1}{2c} = \frac{y-2}{3c} = \frac{z+4}{6c}$$

$$\Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

[dividing denominator by c]

Also, the vector equation of line is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$ **(1)**

33. Find the angle between following pair of lines

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

and $\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$

and check whether the lines are parallel or perpendicular. **HOTS; Delhi 2011**

Firstly, we convert the given lines in standard form and then use the relation

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}},$$

to find the angle between them.

Given equations of two lines are

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

and $\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$

Above equations can be written as

Above equations can be written as

$$\frac{x-2}{2}=\frac{y-1}{7}=\frac{z+3}{-3} \quad \ldots(i)$$

and
$$\frac{x+2}{-1}=\frac{y-4}{2}=\frac{z-5}{4} \quad \ldots(ii) \quad \textbf{(1)}$$

On comparing Eqs. (i) and (ii) with one point form of equation of line

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}, \text{ we get}$$

$$a_1=2, b_1=7, c_1=-3$$

and
$$a_2=-1, b_2=2, c_2=4$$

We know that, angle between two lines is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}} \quad \textbf{(1)}$$

$$\therefore \cos\theta=\frac{(2)(-1)+(7)(2)+(-3)(4)}{\sqrt{(2)^2+(7)^2+(-3)^2}\cdot\sqrt{(-1)^2+(2)^2+(4)^2}}$$

$$\therefore \cos\theta=\frac{-2+14-12}{\sqrt{62}\times\sqrt{21}}=\frac{0}{\sqrt{62}\times\sqrt{21}}=0 \quad \textbf{(1)}$$

$$\Rightarrow \cos\theta=\cos\frac{\pi}{2} \qquad \left[\because 0=\cos\frac{\pi}{2}\right]$$

$$\Rightarrow \theta=\frac{\pi}{2}$$

Hence, angle between them is $\frac{\pi}{2}$. Since, angle between the two lines is $\frac{\pi}{2}$, therefore the given pair of lines are perpendicular to each other. **(1)**

NOTE Please be careful while taking DR's of a line, the line should be in symmetrical or in standard form, otherwise there may be chances of error.

34. Find the shortest distance between lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}.$$

HOTS; All India 2011

> Firstly, convert both the equations in the vector form which is $\vec{r} = \vec{a} + \lambda\vec{b}$. Then, apply the shortest distance formula,
>
> i.e. $$d = \left|\frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right|$$

Given equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \quad \ldots(i)$$

and $$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} \quad \ldots(ii)$$

Firstly, we convert both equations in the vector form as $\vec{r} = \vec{a} + \lambda\vec{b}$ ...(iii)

So, Eq. (i) can be written as

$$\vec{r} = \hat{i} - t\hat{i} + t\hat{j} - 2\hat{j} + 3\hat{k} - 2t\hat{k}$$

$$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \quad \ldots(iv)\ \textbf{(1)}$$

and Eq. (ii) can be written as

$$\vec{r} = s\hat{i} + \hat{i} + 2s\hat{j} - \hat{j} - 2s\hat{k} - \hat{k}$$

$$\Rightarrow \vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \quad \ldots(v)$$

Now, from Eqs. (iii), (iv) and (v), we get

$$\vec{a_1} = \hat{i} - 2\hat{j} + 3\hat{k},\ \vec{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a_2} = \hat{i} - \hat{j} - \hat{k},\ \vec{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$$

Then, $$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= \hat{i}(-2+4) - \hat{j}(2+2) + \hat{k}(-2-1)$$

$$\Rightarrow \vec{b_1} \times \vec{b_2} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(2)^2 + (-4)^2 + (-3)^2}$$

$$= \sqrt{4 + 16 + 9} = \sqrt{29} \qquad \textbf{(1)}$$

Also, $\vec{a_2} - \vec{a_1} = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k})$

$$= \hat{j} - 4\hat{k}$$

We know that, the shortest distance between the lines is given as

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| \qquad \textbf{(1)}$$

Hence, required shortest distance,

$$d = \left| \frac{(2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k})}{\sqrt{29}} \right|$$

$$= \left| \frac{0 - 4 + 12}{\sqrt{29}} \right| = \frac{8}{\sqrt{29}}$$

$$\Rightarrow \quad d = \frac{8\sqrt{29}}{29} \text{ units} \qquad \textbf{(1)}$$

35. Find shortest distance between the lines $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

Foreign 2011; All India 2[illegible]

Do same as Que. 34. $\left[\textbf{Ans. } \frac{3\sqrt{2}}{2} \text{ units}\right]$

36. Find the equation of the perpendicular from point (3, −1, 11) to line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Also, find the coordinates of foot of perpendicular and the length of perpendicular. HOTS; All India 2011C

Firstly, determine any point P on the given line and DR's between given point Q and P, using the relation $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$, where (a_1, b_1, c_1) and (a_2, b_2, c_2) are DR's of PQ and given line.

Given equation of line AB is

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda \qquad \text{[say]}$$

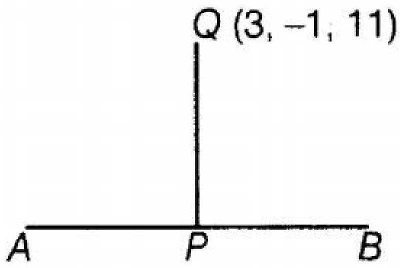
$$\Rightarrow \quad \frac{x}{2} = \lambda, \frac{y-2}{3} = \lambda \text{ and } \frac{z-3}{4} = \lambda$$

$$\Rightarrow \quad x = 2\lambda, y = 3\lambda + 2$$

and $\quad z = 4\lambda + 3$ **(1)**

$\therefore$ Any point P on the given line

$$= (2\lambda, 3\lambda + 2, 4\lambda + 3)$$



Let P be the foot of perpendicular drawn from point $Q(3, -1, 11)$ on line AB. Now, DR's of line

$QP = (2\lambda - 3, 3\lambda + 2 + 1, 4\lambda + 3 - 11)$ **(1)**

$\Rightarrow$ DR's of line $QP = (2\lambda - 3, 3\lambda + 3, 4\lambda - 8)$

Here, $\quad a_1 = 2\lambda - 3, b_1 = 3\lambda + 3, c_1 = 4\lambda - 8,$

and $\quad a_2 = 2, b_2 = 3, c_2 = 4$

Since, $\quad QP \perp AB$

$\therefore$ We have, $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...(i)

$$\Rightarrow \quad 2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$$

$$\Rightarrow \quad 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$$

$$\Rightarrow \quad 29\lambda - 29 = 0$$

$$\Rightarrow \quad 29\lambda = 29 \Rightarrow \lambda = 1$$ **(1)**

$\therefore$ Foot of perpendicular $P = (2, 3 + 2, 4 + 3)$

$$= (2, 5, 7)$$

Now, equation of perpendicular QP, where $Q(3, -1, 11)$ and $P(2, 5, 7)$, is

$$\frac{x-3}{2-3} = \frac{y+1}{5+1} = \frac{z-11}{7-11}$$

$$\left[\begin{array}{l}\text{using two points form of equation of line,} \\ \text{i.e.} \quad \dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}\end{array}\right]$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+1}{6} = \frac{z-11}{-4}$$

Now, length of perpendicular QP = distance between points $Q(3, -1, 11)$ and $P(2, 5, 7)$

$$= \sqrt{(2-3)^2 + (5+1)^2 + (7-11)^2}$$

$$\left[\begin{array}{l}\because \text{Distance} \\ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}\end{array}\right]$$

$$= \sqrt{1+36+16} = \sqrt{53}$$

Hence, length of perpendicular is $\sqrt{53}$. **(1)**

37. Find the perpendicular distance of point $(1, 0, 0)$ from the lines $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$. Also, find the coordinates of foot of perpendicular and equation of perpendicular. **Delhi 2011C**

Do same as Que. 36.

$$\left[\begin{array}{l}\textbf{Ans. } \text{Length of perpendicular is } \sqrt{53}. \\ \text{Coordinates of Food of perpendicular} \\ \qquad = (3, \ -4, \ -2) \\ \therefore \text{Equation of perpendicular} = \dfrac{x-1}{2} = \dfrac{y}{-4} = \dfrac{z}{-2}\end{array}\right]$$

38. Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point $P(1, 3, 3)$. **All India 2010**

Given equation of line is

$$\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}=\lambda \qquad \text{[say]}$$

$$\Rightarrow \quad \frac{x+2}{3}=\lambda, \frac{y+1}{2}=\lambda, \frac{z-3}{2}=\lambda$$

$$\Rightarrow \quad x=3\lambda-2, y=2\lambda-1, z=2\lambda+3$$

So, we have the point

$$Q\,(3\lambda-2, 2\lambda-1, 2\lambda+3) \qquad \text{...(i)} \ \textbf{(1)}$$

Now, given that distance between two points $P\,(1, 3, 3)$ and $Q\,(3\lambda-2,\ 2\lambda-1, 2\lambda+3)$ is 5 units, i.e. $PQ=5$

$$\Rightarrow \quad \sqrt{\begin{bmatrix}(3\lambda-2-1)^2+(2\lambda-1-3)^2\\ +(2\lambda+3-3)^2\end{bmatrix}}=5$$

$$\left[\because \text{distance}=\sqrt{\begin{matrix}(x_2-x_1)^2+(y_2-y_1)^2\\ +(z_2-z_1)^2\end{matrix}}\right]$$

$$\Rightarrow \sqrt{(3\lambda-3)^2+(2\lambda-4)^2+(2\lambda)^2}=5 \qquad \textbf{(1)}$$

On squaring both sides, we get

$$(3\lambda-3)^2+(2\lambda-4)^2+(2\lambda)^2=25$$

$$\Rightarrow \quad 9\lambda^2+9-18\lambda+4\lambda^2+16 -16\lambda+4\lambda^2=25$$

$$\Rightarrow 17\lambda^2-34\lambda=0 \quad \Rightarrow \quad 17\lambda\,(\lambda-2)=0 \qquad \textbf{(1)}$$

$$\Rightarrow \quad \text{Either } 17\lambda=0 \quad \text{or} \quad \lambda-2=0$$

$$\therefore \quad \lambda=0 \text{ or } 2$$

On putting $\lambda=0$ and $\lambda=2$ in Eq. (i), we get the required point as (–2, –1, 3) or (4, 3, 7). **(1)**

39. Find the shortest distance between the lines

$$l_1: \frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-1}{1}$$

$$l_2: \frac{x-2}{2}=\frac{y+1}{1}=\frac{z+1}{2}.$$

All India 2009C

Do same as Que. 25. $\left[\textbf{Ans. } \frac{3}{\sqrt{2}} \text{ units}\right]$

40. Find shortest distance between lines

$$\vec{r} = (1 + 2\lambda)\,\hat{i} + (1 - \lambda)\,\hat{j} + \lambda\hat{k}$$

$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu\,(2\hat{i} + \hat{j} + 2\hat{k}).$$

All India 2009

Do same as Que. 34. $\left[\textbf{Ans. } \frac{3}{\sqrt{29}} \text{ units}\right]$

41. Find the value of λ, so that following lines are perpendicular to each other

$$\frac{x+5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1} \text{ and } \frac{x}{1} = \frac{2y+1}{4\lambda} = \frac{1-z}{-3}.$$

Delhi 2009

Firstly, convert the given equations of lines into one point form of the line, which is of form $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and then use the condition $a_1a_2+b_1b_2+c_1c_2=0$ for perpendicularity of two lines and get value of λ.

Given equation of lines are

$$\frac{x+5}{5\lambda+2}=\frac{2-y}{5}=\frac{1-z}{-1}$$

and $$\frac{x}{1}=\frac{2y+1}{4\lambda}=\frac{1-z}{-3}$$

Above equations can be written as

$$\frac{x+5}{5\lambda+2}=\frac{y-2}{-5}=\frac{z-1}{1} \quad \text{...(i)}$$

and $$\frac{x}{1}=\frac{2\left(y+\frac{1}{2}\right)}{4\lambda}=\frac{z-1}{3}$$

$$\Rightarrow \quad \frac{x}{1}=\frac{y+\frac{1}{2}}{2\lambda}=\frac{z-1}{3} \quad \text{...(ii)} \ \textbf{(1)}$$

On comparing Eqs. (i) and (ii) with one point form of line

$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$, we get

$$a_1=5\lambda+2,\ b_1=-5,\ c_1=1$$

and $$a_2=1,\ b_2=2\lambda,\ c_2=3 \quad \textbf{(1)}$$

Since, the two lines are perpendicular.

$\therefore \quad a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow \quad 1(5\lambda+2)+2\lambda(-5)+3(1)=0$

$\Rightarrow \quad 5\lambda+2-10\lambda+3=0$ **(1)**

$\Rightarrow \quad -5\lambda+5=0$

$\Rightarrow \quad 5\lambda=5$

$\therefore \quad \lambda=1$ **(1)**

42. Find the value of λ, so that lines

$$\frac{1-x}{3}=\frac{y-2}{2\lambda}=\frac{z-3}{2} \text{ and } \frac{x+1}{3\lambda}=\frac{y-1}{1}=\frac{6-z}{7}$$

are perpendicular to each other. **Delhi 2009**

Do same as Que. 41. **[Ans.** $\lambda = -2$]

43. Find the value of λ, so that lines

$$\frac{1-x}{3}=\frac{7y-14}{2\lambda}=\frac{5z-10}{11}$$

and $$\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$$

are perpendicular to each other. **Delhi 2009**

Do same as Que. 41. **[Ans.** $\lambda = 7$]

44. Find the length and foot of perpendicular drawn from the point $(2, -1, 5)$ to line

$$\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}.$$ **All India 2008**

Do same as Que. 36.

[Ans. Length = $\sqrt{14}$ units and foot of prependicular = $(1, 2, 3)$]

6 Marks Questions

45. Find the distance of the point $P(-1, -5, -10)$ from the point of intersection of the line joining the points $A(2, -1, 2)$ and $B(5, 3, 4)$ with the plane $x - y + z = 5$. **Foreign 2014**

The equation of the line passing through the points $A(2, -1, 2)$ and $B(5, 3, 4)$ is given by

$$\frac{x-2}{5-2} = \frac{y+1}{3+1} = \frac{z-2}{4-2}$$

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda \quad \text{[say]} \quad \textbf{(1)}$$

$\Rightarrow \quad x = 3\lambda + 2, y = 4\lambda - 1, z = 2\lambda + 2$ **(1)**

Now, putting the values of x, y and z in the equation of the plane $x - y + z = 5$, we get

$$(3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5 \quad \textbf{(1)}$$

$$\Rightarrow \quad \lambda + 5 = 5$$

$$\therefore \quad \lambda = 0 \quad \textbf{(1)}$$

So, the point of intersection of the line and the plane is $(2, -1, 2)$. **(1)**

$\therefore$ The distance of the point $P(-1, -5, -10)$ and the point of intersection $(2, -1, 2)$ is

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{3^2 + 4^2 + 12^2} = 13 \text{ units} \quad \textbf{(1)}$$

46. Find the vector and cartesian forms of the equation of the plane passing through the point $(1, 2, -4)$ and parallel to the lines

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $\vec{r} = \hat{i} - 3\hat{j} + 5\hat{k} + \mu(\hat{i} + \hat{j} - \hat{k})$. Also, find the distance of the point $(9, -8, -10)$ from the plane thus obtained. **Delhi 2014C**

Let equation of plane through (1, 2, – 4) be

$$a(x-1)+b(y+2)-c(z+4)=0 \quad \ldots(i)$$

Given lines are

$$\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$

and $\vec{r}=\hat{i}-3\hat{j}+5\hat{k}+\lambda(\hat{i}+\hat{j}-\hat{k})$ **(1)**

The plane (i) is parallel to the given lines,

So, $2a+3b+6c=0$ and $a+b-c=0$ **(1)**

For solving these two equations by cross-multiplication, we get

$$\frac{a}{-3-6}=\frac{-b}{-2-6}=\frac{c}{2-3}$$

$$\Rightarrow \quad \frac{a}{-9}=\frac{b}{8}=\frac{c}{-1}=\lambda \quad \text{[say]}$$

$$\therefore \quad a=-9\lambda,\ b=8\lambda,\ c=-\lambda$$

On putting values of a, b and c in Eq. (i), we get $-9\lambda(x-1)+8\lambda(y-2)-\lambda(z+4)=0$

$\therefore$ Equation of plane in cartesian form is

$$-9\lambda(x-1)+8\lambda(y-2)-\lambda(z+4)=0$$

$$\Rightarrow \quad -9x+9+8y-16-z-4=0$$

$$\Rightarrow \quad 9x-8y+z+11=0 \quad \textbf{(1)}$$

Now, vector form of plane is

$$\vec{r}(9\hat{i}-8\hat{j}+\hat{k})=-11 \quad \textbf{(1)}$$

Also, distance of $(9,-8,-10)$ from the above plane

$$=\left|\frac{9-8(-8)+1(-10)+11}{\sqrt{9^2+(-8)^2+1^2}}\right|$$

$$=\left|\frac{72+64-10+11}{\sqrt{81+64+1}}\right|$$

$$\left[\therefore D=\left|\frac{Ax+by+Cz+D}{\sqrt{A^2+B^2+C^2}}\right|\right]$$

$$=\left|\frac{146}{\sqrt{146}}\right|=\sqrt{146}\text{ units} \quad \textbf{(1)}$$

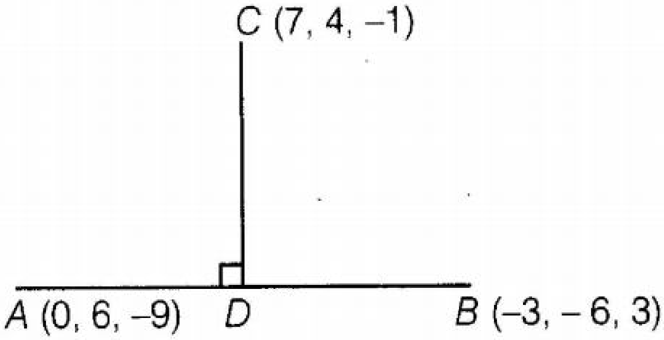
47. Find the equation of line passing through points $A(0,6,-9)$ and $B(-3,-6,3)$. If D is the foot of perpendicular drawn from the point $C(7,4,-1)$ on the line AB, then find the coordinates of point D and equation of line CD. All India 2010C

We know that, equation of line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad \ldots\text{(i)} \; \mathbf{(1)}$$

Here, $A(x_1, y_1, z_1) = (0, 6, -9)$

and $(x_2, y_2, z_2) = (-3, -6, 3)$



∴ Equation of line AB is given by

$$\frac{x - 0}{-3 - 0} = \frac{y - 6}{-6 - 6} = \frac{z + 9}{3 + 9}$$

$$\Rightarrow \quad \frac{x}{-3} = \frac{y - 6}{-12} = \frac{z + 9}{12}$$

$$\Rightarrow \quad \frac{x}{-1} = \frac{y - 6}{-4} = \frac{z + 9}{4} \quad \mathbf{(1)}$$

[dividing denominator by 3]

Next, we have to find coordinates of foot of perpendicular D.

Now, let $\frac{x}{-1} = \frac{y - 6}{-4} = \frac{z + 9}{4} = \lambda$ [say]

$\Rightarrow$ $x = -\lambda$

$y - 6 = -4\lambda$ and $z + 9 = 4\lambda$

$\Rightarrow$ $x = -\lambda$

$y = -4\lambda + 6$

and $z = 4\lambda - 9$ **(1)**

Let coordinates of

$D = (-\lambda, -4\lambda + 6, 4\lambda - 9)$ ...(ii)

Now, DR's of line CD are

$(-\lambda - 7, -4\lambda + 6 - 4, 4\lambda - 9 + 1)$

$(-\lambda - 7, -4\lambda + 6 - 4, 4\lambda - 9 + 1)$

$$= (-\lambda - 7, -4\lambda + 2, 4\lambda - 8)$$

Now, $CD \perp AB$

$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$ **(1)**

where, $a_1 = -\lambda - 7, b_1 = -4\lambda + 2,$

$c_1 = 4\lambda - 8$ [DR's of line CD]

and $a_2 = -1, b_2 = -4, c_2 = 4$

[DR's of line AB]

$\Rightarrow \quad -1(-\lambda - 7) - 4(-4\lambda + 2) + 4(4\lambda - 8) = 0$

$\Rightarrow \quad \lambda + 7 + 16\lambda - 8 + 16\lambda - 32 = 0$

$\Rightarrow \quad 33\lambda - 33 = 0$

$\Rightarrow \quad 33\lambda = 33$

$\therefore \quad \lambda = 1$ **(1)**

On putting $\lambda = 1$ in Eq. (ii), we get required foot of perpendicular,

$$D = (-1, 2, -5)$$

Also, we have to find equation of line CD, where, $C\ (7, 4, -1)$ and $D\ (-1, 2, -5)$.

$\therefore$ Required equation of line is

$$\frac{x-7}{-1-7} = \frac{y-4}{2-4} = \frac{z+1}{-5+1} \text{ [using Eq. (i)]}$$

$$\Rightarrow \quad \frac{x-7}{-8} = \frac{y-4}{-2} = \frac{z+1}{-4}$$

$$\Rightarrow \quad \frac{x-7}{4} = \frac{y-4}{1} = \frac{z+1}{2}$$ **(1)**

[dividing denominator by −2]

48. Find the image of the point (1, 6, 3) on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. Also, write the equation of the line joining the given points and its image and find the length of segment joining given point and its image. **Delhi 2010C**

Firstly, find the coordinates of foot of perpendicular Q. Then, find the image which is point T by using the fact that Q is the mid-point of line PT.

Let T be the image of the point $P(1, 6, 3)$. Q is the foot of perpendicular PQ on the line AB.

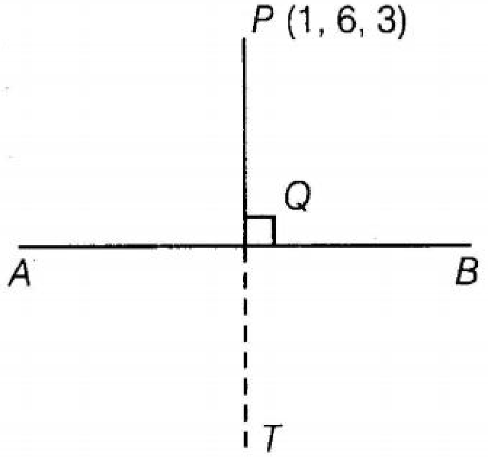
Given equation of line AB is

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} \quad \text{...(i)}$$

Let $$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda \quad \text{[say]}$$

$\Rightarrow$ $x = \lambda, y - 1 = 2\lambda, z - 2 = 3\lambda$

$\Rightarrow$ $x = \lambda, y = 2\lambda + 1, z = 3\lambda + 2$



Then, coordinates of

$$Q = (\lambda, 2\lambda + 1, 3\lambda + 2) \quad \text{...(ii)} \; \mathbf{(1)}$$

Now, DR's of line

$$PQ = (\lambda - 1, 2\lambda + 1 - 6, 3\lambda + 2 - 3)$$

$\Rightarrow$ DR's of $PQ = (\lambda - 1, 2\lambda - 5, 3\lambda - 1)$

Since, line $PQ \perp AB$.

Therefore, $a_1a_2 + b_1b_2 + c_1c_2 = 0$,

where $a_1 = \lambda - 1, b_1 = 2\lambda - 5, c_1 = 3\lambda - 1$

and $a_2 = 1, b_2 = 2, c_2 = 3$

$\therefore$ $1(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$

$\Rightarrow$ $\lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0$

$\Rightarrow$ $14\lambda - 14 = 0$

$\Rightarrow$ $\lambda = 1$ **(1)**

On putting $\lambda = 1$ in Eq. (ii), we get

$$Q\,(1, 2+1, 3+2) = (1, 3, 5)$$

Now, Q is the mid-point of PT.

Let coordinates of $T = (x, y, z)$

By using mid-point formula, **(1)**

$$Q = \text{mid-point of } P\,(1, 6, 3) \text{ and } T\,(x, y, z)$$

$$= \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right)$$

$$\left[\because \text{mid-point} = \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right]$$

But $Q = (1, 3, 5)$

$$\therefore \quad \left(\frac{x+1}{2}, \frac{y+6}{2}, \frac{z+3}{2}\right) = (1, 3, 5)$$

On equating corresponding coordinates, we get

$$\frac{x+1}{2} = 1, \frac{y+6}{2} = 3, \frac{z+3}{2} = 5$$

$$\Rightarrow \quad x = 2-1, y = 6-6, z = 10-3$$

$$\Rightarrow \quad x = 1, y = 0, z = 7$$

$\therefore$ Coordinates of $T = (x, y, z) = (1, 0, 7)$

Hence, coordinates of image of point $P(1, 6, 3)$ is $T\,(1, 0, 7)$. **(1)**

Now, equation of line joining points $P(1, 6, 3)$ and $T(1, 0, 7)$ is

$$\frac{x-1}{1-1} = \frac{y-6}{0-6} = \frac{z-3}{7-3}$$

$$\Rightarrow \quad \frac{x-1}{0} = \frac{y-6}{-6} = \frac{z-3}{4} \quad \textbf{(1)}$$

Also, length of segment PT

$$= \sqrt{(1-1)^2 + (6-0)^2 + (3-7)^2}$$

$$= \sqrt{0+36+16} = \sqrt{52} \text{ units} \quad \textbf{(1)}$$

49. Write the vector equations of following lines and hence find the distance between them

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}, \quad \frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12}.$$

Delhi 2010

Given equations of lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6}$$

and $$\frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12} \quad \textbf{(1)}$$

Now, the vector equation of lines are

$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k}) \quad \ldots(i)$$

[$\because$ vector form of equation of line is $\vec{r}=\vec{a}+\lambda\vec{b}$]

and $\vec{r}=(3\hat{i}+3\hat{j}-5\hat{k})+\mu(4\hat{i}+6\hat{j}+12\hat{k})\ldots(ii)$

Here, $\vec{a_1}=\hat{i}+2\hat{j}-4\hat{k},\ \vec{b_1}=2\hat{i}+3\hat{j}+6\hat{k}$

and $\vec{a_2}=3\hat{i}+3\hat{j}-5\hat{k},\ \vec{b_2}=4\hat{i}+6\hat{j}+12\hat{k}$

Then, $\vec{a_2}-\vec{a_1}=(3\hat{i}+3\hat{j}-5\hat{k})-(\hat{i}+2\hat{j}-4\hat{k})$

$$=2\hat{i}+\hat{j}-\hat{k} \quad \ldots(iii)\ \textbf{(1)}$$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 6\\ 4 & 6 & 12\end{vmatrix}$$

$$=\hat{i}(36-36)-\hat{j}(24-24)+\hat{k}(12-12)$$

$$=0\hat{i}-0\hat{j}+0\hat{k}=\vec{0} \quad \textbf{(1)}$$

$\Rightarrow \quad \vec{b_1}\times\vec{b_2}=\vec{0}$

$\Rightarrow$ Vector $\vec{b_1}$ is parallel to $\vec{b_2}$

[$\because$ if $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} \parallel \vec{b}$]

As, two lines are parallel.

$$\therefore \quad \vec{b} = (2\hat{i} + 3\hat{j} + 6\hat{k}) \quad \text{...(iv)}$$

[since, DR's of given lines are proportional] **(1)**

Since, the two lines are parallel, we use the formula for shortest distance between two parallel lines.

We know that,

$$\text{shortest distance, } d = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right| \quad \text{...(v)}$$

From Eqs. (iii), (iv) and (v), we get

$$d = \left| \frac{(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})}{\sqrt{(2)^2 + (3)^2 + (6)^2}} \right| \text{...(vi)}$$

Now, $(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (2\hat{i} + \hat{j} - \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(-3 - 6) - \hat{j}(-2 - 12) + \hat{k}(2 - 6)$$

$$= -9\hat{i} + 14\hat{j} - 4\hat{k} \quad \textbf{(1)}$$

From Eq. (vi), we get

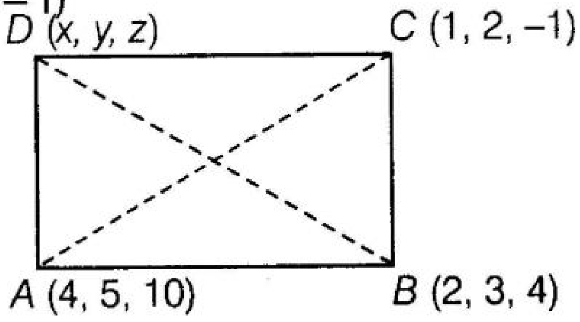
$$d = \left| \frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{\sqrt{49}} \right| = \frac{\sqrt{(-9)^2 + (14)^2 + (-4)^2}}{7}$$

$$\Rightarrow d = \frac{\sqrt{81 + 196 + 16}}{7} = \frac{\sqrt{293}}{7} \text{ units} \quad \textbf{(1)}$$

50. The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of parallelogram $ABCD$. Find the vector equations of sides AB and BC and also find coordinates of point D. HOTS; Delhi 2010

> The vector equation of a side of a parallelogram, when two points are given, is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$. Also, the diagonals of a rectangle intersect each other at mid-point.

Given points are $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$



We know that, two points vector form of line is given by

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \qquad \text{...(i)} \ \mathbf{(1)}$$

where, $\vec{a}$ and $\vec{b}$ are the position vector of points through which the line is passing through. Here, for line AB position vectors

are $\qquad \vec{a} = \vec{OA} = 4\hat{i} + 5\hat{j} + 10\hat{k}$

and $\qquad \vec{b} = \vec{OB} = 2\hat{i} + 3\hat{j} + 4\hat{k} \qquad \mathbf{(1)}$

Using Eq. (i), the required equation of line AB is

$$\vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) + \lambda[2\hat{i} + 3\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + 10\hat{k})]$$

$$\Rightarrow \quad \vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) - \lambda(2\hat{i} + 2\hat{j} + 6\hat{k}) \quad \mathbf{(1)}$$

Similarly, vector equation of line BC, where $B(2, 3, 4)$ and $C(1, 2, -1)$ is

$$\vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \mu[\hat{i} + 2\hat{j} - \hat{k} - (2\hat{i} + 3\hat{j} + 4\hat{k})]$$

$$\Rightarrow \quad \vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - \mu(\hat{i} + \hat{j} + 5\hat{k}) \quad \mathbf{(1)}$$

We know that, mid-point of diagonal BD
= Mid-point of diagonal AC

[∵ diagonal of a parallelogram bisect each other]

$$\therefore \left(\frac{x+2}{2}, \frac{y+3}{2}, \frac{z+4}{2}\right) = \left(\frac{4+1}{2}, \frac{5+2}{2}, \frac{10-1}{2}\right)$$ **(1)**

On comparing corresponding coordinates, we get

$$\frac{x+2}{2} = \frac{5}{2}, \frac{y+3}{2} = \frac{7}{2}$$

and $\frac{z+4}{2} = \frac{9}{2} \Rightarrow x = 3, y = 4$ and $z = 5$

Hence, coordinates of point

$$D(x, y, z) = (3, 4, 5)$$ **(1)**

and vector equations of sides AB and BC are

$\vec{r} = (4\hat{i} + 5\hat{j} + 10\hat{k}) - \lambda(2\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - \mu(\hat{i} + \hat{j} + 5\hat{k})$,

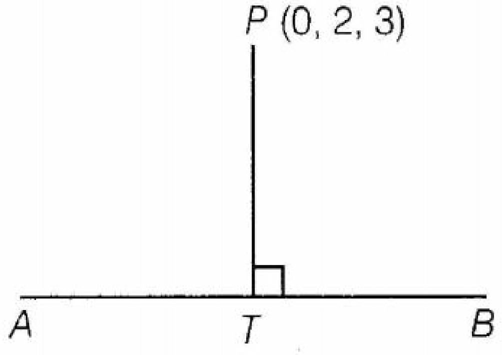
respectively.

51. Find the coordinates of foot of perpendicular drawn from the point (0, 2, 3) on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. Also, find the length of perpendicular. Delhi 2009C

Given equation of line is

$$\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$$

and given point is $P(0, 2, 3)$, let foot of perpendicular PT is T.



Now, $$\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3} = \lambda \quad \text{[say]} \quad \textbf{(1)}$$

$\Rightarrow$ $x = 5\lambda - 3, y = 2\lambda + 1, z = 3\lambda - 4$

$\therefore$ Coordinates of point T are

$(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$ **(1)**

DR's of line

$$PT = (5\lambda - 3 - 0, 2\lambda + 1 - 2, 3\lambda - 4 - 3)$$
$$= (5\lambda - 3, 2\lambda - 1, 3\lambda - 7)$$ **(1)**

Since, $PT \perp AB$

Therefore, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

where, $a_1 = 5\lambda - 3,\ b_1 = 2\lambda - 1,\ c_1 = 3\lambda - 7$

and $\quad a_2 = 5, b_2 = 2, c_2 = 3$ **(1)**

$\therefore\ 5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$

$\Rightarrow \quad 25\lambda - 15 + 4\lambda - 2 + 9\lambda - 21 = 0$

$\Rightarrow \quad 38\lambda - 38 = 0 \Rightarrow 38\lambda = 38$

$\Rightarrow \quad \lambda = 1$ **(1)**

$\therefore$ The foot of perpendicular

$$T = (5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$$

$$= (2, 3, -1) \qquad [\text{put } \lambda = 1]$$ **(1/2)**

Also, length of perpendicular, PT = Distance between points P and T

$\Rightarrow PT = \sqrt{(0-2)^2 + (2-3)^2 + (3+1)^2}$

$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right]$

$$= \sqrt{4 + 1 + 16} = \sqrt{21} \text{ units}$$ **(1/2)**

52. Find the perpendicular distance of the point (2, 3, 4) from the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.
Also, find coordinates of foot of perpendicular. Delhi 2009C

Do same as Que. 51.

$\left[\begin{array}{l}\textbf{Ans. } \text{Perpendicular distance} = \text{Distance} \\ \text{coordinates of foot} = \left(\frac{170}{49}, \frac{78}{49}, \frac{60}{49}\right)\end{array}\right]$

Plane

1 Mark Questions

1. Write the vector equation of the plane passing through the point (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$. **Delhi 2014**

The required plane is passing through the point (a,b,c) whose position vector is $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ and is parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.

So, it is normal to the vector

$$\vec{n} = \hat{i} + \hat{j} + \hat{k}$$

Hence, required equation of plane is

$$(\vec{r} - \vec{p}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{p} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = (a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$$ **(1)**

2. Find the length of the perpendicular drawn from the origin to the plane $2x-3y+6z+21=0$.

All India 2013

> The distance from point (x_1, y_1, z_1) to the plane $Ax+By+Cz+D=0$ is $\dfrac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}$.

Given, equation of plane is

$$2x-3y+6z+21=0 \quad \text{...(i)}$$

$\therefore$ Length of the perpendicular drawn from the origin to this plane

$$=\left|\frac{2\cdot 0-3\cdot 0+6\cdot 0+21}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right|=\left|\frac{0-0+0+21}{\sqrt{4+9+36}}\right|$$

$$=\frac{21}{\sqrt{49}}=\frac{21}{7}$$

$$=3 \text{ units}$$ **(1)**

3. Find distance of the plane $3x-4y+12z=3$ from the origin. **Delhi 2011**

Given equation of plane is

$3x-4y+12z-3=0$ and the point is $(0,0,0)$.

We know that, distance of the plane $Ax+By+Cz+D=0$ from the point (x_1, y_1, z_1) is

$$d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$$

Here, $x_1=y_1=z_1=0$

and $A=3, B=-4, C=12, D=-3$

$\therefore$ Required distance

$$d=\frac{|3(0)-4(0)+12(0)-3|}{\sqrt{(3)^2+(-4)^2+(12)^2}}$$

$$\Rightarrow d=\frac{|-3|}{\sqrt{9+16+144}}=\frac{3}{\sqrt{169}}=\frac{3}{13} \text{ units}$$ **(1)**

4. Write the intercept cut-off by plane $2x + y - z = 5$ on X-axis. **HOTS; Delhi 2011**

> Firstly, we convert the given equation of plane in intercept form, i.e. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, which cut the X-axis at $(a, 0, 0)$.

Given equation of plane is $2x + y - z = 5$.

On dividing both sides by 5, we get

$$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1$$

$$\Rightarrow \quad \frac{x}{\left(\frac{5}{2}\right)} + \frac{y}{5} + \frac{z}{(-5)} = 1$$

On comparing above equation of plane with the intercept form of equation of plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where, a = x-intercept, b = y-intercept and c = z-intercept

We get, $a = \frac{5}{2}$

i.e. intercept cut-off on X-axis $= \frac{5}{2}$ **(1)**

5. Write the distance of following plane from origin, $2x - y + 2z + 1 = 0$. **All India 2010**

Do same as Que. 3 $\left[\textbf{Ans. } \frac{1}{3} \text{ unit}\right]$

6. Find the value of λ, such that the line $\frac{x-2}{6} = \frac{y-1}{\lambda} = \frac{z+5}{-4}$ is perpendicular to the plane $3x - y - 2z = 7$. **All India 2010C**

Given, line $\frac{x-2}{6} = \frac{y-1}{\lambda} = \frac{z+5}{-4}$ is perpendicular to plane $3x - y - 2z = 7$.

Therefore, DR's of the line are proportional to the DR's normal to the plane.

$$\therefore \quad \frac{6}{3} = \frac{\lambda}{-1} = \frac{-4}{-2} \Rightarrow 2 = -\lambda$$

$$\Rightarrow \quad \lambda = -2 \quad \textbf{(1)}$$

4 Marks Questions

7. A plane makes intercepts $-6, 3, 4$ respectively on the coordinate axes. Find the length of the perpendicular from the origin on it. **Delhi 2014C**

Given, intercepts on the coordinate axes are (–6, 3, 4), then equation of plane will be

$$\frac{x}{-6}+\frac{y}{3}+\frac{z}{4}=1$$

or $$\frac{x}{-6}+\frac{y}{3}+\frac{z}{4}-1=0 \quad \textbf{(1)}$$

Distance of a point from plane is given by

$$D=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right| \quad \textbf{(1)}$$

Here, distance of origin from above plane

$$=\left|\frac{\left(\frac{-1}{6}\right)\cdot 0+\left(\frac{1}{3}\right)\cdot 0+\left(\frac{1}{4}\right)\cdot 0-1}{\sqrt{\left(\frac{-1}{6}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{4}\right)^2}}\right|$$

$$=\left|\frac{-1}{\sqrt{\frac{1}{36}+\frac{1}{9}+\frac{1}{16}}}\right|=\left|\frac{-1}{\sqrt{\frac{29}{144}}}\right|=\frac{12}{\sqrt{29}} \quad \textbf{(1)}$$

Hence, required length of the perpendicular from origin to plane is $\frac{12}{\sqrt{29}}$ units. **(1)**

8. Find the distance of the point (–1, – 5, – 10) from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$.

Delhi 2014; All India 2014C, 2011; HOTS

Firstly, convert the given equations of line and plane in cartesian form and then solve them to get their point of intersection, then use distance formula to find the required distance.

Given equations of line and plane are

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda\,(3\hat{i} + 4\hat{j} + 2\hat{k})$$

and $$\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$$

Above equations in cartesian form can be written as

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} \quad \text{...(i)}$$

and $$x - y + z = 5 \quad \text{...(ii)}$$ **(1)**

Let the point of intersection of line (i) and plane (ii) be Q.

Let $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=\lambda$

$\Rightarrow \quad \frac{x-2}{3}=\lambda, \frac{y+1}{4}=\lambda, \frac{z-2}{2}=\lambda$

$\Rightarrow \quad x=3\lambda+2, y=4\lambda-1, z=2\lambda+2$

$\therefore$ Any point Q on the given line is

$Q(3\lambda+2, 4\lambda-1, 2\lambda+2)$ **(1)**

Since, plane also passes through Q, so coordinates of Q will satisfy Eq. (ii).

$\therefore \quad (3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5$

$\Rightarrow \quad 3\lambda+2-4\lambda+1+2\lambda+2=5$

$\Rightarrow \quad \lambda=0$ **(1)**

On putting $\lambda=0$ in $Q(3\lambda+2, 4\lambda-1, 2\lambda+2)$, we get the point of intersection as $Q(2,-1,2)$.

Now, the required distance

$PQ=\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}=13$

$\left[\because \text{distance}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right]$

$=\sqrt{9+16+144}$

$=\sqrt{144+25}$

$=\sqrt{169}$

$=13$ units

Hence, the required distance is 13 units. **(1)**

9. Show that the lines $\frac{5-x}{-4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{2y-8}{2}=\frac{z-5}{3}$ are coplanar.

Delhi 2014

Given lines can be written as

$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$$

and $$\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$$ **(1)**

On comparing both lines with,

$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{b}$ respectively, we get

$x_1=5, y_1=7, z_1=-3, a_1=4, b_1=4, c_1=-5$

and $x_2=8, y_2=4, z_2=5, a_2=7, \quad b_2=1,$

$c_2=3$ **(1)**

If given lines are coplanar, then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$ **(1)**

$$\text{LHS}=\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=\begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$=3(12+5)+3(12+35)+8(4-28)$$

$$=3\times17+3\times47+8(-24)$$

$$=51+141-192=192-192=0=\text{RHS}$$

Therefore, given lines are coplanar. **(1)**

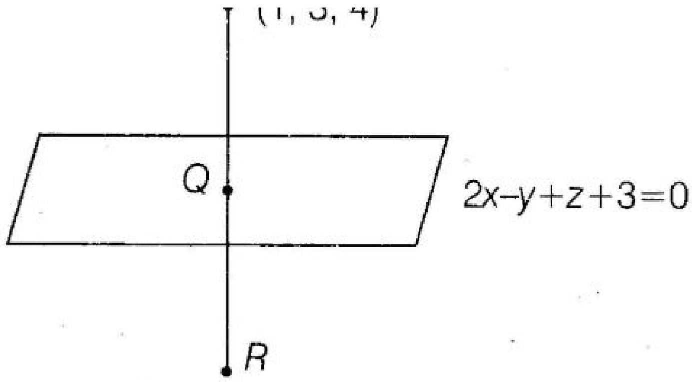
10. Find the image of the point having position vector $\hat{i}+3\hat{j}+4\hat{k}$ in the plane

$\vec{r}(2\hat{i}-\hat{j}+\hat{k})+3=0.$ **All India 2014C**

Given, position vector of point is $(\hat{i}+3\hat{j}+4\hat{k})$.

So, coordinates of point P are $(1, 3, 4)$ and vector equation of plane is $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+3=0$, then cartesian equation of plane is $2x-y+z+3=0$.

Let Q be the foot of perpendicular from P on the plane.



(1)

Since, PQ is perpendicular to the plane. Hence, Dr's of PQ will be 2, –1, 1.

So, equation of PQ will be

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=\lambda \quad \text{[say]}$$

Coordinates of $Q = (2\lambda + 1, -\lambda + 3, \lambda + 4)$, also Q lies on plane, so it will satisfy the equation of plane.

$$\therefore \; 2(2\lambda+1)-1(-\lambda+3)+(\lambda+4)+3=0$$

$$\Rightarrow \quad 4\lambda+2+\lambda-3+\lambda+4+3=0$$

$$\Rightarrow \quad 6\lambda=-6$$

$$\Rightarrow \quad \lambda=-1 \quad \textbf{(1)}$$

So, coordinates of Q will be (–1, 4, 3). Since, Q is the mid-point of PR and let R be (x, y, z),

$$\text{then} \left(\frac{x+1}{2},\frac{y+3}{2},\frac{z+4}{2}\right) \equiv (-1, 4, 3) \quad \textbf{(1)}$$

On comparing corresponding coordinates, we get

$$\frac{x+1}{2}=-1 \Rightarrow x=-2 \Rightarrow x=-3,$$

$$\frac{y+3}{2}=4 \Rightarrow y+3=8 \Rightarrow y=8-3=5$$

$$\text{and } \frac{z+4}{2}=3 \Rightarrow z+4=6 \Rightarrow z=6-4=2$$

Hence, required coordinates of image point R is (–3, 5, 2). **(1)**

11. Find the vector equation of the plane through the points (2, 1 − 1) and (−1, 3, 4) and perpendicular to the plane $x-2y+4z=10$.

All India 2013

The required plane passes through two points $P(2,1,-1)$ and $Q(-1,3,4)$. Let $\vec{a}$ and $\vec{b}$ be the position vectors of points P and Q, respectively.

Then, $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$

and $\vec{b}=-\hat{i}+3\hat{j}+4\hat{k}$

$$\therefore \overrightarrow{PQ}=\vec{b}-\vec{a}=(-\hat{i}+3\hat{j}+4\hat{k})-(2\hat{i}+\hat{j}-\hat{k})$$

$$=-3\hat{i}+2\hat{j}+5\hat{k} \qquad \textbf{(1)}$$

Let $\vec{n_1}$ be the normal vector to the given plane, $x-2y+4z=10$, then $\vec{n_1}=\hat{i}-2\hat{j}+4\hat{k}$.

Let $\vec{n}$ be the normal vector to the required plane. Then,

$$\vec{n}=\vec{n_1}\times\overrightarrow{PQ}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 4\\ -3 & 2 & 5\end{vmatrix}$$

$$=\hat{i}(-10-8)-\hat{j}(5+12)+\hat{k}(2-6)$$

$$=-18\hat{i}-17\hat{j}-4\hat{k} \qquad \textbf{(1)}$$

The required plane passes through a point having position vector $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and is normal to the vector $\vec{n_1}=-18\hat{i}-17\hat{j}-4\hat{k}$. So, its vector equation is

$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n} \qquad \textbf{(1)}$$

$$\Rightarrow \vec{r}\cdot(-18\hat{i}-17\hat{j}-4\hat{k})=(2\hat{i}+\hat{j}-\hat{k})\cdot(-18\hat{i}-17\hat{j}-4\hat{k})$$

$$\Rightarrow \vec{r}\cdot(-18\hat{i}-17\hat{j}-4\hat{k})=-36-17+4$$

$$\therefore \vec{r}\cdot(18\hat{i}+17\hat{j}+4\hat{k})=49 \qquad \textbf{(1)}$$

12. Find the coordinates of the point, where the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ intersects the plane $x-y+z-5=0$. Also, find the angle between the line and the plane. **Delhi 2013**

Given equation of line is

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=\lambda \qquad \text{[say]}$$

$$x=3\lambda+2, y=4\lambda-1, z=2\lambda+2$$

Then, $[(3\lambda+2),(4\lambda-1),(2\lambda+2)]$ be any point on the given line. **(1)**

This point lies on the plane $x-y+z-5=0$

$\therefore \quad (3\lambda+2)-(4\lambda-1)+(2\lambda+2)-5=0$

$\Rightarrow \quad 3\lambda+2-4\lambda+1+2\lambda+2-5=0$

$\Rightarrow \quad \lambda=0 \quad$ ...(i) **(1)**

$\therefore$ Point of intersection of line and the plane

$=(3\times0+2, 4\times0-1, 2\times0+2)=(2,-1,2)$ **(1/2)**

Let θ be the angle between line and plane.

Then,

$$\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

Here, $a=3, b=4, c=2; l=1, m=-1, n=1$.

$$\therefore \quad \sin\theta=\frac{(3)(1)+4(-1)+2(1)}{\sqrt{9+16+4}\sqrt{1+1+1}}$$

$$\Rightarrow \quad \sin\theta=\frac{3-4+2}{\sqrt{29}\sqrt{3}}=\frac{1}{\sqrt{87}}$$

$$\Rightarrow \quad \theta=\sin^{-1}\frac{1}{\sqrt{87}} \qquad \textbf{(1½)}$$

which is the required angle.

13. Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$ and $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$. **HOTS; Delhi 2013**

Let the given equation of plane in cartesian form be $\pi_1 \equiv x+2y+3z-4=0$

and $\pi_2 \equiv 2x+y-z+5=0$

Equation of plane through π_1 and π_2 is

$$(x+2y+3z-4)+k(2x+y-z+5)=0$$

$$\Rightarrow x(1+2k)+y(2+k)+z(3-k)-4+5k=0 \quad \text{...(i)}$$

(1)

This plane is perpendicular to the given plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$ whose cartesian equation is

$$5x+3y-6z+8=0$$

$$\therefore \; 5(1+2k)+3(2+k)-6(3-k)=0$$

$$[\because l_1l_2+m_1m_2+n_1n_2=0]$$

$$\Rightarrow \; 5+10k+6+3k-18+6k=0$$

$$\Rightarrow \; 19k-7=0 \Rightarrow k=\frac{7}{19}$$ **(2)**

On putting $k=\frac{7}{19}$ in Eq. (i), we get the equation of plane as

$$x\left(1+\frac{14}{19}\right)+y\left(2+\frac{7}{19}\right)+z\left(3-\frac{7}{19}\right)-4+\frac{35}{19}=0$$

$$\Rightarrow x\left(\frac{33}{19}\right)+y\left(\frac{45}{19}\right)+z\left(\frac{50}{19}\right)-\frac{41}{19}=0$$

$$\Rightarrow \; 33x+45y+50z-41=0$$ **(1)**

14. Find the equation of plane(s) passing through the intersection of planes $x + 3y + 6 = 0$ and $3x - y - 4z = 0$ and whose perpendicular distance from origin is unity.

HOTS; All India 2010C

Firstly, write the required equation of plane as $(x+3y+6)+\lambda(3x-y-4z)=0$.
Then, convert the above equation in general form of plane which is $ax+by+c+d=0$.
Finally, use the formula for distance from a point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$.

Let the required equation of plane passing through the intersection of planes $x + 3y + 6 = 0$ and $3x - y - 4z = 0$ be

$$(x + 3y + 6) + \lambda (3x - y - 4z) = 0 \quad \text{...(i)}$$

Above equation can be written as

$$x + 3y + 6 + 3\lambda x - \lambda y - 4\lambda z = 0$$

$$\Rightarrow x(1+3\lambda) + y(3-\lambda) - 4\lambda z + 6 = 0 \quad \text{...(ii)} \;\mathbf{(1)}$$

which is the general form of equation of plane.

Also, given that perpendicular distance of plane (i) from origin, i.e. (0, 0, 0) is unity, i.e. one.

$$\therefore \left| \frac{(1+3\lambda)(0) + (3-\lambda)(0) - 4\lambda(0) + 6}{\sqrt{(1+3\lambda)^2 + (3-\lambda)^2 + (-4\lambda)^2}} \right| = 1$$

$$\left[\begin{array}{l} \because \text{distance of point } (x_1, y_1, z_1) \text{ from a} \\ \text{plane } ax + by + cz + d = 0 \text{ is given by} \\ d = \left| \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right| \\ \text{here, } a = 1 + 3\lambda, b = 3 - \lambda, c = -4\lambda, \\ (x_1, y_1, z_1) = (0, 0, 0) \end{array}\right] \quad \mathbf{(1)}$$

$$\Rightarrow \left| \frac{6}{\quad} \right| = 1$$

$$\sqrt{1+9\lambda^2+6\lambda+9+\lambda^2-6\lambda+16\lambda^2}$$

$$\Rightarrow \quad \frac{6}{\sqrt{26\lambda^2+10}}=1$$

$$\Rightarrow \quad 6=\sqrt{26\lambda^2+10}$$

On squaring both sides, we get

$$36=26\lambda^2+10$$

$$\Rightarrow \quad 26\lambda^2=26 \Rightarrow \lambda^2=1 \Rightarrow \lambda=\pm 1 \quad \textbf{(1)}$$

Now, on putting $\lambda=1$ in Eq. (i), we get

$$x+3y+6+3x-y-4z=0$$

$$\Rightarrow \quad 4x+2y-4z+6=0$$

$$\Rightarrow \quad 2x+y-2z+3=0 \quad \text{...(iii)}$$

Again, on putting $\lambda=-1$ in Eq. (i), we get

$$x+3y+6-3x+y+4z=0$$

$$\Rightarrow \quad -2x+4y+4z+6=0$$

$$\Rightarrow \quad x-2y-2z-3=0 \quad \text{...(iv)}$$

Eqs. (iii) and (iv) are the required equations of the plane. **(1)**

15. Find the equation of plane passing through the point $A(1, 2, 1)$ and perpendicular to the line joining points $P(1, 4, 2)$ and $Q(2, 3, 5)$. Also, find distance of this plane from the line $\frac{x+3}{2}=\frac{y-5}{-1}=\frac{z-7}{-1}$. **Delhi 2010C**

Equation of plane passing through the point $A(1, 2, 1)$ is given as

$$a(x-1)+b(y-2)+c(z-1)=0 \quad \text{...(i)} \ \textbf{(1)}$$

[$\because$ equation of plane passing through (x_1, y_1, z_1) having DR's a, b, c is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$]

Now, DR's of line PQ, where $P(1, 4, 2)$ and $Q(2, 3, 5)$ are $2-1, 3-4, 5-2$, i.e. $1, -1, 3$.

Since, plane (i) is perpendicular to line PQ.

$\therefore$ DR's of plane (i) are $1, -1, 3$

i.e. $a = 1, b = -1, c = 3$ **(1)**

On putting values of a, b and c in Eq. (i), we get the required equation of plane as

$$1(x-1) - 1(y-2) + 3(z-1) = 0$$

$$\Rightarrow \quad x - 1 - y + 2 + 3z - 3 = 0$$

$$\Rightarrow \quad x - y + 3z - 2 = 0 \quad \text{...(ii)}$$

Now, the given equation of line is

$$\frac{x+3}{2} = \frac{y-5}{-1} = \frac{z-7}{-1} \quad \text{...(iii)} \ \mathbf{(1)}$$

DR's of this line are (2, −1, −1) and point (−3, 5, 7).

Now, From Eqs. (ii) and (iii), we get

$$\therefore \quad 2(1) - 1(-1) - 1(3) = 2 + 1 - 3 = 0$$

[by using $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$]

So, line (iii) is parallel to plane (i).

∴ The required distance = Distance of the point (−3, 5, 7) from the plane (ii)

$$\Rightarrow \quad d = \left| \frac{(-3)(1) + (5)(-1) + 7(3) - 2}{\sqrt{(1)^2 + (-1)^2 + (3)^2}} \right|$$

$$\left[\begin{array}{l} \because \text{distance of the point } (x_1, y_1, z_1) \\ \text{to the plane } ax + by + cz + d = 0 \text{ is} \\ d = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \end{array} \right]$$

$$= \left| \frac{-3 - 5 + 21 - 2}{\sqrt{1+1+9}} \right| = \left| \frac{11}{\sqrt{11}} \right| = \left| \frac{(\sqrt{11})^2}{\sqrt{11}} \right|$$

$= \sqrt{11}$ units **(1)**

16. Find the cartesian equation of the plane passing through points $A(0, 0, 0)$ and $B(3, -1, 2)$ and parallel to the line $\dfrac{x-4}{1} = \dfrac{y+3}{-4} = \dfrac{z+1}{7}$. HOTS; Delhi 2010

The equation of any plane passing through (x_1, y_1, z_1) is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0.$

This plane is parallel to the line

$$\frac{x-x_2}{a_1}=\frac{y-y_2}{b_1}=\frac{z-z_2}{c_1}.$$

∴ Normal to the plane is perpendicular to the line, i.e. $aa_1+bb_1+cc_1=0$. Use these results and solve it.

Equation of plane passing through the point $A(0, 0, 0)$ is

$$a(x-0)+b(y-0)+c(z-0)=0$$

$$\Rightarrow \quad ax+by+cz=0 \quad \text{...(i)}\textbf{(1)}$$

[using one point form of plane

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$]

Given, the plane (i) passes through the point $B(3, -1, 2)$.

∴ Put $x=3$, $y=-1$ and $z=2$ in Eq. (i), we get

$$3a-b+2c=0 \quad \text{...(ii)}$$

Also, the plane (i) is parallel to the line

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$$

$$\therefore \quad a(1)+b(-4)+c(7)=0$$

$$\left[\begin{array}{l}\text{if plane is parallel to the line, then normal}\\ \text{to the plane is perpendicular to the line.}\\ \therefore a_1a_2+b_1b_2+c_1c_2=0\end{array}\right]$$

$$\Rightarrow \quad a-4b+7c=0 \quad \text{...(iii)}\ \textbf{(1)}$$

Now, on multiplying Eq. (iii) by 3 and subtracting it from Eq. (ii), we get

$$\begin{array}{r}3a-\ \ \ b+\ \ 2c=0\\ 3a-12b+21c=0\\ -\quad+\quad\ -\quad\quad\\ \hline 11b-19c=0\end{array}$$

$$\Rightarrow \quad b=\frac{19}{11}c$$

On putting $b=\frac{19c}{11}$ in Eq. (ii), we get

$$3a-\frac{19c}{11}+2c=0 \Rightarrow 3a+\frac{-19c+22c}{11}=0$$

$$\Rightarrow \quad 3a + \frac{3c}{11} = 0 \Rightarrow 3a = -\frac{3c}{11}$$

$$\therefore \quad a = -\frac{c}{11} \qquad \textbf{(1)}$$

Now, putting $a = -\frac{c}{11}$ and $b = \frac{19}{11}c$ in Eq. (i),

we get the required equation of plane as

$$\frac{-c}{11}x + \frac{19c}{11}y + cz = 0$$

$$\Rightarrow -\frac{x}{11} + \frac{19y}{11} + z = 0$$

[dividing both sides by c]

$$\Rightarrow \quad -x + 19y + 11z = 0$$

[multiplying both sides by 11]

$$\Rightarrow \quad x - 19y - 11z = 0 \qquad \textbf{(1)}$$

17. Find the equation of plane that contains the point (1, −1, 2) and is perpendicular to each of planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.

Delhi 2009C

> The equation of any plane passing through (x_1, y_1, z_1) is
> $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$
> and if it is perpendicular to the planes
> $a_1x+b_1y+c_1z+d_1=0$ and
> $a_2x+b_2y+c_2z+d_2=0$.
> Then, $aa_1+bb_1+cc_1=0$
> and $aa_2+bb_2+cc_2=0$.
> Use these results and solve it.

Equation of plane passing through point (1, −1, 2) is given by

$$a(x-1)+b(y+1)+c(z-2)=0 \quad \ldots(i)\ \mathbf{(1)}$$

Now, given that plane (i) is perpendicular to planes

$$2x+3y-2z=5 \quad \ldots(ii)$$

and $$x+2y-3z=8 \quad \ldots(iii)$$

We know that, when two planes

$a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$ are perpendicular, then

$$a_1a_2+b_1b_2+c_1c_2=0 \quad \mathbf{(1)}$$

$\therefore$ $2a+3b-2c=0$ [from Eqs. (i) and (ii)]

and $a+2b-3c=0$ [from Eqs. (i) and (iii)]

$\Rightarrow$ $$2a+3b=2c \quad \ldots(iv)$$

and $$a+2b=3c \quad \ldots(v)\mathbf{(1)}$$

On multiplying Eq. (v) by 2 and subtracting it from Eq. (iv), we get

$$\begin{array}{r} 2a+3b=2c \\ \underline{\overset{}{-}2a\pm 4b=\overset{}{-}6c} \\ -b=-4c \end{array}$$

$\Rightarrow$ $$b=4c$$

On putting $b=4c$ in Eq. (v), we get

$$a+8c=3c$$

$\Rightarrow$ $$a=-5c$$

Now, on putting $a = -5c$ and $b = 4c$ in Eq. (i), we get the required equation of plane as

$$-5c(x-1) + 4c(y+1) + c(z-2) = 0$$

$$\Rightarrow \quad -5(x-1) + 4(y+1) + (z-2) = 0$$

[dividing both sides by c]

$$\Rightarrow \quad -5x + 5 + 4y + 4 + z - 2 = 0$$

$$\Rightarrow \quad 5x - 4y - z - 7 = 0 \qquad \textbf{(1)}$$

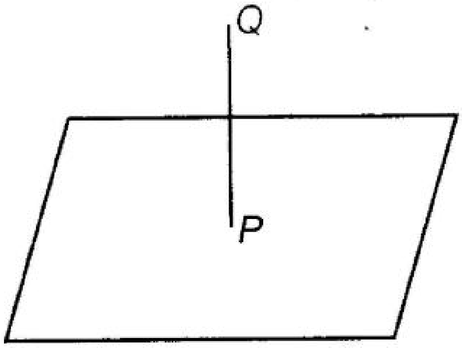
18. Find the coordinates of point, where the line $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ meets the plane $x + y + 4z = 6$. **All India 2008**

Given equation of line is

$$\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4},$$

then $$\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4} = \lambda \quad \text{[say]} \ \textbf{(1)}$$

$$\Rightarrow \quad x = 2\lambda - 1, y = 3\lambda - 2, z = 4\lambda - 3$$



$\therefore$ Any point on the line is

$$P(2\lambda - 1, 3\lambda - 2, 4\lambda - 3) \qquad \textbf{(1)}$$

Now, as point P lies on the plane. So, it will satisfy the given equation of plane which is

$$x + y + 4z = 6$$

$$\therefore \quad (2\lambda - 1) + (3\lambda - 2) + 4(4\lambda - 3) = 6 \qquad \textbf{(1)}$$

$$\Rightarrow \quad 2\lambda - 1 + 3\lambda - 2 + 16\lambda - 12 = 6$$

$$\Rightarrow \quad 21\lambda - 21 = 0 \Rightarrow 21\lambda = 21$$

$$\Rightarrow \quad \lambda = 1$$

On putting $\lambda = 1$ in point P, we get the required point $P(2-1, 3-2, 4-3) = P(1, 1, 1)$.

(1)

6 Marks Questions

19. Find the equation of the plane passing through the line of intersection of the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0$ and parallel to X-axis. **All India 2014C, 2011**

Given equation of planes are

$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0$

Above equations can be written in cartesian form as

$$x+y+z-1=0 \quad \ldots(i)$$

and $$2x+3y-z+4=0 \quad \ldots(ii)$$ **(1)**

Let the required equation of plane passing through the line of intersection of planes (i) and (ii) be

$(x+y+z-1)+\lambda(2x+3y-z+4)=0$

$\Rightarrow x+y+z-1+2\lambda x+3\lambda y-\lambda z+4\lambda=0$

$\Rightarrow x(1+2\lambda)+y(1+3\lambda)+z(1-\lambda)$

$+(-1+4\lambda)=0 \ldots(iii)$ **(1)**

$\therefore$ DR's of the above planes are $1+2\lambda, 1+3\lambda$, $1-\lambda$.

Also, DR's of the X-axis are (1, 0, 0). **(1)**

Also, given that the above plane (iii) is parallel to the X-axis.

$$\therefore \quad 1(1+2\lambda)+0\,(1+3\lambda)+0\,(1-\lambda)=0 \quad \textbf{(1)}$$

$$\left[\begin{array}{ll} \because & a_1a_2+b_1b_2+c_1c_2=0 \\ \text{where,} & a_1=1+2\lambda, b_1=1+3\lambda, c_1=1-\lambda \\ \text{and} & a_2=1, b_2=0, c_2=0 \end{array}\right]$$

$$\Rightarrow \quad 1+2\lambda=0 \Rightarrow 2\lambda=-1$$

$$\Rightarrow \quad \lambda=-\frac{1}{2} \quad \textbf{(1)}$$

On putting $\lambda=-1/2$ in Eq. (iii), we get the required equation of plane as

$$x\left(1-\frac{2\times 1}{2}\right)+y\left(1-\frac{3}{2}\right)+z\left(1+\frac{1}{2}\right)+\left(-1-\frac{4}{2}\right)=0$$

$$\Rightarrow \frac{-y}{2}+\frac{3z}{2}-\frac{6}{2}=0 \Rightarrow y-3z+6=0$$

$\therefore$ The vector equation of plane is

$$\vec{r}\cdot(\hat{j}-3\hat{k})+6=0. \quad \textbf{(1)}$$

20. Find the distance between the point (7, 2, 4) and the plane determined by the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$.

Delhi 2014

Given points are $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$.

Consider $(x_1, y_1, z_1) = A\ (2, 5, -3)$

$(x_2, y_2, z_2) = B\ (-2, -3, 5)$

and $(x_3, y_3, z_3) = C\ (5, 3, -3)$ **(1)**

Now, equation of plane passing through three collinear points is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$ **(1)**

On putting the values of three points , we get

$$\begin{vmatrix} x - 2 & y - 5 & z + 3 \\ -2 - 2 & -3 - 5 & 5 + 3 \\ 5 - 2 & 3 - 5 & -3 + 3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x - 2 & y - 5 & z + 3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$ **(1)**

$$\Rightarrow \quad (x - 2)(0 + 16) - (y - 5)(0 - 24) + (z + 3)(8 + 24) = 0$$

$$\Rightarrow 16x - 32 + 24y - 120 + 32z + 96 = 0$$
$$\Rightarrow \quad 16x + 24y + 32z - 56 = 0$$
$$\Rightarrow \quad 2x + 3y + 4z - 7 = 0 \quad \text{...(i)} \quad \textbf{(1)}$$

Now, distance between the plane (i) and the point (7, 2, 4) is

$$d = \left| \frac{2(7) + 3(2) + 4(4) - 7}{\sqrt{(2)^2 + (3)^2 + (4)^2}} \right| \quad \textbf{(1)}$$

$$\left[\begin{array}{l} \because \text{distance between the plane} \\ \quad ax + by + cz + d = 0 \text{ and the point} \\ \quad (x_1 y_1 z_1) \text{ is } \left| \dfrac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}} \right| \end{array} \right]$$

$$= \left| \frac{14 + 6 + 16 - 7}{\sqrt{29}} \right|$$

$$= \frac{29}{\sqrt{29}}$$

$$= \sqrt{29} \text{ units} \quad \textbf{(1)}$$

21. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$, which is perpendicular to the plane $x - y + z = 0$. Also, find the distance of the plane obtained above, from the origin. **All India 2014**

Equation of any plane through the line of intersection of the given planes $x + y + z = 1$ and $2x + 3y + 4z = 5$, is

$$(x + y + z - 1) + \lambda (2x + 3y + 4z - 5) = 0$$

$$\Rightarrow \quad (1 + 2\lambda) x + (1 + 3\lambda) y + (1 + 4\lambda) z - 1 - 5\lambda = 0 \quad \text{...(i)} \quad \textbf{(1)}$$

The direction ratios a_1, b_1, c_1 of the plane are $(2\lambda + 1), (3\lambda + 1)$ and $(4\lambda + 1)$.

Also, given that the plane, i.e. Eq. (i) is perpendicular to the plane $x - y + z = 0$, whose direction ratios a_2, b_2, c_2 are $1, -1$ and 1. **(1)**

Then, $$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\Rightarrow \quad 1(1 + 2\lambda) - 1(1 + 3\lambda) + 1(1 + 4\lambda) = 0$$

$$\Rightarrow \quad 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$$

$$\Rightarrow \quad 3\lambda = -1 \Rightarrow \lambda = -\frac{1}{3} \quad \textbf{(1)}$$

On substituting the value of λ in Eq. (i), we get the equation required plane as

$$\left(1 - \frac{2}{3}\right) x + \left(1 - \frac{3}{3}\right) y + \left(1 - \frac{4}{3}\right) z - 1 + \frac{5}{3} = 0$$

$$\Rightarrow \quad \frac{1}{3} x - \frac{1}{3} z + \frac{2}{3} = 0$$

$$\Rightarrow \quad x - z + 2 = 0 \quad \textbf{(1)}$$

Now, we know that, distance between a point $P(x_1, y_1, z_1)$ and plane $Ax + By + Cz = D$ is

$$d = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} \quad \textbf{(1)}$$

Here, point is $(0, 0, 0)$ and the plane is $x - z + 2 = 0$.

$\therefore$ Required distance,

$$d = \left| \frac{1 \times 0 + 0 + (-1) \times 0 + 2}{\sqrt{(1)^2 + (-1)^2}} \right| = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ unit}$$

(1)

22. Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$. **All India 2014**

Given equation of line is

$$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad \text{...(i)}$$

and the equation of plane is

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \text{...(ii)}$$

(1)

For point of intersection of Eqs. (i) and (ii), we get

$$[2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})] \cdot [\hat{i} - 2\hat{j} + \hat{k}] = 0 \quad \textbf{(1)}$$

[on putting the value of $\vec{r}$ from Eq. (i) to Eq. (ii)]

$$\Rightarrow \quad 2 + 8 + 2 + 3\lambda - 8\lambda + 2\lambda = 0$$

$$\Rightarrow \quad 12 - 3\lambda = 0$$

$$\Rightarrow \quad -3\lambda = -12 \Rightarrow \lambda = 4 \quad \textbf{(1)}$$

On putting $\lambda = 4$ in Eq. (i), we get

$$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + 4(3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$= 14\hat{i} + 12\hat{j} + 10\hat{k} \quad \textbf{(1)}$$

Since, $\vec{r}$ is the position vector of the point (14, 12, 10). **(1)**

$\therefore$ Distance between the points (2, 12, 5) and (14, 12, 10)

$$= \sqrt{(2-14)^2 + (12-12)^2 + (5-10)^2}$$

$$= \sqrt{(-12)^2 + (0)^2 + (-5)^2}$$

$$= \sqrt{144 + 25}$$

$$= \sqrt{169}$$

$$= 13 \text{ units} \quad \textbf{(1)}$$

23. Find the equation of the plane that contains the point (1, −1, 2) and is perpendicular to both the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$. Hence, find the distance of point $P(-2, 5, 5)$ from the plane obtained above. **Foreign 2014**

Given point is (1, −1, 2) whose position vector is $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$.

Also, given equation of planes are

$2x + 3y - 2z$ and $x + 2y - 3z = 8$ **(1)**

Vector forms of these planes are

$\vec{N_1} = 2\hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{N_2} = \hat{i} + 2\hat{j} - 3\hat{k}$. **(1)**

Now, required planes is prependicular to given planes, so the normal vector of the required plane

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -2 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= \hat{i}(-9 + 4) - \hat{j}(-6 + 2) + \hat{k}(4 - 3)$$

$$= -5\hat{i} + 4\hat{j} + \hat{k}$$ **(1)**

So, the equation of the required plane is

$(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$\Rightarrow [\vec{r} - (\hat{i} - \hat{j} + 2\hat{k})] \cdot (-5\hat{i} + 4\hat{j} + \hat{k}) = 0$ **(1)**

$\Rightarrow \vec{r}(-5\hat{i} + 4\hat{j} + \hat{k}) + 7 = 0$ is the vector equation and $5x - 4y - z = 7$ is the cartesian equation of required plane. **(1)**

Also, the distance of point $P(-2, 5, 5)$ from the plane obtained $= \left| \dfrac{5(-2) - 4(5) - (5) - 7}{\sqrt{25 + 16 + 1}} \right|$

$$= \left| \frac{-42}{\sqrt{42}} \right| = \sqrt{42} \text{ units}$$ **(1)**

24. Show that the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ are coplanar. Also, find the equation of the plane containing them. **Delhi 2013C**

Given lines are $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$

and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

On comparing both equations of lines with $\vec{r} = \vec{a} + \lambda \vec{b}$ respectively, we get

$$\vec{a_1} = \hat{i} + \hat{j} - \hat{k}, \vec{b_1} = 3\hat{i} - \hat{j}$$

and $\vec{a_2} = 4\hat{i} - \hat{k}, \vec{b_2} = 2\hat{i} + 3\hat{k}$ **(1)**

then, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$

$$= \hat{i}(-3 - 0) - \hat{j}(9 - 0) + \hat{k}(0 + 2)$$

$= -3\hat{i} - 9\hat{j} + 2\hat{k}$ **(1)**

and $\vec{a_2} - \vec{a_1} = (4\hat{i} - \hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = 3\hat{i} - \hat{j}$ **(1)**

Now, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (3\hat{i} - \hat{j})$

$$\cdot(-3\hat{i} - 9\hat{j} + 2\hat{k})$$

$$= -9 + 9 = 0$$

Hence, given lines are coplanar. **(1)**

Now, cartesian equations of given lines are

$$\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0} \text{ and}$$

$$\frac{x-4}{2} = \frac{y-0}{0} = \frac{z+1}{3},$$

Then, equation of plane containing them is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-1 & y-1 & z+1 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0$$ **(1)**

$$\Rightarrow \quad (x-1)(-3-0) - (y-1)(9-0) + (z+1)(0+2) = 0$$

$$\Rightarrow \quad -3x + 3 - 9y + 9 + 2z + 2 = 0$$

$$\therefore \quad 3x + 9y - 2z = 14$$ **(1)**

25. Find the distance of the point (1, −2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}$. Delhi 2013C

Let $P(1, -2, 3)$ be the given point.

Given equation of line is

$$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6} = \lambda \qquad \text{[say]}$$

$\Rightarrow \quad x = 2\lambda + 1, y = 3\lambda + 3, z = -6\lambda - 2$

Any point on the line is $Q(2\lambda + 1, 3\lambda + 3, -6\lambda - 2)$.

Now, direction ratios of PQ are

$(2\lambda + 1 - 1, 3\lambda + 3 + 2, -6\lambda - 2 - 3)$

i.e. $\quad (2\lambda, 3\lambda + 5, -6\lambda - 5)$ **(1)**

According to question, the line PQ is parallel to the plane

$$x - y + z = 5$$

Therefore, normal to the plane is perpendicular to the line.

i.e. $\quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$\therefore \; 2\lambda(1) + (3\lambda + 5)(-1) + (-6\lambda - 5)(1) = 0$ **(1)**

$\Rightarrow \quad 2\lambda - 3\lambda - 5 - 6\lambda - 5 = 0$

$\Rightarrow \quad -7\lambda - 10 = 0 \Rightarrow \lambda = \frac{-10}{7}$ **(1)**

∴ Coordinates of Q are

$$Q\left[2\times\left(\frac{-10}{7}\right)+1, 3\left(\frac{-10}{7}\right)+3, -6\left(\frac{-10}{7}\right)-2\right]$$

i.e. $$Q\left(\frac{-13}{7}, \frac{-9}{7}, \frac{46}{7}\right)$$ **(1)**

Hence, distance between the points $P(1, -2, 3)$ and $Q\left(\frac{-13}{7}, \frac{-9}{7}, \frac{46}{7}\right)$

$$= \sqrt{\left(\frac{-13}{7}-1\right)^2+\left(\frac{-9}{7}+2\right)^2+\left(\frac{46}{7}-3\right)^2}$$ **(1)**

$$\left[\because \text{distance} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right]$$

$$= \sqrt{\left(\frac{-20}{7}\right)^2+\left(\frac{5}{7}\right)^2+\left(\frac{25}{7}\right)^2}$$

$$= \frac{1}{7}\sqrt{400+25+625}$$

$$= \frac{\sqrt{1050}}{7} = \frac{5\sqrt{6\times 7}}{7} = 5\sqrt{\frac{6}{7}} \text{ units}$$ **(1)**

26. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(\hat{i}+3\hat{j})-6=0$ and $\vec{r}\cdot(3\hat{i}-\hat{j}-4\hat{k})=0$, whose perpendicular distance from origin is unity.

All India 2013C

Given planes are $\vec{r}\cdot(\hat{i}+3\hat{j})-6=0$ and $\vec{r}\cdot(3\hat{i}-\hat{j}-4\hat{k})=0$. The equation of any plane passing through the line of intersection of these planes is

$$[\vec{r}\cdot(\hat{i}+3\hat{j})-6]+\lambda[\vec{r}\cdot(3\hat{i}-\hat{j}-4\hat{k})]=0$$

$$\Rightarrow \vec{r}\cdot[(3\lambda+1)\hat{i}+(3-\lambda)\hat{j}-4\lambda\hat{k}]-6=0 \quad \text{...(i)} \quad \textbf{(1)}$$

Given, perpendicular distance of origin from this plane is unity.

$$\therefore \frac{|(3\lambda+1)\times 0+(3-\lambda)\times 0-4\lambda\times 0-6|}{\sqrt{(3\lambda+1)^2+(3-\lambda)^2+(-4\lambda)^2}}=1 \quad \textbf{(1)}$$

$$\Rightarrow \quad 6=\sqrt{9\lambda^2+1+6\lambda+9+\lambda^2-6\lambda+16\lambda^2}$$

$$\Rightarrow \quad 6=\sqrt{26\lambda^2+10} \quad \textbf{(1)}$$

On squaring both sides, we get

$$26\lambda^2+10=36 \Rightarrow 26\lambda^2=26$$

$$\Rightarrow \quad \lambda^2=1 \quad \Rightarrow \quad \lambda=\pm 1 \quad \textbf{(1)}$$

On putting $\lambda=1$ in Eq. (i), we get the required equation of plane as

$$\vec{r}\cdot[(3+1)\hat{i}+(3-1)\hat{j}-4\hat{k}]-6=0$$

$$\Rightarrow \quad \vec{r}(4\hat{i}+2\hat{j}-4\hat{k})-6=0 \quad \textbf{(1)}$$

On putting $\lambda=-1$ in Eq. (i), we get the required equation of plane as

$$\vec{r}[(3(-1)+1)\hat{i}+(3+1)\hat{j}-4(-1)\hat{k}]-6=0$$

$$\Rightarrow \quad \vec{r}[-2\hat{i}+4\hat{j}+4\hat{k}]-6=0 \quad \textbf{(1)}$$

27. Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k})=6$.

All India 2013

Suppose the required line is parallel to vector

$\vec{b}$ which is given by $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

The position vector of the point (1, 2, 3) is

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$

The equation of line passing through (1, 2, 3) and parallel to $\vec{b}$ is given by

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad \text{...(i)}$$

(1)

The equation of the given planes are

$$\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5 \quad \text{...(ii)}$$

and $\quad \vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6 \quad$ ...(iii) **(1)**

The line in Eq. (i) and plane in Eq. (ii) are parallel.

Therefore, the normal to the plane of Eq. (ii) and the given line are perpendicular.

$\therefore \ (\hat{i} - \hat{j} + 2\hat{k}) \cdot \lambda (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = 0$ **(1)**

$$\Rightarrow \quad \lambda(b_1 - b_2 + 2b_3) = 0$$

$$\Rightarrow \quad b_1 - b_2 + 2b_3 = 0 \quad \text{...(iv)}$$

Similarly, from Eqs. (i) and (ii), we get

$$(3\hat{i} + \hat{j} + \hat{k}) \cdot \lambda (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = 0$$

$$\Rightarrow \quad \lambda(3b_1 + b_2 + b_3) = 0$$

$\Rightarrow \quad 3b_1 + b_2 + b_3 = 0 \quad$ ...(v)**(1)**

From Eqs. (iv) and (v), we get

$$\frac{b_1}{(-1)\times 1 - 1\times 2} = \frac{b_2}{2\times 3 - 1\times 1} = \frac{b_3}{1\times 1 - 3(-1)}$$

$\Rightarrow \quad \dfrac{b_1}{-3} = \dfrac{b_2}{5} = \dfrac{b_3}{4}$ **(1)**

Therefore, the direction ratios of $\vec{b}$ are $-3, 5$ and 4.

$\therefore\ b = -3i + 5j + 4k$ [$\because\ b = b_1 i + b_2 j + b_3 k$]

On substituting the value of $\vec{b}$ in Eq. (i), we get

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k}) \quad \textbf{(1)}$$

which is the equation of the required line.

28. Find the coordinates of the point, where the line through (3, –4, – 5) and (2, – 3, 1) crosses the plane, passing through the point (2, 2, 1), (3, 0, 1) and (4, – 1, 0). **Delhi 2013**

Equation of the line passing through the points (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$$\left[\because \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}\right]$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \quad \text{[say]} \ \textbf{(1½)}$$

$\Rightarrow \quad x = -\lambda + 3, y = \lambda - 4, z = 6\lambda - 5$

$\therefore$ Any point on this line be

$$(-\lambda + 3, \lambda - 4, 6\lambda - 5). \quad \textbf{(1/2)}$$

Now, equation of plane passes through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} \quad \textbf{(1)}$$

$\therefore$ Equation of plane passes through the points (2, 2, 1), (3, 0, 1) and (4, –1, 0) is

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 1-1 \\ 4-2 & -1-2 & 0-1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(2-0) - (y-2)(-1-0)$$

$$+(z-1)(-3+4)=0$$

$\Rightarrow \quad 2x-4+y-2+z-1=0$

$\Rightarrow \quad 2x+y+z-7=0$ ...(i)**(1)**

Also, the point of line lies on plane (i).

$\therefore \quad 2(-\lambda+3)+(\lambda-4)+(6\lambda-5)-7=0$

$\Rightarrow \quad -2\lambda+6+\lambda-4+6\lambda-5-7=0$

$\Rightarrow \quad 5\lambda-10=0$

$\Rightarrow \quad \lambda=2$ **(1)**

Hence, point of intersection of line and plane is

$(-2+3, 2-4, 12-5)$, i.e. $(1, -2, 7)$. **(1)**

29. Find the vector equation of the plane passing through the three points with position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also, find the coordinates of the point of intersection of this plane and the line $\vec{r}=3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$. **Delhi 2013**

Given position vectors of three points are $(\hat{i}+\hat{j}-2\hat{k}), 2\hat{i}-\hat{j}+\hat{k}$ and $(\hat{i}+2\hat{j}+\hat{k})$.

We know that, the vector equation of the plane passing through the three points is

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0 \qquad \textbf{(1)}$$

Now, $$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 1 & 2 & 1\end{vmatrix}=-3\hat{i}-\hat{j}+5\hat{k} \ \textbf{(1)}$$

∴ Equation of plane is

$$[\vec{r}-(\hat{i}+\hat{j}-2\hat{k})]\cdot(-3\hat{i}-\hat{j}+5\hat{k})=0$$

$$\Rightarrow \quad \vec{r}\cdot(-3\hat{i}-\hat{j}+5\hat{k})=-3-1-10$$

$$\Rightarrow \quad \vec{r}\cdot(-3\hat{i}-\hat{j}+5\hat{k})=-14$$

$$\Rightarrow \quad \vec{r}\cdot(3\hat{i}+\hat{j}-5\hat{k})=14 \qquad \text{...(i)} \ \textbf{(1½)}$$

Also, given equation of line is

$$\vec{r}=3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k}) \qquad \text{...(ii)}$$

This line intersect the plane (i), so

$$[(3+2\lambda)\hat{i}+(-1-2\lambda)\hat{j}$$
$$+(-1+\lambda)\hat{k}]\cdot(3\hat{i}+\hat{j}-5\hat{k})=14$$

$$\Rightarrow \quad 3\cdot(3+2\lambda)+1\cdot(-1-2\lambda)-5(-1+\lambda)=14$$

$$\Rightarrow \quad 9+6\lambda-1-2\lambda+5-5\lambda=14$$

$$\Rightarrow \quad -\lambda=14-13$$

$$\Rightarrow \quad -\lambda=1$$

$$\therefore \quad \lambda=-1 \qquad \textbf{(1½)}$$

On putting $\lambda=-1$ in Eq. (ii), the required point of intersection is

$$\vec{r}=\hat{i}+\hat{j}-2\hat{k} \qquad \textbf{(1)}$$

30. Find the equation of plane determined by points $A(3,-1,2)$, $B(5,2,4)$ and $C(-1,-1,6)$ and hence find the distance between planes and point $(6,5,9)$.

Delhi 2013, 2012; All India 2009, 2008C

Given points are $A(3,-1,2)$ $B(5,2,4)$, and $C(-1,-1,6)$.

Now, $AB = \sqrt{(5-3)^2 + (2+1)^2 + (4-2)^2}$

$= \sqrt{4+9+4} = \sqrt{17}$ units

$BC = \sqrt{(-1-5)^2 + (-1-2)^2 + (6-4)^2}$

$= \sqrt{36+9+4}$

$= \sqrt{49} = 7$ units

and $CA = \sqrt{(3+1)^2 + (-1+1)^2 + (2-6)^2}$

$= \sqrt{16+0+16}$ **(1)**

$= \sqrt{32}$ units.

$\because$ $AB + BC \neq CA$, so given points are non-collinear. Now, equation of plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Here, $(x_1, y_1, z_1) = (3,-1,2)$,

$(x_2, y_2, z_2) = (5,2,4)$ **(1)**

and $(x_3, y_3, z_3) = (-1,-1,6)$

Equation of plane is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$ **(1)**

$\Rightarrow \quad (x-3)[12-0]-(y+1)(8+8)$
$+(z-2)(0+12)=0$

$\Rightarrow \quad 12(x-3)-16(y+1)+12(z-2)=0$

$\Rightarrow \quad 12x-36-16y-16+12z-24=0$

$\Rightarrow \quad 12x-16y+12z-76=0$

On dividing both sides by 4, we get the required equation of plane as

$$3x-4y+3z-19=0 \qquad \text{...(i)} \ \textbf{(1½)}$$

Now, distance of above plane (i) from point $P(6,5,9)$ is

$$d=\frac{|3(6)+(-4)(5)+(3)(9)-19|}{\sqrt{(3)^2+(-4)^2+(3)^2}}$$

$$\left[\because d=\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}\right]$$

$$=\frac{|18-20+27-19|}{\sqrt{9+16+9}}$$

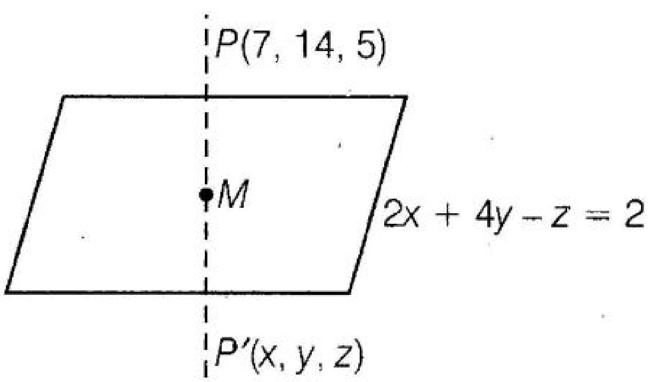
$$=\frac{6}{\sqrt{34}} \text{ units} \qquad \textbf{(1½)}$$

31. Find the length and foot of perpendicular from point $P(7,14,5)$ to plane $2x+4y-z=2$. Also, find the image of point P in the plane.

HOTS; All India 2012

Let $P'(x,y,z)$ be the image of the given point $P(7,14,5)$ and M be the foot of perpendicular lie on the plane $2x+4y-z=2$. The equation of line PM in plane is given by

$$\frac{x-7}{2}=\frac{y-14}{4}=\frac{z-5}{-1}$$



[∵ DR's of a line is proportional to the normal to the plane] **(1)**

Let $\frac{x-7}{2}=\frac{y-14}{4}=\frac{z-5}{-1}=\lambda$ [say]

$\Rightarrow x=2\lambda+7, y=4\lambda+14$ and $z=-\lambda+5$

Let coordinates of point M be

$(2\lambda+7, 4\lambda+14, -\lambda+5)$ ...(i) **(1)**

Since, M lies on the given plane $2x+4y-z=2$.

$$\therefore \quad 2(2\lambda+7)+4(4\lambda+14)-(-\lambda+5)=2$$

$$\Rightarrow \quad 4\lambda+14+16\lambda+56+\lambda-5=2$$

$$\Rightarrow \quad 21\lambda+63=0$$

$$\Rightarrow \quad \lambda=\frac{-63}{21}=-3$$

(1)

On putting $\lambda=-3$ in Eq. (i), we get

$$M=(1,2,8)$$

∴ Foot of perpendicular M is (1, 2, 8). **(1)**

Also, length of perpendicular PM = Distance between points P and M

$$=\sqrt{(1-7)^2+(2-14)^2+(8-5)^2}$$

$$\left[\because \text{distance}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right]$$

$$=\sqrt{36+144+9}$$

$$=\sqrt{189} \text{ units}$$ **(1)**

Now, M = Mid-point of P and P'

$$\Rightarrow \quad (1,2,8)=\left(\frac{x+7}{2}, \frac{y+14}{2}, \frac{z+5}{2}\right)$$

On equating corresponding coordinates, we get

$$\frac{x+7}{2}=1, \frac{y+14}{2}=2 \text{ and } \frac{z+5}{2}=8$$

$\Rightarrow$ $x=2-7, y=4-14$ and $z=16-5$

$\Rightarrow$ $x=-5, y=-10$ and $z=11$

Hence, image of point $P(7,14,5)$ is

$P'(-5,-10,11)$. **(1)**

32. Find the equation of plane which contains the line of intersection of planes

$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0, \vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$

and which is perpendicular to plane

$\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$. **All India 2011**

Given, the required plane contains the line of intersection of planes whose equations are

$$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \quad \text{...(i)}$$

and $$\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \quad \text{...(ii)}$$

Eqs. (i) and (ii) can be written in cartesian form

$$x + 2y + 3z - 4 = 0$$

and $$2x + y - z + 5 = 0 \quad \textbf{(1)}$$

So, the required equation of plane is

$(x + 2y + 3z - 4) + \lambda \quad (2x + y - z + 5) = 0$ ...(iii)

$\Rightarrow x + 2y + 3z - 4 + 2\lambda x + \lambda y - \lambda z + 5\lambda = 0$ **(1/2)**

$\Rightarrow x(1 + 2\lambda) + y(2 + \lambda) + z(3 - \lambda) + (-4 + 5\lambda) = 0$ ...(iv) **(1/2)**

Also, given that plane in Eq. (iv) is perpendicular to the plane

$$\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$$

which in cartesian form can be written as

$$5x + 3y - 6z + 8 = 0 \quad \text{...(v)}$$

$\therefore \quad 5(1 + 2\lambda) + 3(2 + \lambda) - 6(3 - \lambda) = 0$

$$\left[\begin{array}{l} \because a_1a_2 + b_1b_2 + c_1c_2 = 0, \\ \text{where } a_1 = 1 + 2\lambda, b_1 = 2 + \lambda, c_1 = 3 - \lambda \\ \text{and } a_2 = 5, b_2 = 3, c_2 = -6 \end{array}\right]$$

(1)

$\Rightarrow \quad 5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$

$\Rightarrow \quad 19\lambda - 7 = 0 \Rightarrow \lambda = \frac{7}{19}$ **(1)**

On putting $\lambda = \frac{7}{19}$ in Eq. (iii), we get the required equation of plane as

$$(x + 2y + 3z - 4) + \frac{7}{19}(2x + y - z + 5) = 0$$

$$\Rightarrow \quad 19x + 38y + 57z - 76 + 14x$$
$$+ 7y - 7z + 35 = 0 \quad \textbf{(1)}$$

$$\Rightarrow \quad 33x + 45y + 50z - 41 = 0$$

In vector form, the required equation of plane is $\vec{r} \cdot (33\hat{i} + 45\hat{j} + 50\hat{k}) = 41.$ **(1)**

33. Find the equation of plane passing through the line of intersection of planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ and parallel to line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$.

HOTS; All India 2011

Given equations of planes are

$$2x + y - z - 3 = 0 \quad \ldots(i)$$

and $$5x - 3y + 4z + 9 = 0 \quad \ldots(ii)$$ **(1)**

Let the required equation of plane which passes through the line of intersection of planes (i) and (ii) be

$$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0 \quad \ldots(iii)$$

$$\Rightarrow x(2 + 5\lambda) + y(1 - 3\lambda) + z(-1 + 4\lambda) + (-3 + 9\lambda) = 0 \quad \ldots(iv)$$ **(1)**

Here, DR's of plane are $2 + 5\lambda, 1 - 3\lambda, -1 + 4\lambda$. Also, given that the plane (i) is parallel to the line, whose equation is

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$$

DR's of the line are 2, 4, 5.

Since, the plane is parallel to the line.

Therefore, normal to the plane is perpendicular to the line.

i.e. $a_1a_2 + b_1b_2 + c_1c_2 = 0$ **(1)**

Here, $a_1 = 2 + 5\lambda, b_1 = 1 - 3\lambda, c_1 = -1 + 4\lambda$

and $a_2 = 2, b_2 = 4, c_2 = 5$

$\therefore \; 2(2 + 5\lambda) + 4(1 - 3\lambda) + 5(-1 + 4\lambda) = 0$

$\Rightarrow \; 4 + 10\lambda + 4 - 12\lambda - 5 + 20\lambda = 0$

$\Rightarrow \; 18\lambda + 3 = 0 \; \Rightarrow \; \lambda = -\frac{3}{18} = -\frac{1}{6}$ **(1½)**

On putting $\lambda = -\frac{1}{6}$ in Eq. (iii), we get the required equation of plane as

$$(2x + y - z - 3) - \frac{1}{6}(5x - 3y + 4z + 9) = 0$$

$$\Rightarrow 12x + 6y - 6z - 18 - 5x + 3y - 4z - 9 = 0$$

$\therefore \quad 7x + 9y - 10z - 27 = 0$ **(1½)**

34. Find the equation of plane passing through the point (–1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 5$. **Foreign 2011; All India 2009**

Let the required equation of plane passing through (–1, 3,2) be

$a(x+1) + b(y-3) + c(z-2) = 0$ ...(i) **(1)**

$$\left[\because \text{equation of plane passing through } (x_1, y_1, z_1) \text{ is } a(x-x_1) + b(y-y_1) + c(z-z_1) = 0\right]$$

Given that plane (i) is perpendicular to the planes whose equations are

$$x + 2y + 3z = 5 \quad \text{...(ii)}$$

and $$3x + 3y + z = 5 \quad \text{...(iii)}$$

$$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad \textbf{(1)}$$

Using the above result first in Eqs. (i), (ii) and then in Eqs. (i), (iii), we get

$$a + 2b + 3c = 0 \quad \text{...(iv)}$$

and $$3a + 3b + c = 0 \quad \text{...(v)}\textbf{(1)}$$

On multiplying Eq. (iv) by 3 and subtracting it from Eq. (v), we get

$$\begin{array}{r} 3a + 3b + c = 0 \\ 3a + 6b + 9c = 0 \\ - \quad - \quad - \\ \hline -3b - 8c = 0 \end{array}$$

$$\Rightarrow \quad -3b = 8c$$

$$\Rightarrow \quad b = -\frac{8c}{3}$$

On putting $b = -\frac{8c}{3}$ in Eq. (iv), we get

$$a + 2\left(-\frac{8c}{3}\right) + 3c = 0$$

$$\Rightarrow \quad a - \frac{16c}{3} + 3c = 0$$

16c

$$\Rightarrow \quad a = \frac{\cdots}{3} - 3c$$

$$= \frac{16c - 9c}{3}$$

$$= \frac{7c}{3}$$

$$\Rightarrow \quad a = \frac{7c}{3} \quad \textbf{(1½)}$$

Now, on putting $a = \frac{7c}{3}$ and $b = -\frac{8c}{3}$ in Eq. (i), we get the required equation of plane as

$$\frac{7c}{3}(x+1) - \frac{8c}{3}(y-3) + c(z-2) = 0$$

On dividing both sides by c, we get

$$\frac{7}{3}(x+1) - \frac{8}{3}(y-3) + (z-2) = 0$$

$$\Rightarrow \quad 7x + 7 - 8y + 24 + 3z - 6 = 0$$

[multiplying both sides by 3]

$$\therefore \quad 7x - 8y + 3z + 25 = 0 \quad \textbf{(1½)}$$

35. Find the vector equation of plane passing through the points $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$. Also, find the cartesian equation of plane. **Foreign 2011**

First, we check whether the points are collinear or not.

Given points are $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$.

$$\therefore \quad AB = \sqrt{(3-2)^2 + (4-2)^2 + (2+1)^2}$$

$$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right]$$

$$= \sqrt{1 + 4 + 9} = \sqrt{14} \text{ units}$$

$$BC = \sqrt{(7-3)^2 + (0-4)^2 + (6-2)^2}$$

$$= \sqrt{16 + 16 + 16} = \sqrt{48} = 4\sqrt{3} \text{ units}$$

and $CA = \sqrt{(2-7)^2 + (2-0)^2 + (-1-6)^2}$

$$= \sqrt{25+4+49} = \sqrt{78} \text{ units}$$

$\because AB + BC \neq CA$, so points A, B, C are non-collinear. **(1)**

Now, the equation of plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Here , $(x_1, y_1, z_1) = (2, 2, -1)$, $(x_2, y_2, z_2) = (3, 4, 2)$ and $(x_3, y_3, z_3) = (7, 0, 6)$

$\therefore$ Equation of plane is

$$\begin{vmatrix} x-2 & y-2 & z+1 \\ 3-2 & 4-2 & 2+1 \\ 7-2 & 0-2 & 6+1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0 \qquad \textbf{(1½)}$$

Expanding along R_1, we get

$$(x-2)(14+6) - (y-2)(7-15) + (z+1)(-2-10) = 0$$

$\Rightarrow (x-2)\cdot 20 - (y-2)(-8) + (z+1)(-12) = 0$

$\Rightarrow \quad 20x - 40 + 8y - 16 - 12z - 12 = 0$

$\Rightarrow \quad 20x + 8y - 12z - 68 = 0$ **(1½)**

On dividing both sides by 4, we get

$$5x + 2y - 3z = 17$$

which is the required cartesian equation of plane.

Now, we know that, vector form of cartesian equation $ax + by + cz = d$ of plane is given by

$$\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = d \qquad \textbf{(1)}$$

$\therefore$ Required vector equation of plane is

$$\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17 \qquad \textbf{(1)}$$

36. Find the equation of plane passing through point (1, 1, –1) and perpendicular to planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

Foreign 2011

Do same as Que. 34.

[**Ans.** $7x + 2y - 7z - 26 = 0$]

37. Find the vector and cartesian equation of a plane containing the two lines

$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k})$ and

$\vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$.

Also, show that the line

$\vec{r} = (2\hat{i} + 5\hat{j} + 2\hat{k}) + P(3\hat{i} - 2\hat{j} + 5\hat{k})$ lies in the plane.

All India 2011C

Given equations of lines are

$$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \quad \text{...(i)}$$

and $\vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{\imath} - 2\hat{j} + 5\hat{k})$...(ii)**(1)**

On comparing Eqs. (i) and (ii) with the vector equation of line $\vec{r} = \vec{a} + \lambda\vec{b}$ respectively, we get

$$\vec{a}_1 = 2\hat{i} + \hat{j} - 3\hat{k}, \vec{b}_1 = \hat{i} + 2\hat{j} + 5\hat{k}$$

and $\vec{a}_2 = 3\hat{i} + 3j + 2\hat{k}, \vec{b}_2 = 3\hat{i} - 2j + 5\hat{k}$ **(1)**

Now, the required plane which contains the lines (i) and (ii) will pass through $\vec{a}_1 = 2\hat{i} + \hat{j} - 3\hat{k}$. Also, the required plane has $\vec{b}_1$ and $\vec{b}_2$ parallel to it.

∴ The normal vector to the plane

$$\vec{n} = \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 3 & -2 & 5 \end{vmatrix}$$

$$= \hat{i}(10 + 10) - \hat{j}(5 - 15) + \hat{k}(-2 - 6)$$

$$= 20\hat{i} + 10\hat{j} - 8\hat{k}$$ **(1)**

∴ The vector equation of required plane is given by

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot \vec{n} = \vec{a}_1 \cdot \vec{n}$$

[∵ here, $\vec{a} = \vec{a}_1$]

$$\Rightarrow \vec{r} \cdot (20\hat{i} + 10\hat{j} - 8\hat{k}) = (2\hat{i} + \hat{j} - 3\hat{k}) \cdot (20\hat{i} + 10\hat{j} - 8\hat{k})$$ **(1)**

$$\Rightarrow \vec{r} \cdot (20\hat{i} + 10\hat{j} - 8\hat{k}) = 40 + 10 + 24 = 74$$

$$\Rightarrow \vec{r} \cdot (10\hat{i} + 5\hat{j} - 4\hat{k}) = 37 \text{ [dividing by 2]...(iii)}$$

which is the required equation of plane.

Also, its cartesian equation is given by

$$10x + 5y - 4z = 37$$

$$\left[\begin{array}{l} \because \text{ vector formof plane } \vec{r} \cdot (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = d \\ \text{can be written in its cartesian form as} \\ \hfill a_1x + a_2y + a_3z = d \end{array}\right]$$

Now, we have to show that the line

$$\vec{r} = (2\hat{i} + 5\hat{j} + 2\hat{k}) + P\,(3\hat{i} - 2\hat{j} + 5\hat{k}) \quad \text{...(iv)}$$

lies in the plane (iii). **(1)**

The above line will lies on plane (iii) when it passes through the point $\vec{a} = 2\hat{i} + 5\hat{j} + 2\hat{k}$ of line (iv) and it is parallel to line (iv).

$$\therefore \vec{a} \cdot (10\hat{i} + 5\hat{j} - 4\hat{k})$$

$$= (2\hat{i} + 5\hat{j} + 2\hat{k}) \cdot (10\hat{i} + 5\hat{j} - 4\hat{k})$$

$$= 20 + 25 - 8 = 37$$

$\Rightarrow \vec{a}$ lies on plane whose equation is given by Eq. (iii).

Hence, line (iv) lies on the plane (iii). **(1)**

38. Find the equation of plane passing through the point (1, 2, 1) and perpendicular to line joining points (1, 4, 2) and (2, 3, 5).

Also, find the coordinates of foot of the perpendicular and the perpendicular distance of the point (4, 0, 3) from above found plane. **HOTS; Delhi 2011C**

Required equation of plane passing through point $R(1, 2, 1)$ and is perpendicular to line PQ, where $P(1, 4, 2)$ and $Q(2, 3, 5)$.

DR's of the line PQ are

$$= (2-1, 3-4, 5-2) = (1, -1, 3) \qquad \textbf{(1)}$$

Let the required equation of plane which passes through point $R(1, 2, 1)$ be

$$a(x-1) + b(y-2) + c(z-1) = 0 \qquad \ldots\text{(i)}$$

Plane (i) is perpendicular to line PQ.

$$[\therefore a_1 a_2 + b_1 b_2 + c_1 c_2 = 0]$$

$$\therefore \quad 1(x-1) - 1(y-2) + 3(z-1) = 0$$

$$\left[\begin{array}{l}\because \text{ line is perpendicular to the plane, then} \\ \text{Dr's of normal to the plane are proportional} \\ \text{to the Dr's of a line i.e. } a \propto 1, b \propto -1, c \propto 3\end{array}\right]$$

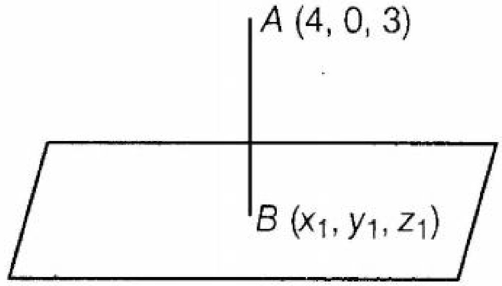
$$\Rightarrow \quad x - 1 - y + 2 + 3z - 3 = 0$$

$$\Rightarrow \quad x - y + 3z - 2 = 0 \qquad \ldots\text{(ii)}\textbf{(1)}$$

which is the required equation of plane.

Now, let $B(x_1, y_1, z_1)$ be the foot of perpendicular on above plane (ii). So, it must satisfies Eq. (ii).

$\therefore \qquad x_1 - y_1 + 3z_1 - 2 = 0 \qquad \ldots\text{(iii)}$



(1)

Also, DR's of line AB normal to above plane (i) are given by

$$x_1 - 4, y_1 - 0, z_1 - 3$$

$$\therefore \qquad \frac{x_1 - 4}{1} = \frac{y_1 - 0}{-1} = \frac{z_1 - 3}{3}$$

[$\because$ DR's of line AB and plane (iii) are proportional]

Let $\quad \frac{x_1 - 4}{1} = \frac{y_1}{-1} = \frac{z_1 - 3}{3} = \lambda \qquad$ [say]

$\Rightarrow \qquad x_1 = \lambda + 4, y_1 = -\lambda, z_1 = 3\lambda + 3 \qquad \ldots\text{(iv)}$

On putting above values of x_1, y_1 and z_1 in Eq. (iii), we get

$$\lambda + 4 + \lambda + 9\lambda + 9 - 2 = 0$$

$\Rightarrow \qquad 11\lambda + 11 = 0 \Rightarrow 11\lambda = -11$

$\therefore \qquad \lambda = -1 \qquad$ **(1½)**

On putting $\lambda = -1$ in Eq. (iv), we get the required foot of perpendicular as

$$B(x_1, y_1, z_1) = B(\lambda + 4, -\lambda, 3\lambda + 3) = B(3, 1, 0)$$

Also, perpendicular distance AB, where $A(4, 0, 3)$ and $B(3, 1, 0)$

$$AB = \sqrt{(3-4)^2 + (1-0)^2 + (0-3)^2}$$

$$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right]$$

$= \sqrt{1 + 1 + 9} = \sqrt{11}$ units $\qquad$ **(1½)**

NOTE If line is perpendicular to the plane, then DR's of normal to the plane are proportional to the DR's of a line.

39. Find the equation of plane passing through point $P(1, 1, 1)$ and containing the line $\vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda(3\hat{i} - \hat{j} - 5\hat{k})$. Also, show that plane contains the line $\vec{r} = (-\hat{i} + 2\hat{j} + 5\hat{k}) + \mu(\hat{i} - 2\hat{j} - 5\hat{k})$.

All India 2010

Equation of plane passing through point $P(1, 1, 1)$ is given by

$a(x-1) + b(y-1) + c(z-1) = 0$ ...(i)**(1)**

[∵ equation of plane passing through (x_1, y_1, z_1) is given as $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$]

Given equation of line is

$$\vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda(3\hat{i} - \hat{j} - 5\hat{k}) \quad \text{...(ii)}$$

DR's of the line are 3, −1 and −5 and the line passes through point (−3, 1, 5).

Now, as the plane (i) contains line (ii), so

$$a(-3-1) + b(1-1) + c(5-1) = 0$$

[as plane contain a line, it means point of line lie on a plane.]

$$\Rightarrow \quad -4a + 4c = 0 \quad \text{...(iii)} \ \textbf{(1)}$$

$$\Rightarrow \quad 4a = 4c$$

$$\Rightarrow \quad a = c$$

Also, since DR's of plane are normal to that of line.

$$\therefore \quad 3a - b - 5c = 0 \quad \text{...(iv)}$$

[∵ plane contains line, it means DR's of plane are perpendicular to the line, i.e.

$aa_1 + bb_1 + cc_1 = 0$]

On putting $a = c$ in Eq. (iv), we get

$$3c - b - 5c = 0 \quad \textbf{(1)}$$

$$\Rightarrow \quad -b - 2c = 0$$

$$\Rightarrow \quad b = -2c$$

On putting $a = c$ and $b = -2c$ in Eq.(i), we get the required equation of plane as

$c(x-1) - 2c(y-1) + c(z-1) = 0$

On dividing both sides by c, we get

$$x - 1 - 2y + 2 + z - 1 = 0$$

$$\Rightarrow \quad x - 2y + z = 0 \quad \text{...(v)} \textbf{(1½)}$$

Now, we have to show that the above plane (v) contains the line

$$\vec{r} = (-\hat{i} + 2\hat{j} + 5\hat{k}) + \mu(\hat{i} - 2\hat{j} - 5\hat{k}) \quad \text{...(vi)}$$

Vector equation of plane (v) is

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \text{...(vii)}$$

The plane (vii) will contains line (vi), if

$$(\hat{i} - 2\hat{j} - 5\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad [\because \vec{b_1} \cdot \vec{b_2} = 0]$$

$$\Rightarrow (1)(1) - 2(-2) - 5(1) = 0$$

$$\Rightarrow \quad 1 + 4 - 5 = 0$$

$$\Rightarrow \quad 0 = 0, \text{ which is true. } \textbf{(1½)}$$

Hence, the plane contains the given line.

40. Find the coordinates of the foot of perpendicular and the perpendicular distance of point $P(3, 2, 1)$ from the plane $2x - y + z + 1 = 0$. Also, find image of the point in the plane. **All India 2010**

Do same as Q. 31.

$$\left[\begin{array}{l} \textbf{Ans. } \text{Foot of perpendicular } = (1, 3, 0), \\ \text{perpendicular distance } = \sqrt{6} \text{ units} \\ \text{Image point } = (-1, 4, -1) \end{array}\right]$$

41. Find the distance of the point (2, 3, 4) from the line $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$ measured parallel to the plane $3x + 2y + 2z - 5 = 0$.

HOTS ; All India 2009C

Let $P(2, 3, 4)$ be the given point and given equation of line be

$$\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$$

Let $$\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2} = \lambda \quad \text{[say]}$$

$\Rightarrow \quad x = 3\lambda - 3, y = 6\lambda + 2, z = 2\lambda$

$\therefore$ Coordinates of any point T on given line are $(3\lambda - 3, 6\lambda + 2, 2\lambda)$. **(1½)**

Now, DR's of line PT

$$= (3\lambda - 3 - 2, 6\lambda + 2 - 3, 2\lambda - 4)$$
$$= (3\lambda - 5, 6\lambda - 1, 2\lambda - 4) \quad \textbf{(1)}$$

Since, the line PT is parallel to the plane

$$3x + 2y + 2z - 5 = 0$$

$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

[$\because$ line is parallel to the plane, therefore normal to the plane is perpendicular to the line, where $a_1 = 3\lambda - 5, b_1 = 6\lambda - 1, c_1 = 2\lambda - 4$ and $a_2 = 3, b_2 = 2, c_2 = 2$]

$\Rightarrow 3(3\lambda - 5) + 2(6\lambda - 1) + 2(2\lambda - 4) = 0$ **(1½)**

$\Rightarrow \quad 9\lambda - 15 + 12\lambda - 2 + 4\lambda - 8 = 0$

$\Rightarrow \quad 25\lambda - 25 = 0 \Rightarrow 25\lambda = 25$

$\Rightarrow \quad \lambda = 1$ **(1)**

$\therefore$ Coordinates of $T = (3\lambda - 3, 6\lambda + 2, 2\lambda)$

$= (0, 8, 2) \quad [\because \text{put } \lambda = 1]$

Now, the required distance between points $P(2, 3, 4)$ and $T(0, 8, 2)$ is given by

$$PT = \sqrt{(0-2)^2 + (8-3)^2 + (2-4)^2}$$

$[\because (x_1, y_1, z_1) = (2, 3, 4)$ and $(x_2, y_2, z_2) = (0, 8, 2)]$

$$= \sqrt{4 + 25 + 4} = \sqrt{33} \text{ units} \quad \textbf{(1)}$$

42. Find the distance of point $(-2, 3, -4)$ from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

HOTS; All India 2009C, 2008C, 2008

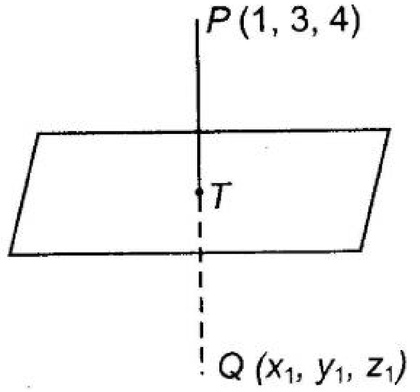
Do same as Que. 41. $\left[\textbf{Ans. } \frac{17}{2} \text{ units}\right]$

43. Find the coordinates of image of point (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

All India 2008

Let $Q(x_1, y_1, z_1)$ be the image of the point $P(1, 3, 4)$ on the plane whose equation is

$$2x - y + z + 3 = 0 \quad \dots(i)$$



(1)

Now, the line PT is normal to the plane, so the DR's of PT are proportional to the DR's of plane which are 2, −1 and 1.

∴ Equation of line PT, where $P(1, 3, 4)$ is given by

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} \quad \dots\text{(ii)}\textbf{(1)}$$

$\left[\because \text{ equation of line passing through one point is } \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}\right]$

Let $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = \lambda$

$\Rightarrow \quad x = 2\lambda + 1, y = 3 - \lambda, z = 4 + \lambda$

∴ Coordinates of the point T

$$= (2\lambda + 1, 3 - \lambda, 4 + \lambda) \quad \dots\text{(iii)}$$

Since, the point T lies on the plane, so we put $x = 2\lambda + 1, y = 3 - \lambda$ and $z = 4 + \lambda$ in Eq. (i),

we get $2(2\lambda + 1) - (3 - \lambda) + (4 + \lambda) + 3 = 0$

$$\Rightarrow \quad 4\lambda + 2 - 3 + \lambda + 4 + \lambda + 3 = 0$$

$$\Rightarrow \quad 6\lambda + 6 = 0$$

$$\Rightarrow \quad 6\lambda = -6$$

$$\therefore \quad \lambda = -1 \quad \textbf{(1½)}$$

On putting $\lambda = -1$ in Eq. (iii), we get the point $T(-1, 4, 3)$.

Now, T is the mid-point of line PQ. So, using mid-point formula, we get

$$\left(\frac{x_1+1}{2}, \frac{y_1+3}{2}, \frac{z_1+4}{2}\right) = (-1, 4, 3) \quad \textbf{(1)}$$

$$\left[\because \text{mid-point} = \left(\frac{x_1+x_2}{2} = \frac{y_1+y_2}{2} = \frac{z_1+z_2}{2}\right)\right]$$

On equating corresponding coordinates, we get

$$\frac{x_1+1}{2} = -1, \frac{y_1+3}{2} = 4, \frac{z_1+4}{2} = 3$$

$$\Rightarrow \quad x_1 = -2-1, y_1 = 8-3, z_1 = 6-4$$

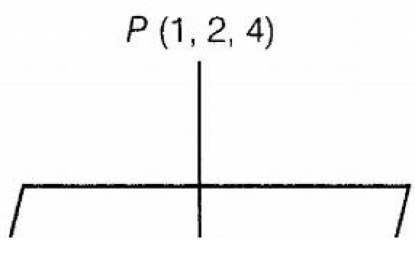
$$\Rightarrow \quad x_1 = -3, y_1 = 5, z_1 = 2$$

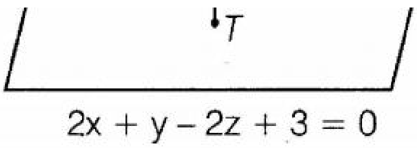
Hence, image Q of the point $P(1, 3, 4)$ is $Q(-3, 5, 2)$. **(1½)**

44. From the point $P\ (1, 2, 4)$, a perpendicular is drawn on the plane $2x + y - 2z + 3 = 0$. Find the equation, the length and the coordinates of foot of perpendicular. **All India 2008**

Let PT be the perpendicular drawn from the point $P(1, 2, 4)$ to the plane whose equation is given by

$$2x + y - 2z + 3 = 0 \quad \ldots(i)$$





From Eq. (i), DR's of plane are 2, 1, −2.
Since, the line PT is normal to the plane, so DR's of line normal to plane are 2, 1, −2. **(1)**

$\therefore$ Equation of line PT,
where $P(1, 2, 4)$is given as

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-4}{-2} \qquad \text{...(ii)}$$

$$\left[\because \text{ equation of line passing through one point is } \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}\right]$$

Let $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-4}{-2} = \lambda$ [say]

$\Rightarrow \frac{x-1}{2} = \lambda, \frac{y-2}{1} = \lambda, \frac{z-4}{-2} = \lambda$

$\Rightarrow x = 2\lambda + 1, y = \lambda + 2, z = -2\lambda + 4$

$\therefore$ Coordinates of any random point on plane are

$T\ (2\lambda + 1, \lambda + 2, -2\lambda + 4)$ ...(iii) **(1½)**

Since, T lies on the given plane, so we put $x = 2\lambda + 1, y = \lambda + 2$ and $z = -2\lambda + 4$ in Eq. (i), we get

$2(2\lambda + 1) + (\lambda + 2) - 2(-2\lambda + 4) + 3 = 0$

$\Rightarrow 4\lambda + 2 + \lambda + 2 + 4\lambda - 8 + 3 = 0$

$\Rightarrow 9\lambda - 1 = 0$

$\therefore \lambda = \frac{1}{9}$ **(1½)**

On putting value of λ in Eq. (iii), we get the foot of perpendicular

$$= T\left(\frac{2}{9} + 1, \frac{1}{9} + 2, -\frac{2}{9} + 4\right)\left[\therefore \text{ put } \lambda = \frac{1}{9}\right]$$

$$= T\left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$$

Also, length of perpendicular PT = Distance between points P and T

$$= \sqrt{\left(1-\frac{11}{9}\right)^2 + \left(2-\frac{19}{9}\right)^2 + \left(4-\frac{34}{9}\right)^2}$$

$$\left[\begin{array}{l} \because (x_1, y_1, z_1) = (1, 2, 4) \\ \text{and } (x_2, y_2, z_2) = \left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right) \end{array}\right]$$

$$= \sqrt{\left(-\frac{2}{9}\right)^2 + \left(-\frac{1}{9}\right)^2 + \left(\frac{2}{9}\right)^2}$$

$$= \sqrt{\frac{4}{81} + \frac{1}{81} + \frac{4}{81}} = \sqrt{\frac{9}{81}} = \sqrt{\frac{1}{9}} = \frac{1}{3} \text{ unit} \quad \textbf{(1)}$$

Now, the equation of perpendicular line PT, where $P(1, 2, 4)$ and $T\left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$ is given as

$$\frac{x-1}{\frac{11}{9}-1} = \frac{y-2}{\frac{19}{9}-2} = \frac{z-4}{\frac{34}{9}-4}$$

$$\left[\begin{array}{l} \because \text{using two points form of a line} \\ \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \end{array}\right]$$

$$\Rightarrow \quad \frac{x-1}{\left(\frac{2}{9}\right)} = \frac{y-2}{\left(\frac{1}{9}\right)} = \frac{z-4}{\left(-\frac{2}{9}\right)}$$

$$\therefore \quad \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-4}{-2} \quad \textbf{(1)}$$

45. Find the equation of plane passing through the point $(-1, -1, 2)$ and perpendicular to each plane $2x + 3y - 3z = 2$ and $5x - 4y + z = 6$. **Delhi 2008**

Do same as Que. 34.

[**Ans.** $9x + 17y + 23z - 20 = 0$]

46. Find the equation of plane passing through points $(3, 4, 1)$ and $(0, 1, 0)$ and parallel to line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$. **HOTS; Delhi 2008**

Equation of plane passing through the point (3, 4, 1) is given as

$$a(x-3)+b(y-4)+c(z-1)=0 \quad \ldots(i)$$

[$\because$ using one point form of a plane

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0]$$

Since, given plane (i) is also passing through point (0, 1, 0).

So, this point also satisfies equation of plane.

$$\therefore \quad a(0-3)+b(1-4)+c(0-1)=0$$

$$\Rightarrow \quad -3a-3b-c=0$$

$$\Rightarrow \quad 3a+3b+c=0 \quad \ldots(ii)$$ **(1)**

Also, given that plane (i) is parallel to the line

$$\frac{x+3}{2}=\frac{y-3}{7}=\frac{z-2}{5}$$

$$\left[\begin{array}{l}\therefore \text{line is parallel to the plane, therefore} \\ \text{normal to the plane is perpendicular to the} \\ \text{line, i.e. } a_1a_2+b_1b_2+c_1c_2=0 \\ \text{where, } a_1=a, b_1=b, c_1=c \\ \text{and } a_2=2, b_2=7 \text{ and } c_2=5\end{array}\right]$$

$$\therefore \quad 2a+7b+5c=0 \quad \ldots(iii)$$ **(1½)**

On multiplying Eq. (ii) by 2 and Eq. (iii) by 3 and then subtracting, we get

$$\begin{array}{r} 6a+6b+2c=0 \\ \underset{-}{}6a+\underset{-}{}21b+\underset{-}{}15c=0 \\ \hline -15b-13c=0 \end{array}$$

$$\Rightarrow \quad -15b=13c$$

$$\therefore \quad b=\frac{-13}{15}c$$

On putting $b=\frac{-13c}{15}$ in Eq. (ii), we get

$$3a+3\left(\frac{-13c}{15}\right)+c=0$$

$$\Rightarrow \quad 3a-\frac{13}{5}c+c=0$$

$$8c$$

$$\Rightarrow \quad 3a - \frac{}{5} = 0$$

$$\Rightarrow \quad 3a = \frac{8c}{5}$$

$$\therefore \quad a = \frac{8c}{15} \qquad \textbf{(1½)}$$

On putting $a = \frac{8c}{15}$ and $b = -\frac{13c}{15}$ in Eq. (i),

we get the required equation of plane as

$$\frac{8c}{15}(x - 3) - \frac{13c}{15}(y - 4) + c(z - 1) = 0 \quad \textbf{(1)}$$

On dividing both sides by c, we get

$$\frac{8}{15}(x - 3) - \frac{13}{15}(y - 4) + z - 1 = 0$$

$$\Rightarrow \quad 8x - 24 - 13y + 52 + 15z - 15 = 0$$

$$\therefore \quad 8x - 13y + 15z + 13 = 0 \quad \textbf{(1)}$$