3. Binary Operations

Exercise 3A

1. Question

Let * be a binary operation on the set I of all integers, defined by a * b = 3a + 4b - 2. Find the value of 4 * 5.

Answer

To find: 4*5

a*b = 3a + 4b - 2

Here a = 4 and b = 5

⇒ 4*5 = 3 × 4 + 4 × 5 - 2 = 12 + 20 - 2 = 30

⇒ 4*5 = 30

2. Question

The binary operation * on R is defined by a * b = 2a + b. Find (2 * 3) * 4.

Answer

To find: (2*3)*4

Given: a*b = 2a + b

⇒ 2*3 = 2 × 2 + 3 = 7

Now 7*4 = 2 × 7 + 4 = 14 + 4 = 18

⇒ (2*3)*4 = 18

3. Question

Let * be a binary operation on the set of all nonzero real numbers, defined by $a * b = \frac{ab}{5}$. Find the value of x given that2 * (x * 5) = 10.

Answer

To find: value of x

Given: a*b = $\frac{ab}{5}$

⇒ x*5 = $\frac{5x}{5}$ = x

Now $(2*x) = \frac{2x}{5}$

$\Rightarrow \frac{2x}{5} = 10 \Rightarrow x = 25$

4. Question

Let *: R × R → R be a binary operation given by $a * b = a + 4b^2$. Then, compute$(- 5) * (2 * 0)$.

Answer

To find: $(- 5)*(2*0)$

Given: $a*b = a + 4b^2$

$\Rightarrow (2*0) = 2 + 4 \times 0^2 = 2$

Now $(- 5)*2 = - 5 + 4 \times 2^2 = - 5 + 16 = 11$

5. Question

Let be a binary operation on the set Q of all rational numbers given as $a * b = (2a - b)^2$ for all a, b ∈ Q. Find 3 * 5 and 5 * 3. Is 3 * 5 = 5 * 3?

Answer

To find: 3*5 and 5*3

Given:$a*b = (2a - b)^2$

$\Rightarrow 3*5 = (6 - 5)^2 = 1$

Now $5*3 = (10 - 3)^2 = 49$

⇒ 3*5 is not equal to 5*3

6. Question

Let * be a binary operation on N given by a *b = 1 cm of a and b. Find the value of20 * 16.

Is * (i) commutative, (ii) associative?

Answer

To find: LCM of 20 and 16

Prime factorizing 20 and 16 we get.

$20 = 2^2 \times 5$

$16 = 2^4$

$\Rightarrow$ LCM of 20 and 16 $= 2^4 \times 5 = 80$

(i) To find LCM highest power of each prime factor has been taken from both the numbers and multiplied.

So it is irrelevant in which order the number are taken as their prime factors will remain the same.

So LCM(a,b) = LCM(b,a)

So * is commutative

(ii) Let us assume that * is associative

$\Rightarrow$ LCM[LCM(a,b),c] = LCM[a,LCM(b,c)]

Let the prime factors of a be p_1, p_2

Let the prime factors of b be p_2, p_3

Let the prime factors of c be p_3, p_4

Let the higher factor of p_i be q_i for i = 1,2,3,4

LCM (a,b) = $p_1^{q1} \times p_2^{q2} \times p_3^{q3}$

LCM[LCM(a,b),c] = $p_1^{q1} \times p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

LCM (b,c) = $p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

LCM[a,LCM(b,c)] = $p_1^{q1} \times p_2^{q2} \times p_3^{q3} \times p_4^{q4}$

$\Rightarrow$ * is associative

7. Question

If * be the binary operation on the set Z of all integers defined by a * b (a + $3b^2$), find 2 * 4

Answer

To find: 2*4

Given: a*b = a + $3b^2$

$\Rightarrow$ 2*4 = $(2 + 3 \times 4^2)$ = 2 + 48 = 50

8. Question

Show that * on Z^+ defined by a *b = |a −b| is not a binary operation.

Answer

To prove: * is not a binary operation

Given: a and b are defined on positive integer set

And a*b = |a - b|

$\Rightarrow$ a*b = (a - b), when a>b

= b - a when b>a

= 0 when a = b

But 0 is neither positive nor negative.

So 0 does not belong to Z^+ .

So $a*b = |a - b|$ does not belong to Z^+ for all values of a and b

So * is not a binary operation.

Hence proved

9. Question

Let * be a binary operation on N, defined by $a * b = a^b$ for all a. b ∈ N.

Show that * is neither commutative nor associative.

Answer

To prove: * is neither commutative nor associative

Let us assume that * is commutative

⇒ $a^b = b^a$ for all a,b ∈ N

This is valid only for a = b

For example take a = 1, b = 2

$1^2 = 1$ and $2^1 = 2$

So * is not commutative

Let us assume that * is associative

⇒ $(a^b)^c = a^{b^c}$ for all a,b,c ∈ N

⇒ $a^{bc} = a^{b^c}$ for all a,b,c ∈ N

This is valid in the following cases:

(i) a = 1

(ii) b = 0

(iii) $bc = b^c$

For example, let a = 2,b = 1,c = 3

$a^{bc} = 2^{(1 \times 3)} = 2^3 = 8$

$a^{b^c} = 2^{1^3} = 2$

So * is not associative

10. Question

Let a * b = 1 cm (a, b) for all values ofa, b ∈ N.

(i) Find (12 * 16).

(ii) Show that * is commutative on N.

(iii) Find the identity element in N.

(iv) Find all invertible elements in N.

Answer

To find: (i)

LCM of 12 and 16

Prime factorizing 12 and 16 we get.

$20 = 2^2 \times 3$

$16 = 2^4$

⇒ LCM of 20 and 16 = $2^4 \times 3 = 48$

(ii) To find LCM highest power of each prime factor has been taken from both the numbers and multiplied.

So it is irrelevant in which order the number are taken as their prime factors will remain the same.

So LCM(a,b) = LCM(b,a)

So * is commutative.

(iii)let $x \in N$ and x*1 = lcm(x,1) = x = lcm(1,x)

1 is the identity element.

(iv)let there exist y in n such that x*y = e = y*x

Here e = 1,

Lcm(x,y) = 1

This happens only when x = y = 1.

1 is the invertible element of n with respect to *.

11. Question

Let Q be the set of all positive rational numbers.

(i) Show that the operation * on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.

(ii) Show that * is commutative.

(iii) Show that * is not associative.

Answer

(i)Let a = 1, b = $2 \in Q^+$

a*b = $\frac{1}{2}(1 + 2)$ = 1.5$\in$Q $^+$

* is closed and is thus a binary operation on Q $^+$

(ii) a*b = $\frac{1}{2}(1 + 2)$ = 1.5

And b*a = $\frac{1}{2}(2 + 1)$ = 1.5

Hence * is commutative.

(iii)let c = 3.

(a*b)*c = 1.5*c = $\frac{1}{2}(1.5 + 3) = 2.75$

a*(b*c) = a*$\frac{1}{2}(2 + 3)$ = 1*2.5 = $\frac{1}{2}(1 + 2.5)$ = 1.75

hence * is not associative.

12. Question

Show that the set A = { - 1, 0, 1) is not closed for addition.

Answer

For a set to be closed for addition,

For any 2 elements of the set,say a and b, a + b must also be a member of the given set, where a and b may be same or distinct

In the given problem let a = 1 and b = 1

a + b = 2 which is not in the given in set

So the set is not closed for addition.

Hence proved.

13. Question

Show that * on R –{ - 1}, defined by $(a * b) = \frac{a}{(b+1)}$ is neither commutative nor associative.

Answer

let a = 1,b = 0$\in$R - { - 1}

a*b = $\frac{1}{0 + 1}$ = 1

And b*a = $\frac{0}{1 + 1}$ = 0

Hence * is not commutative.

Let c = 3.

$(a*b)*c = 1*c = \frac{1}{3+1} = \frac{1}{4}$

$a*(b*c) = a*\frac{0}{3+1} = 1*0 = \frac{1}{0+1} = 1$

Hence * is not associative.

14. Question

For all a, b ∈ R, we define a * b = |a − b|.

Show that * is commutative but not associative.

Answer

a*b = a - b if a>b

= - (a - b) if b>a

b*a = a - b if a>b

= - (a - b) if b>a

So a*b = b*a

So * is commutative

To show that * is associative we need to show

(a*b)*c = a*(b*c)

Or ||a - b| - c| = |a - |b - c||

Let us consider c>a>b

Eg a = 1,b = - 1,c = 5

LHS:

|a - b| = |1 + 1| = 2

||a - b| - c| = |2 - 5| = 3

RHS

|b - c| = | - 1 - 5| = 6

|a - |b - c|| = |1 - 6| = | - 5| = 5

As LHS is not equal to RHS * is not associative

15. Question

For all a, b ∈ N, we define a * b = $a^3 + b^3$.

Show that * is commutative but not associative.

Answer

let a = 1,b = 2∈N

$a*b = 1^3 + 2^3 = 9$

And $b*a = 2^3 + 1^3 = 9$

Hence * is commutative.

Let $c = 3$

$(a*b)*c = 9*c = 9^3 + 3^3$

$a*(b*c) = a*(2^3 + 3^3) = 1*35 = 1^3 + 35^3$

$(a*b)*c \neq a*(b*c)$

Hence * is not associative.

16. Question

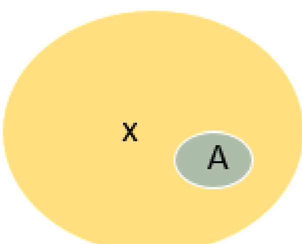
Let X be a nonempty set and * be a binary operation on P(X), the power set of X, defined by $A * B = A \cap B$ for all $A, B \in P(X)$.

(i) Find the identity element in P(X).

(ii) Show that X is the only invertible element in P(X).

Answer

e is the identity of * if $e*a = a$



From the above Venn diagram,

$A*X = A \cap X = A$

$X*A = X \cap A = A$

$\Rightarrow$ X is the identity element for binary operation *

Let B be the invertible element

$\Rightarrow A*B = X$

$\Rightarrow A \cap B = X$

This is only possible if $A = B = X$

Thus X is the only invertible element in P(X)

Hence proved.

17. Question

A binary operation * on the set (0, 1, 2, 3, 4, 5) is defined as

$$a * b = \begin{cases} a+b; & \text{if } a+b<6 \\ a+b-6; & \text{if } a+b \geq 6 \end{cases}$$

Show that 0 is the identity for this operation and each element a has an inverse (6 - a)

Answer

To find: identity and inverse element

For a binary operation if a*e = a, then e s called the right identity

If e*a = a then e is called the left identity

For the given binary operation,

e*b = b

⇒ e + b = b

⇒ e = 0 which is less than 6.

b*e = b

⇒ b + e = b

⇒ e = 0 which is less than 6

For the 2^{nd} condition,

e*b = b

⇒ e + b - 6 = b

⇒ e = 6

But e = 6 does not belong to the given set (0,1,2,3,4,5)

So the identity element is 0

An element c is said to be the inverse of a, if a*c = e where e is the identity element (in our case it is 0)

a*c = e

⇒ a + c = e

⇒ a + c = 0

⇒ c = - a

a belongs to (0,1,2,3,4,5)

- a belongs to (0, - 1, - 2, - 3, - 4, - 5)

So c belongs to (0, - 1, - 2, - 3, - 4, - 5)

So c = - a is not the inverse for all elements a

Putting in the 2nd condition

a*c = e

⇒ a + c - 6 = 0

⇒ c = 6 - a

$0 \le a < 6$

⇒ $-6 \le -a < 0 \Rightarrow 0 \le 6 - a < 6 0 \le c < 5$

So c belongs to the given set

Hence the inverse of the element a is (6 - a)

Hence proved

Exercise 3B

1. Question

Define * on N by m * n = 1 cm (m, n). Show that * is a binary operation which is commutative as well as associative.

Answer

* is an operation as m*n = LCM (m, n) where m, n ∈ N. Let m = 2 and b = 3 two natural numbers.

m*n = 2*3

= LCM (2, 3)

= 6∈ N

So, * is a binary operation from $N \times N \to N$.

For commutative,

n*m = 3*2

= LCM (3, 2)

= 6∈ N

Since m*n = n*m, hence * is commutative operation.

Again, for associative, let p = 4

m*(n*p) = 2*LCM (3, 4)

= 2*12

= LCM (2, 12)

= 12∈ N

(m*n) *p = LCM (2, 3) *4

= 6*4

= LCM (6, 4)

= 12∈ N

As m*(n*p) = (m*n) *p, hence * an associative operation.

2. Question

Define * on Z by a * b = a – b + ab. Show that * is a binary operation on Z which is neither commutative nor associative.

Answer

* is an operation as a*b = a-b + ab where a, b ∈ Z. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 = \frac{1}{2} - 2 + \frac{1}{2}.2 \Rightarrow \frac{1-4}{2} + 1 = \frac{-3+2}{2} \Rightarrow \frac{-1}{2} \in Z$$

So, * is a binary operation from $Z \times Z \to Z$.

For commutative,

$$b*a = 2 - \frac{1}{2} + 2.\frac{1}{2} = \frac{4-1}{2} + 1 \Rightarrow \frac{3+2}{2} = \frac{5}{2} \in Z$$

Since a*b ≠ b*a, hence * is not commutative operation.

Again for associative,

a*(b*c) = a*(b-c+ bc)

= a- (b- c+ bc) +a (b- c+ bc)

= a-b+ c- bc+ ab- ac+ abc

(a*b) *c = (a-b+ ab) *c

= a- b+ ab-c+ (a- b+ ab) c

= a-b-c+ ab+ ac- bc+ abc

As a*(b*c) ≠ (a*b) *c, hence * not an associative operation.

3. Question

Define * on Z by a * b = a + b - ab. Show that * is a binary operation on Z which is commutative as well as associative.

Answer

* is an operation as a*b = a+ b - ab where a, b ∈ Z. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 = \frac{1}{2} + 2 - \frac{1}{2}.2 \Rightarrow \frac{1+4}{2} - 1 = \frac{5-2}{2} \Rightarrow \frac{3}{2} \in Z$$

So, * is a binary operation from $Z \times Z \to Z$.

For commutative,

$$b*a = 2+\frac{1}{2}-2.\frac{1}{2} = \frac{4+1}{2}-1 \Rightarrow \frac{5-2}{2} = \frac{3}{2} \in Z$$

Since a*b = b*a, hence * is a commutative binary operation.

Again for associative,

a*(b*c) = a*(b+ c- bc)

= a+ (b+ c- bc) -a (b+ c- bc)

= a+ b+ c- bc- ab- ac+ abc

(a*b) *c = (a+ b- ab) *c

= a+ b- ab+ c- (a+ b- ab) c

= a+ b+ c- ab- ac- bc+ abc

As a*(b*c) = (a*b) *c, hence * an associative binary operation.

4. Question

Consider a binary operation on Q – {1}, defined by a * b = a + b - ab.

(i) Find the identity element in Q – {1}.

(ii) Show that each a ∈ Q - {1} has its inverse.

Answer

(i) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

a*e = a+ e- ae (1)

e*a = e+ a- e a (2)

using a*e = a

a+ e- ae = a

e-ae = 0

e(1-a) = 0

either e = 0 or a = 1 as operation is on Q excluding 1 so a≠1, hence e = 0.

So identity element e = 0.

(ii) for a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 0

a+ b- ab = 0

b(1-a) = -a

$$b = \frac{-a}{(1-a)} \Rightarrow \frac{a}{(a-1)}$$

$$a^{-1} = \frac{a}{(a-1)}$$

5. Question

Let Q_0 be the set of all nonzero rational numbers. Let * be a binary operation on Q_0, defined by $a * b = \frac{ab}{4}$ for all a, b ∈ Q_0.

(i) Show that * is commutative and associative.

(ii) Find the identity element in Q_0.

(iii) Find the inverse of an element a in Q_0.

Answer

(i) For commutative binary operation, a*b = b*a.

$$a*b = \frac{ab}{4} \text{ and } b*a = \frac{ba}{4}$$

as multiplication is commutative ab = ba so a*b = b*a. Hence * is commutative binary operation.

For associative binary operation, a*(b*c) = (a*b) *c

$$a*(b*c) = a*\frac{bc}{4} \Rightarrow \frac{a.\frac{bc}{4}}{4} = \frac{abc}{16}$$

$$(a*b)*c = \frac{ab}{4}*c \Rightarrow \frac{\frac{ab}{4}.c}{4} = \frac{abc}{16}$$

Since a*(b*c) = (a*b) *c, hence * is an associative binary operation.

(ii) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

$$a*e = \frac{ae}{4} \text{ (1)}$$

$$e*a = \frac{ea}{4} \text{ (2)}$$

using a*e = a

$$\frac{ae}{4} = a \Rightarrow \frac{ae}{4} - a = 0 \Rightarrow \frac{a}{4}(e-4) = 0$$

Either a = 0 or e = 4 as given a≠0, so e = 4.

Identity element e = 4.

(iii) For a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 4

$$\frac{ab}{4} = 4 \Rightarrow b = \frac{16}{a}$$

$$a^{-1} = \frac{16}{a}$$

6. Question

On the set Q^+ of all positive rational numbers, define an operation * on Q^+ by $a * b = \frac{ab}{2}$ for all a, b ∈ Q^+. Show that

(i) * is a binary operation on Q^+,

(ii) * is commutative,

(iii) * is associative.

Find the identity element in Q^+ for *. What is the inverse of a ∈ Q^+?

Answer

(i) * is an operation as $a*b = \frac{ab}{2}$ where a, b ∈ Q^+. Let $a = \frac{1}{2}$ and b = 2 two integers.

$$a*b = \frac{1}{2}*2 \Rightarrow 1 \in Q^+$$

So, * is a binary operation from $Q^+ \times Q^+ \rightarrow Q^+$.

(ii) For commutative binary operation, a*b = b*a.

$$b*a = 2.\frac{1}{2} \Rightarrow 1 \in Q^+$$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c.

$$a*(b*c) = a*\frac{bc}{2} \Rightarrow \frac{a.\frac{bc}{2}}{2} = \frac{abc}{4}$$

$$(a*b)*c = \frac{ab}{2}*c \Rightarrow \frac{\frac{ab}{2}.c}{2} = \frac{abc}{4}$$

As a*(b*c) = (a*b) *c, hence * is an associative binary operation.

For a binary operation *, e identity element exists if a*e = e*a = a.

$a*e = \frac{ae}{2}$ (1)

$e*a = \frac{ea}{2}$ (2)

using a*e = a

$\frac{ae}{2} = a \Rightarrow \frac{ae}{2} - a = 0 \Rightarrow \frac{a}{2}(e-2) = 0$

Either a = 0 or e = 2 as given a≠0, so e = 2.

For a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 2

$\frac{ab}{2} = 2 \Rightarrow b = \frac{4}{a}$

$a^{-1} = \frac{4}{a}$

7. Question

Let Q^+ be the set of all positive rational numbers.

(i) Show that the operation * on Q^+ defined by $a * b = \frac{1}{2}(a + b)$ is a binary operation.

(ii) Show that * is commutative.

(iii) Show that * is not associative.

Answer

(i) * is an operation as $a*b = \frac{1}{2}(a+b)$ where a, b ∈ Q^+. Let $a = 1$ and b = 2 two integers.

$a*b = \frac{1}{2}(1+2) \Rightarrow \frac{3}{2} \in Q^+$

So, * is a binary operation from $Q^+ \times Q^+ \rightarrow Q^+$.

(ii) For commutative binary operation, a*b = b*a.

$b*a = \frac{1}{2}(2+1) \Rightarrow \frac{3}{2} \in Q^+$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c.

$$a*(b*c) = a*\frac{1}{2}(b+c) \Rightarrow \frac{1}{2}\left(a+\frac{b+c}{2}\right) = \frac{1}{4}(2a+b+c)$$

$$(a*b)*c = \frac{1}{2}(a+b)*c \Rightarrow \frac{1}{2}\left(\frac{a+b}{2}+c\right) = \frac{1}{4}(a+b+2c)$$

As a*(b*c) ≠(a*b) *c, hence * is not associative binary operation.

8. Question

Let Q be the set of all rational numbers. Define an operation on Q – {-1} by a * b = a + b + ab.

Show that

(i) * is a binary operation on Q – {-1},

(ii) * is Commutative,

(iii) * is associative,

(iv) zero is the identity element in Q – {-1} for *,

(v) $a^{-1} = \left(\frac{-a}{1+a}\right)$, where a ∈ Q – {-1}.

Answer

(i) * is an operation as a*b = a+ b+ ab where a, b ∈ Q- {-1}. Let $a=1$ and $b=\frac{-3}{2}$ two rational numbers.

$$a*b = 1+\frac{-3}{2}+1.\frac{-3}{2} \Rightarrow \frac{2-3}{2}-\frac{3}{2} = \frac{-1-3}{2} \Rightarrow \frac{-4}{2} = -2 \in Q-\{-1\}$$

So, * is a binary operation from $Q-\{-1\}\times Q-\{-1\} \rightarrow Q$ - {-1}.

(ii) For commutative binary operation, a*b = b*a.

$$b*a = \frac{-3}{2}+1+\frac{-3}{2}.1 \Rightarrow \frac{-3+2}{2}-\frac{3}{2} = \frac{-1-3}{2} \Rightarrow \frac{-4}{2} = -2 \in Q-\{-1\}$$

Since a*b = b*a, hence * is a commutative binary operation.

(iii) For associative binary operation, a*(b*c) = (a*b) *c

a+(b*c) = a*(b+ c+ bc) = a+ (b+ c+ bc) +a(b+ c+ bc)

= a+ b+ c+ bc+ ab+ ac+ abc

(a*b) *c = (a+ b+ ab) *c = a+ b+ ab+ c+ (a+ b+ ab) c

= a+ b+ c+ ab+ ac+ bc+ abc

Now as a*(b*c) = (a*b) *c, hence an associative binary operation.

(iv) For a binary operation *, e identity element exists if a*e = e*a = a. As a*b = a+ b- ab

a*e = a+ e+ ae (1)

e*a = e+ a +e a (2)

using a*e = a

a+ e+ ae = a

e+ ae = 0

e(1+a) = 0

either e = 0 or a = -1 as operation is on Q excluding -1 so a≠-1, hence e = 0.

So identity element e = 0.

(v) for a binary operation * if e is identity element then it is invertible with respect to * if for an element b, a*b = e = b*a where b is called inverse of * and denoted by a^{-1}.

a*b = 0

a+ b+ ab = 0

b(1+a) = -a

$$b = \frac{-a}{(1+a)}$$

$$a^{-1} = \frac{-a}{(a+1)}$$

9. Question

Let A = N × N. Define * on A by (a, b) * (c, d) = (a + c, b + d).

Show that

(i) A is closed for *,

(ii) * is commutative,

(iii) * is associative,

(iv) identity element does not exist in A.

Answer

(i) A is said to be closed on * if all the elements of (a, b) *(c, d) = (a+ c, b+ d) belongs to N×N for A = N×N.

Let a = 1, b = 3, c = 8, d = 2

(1, 3) * (8, 2) = (1+8, 3+2)

= (9, 5) ∈N×N

Hence A is closed for *.

(ii) For commutative,

(c, d) *(a, b) = (c+ a, d+ b)

As addition is commutative a+ c = c+ a and b+ d = d+ b, hence * is commutative binary operation.

(iii) For associative,

(a, b) *((c, d) *(e, f)) = (a, b) *(c+ e, d+ f)

= (a+ c+ e, b+ d+ f)

((a, b) *(c, d)) *(e, f) = (a+ c, b+ d) *(e, f)

= (a+ c+ e, b+ d+ f)

As (a, b) *((c, d) *(e, f)) = ((a, b) *(c, d)) *(e, f), hence * is an associative binary operation.

(iv) For identity element (e_1, e_2), (a, b) *(e_1, e_2) = (e_1, e_2) *(a, b) = (a, b) in a binary operation.

(a, b) *(e_1, e_2) = (a, b)

(a+e_1, b+e_2) = (a, b)

(e_1, e_2) = (0, 0)

As (0,0) ∉N×N, hence identity element does not exist for *.

10. Question

Let A = (1, -1, i, -i) be the set of four 4th roots of unity. Prepare the composition table for multiplication on A and show that

(i) A is closed for multiplication,

(ii) multiplication is associative on A,

(iii) multiplication is commutative on A,

(iv) 1 is the multiplicative identity,

(v) every element in A has its multiplicative inverse.

Answer

(i) A is said to be closed on * if all the elements of a*b ∈A. composition table is

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

(as $i^2 = -1$)

As table contains all elements from set A, A is close for multiplication operation.

(ii) For associative, a× (b× c) = (a× b) ×c

1× (-i× i) = 1×1 = 1

(1× -i) ×i = -i× i = 1

a× (b× c) = (a× b) ×c, so A is associative for multiplication.

(iii) For commutative, a× b = b× a

1× -1 = -1

-1× 1 = -1

a× b = b× a, so A is commutative for multiplication.

(iv) For multiplicative identity element e, a× e = e× a = a where a ∈A.

a× e = a

a(e-1) = 0

either a = 0 or e = 1 as a≠0 hence e = 1.

So, multiplicative identity element e = 1.

(v) For multiplicative inverse of every element of A, a*b = e where a, b∈A.

$1 \times b_1 = 1$

$b_1 = 1$

$-1 \times b_2 = 1$

$b_2 = -1$

$i \times b_3 = 1$

$$b_3 = \frac{1}{i} \Rightarrow \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} \Rightarrow \frac{i}{-1} = -i$$

$-i \times b_4 = 1$

$$b_4 = \frac{1}{-i} \Rightarrow \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} \Rightarrow \frac{i}{-(-1)} = i$$

So, multiplicative inverse of A = {1, -1, -i, i}