Chapter 5

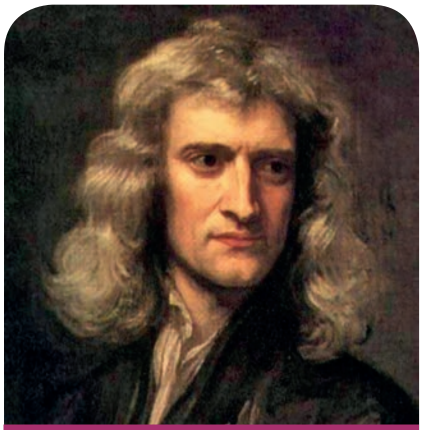
DIFFERENTIAL CALCULUS

Learning Objectives

After studying this chapter, the students will be able to understand

- the concept of limit continuity
- formula of limit and definition of continuity
- basic concept of derivative
- how to find derivative of a function by first principle
- derivative of a function by certain direct formula and its respective application
- how to find higher order derivatives


G.W. Leibnitz


Isaac Newton

Introduction

Calculus is a Latin word which means that *Pebble or a small* stone used for calculation. The word calculation is also derived from the same Latin word. Calculus is primary mathematical tool for dealing with change. The concept of derivative is the basic tool in science of calculus. Calculus is essentially concerned with the rate of change of dependent variable with respect to an independent variable. Sir Issac Newton (1642 – 1727 CE) and the German mathematician G.W. Leibnitz (1646 – 1716 CE) invented and developed the subject independently and almost simultaneously.

In this chapter we will study about functions and their graphs, limits, derivatives and differentiation technique.

5.1 Functions and their Graphs

Some basic concepts

5.1.1 Quantity

Anything which can be performed on basic mathematical operations like addition, subtraction, multiplication and division is called a quantity.

5.1.2 Constant

A quantity which retains the same value throughout a mathematical investigation is called a constant.

Basically constant quantities are of two types

(i) **Absolute constants** are those which do not change their values in any mathematical investigation. In other words, they are fixed for ever.

Examples: 3, $\sqrt{3}$, π, ...

(ii) **Arbitrary constants** are those which retain the same value throughout a problem, but we may assign different values to get different solutions. The arbitrary constants are usually denoted by the letters *a, b, c, ...* **Example:** In an equation $y=mx+4$, m is called arbitrary constant.

5.1.3 Variable

A variable is a quantity which can assume different values in a particular problem. Variables are generally denoted by the letters $x, y, z, \ldots$

Example: In an equation of the straight line $\frac{x}{a}+\frac{y}{b}=1$,

x and y are variables because they assumes the co-ordinates of a moving point in a straight line and thus changes its value from point to point. a and b are intercept values on the axes which are arbitrary.

There are two kinds of variables

(i) A variable is said to be an **independent variable** when it can have any arbitrary value.

(ii) A variable is said to be a **dependent variable** when its value depend on the value assumed by some other variable.

Example: In the equation
$y = 5x^2 - 2x + 3$,

"x" is the independent variable,

"y" is the dependent variable and

"3" is the constant.

5.1.4 Intervals

The real numbers can be represented geometrically as points on a number line called real line. The symbol R denotes either the real number system or the real line. A subset of the real line is an interval. It contains atleast two numbers and all the real numbers lying between them.

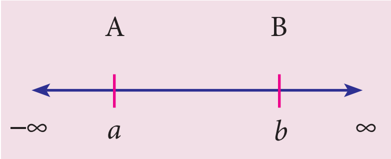


Fig. 5.1

(i) Open interval

The set $\{x : a < x < b\}$ is an open interval, denoted by (a, b).

In this interval, the boundary points a and b are not included.

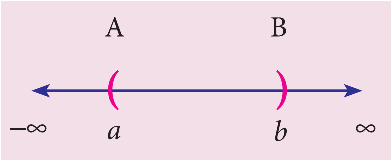


Fig. 5.2

For example, in an open interval (4, 7), 4 and 7 are not elements of this interval.

But, 4.001 and 6.99 are elements of this interval.

(ii) Closed interval

The set $\{x : a \leq x \leq b\}$ is a closed interval and is denoted by $[a, b]$.

In the interval $[a, b]$, the boundary points a and b are included.

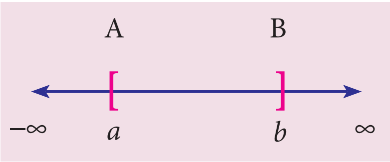


Fig. 5.3

For example, in an interval [4, 7], 4 and 7 are also elements of this interval.

Also we can make a mention about semi closed or semi open intervals.

$(a, b] = \{x : a < x \leq b\}$ is called left open interval and

$[a, b) = \{x : a \leq x < b\}$ is called right open interval.

In all the above cases, $b - a = h$ is called the length of the interval.

5.1.5 Neighbourhood of a point

Let 'a' be any real number. Let $\varepsilon > 0$ be arbitrarily small real number.

Then $(a - \varepsilon, a + \varepsilon)$ is called an ε - neighbourhood of the point 'a' and denoted by $N_{a,\varepsilon}$

For examples,

$$N_{5,\frac{1}{4}} = \left(5 - \frac{1}{4}, 5 + \frac{1}{4}\right) = \left\{x: \frac{19}{4} < x < \frac{21}{4}\right\}$$

$$N_{2,\frac{1}{7}} = \left(2 - \frac{1}{7}, 2 + \frac{1}{7}\right) = \left\{x: \frac{13}{7} < x < \frac{15}{7}\right\}$$

5.1.6 Function

Let X and Y be two non-empty sets of real numbers. If there exists a rule f which associates to every element $x \in X$, a unique element $y \in Y$, then such a rule f is called a function (or mapping) from the set X to the set Y. We write $f : X \longrightarrow Y$.

DO YOU KNOW?

The function notation $y = f(x)$ was first used by Leonhard Euler in 1734 - 1735

The set X is called the domain of f, Y is called the co-domain of f and the range of f is defined as $f(X) = \{f(x) \,/\, x \in X\}$. Clearly $f(X) \subseteq Y$.

A function of x is generally denoted by the symbol $f(x)$, and read as *"function of x" or "f of x".*

5.1.7 Classification of functions

Functions can be classified into two groups.

(i) Algebraic functions

Algebraic functions are algebraic expressions using a finite number of terms, involving only the algebraic operations addition, subtraction, multiplication, division and raising to a fractional power.

Examples : $y = 3x^4 - x^3 + 5x^2 - 7$,

$y = \dfrac{2x^3 + 7x - 3}{x^3 + x^2 + 1}$, $y = \sqrt{3x^2 + 6x - 1}$ are algebraic functions.

(a) $y = 3x^4 - x^3 + 5x^2 - 7$ is a polynomial or rational integral function.

(b) $y = \dfrac{2x^3 + 7x - 3}{x^3 + x^2 + 1}$ is a rational function.

(c) $y = \sqrt{3x^2 + 6x - 1}$ is an irrational function.

(ii) Transcendental functions

A function which is not algebraic is called transcendental function.

Examples: $\sin x$, $\sin^{-1} x$, e^x, $\log_a x$ are transcendental functions.

(*a*) $\sin x$, $\tan 2x$, ... are trigonometric functions.

(*b*) $\sin^{-1} x$, $\cos^{-1} x$, ... are inverse trigonometric functions.

(*c*) e^x, 2^x, x^x, ... are exponential functions.

(*d*) $\log_a x$, $\log_e (\sin x)$, ... are logarithmic functions.

5.1.8 Even and odd functions

A function $f(x)$ is said to be an even function of x, if $f(-x) = f(x)$.

A function $f(x)$ is said to be an odd function of x, if $f(-x) = -f(x)$.

Examples:

$f(x) = x^2$ and $f(x) = \cos x$ are even functions.

$f(x) = x^3$ and $f(x) = \sin x$ are odd functions.

NOTE

$f(x) = x^3 + 5$ is neither even nor odd function

5.1.9 Explicit and implicit functions

A function in which the dependent variable is expressed explicitly in terms of some independent variables is known as **explicit function**.

Examples: $y = x^2 + 3$ and $y = e^x + e^{-x}$ are explicit functions of x.

If two variables x and y are connected by the relation or function $f(x, y) = 0$ and none of the variable is directly expressed in terms of the other, then the function is called an **implicit function**.

Example: $x^3 + y^3 - xy = 0$ is an implicit function.

5.1.10 Constant function

If k is a fixed real number then the function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.

Examples: $y = 3$, $f(x) = -5$ are constant functions.

DO YOU KNOW?

Graph of a constant function $y = f(x)$ is a straight line parallel to x-axis

5.1.11 Identity function

A function that associates each real number to itself is called the identity function and is denoted by I.

i.e. A function defined on R by $f(x) = x$ for all $x \in R$ is an identity function.

Example: The set of ordered pairs $\{(1, 1), (2, 2), (3, 3)\}$ defined by $f : A \to A$ where $A = \{1, 2, 3\}$ is an identity function.

DO YOU KNOW?

Graph of an identity function on R is a straight line passing through the origin and makes an angle of 45° with positive direction of x-axis

5.1.12 Modulus function

A function $f(x)$ defined by $f(x) = |x|$, where $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ is called the modulus function.

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It is also called an absolute value function.

Remark Domain is R and range set is $[0, \infty)$

NOTE

$|5| = 5, |-5| = -(-5) = 5$

5.1.13 Signum function

A function $f(x)$ is defined by $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is called the Signum function.

Remark Domain is R and range set is $\{-1, 0, 1\}$

5.1.14 Step function

(i) Greatest integer function

The function whose value at any real number x is the greatest integer less than or equal to x is called the greatest integer function. It is denoted by $\lfloor x \rfloor$.

i.e. $f : R \longrightarrow R$ defined by $f(x) = \lfloor x \rfloor$ is called the greatest integer function.

NOTE

$\lfloor 2.5 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$,
$\lfloor 0.74 \rfloor = 0$, $\lfloor -0.3 \rfloor = -1$, $\lfloor 4 \rfloor = 4$.

(ii) Least integer function

The function whose value at any real number x is the smallest integer greater than or equal to x is called the least integer function. It is denoted by $\lceil x \rceil$.

i.e. $f : R \longrightarrow R$ defined by $f(x) = \lceil x \rceil$ is called the least integer function.

Remark Function's domain is R and range is Z [set of integers]

NOTE

$\lceil 4.7 \rceil = 5, \lceil -7.2 \rceil = -7, \lceil 5 \rceil = 5, \lceil 0.75 \rceil = 1.$

5.1.15 Rational function

A function $f(x)$ is defined by the rule $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ is called rational function.

Example: $f(x) = \frac{x^2+1}{x-3}, x \neq 3$ is a rational function.

5.1.16 Polynomial function

For the real numbers $a_0, a_1, a_2, ..., a_n$; $a_0 \neq 0$ and n is a non-negative integer, a function $f(x)$ given by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ is called as a polynomial function of degree n.

Example: $f(x) = 2x^3 + 3x^2 + 2x - 7$ is a polynomial function of degree 3.

5.1.17 Linear function

For the real numbers a and b with $a \neq 0$, a function $f(x) = ax + b$ is called a linear function.

Example: $y = 2x + 3$ is a linear function.

5.1.18 Quadratic function

For the real numbers a, b and c with $a \neq 0$, a function $f(x) = ax^2 + bx + c$ is called a quadratic function.

Example: $f(x) = 3x^2 + 2x - 7$ is a quadratic function.

5.1.19 Exponential function

A function $f(x) = a^x$, $a \neq 1$ and $a > 0$, for all $x \in R$ is called an exponential function

Remark Domain is R and range is $(0, \infty)$ and $(0, 1)$ is a point on the graph

Examples: e^{2x}, e^{x^2+1} and 2^x are exponential functions.

5.1.20 Logarithmic function

For $x > 0$, $a > 0$ and $a \neq 1$, a function $f(x)$ defined by $f(x) = \log_a x$ is called the logarithmic function.

Remark	Domain is $(0, \infty)$, range is R and (1, 0) is a point on the graph

Example: $f(x) = \log_e(x+2)$ and $f(x) = \log_e(\sin x)$ are logarithmic functions.

5.1.21 Sum, difference, product and quotient of two functions

If $f(x)$ and $g(x)$ are two functions having same domain and co-domain, then

(i) $(f \pm g)(x) = f(x) \pm g(x)$

(ii) $(fg)(x) = f(x)\, g(x)$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$

(iv) $(cf)(x) = cf(x)$, c is a constant.

5.1.22 Graph of a function

The graph of a function is the set of points $(x, f(x))$ where x belongs to the domain of the function and $f(x)$ is the value of the function at x.

To draw the graph of a function, we find a sufficient number of ordered pairs $(x, f(x))$ belonging to the function and join them by a smooth curve.

NOTE

It is enough to draw a diagram for graph of a function in white sheet itself.

Example 5.1

Find the domain for which the functions $f(x) = 2x^2-1$ and $g(x) = 1-3x$ are equal.

Solution

Given that $f(x) = g(x)$

$$\Rightarrow \quad 2x^2 - 1 = 1-3x$$

$$2x^2 + 3x - 2 = 0$$

$$(x+2)(2x-1) = 0$$

$$\therefore x = -2,\ x = \frac{1}{2}$$

Domain is $\left\{-2, \frac{1}{2}\right\}$

Example 5.2

$f = \{(1, 1), (2, 3)\}$ be a function described by the formula $f(x) = ax+b$. Determine a and b ?

Solution

Given that $f(x) = ax + b$, $f(1) = 1$ and $f(2) = 3$

$$\therefore \quad a + b = 1 \text{ and } 2a + b = 3$$

$$\Rightarrow \quad a = 2 \text{ and } b = -1$$

Example 5.3

If $f(x) = x + \frac{1}{x}$, $x > 0$, then show that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$

Solution

$f(x) = x + \frac{1}{x}$, $f(x^3) = x^3 + \frac{1}{x^3}$

and $f\left(\frac{1}{x}\right) = f(x)$

$$\begin{aligned}
\text{Now, } LHS &= [f(x)]^3 \\
&= \left[x + \frac{1}{x}\right]^3 \\
&= x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) \\
&= f(x^3) + 3f(x) \\
&= f(x^3) + 3f\left(\frac{1}{x}\right) \\
&= RHS
\end{aligned}$$

Example 5.4

Let f be defined by $f(x) = x - 5$ and g be defined by

$$g(x) = \begin{cases} \frac{x^2-25}{x+5} & \text{if } x \neq -5 \\ \lambda & \text{if } x = -5 \end{cases} \quad \text{Find } \lambda$$

such that $f(x) = g(x)$ for all $x \in R$

Solution

Given that $f(x) = g(x)$ for all $x \in R$

$$\therefore \quad f(-5) = g(-5)$$

$$-5 - 5 = \lambda$$

$$\therefore \quad \lambda = -10$$

Example 5.5

If $f(x) = 2^x$, then show that $f(x) \cdot f(y) = f(x+y)$

Solution

$$f(x) = 2^x \text{ (Given)}$$

$$\therefore \quad f(x+y) = 2^{x+y}$$

$$= 2^x \cdot 2^y$$

$$= f(x) \cdot f(y)$$

Example 5.6

If $f(x) = \frac{x-1}{x+1}$, $x > 0$, then show that $f[f(x)] = -\frac{1}{x}$

Solution

$$f(x) = \frac{x-1}{x+1} \text{ (Given)}$$

$$\therefore f[f(x)] = \frac{f(x)-1}{f(x)+1} = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}$$

$$= \frac{x-1-x-1}{x-1+x+1} = -\frac{2}{2x} = -\frac{1}{x}$$

Example 5.7

If $f(x) = \log\frac{1+x}{1-x}$, $0 < x < 1$, then show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

Solution

$$f(x) = \log\frac{1+x}{1-x} \quad \text{(Given)}$$

$$\therefore f\left(\frac{2x}{1+x^2}\right) = \log\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}$$

$$= \log\frac{1+x^2+2x}{1+x^2-2x} = \log\frac{(1+x)^2}{(1-x)^2}$$

$$= 2\log\frac{1+x}{1-x} = 2f(x)$$

Example 5.8

If $f(x) = x$ and $g(x) = |x|$, then find

(i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(fg)(x)$

Solution

(i) $(f+g)(x) = f(x) + g(x) = x + |x|$

$$= \begin{cases} x + x & \text{if } x \geq 0 \\ x - x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(ii) $(f-g)(x) = f(x) - g(x) = x - |x|$

$$= \begin{cases} x - x & \text{if } x \geq 0 \\ x - (-x) & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases}$$

(iii) $(fg)(x) = f(x)\,g(x) = x\,|x|$

$$= \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Example 5.9

A group of students wish to charter a bus for an educational tour which can accommodate atmost 50 students. The bus company requires at least 35 students for that trip. The bus company charges ₹ 200 per student up to the strength of 45. For more than 45 students, company charges each student ₹ 200 less $\frac{1}{5}$ times the number more than 45. Consider the number of students who participates the tour as a function, find the total cost and its domain.

Solution

Let 'x' be the number of students who participate the tour.

Then $35 \leq x \leq 50$ and x is a positive integer.

Formula : *Total cost = (cost per student) × (number of students)*

(i) If the number of students are between 35 and 45, then

the cost per student is ₹ 200

$\therefore$ The total cost is $y = 200x$.

(ii) If the number of students are between 46 and 50, then

the cost per student is ₹ $\left\{200 - \frac{1}{5}(x - 45)\right\}$ $= 209 - \frac{x}{5}$

$\therefore$ The total cost is $y = \left(209 - \frac{x}{5}\right)x = 209x - \frac{x^2}{5}$

$$\therefore y = \begin{cases} 200x & ; \ 35 \leq x \leq 45 \\ 209x - \frac{x^2}{5} & ; \ 46 \leq x \leq 50 \end{cases}$$ and

the domain is {35, 36, 37, ..., 50}.

Example 5.10

Draw the graph of the function

$f(x) = |x|$

Solution

$$y = f(x) = |x|$$
$$= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Choose suitable values for x and determine y.

Thus we get the following table.

x	−2	−1	0	1	2
y	2	1	0	1	2

Table : 5.1

Plot the points (−2, 2), (−1, 1), (0, 0), (1, 1), (2, 2) and draw a smooth curve.

The graph is as shown in the figure

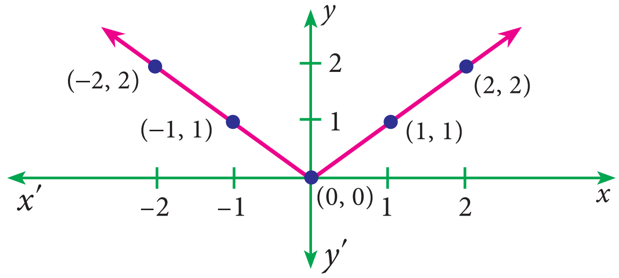

Fig. 5.4

Example 5.11

Draw the graph of the function $f(x) = x^2 - 5$

Solution

Let $y = f(x) = x^2 - 5$

Choose suitable values for x and determine y.

Thus we get the following table.

x	−3	−2	−1	0	1	2	3
y	4	−1	−4	−5	−4	−1	4

Table : 5.2

Plot the points (−3, 4), (−2, −1), (−1, −4), (0, −5), (1, −4), (2, −1), (3, 4) and draw a smooth curve.

The graph is as shown in the figure

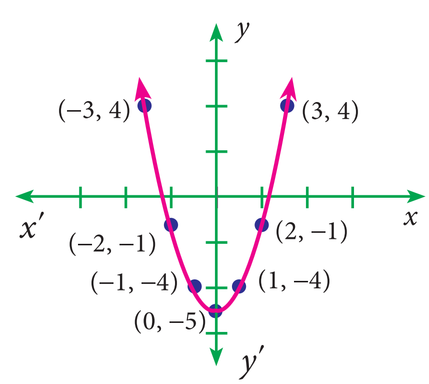

Fig 5.5

Example 5.12

Draw the graph of $f(x) = a^x$, $a \neq 1$ and $a > 0$

Solution

We know that, domain set is R, range set is $(0, \infty)$ and the curve passing through the point (0, 1)

Case (i) when $a > 1$

$$y = f(x) = a^x = \begin{cases} < 1 & \text{if } x < 0 \\ = 1 & \text{if } x = 0 \\ > 1 & \text{if } x > 0 \end{cases}$$

We noticed that as x increases, y is also increases and $y > 0$.

$\therefore$ The graph is as shown in the figure.

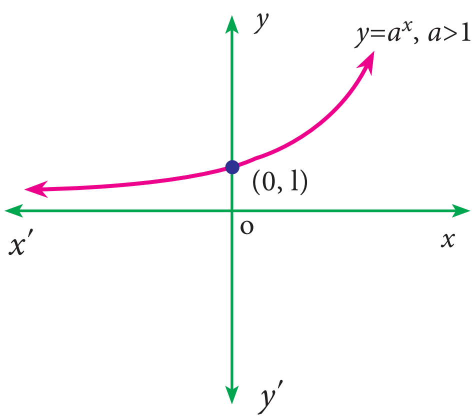

Fig 5.6

Case (ii) when $0 < a < 1$

$$y = f(x) = a^x = \begin{cases} > 1 & \text{if } x < 0 \\ = 1 & \text{if } x = 0 \\ < 1 & \text{if } x > 0 \end{cases}$$

We noticed that as x increases, y decreases and $y > 0$.

$\therefore$ The graph is as shown in the figure.

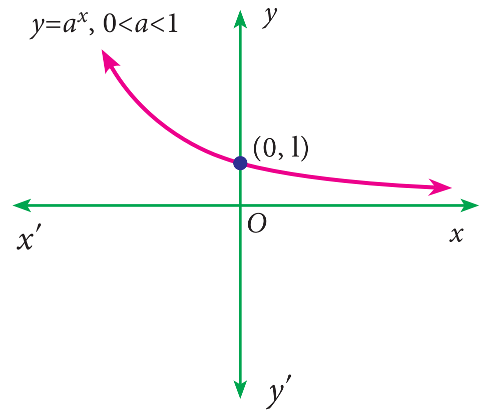

Fig 5.7

NOTE

(i) *The value of e lies between 2 and 3. i.e.* $2 < e < 3$

(ii) *The graph of* $f(x) = e^x$ *is identical to that of* $f(x) = a^x$ *if* $a > 1$ *and the graph of* $f(x) = e^{-x}$ *is identical to that of* $f(x) = a^x$ *if* $0 < a < 1$

Example 5.13

Draw the graph of $f(x) = \log_a x$; $x > 0$, $a > 0$ and $a \neq 1$.

Solution

We know that, domain set is $(0, \infty)$, range set is R and the curve passing the point (1, 0)

Case (i) when $a > 1$ and $x > 0$

$$f(x) = \log_a x = \begin{cases} < 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ > 0 & \text{if } x > 1 \end{cases}$$

We noticed that as x increases, $f(x)$ is also increases.

$\therefore$ The graph is as shown in the figure.

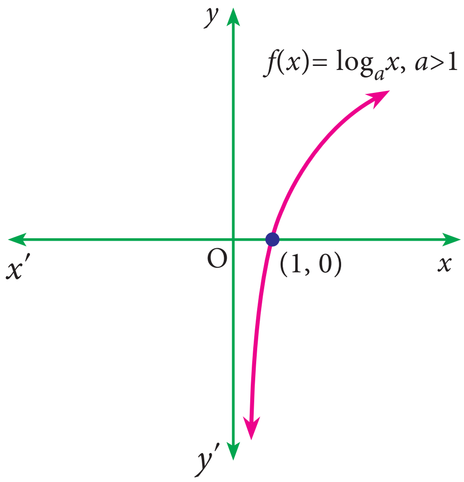

Fig 5.8

Case (ii) when $0 < a < 1$ and $x > 0$

$$f(x) = \log_a x = \begin{cases} > 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ < 0 & \text{if } x > 1 \end{cases}$$

We noticed that x increases, the value of $f(x)$ decreases.

$\therefore$ The graph is as shown in the figure.

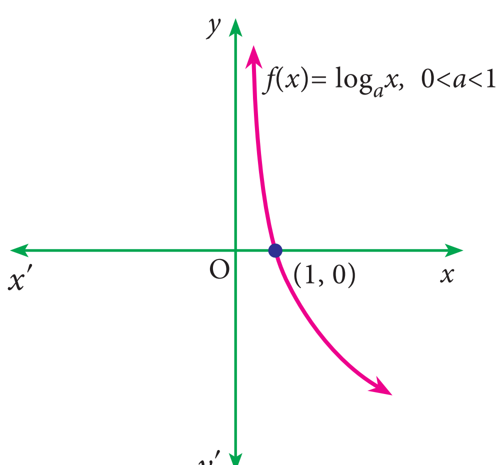

Fig 5.9

Exercise 5.1

1. Determine whether the following functions are odd or even?

   (i) $f(x) = \left(\dfrac{a^x - 1}{a^x + 1}\right)$

   (ii) $f(x) = \log\left(x^2 + \sqrt{x^2 + 1}\right)$

   (iii) $f(x) = \sin x + \cos x$

   (iv) $f(x) = x^2 - |x|$

   (v) $f(x) = x + x^2$

2. Let f be defined by $f(x) = x^3 - kx^2 + 2x$, $x \in R$. Find k, if 'f' is an odd function.

3. If $f(x) = x^3 - \dfrac{1}{x^3}$, $x \neq 0$, then show that $f(x) + f\left(\dfrac{1}{x}\right) = 0$

4. If $f(x) = \dfrac{x+1}{x-1}$, $x \neq 1$, then prove that $f(f(x)) = x$

5. For $f(x) = \dfrac{x-1}{3x+1}$, $x > 1$, write the expressions of $f\left(\dfrac{1}{x}\right)$ and $\dfrac{1}{f(x)}$

6. If $f(x) = e^x$ and $g(x) = \log_e x$, then find

   (i) $(f+g)(1)$ (ii) $(fg)(1)$

   (iii) $(3f)(1)$ (iv) $(5g)(1)$

7. Draw the graph of the following functions:

   (i) $f(x) = 16 - x^2$

   (ii) $f(x) = |x - 2|$

   (iii) $f(x) = x|x|$

   (iv) $f(x) = e^{2x}$

   (v) $f(x) = e^{-2x}$

   (vi) $f(x) = \dfrac{|x|}{x}$

5.2 Limits and Derivatives

In this section, we will discuss about limits, continuity of a function, differentiability and differentiation from first principle. Limits are used when we have to find the value of a function near to some value.

Definition 5.1

Let f be a real valued function of x. Let l and a be two fixed numbers. If $f(x)$ approaches the value l as x approaches to a, then we say that l is the limit of the function $f(x)$ as x tends to a, and written as

$$\lim_{x \to a} f(x) = l$$

Notations:

(i) $L[f(x)]_{x=a} = \lim_{x \to a^-} f(x)$ is known as left hand limit of $f(x)$ at $x=a$.

(ii) $R[f(x)]_{x=a} = \lim_{x \to a^+} f(x)$ is known as right hand limit of $f(x)$ at $x=a$.

5.2.1 Existence of limit

$\lim_{x \to a} f(x)$ exist if and only if both $L[f(x)]_{x=a}$ *and* $R[f(x)]_{x=a}$ exist and are equal.

i.e., $\lim_{x \to a} f(x)$ exist if and only if $L[f(x)]_{x=a} = R[f(x)]_{x=a}$.

NOTE

$L[f(x)]_{x=a}$ and $R[f(x)]_{x=a}$ are shortly written as $L[f(a)]$ and $R[f(a)]$

5.2.2 Algorithm of left hand limit: $L[f(x)]_{x=a}$

(i) Write $\lim_{x \to a^-} f(x)$.

(ii) Put $x=a-h$, $h>0$ and replace $x \to a^-$ by $h \to 0$.

(iii) Obtain $\lim_{h \to 0} f(a-h)$.

(iv) The value obtained in step (iii) is called as the value of left hand limit of the function $f(x)$ at $x=a$.

5.2.3 Algorithm of right hand limit: $R[f(x)]_{x=a}$

(i) Write $\lim_{x \to a^+} f(x)$.

(ii) Put $x = a+h$, $h>0$ and replace $x \to a^+$ by $h \to 0$.

(iii) Obtain $\lim_{h \to 0} f(a+h)$.

(iv) The value obtained in step (iii) is called as the value of right hand limit of the function $f(x)$ at $x=a$.

Example 5.14

Evaluate the left hand and right hand limits of the function

$$f(x) = \begin{cases} \dfrac{|x-3|}{x-3} & if\ x \neq 3 \\ 0 & if\ x = 3 \end{cases} \quad \text{at } x = 3$$

Solution

$$\begin{aligned} L[f(x)]_{x=3} &= \lim_{x \to 3^-} f(x) \\ &= \lim_{h \to 0} f(3-h),\ x = 3-h \\ &= \lim_{h \to 0} \frac{|(3-h)-3|}{(3-h)-3} \\ &= \lim_{h \to 0} \frac{|-h|}{-h} \\ &= \lim_{h \to 0} \frac{h}{-h} \end{aligned}$$

$$= \lim_{h \to 0} -1 = -1$$

$$R[f(x)]_{x=3} = \lim_{x \to 3^+} f(x)$$

$$= \lim_{h \to 0} f(3+h)$$

$$= \lim_{h \to 0} \frac{|(3+h)-3|}{(3+h)-3}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= \lim_{h \to 0} 1 = 1$$

NOTE

Here,

$L[f(3)] \neq R[f(3)]$

$\therefore \lim_{x \to 3} f(x)$ does not exist.

Example 5.15

Verify the existence of the function

$$f(x) = \begin{cases} 5x - 4 & \text{if } 0 < x \leq 1 \\ 4x^3 - 3x & \text{if } 1 < x < 2 \end{cases} \quad \text{at } x = 1.$$

Solution

$$L[f(x)]_{x=1} = \lim_{x \to 1^-} f(x)$$

$$= \lim_{h \to 0} f(1-h), \; x = 1-h$$

$$= \lim_{h \to 0} [5(1-h)-4]$$

$$= \lim_{h \to 0} (1-5h) = 1$$

$$R[f(x)]_{x=1} = \lim_{x \to 1^+} f(x)$$

$$= \lim_{h \to 0} f(1+h), \; x = 1+h$$

$$= \lim_{h \to 0} [4(1+h)^3 - 3(1+h)]$$

$$= 4(1)^3 - 3(1) = 1$$

Clearly, $L[f(1)] = R[f(1)]$

$\therefore \lim_{x \to 1} f(x)$ exists and equal to 1

NOTE

Let a be a point and $f(x)$ be a function, then the following stages may happen

(i) $\lim_{x \to a} f(x)$ exists but $f(a)$ does not exist.

(ii) The value of $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist.

(iii) Both $\lim_{x \to a} f(x)$ and $f(a)$ exist but are unequal.

(iv) Both $\lim_{x \to a} f(x)$ and $f(a)$ exist and are equal.

5.2.4 Some results of limits

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists, then

(i) $\lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

(iii) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, whenever $\lim_{x \to a} g(x) \neq 0$

(iv) $\lim_{x \to a} k\, f(x) = k \lim_{x \to a} f(x)$, where k is a constant.

5.2.5 Indeterminate forms and evaluation of limits

Let $f(x)$ and $g(x)$ are two functions in which the limits exist.

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then $\frac{f(a)}{g(a)}$ takes $\frac{0}{0}$ form which is meaningless but does not imply that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is meaningless.

In many cases, this limit exists and have a finite value. Finding the solution of such limits are called evaluation of the indeterminate form.

The indeterminate forms are $\frac{0}{0}, \frac{\infty}{\infty}, \frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, \infty^0$ and 1^∞.

Among all these indeterminate forms, $\frac{0}{0}$ is the fundamental one.

5.2.6 Methods of evaluation of algebraic limits

(i) Direct substitution

(ii) Factorization

(iii) Rationalisation

(iv) By using some standard limits

5.2.7 Some standard limits

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ if $n \in Q$

(ii) $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$, θ is in radian.

(iii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$ if $a > 0$

(iv) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(v) $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

(vi) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(vii) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Example 5.16

Evaluate: $\lim_{x \to 1} (3x^2 + 4x - 5)$.

Solution

$$\lim_{x \to 1} (3x^2 + 4x - 5)$$

$$= 3(1)^2 + 4(1) - 5 = 2$$

Example 5.17

Evaluate: $\lim_{x \to 2} \frac{x^2 - 4x + 6}{x + 2}$

Solution

$$\lim_{x \to 2} \frac{x^2 - 4x + 6}{x + 2} = \frac{\lim_{x \to 2}(x^2 - 4x + 6)}{\lim_{x \to 2}(x + 2)}$$

$$= \frac{(2)^2 - 4(2) + 6}{2 + 2} = \frac{1}{2}$$

Example 5.18

Evaluate : $\lim_{x \to \frac{\pi}{4}} \frac{5 \sin 2x - 2 \cos 2x}{3 \cos 2x + 2 \sin 2x}$

Solution

$$\lim_{x \to \frac{\pi}{4}} \frac{5 \sin 2x - 2 \cos 2x}{3 \cos 2x + 2 \sin 2x}$$

$$= \frac{\lim_{x \to \frac{\pi}{4}} (5 \sin 2x - 2 \cos 2x)}{\lim_{x \to \frac{\pi}{4}} (3 \cos 2x + 2 \sin 2x)}$$

$$= \frac{5 \sin \frac{\pi}{2} - 2 \cos \frac{\pi}{2}}{3 \cos \frac{\pi}{2} + 2 \sin \frac{\pi}{2}}$$

$$= \frac{5(1) - 2(0)}{3(0) + 2(1)} = \frac{5}{2}$$

Example 5.19

Evaluate: $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Solution

$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$ is of the type $\frac{0}{0}$

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{(x - 1)}$$

$$= \lim_{x \to 1} (x^2 + x + 1)$$

$$= (1)^2 + 1 + 1$$

$$= 3$$

Example 5.20

Evaluate: $\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

Solution

$$\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$$

$$= \lim_{x \to 0} \frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{2+x-2}{x(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{2+x}+\sqrt{2})}$$

$$= \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Example 5.21

Evaluate: $\lim_{x \to a} \frac{x^{\frac{3}{5}}-a^{\frac{3}{5}}}{x^{\frac{1}{5}}-a^{\frac{1}{5}}}$

Solution

$$\lim_{x \to a} \frac{x^{\frac{3}{5}}-a^{\frac{3}{5}}}{x^{\frac{1}{5}}-a^{\frac{1}{5}}} = \lim_{x \to a} \frac{\frac{x^{\frac{3}{5}}-a^{\frac{3}{5}}}{x-a}}{\frac{x^{\frac{1}{5}}-a^{\frac{1}{5}}}{x-a}}$$

$$= \frac{\frac{3}{5}(a)^{-\frac{2}{5}}}{\frac{1}{5}(a)^{-\frac{4}{5}}} = 3a^{\frac{-2}{5}+\frac{4}{5}} = 3a^{\frac{2}{5}}$$

Example 5.22

Evaluate: $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$

Solution

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \left\{ \frac{3x \times \frac{\sin 3x}{3x}}{5x \times \frac{\sin 5x}{5x}} \right\}$$

$$= \frac{3}{5} \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 5x}{5x}}$$

$$= \frac{3}{5}\left(\frac{1}{1}\right) = \frac{3}{5}$$

Example 5.23

Evaluate: $\lim_{x \to \infty} \frac{6-5x^2}{4x+15x^2}$

Solution

$$\lim_{x \to \infty} \frac{6-5x^2}{4x+15x^2} = \lim_{x \to \infty} \frac{\frac{6}{x^2}-5}{\frac{4}{x}+15}$$

$$= \frac{0-5}{0+15} = -\frac{1}{3}$$

Example 5.24

Evaluate: $\lim_{x \to \infty} x \tan\left(\frac{1}{x}\right)$

Solution

Let $y = \frac{1}{x}$ and $y \to 0$ as $x \to \infty$

$$\lim_{x \to \infty} x \tan\left(\frac{1}{x}\right) = \lim_{y \to 0} \frac{\tan y}{y} = 1$$

Example 5.25

Evaluate: $\lim_{n \to \infty} \frac{\sum n^2}{n^3}$

Solution

$$\lim_{n \to \infty} \frac{\sum n^2}{n^3} = \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n \to \infty} \frac{1}{6}\left\{\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right\}$$

$$= \frac{1}{6} \lim_{n \to \infty} \left\{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right\}$$

$$= \frac{1}{6}(1)(2) = \frac{1}{3}$$

Example 5.26

Show that $\lim_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x} = 1$

Solution

$$\lim_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x}$$

$$= \lim_{x \to 0} \left\{ \frac{\log(1+x^3)}{x^3} \times \frac{x^3}{\sin^3 x} \right\}$$

$$= \lim_{x \to 0} \frac{\log(1+x^3)}{x^3} \times \frac{1}{\lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)^3}$$

$$= 1 \times \frac{1}{1} = 1$$

Exercise 5.2

1. Evaluate the following

(i) $\lim_{x \to 2} \frac{x^3+2}{x+1}$

(ii) $\lim_{x \to \infty} \frac{2x+5}{x^2+3x+9}$

(iii) $\lim_{x \to \infty} \frac{\sum n}{n^2}$

(iv) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

(v) $\lim_{x \to a} \frac{x^{\frac{5}{8}} - a^{\frac{5}{8}}}{x^{\frac{2}{3}} - a^{\frac{2}{3}}}$

(vi) $\lim_{x \to 0} \frac{\sin^2 3x}{x^2}$

2. If $\lim_{x \to -a} \frac{x^9+a^9}{x+a} = \lim_{x \to 3}(x+6)$, then find the values of a.

3. If $\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 448$, then find the least positive integer n.

4. If $f(x) = \frac{x^7 - 128}{x^5 - 32}$, then find $\lim_{x \to 2} f(x)$

5. Let $f(x) = \frac{ax+b}{x+1}$, If $\lim_{x \to 0} f(x) = 2$ and $\lim_{x \to \infty} f(x) = 1$, then show that $f(-2) = 0$.

Derivative

Before going into the topic, let us first discuss about the continuity of a function. A function $f(x)$ is continues at $x = a$ if its graph has no break at $x = a$

If there is a break at the point $x = a$ then we say that the function is not continuous at the point $x = a$.

If a function is continuous at all the point in an interval, then it is said to be a continuous in that interval.

5.2.8 Continuous function

A function $f(x)$ is continuous at $x = a$ if

(i) $f(a)$ exists

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

Observation

In the above statement, if atleast any one condition is not satisfied at a point $x = a$ by the function $f(x)$, then it is said to be a discontinuous function at $x = a$

5.2.9 Some properties of continuous functions

If $f(x)$ and $g(x)$ are two real valued continuous functions at $x = a$, then

(i) $f(x) \pm g(x)$ is also continuous at $x = a$.

(ii) $f(x)\, g(x)$ is also continuous at $x = a$.

(iii) $kf(x)$ is also continuous at $x = a$, k be a real number.

(iv) $\frac{1}{f(x)}$ is also continuous at $x = a$, if $f(a) \neq 0$.

(v) $\frac{f(x)}{g(x)}$ is also continuous at $x = a$, if $g(a) \neq 0$.

(vi) $|f(x)|$ is also continuous at $x = a$.

Example 5.27

Show that

$$f(x) = \begin{cases} 5x-4, & \text{if } 0 < x \leq 1 \\ 4x^3 - 3x, & \text{if } 1 < x < 2 \end{cases} \text{ is}$$

continuous at $x = 1$

Solution

$$L[f(x)]_{x=1} = \lim_{x \to 1^-} f(x)$$

$$= \lim_{h \to 0} f(1-h),\ x = 1-h$$

$$= \lim_{h \to 0} [5(1-h)-4]$$

$$= 5(1)-4 = 1$$

$$R[f(x)]_{x=1} = \lim_{x \to 1^+} f(x)$$

$$= \lim_{h \to 0} f(1+h),\ x = 1+h$$

$$= \lim_{h \to 0} [4(1+h)^3 - 3(1+h)]$$

$$= 4(1)^3 - 3(1)$$

$$= 4 - 3 = 1$$

Now, $f(1) = 5(1) - 4 = 5-4 = 1$

$$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1),$$

$f(x)$ is continuous at $x = 1$

Example 5.28

Verify the continuity of the function $f(x)$ given by

$$f(x) = \begin{cases} 2-x \text{ if } x < 2 \\ 2+x \text{ if } x \geq 2 \end{cases} \quad \text{at } x = 2$$

Solution

$$L[f(x)]_{x=2} = \lim_{x \to 2^-} f(x)$$

$$= \lim_{h \to 0} f(2-h),\ x = 2-h$$

$$= \lim_{h \to 0} \{2-(2-h)\}$$

$$= \lim_{h \to 0} h = 0$$

$$R[f(x)]_{x=2} = \lim_{x \to 2^+} f(x)$$

$$= \lim_{h \to 0} f(2+h),\ x = 2+h$$

$$= \lim_{h \to 0} \{2+(2+h)\}$$

$$= \lim_{h \to 0} (4+h) = 4$$

Now, $f(2) = 2+2 = 4$

Here $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

Hence $f(x)$ is not continuous at $x = 2$

Observations

(i) Constant function is everywhere continuous.

(ii) Identity function is everywhere continuous.

(iii) Polynomial function is everywhere continuous.

(iv) Modulus function is everywhere continuous.

(v) Exponential function a^x, $a>0$ is everywhere continuous.

(vi) Logarithmic function is continuous in its domain.

(vii) Rational function is continuous at every point in its domain.

DO YOU KNOW?

If $f(x) = \begin{cases} \alpha(x) & if \quad x < a \\ k & if \quad x = a \\ \beta(x) & if \quad x > a \end{cases}$, then

$L[f(x)]_{x=a} = \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a} \alpha(x)$

$R[f(x)]_{x=a} = \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a} \beta(x)$

It is applicable only when the function has different definition on both sides of the given point $x = a$.

Exercise 5.3

1. Examine the following functions for continuity at indicated points

   (a) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & if \quad x \neq 2 \\ 0, & if \quad x = 2 \end{cases}$ at $x = 2$

   (b) $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & if \quad x \neq 3 \\ 6, & if \quad x = 3 \end{cases}$ at $x = 3$

2. Show that $f(x) = |\, x\,|$ is continuous at $x = 0$

5.2.10 Differentiability at a point

Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$. $f(x)$ is said to be differentiable or derivable at $x = c$ if and only if $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely.

This limit is called the derivative (or) differential co-efficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ (or) $D\,f(c)$ (or) $\left[\dfrac{d}{dx}(f(x))\right]_{x=c}$.

Thus $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

5.2.11 Left hand derivative and right hand derivative

(i) $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$ or

$\lim\limits_{h \to 0} \dfrac{f(c-h) - f(c)}{-h}$ is called the left hand derivative of $f(x)$ at $x = c$ and it denoted by $f'(c^-)$ or $L[f'(c)]$.

(ii) $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c}$ or

$\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ is called the right hand derivative of $f(x)$ at $x = c$ and it denoted by $f'(c^+)$ or $R[f'(c)]$.

Result	$f(x)$ is differentiable at $x = c \Leftrightarrow L[f'(c)] = R[f'(c)]$

Remarks:

(i) If $L[f'(c)] \neq R[f'(c)]$, then we say that $f(x)$ is not differentiable at $x=c$.

(ii) $f(x)$ is differentiable at $x = c \Rightarrow f(x)$ is continuous at $x = c$.

DO YOU KNOW?

A function may be continuous at a point but may not be differentiable at that point.

Example 5.29

Show that the function $f(x) = |\, x\,|$ is not differentiable at $x = 0$.

Solution

$$f(x) = |\, x\,| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

$$L[f'(0)] = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0},$$

$$= \lim_{h \to 0} \frac{f(0-h)-f(0)}{0-h-0},\ x = 0-h$$

$$= \lim_{h \to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h|-|0|}{-h}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0}(-1) = -1$$

$$R[f'(0)] = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h \to 0} \frac{f(0+h)-f(0)}{0+h-0},\ x = 0+h$$

$$= \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h|-|0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1 = 1$$

Here, $L[f'(0)] \neq R[f'(0)]$

$\therefore$ *f(x)* is not differentiable at *x=0*

Example 5.30

Show that $f(x) = x^2$ differentiable at $x=1$ and find $f'(1)$.

Solution

$$f(x) = x^2$$

$$L[f'(1)] = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{h \to 0} \frac{f(1-h)-f(1)}{1-h-1},\ x=1-h$$

$$= \lim_{h \to 0} \frac{(1-h)^2-1}{-h}$$

$$= \lim_{h \to 0} \frac{1+h^2-2h-1}{-h}$$

$$= \lim_{h \to 0}(-h+2) = 2$$

$$R[f'(1)] = \lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{h \to 0} \frac{f(1+h)-f(1)}{1+h-1},\ x = 1+h$$

$$= \lim_{h \to 0} \frac{(1+h)^2-1}{h}$$

$$= \lim_{h \to 0} \frac{1+h^2+2h-1}{h}$$

$$= \lim_{h \to 0}(h+2) = 2$$

Here, $L[f'(1)] = R[f'(1)]$

$\therefore$ $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$

5.2.12 Differentiation from first principle

The process of finding the derivative of a function by using the definition that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$ is called

the differentiation from first principle. It is convenient to write $f'(x)$ as $\frac{dy}{dx}$.

5.2.13 Derivatives of some standard functions using first principle

1. For $x \in R, \frac{d}{dx}(x^n) = n\,x^{n-1}$

Proof:

Let $f(x) = x^n$

$\therefore f(x+h) = (x+h)^n$

$$\frac{d}{dx}\big(f(x)\big) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

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$$= \lim_{h \to 0} \frac{(x+h)^n - (x)^n}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - (x)^n}{(x+h) - x}$$

$$= \lim_{z \to x} \frac{z^n - x^n}{z - x}, \text{ where } z = x + h \text{ and}$$

$$z \to x \text{ as } h \to 0$$

$$= nx^{n-1} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

i.e., $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(e^x) = e^x$

Proof:

Let $f(x) = e^x$

$\therefore f(x+h) = e^{x+h}$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \to 0} e^x \left[\frac{e^h - 1}{h}\right]$$

$$= e^x \lim_{h \to 0} \left[\frac{e^h - 1}{h}\right]$$

$$= e^x \times 1 \qquad \left[\because \lim_{x \to 0} \left[\frac{e^x - 1}{x}\right] = 1\right]$$

$$= e^x$$

i.e., $\frac{d}{dx}(e^x) = e^x$.

3. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

Proof:

Let $f(x) = \log_e x$

$\therefore f(x+h) = \log_e(x+h)$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\log_e(x+h) - \log_e x}{h}$$

$$= \lim_{h \to 0} \frac{\log_e\left(1 + \frac{h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\log_e\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \frac{1}{x}$$

$$= \frac{1}{x} \lim_{h \to 0} \frac{\log_e\left(1 + \frac{h}{x}\right)}{\frac{h}{x}}$$

$$= \frac{1}{x} \times 1$$

$$\left[\because \lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1\right]$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

Example 5.31

Find $\frac{d}{dx}(x^3)$ from first principle.

Solution

Let $f(x) = x^3$

$\therefore f(x+h) = (x+h)^3$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 3x(0) + (0)^2$$

$$= 3x^2$$

$$\frac{d}{dx}(x^3) = 3x^2$$

Example 5.32

Find $\frac{d}{dx}(e^{3x})$ from first principle.

Solution

Let $f(x) = e^{3x}$

$\therefore f(x+h) = e^{3(x+h)}$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{3(x+h)} - e^{3x}}{h}$$

$$= \lim_{h \to 0} \frac{e^{3x} \cdot e^{3h} - e^{3x}}{h}$$

$$= \lim_{h \to 0} \frac{e^{3x}(e^{3h} - 1)}{h}$$

$$= 3e^{3x} \lim_{h \to 0} \frac{e^{3h} - 1}{3h}$$

$$= 3e^{3x} \times 1 \quad [\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1]$$

$$= 3e^{3x}$$

Exercise 5.4

1. Find the derivative of the following functions from first principle.

   (i) x^2 (ii) e^{-x} (iii) $\log(x+1)$

5.3 Differentiation Techniques

In this section we will discuss about different techniques to obtain the derivatives of the given functions.

5.3.1 Some standard results [formulae]

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(k) = 0$, k is a constant

4. $\frac{d}{dx}(kx) = k$, k is a constant

5. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

6. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

7. $\frac{d}{dx}(e^x) = e^x$

8. $\frac{d}{dx}(e^{ax}) = ae^{ax}$

9. $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$

10. $\frac{d}{dx}(e^{-x^2}) = -2x\,e^{-x^2}$

11. $\frac{d}{dx}(a^x) = a^x \log_e a$, $a > 0$

12. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

13. $\frac{d}{dx}\log_e(x+a) = \frac{1}{x+a}$

14. $\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{(ax+b)}$

15. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$, a>0 and a≠1

16. $\frac{d}{dx}(\sin x) = \cos x$

17. $\frac{d}{dx}(\cos x) = -\sin x$

18. $\frac{d}{dx}(\tan x) = \sec^2 x$

19. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

20. $\frac{d}{dx}(\sec x) = \sec x \tan x$

21. $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec} x \cot x$

5.3.2 General rules for differentiation

(i) Addition rule

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]$$

(ii) Subtraction rule (or) Difference rule

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]$$

(iii) Product rule

$$\frac{d}{dx}[f(x)\cdot g(x)] = \frac{d}{dx}[f(x)]g(x)+f(x)\frac{d}{dx}[g(x)]$$

(or) If $u = f(x)$ and $v = g(x)$ then,

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$$

(iv) Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2},$$

$g(x) \neq 0$

(or) If $u = f(x)$ and $v = g(x)$ then,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}$$

(v) Scalar product

$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$, where c is a constant.

(vi) Chain rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$$

(or) If $y = f(t)$ and $t = g(x)$ then,

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

Here we discuss about the differentiation for the explicit functions using standard results and general rules for differentiation.

Example 5.33

Differentiate the following functions with respect to x.

(i) $x^{\frac{3}{2}}$ (ii) $7e^x$

(iii) $\frac{1-3x}{1+3x}$ (iv) $x^2 \sin x$

(v) $\sin^3 x$ (vi) $\sqrt{x^2+x+1}$

Solution

(i) $\frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{3}{2}-1}$

$= \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

(ii) $\frac{d}{dx}(7e^x) = 7\frac{d}{dx}(e^x) = 7e^x$

(iii) Differentiating $y = \frac{1-3x}{1+3x}$ with respect to x

$$\frac{dy}{dx} = \frac{(1+3x)\frac{d}{dx}(1-3x)-(1-3x)\frac{d}{dx}(1+3x)}{(1+3x)^2}$$

$$= \frac{(1+3x)(-3)-(1-3x)(3)}{(1+3x)^2}$$

$$= \frac{-6}{(1+3x)^2}$$

(iv) Differentiating $y = x^2 \sin x$ with respect to x

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x^2)$$

$$= x^2\cos x+2x\sin x$$

$$= x(x\cos x+2\sin x)$$

(v) $y = \sin^3 x$ (or) $(\sin x)^3$

Let $u = sin\ x$ then $y = u^3$

Differentiating $u = sin\ x$ with respect to x

We get, $\frac{du}{dx} = \cos x$

Differentiating $y = u^3$ with respect to u

We get, $\dfrac{dy}{du} = 3u^2 = 3\sin^2 x$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3\sin^2 x\cdot\cos x$$

(vi) $y = \sqrt{x^2+x+1}$

Let $u = x^2 + x + 1$ then $y = \sqrt{u}$

Differentiating $u = x^2 + x + 1$ with respect to x

We get, $\dfrac{du}{dx} = 2x + 1$

Differentiating $y = \sqrt{u}$ with respect to u.

We get, $\dfrac{dy}{du} = \dfrac{1}{2\sqrt{u}}$

$$= \frac{1}{2\sqrt{x^2+x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{x^2+x+1}}\cdot(2x+1)$$

$$= \frac{2x+1}{2\sqrt{x^2+x+1}}$$

DO YOU KNOW?

★ $\dfrac{d}{dx}(\sin(\log x))$

$$= \frac{d}{d(\log x)}(\sin(\log x))\frac{d}{dx}(\log x)$$

★ $\dfrac{d}{dx}(x^2+3x+1)^5$

$$= \frac{d}{d(x^2+3x+1)}(x^2+3x+1)^5 \frac{d}{dx}(x^2+3x+1)$$

★ $\dfrac{d}{dx}\left(e^{(x^3+3)}\right)$

$$= \frac{d}{d(x^3+3)}\left(e^{(x^3+3)}\right)\frac{d}{dx}(x^3+3)$$

Example 5.34

If $f(x) = x^n$ and $f'(1) = 5$, then find the value of n.

Solution

$$f(x) = x^n$$

$$\therefore\ f'(x) = nx^{n-1}$$

$$f'(1) = n(1)^{n-1}$$

$$f'(1) = n$$

$$f'(1) = 5 \text{ (given)}$$

$$\Longrightarrow n = 5$$

Example 5.35

If $y = \dfrac{1}{u^2}$ and $u = x^2 - 9$, then find $\dfrac{dy}{dx}$.

Solution

$$y = \frac{1}{u^2} = u^{-2}$$

$$\frac{dy}{du} = -\frac{2}{u^3}$$

$$\frac{dy}{du} = -\frac{2}{(x^2-9)^3} \quad (\because u = x^2 - 9)$$

$$u = x^2 - 9$$

$$\frac{du}{dx} = 2x$$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$= -\frac{2}{(x^2-9)^3}\cdot 2x$$

$$= -\frac{4x}{(x^2-9)^3}$$

Exercise 5.5

1. Differentiate the following with respect to x.

(i) $3x^4 - 2x^3 + x + 8$

(ii) $\dfrac{5}{x^4} - \dfrac{2}{x^3} + \dfrac{5}{x}$

(iii) $\sqrt{x}+\frac{1}{\sqrt[3]{x}}+e^x$

(iv) $\frac{3+2x-x^2}{x}$

(v) $x^3 e^x$

(vi) $(x^2-3x+2)(x+1)$

(vii) $x^4-3\sin x+\cos x$

(viii) $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

2. Differentiate the following with respect to x.

(i) $\frac{e^x}{1+x}$ (ii) $\frac{x^2+x+1}{x^2-x+1}$

(iii) $\frac{e^x}{1+e^x}$

3. Differentiate the following with respect to x.

(i) $x\sin x$ (ii) $e^x\sin x$

(iii) $e^x(x+\log x)$ (iv) $\sin x\cos x$

(v) $x^3 e^x$

4. Differentiate the following with respect to x.

(i) $\sin^2 x$ (ii) $\cos^2 x$

(iii) $\cos^3 x$ (iv) $\sqrt{1+x^2}$

(v) $(ax^2+bx+c)^n$ (vi) $\sin(x^2)$

(vii) $\frac{1}{\sqrt{1+x^2}}$

5.3.3 Derivative of implicit functions

For the implicit function $f(x,y)=0$, differentiate each term with respect to x treating y as a function of x and then collect the terms of $\frac{dy}{dx}$ together on left hand side and remaining terms on the right hand side and then find $\frac{dy}{dx}$.

NOTE

If the function $f(x, y) = 0$ is an implicit function, then $\frac{dy}{dx}$ contains both x and y.

Example 5.36

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then find $\frac{dy}{dx}$.

Solution

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Differentiating both side with respect to x,

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$(2ax+2hy+2g)+(2hx+2by+2f)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{(ax+hy+g)}{(hx+by+f)}.$$

Example 5.37

If $x^3+y^3=3axy$, then find $\frac{dy}{dx}$.

Solution

$$x^3+y^3=3axy$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(3axy)$$

$$3x^2+3y^2\frac{dy}{dx}=3a\left[x\frac{dy}{dx}+y\right]$$

$$x^2+y^2\frac{dy}{dx}=ax\frac{dy}{dx}+ay$$

$$(y^2-ax)\frac{dy}{dx}=(ay-x^2)$$

$$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$$

Example 5.38

Find $\frac{dy}{dx}$ at (1,1) to the curve

$2x^2 + 3xy + 5y^2 = 10$

Solution

$2x^2 + 3xy + 5y^2 = 10$

Differentiating both sides with respect to x,

$$\frac{d}{dx}\left[2x^2 + 3xy + 5y^2\right] = \frac{d}{dx}[10]$$

$$4x + 3x\frac{dy}{dx} + 3y + 10y\frac{dy}{dx} = 0$$

$$(3x+10y)\frac{dy}{dx} = -3y - 4x$$

$$\frac{dy}{dx} = -\frac{(3y+4x)}{(3x+10y)}$$

$$\text{Now, } \frac{dy}{dx} \text{ at } (1, 1) = -\frac{3+4}{3+10} = -\frac{7}{13}$$

Example 5.39

If $\sin y = x \sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Solution

$\sin y = x \sin(a+y)$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating with respect to y,

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cdot \cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Exercise 5.6

1. Find $\frac{dy}{dx}$ for the following functions

   (i) $xy = \tan(xy)$

   (ii) $x^2 - xy + y^2 = 7$

   (iii) $x^3 + y^3 + 3axy = 1$

2. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, then prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$

3. If $4x + 3y = \log(4x - 3y)$, then find $\frac{dy}{dx}$

5.3.4 Logarithmic differentiation

Some times, the function whose derivative is required involves products, quotients, and powers. For such cases, differentiation can be carried out more conveniently if we take logarithms and simplify before differentiation.

Example 5.40

Differentiate the following with respect to x.

(i) x^x (ii) $(\log x)^{\cos x}$

Solution

(i) Let $y = x^x$

Taking logarithm on both sides

$\log y = x \log x$

Differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\frac{dy}{dx} = y[1 + \log x]$$

$$\therefore \frac{dy}{dx} = x^x[1 + \log x]$$

(ii) Let $y = (\log x)^{\cos x}$

Taking logarithm on both sides

$\log y = \cos x \; \log(\log x)$

Differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \frac{1}{\log x}\cdot\frac{1}{x} + [\log(\log x)](-\sin x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x\log x} - \sin x \; \log(\log x)\right]$$

$$= (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x\,\log(\log x)\right]$$

Example 5.41

If $x^y = e^{x-y}$, then prove that

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Solution

$x^y = e^{x-y}$

Taking logarithm on both sides,

$$y \log x = (x-y)$$

$$y(1+\log x) = x$$

$$y = \frac{x}{1+\log x}$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{(1+\log x)(1) - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2}$$

Example 5.42

Differentiate: $\sqrt{\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Solution

Let $y = \sqrt{\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}}$

$$= \left[\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right]^{\frac{1}{2}}$$

Taking logarithm on both sides,

$$\log y = \frac{1}{2}\left[\log(x-3)+\log(x^2+4) - \log(3x^2+4x+5)\right]$$

$$\left[\because \log ab = \log a + \log b \quad \text{and} \quad \log\frac{a}{b} = \log a - \log b\right]$$

Differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

Exercise 5.7

1. Differentiate the following with respect to x,

   (i) $x^{\sin x}$ (ii) $(\sin x)^x$

   (iii) $(\sin x)^{\tan x}$

   (iv) $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x^2+x+1)}}$

2. If $x^m \cdot y^n = (x+y)^{m+n}$, then show that $\dfrac{dy}{dx} = \dfrac{y}{x}$

5.3.5 Differentiation of parametric functions

If the variables x and y are functions of another variable namely t, then the functions are called a parametric functions. The variable t is called the parameter of the function.

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

5.3.6 Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of x. The derivative of $f(x)$ with respect to $g(x)$ is given by the formula,

$$\frac{d\left(f(x)\right)}{d\left(g(x)\right)} = \frac{du/dx}{dv/dx}$$

Example 5.43

Find $\frac{dy}{dx}$ if

(i) $x = at^2$, $y = 2at$

(ii) $x = a\cos\theta, \ y = a\sin\theta$

Solution

(i)

$x = at^2$	$y = 2at$
$\frac{dx}{dt} = 2at$	$\frac{dy}{dt} = 2a$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

(ii) $x = a\cos\theta, \ y = a\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$	$\frac{dy}{d\theta} = a\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

Example 5.44

If $x = a\theta$ and $y = \frac{a}{\theta}$, then prove that $\frac{dy}{dx} + \frac{y}{x} = 0$

Solution

$x = a\theta$	$y = \frac{a}{\theta}$
$\frac{dx}{d\theta} = a$	$\frac{dy}{d\theta} = \frac{-a}{\theta^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\left(\frac{-a}{\theta^2}\right)}{a} = -\frac{1}{\theta^2} = -\frac{y}{x}$$

i.e. $\frac{dy}{dx} + \frac{y}{x} = 0$

Aliter :

Take $xy = a\theta \cdot \frac{a}{\theta}$

$$xy = a^2$$

Differentiating with respect to x,

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

Example 5.45

Differentiate $\frac{x^2}{1+x^2}$ with respect to x^2

Solution

Let $u = \frac{x^2}{1+x^2}$	and $v = x^2$
$\frac{du}{dx} = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2}$	$\therefore \frac{dv}{dx} = 2x$
$= \frac{2x}{(1+x^2)^2}$	

$$\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\left[\frac{2x}{(1+x^2)^2}\right]}{2x}$$

$$= \frac{1}{(1+x^2)^2}$$

Exercise 5.8

1. Find $\frac{dy}{dx}$ of the following functions

   (i) $x = ct, \ y = \frac{c}{t}$

   (ii) $x = \log t, \ y = \sin t$

   (iii) $x = a\cos^3\theta, \ y = a\sin^3\theta$

   (iv) $x = a(\theta - \sin\theta), \ y = a(1-\cos\theta)$

2. Differentiate $\sin^3 x$ with respect to $\cos^3 x$

3. Differentiate $\sin^2 x$ with respect to x^2

5.3.7 Successive differentiation

The process of differentiating the same function again and again is called successive differentiation.

(i) The derivative of y with respect to x is called the first order derivative and is denoted by $\frac{dy}{dx}$ (or) y_1 (or) $f'(x)$

(ii) If $f'(x)$ is differentiable, then the derivative of $f'(x)$ with respect to x is called the second order derivative and is denoted by $\frac{d^2y}{dx^2}$ (or) y_2 (or) $f''(x)$

(iii) Further $\frac{d^ny}{dx^n}$ (or) y_n (or) $f^{(n)}(x)$ is called n^{th} order derivative of the function $y = f(x)$

Remarks

(i) If $y = f(x)$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

(ii) If $x = f(t)$ and $y = g(t)$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left\{\frac{g'(t)}{f'(t)}\right\}\cdot\frac{dt}{dx}$$

Example 5.46

Find the second order derivative of the following functions with respect to x,

(i) $3\cos x + 4\sin x$

(ii) $x = at^2, \quad y = 2at$

(iii) $x\sin x$

Solution

(i) Let $y = 3\cos x + 4\sin x$

$y_1 = -3\sin x + 4\cos x$

$y_2 = -3\cos x - 4\sin x$

$y_2 = -(3\cos x + 4\sin x)$

$y_2 = -y$ (or) $y_2 + y = 0$

(ii) $x = at^2, \qquad y = 2at$

$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{1}{t}$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{1}{2at}$$

$$= -\frac{1}{t^2}\cdot\frac{1}{2at}$$

$$= -\frac{1}{2at^3}$$

(iii) $y = x \sin x$

$$\frac{dy}{dx} = x\cos x + \sin x$$

$$\frac{d^2y}{dx^2} = -x\sin x + \cos x + \cos x$$

$$= 2\cos x - x\sin x$$

Example 5.47

If $y = A\sin x + B\cos x$, then prove that $y_2 + y = 0$

Solution

$$y = A\sin x + B\cos x$$

$$y_1 = A\cos x - B\sin x$$

$$y_2 = -A\sin x - B\cos x$$

$$y_2 = -y$$

$$y_2 + y = 0$$

Exercise 5.9

1. Find y_2 for the following functions

   (i) $y = e^{3x+2}$

   (ii) $y = \log x + a^x$

   (iii) $x = a\cos\theta,\ y = a\sin\theta$

2. If $y = 500e^{7x} + 600e^{-7x}$, then show that $y_2 - 49y = 0$

3. If $y = 2 + \log x$, then show that $xy_2 + y_1 = 0$

4. If $y = a\cos mx + b\sin mx$, then show that $y_2 + m^2y = 0$

5. If $y = (x + \sqrt{1+x^2})^m$, then show that $(1+x^2)y_2 + xy_1 - m^2y = 0$

6. If $y = \sin(\log x)$, then show that $x^2y_2 + xy_1 + y = 0$.

Exercise 5.10

Choose the correct answer

1. If $f(x) = x^2 - x + 1$, then $f(x+1)$ is

   (a) x^2 (b) x

   (c) 1 (d) $x^2 + x + 1$

2. If $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x+2 & \text{if } x < 2 \end{cases}$, then $f(5)$ is

   (a) –1 (b) 2

   (c) 5 (d) 7

3. If $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x+2 & \text{if } x < 2 \end{cases}$, then $f(0)$ is

   (a) 2 (b) 5

   (c) –1 (d) 0

4. If $f(x) = \frac{1-x}{1+x}$, $x > 1$, then $f(-x)$ is equal to

   (a) $-f(x)$ (b) $\frac{1}{f(x)}$

   (c) $-\frac{1}{f(x)}$ (d) $f(x)$

5. The graph of the line $y = 3$ is

   (a) Parallel to x-axis

   (b) Parallel to y-axis

   (c) Passing through the origin

   (d) Perpendicular to x-axis

6. The graph of $y = 2x^2$ is passing through

   (a) (0,0) (b) (2,1)

   (c) (2,0) (d) (0,2)

7. The graph of $y = e^x$ intersect the y-axis at

(a) (0, 0) (b) (1, 0)
(c) (0, 1) (d) (1, 1)

8. The minimum value of the function $f(x)=|x|$ is

(a) 0 (b) –1
(c) +1 (d) $-\infty$

9. Which one of the following functions has the property $f(x) = f\left(\frac{1}{x}\right)$, provided $x \neq 0$

(a) $f(x) = \frac{x^2 - 1}{x}$

(b) $f(x) = \frac{1 - x^2}{x}$

(c) $f(x) = x$

(d) $f(x) = \frac{x^2 + 1}{x}$

10. If $f(x) = 2^x$ and $g(x) = \frac{1}{2^x}$, then $(fg)(x)$ is

(a) 1 (b) 0
(c) 4^x (d) $\frac{1}{4^x}$

11. Which of the following function is neither even nor odd?

(a) $f(x) = x^3 + 5$ (b) $f(x) = x^5$
(c) $f(x) = x^{10}$ (d) $f(x) = x^2$

12. $f(x) = -5$, for all $x \in R$ is a

(a) an identity function

(b) modulus function

(c) exponential function

(d) constant function

13. The range of $f(x) = |x|$, for all $x \in R$ is

(a) $(0, \infty)$ (b) $[0, \infty)$
(c) $(-\infty, \infty)$ (d) $[1, \infty)$

14. The graph of $f(x) = e^x$ is identical to that of

(a) $f(x) = a^x, a > 1$

(b) $f(x) = a^x, a < 1$

(c) $f(x) = a^x, 0 < a < 1$

(d) $y = ax + b, a \neq 0$

15. If $f(x) = x^2$ and $g(x) = 2x+1$, then $(fg)(0)$ is

(a) 0 (b) 2
(c) 1 (d) 4

16. $\lim\limits_{\theta \to 0} \frac{\tan \theta}{\theta} =$

(a) 1 (b) ∞
(c) $-\infty$ (d) θ

17. $\lim\limits_{x \to 0} \frac{e^x - 1}{x} =$

(a) e (b) nx^{n-1}
(c) 1 (d) 0

18. For what value of x, $f(x) = \frac{x+2}{x-1}$ is not continuous?

(a) –2 (b) 1
(c) 2 (d) –1

19. If the function $f(x)$ is continuous at $x = a$, then $\lim\limits_{x \to a} f(x)$ is equal to

(a) $f(-a)$ (b) $f\left(\frac{1}{a}\right)$
(c) $2f(a)$ (d) $f(a)$

20. $\frac{d}{dx}\left(\frac{1}{x}\right)$ is equal to

(a) $-\frac{1}{x^2}$ (b) $-\frac{1}{x}$
(c) $\log x$ (d) $\frac{1}{x^2}$

21. $\frac{d}{dx}(5e^x - 2\log x)$ is equal to

(a) $5e^x - \frac{2}{x}$

(b) $5e^x - 2x$

(c) $5e^x - \frac{1}{x}$

(d) $2\log x$

22. If $y = x$ and $z = \frac{1}{x}$,then $\frac{dy}{dz}$ =

(a) x^2 (b) 1

(c) $-x^2$ (d) $-\frac{1}{x^2}$

23. If $y = e^{2x}$,then $\frac{d^2y}{dx^2}$ at $x = 0$ is

(a) 4 (b) 9

(c) 2 (d) 0

24. If $y = \log x$,then y_2 =

(a) $\frac{1}{x}$ (b) $-\frac{1}{x^2}$

(c) $-\frac{2}{x^2}$ (d) e^2

25. $\frac{d}{dx}(a^x)$ =

(a) $\frac{1}{x\log_e a}$ (b) a^a

(c) $x\log_e a$ (d) $a^x\log_e a$

Miscellaneous Problems

1. If $f(x) = \frac{1}{2x+1}$, $x > -\frac{1}{2}$, then show that $f(f(x)) = \frac{2x+1}{2x+3}$.

2. Draw the graph of $y = 9 - x^2$

3. If $f(x) = \begin{cases} \frac{x-|x|}{x} & if \;\; x \neq 0 \\ 2 & if \;\; x = 0 \end{cases}$ then show that $\lim_{x\to 0} f(x)$ does not exist.

4. Evaluate: $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

5. Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$

6. Verify the continuity and differentiability of $f(x) = \begin{cases} 1-x & if \;\; x<1 \\ (1-x)(2-x) & if \;\; 1\le x \le 2 \\ 3-x & if \;\; x>2 \end{cases}$ at $x = 1$ and $x = 2$

7. If $x^y = y^x$, then prove that $\frac{dy}{dx} = \frac{y}{x}\left(\frac{x\log y - y}{y\log x - x}\right)$

8. If $xy^2 = 1$, then prove that $2\frac{dy}{dx} + y^3 = 0$.

9. If $y = \tan x$, then prove that $y_2 - 2yy_1 = 0$.

10. If $y = 2\sin x + 3\cos x$, then show that $y_2 + y = 0$.

Summary

- Let A and B be two non empty sets, then a function f from A to B, associates every element of A to an unique element of B.
- $\lim_{x\to a} f(x)$ exists $\Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.
- For the function $f(x)$ and a real number a, $\lim_{x\to a} f(x)$ and $f(a)$ may not be the same.
- A function $f(x)$ is a continuous function at $x = a$ if only if $\lim_{x\to a} f(x) = f(a)$.
- A function $f(x)$ is continuous, if it is continuous at every point of its domain.
- If $f(x)$ and $g(x)$ are continuous on their common domain, then $f \pm g$, $f \cdot g$, kf (k is a constant) are continuous and if $g \neq 0$ then $\frac{f}{g}$ is also continuous.

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- $L[f'(c)] = \lim_{h \to 0} \frac{f(c-h)-f(c)}{-h}$ and $R[f'(c)] = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$
- A function $f(x)$ is said to be differentiable at $x = c$ if and only if $L[f'(c)] = R[f'(c)]$.
- A function $f(x)$ is differentiable at $x = c$ if and only if $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists finitely and denoted by $f'(c)$.
- Every differentiable function is continuous but, the converse is not necessarily true.
- If $y = f(x)$ then $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called second order derivative of y with respect to x.
- If $x = f(t)$ and $y = g(t)$ then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{g'(t)}{f'(t)}\right\}$ (or) $\frac{d^2y}{dx^2} = \frac{d}{dt}\left\{\frac{g'(t)}{f'(t)}\right\}\cdot\frac{dt}{dx}$

GLOSSARY (கலைச்சொற்கள்)	
Absolute constants	முழுமையான மாறிலிகள்
Algebric functions	இயற்கணித சார்புகள்
Arbitrary constants	தன்னிச்சை மாறிலிகள்
Chain rule	சங்கிலி விதி
Closed interval	மூடிய இடைவெளி
Constant	மாறிலி
Continuous	தொடர்ச்சி
Dependent variable	சார்ந்த மாறி
Derivative	வகையீடு
Domain	சார்பகம் / அரங்கம்
Explict	வெளிபடு
Exponential	அடுக்கு
Function	சார்பு
Identity	சமனி
Implict	உட்படு
Independent variable	சாரா மாறி
Interval	இடைவெளி
Left limit	இடக்கை எல்லை
Limit	எல்லை
Logarithmic	மடக்கை
Modulus	மட்டு
Neighbourhood	அண்மையகம்
Open interval	திறந்த இடைவெளி
Parametric functions	துணையலகு சார்புகள்
Range	வீச்சகம்
Right limit	வலக்கை எல்லை
Signum function	குறிச் சார்பு
Successive differentiation	தொடர் வகையிடல்
Transcendental functions	விஞ்சிய சார்புகள்
Variable	மாறி