Chapter 4

Linear Algebra

CHAPTER HIGHLIGHTS

- *Introduction*
- *Determinants*
- *Systems of linear equations*

Introduction

A set of 'mn' elements arranged in the form of rectangular array having 'm' rows and 'n' columns is called an $m \times n$ matrix (read as 'm by n matrix') and is denoted by $A = [a_{ij}]$ where $1 \le i \le m$; $1 \le j \le n$

or
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{pmatrix}$$

The element a_{ij} lies in the ith row and jth column.

Type of Matrices

Square Matrix A matrix $A = [a_{ij}]_{m\times n}$ is said to be a square matrix, if $m = n$ (i.e., Number of rows of A = Number of columns of A)

The elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called 'DIAGONAL ELEMENTS'.

The line containing the diagonal elements is the 'PRINCIPAL DIAGONAL'.

The sum of the diagonal elements of 'A' is the 'TRACE' of A.

Row Matrix A matrix $A = [a_{ij}]_{m\times n}$ is said to be row matrix, if $m = 1$ (i.e., the matrix has only one row)

General form is $A = [a_1, a_2, \ldots, a_n]$ or $[a_{ij}]_{1\times n}$

Column Matrix A matrix which has only one column

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ or } [a_{ij}]_{n\times 1}$$

Diagonal Matrix A square matrix is said to be a diagonal matrix if all its elements except those in the principal diagonal are zeros. That is, if

1. $m = n$ (A is a square matrix) and
2. $a_{ij} = 0$ if $i \ne j$ (The non-diagonal elements are zeros)

A diagonal matrix of order 'n' with diagonal elements d_1, $d_2, \ldots, d_n$ is denoted by Diag $[d_1\ d_2 \ldots d_n]$.

Scalar Matrix A diagonal matrix whose diagonal elements are all equal is called a scalar matrix. That is, if

1. $m = n$
2. $a_{ij} = 0$ if $i \ne j$
3. $a_{ij} = k$ if $i = j$ for some constant 'k'.

Unit or Identity Matrix A scalar matrix of order 'n' in which each diagonal element is '1' (unity) is called a unit matrix or identity matrix of order 'n' and is denoted by I_n. That is,

1. $m = n$
2. $a_{ij} = 0$ if $i \neq j$
3. $a_{ij} = 1$ if $i = j$

Example: $I_1 = [1],\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Null Matrix or Zero Matrix A matrix is a 'null matrix' or zero matrix if all its elements are zeros.

Upper Triangular Matrix A square matrix is said to be an upper triangular matrix, if each element below the principal diagonal is zero. That is,

1. $m = n$
2. $a_{ij} = 0$ if $i > j$

For example, $\begin{pmatrix} 1 & 4 & 3 & 2 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 9 \end{pmatrix}_{4\times 4}$

Lower Triangular Matrix A square matrix is said to be a lower triangular matrix, if each element above the principal diagonal is zero, i.e., if

1. $m = n$
2. $a_{ij} = 0$ if $i < j$

For example, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 7 & 8 & 0 \\ 5 & 4 & 2 & 1 \end{pmatrix}$

Horizontal Matrix If the number of rows of a matrix is less than the number of columns, i.e., $m < n$, then the matrix is called horizontal matrix.

Vertical Matrix If the number of columns in a matrix is less than the number of rows, i.e., if $m > n$, then the matrix is called a vertical matrix.

Comparable Matrices Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ are said to be comparable, if they are of same order, i.e., $m = p$; $n = q$.

Equal Matrices Two comparable matrices are said to be 'equal', if the corresponding elements are equal, i.e., $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ are equal if

1. $m = p$; $n = q$ (i.e., they are of the same order)
2. $a_{ij} = b_{ij}\ \forall\ i, j$ (i.e., the corresponding elements are equal)

Transpose of a Matrix

The matrix obtained by interchanging the rows and the columns of a given matrix 'A' is called the 'transpose' of A and is denoted by A^T or A'. If A is an $(m \times n)$ matrix, A^T will be an $(n \times m)$ matrix. Thus if $A = [a_{ij}]_{m\times n}$ then $A^T = [u_{ij}]_{n\times m}$, where $u_{ij} = a_{ji}$.

Properties of Transpose

$T-1$: $(A')' = A$, for any matrix A

$T-2$: $(A + B)' = A' + B'$, for any two matrices A, B of same order

$T-3$: $(KA)' = KA'$, for any matrix A

$T-4$: $(AB)' = B'A'$, for any matrices A, B such that number of columns of A = number of rows of B (REVERSAL LAW)

$T-5$: $(A^n)' = (A')^n$, for any square matrix A

Trace of a Matrix

Let 'A' be a square matrix. The trace of A is defined as the sum of elements of 'A' lying in the principal diagonal.

Thus if $A = [a_{ij}]_{n\times n}$ then trace of 'A' denoted by $t_r A = a_{11} + a_{22} + \cdots + a_{nn}$.

Properties of Trace of a Matrix Let A and B be any two square matrices and K any scalar then,

1. $t_r(A + B) = t_r A + t_r B$
2. $t_r(KA) = K t_r A$
3. $t_r(AB) = t_r(BA)$

Conjugate of a Matrix

A matrix obtained by replacing each element of a matrix 'A' by its complex conjugate is called the 'conjugate matrix' of A and is denoted by $\bar{A}$. If $A = [a_{ij}]_{m\times n}$, then $\bar{A} = \left[\overline{a_{ij}}\right]$ where $\bar{a}_{ij}$ is the conjugate of 'a_{ij}'.

Properties of Conjugate of a Matrix

$C-1$: $\overline{(\overline{A})} = A$ for any matrix 'A'

$C-2$: $\overline{(A+B)} = \bar{A} + \bar{B}$ for any matrices A, B of same order.

$C-3$: $\overline{(KA)} = \bar{K}\,\bar{A}$ for any matrix 'A' and any Scalar K.

$C-4$: $\overline{(AB)} = (\bar{A})\cdot\bar{B}$ for any matrices A and B with the condition that number of columns of A = number of rows of B.

$C-5$: $(\bar{A})^n = \overline{(A^n)}$ for any square matrix 'A'.

Tranjugate or Transposed Conjugate of a Matrix

Tranjugate of a matrix 'A' is obtained by transposing the conjugate of A and is denoted by A^θ. Thus $A^\theta = (\bar{A})^T$.

Properties of Tranjugate of a Matrix

TC – 1: $(A^\theta)^\theta = A$ for any matrix A

TC – 2: $(A + B)^\theta = A^\theta + B^\theta$ for any matrices A, B of the same order.

TC – 3: $(KA)^\theta = KA^\theta$ for any matrix A and any scalar K.

TC – 4: $(BA)^\theta = B^\theta A^\theta$ for any matrix A, B with the condition that number of columns of A = number of rows of B.

TC – 5: $(A^n)^\theta = (A^\theta)^n$ for any square matrix 'A'.

Symmetric Matrix A matrix A is said to be symmetric, if $A^T = A$ (i.e., transpose of $A = A$).

NOTE

A symmetric matrix must be a square matrix.

Skew-symmetric Matrix A matrix 'A' is said to be skew-symmetric matrix, if $A^T = (-A)$, i.e., $A = [a_{ij}]_{m\times n}$ is skew symmetric if

1. $m = n$
2. $a_{jI} = -a_{ij} \forall\, i, j$

NOTE

In a skew-symmetric matrix, all the elements of the principal diagonal are zero.

Orthogonal Matrix A square matrix 'A' of order $n \times n$ is said to be an orthogonal matrix, if $AA^T = A^TA = I_n$.

Involutory Matrix A square matrix 'A' is said to be involutory matrix, if $A^2 = I$ (where I is identity matrix).

Idempotent Matrix A square matrix 'A' is said to be an idempotent matrix, if $A^2 = A$.

Nilpotent Matrix A square matrix 'A' is said to be nilpotent matrix, if there exists a natural number 'n' such that $A^n = O$. If 'n' is the least natural number such that $A^n = O$, then 'n' is called the index of the nilpotent matrix 'A'. (Where O is the null matrix).

Unitary Matrix A square matrix 'A' is said to be a unitary matrix if, $AA^\theta = A^\theta A = I$. (Where A^θ is the transposed conjugate of A.)

Hermitian Matrix A matrix 'A' is said to be a hermitian matrix, if $A^\theta = A$, i.e., $A = [a_{ij}]_{m\times n}$ is hermitian if

1. $m = n$
2. $a_{ij} = \bar{a}_{ij} \forall i, j$

NOTE

The diagonal elements in a hermitian matrix are real numbers.

Skew-hermitian Matrix A matrix 'A' is said to be a skew-hermitian matrix, if $A^\theta = -A$.

Operations on Matrices

Scalar Multiplication of Matrices

If A is a matrix of order $m \times n$ and 'K' be any scalar (a real or complex number), then KA is defined to be a $m \times n$ matrix whose elements are obtained by multiplying each element of 'A' by K, i.e., if $A = [a_{ij}]_{m\times n}$ then $KA = [Ka_{ij}]_{m\times n}$ in particular if $K = -1$; then $KA = -A$ is called the negative of A and is such that,

$A + (-A) = [a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}] = [0] = O$ (zero matrix)

$(-A) + A = [-a_{ij}] + [a_{ij}] = [-a_{ij} + a_{ij}] = [0] = O$

That is, $A + (-A) = (-A) + A = O$.

Properties of Scalar Multiplication

Let A, B are two matrices of same order and α, β are any scalars, then

$S-1$: $\alpha(A + B) = \alpha A + \alpha B$

$S-2$: $(\alpha + \beta)A = \alpha A + \beta A$

$S-3$: $\alpha(\beta A) = (\alpha\beta)A$

$S-4$: $1A = A$

Addition of Matrices

If A and B are two matrices of the same order, then they are 'conformable' for addition and their sum '$A + B$' is obtained by adding the corresponding elements of A and B, i.e., if $A = [a_{ij}]_{m\times n}$; $B = [b_{ij}]_{m\times n}$, then $A + B = [a_{ij} + b_{ij}]_{m\times n}$.

Properties of Addition Let A, B and C be three matrices of same order say $m \times n$, then

$A-1$: $A + B$ is also a $m \times n$ matrix (CLOSURE)

$A-2$: $(A + B) + C = A + (B + C)$ (ASSOCIATIVITY)

$A-3$: If 'O' is the $m \times n$ zero (null) matrix, then $A + O = O + A = A$ ('O' is the ADDITIVE IDENTITY)

$A-4$: $A + (-A) = (-A) + A = O$ ($-A$ is the ADDITIVE INVERSE)

$A-5$: $A + B = B + A$ (COMMUTATIVITY)

NOTE

The set of matrices of same order form an 'Abelian Group' under addition.

Multiplication of Matrices

Let A and B be two matrices. A and B are conformable for multiplication, only if the number of columns of A is equal to the number of rows of B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix, $B = [b_{jk}]$ be an $n \times p$ matrix. Then the product 'AB' is defined as the matrix $C = [c_{ik}]$ of order $m \times p$ where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$

$$= \sum_{j=1}^{n} a_{ij}b_{jk}.$$

c_{ij} calculated for $i = 1, 2, \ldots m$ and $k = 1, 2, \ldots, p$ will give all the elements of the matrix C.

Properties of Multiplication

$M-1$: If A, B, C be $m \times n$, $n \times p$, $p \times q$ matrices respectively, then $(AB)C = A(BC)$ (ASSOCIATIVITY).

$M-2$: If A is a $m \times n$ matrix, then $A\,I_n = A$ and $I_m\,A = A$ and if A is a square matrix, i.e., $m = n$, then $AI = IA = A$ (I is the MULTIPLICATIVE IDENTITY).

$M-3$: If A, B, C be $m\times n$, $n\times p$, $p\times q$ matrices respectively, then $A(B + C) = AB + AC$ (DISTRIBUTIVE LAW).

$M-4$: Matrix multiplication is NOT COMMUTATIVE in general.

$M-5$: The INVERSE of a given matrix may not always exist.

Determinants

Let $A = [a_{ij}]$ be a square matrix of order 'n'. Then the determinant of order 'n' associated with 'A' is denoted by $|A|$ or $|a_{ij}|$ or Det(A) or Δ.

NOTES

1. Determinant of a matrix exists, only if it is a square matrix.
2. The value of a determinant is a single number.

Determinant of Order 1 (or First Order Determinant)

If 'a' be any number, then determinant of 'a' is of order '1' and is denoted by $|a|$. The value of $|a| = a$.

Determinant of Order 2 (or Second Order Determinant)

If 'A' is a square matrix of order 2 given by $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ then $\text{Det}(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is determinant of order 2 and its value is $\Delta = a_1b_2 - a_2b_1$

Minor and Cofactor of a Matrix

Let $$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ be a } 3\times 3 \text{ matrix}$$

Then the minor of an element a_{ij} of 'A' is the determinant of the 2×2 matrix obtained after deleting the i-th row and j-th column of A and is denoted by M_{ij}.

The cofactor of a_{ij} is denoted by A_{ij} and is defined as $(-1)^{i+j} M_{ij}$, i.e., $A_{ij} = (-1)^{i+j} M_{ij}$

Determinant of Order 3 (Third Order Determinant)

If A is a square matrix of order '3', given by $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. Then the determinant of 'A' is given by

$$\Delta = \text{Det } A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ is a determinant of order 3 and the value is obtained by taking the sum of the products of the elements of any row (or column) by their corresponding cofactors.

Thus for A, $\Delta = a_1A_1 + b_1B_1 + c_1C_1$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

or also $\Delta = a_1A_1 + a_2A_2 + a_3A_3$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

(This is by expanding by C_1) and so on.

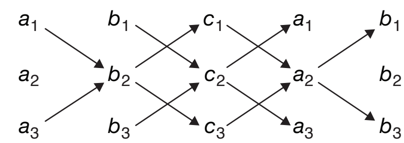
The sign to be used before a particular element can be judged by using the following rule:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

The value of the determinants of order 3 can also be evaluated by using 'Sarrus' method given as follows:

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Enter the first column and then the second column after the third column and take the product of numbers as shown by the arrows, taking care of signs indicated



Then

$$\Delta = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

We can now define the cofactor of an element a_{ij} in a 4×4 matrix as $(-1)^{i+j} \times$ (Determinant of the 3×3 matrix obtained by deleting the i-th row and j-th column) and determinant of a 4×4 matrix to be the sum of products of elements of any row (or column) by their corresponding cofactors. We can similarly define determinant of a square matrix of any order.

Properties of Determinant

1. If two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by (-1).
2. If the rows and columns of a determinant are interchanged, the value of the determinant remains unchanged, i.e., $\text{Det}(A) = \text{Det}(A^T)$.
3. If all the elements of a row (or column) of a determinant are multiplied by a scalar (say 'K'), the value of the new determinant is equal to 'K' times the value of the original determinant.
4. If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
5. If the elements of a row (or a column) in a determinant are proportional to the elements of any other row (or column), then the determinant is '0'.
6. If every element of any row (or column) is zero, then determinant is '0'.
7. If each element in a row (or column) of a determinant is the sum of two terms, then its determinant can be expressed as the sum of two determinants of the same order.
8. (The theorem of 'false cofactor') The sum of products of elements of a row (or column) with the cofactors of any other row (or column) is zero.

   Thus in $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

   $a_1A_2 + b_1B_2 + c_1C_2 = 0$

   $a_2A_1 + b_2B_1 + c_2C_1 = 0$ and so on in general

   $a_rA_s + b_rB_s + c_rC_s = 0$ if $r \neq s$

9. If the elements of a determinant are polynomials in x and the determinant vanishes for $x = a$, then $x - a$ is a factor of the determinant.

Singular and Non-singular Matrices

A square matrix 'A' is said to be singular, if $\text{Det}(A) = 0$ and is non-singular, if $\text{Det}(A) \neq 0$.

NOTES

1. A unit matrix is non-singular (since its Det = 1)
2. If A and B are non-singular matrices of the same 'type', then AB is non-singular of the same 'type'.

Inverse of a Matrix

Let 'A' be a square matrix. A matrix 'B' is said to be an inverse of 'A', if $AB = BA = I$.

NOTE

If B is the inverse of 'A', then 'A' is the inverse of 'B'.

Some Results of Inverse

1. Inverse of a square matrix, when it exists, is unique.
2. The inverse of a square matrix exists, if and only if it is non-singular.
3. If 'A' and 'B' are square matrices of the same order, then 'AB' is invertible (i.e., inverse of AB exists) if 'A' and 'B' are both invertible.
4. If 'A' and 'B' are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.
5. If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.
6. If A is invertible, then so is A^θ and $(A^\theta)^{-1} = (A^{-1})^\theta$.

Adjoint of a Matrix

The adjoint of a square matrix 'A' is the transpose of the matrix obtained by replacing the elements of 'A' by their corresponding cofactors.

NOTE

The adjoint is defined only for square matrices and the adjoint of a matrix 'A' is denoted by Adj(A). If

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots \\ l_1 & l_2 & \cdots & l_n \end{pmatrix}$$

$$\text{Adj } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_n \\ B_1 & B_2 & \cdots & B_n \\ \vdots & \vdots & \vdots & \vdots \\ L_1 & L_2 & \cdots & L_n \end{pmatrix}^T = \begin{pmatrix} A_1 & B_1 & \cdots & L_1 \\ A_2 & B_2 & \cdots & L_2 \\ \vdots & \vdots & \vdots & \vdots \\ A_n & B_n & \cdots & L_n \end{pmatrix}$$

Results

1. If 'A' is of order 3×3 and K is any number, then $\text{Adj}(KA) = K^2(\text{Adj } A)$.
2. $A(\text{Adj } A) = (\text{Adj } A)A = |A|\, I$ for any square matrix 'A'.
3. $\text{Adj } I = I$; $\text{Adj } O = O$ where I is the identity matrix and O is the null matrix.
4. $\text{Adj}(AB) = (\text{Adj } B)(\text{Adj } A)$ if A, B are non-singular and are of same type.
5. If $A = A_{n\times n}$, then
   $\det(\text{Adj } A) = (\det A)^{n-1}$.
   $\text{Adj}(\text{Adj } A) = (\det A)^{n-2}(A)$.
   $|\text{Adj}(\text{Adj } A)| = (\det A)^{(n-1)^2}$

Evaluating Inverse of a Square Matrix

If A is a square matrix, then $A^{-1} = \dfrac{1}{|A|}(\text{Adj } A)$

NOTES

1. The inverse of an identity matrix is itself.
2. $(\text{Adj } A)^{-1} = \dfrac{1}{|A|}A$
3. If A is a non-singular square matrix (say of order 3) and K is any non-zero number, then

$$(KA)^{-1} = \frac{1}{K}A^{-1}$$

Rank and Nullity of a Matrix

Rank of a Matrix The Matrix 'A' is said to be of rank 'r', if and only if it has at least one non-singular square sub-matrix of order 'r' and all square sub-matrices of order $(r + 1)$ and higher orders are singular. The rank of a matrix A is denoted by rank (A) or $\rho(A)$.

Nullity of a Matrix If A is a square matrix of order 'n', then $n - \rho(A)$, i.e., n – rank (A) is defined as nullity of matrix 'A' and is denoted by $N(A)$.

Remark 1: If there is a non-singular square sub-matrix of order 'K', then $\rho(A) \geq K$.

Remark 2: If there is no non-singular square sub-matrix of order 'K', then $\rho(A) < K$.

Remark 3: If A' is the transpose of A, then $\rho(A) = \rho(A')$.

Remark 4: The rank of a null matrix is '0'.

Remark 5: The rank of a non-singular square matrix of order 'n' is 'n' and its nullity is '0'.

Remark 6: Elementary operations do not change the rank of a matrix.

Remark 7: If the product of two matrices A and B is defined, then $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$. That is, the rank of product of two matrices cannot exceed the rank of either of them.

Elementary Operations or Elementary Transformations

1. Elementary row operations
   (a) $R_i \leftrightarrow R_j$: Interchanging of ith and jth rows
   (b) $R_i \to KR_i$: Multiplication of every element of ith row with a non-zero scalar K
   (c) $R_i \to R_i + kR_j$: Addition of k times the elements of jth row to the corresponding elements of ith row.
2. Elementary column operations
   (a) $C_i \leftrightarrow C_j$: Interchanging of ith and jth columns
   (b) $C_i \to KC_i$: Multiplication of every element of ith column with a non-zero scalar K.
   (c) $C_i \to C_i + KC_j$: Addition of K times the elements of jth column to the corresponding elements of ith column.

Example: Consider the matrix $A = \begin{bmatrix} 2 & 3 & -4 & 1 \\ 3 & 0 & 1 & 5 \\ 4 & 7 & 1 & 2 \end{bmatrix}$

$$R_2 \to 2R_2 \sim \begin{bmatrix} 2 & 3 & -4 & 1 \\ 6 & 0 & 2 & 10 \\ 4 & 7 & 1 & 2 \end{bmatrix}$$

$$C_2 \leftrightarrow C_3 \sim \begin{bmatrix} 2 & -4 & 3 & 1 \\ 3 & 1 & 0 & 5 \\ 4 & 1 & 7 & 2 \end{bmatrix}$$

$$C_1 \to C_1 - 2C_4 \sim \begin{bmatrix} 0 & -4 & 3 & 1 \\ -7 & 1 & 0 & 5 \\ 0 & 1 & 7 & 2 \end{bmatrix}$$

NOTE

The rank of a matrix is invariant under elementary operations

Row and Column Equivalence Matrices

Row Equivalence Matrix If B is a matrix obtained by applying a finite number of elementary row operations successively on matrix A, then matrix B is said to be row equivalent to A (or a row equivalent matrix of A).

Column Equivalence Matrix If B is obtained by applying a finite number of elementary column operations successively on matrix A, then matrix B is said to be column equivalent to A (or a column equivalent matrix of A).

Example: $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & -2 \\ 1 & 4 & -3 \end{bmatrix}$

$$R_2 - 2R_1, R_3 - R_1 \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -10 \\ 0 & 1 & -7 \end{bmatrix} = B \text{ (say)}$$

B is a row equivalent matrix of A.

Example: $B = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & -4 \\ 1 & 1 & 6 \end{bmatrix}$

$$C_2 - 3C_1, \frac{1}{2}C_3 \sim \begin{bmatrix} 1 & 0 & 1 \\ 3 & -5 & -2 \\ 1 & -2 & 6 \end{bmatrix} = C \text{ (say)}$$

C is a column equivalent to B.

Row Reduced Matrix A matrix A of order $m \times n$ is said to be row reduced if,

1. The first non-zero element of a non-zero row is 1.
2. Every other element in the column in which such 1's occur is 0.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \text{ is a row reduced matrix}$$

$$B = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not a row reduced matrix.}$$

Row Reduced Echelon Matrix A matrix 'X' is said to be row reduced echelon matrix if,

1. X is row reduced.
2. There exists integer $\underline{P}(0 \le p \le m)$ such that first 'p' rows of X are non-zero and all the remaining rows are zero rows.
3. For the ith non-zero row, if the first non-zero element of the row (i.e., 1) occurs in the jth column then, $j_1 < j_2 < j_3 < \cdots < j_p$.

Example: $P = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $Q = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

are echelon matrices. The number of non-zero rows (i.e., value of P and Q) are 3 and 2 respectively. The value of i and j are tabulated below

P:	i	1	2	3
	j	1	2	3

Q:	i	1	2
	j	2	4

Normal form of a Matrix

By means of elementary transformations, every matrix 'A' of order $m \times n$ and rank r (> 0) can be reduced to one of the following forms.

1. $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ **2.** $[I_r/0]$ **3.** $[\underline{I_r}]$ **4.** $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

and these are called the normal forms. I_r is the unit matrix of order 'r'.

NOTE

If a $m \times n$ matrix 'A' has been reduced to the normal form say $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ then 'r' is the rank of A.

Systems of Linear Equations

Let
$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}X_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}X_n = b_2 \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \ a_{nn}x_n = b_n \end{matrix}\right\} \qquad (1)$$

be a system of 'n' linear equations in 'n' variables $x_1, x_2, \ldots, x_n$. The above system of equations can be written as

$$\begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \text{or} \quad AX = B$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

A is called the co-efficient matrix.

Any set of values of $x_1, x_2, x_3, \ldots$ which simultaneously satisfy these equations is called a solution of the system. When the system of equations has one or more solutions, the equations are said to be CONSISTENT and the system of equations are said to be INCONSISTENT if it does not admit any solution. The system of equations (1) is said to be

HOMOGENEOUS, if $B = 0$

NON-HOMOGENEOUS, if $B \neq 0$

Let the system of equations be

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned}$$

This is a system of 'm' equations in 'n' variables $x_1, x_2, \ldots, x_n$. The system of equations can be written as $AX = B$ where

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix $\begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} \cdots & a_{mn} & b_m \end{pmatrix}$ is called the augmented matrix of the system of equations and is denoted by $[A : B]$.

Let $AX = B$ represents 'm' linear equations with 'n' variables. Let rank of $A = r$ and rank $(A, B) = r_1$ [where (A, B) is an augmented matrix]. If $r_1 \neq r$, then the system of equations are inconsistent.

If $r_1 = r$, the table follows:

	$m = n$		$m > n$		$m < n$	
	$r = n$	$r < n$	$r = n$	$r < n$	$r = m$	$r < m$
Homo-geneous	Only trivial solution	Infinite solutions	Only trivial solution	Infinite solutions	Infinite solutions	Infinite solutions
Non-homo geneous	Unique solution	Infinite solutions	Unique solution	Infinite solutions	Infinite solutions	Infinite solutions

Solving System of Linear Equations

The following methods of solving system of linear equations (1) is applicable only when the co-efficient matrix 'A' is non singular, i.e., $|A| \neq 0$.

Cramers Method

Let $AX = B$ represent the system of equations (1) where A, X and B are as defined earlier.

Let Δ be $|A|$ and $\Delta_1, \Delta_2, \ldots, \Delta_n$ be the determinants obtained by replacing the elements of 1st, 2nd, $\ldots$, nth column of A by the elements of B. Then if $\Delta \neq 0$, we have

$$x_1 = \Delta_1/\Delta;\ x_2 = \Delta_2/\Delta;\ x_3 = \Delta_3/\Delta;\ \ldots;\ x_n = \Delta_n/\Delta.$$

Inverse Method

Let the system of linear equations be $AX = B$, where A, X, B are as defined earlier.

If $|A| \neq 0$ then pre-multiplying with A^{-1}, we get $A^{-1}(AX) = A^{-1}B$.

$\Rightarrow \quad X = A^{-1}B$ which gives the values of the variables.

Gauss–Jordan Method

Consider the augmented matrix $[A : B]$ of the system of 'n' non-homogeneous equations (1) in n-variables

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Reduce this augmented matrix to the standard form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & d_1 \\ 0 & 1 & \cdots & 0 & d_2 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & d_n \end{bmatrix}$$

By applying the elementary operations, the solution of the equations is $x_1 = d_1, x_2 = d_2, \ldots, x_n = d_n$.

Gauss Elimination Method

Let the system of linear equations given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = c_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = c_n \end{array}\right\} \quad (1)$$

Let $a_{11} \neq 0$ write the above equations in the matrix form $AX = B$

Write the augmented matrix $[A\ B]$.

Using elementary row operations, eliminate the unknown x_1 from all the equations except the first. Eliminate the unknown x_2 from all the equations except from first and second rows, continuing in this way we finally get the following equivalent system of equations at the $(n-1)$th step.

$$\begin{array}{r} a'_{11}x_1 + a'_{12}x_2 + a'_{13}x_3 + \cdots + a'_{1n}x_n = c'_1 \\ a'_{22}x_2 + \cdots + a'_{2n}x_n = c'_2 \\ a'_{33}x_3 + \cdots + a'_{3n}x_n = c'_3 \\ a'_{nn}x_n = c'_n \end{array}$$

From the above system of equations we can find the values of the unknowns.

Linear Dependence

A set of vectors of n dimensions is said to be linearly dependent if one of these vectors can be expressed as a linear combination of some other vectors in the set.

If no vector can be expressed as a linear combination of the others, then the set of vectors is said to be linearly independent.

NOTE

The maximum number of linearly independent rows or columns of a matrix is called the rank of the matrix.

LU Decomposition Method of Factorisation or Method of Triangularization

Consider the system of equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}\right\} \quad (1)$$

In matrix notation, Eq. (1) can be written as $AX = B$ (2)

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Step 1: Write $A = LU$, where $L \to$ Lower triangular matrix with principal diagonal elements being equal to 1 and $U \to$ Upper triangular matrix.

$$\text{That is, } L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Step 2: Now Eq. (2) becomes $LUX = B$ (3)

Step 3: Let $UX = Y$ (4)

$$\text{where} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Step 4: Combining Eqs. (3) and (4), we get $LY = B$ (5)
On solving Eq. (5) we get y_1, y_2, y_3.

Step 5: Substituting Y in Eq. (4), we get $UX = Y$
On solving, we get X, i.e., x_1, x_2, x_3.

The Characteristic Equation of a Matrix

Characteristic Matrix If A is any square matrix, the matrix $A - \lambda I$ where λ is a scalar, is called the characteristic matrix of A.

Characteristic Polynomial If A is any square matrix of order n, then the determinant $|A - \lambda I|$ yields a polynomial $\phi(\lambda)$ of degree n in λ which is known as the characteristic polynomial of the matrix A.

Characteristic Equation If $\phi(\lambda)$ is the characteristic polynomial of a matrix A, then $\phi(\lambda) = 0$, is called the characteristic equation of A.

And the roots of this equation, say $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the characteristic roots or latent roots or **eigen values**. If λ is a characteristic root of ordert, then t is called the algebraic multiplicity of λ.

Characteristic Vectors Corresponding to each characteristic root λ, there is a non-zero vector which satisfies the characteristic equation $|A - \lambda I| = 0$. These non-zero vectors are called the characteristic vectors or **eigen vectors** or latent vectors.

NOTES

1. The characteristic roots of a matrix and its transpose are the same.
2. 0 is a characteristic roots of a matrix, if the matrix is singular.
3. The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.
4. If K is any scalar, the characteristic roots of matrix KA are K times the characteristic roots of matrix A.
5. If $a_1, a_2, a_3, \ldots, a_n$ are characteristic roots of matrix A and K is a scalar, then the characteristic roots of matrix $A - KI$ are $a_1 - K, a_2 - K, \ldots, a_n - K$.
6. If λ is a characteristic root of a non-singular matrix, then λ^{-1} is a characteristic root of A^{-1}.
7. If the eigen values of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$.

Cayley–Hamilton Theorem

Every square matrix satisfies its characteristic equation.

Inverse by Cayley–Hamilton Theorem

Let A be non-singular square matrix of order n

Let the characteristic equation of A be

$$|A - \lambda I| = (-1)^n \lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + \cdots + C_{n-1}\lambda + C_n = 0$$

Where $C_1, C_2, \ldots, C_n$ are all scalar constants

Then by Cayley–Hamilton theorem

$$(-1)^n A^n + C_1 A^{n-1} + C_2 A^{n-2} + \cdots + C_{n-1}A + C_n I = O \qquad (1)$$

Multiplying Eq. (1) throughout by A^{-1}, we have

$$A^{-1}[(-1)^n A^{n-1} + C_1 A^{n-1} + C_2 A^{n-2} + \cdots + C_{n-1} A + C_n I] = A^{-1} \cdot 0$$

$$\Rightarrow \quad (-1)^n A^{n-1} + C_1 A^{n-2} + C_2 A^{n-3} + \cdots + C_{n-1} I + C_n A^{-1}$$

$$\Rightarrow \quad A^{-1} = \frac{-1}{C_n}[(-1)^n A^{n-1} + C_1 A^{n-2} + C_2 A^{n-3} + \cdots + C_{n-1} I]$$

NOTE

Similarly, we can find $A^{-2}, A^{-3}, \ldots$ for the matrix A_x provided A is non-singular.

Power of a Matrix by Cayley–Hamilton Theorem

Cayley–Hamilton theorem is also helpful in finding higher powers of a square matrix with least possible number of matrix multiplications. This is explained in Examples 11 and 12.

Reduction to Diagonal Form

If A is a square matrix of order n with n linearly independent eigen vectors, then A can be reduced to a diagonal matrix, called diagonal form of A.

Procedure to Reduce a Square Matrix into Diagonal Form

Let A be a square matrix of order n that can be reduced to diagonal form

1. Find the eigen values and their corresponding eigen vectors of A. Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the eigen values and let $X_1, X_2, X_3, \ldots, X_n$ be their corresponding eigen vectors that are linearly independent.
2. Form the matrix P with $X_1, X_2, X_3, \ldots, X_n$ as its columns i.e., $P = [X_1\ X_2\ X_3 \ldots X_n]$ it can be easily observed that P is invertible.
3. Find the inverse of P (i.e., find P^{-1})
4. The diagonal form of A is given by $D = P^{-1}$ AP.

Where $D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ is a diagonal matrix with eigen values of A as its principal diagonal elements.

NOTE

Here P is called the modal matrix and D is the spectral matrix of the matrix A

Power of a Matrix by Using its Diagonal Form

If D is the diagonal form of a square matrix A, then for any positive integer n, we have $A^n = P D^n P^{-1}$.

Where P is the modal matrix of A.

SOLVED EXAMPLES

Example 1

Find the value of

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}.$$

Solution

$c_1 \to c_1 + c_2 + c_3$

$$\begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$R_2 \to R_2 - R_1 \qquad R_3 \to R_3 - R_1$

$$2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)^3\begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2(a+b+c)^3.$$

Example 2

Find the rank of the matrix $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & -1 \\ 1 & 4 & 1 \end{bmatrix}$.

Solution

Given

$$\begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & -1 \\ 1 & 4 & 1 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 4 & 1 \\ 2 & 0 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & -8 & -3 \\ 0 & -11 & -5 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{-11}{8} R_2 \sim \begin{bmatrix} 1 & 4 & 1 \\ 0 & -8 & -3 \\ 0 & 0 & \frac{-7}{8} \end{bmatrix}$$

which is a row echelon form. The number of non zero rows = 3.

The rank of the matrix = The number of non-zero rows in it = 3

∴ Rank of the matrix = 3.

Example 3

Find whether the vectors given below are linearly dependent or independent $\{(1, 3, 2), (1, -4, 1), (-1, 2, 5)\}$.

Solution

Let $x, y, z \in R$ such that $x(1, 3, 2) + y(1, -4, 1) + z(-1, 2, 5) = (0, 0, 0)$

$$\Rightarrow \left.\begin{aligned} x + y - z &= 0 \\ 3x - 4y + 2z &= 0 \\ 2x + y + 5z &= 0 \end{aligned}\right\} \quad (1)$$

The above system of equations when expressed in determinant form, we have

$$\begin{vmatrix} 1 & 1 & -1 \\ 3 & -4 & 2 \\ 2 & 1 & 5 \end{vmatrix} \xrightarrow{R_2 - 3R_1,\, R_3 - 2R_1} \begin{vmatrix} 1 & 1 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 7 \end{vmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{7}R_2} \begin{vmatrix} 1 & 1 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & \frac{44}{7} \end{vmatrix}$$

∴ Rank = 3 = number of unknowns

∴ There exists a unique solution $x = 0$, $y = 0$ and $z = 0$

$\Rightarrow \quad x(1, 3, 2) + y(1, -4, 1) + z(-1, 2, 5)$
$= (0, 0, 0)$ only when $x = 0, y = 0, z = 0$.

∴ The set of vectors are linearly independent.

Example 4

Show that the set of vectors $\{(2, 3, 9), (3, -2, -6), (-1, 5, 15)\}$ are linearly dependent.

Solution

Let $x, y, z \; \varepsilon \; R$ such that

$x(2, 3, 9) + y(3, -2, -6) + z(-1, 5, 15) = (0, 0, 0)$

$$\Rightarrow \begin{aligned} 2x + 3y - z &= 0 \\ 3x - 2y + 5z &= 0 \\ 9x - 6y + 15z &= 0 \end{aligned}$$

The above system when expressed in matrix form we have the coefficient matrix

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & -2 & 5 \\ 9 & -6 & 15 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 5 \\ 9 & -6 & 15 \end{vmatrix} = 0$$

as $\quad R_3 = 3R_2$ and $\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \neq 0$

∴ Rank of $A = 2 <$ the number of variables which is 3.

∴ The system will possess a non-zero solution, i.e.,

$$\begin{aligned} 2x + 3y - z &= 0 \\ 3x - 2y + 5z &= 0 \end{aligned}$$

$$\frac{x}{15 - 2} = \frac{y}{-3 - 10} = \frac{z}{-4 - 9} = k \text{ (say)}$$

$\Rightarrow \quad x = 13k$, $y = -13k$ and $z = -13k$

Let $k = 1 \Rightarrow x = 13, y = -13, z = -13$

∴ There exists a non-zero solution such that $x, y, z \; \varepsilon \; R$

$x(2, 3, 9) + y(3, -2, -6) + z(-1, 5, 15) = (0, 0, 0)$

∴ The set of given vectors are linearly dependent.

Example 5

How many solutions are there for the system of linear equations $x + 2y + z = 0$, $3x + 2y - z = 0$ and $4x + y - 3z = 0$?

Solution

Determinant of the co-efficient matrix of the given equations

is $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & -1 \\ 4 & 1 & -3 \end{vmatrix}$

$$= 1(-6 + 1) - 2(-9 + 4) + 1(3 - 8) = 0$$

∴ The system has infinite number of solutions.

Example 6

Solve the system of equations

$x_1 + x_2 + x_3 = 1$, $3x_1 + x_2 - 3x_3 = 5$ and $x_1 - 2x_2 - 5x_3 = 10$ by LU decomposition method.

Solution

$$AX = B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

Step 1: $LU = A$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

Expanding and on solving we get, $u_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$,

$u_{22} = -2$, $u_{23} = -6$, $u_{33} = 3$, $l_{21} = 3$, $l_{31} = 1$, $l_{32} = \frac{3}{2}$

Step 2: Now $LUX = B$

Step 3: Let $UX = Y$

Step 4: $\therefore$ $LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

On solving, $y_1 = 1$, $y_2 = 2$ and $y_3 = 6$.

Step 5: $UX = Y$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

On solving we get $x_1 = 6$, $x_2 = -7$ and $x_3 = 2$

$\therefore$ The solution is (6, –7, 2).

Example 7

Solve: $x + y + z = 6$, $3x - 2y - z = -4$ and $2x + 3y - 2z = 2$.

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 3 & -2 & -1 \\ 2 & 3 & -2 \end{vmatrix} = 1(7) - 1(-4) + 1(9+4) \neq 0$$

$\therefore$ The set of given equations are non-homogeneous and the number of equations is equal to the number of variables.

$\therefore$ The given system of equations is consistent and has a unique solution.

Augmented matrix,

$$[AB] \text{ is } \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & -2 & -1 & -4 \\ 2 & 3 & -2 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, and $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -5 & -4 & -22 \\ 0 & 1 & -4 & -10 \end{bmatrix}$$

$R_1 \to R_1 + \frac{1}{5}R_2$ and $R_3 \to R_3 + \frac{1}{5}R_2$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{8}{5} \\ 0 & -5 & -4 & -22 \\ 0 & 0 & -\frac{24}{5} & -\frac{72}{5} \end{bmatrix}$$

$R_2 \to \frac{-5}{6}R_3 + R_2$; $R_3 \to \frac{-5}{24}R_3$

$$\sim \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{8}{5} \\ 0 & -5 & 0 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$R_1 \to -\frac{1}{5}R_3$ and $R_2 \to \frac{-1}{5}R_2$

$$R_2 \to -\frac{1}{5}R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore$ Solution is $x = 1$, $y = 2$ and $z = 3$.

Example 8

Solve $3x + 2y - z = 0$, $4x + y + 2z = 0$ and $x - 5y + 7z = 0$.

Solution

Determinant of the co-efficient matrix of the equations when written in matrix form is $\begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 2 \\ 1 & -5 & 7 \end{vmatrix}$

$$= 3(7 + 10) - 2(28 - 2) - 1(-20 - 1)$$
$$= 51 - 52 + 21 = 20$$

$\therefore$ The given system of equations have only one solution, i.e., $x = y = z = 0$.

Example 9

Determine the eigen values and eigen vectors of $A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$.

Solution

Characteristic equation of the given matrix is $|A - \lambda| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda - 6 = 0$$
$$(\lambda - 6)(\lambda + 1) = 0$$

$\Rightarrow \quad \lambda = -1$ and $\lambda = 6$ are the eigen values. Eigen vector corresponding to $\lambda = -1$ is obtained as follows:

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} + 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 3x_1 + 4x_2 = 0$$

$$3x_1 + 4x_2 = 0 \Rightarrow x_1 = -\frac{4}{3}x_2$$

$\therefore$ Eigen vector corresponding to $\lambda = -1$ is,

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{-4}{3}x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} \frac{-4}{3} \\ 1 \end{pmatrix}$$

Similarly eigen vector corresponding to $\lambda = 6$ is obtained as follows:

$$\left[\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -4x_1 + 4x_2 = 0 \text{ and } 3x_1 - 3x_2 = 0$$
$$\Rightarrow \quad x_1 = x_2$$

Eigen vector corresponding to $\lambda = 6$ is,

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example 10

Find the eigen values of the matrix

$$A = \begin{pmatrix} 6 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

Solution

Characteristic equation of the given matrix is $|A - \lambda| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 2 & 2 \\ 2 & 3-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$
$$\lambda = 2, 2, 8$$

$\therefore$ Eigen values are 2, 2, 8.

Example 11

If $A = \begin{bmatrix} 4 & 2 \\ -7 & -4 \end{bmatrix}$, then find A^{16} by using Cayley–Hamilton theorem.

Solution

The characteristic equation of

$$A = \begin{bmatrix} 4 & 2 \\ -7 & -4 \end{bmatrix} \text{ is } |A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 2 \\ -7 & -4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (4 - \lambda)(-4 - \lambda) + 14 = 0$$
$$\Rightarrow \quad -16 - 4\lambda + 4\lambda + \lambda^2 + 14 = 0$$
$$\Rightarrow \quad \lambda^2 - 2 = 0 \qquad (1)$$

By Cayley–Hamilton theorem, the matrix A satisfies its characteristic equation (1).

$\therefore A^2 - 2I = O$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad A^2 = 2I \qquad (2)$$

Now $A^{16} = (A^2)^8 = (2I)^8$ (From Eq. (2))

$$= 2^8 I^8 = 256I = 256\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \qquad A^{16} = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$$

Example 12

If $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & -5 \\ 0 & 1 & 0 \end{bmatrix}$; then find the value of the matrix polynomial $3A^9 - 18A^8 + 39A^7 - 32A^6 + 12A^5 - 26A^4 + 16A^3 + 24A^2 - 50A + 40I$.

Solution

The characteristic equation of

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & -5 \\ 0 & 1 & 0 \end{bmatrix} \text{ is } |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & 4-\lambda & -5 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda)\{(4-\lambda)(-\lambda)+5\}=0$

$\Rightarrow \quad (2-\lambda)\{(\lambda^2-4\lambda+5)\}=0$

$\Rightarrow 2\lambda^2-8\lambda+10-\lambda^3+4\lambda^2-5\lambda=0.$

$\Rightarrow \quad -\lambda^3+6\lambda^2-13\lambda+10=0$

$\Rightarrow \quad \lambda^3-6\lambda^2+13\lambda-10=0 \qquad (1)$

By Cayley–Hamilton theorem, the matrix A will satisfy its characteristic Eq. (1)

$\therefore\ A^3-6A^2+13A-10I=O,$

where $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ and $O=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

$\therefore \quad A^3-6A^2+13A-10I=0 \qquad (2)$

Now consider the given matrix polynomial

$3A^9-18A^8+39A^7-32A^6+12A^5-26A^4+16A^3+24A^2-50A+40I$

$=3A^9-18A^8+39A^7-30A^6-2A^6+12A^5-26A^4+20A^3-4A^3+24A^2-52A+2A+40I$

$=3A^6(A^3-6A^2+13A-10I)-2A^3(A^3-6A^2+13A-10I)-4(A^3-6A^2+13A-10I)+2A$

$=3A^6\times 0-2A^3\times 0-4\times 0+2A$

(From Eq. (2))

$$=2A=2\begin{bmatrix}2&0&3\\0&4&-5\\0&1&0\end{bmatrix}=\begin{bmatrix}4&0&6\\0&8&-10\\0&2&0\end{bmatrix}.$$

Exercises

1. Which of the following is false?

(A) Every diagonal matrix is a square matrix.
(B) Every unit matrix is a scalar matrix.
(C) Every square matrix is a diagonal matrix.
(D) Every scalar matrix is a diagonal matrix.

2. If the trace of the matrix $\begin{pmatrix}1&a_{12}&\cdots&a_{1n}\\a_{21}&2&\cdots&a_{2n}\\\vdots&\vdots&\vdots&\vdots\\a_{n1}&a_{n2}&\cdots&n\end{pmatrix}$ is 55 then the value of n is

(A) 10 (B) 11
(C) 9 (D) Cannot be determined

3. Which of the following statement is/are false?

(A) $A^T\cdot B^T$ always defined for square matrices of same order.
(B) $A^T\cdot B$ is defined for matrices of the same order.
(C) $t_r(A^T)+t_r(B^T)$ is always defined for matrices A, B of same order.
(D) A^T+B^T is always defined for matrices A, B of same order.

4. Consider the following statements about two square matrices A and B of the same order:

P: $(A+B)^2=A^2+2AB+B^2$
Q: $(A+B)(A-B)=A^2-B^2$

Then,

(A) both P and Q are true.
(B) both P and Q are false
(C) both P and Q are true if A and B commute
(D) P is true but Q is false.

5. If $\begin{pmatrix}2&1&2\\1&0&1\\2&2&1\end{pmatrix}\begin{pmatrix}-2x&3&x\\x&-2&0\\2&-2x&-x\end{pmatrix}=I_{3\times3}$, then $x=$

(A) -1 (B) 1
(C) $\frac{1}{2}$ (D) 2

6. If $\Delta=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}$, then which of the following is true? (Here, A_{ij} is the cofactor of the element a_{ij})

(A) $a_{11}A_{11}+a_{21}A_{12}+a_{23}A_{32}=\Delta$
(B) $a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=\Delta$
(C) $a_{21}A_{12}+a_{23}A_{32}+a_{12}A_{21}=\Delta$
(D) $a_{12}A_{21}+a_{21}A_{12}+a_{31}A_{13}=\Delta$

7. The determinant value of $\begin{pmatrix}2&3&-3\\1&-2&2\\7&4&-4\end{pmatrix}$ is

(A) 0 (B) 10
(C) -10 (D) 15

8. The value of $\begin{vmatrix}n!&(n+1)!&(n+2)!\\(n+1)!&(n+2)!&(n+3)!\\(n+2)!&(n+3)!&(n+4)!\end{vmatrix}$ is

(A) $2n!\,(n+1)!$
(B) $2n!\,(n+1)!\,(n+2)!$
(C) $(2n)!\,(n+1)!\,(n+2)!$
(D) $2n!\,(n+3)!$

9. If $f(x)=\begin{vmatrix}{}^xC_0&{}^xC_1&{}^{x+1}C_1\\2^xC_1&2^xC_2&2^{(x+1)}C_2\\6^xC_2&6^xC_3&6^{(x+1)}C_3\end{vmatrix}$, then $f(200)$ is

(A) 200 (B) -200
(C) 0 (D) -2001

10. The determinant $\begin{vmatrix}2&3+i&-1\\3-i&0&-1+i\\-1&-1-i&1\end{vmatrix}$ is

(A) purely imaginary (B) zero
(C) real (D) 10

11. If $A = \begin{pmatrix} x & y & z \\ 2x & y & 3z \\ \frac{x}{2} & \frac{y}{2} & \frac{z}{2} \end{pmatrix}$, then $|A| =$ ______.

(A) $10xyz$ (B) 1

(C) 0 (D) $\frac{1}{2}(x^3 + y^3 + z^3 - 3xyz)$

12. If the elements of a row or column of a given square matrix is multiplied by 2, then the value of determinant is ______ times the original determinant.

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

13. If A is a square matrix of order k and det(kA) = 27 det(A), then $k =$ ______.

(A) 9 (B) 1

(C) 2 (D) 3

14. If A and B are two square matrices of order 4 such that $|A| = -2$ and $|B| = 5$, then $|4AB|$ is

(A) −80 (B) −160

(C) −2560 (D) −256

15. I. $(a - b)$, $(b - c)$, $(c - a)$ are factors of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

II. If the elements of a determinant are functions of x and its two rows or columns become identical (i.e., determinant equals zero,) when we substitute $x = k$, then $(x - k)$ is a factor of the determinant.

Which of the following is correct?

(A) Both I and II true. (B) Both I and II false.

(C) I is true, II is false (D) I is false, II is true

16. A lower triangular matrix $A = (a_{ij})_{n \times n}$ is singular if and only if

(A) $a_{ii} = 0$ for all $i = 1, 2, \ldots n$

(B) $a_{ii} = 0$ for atleast one $i = 1, 2, \ldots n$

(C) $a_{ii} \neq 0$ for all $i = 1, 2, \ldots n$

(D) $a_{ii} \neq 0$ for atleast one i, $i = 1, 2, \ldots n$

17. Inverse of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \\ -4 & 1 & -1 \end{pmatrix}$ is

(A) $\begin{pmatrix} -5 & -11 & 9 \\ -1 & -2 & 2 \\ -3 & -6 & 5 \end{pmatrix}$ (B) $\begin{pmatrix} -5 & -1 & -3 \\ -11 & -2 & -6 \\ 9 & 2 & 5 \end{pmatrix}$

(C) $\begin{pmatrix} 5 & 11 & 9 \\ 1 & 2 & -2 \\ 3 & 6 & -5 \end{pmatrix}$ (D) $\begin{pmatrix} 5 & 1 & 3 \\ 11 & 2 & 6 \\ 9 & -2 & -5 \end{pmatrix}$

18. I. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$, then A^{-1} is symmetric.

II. If a non-singular matrix A is symmetric, then A^{-1} is also symmetric.

Which of the following is correct?

(A) Both I and II true. (B) Both I and II false.

(C) I is true, II is false. (D) I is false, II is true.

19. A is a third order matrix. If the value of the square of the determinant of the matrix of co-factors of A is 28561, then $|A|$ equals

(A) 25 (B) ±13

(C) 120 (D) ±169

20. If A is a square matrix of order 3, then the product of A and its transpose is

(A) unit matrix. (B) zero matrix.

(C) identity matrix. (D) symmetric matrix.

21. If A and B are two skew symmetric matrices of the same order then AB is skew symmetric if and only if

(A) $AB + BA = O$ (B) $AB - BA = O$

(C) $AB + BA = I$ (D) $AB - BA = I$

22. Rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is

(A) 1 (B) 2

(C) 3 (D) 4

23. The rank of the matrix $\begin{pmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \\ -10 & 5 & 15 \end{pmatrix}$ is

(A) 0 (B) 1

(C) 2 (D) 3

24. If $A = (1\ \ 2\ \ 3)$ and $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then $\rho(AB)$ is

(A) 0 (B) 1

(C) 2 (D) 4

25. Which of the following matrix is row echelon form?

(A) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

26. Which of the following set of vectors are linearly dependent?
(A) (2, 3, 3), (3, −1, 3), (4, −2, 5)
(B) (3, 4, −1), (−1, 3, 1), (−2, −7, −2)
(C) (2, 1, 4), (1, −2, 2), (−3, 1, −6)
(D) (1, 3, −5), (−5, −1, 3), (4, −2, −2)

27. The system of equations $2x - y + 3z = 9$, $x + y + z = 0$ and $x - y + z = 0$ has/is
(A) unique solution.
(B) infinite solutions.
(C) only zero solution.
(D) inconsistent.

28. The system of equations $6x + 7y + 8z = 1$, $13x + 14y + 15z = 2$ and $x + 2y + 3z = 2$ is
(A) consistent with unique solution.
(B) consistent with infinite solutions.
(C) inconsistent.
(D) None of these

29. The value of λ for which the following system of equation does not have a solution is
$x + y + z = 6$
$4x + \lambda y - \lambda z = 0$
$3x + 2y - 4z = -8$
(A) 3 (B) −3
(C) 0 (D) 1

30. If the number of variables in the linear homogeneous system $AX = O$ is n, then the system will have exactly one solution $X = O$, if the rank of the matrix A is
(A) 1 (B) $< n$
(C) $\le n$ (D) n

31. If the equations $2x - y - z = 0$, $kx - 3y + 2z = 0$ and $-3x + 2y + kz = 0$ have a non-zero solution, then the value of k is
(A) 2 (B) 1
(C) 7 (D) Both 1 and 7

32. The system of equations $\alpha + 3y + 5z = 0$, $2x - 4\alpha y + \alpha z = 0$, $-4x + 18y + 7z = 0$ has only trivial solution if α is
(A) −1 or −3 (B) 1 or −3
(C) not equal to 1, −3 (D) not equal to −1 and 3

33. The eigen values of $\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ is
(A) 0, 0, 0 (B) 0, 1, 0
(C) 2, 1, 3 (D) −2, −1, −3

34. The characteristic roots of the inverse of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are
(A) −1, −1, 5 (B) 1, 1, 5
(C) $1, 1, \frac{1}{5}$ (D) $-1, -1, \frac{1}{5}$

35. The sum and product of the eigen values of the matrix $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & -2 \\ 1 & 3 & -5 \end{bmatrix}$ is respectively
(A) 0, 24 (B) 1, −24
(C) 2, 20 (D) 4, −24

36. The eigen values of a matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & p \\ 1 & 0 & q \end{bmatrix}$ are 1, 2, and 3. Then the values of p and q are _______.
(A) $p = 0, q = 0$
(B) $p =$ any real number, $q = 2$
(C) $p = 2, q = 0$
(D) $p = 2, q = 2$

37. The eigen values of the matrix $\begin{bmatrix} 0 & -1 & 2 & -3 \\ 1 & 0 & 4 & 6 \\ -2 & -4 & 0 & 5 \\ 3 & -6 & -5 & 0 \end{bmatrix}$ is
(A) real only (B) imaginary
(C) zero only (D) imaginary or zero

38. The number of linearly independent eigen vectors of $\begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}$ is _______.
(A) 0 (B) 1 (C) 2 (D) infinite

39. Which of the following is an eigen vector for the matrix $\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$?
(A) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (B) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
(C) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (D) $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$

40. For a matrix $A = \begin{bmatrix} 6 & -6 & 2 \\ -6 & 5 & -4 \\ 2 & -4 & 1 \end{bmatrix}$, $X = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ is an eigen vector. The corresponding eigen value is _______.
(A) −2 (B) 1
(C) 2 (D) 13

41. Let A be a 2 × 2 square matrix with $\lambda_1 = -2$ and $\lambda_2 = -3$ as its eigen values and $x_1 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$, $x^2 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$ as its eigen vectors then A is given by
(A) $\begin{bmatrix} 0 & 2 \\ 4 & -5 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & -6 \\ 7 & -9 \end{bmatrix}$
(C) $\begin{bmatrix} -2 & 6 \\ 7 & -3 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & 6 \\ -4 & 3 \end{bmatrix}$

42. Consider the matrix $A = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 0 \\ 0 & -3 & -2 \end{bmatrix}$ let $B = A^{-1}$, then $B =$ ______

(A) $\frac{-1}{4}[A^2 - A - 4I]$ (B) $\frac{1}{4}[A^2 - A - 4I]$

(C) $\frac{1}{4}[A^2 + A - 4I]$ (D) $\frac{-1}{4}[A^2 - A + 4I]$

43. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, then $A^{15} =$

(A) $8^{14}A$ (B) $8^{15}A$

(C) $8^{16}A$ (D) $15A$

44. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 6 & 7 \\ 9 & 0 & 1 \end{bmatrix}$, then the value of the matrix polynomial $2A^{10} - 18A^9 + 40A^8 - 25A^7 + 9A^6 - 20A^5 + 13A^4 - 9A^3 + 20A^2 - 10A$ is ______.

(A) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 6 & 7 \\ 9 & 7 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4 & 0 & 0 \\ 6 & 12 & 14 \\ 18 & 0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

45. For the matrix $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ 1 & 1 & -1 \end{bmatrix}$, consider the following statements

(P) The characteristic equation of A is $\lambda^3 - 5\lambda^2 + 4\lambda = 0$

(Q) A^{-1} exists

(R) The matrix A is diagonalizable

Which of the above statements are TRUE?

(A) P, Q and R

(B) P and R but not Q

(C) P and Q but not R

(D) Q and R but not P

46. If P is a modal matrix and D is a spectral matrix of a diagonalizable matrix A, then which of the following relations is NOT TRUE among A, P and D?

(A) $PD = AP$ (B) $DP^{-1} = P^{-1}A$

(C) $A^2P = PD^2$ (D) $DP = PA$

47. If A is a 3×3 square matrix with eigen values 0, 2, 3 with P as its modal matrix, then the eigen values of the matrix $P^{-1}AP$ are ______.

(A) 0, 2, 3

(B) 0, 4, 6

(C) $0, \frac{1}{2}, \frac{1}{3}$

(D) $1, \frac{1}{2}, \frac{1}{3}$

Previous Years' Questions

1. For what value of α and β, the following simultaneous equations have an infinite number of solutions?

$x + y + z = 5$; $x + 3y + 3z = 9$; $x + zy + az = b$

[GATE, 2007]

(A) 2, 7 (B) 3, 8

(C) 8, 3 (D) 7, 2

2. The product of matrices $(PQ)^{-1}P$ is **[GATE, 2008]**

(A) P^{-1} (B) Q^{-1}

(C) $P^{-1}Q^{-1}P$ (D) PQP^{-1}

3. The following simultaneous equations

$x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + kz = 6$

will NOT have a unique solution for k equal to

[GATE, 2008]

(A) 0 (B) 5

(C) 6 (D) 7

4. A square matrix B is skew-symmetric if

[GATE, 2009]

(A) $B^T = -B$ (B) $B^T = B$

(C) $B^{-1} = B$ (D) $B^{-1} = BT$

5. The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is

[GATE, 2010]

(A) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ -i & 3-2i \end{bmatrix}$

(B) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(C) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$

(D) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

6. $[A]$ is a square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$, respectively. Which of the following statements is TRUE? **[GATE, 2011]**

(A) Both [S] and [D] are symmetric.
(B) Both [S] and [D] are skew-symmetric.
(C) [S] is skew-symmetric and [D] is symmetric.
(D) [S] is symmetric and [D] is skew-symmetric.

7. The eigen vales of matrix $\begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$ are **[GATE, 2012]**

(A) −2.42 and 6.86 (B) 3.48 and 13.53
(C) 4.70 and 6.86 (D) 6.86 and 9.50

8. What is the minimum number of multiplications involved in computing the matrix product PQR? Matrix P has 4 rows and 2 columns, matrix Q has 2 rows and 4 columns, a matrix R has 4 rows and 1 column?

[GATE, 2013]

9. Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and $K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ the product $K^T JK$ is. **[GATE, 2014]**

10. The determinant of matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is.

[GATE, 2014]

11. The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is.

[GATE, 2014]

12. The sum of Eigen values of the matrix [M] is.

Where $[M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$ **[GATE, 2014]**

(A) 915 (B) 1355
(C) 1640 (D) 2180

13. Let $A = [a_{ij}]$, $1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i \cdot j$. The rank of A is **[GATE, 2015]**

(A) 0 (B) 1
(C) $n - 1$ (D) n

14. For what value of p the following set of equations will have no solution?

$2x + 3y = 5$
$3x + py = 10$

[GATE, 2015]

15. The smallest and largest eigen values of the following matrix are: $\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$ **[GATE, 2015]**

(A) 1.5 and 2.5 (B) 0.5 and 2.5
(C) 1.0 and 3.0 (D) 1.0 and 2.0

16. The two eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3 : 1 for $p = 2$. What is another value of p for which the eigen values have the same ratio of 3 : 1?

[GATE, 2015]

(A) −2 (B) 1
(C) $\frac{7}{3}$ (D) $\frac{14}{3}$

17. Consider the following linear systems:

$x + 2y - 3z = a$
$2x + 3y + 3z = b$
$5x + 9y - 6z = c$

This system is consistent if a, b and c satisfy the equation **[GATE, 2016]**

(A) $7a - b - c = 0$ (B) $3a + b - c = 0$
(C) $3a - b + c = 0$ (D) $7a - b + c = 0$

18. If the entries in each column of a square matrix M add up to 1, then an eigen value of M is **[GATE, 2016]**

(A) 4 (B) 3
(C) 2 (D) 1

Answer Keys

Exercises

1. C	**2.** A	**3.** C	**4.** C	**5.** B	**6.** B	**7.** A	**8.** B	**9.** C	**10.** C
11. C	**12.** C	**13.** D	**14.** C	**15.** A	**16.** B	**17.** B	**18.** D	**19.** B	**20.** D
21. A	**22.** B	**23.** B	**24.** B	**25.** A	**26.** C	**27.** A	**28.** C	**29.** A	**30.** D
31. D	**32.** C	**33.** C	**34.** C	**35.** B	**36.** B	**37.** B	**38.** B	**39.** B	**40.** D
41. B	**42.** A	**43.** A	**44.** B	**45.** B	**46.** D	**47.** A			

Previous Years' Questions

1. A	**2.** B	**3.** D	**4.** A	**5.** B	**6.** D	**7.** B	**8.** 16	**9.** 23	**10.** 88
11. 2	**12.** A	**13.** B	**14.** 4.49 to 4.51		**15.** D	**16.** D	**17.** B	**18.** D	