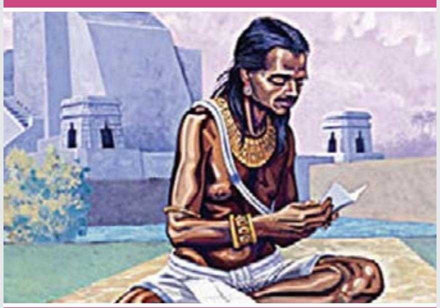
1 Applications of Matrices and Determinants



Brahmagupta

(c.598 AD(CE) - c.668 AD(CE))

Introduction

In our daily life, We use Matrices for taking seismic surveys. They are used for plotting graphs, statistics and also to do Scientific studies in almost different fields. Matrices are used in representing the real world data like the traits of people's population, habits etc..

Determinants have wonderful algebraic properties and occupied their proud place in linear algebra, because of their role in higher level algebraic thinking.

Brahmagupta (born c.598 AD(CE) died c.668 AD(CE)) was an Indian Mathematician and astronomer. He was the first to give rules to compute with zero. His contribution in Matrix is called as Brahmagupta Matrix.

$$B(x\,,\,y) = \begin{pmatrix} x & y \\ \pm ty & \pm x \end{pmatrix}.$$

Learning Objectives

On Completion of this chapter , the students are able to understand

- the concept of rank of a matrix.
- elementary transformations and equivalent matrices.
- echelon form of a matrix.
- the rank of the matrix.
- testing the consistency of a non- homogeneous linear equations.
- applications of linear equations
- the concept of Cramer's rule to solve non- homogenous linear equations.
- forecasting the succeeding state when the initial market share is given.

1.1 Rank of a Matrix

Matrices are one of the most commonly used tools in many fields such as Economics, Commerce and Industry.

We have already studied the basic properties of matrices. In this chapter we will study about the elementary transformations to develop new methods for various applications of matrices.

1.1.1 Concept

With each matrix, we can associate a non-negative integer called its rank.

Definition 1.1

The rank of a matrix A is the order of the largest non-zero minor of A and is denoted by $\rho(A)$

In other words, A positive integer 'r' is said to be the rank of a non- zero matrix A, if

(i) there is atleast one minor of A of order 'r' which is not zero and

(ii) every minor of A of order greater than 'r' is zero.

Note

(i) $\rho(A) \geq 0$

(ii) If A is a matrix of order $m \times n$, then $\rho(A) \leq$ minimum of $\{m,n\}$

(iii) The rank of a zero matrix is '0'

(iv) The rank of a non- singular matrix of order $n \times n$ is 'n'

Example 1.1

Find the rank of the matrix $\begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 5 \\ 3 & 9 \end{pmatrix}$

Order of A is 2×2 $\quad \therefore \rho(A) \leq 2$

Consider the second order minor

$$\begin{vmatrix} 1 & 5 \\ 3 & 9 \end{vmatrix} = -6 \neq 0$$

There is a minor of order 2, which is not zero. $\therefore \rho(A) = 2$.

DO YOU KNOW?

In cryptography, we are using matrix concepts.

Example 1.2

Find the rank of the matrix $\begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} -5 & -7 \\ 5 & 7 \end{pmatrix}$

Order of A is 2×2 $\quad \therefore \rho(A) \leq 2$

Consider the second order minor $\begin{vmatrix} -5 & -7 \\ 5 & 7 \end{vmatrix} = 0$

Since the second order minor vanishes, $\rho(A) \neq 2$

Consider a first order minor $|-5| \neq 0$

There is a minor of order 1, which is not zero

$$\therefore \rho\ (A) = 1$$

Example 1.3

Find the rank of the matrix $\begin{pmatrix} 0 & -1 & 5 \\ 2 & 4 & -6 \\ 1 & 1 & 5 \end{pmatrix}$

Solution:

Let $A=\begin{pmatrix}0&-1&5\\2&4&-6\\1&1&5\end{pmatrix}$

Order of A is 3×3.

$\therefore \rho(A)\leq 3$

Consider the third order minor

$\begin{vmatrix}0&-1&5\\2&4&-6\\1&1&5\end{vmatrix}=6\neq0$

There is a minor of order 3, which is not zero

$\therefore \rho(A)=3.$

Example 1.4

Find the rank of the matrix $\begin{pmatrix}5&3&0\\1&2&-4\\-2&-4&8\end{pmatrix}$

Solution:

Let $A=\begin{pmatrix}5&3&0\\1&2&-4\\-2&-4&8\end{pmatrix}$

Order of A is 3×3.

$\therefore \rho(A)\leq 3.$

Consider the third order minor $\begin{vmatrix}5&3&0\\1&2&-4\\-2&-4&8\end{vmatrix}=0$

Since the third order minor vanishes, therefore $\rho(A)\neq3$

Consider a second order minor $\begin{vmatrix}5&3\\1&2\end{vmatrix}=7\neq0$

There is a minor of order 2, which is not zero.

$\therefore \rho(A)=2.$

Example 1.5

Find the rank of the matrix $\begin{pmatrix}1&2&-1&3\\2&4&1&-2\\3&6&3&-7\end{pmatrix}$

Solution:

Let $A=\begin{pmatrix}1&2&-1&3\\2&4&1&-2\\3&6&3&-7\end{pmatrix}$

Order of A is 3×4

$\therefore \rho(A)\leq 3.$

Consider the third order minors

$\begin{vmatrix}1&2&-1\\2&4&1\\3&6&3\end{vmatrix}=0,\quad \begin{vmatrix}1&-1&3\\2&1&-2\\3&3&-7\end{vmatrix}=0$

$\begin{vmatrix}1&2&3\\2&4&-2\\3&6&-7\end{vmatrix}=0,\quad \begin{vmatrix}2&-1&3\\4&1&-2\\6&3&-7\end{vmatrix}=0$

Since all third order minors vanishes, $\rho(A)\neq3.$

Now, let us consider the second order minors,

Consider one of the second order minors

$\begin{vmatrix}2&-1\\4&1\end{vmatrix}=6\neq0$

There is a minor of order 2 which is not zero.

$\therefore \rho(A)=2.$

DO YOU KNOW?

A is a square matrix of order 3. If A is of rank 2, then *adj* A is of rank 1.

1.1.2 Elementary Transformations and Equivalent matrices

Elementary transformations of a matrix

(i) Interchange of any two rows (or columns): $R_i \leftrightarrow R_j$ $(or\ C_i \leftrightarrow C_j)$.

(ii) Multiplication of each element of a row (or column) by any non-zero scalar k: $R_i \to kR_i$ $(or\ C_i \to kC_i)$

(iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column):

$$R_i \to R_i + kR_j.(or\ C_i \to C_i + kC_j)$$

Equivalent Matrices

Two matrices A and B are said to be equivalent if one is obtained from the another by applying a finite number of elementary transformations and we write it as $A \sim B$ or $B \sim A$.

1.1.3 Echelon form and finding the rank of the matrix (upto the order of 3×4)

A matrix A of order $m \times n$ is said to be in echelon form if

(i) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.

(ii) The number of zeros before the first non-zero element in a row is less then the number of such zeros in the next row.

Example 1.6

Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$

Solution :

The order of A is 3×3.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form by using elementary transformations.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ The above matrix is in echelon form	$R_3 \to R_3 - R_2$

The number of non zero rows is 2

$\therefore$ Rank of A *is* 2.

$\rho(A) = 2$.

Note

A row having atleast one non-zero element is called as non-zero row.

Example 1.7

Find the rank of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

Solution:

The order of A is 3×4.

$\therefore \rho(A) \leq 3$.

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	
$A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$	$R_3 \to R_3 + 5R_2$

The number of non zero rows is 3. $\therefore \rho(A) = 3.$

Example 1.8

Find the rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$

Solution:

The order of A is 3×4.

$$\therefore \rho(A) \leq 3.$$

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The number of non zero rows is 3.

$$\therefore \rho(A) = 3.$$

Consistency of Equations

System of linear equations in two variables

We have already studied , how to solve two simultaneous linear equations by matrix inversion method.

> **Recall**
>
> Linear equations can be written in matrix form AX=B, then the solution is $X = A^{-1}B$, provided $|A| \neq 0$

Consider a system of linear equations with two variables,

$$\left.\begin{aligned} ax + by &= h \\ cx + dy &= k \end{aligned}\right\} \quad (1)$$

Where a, b, c, d, h and k are real constants and neither a and b nor c and d are both zero.

For any tow given lines L_1 and L_2, one and only one of the following may occur.

L_1 and L_2 intersect at exactly one point

L_1 and L_2 are coincident

L_1 and L_2 are parallel and distinct.

(see Fig 1.1) In the first case, the system has a unique solution corresponding to the single point of intersection of the two lines.

In the second case, the system has infinitely many solutions corresponding to the points lying on the same line

Finally in the third case, the system has no solution because the two lines do not intersect.

Let us illustrate each of these possibilities by considering some specific examples.

(a) A system of equations with exactly one solution

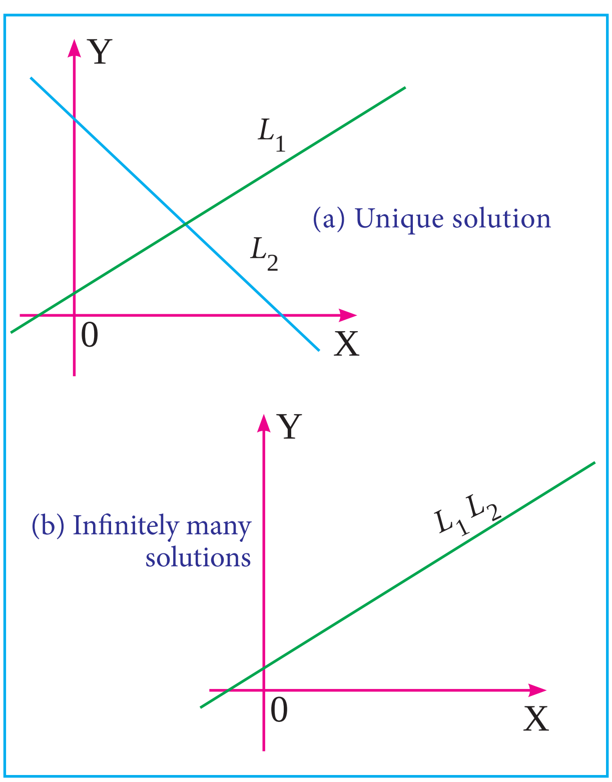
Consider the system $2x - y = 1,\ 3x + 2y = 12$ which represents two lines intersecting at (2, 3) i.e (2, 3) lies on both lines. So the equations are consistent and have unique solution.

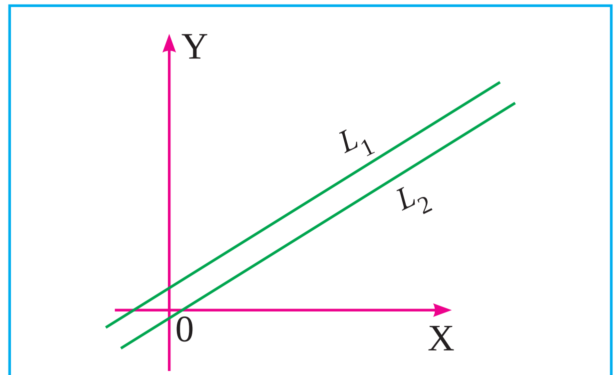
(b) A system of equations with infinitely many solutions

Consider the system $2x - y = 1,\ 6x - 3y = 3$ which represents two coincident lines. We find that any point on the line is a solution. The equations are consistent and have infinite sets of solutions such as $(0, -1), (1, 1)$ and so on. Such a system is said to be dependent.

(c) A system of equations that has no solution

Consider the system $2x - y = 1$, $6x - 3y = 12$ which represents two parallel straight lines. The equations are inconsistent and have no solution.





(c) No solution

Fig. 1.1

System of non Homogeneous Equations in three variables

A linear system composed of three linear equations with three variables x, y and z has the general form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad (2)$$

A linear equation $ax + by + cz = d$ (a, b and c not all equal to zero) in three variables represents a plane in three dimensional space. Thus, each equation in system (2) represents a plane in three dimensional space, and the solution(s) of the system is precisely the point(s) of intersection of the three planes defined by the three linear equations that make up the system. This system has one and only one solution, infinitely many solutions, or no solution, depending on whether and how the planes intersect one another. Figure 1.2 illustrates each of these possibilities.

In Figure 1.2(a), the three planes intersect at a point corresponding to the situation in which system (2) has a unique solution.

Figure 1.2 (b) depicts a situation in which there are infinitely many solutions to the system. Here the three planes intersect along a line, and the solutions are represented by the infinitely many points lying on this line.

In Figure 1.2 (c), the three planes are parallel and distinct, so there is no point in common to all three planes; system (2) has no solution in this case.

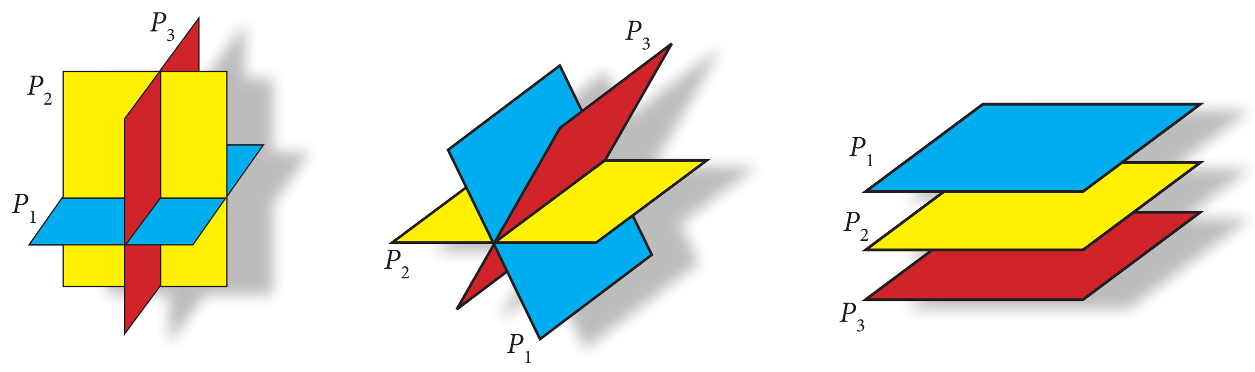


Fig.1.2

(a) A Unique solution (b) Infinitely many solutions (c) No solution

Note

Every system of linear equations has no solution, or has exactly one solution or has infinitely many solutions.

An arbitrary system of 'm' linear equations in 'n' unknowns can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where $x_1, x_2, .., x_n$ are the unknowns and the subscripted a's and b's denote the constants.

Augmented matrices

A system of 'm' linear equations in 'n' unknowns can be abbreviated by writing only the rectangular array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_n \end{bmatrix}$$

This is called the augmented matrix for the system and $\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$ is the coefficient matrix.

Consider the following system of equations

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

$$A \quad X = B$$

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$ is the coefficient matrix

and $[A, B] = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$ is the augmented matrix.

1.1.4 Testing the consistency of non homogeneous linear equations (two and three variables) by rank method.

Consider the equations $A X = B$ in 'n' unknowns.

(i) If $\rho([A, B]) = \rho(A)$, then the equations are consistent.

(ii) If $\rho([A, B]) = \rho(A) = n$, then the equations are consistent and have unique solution.

(iii) If $\rho([A, B]) = \rho(A) < n$, then the equations are consistent and have infinitely many solutions.

(iv) If $\rho([A, B]) \neq \rho(A)$ then the equations are inconsistent and has no solution.

Example 1.9

Show that the equations $x + y = 5$, $2x + y = 8$ are consistent and solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\text{A} \quad \text{X} \quad = \quad \text{B}$$

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 1 & 8 \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	$\sim\begin{pmatrix} 1 & 1 & 5 \\ 0 & -1 & -2 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A) = 2$	$\rho([A,B]) = 2$	

Number of non-zero rows is 2.

$\rho(A) = \rho([A,B]) = 2 =$ Number of unknowns.

The given system is consistent and has unique solution.

Now, the given system is transformed into

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$x + y = 5 \qquad (1)$$

$$y = 2$$

$$\therefore (1) \Rightarrow x + 2 = 5$$

$$x = 3$$

Solution is $x = 3,\ y = 2$

Example 1.10

Show that the equations $2x + y = 5$, $4x + 2y = 10$ are consistent and solve them.

Solution:

The matrix equation corresponding to the system is

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$A \quad X \quad = \quad B$$

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \end{pmatrix}$	
$\sim\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$	$\sim\begin{pmatrix} 2 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\rho(A) = 1$	$\rho([A,B]) = 1$	

$\rho(A) = \rho([A,B]) = 1 <$ number of unknowns

$\therefore$ The given system is consistent and has infinitely many solutions.

Now, the given system is transformed into the matrix equation.

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x + y = 5$$

Let us take $y = k,\ k \in R$

$$\Rightarrow 2x+k=5$$

$$x=\frac{1}{2}(5-k)$$

$$x=\frac{1}{2}(5-k),\ y=k \text{ for all } k\in R$$

Thus by giving different values for k, we get different solution. Hence the system has infinite number of solutions.

Example 1.11

Show that the equations $3x-2y=6$, $6x-4y=10$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix}3 & -2\\ 6 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}6\\ 10\end{pmatrix}$$

AX = B

Matrix A	Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix}3 & -2\\ 6 & -4\end{pmatrix}$	$\begin{pmatrix}3 & -2 & 6\\ 6 & -4 & 10\end{pmatrix}$	
$\sim\begin{pmatrix}3 & -2\\ 0 & 0\end{pmatrix}$	$\sim\begin{pmatrix}3 & -2 & 6\\ 0 & 0 & -2\end{pmatrix}$	$R_2\to R_2-2R_1$
$\rho(A)=1$	$\rho([A,B])=2$	

$$\therefore \rho([A,B])=2,\qquad \rho(A)=1$$

$$\rho(A)\neq\rho([A,B])$$

$\therefore$ The given system is inconsistent and has no solution.

Example 1.12

Show that the equations $2x+y+z=5,\ x+y+z=4,\ x-y+2z=1$ are consistent and hence solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix}2 & 1 & 1\\ 1 & 1 & 1\\ 1 & -1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}5\\ 4\\ 1\end{pmatrix}$$

$$A \qquad X \quad = \quad B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix}2 & 1 & 1 & 5\\ 1 & 1 & 1 & 4\\ 1 & -1 & 2 & 1\end{pmatrix}$	
$\sim\begin{pmatrix}1 & 1 & 1 & 4\\ 2 & 1 & 1 & 5\\ 1 & -1 & 2 & 1\end{pmatrix}$	$R_1\leftrightarrow R_2$
$\sim\begin{pmatrix}1 & 1 & 1 & 4\\ 0 & -1 & -1 & -3\\ 0 & -2 & 1 & -3\end{pmatrix}$	$R_2\to R_2-2R_1$ $R_3\to R_3-R_1$
$\sim\begin{pmatrix}1 & 1 & 1 & 4\\ 0 & -1 & -1 & -3\\ 0 & 0 & 3 & 3\end{pmatrix}$	$R_3\to R_3-2R_2$
$\rho(A)=3,\ \rho([A,B])=3$	

Obviously the last equivalent matrix is in the echelon form. It has three non-zero rows.

$\rho(A)=\rho([A,B])=3=$ Number of unknowns .

The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix}1 & 1 & 1\\ 0 & -1 & -1\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}4\\ -3\\ 3\end{pmatrix}$$

$$x+y+z=4 \quad (1)$$
$$y+z=3 \quad (2)$$
$$3z=3 \quad (3)$$
$$(3) \Rightarrow \quad z=1$$
$$(2) \Rightarrow \quad y=3-z=2$$
$$(1) \Rightarrow x=4-y-z$$
$$x=1$$
$$\therefore\ x=1, \quad y=2, \quad z=1$$

Example 1.13

Show that the equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ are consistent and solve them.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 30 \end{pmatrix}$$
$$A \qquad X \quad = \quad B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 4 & 16 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_2$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$
$\rho(A)=2,\ \rho([A,B])=2$	

Obviously the last equivalent matrix is in the echelon form. It has two non-zero rows.

$$\therefore \quad \rho([A,B])=2,\ \rho(A)=2$$

$\rho(A)=\rho([A,\ B])=2<$ Number of unknowns.

The given system is consistent and has infinitely many solutions.

The given system is equivalent to the matrix equation,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}$$

$$x+y+z=6 \quad (1)$$
$$y+2z=8 \quad (2)$$

$(2) \Rightarrow \quad y=8-2z,$

$(1) \Rightarrow \quad x=6-y-z=6-(8-2z)-z=z-2$

Let us take $z=k, k\in R$, we get $x=k-2, y=8-2k$, Thus by giving different values for k we get different solutions. Hence the given system has infinitely many solutions.

Example 1.14

Show that the equations are inconsistent: $x-4y+7z=14$, $3x+8y-2z=13$, $7x-8y+26z=5$

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 5 \end{pmatrix}$$
$$A \qquad X \quad = \quad B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{pmatrix}$	

$\sim\begin{pmatrix}1&-4&7&14\\0&20&-23&-29\\0&20&-23&-93\end{pmatrix}$	$R_2\to R_2-3R_1$ $R_3\to R_3-7R_1$
$\sim\begin{pmatrix}1&-4&7&14\\0&20&-23&-29\\0&0&0&64\end{pmatrix}$	$R_3\to R_3-R_2$
$\rho(A)=2,\ \rho([A,B])=3$	

The last equivalent matrix is in the echelon form. $[A, B]$ has 3 non-zero rows and $[A]$ has 2 non-zero rows.

$\therefore \quad \rho([A, B])=3, \qquad \rho(A)=2$

$\rho(A)\neq\rho([A, B])$

The system is inconsistent and has no solution.

Example 1.15

Find k, if the equations $x+2y-3z=-2$, $3x-y-2z=1$, $2x+3y-5z=k$ are consistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix}1&2&-3\\3&-1&-2\\2&3&-5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2\\1\\k\end{pmatrix}$$

$$A \qquad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix}1&2&-3&-2\\3&-1&-2&1\\2&3&-5&k\end{pmatrix}$	
$\sim\begin{pmatrix}1&2&-3&-2\\0&-7&7&7\\0&-1&1&4+k\end{pmatrix}$	$R_2\to R_2-3R_1$ $R_3\to R_3-2R_1$
$\sim\begin{pmatrix}1&2&-3&-2\\0&-7&7&7\\0&0&0&21+7k\end{pmatrix}$	$R_3\to 7R_3-R_2$
$\rho(A)=2,\ \rho([A, B])=2$ or 3	

For the equations to be consistent, $\rho([A, B])=\rho(A)=2$

$\therefore \qquad 21+7k=0$

$7k=-21.$

$k=-3$

Example 1.16

Find k, if the equations $x+y+z=7$, $x+2y+3z=18$, $y+kz=6$ are inconsistent.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix}1&1&1\\1&2&3\\0&1&k\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}7\\18\\6\end{pmatrix}$$

$$A \qquad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix}1&1&1&7\\1&2&3&18\\0&1&k&6\end{pmatrix}$	
$\sim\begin{pmatrix}1&1&1&7\\0&1&2&11\\0&1&k&6\end{pmatrix}$	$R_2\to R_2-R_1,$
$\sim\begin{pmatrix}1&1&1&7\\0&1&2&11\\0&0&k-2&-5\end{pmatrix}$	$R_3\to R_3-R_2$
$\rho(A)=2$ or 3, $\rho([A,B])=3$	

For the equations to be inconsistent

$$\rho([A, B])\neq\rho(A)$$

It is possible if $k-2=0$.

$\therefore \boxed{k=2}$

Example 1.17

Investigate for what values of 'a' and 'b' the following system of equations

$x+y+z=6, \quad x+2y+3z=10,$
$x+2y+az=b$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Solution:

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ b \end{pmatrix}$$

$$A \qquad X \quad = \quad B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{pmatrix}$	$R_2 \to R_2 - R_1,$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Case (i) For no solution:

The system possesses no solution only when $\rho(A) \neq \rho([A,B])$ which is possible only when $a-3=0$ and $b-10 \neq 0$

Hence for $a=3$, $b \neq 10$, the system possesses no solution.

Case (ii) For a unique solution:

The system possesses a unique solution only when $\rho(A) = \rho([A,B]) =$ number of unknowns.

i.e when $\rho(A) = \rho([A,B]) = 3$

Which is possible only when $a-3 \neq 0$ and b may be any real number as we can observe .

Hence for $a \neq 3$ and $b \in R$, the system possesses a unique solution.

Case (iii) For an infinite number of solutions:

The system possesses an infinite number of solutions only when

$$\rho(A) = \rho([A,B]) < \text{number of unknowns}$$

i,e when $\rho(A) = \rho([A,B]) = 2 < 3$ (number of unknowns) which is possible only when $a-3=0, \quad b-10=0$

Hence for $a = 3$, $b = 10$, the system possesses infinite number of solutions.

Example 1.18

The total number of units produced (P) is a linear function of amount of over times in labour (in hours) (l), amount of additional machine time (m) and fixed finishing time (a)

i.e, $P = a + bl + cm$

From the data given below, find the values of constants a, b and c

Day	Production (in Units P)	Labour (in Hrs l)	Additional Machine Time (in Hrs m)
Monday	6,950	40	10
Tuesday	6,725	35	9
Wednesday	7,100	40	12

Estimate the production when overtime in labour is 50 hrs and additional machine time is 15 hrs.

Solution:

We have, $P = a + bl + cm$

Putting above values we have

$$6{,}950 = a + 40b + 10c$$

$$6{,}725 = a + 35b + 9c$$

$$7{,}100 = a + 40b + 12c$$

The Matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 40 & 10 \\ 1 & 35 & 9 \\ 1 & 40 & 12 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6950 \\ 6725 \\ 7100 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix $[A,B]$	Elementary Transformation
$\begin{pmatrix} 1 & 40 & 10 & 6950 \\ 1 & 35 & 9 & 6725 \\ 1 & 40 & 12 & 7100 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 40 & 10 & 6950 \\ 0 & -5 & -1 & -225 \\ 0 & 0 & 2 & 150 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\rho(A) = 3, \rho([A,B]) = 3$	

$\therefore$ The given system is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 40 & 10 \\ 0 & -5 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 6950 \\ -225 \\ 150 \end{pmatrix}$$

$$a + 40b + 10c = 6950 \quad (1)$$

$$-5b - c = -225 \quad (2)$$

$$2c = 150 \quad (3)$$

$$\boxed{c = 75}$$

Now, $(2) \Rightarrow -5b - 75 = -225$

$$\boxed{b = 30}$$

and $(1) \Rightarrow a + 1200 + 750 = 6950$

$$a = 5000$$

$$a = 5000, \; b = 30, \; c = 75$$

$\therefore$ The production equation is

$$P = 5000 + 30l + 75m$$

$$P_{\text{at } l=50, \; m=15} = 5000 + 30(50) + 75(15)$$

$$= 7625$$

$\therefore$ The production $= 7{,}625$ units.

Exercise 1.1

1. Find the rank of each of the following matrices

i) $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$

iii) $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ iv) $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$

v) $\begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$ vi) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

vii) $\begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$

viii) $\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

2. If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$, then find the rank of AB and the rank of BA.

3. Solve the following system of equations by rank method: $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x - y - z = 0$

4. Show that the equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$ are consistent and solve them by rank method.

5. Show that the following system of equations have unique solution: $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$ by rank method.

6. For what values of the parameter λ, will the following equations fail to have unique solution: $3x - y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$ by rank method.

7. The price of three commodities X,Y and Z are x,y and z respectively Mr.Anand purchases 6 units of Z and sells 2 units of X and 3 units of Y. Mr.Amar purchases a unit of Y and sells 3 units of X and 2units of Z. Mr.Amit purchases a unit of X and sells 3 units of Y and a unit of Z. In the process they earn ₹5,000/-, ₹2,000/- and ₹5,500/- respectively Find the prices per unit of three commodities by rank method.

8. An amount of ₹5,000/- is to be deposited in three different bonds bearing 6%, 7% and 8% per year respectively. Total annual income is ₹358/-. If the income from first two investments is ₹70/- more than the income from the third, then find the amount of investment in each bond by rank method.

1.2 Cramer's Rule

Gabriel Cramer, a Swiss mathematician born in the city Geneva in 31 July 1704. He edited the works of the two elder Bernoulli's, and wrote on the physical cause of the spheriodal shape of the planets and the motion of their apsides (1730), and on Newton's treatment of cubic curves (1746).

In 1750 he published Cramer's Rule, giving a general formula for the solution of certain linear system of n equations in n unknowns having a unique solution in terms of determinants. Advantages of Cramer's rule is that we can find the value of x, y or z without knowing any of the other values of x, y or z. Cramer's rule is applicable only when $\Delta \neq 0$ (Δ is the determinant value of the coefficient matrix) for unique solution.

Theorem (without proof) Cramer's Rule:

Consider, $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $AX = B$ is a system of n linear equations in 'n' unknowns such that $\det(A) \neq 0$, then the system has a unique solution.

This solution is,

$$x_1 = \frac{\det(A_1)}{\det A}, \quad x_2 = \frac{\det(A_2)}{\det A}, \ \ldots\ , \ x_n = \frac{\det(A_n)}{\det A}$$

where A_j is the matrix obtained by replacing the entries in the j^{th} column of A by the entries in the matrix

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

1.2.1 Non Homogeneous linear equations upto three variables

(a) Consider the system of two linear equations with two unknowns.

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \Delta_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix} \quad \Delta_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

The solution of unknown by Cramer's rule is

$$x = \frac{\Delta x}{\Delta}, \qquad y = \frac{\Delta y}{\Delta} \qquad \text{provided } \Delta \neq 0$$

(b) Consider the system of three linear equations with three unknowns

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Solution of unknown by Cramer Rule is

$$x = \frac{\Delta x}{\Delta},\ y = \frac{\Delta y}{\Delta}, z = \frac{\Delta z}{\Delta}$$

Example 1.19

Solve the equations $2x + 3y = 7$, $3x + 5y = 9$ by Cramer's rule.

Solution:

The equations are

$$2x+3y=7$$
$$3x+5y=9$$

Here $\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1 \neq 0$

$\therefore$ we can apply Cramer's Rule

Now $\Delta_x = \begin{vmatrix} 7 & 3 \\ 9 & 5 \end{vmatrix} = 8$ $\quad \Delta_y = \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} = -3$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{8}{1} = 8 \quad y = \frac{\Delta y}{\Delta} = \frac{-3}{1} = -3$$

$\therefore$ Solution is $x = 8, y = -3$

Example 1.20

The following table represents the number of shares of two companies A and B during the month of January and February and it also gives the amount in rupees invested by Ravi during these two months for the purchase of shares of two companies. Find the the price per share of A and B purchased during both the months.

Months	Number of Shares of the company		Amount invested by Ravi (in ₹)
	A	B	
January	10	5	125
February	9	12	150

Solution:

Let the price of one share of A be x

Let the price of one share of B be y

$\therefore$ By given data, we get the following equations

$$10x+5y=125$$
$$9x+12y=150$$

$$\Delta = \begin{vmatrix} 10 & 5 \\ 9 & 12 \end{vmatrix} = 75 \neq 0$$

$\Delta_x = \begin{vmatrix} 125 & 5 \\ 150 & 12 \end{vmatrix} = 750$ $\quad \Delta_y = \begin{vmatrix} 10 & 125 \\ 9 & 150 \end{vmatrix} = 375$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{750}{75} = 10 \quad y = \frac{\Delta y}{\Delta} = \frac{375}{75} = 5$$

The price of the share A is ₹10 and the price of the share B is ₹5.

Example 1.21

The total cost of 11 pencils and 3 erasers is ₹ 64 and the total cost of 8 pencils and 3 erasers is ₹49. Find the cost of each pencil and each eraser by Cramer's rule.

Solution:

Let 'x' be the cost of a pencil

Let 'y' be the cost of an eraser

$\therefore$ By given data, we get the following equations

$$11x+3y=64$$
$$8x+3y=49$$

$\Delta = \begin{vmatrix} 11 & 3 \\ 8 & 3 \end{vmatrix} = 9 \neq 0$. It has unique solution.

$\Delta_x = \begin{vmatrix} 64 & 3 \\ 49 & 3 \end{vmatrix} = 45$ $\quad \Delta_y = \begin{vmatrix} 11 & 64 \\ 8 & 49 \end{vmatrix} = 27$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{45}{9} = 5 \quad y = \frac{\Delta y}{\Delta} = \frac{27}{9} = 3$$

$\therefore$ The cost of a pencil is ₹ 5 and the cost of an eraser is ₹ 3.

Example 1.22

Solve by Cramer's rule $x+y+z=4$, $2x-y+3z=1$, $3x+2y-z=1$

Solution:

Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix} = 13 \neq 0$

$\therefore$ We can apply Cramer's Rule and the system is consistent and it has unique solution.

$$\Delta_x = \begin{vmatrix} 4 & 1 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = -13$$

$$\Delta_y = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & -1 \end{vmatrix} = 39 \quad \Delta_z = \begin{vmatrix} 1 & 1 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 26$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{-13}{13} = -1$$

$$y = \frac{\Delta y}{\Delta} = \frac{39}{13} = 3 \qquad z = \frac{\Delta z}{\Delta} = \frac{26}{13} = 2$$

$\therefore$ The solution is $(x, y, z) = (-1, 3, 2)$

DO YOU KNOW?

If $|A|=0$, then the system of equations has either no solution or infinitely many solutions.

Example 1.23

The price of 3 Business Mathematics books, 2 Accountancy books and one Commerce book is ₹ 840. The price of 2 Business Mathematics books, one Accountancy book and one Commerce book is ₹ 570. The price of one Business Mathematics book, one Accountancy book and 2 Commerce books is ₹ 630. Find the cost of each book by using Cramer's rule.

Solution:

Let 'x' be the cost of a Business Mathematics book

Let 'y' be the cost of a Accountancy book.

Let 'z' be the cost of a Commerce book.

$$\therefore \quad \begin{aligned} 3x + 2y + z &= 840 \\ 2x + y + z &= 570 \\ x + y + 2z &= 630 \end{aligned}$$

Here

$$\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2 \neq 0$$

$$\Delta_x = \begin{vmatrix} 840 & 2 & 1 \\ 570 & 1 & 1 \\ 630 & 1 & 2 \end{vmatrix} = -240$$

$$\Delta_y = \begin{vmatrix} 3 & 840 & 1 \\ 2 & 570 & 1 \\ 1 & 630 & 2 \end{vmatrix} = -300$$

$$\Delta_z = \begin{vmatrix} 3 & 2 & 840 \\ 2 & 1 & 570 \\ 1 & 1 & 630 \end{vmatrix} = -360$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{-240}{-2} = 120$$

$$y = \frac{\Delta y}{\Delta} = \frac{-300}{-2} = 150 \quad z = \frac{\Delta z}{\Delta} = \frac{-360}{-2} = 180$$

$\therefore$ The cost of a Business Mathematics book is ₹120, the cost of a Accountancy book is ₹150 and the cost of a Commerce book is ₹180.

Example 1.24

An automobile company uses three types of Steel S_1, S_2 and S_3 for providing three different types of Cars C_1, C_2 and C_3. Steel requirement R (in tonnes) for each type of car and total available steel of all the three types are summarized in the following table.

Types of Steel	Types of Car			Total Steel available
	C_1	C_2	C_3	
S_1	3	2	4	28
S_2	1	1	2	13
S_3	2	2	1	14

Determine the number of Cars of each type which can be produced by Cramer's rule.

Solution:

Let 'x' be the number of cars of type C_1

Let 'y' be the number of cars of type C_2

Let 'z' be the number of cars of type C_3

$$3x+2y+4z=28$$
$$x+y+2z=13$$
$$2x+2y+z=14$$

Here $$\Delta = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -3 \neq 0$$

$$\Delta_x = \begin{vmatrix} 28 & 2 & 4 \\ 13 & 1 & 2 \\ 14 & 2 & 1 \end{vmatrix} = -6$$

$$\Delta_y = \begin{vmatrix} 3 & 28 & 4 \\ 1 & 13 & 2 \\ 2 & 14 & 1 \end{vmatrix} = -9$$

$$\Delta_z = \begin{vmatrix} 3 & 2 & 28 \\ 1 & 1 & 13 \\ 2 & 2 & 14 \end{vmatrix} = -12$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta x}{\Delta} = \frac{-6}{-3} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{-9}{-3} = 3 \qquad z = \frac{\Delta z}{\Delta} = \frac{-12}{-3} = 4$$

$\therefore$ The number of cars of each type which can be produced are 2, 3 and 4.

Exercise 1.2

1. Solve the following equations by using Cramer's rule

(i) $2x + 3y = 7;\quad 3x + 5y = 9$

(ii) $5x + 3y = 17;\quad 3x + 7y = 31$

(iii) $2x + y - z = 3,\quad x + y + z = 1,$
$x - 2y - 3z = 4$

(iv) $x + y + z = 6,\quad 2x + 3y - z = 5,$
$6x - 2y - 3z = -7$

(v) $x + 4y + 3z = 2,\quad 2x - 6y + 6z = -3,$
$5x - 2y + 3z = -5$

2. A commodity was produced by using 3 units of labour and 2 units of capital, the total cost is ₹ 62. If the commodity had been produced by using 4 units of labour and one unit of capital, the cost is ₹ 56. What is the cost per unit of labour and capital? (Use determinant method).

3. A total of ₹ 8,600 was invested in two accounts. One account earned $4\frac{3}{4}\%$ annual interest and the other earned $6\frac{1}{2}\%$ annual interest. If the total interest for one year was ₹ 431.25, how much was invested in each account? (Use determinant method).

4. At marina two types of games viz., Horse riding and Quad Bikes riding are available on hourly rent. Keren and Benita spent ₹780 and ₹560 during the month of May.

Name	Number of hours		Total amount spent (in ₹)
	Horse Riding	Quad Bike Riding	
Keren	3	4	780
Benita	2	3	560

Find the hourly charges for the two games (rides). (Use determinant method).

5. In a market survey three commodities A, B and C were considered. In finding out the index number some fixed weights were assigned to the three varieties in each of the commodities. The table below provides the information regarding the consumption of three commodities according to the three varieties and also the total weight received by the commodity.

Commodity Variety	Variety			Total weight
	I	II	III	
A	1	2	3	11
B	2	4	5	21
C	3	5	6	27

Find the weights assigned to the three varieties by using Cramer's Rule.

6. A total of ₹ 8,500 was invested in three interest earning accounts. The interest rates were 2%, 3% and 6% if the total simple interest for one year was ₹380 and the amount invested at 6% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account? (use Cramer's rule).

1.3 Transition Probability Matrices

1.3.1 Forecasting the succeeding state when the initial market share is given

One stage Transition Probability

The Occurrence of an event at a specified point in time, put the system in state S_n; if after the passage of one unit of time, another event occurs, that is the system moved from the state S_n to S_{n+1}. This movement is related to a probability distribution, there is a probability associated with each (move) transition from event S_n to S_{n+1}. This probability distribution is called one stage transition probability.

Transition Matrix

The transition Probabilities P_{jk} satisfy $P_{jk} > 0$, $\sum_k P_{jk} = 1$ for all j

These probabilities may be written in the matrix form

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

This is called the transition probability matrix.

Example 1.25

Consider the matrix of transition probabilities of a product available in the market in two brands A and B.

$$\begin{array}{cc} & \begin{array}{cc} A & \;\;B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

Determine the market share of each brand in equilibrium position.

Solution:

Transition probability matrix

$$T = \begin{array}{cc} & \begin{array}{cc} A & \;\;B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

At equilibrium, $\begin{pmatrix} A & B \end{pmatrix} T = \begin{pmatrix} A & B \end{pmatrix}$ where $A + B = 1$

$$\begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

$$0.9A + 0.3B = A$$

$$0.9A + 0.3(1 - A) = A$$

$$0.9A - 0.3A + 0.3 = A$$

$$0.6A + 0.3 = A$$

$$0.4A = 0.3$$

$$A = \frac{0.3}{0.4} = \frac{3}{4}$$

$$B = 1 - \frac{3}{4} = \frac{1}{4}$$

Hence the market share of brand A is 75% and the market share of brand B is 25%

Example 1.26

Parithi is either sad (S) or happy (H) each day. If he is happy in one day, he is sad on the next day by four times out of five. If he is sad on one day, he is happy on the next day by two times out of three. Over a long run, what are the chances that Parithi is happy on any given day?

Solution:

The transition porbability matrix is

$$T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

At equilibrium, $(S \quad H)\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix} = (S \quad H)$

where $S + H = 1$

$$\frac{1}{3}S + \frac{4}{5}H = S$$

$$\frac{1}{3}(1-H) + \frac{4}{5}H = 1\text{-}H$$

On solving this, we get

$$S = \frac{6}{11} \text{ and } H = \frac{5}{11}$$

In the long run, on a randomly selected day, his chances of being happy is $\frac{5}{11}$.

Example 1.27

Akash bats according to the following traits. If he makes a hit (S), there is a 25% chance that he will make a hit his next time at bat. If he fails to hit (F), there is a 35% chance that he will make a hit his next time at bat. Find the transition probability matrix for the data and determine Akash's long- range batting average.

Solution:

The Transition probability matrix is

$$T = \begin{pmatrix} 0.25 & 0.75 \\ 0.35 & 0.65 \end{pmatrix}$$

At equilibrium, $(S \quad F)\begin{pmatrix} 0.25 & 0.75 \\ 0.35 & 0.65 \end{pmatrix} = (S \quad F)$

where $S + F = 1$

$0.25\,S + 0.35\,F = S$

$0.25\,S + 0.35\,(1 - S) = S$

On solving this, we get, $S = \dfrac{0.35}{1.10}$

$$\Rightarrow S = 0.318 \text{ and } F = 0.682$$

$\therefore$ Akash's batting average is 31.8%

Example 1.28

80% of students who do maths work during one study period, will do the maths work at the next study period. 30% of students who do english work during one study period, will do the english work at the next study period.

Initially there were 60 students do maths work and 40 students do english work. Calculate,

(i) The transition probability matrix

(ii) The number of students who do maths work, english work for the next subsequent 2 study periods.

Solution

(i) Transition probability matrix

$$T = \begin{array}{c} \\ M \\ E \end{array}\!\!\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array}$$

After one study period,

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 60 & 40 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\!\!\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 76 & 24 \end{pmatrix} \end{array}$$

So in the very next study period, there will be 76 students do maths work and 24 students do the English work.

After two study periods,

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 76 & 24 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\!\!\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} 60.8 + 16.8 & 15.2 + 7.2 \end{pmatrix}$$

$$= \begin{pmatrix} 77.6 & 22.4 \end{pmatrix}$$

After two study periods there will be 78 (approx) students do maths work and 22 (approx) students do English work.

Aliter

$$\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 60 & 40 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\!\!\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.7 & 0.3 \end{pmatrix}^2 \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 60 & 40 \end{pmatrix} \end{array} \begin{array}{c} \\ M \\ E \end{array}\!\!\begin{array}{c} \begin{array}{cc} M & E \end{array} \\ \begin{pmatrix} 0.78 & 0.22 \\ 0.77 & 0.23 \end{pmatrix} \end{array}$$

$= (46.8 + 30.8 \quad 13.2 + 9.2)$

$= (77.6 \quad 22.4)$

Exercise 1.3

1. The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 45% of those who already subscribe will subscribe again while 30% of those who do not now subscribe will subscribe. On the last letter, it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

2. A new transit system has just gone into operation in Chennai. Of those who use the transit system this year, 30% will switch over to using metro train next year and 70% will continue to use the transit system. Of those who use metro train this year, 70% will continue to use metro train next year and 30% will switch over to the transit system. Suppose the population of Chennai city remains constant and that 60% of the commuters use the transit system and 40% of the commuters use metro train next year.
   (i) What percent of commuters will be using the transit system year after the next year?
   (ii) What percent of commuters will be using the transit system in the long run?

3. Two types of soaps A and B are in the market. Their present market shares are 15% for A and 85% for B. Of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after one year and when is the equilibrium reached?

4. Two products A and B currently share the market with shares 50% and 50% each respectively. Each week some brand switching takes place. Of those who bought A the previous week, 60% buy it again whereas 40% switch over to B. Of those who bought B the previous week, 80% buy it again where as 20% switch over to A. Find their shares after one week and after two weeks. If the price war continues, when is the equilibrium reached?

Exercise 1.4

Choose the correct answer

1. If $A=(1\ \ 2\ \ 3)$, then the rank of AA^T is
   (a) 0 (b) 2
   (c) 3 (d) 1

2. The rank of $m\times n$ matrix whose elements are unity is
   (a) 0 (b) 1
   (c) m (d) n

3. If $T = \begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$ is a transition probability matrix, then at equilibrium A is equal to
   (a) $\frac{1}{4}$ (b) $\frac{1}{5}$
   (c) $\frac{1}{6}$ (d) $\frac{1}{8}$

4. If $A=\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$, then $\rho(A)$ is
   (a) 0 (b) 1
   (c) 2 (d) n

5. The rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ is
   (a) 0 (b) 1
   (c) 2 (d)3

6. The rank of the unit matrix of order n is
   (a) $n-1$ (b) n
   (c) $n+1$ (d) n^2

7. If $\rho(A)=r$ then which of the following is correct?
   (a) all the minors of order r which does not vanish
   (b) A has at least one minor of order r which does not vanish
   (c) A has at least one $(r+1)$ order minor which vanishes
   (d) all $(r+1)$ and higher order minors should not vanish

8. If A $=\begin{pmatrix}1\\2\\3\end{pmatrix}$ then the rank of AA^T is
   (a) 0 (b) 1
   (c) 2 (d) 3

9. If the rank of the matrix $\begin{pmatrix}\lambda & -1 & 0\\0 & \lambda & -1\\-1 & 0 & \lambda\end{pmatrix}$ is 2, then λ is
   (a) 1 (b) 2
   (c) 3 (d) only real number

10. The rank of the diagonal matrix $\begin{pmatrix}1 &&&&&\\ & 2 &&&&\\ && -3 &&&\\ &&& 0 &&\\ &&&& 0 &\\ &&&&& 0\end{pmatrix}$
   (a) 0 (b) 2
   (c) 3 (d) 5

11. If $T=\begin{array}{c}\\A\\B\end{array}\begin{array}{c}\begin{array}{cc}A & B\end{array}\\\begin{pmatrix}0.7 & 0.3\\0.6 & x\end{pmatrix}\end{array}$ is a transition probability matrix, then the value of x is
   (a) 0.2 (b) 0.3
   (c) 0.4 (d) 0.7

12. Which of the following is not an elementary transformation?
   (a) $R_i \leftrightarrow R_j$ (b) $R_i \to 2R_i + 2C_j$
   (c) $R_i \to 2R_i - 4R_j$ (d) $C_i \to C_i + 5C_j$

13. If $\rho(A)=\rho(A,B)$ then the system is
   (a) Consistent and has infinitely many solutions
   (b) Consistent and has a unique solution
   (c) Consistent
   (d) inconsistent

14. If $\rho(A)=\rho(A,B)=$ the number of unknowns, then the system is
   (a) Consistent and has infinitely many solutions
   (b) Consistent and has a unique solution
   (c) inconsistent
   (d) consistent

15. If $\rho(A)\neq\rho(A,B)$, then the system is
   (a) Consistent and has infinitely many solutions
   (b) Consistent and has a unique solution
   (c) inconsistent
   (d) consistent

16. In a transition probability matrix, all the entries are greater than or equal to
   (a) 2 (b) 1
   (c) 0 (d) 3

17. If the number of variables in a non-homogeneous system $AX=B$ is n, then the system possesses a unique solution only when
   (a) $\rho(A)=\rho(A,B)>n$
   (b) $\rho(A)=\rho(A,B)=n$
   (c) $\rho(A)=\rho(A,B)<n$
   (d) none of these

18. The system of equations $4x+6y=5,\ 6x+9y=7$ has
   (a) a unique solution
   (b) no solution
   (c) infinitely many solutions
   (d) none of these

19. For the system of equations $x+2y+3z=1, \quad 2x+y+3z=2$ $5x+5y+9z=4$
(a) there is only one solution
(b) there exists infinitely many solutions
(c) there is no solution
(d) None of these

20. If $|A| \neq 0$, then A is
(a) non- singular matrix
(b) singular matrix
(c) zero matrix
(d) none of these

21. The system of linear equations $x+y+z=2,\ 2x+y-z=3,\ 3x+2y+k=4$ has unique solution, if k is not equal to
(a) 4 (b) 0
(c) –4 (d) 1

22. Cramer's rule is applicable only to get an unique solution when
(a) $\Delta_z \neq 0$ (b) $\Delta_x \neq 0$
(c) $\Delta \neq 0$ (d) $\Delta_y \neq 0$

23. If $\dfrac{a_1}{x}+\dfrac{b_1}{y}=c_1,\ \dfrac{a_2}{x}+\dfrac{b_2}{y}=c_2,$

$$\Delta_1=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},\ \Delta_2=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\ \Delta_3=\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$$

then (x,y) is
(a) $\left(\dfrac{\Delta_2}{\Delta_1},\dfrac{\Delta_3}{\Delta_1}\right)$ (b) $\left(\dfrac{\Delta_3}{\Delta_1},\dfrac{\Delta_2}{\Delta_1}\right)$
(c) $\left(\dfrac{\Delta_1}{\Delta_2},\dfrac{\Delta_1}{\Delta_3}\right)$ (d) $\left(\dfrac{-\Delta_1}{\Delta_2},\dfrac{-\Delta_1}{\Delta_3}\right)$

24. If $|A_{n\times n}|=3$ and $|adjA|=243$ then the value n is
(a) 4 (b) 5
(c) 6 (d) 7

25. Rank of a null matrix is
(a) 0 (b) –1
(c) ∞ (d) 1

Miscellaneous problems

1. Find the rank of the matrix
$$A=\begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}.$$

2. Find the rank of the matrix
$$A=\begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix}.$$

3. Find the rank of the matrix
$$A=\begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}.$$

4. Examine the consistency of the system of equations:
$x+y+z=7$, $x+2y+3z=18$, $y+2z=6$

5. Find k if the equations $2x+3y-z=5$, $3x-y+4z=2$, $x+7y-6z=k$ are consistent.

6. Find k if the equations $x+y+z=1$, $3x-y-z=4$, $x+5y+5z=k$ are inconsistent.

7. Solve the equations $x+2y+z=7$, $2x-y+2z=4$, $x+y-2z=-1$ by using Cramer's rule.

8. The cost of 2kg. of wheat and 1kg. of sugar is ₹100. The cost of 1kg. of wheat and 1kg. of rice is ₹80. The cost of 3kg. of wheat, 2kg. of sugar and 1kg of rice is ₹220. Find the cost of each per kg., using Cramer's rule.

9. A salesman has the following record of sales during three months for three items A, B and C, which have different rates of commission.

Months	Sales of units			Total commission drawn (in ₹)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rate of commission on the items A, B and C by using Cramer's rule.

10. The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 60% of those who already subscribe will subscribe again while 25% of those who do not now subscribe will subscribe. On the last letter it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

Summary

In this chapter we have acquired the knowledge of:

- **Rank of a matrix**

  The rank of a matrix A is the order of the largest non-zero minor of A
- The rank of a matrix A is the order of the largest non-zero minor of A
- $\rho(A) \geq 0$
- If A is a matrix of order $m \times n$, then $\rho(A) \leq$ minimum of $\{m, n\}$
- The rank of a zero matrix is '0'
- The rank of a non- singular matrix of order $n \times n$ is 'n'
- **Equivalent Matrices**

  Two Matrices A and B are said to be equivalent if one can be obtained from another by a finite number of elementary transformations and we write it as $A \sim B$.
- **Echelon form**

  A matrix of order $m \times n$ is said to be in echelon form if the row having all its entries zero will lie below the row having non-zero entry.
- A system of equations is said to be consistent if it has at least one set of solution. Otherwise, it is said to be inconsistent
  - If $\rho\left(\left[A, B\right]\right) = \rho\left(A\right)$, then the equations are consistent.
  - If $\rho\left(\left[A, B\right]\right) = \rho\left(A\right) = n$, then the equations are consistent and have unique solution.
  - If $\rho\left(\left[A, B\right]\right) = \rho\left(A\right) < n$, then the equations are consistent and have infinitely many solutions.
  - If $\rho\left(\left[A, B\right]\right) \neq \rho\left(A\right)$ then the equations are inconsistent and has no solution.
- $|adjA| = |A|^{n-1}$
- If $|A| = 0$ then A is a singular matrix. Otherwise, A is a non singular matrix.
- In $AX = B$ if $|A| \neq 0$ then the system is consistent and it has unique solution.
- Cramer's rule is applicable only when $\Delta \neq 0$.

GLOSSARY

Augmented	மிகுதிப்படுத்திய, விரிவுபடுத்தப்பட்ட
Commodities	பொருட்கள்
Commuters	பயணிகள்
Consistent	ஒருங்கமைவு
Determinant	அணிக்கோவை
Echelon form	ஏறுபடி வடிவம்
Elementary Transformations	அடிப்படை உருமாற்றங்கள்
Equilibrium	சமநிலை
Homogenerous	சமப் படித்தான
Inconsistent	ஒருங்கமைவு அற்ற
Intersecting	வெட்டிக்கொள்ளும்
Linear equations	நேரிய சமன்பாடுகள்
Matrix	அணி
Matrix equation	அணி சமன்பாடு
Non homogeneous	அசமப் படித்தான
Non- Singular matrix	பூஜ்ஜியமற்ற கோவை அணி
Non trivial solution	வெளிப்படையற்ற தீர்வு
Order	வரிசை
Production	உற்பத்தி
Rank	தரம்
Singular matrix	பூஜ்ஜியக் கோவை அணி
Square matrix	சதுர அணி
Subsequent	தொடர்ச்சியான
Transit system	போக்குவரத்து அமைப்பு
Transition	நிலை மாற்றம்
Trivial solution	வெளிப்படைத் தீர்வு
Unique solution	ஒரே ஒரு தீர்வு
Unknown	தெரியாத
Variables	மாறிகள்