Chapter 3

Vibrations

CHAPTER HIGHLIGHTS

Introduction

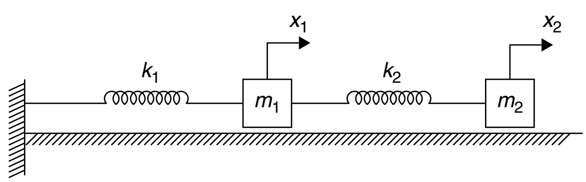
The to and fro motion of a body (or a system of bodies) **about a mean position** (which is the **stable equilibrium position** of the system where the **total potential energy of the system** in all forms **is minimum**) is called vibration. Hence, only those bodies or system of bodies that have a stable equilibrium position can vibrate.

When an **external force** (called as a **disturbing force** or **exciting force**), slowly shifts the system from its mean position (i.e. without any change in its kinetic energy) and takes it to another position, the **potential energy** (or **strain energy**) of the system increases as work has been done against the **internal elastic forces** of the system (which are **conservative forces** within the **elastic limit**). The change in position of the body or system from its mean position to the new position is called the **displacement**. When the body or the system is now released from its new position, neglecting all forces like friction, viscous forces, etc. that offer resistance to motion, the **internal forces** of the system (**elastic forces**) try to reduce the potential energy of the system by bringing it back to the original mean position. This internal force is called **restoring force**.

When the system returns to the mean position, though the potential energy of the system has decreased, because positive work is done by the restoring force (internal forces), the kinetic energy of the system increases. Hence, at the mean position, the system has maximum kinetic energy. Because of kinetic energy, the system overshoots the mean position and gets displaced in the opposite direction till its kinetic energy becomes zero. Again the elastic forces bring the system back to the mean position where the kinetic energy becomes maximum and this process goes on. If there is no dissipation of energy (due to viscosity, friction, etc.), the process is repeated infinitely and it is called **undamped free vibrations**.

If only one co-ordinate is needed to describe the position of a vibrating system, then it is called a **one-dimensional vibration**. It can be either **linear** (in which case only one linear co-ordinate like x or y but not both, is required to describe the system) or **angular** (in which case only one angular position, θ, is required to describe the system). Horizontal oscillation of a spring mass-system and vertical oscillation of a spring-mass system are examples of one-dimensional linear oscillations. Oscillation of a simple pendulum is an example of one-dimensional angular oscillation. If a spring-mass system oscillates up and down and also sways like a simple pendulum it becomes a two-dimensional vibration.

If there are two masses, with two spring along the same line and oscillating in the same direction as shown in figure, it is also a two dimensional vibration (because two co-ordinates x_1 and x_2 need to be specified)



The discussion in this chapter is for one-dimensional vibrations only.

A motion which repeats itself at regular intervals of time is called a **periodic motion**. For example, the orbital motion of the Earth around the Sun. The to and fro motion of a body about a mean position is called **oscillation** or **vibration**. All undamped vibrations are periodic but all periodic motions need not be oscillatory. **Simple harmonic motion (SHM)** is a special case of oscillatory motion (hence periodic motion) in which the **restoring force is directly proportional to**

displacement from mean position (for linear oscillations) or **restoring torque is directly proportional to angular displacement (for angular oscillations)**.

Important Definitions for Oscillatory/Vibrating Motion

Some of the commonly used terms for vibrating motions are listed below.

1. **Time period (T) or period of vibration:** The time taken for the motion to **repeat** itself is called time period. Its SI unit is second (s) and symbol is T.
2. **Cycle:** The motion executed by the system during one time period is called a cycle.
3. **Frequency (f):** The number of **cycles completed** by the vibrating system **in one second** is called frequency. Its SI unit is per second, known as **hertz** (Hz). $f = \dfrac{1}{T}$
4. **Circular frequency** or **angular frequency** of vibration is defined as 2π times frequency. Its symbol is ω(omega) and unit is **radian per second (rad/s)**
$$\omega = 2\pi f = \frac{2\pi}{T}$$
5. **Resonance:** This is a term used for only **forced vibrations,** to describe a state when the frequency of the external force on the system is equal to the natural frequency of free vibrations of the system. Resonance results in very large amplitudes of vibrations and can be dangerous.
6. **Amplitude:** It simply means maximum value but it is used often in the context of displacement amplitude. Displacement amplitude or simply amplitude is the **maximum value of displacement from the mean position. Velocity amplitude** is the maximum value of velocity, which occurs at the mean position. **Acceleration amplitude** is the maximum value of acceleration, etc.
7. **Phase:** This is a term used to represent how far the system has been displaced from the mean position and whether it is moving towards the mean position or away from it. It is usually expressed as angle in radian or in terms of time (as a fraction of time period T) etc.

Types of Vibrations

There are three important types of vibrations.

1. Natural vibrations or Free vibrations
2. Forced vibrations
3. Damped vibrations (which can be free or forced vibrations as well)

These are explained as given below:

Natural Vibrations or Free Vibrations

After initial displacement and release of the body, if the vibratory motion is **maintained only by the internal elastic forces** of the body and no **external force** (including friction and other resistance to the motion of body) acts on the body, then the vibration of the body is called **Free vibration** or **Natural vibration**.

Forced Vibrations

When an external periodic disturbing force is applied continuously on the body to maintain its vibratory motion and the vibratory motion of the body has the same frequency as the frequency of the applied external force, such type of vibration is called **forced vibration**.

Damped Vibrations

A vibration in which **the energy of the vibrating system gradually gets dissipated** by friction and other resistances offered to the motion, is called **damped oscillation**. If the amplitude of oscillation of the body keeps on decreasing over every cycle of vibration and finally the body comes to rest, it is called **Free Damped Vibrations**. If a **periodic external force** is acting on the body which is **executing damped oscillations**, then it is called **Forced-damped oscillations**.

We will discuss undamped free vibrations now.

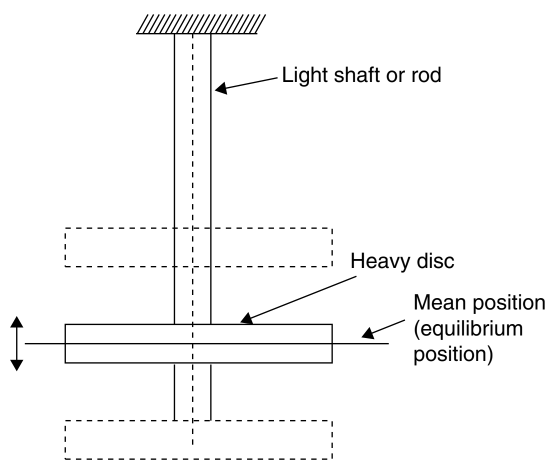
Undamped Free Vibrations (or Undamped Natural Vibrations)

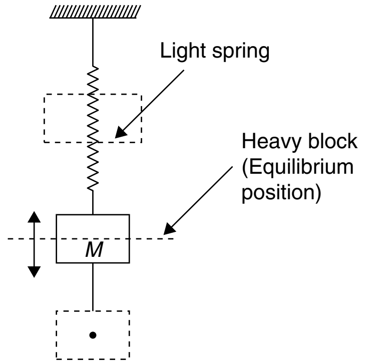
This is an ideal (or hypothetical) vibration in which there is no external force (including friction) acting on the vibrating body, energy of vibration remains constant and so the amplitude of vibration remains constant. There are three types of undamped free-vibrations.

1. Longitudinal vibrations
2. Transverse vibrations and
3. Torsional vibrations

Longitudinal Vibrations

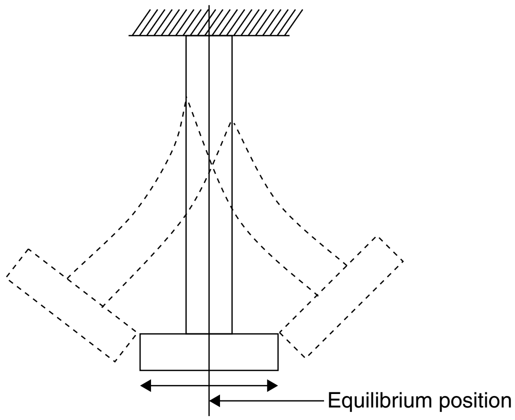
In longitudinal vibrations, the particles of the system vibrate along the axis or length of the system. There is expansion and contraction along the axial (lengthwise) direction, subjecting the system to **axial tensile** and **axial compressive stresses**.





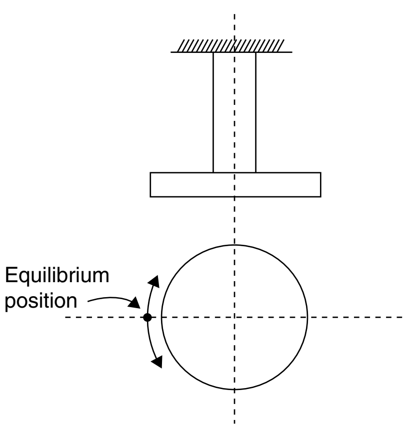
Transverse Vibrations

In transverse vibration, the particles of the system **vibrate in a direction perpendicular to the axis** (or length) of the shaft/rod. This results in alternate bending and straightening of the shaft producing tensile and compressive stresses in the shaft/rod due to bending.



Torsional Vibrations

In torsional vibrations, the particles of the system **vibrate along circular arcs** (of different radii) **whose centres lie on the axis of the shaft**. The shaft gets periodically **twisted and untwisted** which produces **torsional shear stresses** in the shaft.



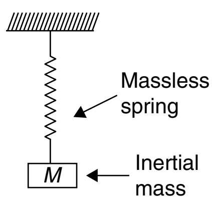
In all the three cases of free vibrations, if the **stresses** on the vibrating body **do not exceed the proportionality limit** of the material of the body (i.e. Hooke's law can be applied), the restoring force (in the case of longitudinal free vibration) and restoring torque (in the case of transverse and torsional vibrations) are directly proportional to displacement (linear or angular) from the mean position. Also, the acceleration of the vibrating body is directed towards mean position. Hence, for small amplitudes of vibration, the **motion of the vibrating body will be simple harmonic**. (i.e. SHM)

Methods of Finding the Natural Frequency of Free Longitudinal Vibrations

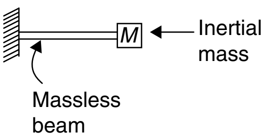
The methods used are (i) Equilibrium method (also known as Force/Torque method) (ii) Energy method and (iii) Rayleigh's method.

For the mathematical analysis of a vibrating system, an **idealised model** which approximately represents the system is needed. For a system to vibrate, with damping or without damping, it **must have inertial** and **restoring elements**. In the case of **vibration with damping**, some **damping element** responsible for dissipation of energy is also required.

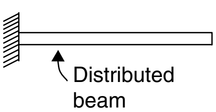
The **inertial elements** are represented by **lumped masses** for rectilinear motion and by **lumped moment of inertia** for angular motion. The lumping of qualities (mass or moment of inertia) depends upon the distribution of these qualities in the system. In a spring-mass system, if the mass of the spring is negligible compared to the mass connected at its end, the spring can be considered as massless. Such a system (without damping) can be represented as shown below.



Similarly for a beam, if the mass of the beam is negligible compared to the mass at its end, the beam can be considered as massless and the system can be represented as shown below.



If the beam cannot be considered massless, then lumping of mass is not possible and the system will be represented as shown below.



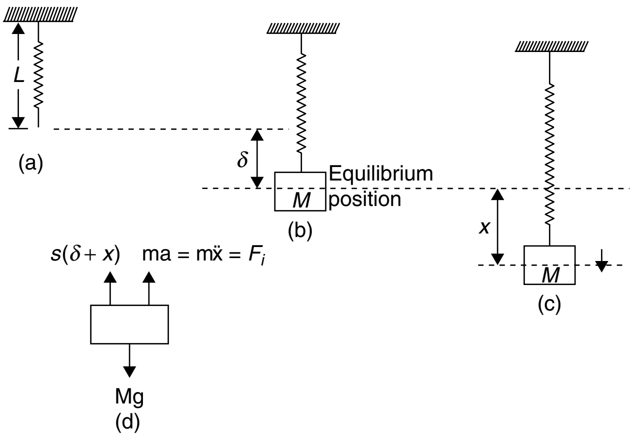
Restoring elements are usually massless linear springs for rectilinear motion and massless torsional springs for torsional motions, respectively.

Equilibrium Method of Finding the Natural Frequency for Free Longitudinal Vibrations

This method makes use of *D*' Alembert's principle for dynamic equilibrium of a system, i.e. for the dynamic equilibrium of a system, the sum of the resultant external force on the system and the inertia force is equal to zero.

i.e. $F_r + F_i = 0$, where F_r = resultant external force and F_i = inertia force

Let us apply this for a spring mass system.



Consider a light spring of stiffness s and natural length *L*, fixed at the upper end and hanging vertically (see Figure a). A block of mass *M*(so weight $W = Mg$) is connected to its lower end and lowered very slowly to its equilibrium position. The extension of the spring is δ (see Figure b). As the mass is in equilibrium, downward force = upward force

$$\Rightarrow \qquad Mg = s\delta \qquad (1)$$

Let this mass be disturbed from its equilibrium position and made to vibrate freely. When this mass is displaced by x downwards from its equilibrium position and moving downwards, (see Figure c), its velocity is $\dot{x}$ and acceleration is $\ddot{x}$, both in the direction of x(i.e. downwards). At this instant, the forces on the mass are

Inertial Force $F_i = -$mass $\times$ acceleration

$= -M\ddot{x}$ (i.e. $M\ddot{x}$ upwards)

Weight, $W = Mg$(downwards)

Spring force, $F_s = s(\delta + x)$

$= s\delta + sx \quad [\because s\delta = Mg$ from (1)]

$= (Mg + sx)$ upwards.

$\therefore$ Net external force on mass, $F_r = F_s - W$

$= Mg + sx - Mg$

$= sx$, upwards

The forces on the mass are shown in Figure d.

As per *D*' Alembert's principle, for dynamic equilibrium of mass.

$$F_r + F_i = 0 \Rightarrow sx + M\ddot{x} = 0$$

i.e. $M\ddot{x} + sx = 0$ represents the differential equation for free longitudinal vibration of the spring-mass system.

$\Rightarrow \ddot{x} + \left(\dfrac{s}{M}\right)x = 0 \rightarrow (2)$, represents a simple harmonic motion (SHM) of **natural circular frequency** (ω_n) given by

$$\omega_n = \sqrt{\left(\frac{s}{M}\right)} \qquad (3)$$

SI unit of ω_n is radian per second (rad/s).

The solution to the differential equation (ii) is given by

$x(t) = X\sin(\omega_n t + \phi)$ where

$x(t)$ = displacement of vibrating mass from the equilibrium position at time, t

X = displacement amplitude of oscillation

ω_n = natural circular frequency (radian/second)

$(\omega_n t + \phi)$ = phase of vibration at time t (in radian)

ϕ = initial phase (at time $t = 0$) in radian, also known as **epoch**

NOTE

The fundamental differential equation for SHM is $\ddot{x} + \omega_n^2 x = 0$

The natural linear frequency,

$$(f_n) = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{M}} \qquad (4)$$

From equation (1) we have $s = \dfrac{Mg}{\delta}$. Substituting this value in equation

$$(4) \rightarrow f_n = \frac{1}{2\pi}\sqrt{\frac{Mg}{\delta M}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$\therefore \ f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}, \text{ where}$$

δ = static deflection = $\dfrac{Mg}{s}$

SI unit of f_n is hertz (Hz)

As $g = 9.81$ m/s^2, we have

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}}$$

or
$$f_n = \frac{0.4985}{\sqrt{\delta}} \text{ Hz}$$

NOTE

Unit of δ must be metre in SI.

If the spring is replaced by a uniform solid shaft of length *L*, cross sectional area *A*, negligible mass and made of a material of Young's modulus *E* and carrying the mass *M*(or weight $W = Mg$) at its end, then the elongation of shaft

$$\delta = \frac{WL}{AE} = \frac{MgL}{AE}$$

If the mass of the spring is not negligible and say it is equal to M_s, then one third of this mass shall be added to the mass *M* at the end of the spring.

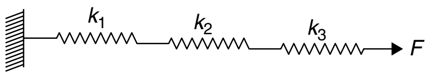
Accordingly, $\omega_n = \sqrt{\dfrac{s}{\left(M + \dfrac{M_s}{3}\right)}}$, where

M_s = mass of spring (or shaft) and

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{\left(M + \dfrac{M_s}{3}\right)}}$$

NOTES

1. When springs are connected in series, the spring force is the same in all springs but extensions are different in each spring.



If x_1, x_2 and x_3 are the extensions of each spring, of stiffnesses k_1, k_2 and k_3, respectively when subjected to an axial force F, total extension $x = x_1 + x_2 + x_3$

If k_s is the equivalent stiffness, then

$$x = \frac{F}{k_s},\ x_1 = \frac{F}{k_1},\ x_2 = \frac{F}{k_2} \text{ and } x_3 = \frac{F}{k_3}$$

$$\Rightarrow \frac{F}{k_s} = \frac{F}{k_1} + \frac{F}{k_2} + \frac{F}{k_3} \Rightarrow \frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

When n identical springs, each of stiffness k, are connected in series, equivalent stiffness $k_s = \dfrac{k}{n}$

2. Stiffness of spring

$$\propto \frac{1}{\text{Length}} \propto \frac{1}{\text{Number of turns}}$$

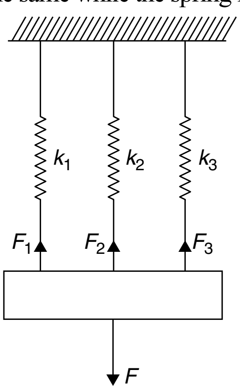
3. If a spring of stiffness k and length L is cut into two parts whose lengths are in the ratio $\dfrac{L_1}{L_2} = \dfrac{m}{n}$, then the corresponding stiffness k_1 and k_2 of the parts are given by

$$k_1L_1 = k_2L_2 = kL$$

$$\Rightarrow k_1 = \frac{(m+n)k}{m} \text{ and } k_2 = \frac{(m+n)k}{n}$$

If a uniform spring of stiffness k is cut into n equal parts, the stiffness of each part = nk.

4. When springs are connected in parallel, the extension of each spring is the same while the spring forces are different.



Extension = x for each spring

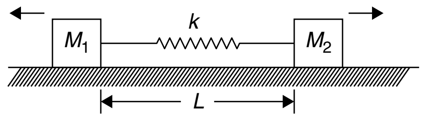
$$F_1 = k_1x$$
$$F_2 = k_2x$$
$$F_3 = k_3x$$

$\therefore F = F_1 + F_2 + F_3$

If k_p is the equivalent stiffness, $k_px = F$

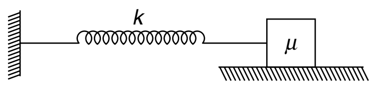
$$\Rightarrow k_px = k_1x + k_2x + k_3x$$
$$\Rightarrow k_p = k_1 + k_2 + k_3$$

$\therefore$ When n identical springs, each of stiffness k, are connected in parallel, equivalent stiffness is $k_p = nk$

5. 

A massless spring of length L and stiffness k, has two masses M_1 and M_2 at each end and placed on a smooth horizontal floor.

When the masses are pulled on either side (along the length of the spring) and released, both the masses execute SHM with same circular frequency ω_n. The centre of mass of the system remains at rest. We can show that this system is equivalent to a spring mass system with one end fixed and other end carrying a mass $\mu = \dfrac{M_1M_2}{(M_1 + M_2)}$ (called the reduced mass of 2 mass system).



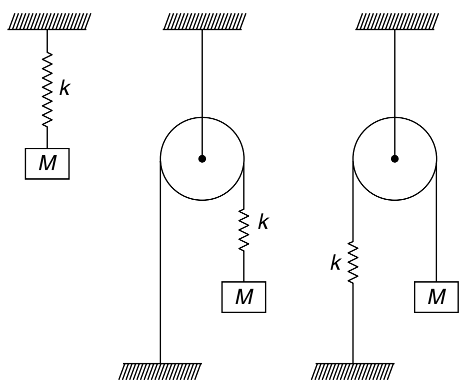
Equivalent to 2 Mass Spring System

$$\therefore \omega_n = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{k(M_1 + M_2)}{M_1M_2}}$$

$$f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{k(M_1 + M_2)}{M_1M_2}}$$

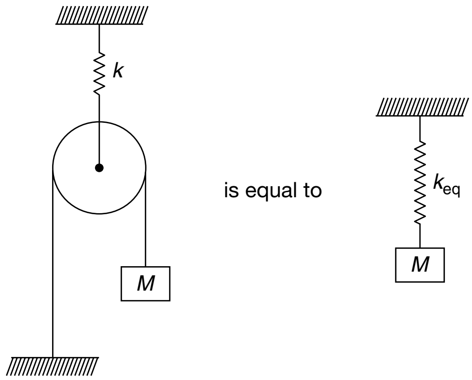
T_n = Time period of free vibration = $\dfrac{1}{f_n}$

$$= 2\pi\sqrt{\frac{M_1M_2}{k(M_1 + M_2)}}.$$

6. 

The pulleys shown in figures are smooth and massless. The circular frequency of vibration in all the three cases are same and equal to $\omega_n = \sqrt{\frac{k}{M}}$.

7.



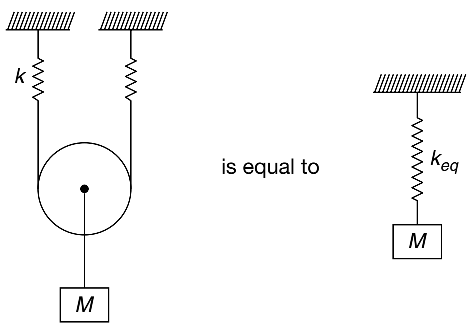
When movable pulley is used with spring configuration as shown, it is equivalent to a spring mass system where $k_{eq} = \frac{k}{4}$

$$\therefore\ \omega_n = \sqrt{\frac{k_{eq}}{M}} = \sqrt{\frac{k}{4M}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{4M}}$$

$$T_n = 2\pi\sqrt{\frac{4M}{k}}$$

8.



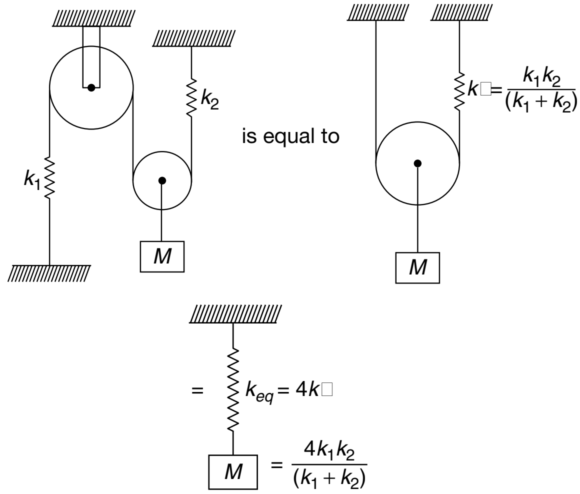
For this movable pulley spring configuration, $k_{eq} = 4k$

$$\therefore\ \omega_n = \sqrt{\frac{k_{eq}}{M}} = \sqrt{\frac{4k}{M}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{4k}{M}}$$

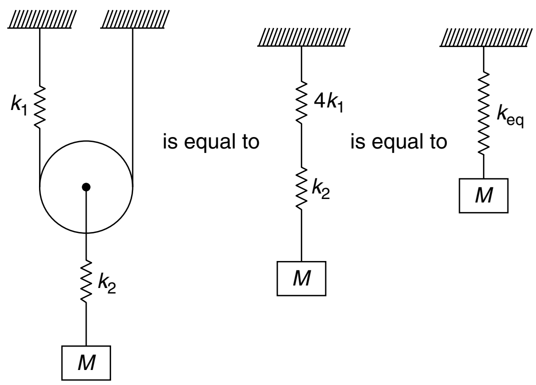
$$T_n = \frac{1}{f_n} = 2\pi\sqrt{\frac{M}{4K}}$$

9.



$$\therefore\ \omega_n = \sqrt{\frac{k_{eq}}{M}} = \sqrt{\frac{4k_1k_2}{(k_1+k_2)M}}$$

10.



Here, $k_{eq} = \frac{4k_1k_2}{(4k_1+k_2)}$

($\because$ $4k_1$ and k_2 are in series connection)

$$\omega_n = \sqrt{\frac{k_{eq}}{M}} = \sqrt{\frac{4k_1k_2}{(4k_1+k_2)M}}$$

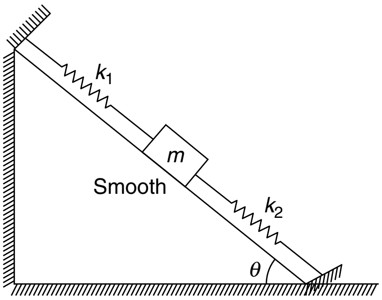
11. When a massless spring of stiffness k is stretched or compressed by x, the elastic potential energy stored in the spring is $\frac{1}{2}kx^2$.

12. The time period of a spring mass sytem for undamped free oscillations is

$T_n = 2\pi\sqrt{\frac{M}{k}}$, where M is the mass of oscillating system and k = stiffness of spring. This time period, unlike a simple pendulum, is independent of acceleration due to gravity. Hence, spring mass system can be used for time measurement in variable acceleration due to gravity (g) situation.

Solved Examples

Example 1:



The contact between the block of mass m and the inclined plane is frictionless. The stiffness of the springs are k_1 and k_2 respectively. The springs are parallel to the inclined plane. The natural frequency of spring mass system is

(A) $\sqrt{\dfrac{k_1+k_2}{2m}}$ (B) $\sqrt{\dfrac{k_1+k_2}{4m}}$

(C) $\sqrt{\dfrac{k_1-k_2}{m}}$ (D) $\sqrt{\dfrac{k_1+k_2}{m}}$

Solution:

No friction means it is free undamped oscillation.

$\therefore\ \omega_n = \sqrt{\dfrac{k_{eq}}{m}}$, where

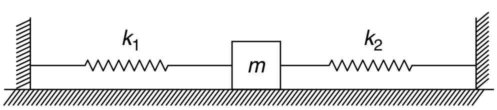
k_{eq} = equivalent stiffness of springs

The springs are connected in parallel ($\because$ one is fixed and other end is connected to the same mass)

$\therefore\ k_{eq} = k_1 + k_2$ (for parallel combination of springs)

$\therefore\ \omega_n = \sqrt{\dfrac{k_1+k_2}{m}}$.

Example 2:



Smooth floor

In the figure shown, the springs are massless, block is of mass $m = 1.4$ kg, $k_1 = 4000$ N/m and $k_2 = 1600$ N/m respectively. The natural frequency of free oscillation of the system is nearly

(A) 8 Hz (B) 10 Hz
(C) 12 Hz (D) 14 Hz

Solution:

Springs are connected in parallel

$$K_{eq} = k_1 + k_2 = 4000 + 1600 = 5600 \text{ N/m}$$
$$m = 1.4 \text{ kg}$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{5600}{1.4}}$$
$$= \frac{\sqrt{4000}}{2\pi}$$
$$= 10.07 \text{ Hz.}$$
$$\approx 10 \text{ Hz.}$$

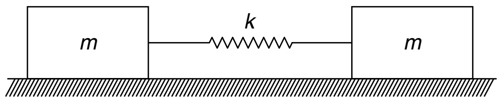
Example 3: The natural frequency of a spring mass system on Earth is ω_n. The natural frequency of this system on the Moon $\left(g_{Moon} = \dfrac{g_{Earth}}{6}\right)$ is

(A) ω_n (B) $0.408\ \omega_n$
(C) $0.204\ \omega_n$ (D) $0.167\ \omega_n$

Solution:

The spring-mass system has an undamped natural frequency $= \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$, independent of g. So the natural frequency of the system on Earth and Moon will be the same and equal to ω_n.

Example 4:



Smooth floor

Consider the system of two blocks, each of mass m, placed on a smooth floor and connected by a massless spring of stiffness k. The natural frequencies of this system are

(A) $0, \sqrt{\dfrac{2k}{m}}$ (B) $\sqrt{\dfrac{k}{m}}, \sqrt{\dfrac{2k}{m}}$

(C) $\sqrt{\dfrac{k}{m}}, \sqrt{\dfrac{k}{2m}}$ (D) $0, \sqrt{\dfrac{k}{2m}}$

Solution:

There are two possible movements for the system. The complete system can have a translation with both blocks having same velocity and same acceleration and the centre of mass of system also moving in the same direction.

$\Rightarrow$ No vibration and so natural frequency is zero

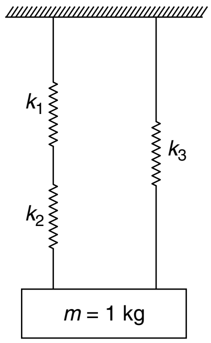
In the second case, the centre of mass of system remains fixed and the masses oscillate along the length of spring with same frequency $\omega_n = \sqrt{\dfrac{k}{\mu}}$, where μ = reduced mass of 2-particle system

$$\mu = \frac{m_1 m_2}{(m_1+m_2)} = \frac{m^2}{2m} = \frac{m}{2}$$

$$\therefore\ \omega_n = \sqrt{\frac{k}{\left(m/2\right)}} = \sqrt{\frac{2k}{m}}$$

$\therefore$ Possible natural frequencies are $0, \sqrt{\dfrac{2k}{m}}$.

Example 5:



$k_1 = 1$ kN/m
$k_2 = 3$ kN/m
$k_3 = 2$ kN/m

A mass of 1 kg is suspended by means of 3 springs as shown in figure. The spring constants k_1, k_2 and k_3 are respectively 1 kN/m, 3 kN/m and 2 kN/m. The natural frequency of the system is approximately

(A) 46.90 rad/s (B) 52.44 rad/s
(C) 60.55 rad/s (D) 77.46 rad/s

Solution:

Springs k_1 and k_2 are in series. So their equivalent spring constant is

$$k_s = \frac{k_1 k_2}{(k_1 + k_2)} = \frac{1 \times 3}{(1+3)} = \frac{3}{4} \text{ kN/m}$$

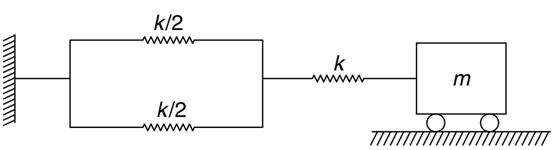
Now, k_s and k_3 are in parallel. Hence, their equivalent spring constant is

$$k_p = k_s + k_3 = \frac{3}{4} + 2 = \frac{11}{4} \text{ kN/m}$$

$$= \frac{11000}{4} \text{ N/m} = 2750 \text{ N/m}$$

$$\therefore \omega_n = \sqrt{\frac{k_p}{m}} = \sqrt{\frac{2750}{1}} = 52.44 \text{ rad/s}$$

Example 6:



The natural frequency of the system shown in figure is

(A) $\sqrt{\frac{k}{2m}}$ (B) $\sqrt{\frac{k}{m}}$

(C) $\sqrt{\frac{2k}{m}}$ (D) $\sqrt{\frac{3k}{m}}$

Solution:

The parallel springs can be reduced to a single spring of

$$k_p = \frac{k}{2} + \frac{k}{2} = k$$

Now two springs of stiffness k and k are connected in series

$$\rightarrow k_s = \frac{k}{2}$$

$$\therefore \omega_n = \sqrt{\frac{k_s}{m}} = \sqrt{\frac{k}{2m}}.$$

Example 7: The differential equation for free vibrations of a spring mass system is $4\frac{d^2x}{dt^2} + 49x = 0$. The time period of natural vibration is (x is in metre and t in second)

(A) 0.893 s (B) 1.284 s
(C) 1.795 s (D) 0.982 s

Solution:

Given equation can be written as $\frac{d^2x}{dt^2} + \frac{49}{4}x = 0$ which is of the form $\ddot{x} + \omega_n^2 x = 0$

$$\Rightarrow \omega_n^2 = \frac{49}{4} \Rightarrow \omega_n = \sqrt{\frac{49}{4}} = \frac{7}{2} = 3.5$$

$$\therefore T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.5} = 1.795 \, s.$$

Let us now recap some important properties and characteristics of simple harmonic motion (SHM).

Properties/Characteristics of SHM

Consider a particle of mass m, executing a linear SHM along the X-direction with equilibrium position at the origin. At time $t = 0$, the particle is at origin, moving towards the $+x$ direction.

1. At time t, position $x = A \sin\omega t$, where A = amplitude of SHM, ω = natural circular frequency of SHM
2. Velocity $v = \frac{dx}{dt} = A\omega \cos\omega t = \omega\sqrt{A^2 - x^2}$

   Maximum velocity $v_{max} = A\omega$ where $x = 0$ i.e. at the equilibrium position. $v = 0$ when $x = \pm A$ (i.e. at displacement amplitude)
3. Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \sin\omega t$
   $= -\omega^2 x$

   |Maximum acceleration| = $\omega^2 A$ when
   $x = \pm A$ (at extreme position)
   At $x = 0$, $a = 0$

   $$\therefore \frac{|a_{max}|}{v_{max}} = \frac{A\omega^2}{A\omega} = \omega \quad \text{for SHM}$$
4. Restoring force F at displacement $\pm x$ is of magnitude $= ma = m\omega^2 x$

   $\therefore$ Average force acting on particle upto x

   $$= \frac{0 + F}{2} = \frac{1}{2}m\omega^2 x$$
5. Potential energy at displacement x,
   $PE = F_{\text{Average}} \times x$

   $$= \frac{1}{2}m\omega^2 x^2$$

At $x = 0$ (equilibrium position),
$PE = 0$ (minimum)
At $x = \pm A$(extreme position),

$$PE = \frac{1}{2} m\omega^2 A^2 \text{ (maximum)}$$

$$\therefore PE \text{ at any position} = \frac{1}{2} m\omega^2 x^2$$
$$= \frac{1}{2} m\omega^2 A^2 \sin^2 \omega t$$
$$= \frac{1}{2} m\omega^2 A^2 \left[\frac{1 - \cos 2\omega t}{2}\right]$$
$$= \frac{1}{2} m\omega^2 A^2 - \frac{1}{4} m\omega^2 A^2 \cos 2\omega t$$

$\therefore$ PE in SHM is also simple harmonic with circular frequency 2ω and shifted origin.

6. Kinetic energy at displacement x,

$$KE = \frac{1}{2} mv^2$$
$$= \frac{1}{2} m\omega^2 (A^2 - x^2) = \frac{1}{2} m\omega^2 A^2 \cos^2 \omega t$$
$$= \frac{1}{2} m\omega^2 A^2 \left[\frac{1 + \cos 2\omega t}{2}\right]$$

$\therefore$ KE in SHM is also simple harmonic with circular frequency 2ω and shifted origin.

7. Total energy in SHM is the sum of KE and PE

$$\therefore E = KE + PE$$
$$= \frac{1}{2} m\omega^2 A^2 \cos^2 \omega t + \frac{1}{2} m\omega^2 A^2 \sin^2 \omega t$$
$$= \frac{1}{2} m\omega^2 A^2 = \text{constant}$$

Hence, in undamped SHM, the total energy of oscillation is constant and it is equal to $\frac{1}{2} m\omega^2 A^2$.

i.e. $E = \frac{1}{2} m\omega^2 A^2 =$ constant, has **zero frequency** of oscillation.

For a spring-mass system, $\omega = \sqrt{\frac{k}{m}}$

$$\Rightarrow E = \frac{1}{2} m \frac{k}{m} A^2$$
$$\Rightarrow E = \frac{1}{2} kA^2$$

i.e. $k = m\omega^2$

8. When displacement is x, $\frac{PE}{kE} = \frac{\sin^2 \omega t}{\cos^2 \omega t}$

$$= \tan^2 \omega t$$

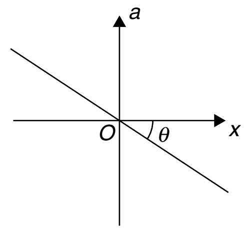
Also, $\frac{PE}{KE} = \frac{x^2}{(A^2 - x^2)}$

when $x = \frac{A}{2}$, $\frac{PE}{KE} = \frac{1}{3}$

when $x = \frac{A}{\sqrt{2}}$, $\frac{PE}{KE} = \frac{1}{2}$:

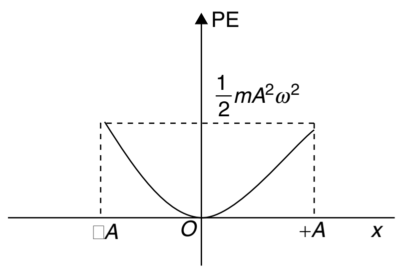
$$PE_{max} = KE_{max} = \frac{1}{2} m\omega^2 A^2$$

9. In SHM, velocity leads displacement by a phase of $\frac{\pi}{2}$ radian (90°), acceleration leads velocity by a phase of $\frac{\pi}{2}$ radian (90°), so acceleration leads displacement by a phase of π radian (180°).
10. The graph between displacement and acceleration in SHM is a straight line with negative slope, passing through origin.

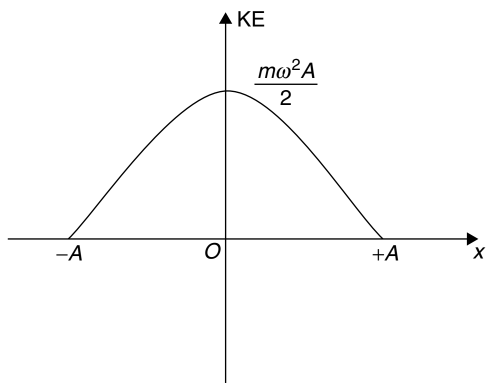


$|\tan\ \theta| = \omega^2$, where θ is the angle which the $a - x$ diagram makes with the x-axis.

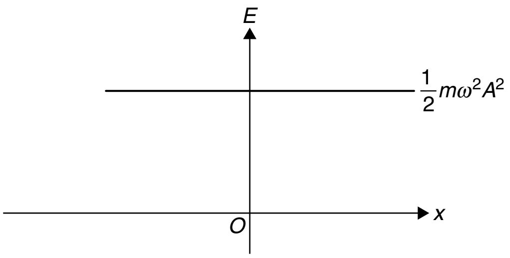
11. The graph between velocity and displacement in SHM is an ellipse. It becomes a circle only when $\omega = 1$ rad/s.
12. The graph between acceleration and velocity in SHM is also an ellipse. It becomes a circle only when $\omega = 1$ rad/s.
13. The graph between PE and displacement in SHM is a parabola which has zero value at equilibrium position ($x = 0$) and maximum values at amplitude positions.



14. The graph between KE and displacement in SHM is an inverted parabola with maximum KE at $x = 0$ and zero KE at $x = \pm A$.



15. The graph between total energy of oscillation E and displacement x is a straight line parallel to the displacement axis.



Example 8: A point mass is executing simple harmonic motion with an amplitude of 10 mm and frequency of 4 Hz. The maximum acceleration (m/s^2) of the mass is ________

Solution: $A = 10$ mm $= 0.01$ m

$$f = 4 \text{ Hz} \to \omega = 2\pi f = 2\pi \times 4$$
$$= 8\pi \text{ rad s}^{-1}$$

Maximum acceleration, $a_{max} = \omega^2 A$

$$= (8\pi)^2 \times 0.01$$
$$= 6.317 \text{ m/s}^2$$

Example 9: A single degree of freedom system having mass 1 kg and stiffness 10 kN/m, initially at rest is subjected to an impulse force of magnitude 5 kN for 10^{-4} second. The amplitude in mm of the resulting free vibration is

(A) 0.5 (B) 1.0
(C) 5.0 (D) 10.0

Solution:

Mass $m = 1$ kg ; $k = 10 \times 10^3$
$= 10^4$ N/m

Impulse given $J = Ft = (5 \times 10^3) \times 10^{-4}$
$= 0.5$ Ns

We know that impulse J
= change in linear momentum
= Δp (by impulse - momentum theorem)
$= p_2 - p_1$
$= p_2$ ($\because$ $p_1 = 0$ as initially system is at rest)

$\therefore$ $p_2 = 0.5$ Ns $= 0.5$ kg ms^{-1}

$$\therefore \text{ Maximum } KE = \frac{(p_2)^2}{2m}$$

$$\left(\because KE = \frac{p^2}{2m}\right)$$

$$= \frac{(0.5)^2}{2\times 1} = 0.125\,J$$

In SHM, $PE_{max} = KE_{max}$ for undamped oscillation.

$$\Rightarrow \frac{1}{2}kA^2 = KE_{max}$$

$$\Rightarrow A = \sqrt{\frac{2KE_{max}}{k}} = \sqrt{\frac{2\times 0.125}{10^4}}$$
$$= 5\times 10^{-3}\,m = 5 \text{ mm}$$

Aliter

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4}{1}} = 100 \text{ rad/s}$$

Impulse $J = Ft = 5 \times 10^3 \times 10^{-4}$
$= 0.5$ Ns.

But $\Delta p = J$ (impulse momentum theorem)

$\Rightarrow (p_2 - p_1) = J$

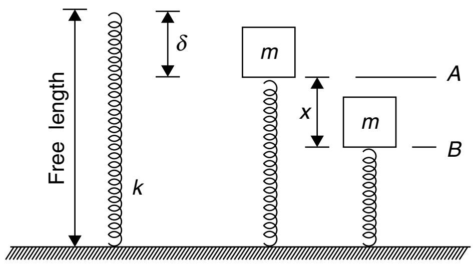
$\Rightarrow (p_2 - 0) = J$ ($\because$ $p_1 = 0$ as system is initially at rest)

$\Rightarrow p_2 = J$ or $mV_{max} = J$

$$\Rightarrow V_{max} = \frac{J}{m} = \frac{0.5}{1} = 0.5 \text{ m/s}$$

$$\therefore a = \frac{V_{max}}{\omega_n} = \frac{0.5}{100} = 0.005 \text{ m} = 5 \text{ mm}$$

Example 10:



In the figure shown, the spring deflects by δ to position A (the equilibrium position) when a mass m is kept on it. During free vibration, the mass is at position B (distant x below A) at some instant. The change in potential energy of spring-mass system from position A to position B is

(A) $\frac{1}{2}kx^2$ (B) $\frac{1}{2}kx^2 - mgx$
(C) $\frac{1}{2}k(x+\delta)^2$ (D) $\frac{1}{2}kx^2 + mgx$

Solution:

There are two forms of potential energy in this system (1) Elastic potential energy of spring (2) Gravitational potential energy of mass m.

The gravitational potential energy at A (equilibrium position) is taken as zero.

Hence, gravitational potential energy at B = $-mgx$ (–, because mass is lowered by x)

At the equilibrium position A, spring is compressed by δ

Hence, elastic potential energy at

$$A = \frac{1}{2}k\delta^2$$

At position B, spring is compressed by $(\delta + x)$

$\therefore$ Elastic potential energy at

$$B = \frac{1}{2}k(\delta + x)^2 = \frac{1}{2}k\delta^2 + \frac{1}{2}kx^2 + k\delta x$$

∴ At A (equilibrium position), spring force = weight

$\Rightarrow k\delta = \text{mg}$

PE_A = Total potential energy at

$$A = 0 + \frac{1}{2}k\delta^2 = \frac{1}{2}k\delta^2$$

PE_B = Total potential energy at B

$$= -mgx + \frac{1}{2}k\delta^2 + \frac{1}{2}kx^2 + k\delta x$$

$$= \frac{1}{2}k\delta^2 + \frac{1}{2}kx^2\ (\because mgx = k\delta x)$$

$$\therefore \Delta PE\ (A \text{ to } B) = PE_B - PE_A$$

$$= \left(\frac{1}{2}k\delta^2 + \frac{1}{2}kx^2\right) - \frac{1}{2}k\delta^2$$

$$= \frac{1}{2}kx^2.$$

NOTE

If we have to find the change in PE of spring alone, it is

$$\left(\frac{1}{2}k\delta^2 + \frac{1}{2}kx^2 + k\delta x\right) - \frac{1}{2}k\delta^2$$

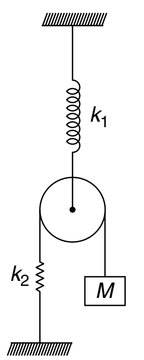
$$= \frac{1}{2}kx^2 + k\delta x$$

$$= \frac{1}{2}kx^2 + mgx$$

$$(\because k\delta = mg)$$

If we have to find the change in PE of mass alone, it is $-mgx$.

Example 11:

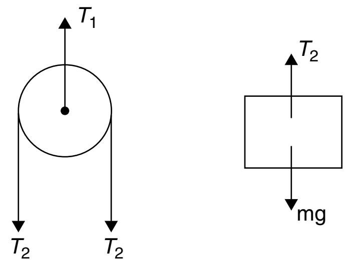


A block of mass m is suspended by the spring arrangement as shown in figure. The springs, strings and pulley are of negligible mass and the pulley is smooth. If the block is moved vertically from its equilibrium position and released, the natural frequency of vibration is

(A) $\sqrt{\dfrac{k_1k_2}{(k_1+k_2)m}}$ (B) $\sqrt{\dfrac{k_1k_2}{(4k_1+k_2)m}}$

(C) $\sqrt{\dfrac{4k_1k_2}{(4k_1+k_2)m}}$ (D) $\sqrt{\dfrac{k_1k_2}{(k_1+4k_2)m}}$

Solution:



Forces on pulley **Forces on block**

For equilibrium of block, $T_2 = mg$

Here T_2 = spring force of spring of stiffness k_2

$= k_2x_2$, where x_2 is the extension of spring 2

$$\therefore T_2 = k_2x_2 = mg \rightarrow x_2 = \frac{mg}{k_2} \quad (1)$$

For the equilibrium of pulley,

$T_1 = 2T_2 = 2mg$

But $T_1 = k_1x_1$, where

x_1 = extension of spring 1

$$\Rightarrow k_1x_1 = 2mg \rightarrow x_1 = \frac{2mg}{k_1} \quad (2)$$

The total distance x though which the block of mass m gets lowered to reach equilibrium position is

$x = 2x_1 + x_2$ [∵ when upper spring stretches by x_1, centre of pulley gets lowered by x_1 and mass m gets lowered by $2x_1$. This is to be added to extension of lower spring]

$$= 2 \times \frac{2mg}{k_1} + \frac{mg}{k_2}$$

$$\text{i.e. } x = mg\left(\frac{4}{k_1} + \frac{1}{k_2}\right) \quad (3)$$

The system can be reduced to a single spring mass system of stiffness k_{eff} and mass m such that

$$x = \frac{mg}{k_{eff}} \quad (4)$$

Comparing (3) and (4),

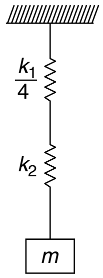
$$\frac{1}{k_{eff}} = \frac{4}{k_1} + \frac{1}{k_2}$$

$$\Rightarrow k_{eff} = \frac{k_1k_2}{(k_1+4k_2)}$$

$$\therefore \omega_n = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{k_1k_2}{(k_1+4k_2)m}}$$

Aliter

The upper spring can be replaced by a spring of stiffness $\dfrac{k_1}{4}$ which is now connected in series with the lower spring of stiffness k_2. So, the equivalent system is as shown.



$$\therefore \frac{1}{k_{eff}} = \frac{4}{k_1} + \frac{1}{k_2} \Rightarrow k_{\text{eff}} = \frac{k_1 k_2}{(k_1 + 4k_2)}$$

$$\therefore \omega_n = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{k_1 k_2}{(k_1 + 4k_2)m}}.$$

Determination of Natural Frequency of Free Angular Vibrations by Equilibrium Method

Using D'Alembert's principle, the equation for dynamic equilibrium will be the sum of all external torques plus the sum of inertial torque is equal to zero.

i.e. Resultant torque + Inertial torque = zero

If θ is the instantaneous angular displacement, $\dot{\theta}\left(\frac{d\theta}{dt}\right)$ is the instantaneous angular velocity (ω) and $\ddot{\theta}\left(\frac{d^2\theta}{dt^2}\right)$ is the instantaneous angular acceleration (α).

τ_γ = Resultant torque = Restoring torque due to elastic forces $= s\theta$,

where θ = angular displacement of system from equilibrium position

s = elastic stiffness (i.e. restoring torque per unit angular displacement)

$\tau_i = I\alpha = I\ddot{\theta}$, where

I = moment of inertia of the system about the axis of angular displacement

$\therefore\ I\ddot{\theta} + s\theta = 0$ is the differential equation for free angular oscillations. The solution for this equation is $\therefore\ \theta(t) = \theta_0 \sin(\omega_n t + \phi)$, which is SHM.

Here, θ_0 = amplitude of angular displacement

ω_n = natural circular frequency of vibration

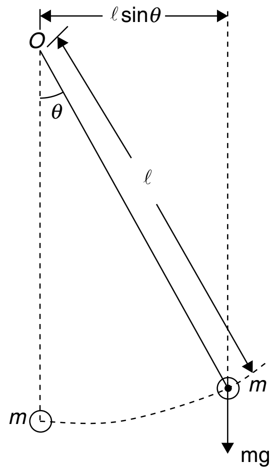
$\omega_n = \sqrt{\frac{s}{I}}$ and

ϕ = initial phase (at time $t = 0$)

Angular acceleration $\alpha = -\omega_n^2\theta$ for angular SHM

Simple Pendulum

A particle of mass m is connected to one end of a light, inextensible string of length ℓ. The other end of the string is pivoted at fixed point O as shown in the following figure.



An external torque moves the bob of mass m through a small anticlockwise angular displacement and the bob is released. When the bob is at an angular displacement θ (counter clockwise),

T_R = Restoring torque $= mg \times \ell \sin\theta$

$= mg\ \ell\ \theta$, clockwise

($\because$ For small angles, $\sin\theta = \theta$ in radian)

T_i = Inertial torque $= I\ \ddot{\theta}$ (clockwise)

$= m\ell^2\ddot{\theta}$ ($\because I = m\ell^2$ about O)

$\therefore\ T_i + T_R = 0$ (for dynamic equilibrium)

$$\Rightarrow m\ell^2\ddot{\theta} + mg\ell\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left(\frac{g}{\ell}\right)\theta = 0 \Rightarrow \text{SHM}$$

$$\therefore \omega_n^2 = \frac{g}{\ell} \Rightarrow \omega_n = \sqrt{\frac{g}{\ell}};$$

$$\therefore f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$$

$T_n = 2\pi\sqrt{\frac{\ell}{g}}$ for a simple pendulum

NOTES

1. The time period of a simple pendulum is independent of the mass of the bob.
2. A simple pendulum having a time period $T = 2$ second is called a **second's pendulum**. The length of the string for a second's pendulum on Earth is nearly 1 m.
3. The time period of a simple pendulum is

   (a) $T = 2\pi\sqrt{\frac{\ell}{g+a}}$, when it is in a lift moving vertically up with an acceleration a ($a < g$)

   (b) $T = 2\pi\sqrt{\frac{\ell}{g-a}}$, when it is in a lift moving vertically down with an acceleration a ($a < g$)

(c) $T = \infty$ (infinite), when it is in a lift under free-fall ($a = g$) such a pendulum will make no oscillation i.e. $f_n = 0$ eg. simple pendulum in an artificial satellite of Earth.

(d) $T = 2\pi\sqrt{\dfrac{\ell}{\sqrt{g^2 + a^2}}}$, when it is in a vehicle moving horizontally with an acceleration a

(e) $T = 2\pi\sqrt{\dfrac{\ell}{g\cos\theta}}$, when it is on a frame which slides on a smooth inclined plane, making angle θ with horizontal.

(f) The time period of a simple pendulum of length ℓ on a planet of radius R and acceleration due to gravity g is

$$T = 2\pi\sqrt{\frac{1}{\left[\dfrac{g}{\ell} + \dfrac{g}{R}\right]}}$$

If $\ell << R$, then $\dfrac{g}{R}$ can be neglected which gives $T = 2\pi\sqrt{\dfrac{\ell}{g}}$.

If $\ell >> R$ (i.e. an infinitely long pendulum), then $\dfrac{g}{\ell}$ can be neglected, which gives $T = 2\pi\sqrt{\dfrac{R}{g}}$. Hence, the time period of an infinitely long pendulum near surface of Earth,

$$T = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.81}}$$

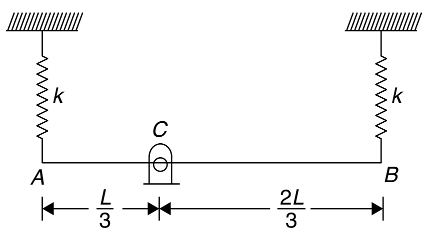
($\because$ R of Earth $= 6.4 \times 10^6$ m

$g = 9.81$ m/s^2)

$= 5075$ second

$= 84.6$ minute, which is finite (and not infinite)

Example 12:

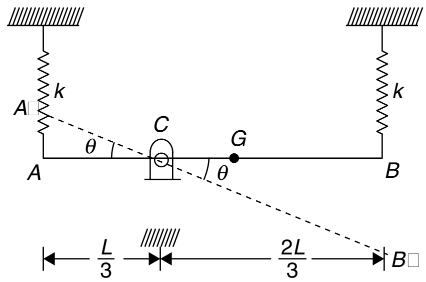


A rigid uniform rod AB of length L and mass m is hinged at C such that $AC = \dfrac{L}{3}$, $CB = \dfrac{2L}{3}$. Ends A and B are supported by springs of spring constant k. The natural frequency of the system is given by

(A) $\sqrt{\dfrac{k}{2m}}$ (B) $\sqrt{\dfrac{k}{m}}$

(C) $\sqrt{\dfrac{2k}{m}}$ (D) $\sqrt{\dfrac{5k}{m}}$

Solution:



G is the centre of mass of rod AB.

$\therefore\ CG = AG - AC = \dfrac{L}{2} - \dfrac{L}{3} = \dfrac{L}{6}$

Moment of inertia of rod about an axis through C and perpendicular to AB,

$$I_C = I_G + m(CG)^2$$

($\because$ $I_G = \dfrac{mL^2}{12}$ for uniform rod)

$$= \frac{mL^2}{12} + m\left(\frac{L}{6}\right)^2$$

$$= \frac{mL^2}{12} + \frac{mL^2}{36}$$

$$= \frac{4mL^2}{36} = \frac{mL^2}{9}$$

$\therefore\ I_C = \dfrac{mL^2}{9}$ ——— (i)

From the equilibrium position, let the rod AB be rotated clockwise by a small angle θ. Point B moves vertically down to B' such that $BB' = CB\ \theta = \dfrac{2L}{3}\theta$

$$F_B = \text{The spring force at } B = k(BB')$$

$$= \frac{2kL\theta}{3}, \text{ vertically upwards}$$

Torque about C due to spring force at B,

$$T_B = F_B(CB) = \frac{2kL\theta}{3}.\frac{2L}{3}$$

$$= \frac{4kL^2\theta}{9}, \text{ anti-clockwise}$$

The point A moves vertically up to A' such that $AA' = (CA)$ $\theta = \dfrac{L}{3}\theta$

$F_A =$ The spring force at $A = k(AA')$

$= \dfrac{kL\theta}{3}$, vertically downwards

Torque about C due to spring force at A,

$$T_A = F_A(CA) = \frac{kL\theta}{3}.\frac{L}{3} = \frac{kL^2\theta}{9}\text{, anti-clockwise}$$

$\therefore$ Total restoring torque,

$$T_R = T_A + T_B$$

$$= \frac{4kL^2\theta}{9} + \frac{kL^2\theta}{9}$$

$$= \left(\frac{5kL^2}{9}\right)\theta\text{, anticlockwise}$$

If α is the angular acceleration (which is anticlockwise), inertial torque $T_i = I_C\alpha$

As per D' Alembert's principle, $T_R + T_i = 0$

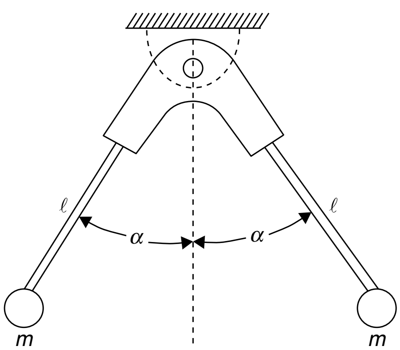
$$\Rightarrow \left(\frac{5kL^2}{9}\right)\theta + I_C\alpha = 0$$

$$\alpha = -\frac{\left(\frac{5kL^2}{9}\right)\theta}{I_C} = \frac{-\left(\frac{5kL^2}{9}\right)\theta}{\left(\frac{mL^2}{9}\right)}$$

i.e. $\alpha = -\left(\frac{5k}{m}\right)\theta \Rightarrow$ angular SHM

$$\therefore \omega_n^2 = \frac{5k}{m} \Rightarrow \omega_n = \sqrt{\frac{5k}{m}}$$

Example 13:

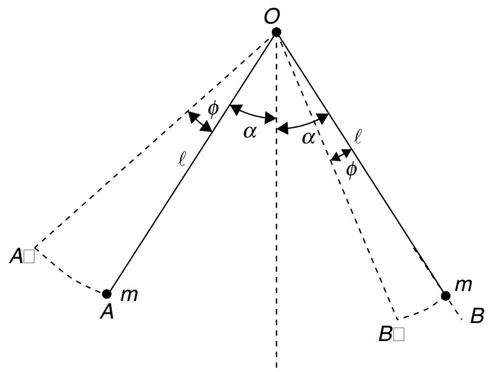


The assembly shown in figure is composed of two massless rods of length ℓ, with two particles, each of mass m. The natural frequency of this assembly for small oscillations is

(A) $\sqrt{\frac{g}{\ell}}$ (B) $\sqrt{\frac{2g}{\ell\cos\alpha}}$

(C) $\sqrt{\frac{g}{\ell\cos\alpha}}$ (D) $\sqrt{\frac{g\cos\alpha}{\ell}}$

Solution:



O is the hinge. The moment of inertia of the system about hinge O is

$$I_o = m\ell^2 + m\ell^2 = 2m\ell^2$$

From the equilibrium position, rotate the system by a small angle in the clockwise sense and release. When the system has moved by a small angle ϕ in the clockwise sense, right arm makes an angle $(\alpha - \phi)$ with the vertical and the left arm makes an angle of $(\alpha + \phi)$ with vertical

Restoring torque on left arm about O,

$$T_L = W\ell\sin(\alpha+\phi)\text{, anticlockwise } [W = mg]$$

Restoring torque on right arm about O,

$$T_R = W\ell\sin(\alpha-\phi)\text{, clockwise}$$

$\therefore$ Net restoring torque about O,

$$T = T_L - T_R$$

$$= W\ell\sin(\alpha+\phi) - W\ell\sin(\alpha-\phi)$$

$$= W\ell[\sin(\alpha+\phi) - \sin(\alpha-\phi)]$$

$$= W\ell[(\sin\alpha\cos\alpha + \cos\alpha\sin\phi) - (\sin\alpha\cos\phi - \cos\alpha\sin\phi)]$$

$$= W\ell[2\cos\alpha\sin\phi]$$

$$= (2W\ell\cos\alpha)\phi\text{, anticlockwise}$$

$$[\because \sin\phi = \phi \text{ for small value of } \phi]$$

If β is the angular acceleration, then inertial torque about O

$T_i = I_0\theta$, anticlockwise

$$= 2m\ell^2\beta$$

As per D'Alembert's principle,

$T + T_i = 0$

$$\Rightarrow (2W\ell\cos\alpha)\phi + 2m\ell^2\beta = 0$$

$$\Rightarrow \beta = \frac{-(2W\ell\cos\alpha)\phi}{2m\ell^2}$$

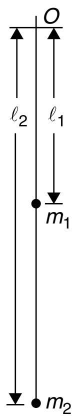
$$= \frac{-(2mg\ell\cos\alpha)\phi}{2m\ell^2}$$

$$= -\left(\frac{g}{\ell}\cos\alpha\right)\phi$$

$$\therefore \beta = -\left(\frac{g}{\ell}\cos\alpha\right)\phi \Rightarrow \text{angular SHM}$$

$$\omega_n^2 = \sqrt{\frac{g\cos\alpha}{\ell}}.$$

Example 14:



A compound pendulum consists of two point masses m_1 and m_2, connected to the same light, inextensible string of length ℓ_2 and upper end connected to fixed pivot O. Mass m_1 is at a distance ℓ_1 from O and mass m_2 is at a distance ℓ_2 from O as shown in figure. Keeping the string taut, mass m_2 is pulled to the right and released so that the system makes to and fro motion of small angular amplitude in the vertical plane. The natural time period of oscillation of this pendulum is

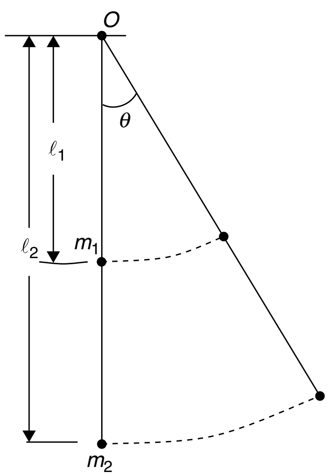
(A) $2\pi\sqrt{\dfrac{(m_1\ell_1^2 + m_2\ell_2^2)}{(m_1\ell_1 + m_2\ell_2)g}}$

(B) $2\pi\sqrt{\dfrac{\ell_2}{g}}$

(C) $2\pi\sqrt{\dfrac{(m_1\ell_1 + m_2\ell_2)}{(m_1 + m_2)g}}$

(D) $2\pi\sqrt{\dfrac{(m_1\ell_1^2 + m_2\ell_2^2)}{(m_1 + m_2)(\ell_1 + \ell_2)g}}$

Solution:



Moment of inertia of system about pivot O,

$$I_0 = m_1\ell_1^2 + m_2\ell_2^2$$

when the angular displacement is a small value θ to the right, angular acceleration is $\alpha = \ddot{\theta}$ and inertial torque $T_i = I\alpha = (m_1\ell_1^2 + m_2\ell_2^2)\alpha$, clockwise. For small angle θ, $\sin\theta = \theta$

∴ Restoring torque (about O),

$$T_R = (m_1 g\,\ell_1 \sin\theta) + m_2 g\,\ell_2 \sin\theta$$
$$= (m_1\ell_1 + m_2\ell_2)g\sin\theta$$
$$= (m_1\ell_1 + m_2\ell_2)g\theta$$

As per D′ Alembart's principle,

$$T_R + T_i = 0$$

$$\therefore (m_1\ell_1 + m_2\ell_2)g\theta + (m_1\ell_1^2 + m_2\ell_2^2)\alpha = 0$$

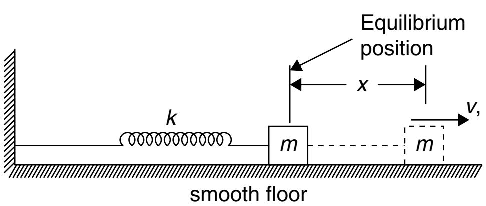
$$\Rightarrow \alpha = \frac{-(m_1\ell_1 + m_2\ell_2)g}{(m_1\ell_1^2 + m_2\ell_2^2)}\theta \Rightarrow \text{SHM}$$

$$\therefore \omega_n^2 = \frac{(m_1\ell_1 + m_2\ell_2)g}{(m_1\ell_1^2 + m_2\ell_2^2)}$$

$$\therefore \omega_n = \sqrt{\frac{(m_1\ell_1 + m_2\ell_2)g}{(m_1\ell_1^2 + m_2\ell_2^2)}}$$

$$\therefore T_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m_1\ell_1^2 + m_2\ell_2^2}{(m_1\ell_1 + m_2\ell_2)g}}$$

Determination of Natural Frequency of Free Undamped Vibration by Energy Method



Consider a light spring of spring constant k and a block of mass m, executing free longitudinal vibrations. When the block is displaced by x to the right from the mean position, let its velocity and acceleration be v and a respectively. We have $v = \dfrac{dx}{dt} = \dot{x}$ and $a = \dfrac{d^2x}{dt^2} = \ddot{x}$

Kinetic energy of system, $KE_x = \dfrac{1}{2}mv^2$

Potential energy of system, $PE_x = \dfrac{1}{2}kx^2$

Total energy of system, $E = KE_x + PE_x$

$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

The total energy of vibration system (i.e. energy of vibration) is constant for undamped vibrations.

$$\therefore \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E = \text{constant}$$

Differentiating the above equation with respect to time (t), we get $\frac{1}{2}m.2v\frac{dv}{dt}+\frac{1}{2}k.2x\frac{dx}{dt}=0$

$$\left(\because \frac{dE}{dt}=0\right)$$

$$\Rightarrow mva+kxv=0 \left(\because \frac{dv}{dt}=a, \frac{dx}{dt}=v\right)$$

$$\Rightarrow ma+kx=0$$

or $a=-\left(\frac{k}{m}\right)x \Rightarrow$ SHM

$\therefore \omega_n=\sqrt{\frac{k}{m}}$, same result obtained earlier by equilibrium method.

NOTES

1. When a system of mass m is having a pure translation with a velocity V, its kinetic energy $=\frac{1}{2}mV^2$
2. When a system of mass m is having a pure rotation about an axis with an angular velocity ω, its rotational kinetic energy $=\frac{1}{2}I\omega^2$, where I = moment of inertia of system about axis of rotation $= mk^2$, where k = radius of gyration of the system about that axis.
3. When a body of mass m and radius R is doing pure rolling with an angular velocity ω and translational velocity V_{CM} (of its centre of mass), then $V_{CM}=R\omega$ and translational $KE=\frac{1}{2}mV_{CM}^2$, rotational $KE=\frac{1}{2}I_{CM}\omega^2$, where $I_{CM}=mK_{CM}^2$ (here K_{CM} = radius of gyration of the round body about rotational axis through centre of mass)

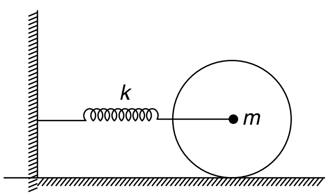
   $\therefore$ Total energy of rotating body

   $= KE$ (translation) $+ KE$ (rotation)

   $=\frac{1}{2}mV_{CM}^2+\frac{1}{2}I_{CM}\omega^2$

   i.e. $E=\frac{1}{2}mV_{CM}^2\left[1+\frac{K_{CM}^2}{R^2}\right]$ in pure rolling.

Example 15:



A thin uniform disc of mass m is attached to a light spring of stiffness k as shown in figure. The disc rolls without slipping on a horizontal surface. The natural frequency of vibration of the system is

(A) $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (B) $\frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

(C) $\frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$ (D) $\frac{1}{2\pi}\sqrt{\frac{3k}{2m}}$

Solution:

Radius of gyration of disc, $K_{CM}=\frac{R}{\sqrt{2}}$

$$\left(\because I_{CM}=\frac{MR^2}{2}=MK_{CM}^2\right)$$

When the centre of mass is displaced by x_{CM} to the right from equilibrium position, let the velocity of centre of mass be V_{CM} and its acceleration be a_{CM}.

$\therefore V_{CM}=\dot{x}_{CM}$ and $a_{CM}=\ddot{x}_{CM}$

PE of system, $PE_x=\frac{1}{2}kx_{CM}^2$

KE of system,

$KE_X=\frac{1}{2}mV_{CM}^2\left[1+\frac{K_{CM}^2}{R^2}\right]$ [For body in pure rolling]

$$=\frac{1}{2}mV_{CM}^2\left[1+\frac{1}{2}\right]$$

$$\left[\because \frac{K_{CM}^2}{R^2}=\frac{1}{2}\text{ for disc}\right]$$

$$=\frac{3}{4}mV_{CM}^2$$

$\therefore KE_x+PE_x=$ Total energy of oscillation, E = constant

$$\therefore \frac{3}{4}mV_{CM}^2+\frac{1}{2}kx_{CM}^2=E$$

Differentiating with respect to time (t), we get

$$\frac{3}{4}m.2V_{CM}\left(\frac{dV_{CM}}{dt}\right)+\frac{1}{2}k.2x_{CM}\left(\frac{dx_{CM}}{dt}\right)=0$$

$$(\because E = \text{constant})$$

$$\Rightarrow \frac{3}{2}mV_{CM}a_{CM}+kx_{CM}V_{CM}=0$$

$$\left(\because \frac{dV_{CM}}{dt}=a_{CM}, \frac{dx_{CM}}{dt}=V_{CM}\right)$$

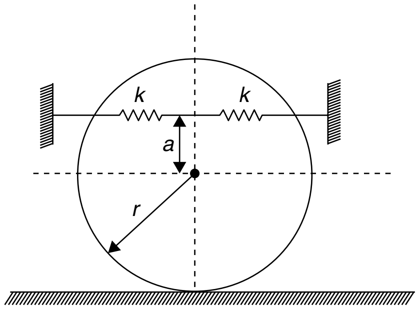
$$\Rightarrow \frac{3}{2}ma_{CM}+kx_{CM}=0$$

$$\Rightarrow a_{CM}=\frac{-kx_{CM}}{\left(\frac{3}{2}m\right)}=-\left(\frac{2k}{3m}\right)x_{CM} \Rightarrow \text{SHM}$$

$$\therefore \omega_n^2=\frac{2k}{3m} \Rightarrow \omega_n=\sqrt{\frac{2k}{3m}}$$

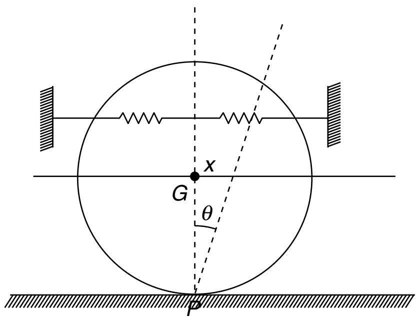
$$\therefore f_n=\frac{\omega_n}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{2k}{3m}}.$$

Example 16:



A cylinder of mass 1 kg and radius $r = 1$ m is connected by two identical springs at a height of $a = 0.5$ m above the centre as shown in the figure. The cylinder rolls without slipping. If the spring constant is 30 kN/m for each spring, the natural frequency of the system for small oscillations (in Hz) is ____________ .

Solution:



$k = 30 \times 10^3$ N/m
$r = 1\ m$
$m = 1$ kg
$a = 0.5\ m$

Point of contact P is the instantaneous centre. The cylinder can be considered to be under pure rotation about point P. If the cylinder rotates clockwise by a small angle θ about P, centre of mass G gets shifted by $x = r\dfrac{d\theta}{dt} = r\omega$ and the change in length of the springs is $(r + a)\theta$, with one spring getting compressed and another getting stretched.

$\therefore$ PE in the springs

$$= 2 \times \frac{1}{2}k\left[(r+a)\theta\right]^2$$

$$= k(r+a)^2\theta^2$$

$$KE \text{ of cylinder} = \frac{1}{2}I_P\omega^2$$

$$= \frac{1}{2}\left[I_G + mr^2\right]\omega^2$$

$$= \frac{1}{2}\left[\frac{mr^2}{2} + mr^2\right]\omega^2$$

$$= \frac{3}{4}mr^2\omega^2$$

$KE + PE = E =$ constant

$$\Rightarrow \frac{3}{4}mr^2\omega^2 + k(r+a)^2\theta^2 = E$$

Taking derivative with respect to time,

$$\frac{3}{4}mr^2\omega\left(\frac{d\omega}{dt}\right) + k(r+a)^2 2\theta\left(\frac{d\theta}{dt}\right) = 0$$

$$\left(\because \frac{dE}{dt} = 0\right)$$

$$\Rightarrow \frac{3}{2}mr^2\left(\frac{d\omega}{dt}\right) + k(r+a)^2 2\theta = 0$$

$$\left[\because \frac{d\theta}{dt} = \omega\right]$$

$$\Rightarrow \frac{3}{2}mr^2\alpha + k(r+a)^2 2\theta = 0$$

$$\left[\because \frac{d\omega}{dt} = \alpha\right]$$

$$\Rightarrow \alpha = \frac{-2k(r+a)^2 \times 2\theta}{3mr^2}$$

$$= -\left(\frac{4}{3}\frac{k(r+a)^2}{mr^2}\right)\theta \Rightarrow \text{SHM}$$

$$\therefore \omega_n = \sqrt{\frac{4}{3}\frac{k(r+a)^2}{mr^2}}$$

$$\Rightarrow f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{4}{3}\frac{k(r+a)^2}{mr^2}}$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{4}{3} \times \frac{30 \times 1000(1+0.5)^2}{1 \times 1^2}}$$

$= 47.75$ Hz

Hence, frequency of natural vibration is 47.75 Hz.

NOTE

The radius of gyration of various round bodies of radius R and mass m, about an axis through their centres of mass are given below.

Sl. No.	Body	I_{CM}	K_{CM}
1	Thin ring	mR^2	R
2	Uniform disc	$\dfrac{mR^2}{2}$	$\dfrac{R}{\sqrt{2}}$
3	Solid cylinder	$\dfrac{mR^2}{2}$	$\dfrac{R}{\sqrt{2}}$
4	Solid sphere	$\dfrac{2}{5}mR^2$	$\sqrt{\dfrac{2}{5}}R$
5	Thin spherical shell	$\dfrac{2}{3}mR^2$	$\sqrt{\dfrac{2}{3}}R$

Determination of Natural Frequency of Vibration of Free Vibrations by Rayleigh's Method

Rayleigh's method takes into consideration that for a system in SHM, the maximum kinetic energy of vibration (which occurs at the equilibrium position, where potential energy is zero) is equal to the maximum potential energy of vibration (which occurs at the maximum amplitude positions, where the kinetic energy is zero)

$\therefore KE_{max} = PE_{max}$

For a spring-mass system, if m = mass of oscillating body, k = spring constant and A = amplitude of vibration, maximum velocity, $v_{max} = A\omega_n$

$$\therefore KE_{max} = \frac{1}{2}mv_{max}^2 = \frac{1}{2}mA^2\omega_n^2$$

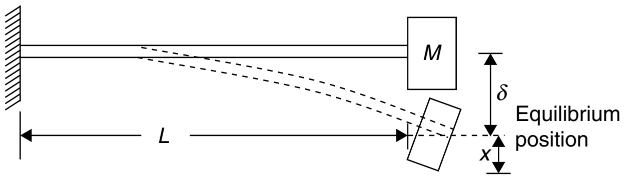
$$PE_{max} = \frac{1}{2}kA^2$$

$$KE_{max} = PE_{max} \text{ (for SHM)}$$

$$\Rightarrow \frac{1}{2}mA^2\omega_n^2 = \frac{1}{2}kA^2$$

$\Rightarrow \omega_n^2 = \dfrac{k}{m} \Rightarrow \omega_n = \sqrt{\dfrac{k}{m}}$, same result as obtained by equilibrium method or by energy method.

Determination of Natural Frequency of Free Undamped Transverse Vibrations of Shafts Carrying a Concentrated load.



Consider a horizontal, uniform rod (or a shaft) of negligible mass, fixed at one end and free at the other end.

A concentrated mass M is connected at the free end (weight of mass, $W = Mg$). The rod (or shaft) undergoes a static lateral deflection δ, when the mass M is slowly released and comes to rest in the equilibrium position.

If s is the stiffness of the rod (or shaft) (i.e. force required to produce unit lateral deflection), then in the equilibrium position,

$$s\delta = W = Mg \qquad (1)$$

If the rod or shaft is deflected further from the equilibrium position by an external force and released, the rod (or shaft) will execute transverse vibrations. At an instant when the rod (or shaft) is deflected downwards from the equilibrium position by x, the net force acting on rod (or shaft)

= weight of body – upward force on rod due to deflection

$= Mg - s(\delta + x) \qquad (\because Mg = s\delta)$

$= -sx$

But net force = mass × acceleration

$$= m\frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -sx \text{ or } m\frac{d^2x}{dt^2} + sx = 0$$

Or $\dfrac{d^2x}{dt^2} + \left(\dfrac{s}{m}\right)x = 0$ which is SHM

This equation is the same as for longitudinal vibrations.

$$\therefore \omega_n = \sqrt{\frac{s}{m}} \Rightarrow \omega_n = \sqrt{\frac{mg}{\delta m}} = \sqrt{\frac{g}{\delta}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{mg}{\delta m}}$$

i.e. $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{s}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{\delta}} = \dfrac{0.4985}{\sqrt{\delta}}$ Hz.

Here, δ = static deflection of rod (or shaft) in m

If E = Young's modulus for material of shaft (or rod)

I = second moment of area of the shaft (or rod)

L = length of shaft (or rod),

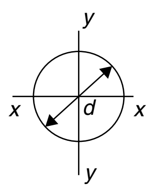
$\delta = \dfrac{MgL^3}{3EI}$, for cantilever beam, $W = Mg$

$\delta = \dfrac{WL^3}{3EI}$, for cantilever beam with concentrated load W at free end.

NOTES

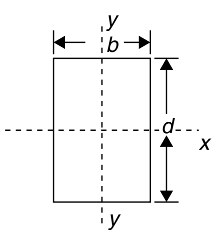
1. **The values of I for various cross sections are**

(a) **Circular**



$$I_{xx} = I_{yy} = \frac{\pi}{64}d^4 \text{, where } d = \text{shaft diameter}$$

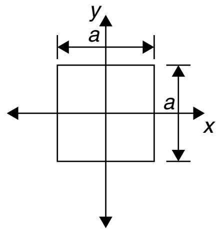
(b) **Rectangular**



$$I_{xx} = \frac{bd^3}{12}$$

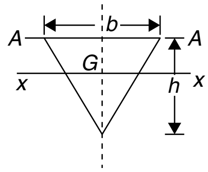
$$I_{yy} = \frac{db^3}{12}$$

(c) **Square**



$$I_{xx} = I_{yy} = \frac{a^4}{12}$$

(d) **Triangular**

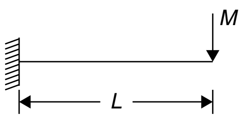


$$I_{xx} = I_G = \frac{bh^3}{36}$$

$$I_{AA} = I_{Base} = \frac{bh^3}{12}$$

2. The static deflections δ for various types of concentrated loads ($W = Mg$) are

(a) **Cantilever**



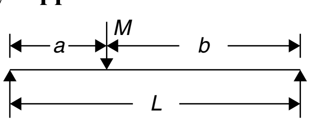
$$\delta = \frac{MgL^3}{3EI}$$

$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{M_{eff}}}$; If mass of shaft is negligible, $M_{eff} = M$

If mass of shaft is m_s (not negligible),

$$M_{eff} = M + \frac{33m_s}{140}$$

(b) **Simply supported**



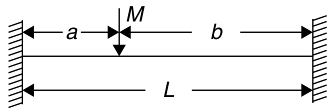
$\delta = \frac{Mga^2b^2}{3EIL}$ (δ is below point of application of Mg)

If $a = b = \frac{L}{2}$, $\delta_{\text{centre}} = \frac{MgL^3}{48EI}$ and

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{M_{eff}}}, \quad \mathrm{M_{eff}} = \mathrm{M} + \frac{17m_s}{35}$$

where m_s = mass of shaft

(c) **Fixed ends**



$\delta = \frac{Mga^3b^3}{3EIL^3}$ (δ is below point of application of Mg)

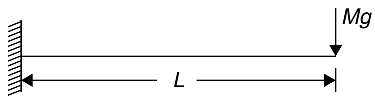
If $a = b = \frac{L}{2}$, $\delta_{\text{centre}} = \frac{MgL^3}{192EI}$ and

$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{M_{eff}}}$, where

$M_{eff} = M + \frac{13m_s}{35}$, where m_s = mass of shaft

Example 17: Consider a cantilever beam, having negligible mass and uniform flexural rigidity, with length 0.01 m. The frequency of vibration of the beam, with a 0.5 kg mass attached at the free tip, is 100 Hz. The flexural rigidity (in Nm^2) of the beam is ___________ .

Solution:



The natural frequency of transverse vibration of a cantilever beam of negligible mass, loaded as shown is,

$f_n = \frac{0.4985}{\sqrt{\delta}}$ Hz, where δ = static deflection, f_n = 100 Hz

$$\rightarrow \delta = \frac{(0.4985)^2}{(100)^2} = 2.485 \times 10^{-5} \text{ m}$$

But $\delta = \frac{MgL^3}{3EI}$

$\therefore$ Flexural rigidity, $EI = \frac{MgL^3}{3\delta}$

$$= \frac{0.5 \times 9.81 \times (0.01)^3}{2.485 \times 10^{-5}}$$

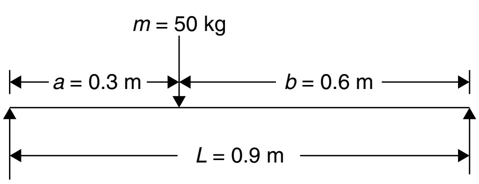
$$= 0.0658 \text{ Nm}^2$$

Hence, the flexural rigidity of the beam is 0.0658 Nm^2.

Example 18: A simply supported shaft of length 900 mm carries a mass of 50 kg placed 300 mm from left end. If $E = 200$ GN/m^2 and diameter of shaft is 40 mm, the natural frequency of undamped natural transverse vibrations is

(A) 86.33 Hz (B) 52.17 Hz
(C) 32.57 Hz (D) 24.38 Hz

Solution:



Shaft dia, $d = 40$ mm $= 0.04$ m

$E = 200 \times 10^9$ N/m^2

$$\therefore I = \frac{\pi}{64} d^4$$

$$= \frac{\pi}{64} \times (0.04)^4 = 1.2566 \times 10^{-7}\,\text{m}^4$$

$$\delta = \frac{Mga^2b^2}{3EIL}$$

$$= \frac{50 \times 9.81 \times 0.3^2 \times 0.6^2}{3 \times 200 \times 10^9 \times 1.2566 \times 10^{-7} \times 0.9}$$

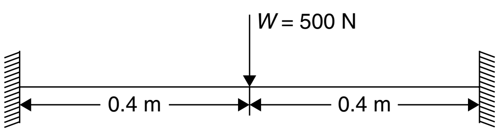
$= 2.342 \times 10^{-4}$ m

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}$$

$$= \frac{0.4985}{\sqrt{2.342 \times 10^{-4}}}$$

$= 32.57$ Hz.

Example 19:



The natural frequency of transverse free vibration of the beam shown in figure is 125 Hz. The flexural rigidity of the beam (in N m^2) is

(A) 83,836.35 (B) 61,937.48
(C) 94,328.37 (D) 75,821.64

Solution:

For transverse vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\ \text{Hz.}$$

$$\Rightarrow \delta = \frac{(0.4985)^2}{f_n^2} = \frac{(0.4985)^2}{(125)^2}$$

$$= 1.5904 \times 10^{-5}\,\text{m}$$

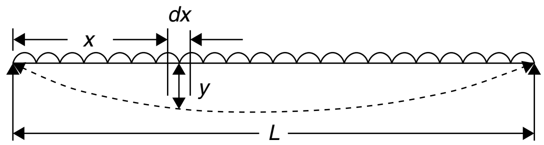
For the fixed beam with concentrated load at centre, $\delta = \dfrac{WL^3}{192EI}$

$$\Rightarrow EI = \frac{WL^3}{192\delta} = \frac{500 \times (0.8)^3}{192 \times 1.5904 \times 10^{-5}}$$

$= 83{,}836.35$ Nm^2.

Determination of Natural Frequency of Free Undamped Transverse Vibrations of Shafts Carrying a Uniformly Distributed Load.

Case 1 Simply supported beam



Consider a simply supported horizontal shaft (or beam) of length L, made of a material of Young's modulus E and having a second moment of area about the bending axis equal to I. Let this beam carry a uniformly distributed load of m kg per metre (or weight mg newton per metre). Due to the load, the shaft (or beam) will bend and come to rest in an equilibrium position, where the deflection at the centre of the shaft (or beam) is $\delta = \dfrac{mgL^4}{384EI}$ This is the **static deflection** at the centre. If the shaft (or beam) is pulled vertically down from this equilibrium position and released, it will execute transverse vibrations.

Consider an elemental length dx of the beam, at a distance x from the left end. Let the displacement of this elemental length from the equilibrium position be y. From the theory of bending of beams (refer to study material on Strength of Materials), we know $EI\dfrac{d^2y}{dx^2} = M_x$, where M_x is the bending moment at distance x from the beam end.

$\Rightarrow EI\dfrac{d^3y}{dx^3} = \dfrac{dM_x}{dx} = F_x$, where F_x is the shear force at distance x from the beam end

$\Rightarrow EI\dfrac{d^4y}{dx^4} =$ dynamic load per unit length on beam at distance x from the beam end

But the dynamic load per unit length of vibrating beam (at distance x) is equal to the centrifugal force per unit length $= m\omega^2 y$.

$\therefore EI\dfrac{d^4y}{dx^4} = m\omega^2 y$, where

$\omega =$ circular frequency of transverse vibration of beam

$$\Rightarrow \frac{d^4y}{dx^4} - \left(\frac{m}{\text{EI}}\right)\omega^2 y = 0$$

i.e. $\dfrac{d^4y}{dx^4} - (m^*)^4\, y = 0$

is the differential equation for this transverse vibration, where $(m^*)^4 = \dfrac{m\omega^2}{EI}$ (1)

The solution to this differential equation is of the form

$y = A\cos m^*x + B\sin m^*x + C\cosh m^*x + D\sinh m^*x$, where (2)

A, B, C and D are constants of integration obtained from end conditions.

The end conditions for simply supported beam are

(i) At $x = 0$ and $x = L$, $y = 0$
($\because$ No deflection at supports)

(ii) At $x = 0$ and $x = L$, $\dfrac{d^2y}{dx^2} = 0$
($\because$ No bending moment at the support locations)

On applying the end conditions in equation 2 and solving for A, B, C and D we find that there are more than one value for m^* which is given by

$$m^* = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L} \text{ etc}$$

Using the smallest value of m^*, equation (1) becomes

$$\left(\frac{\pi}{L}\right)^4 = \frac{m\omega^2}{EI}$$

$$\Rightarrow \omega = \sqrt{\frac{EI}{m}} \times \left(\frac{\pi}{L}\right)^2 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

Hence, the smallest natural frequency of transverse vibrations of a simply supported beam carrying a uniformly distributed load is given by

$$(f_n)_1 = \frac{\omega}{2\pi} = \frac{1}{2\pi} \times \frac{\pi^2}{L^2} \times \sqrt{\frac{EI}{m}}$$

$$\Rightarrow (f_n)_1 = \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}}$$

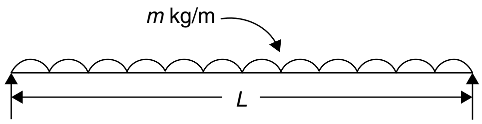
But $\dfrac{EI}{mL^4} = \dfrac{5}{384}\dfrac{g}{\delta}$, where δ = static deflection at the centre of simply supported beam.

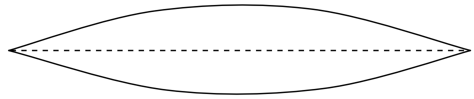
$\therefore$ $(f_n)_1 = \dfrac{\pi}{2}\sqrt{\dfrac{5}{384} \times \dfrac{g}{\delta}} = \dfrac{0.5615}{\sqrt{\delta}}$ is the **fundamental frequency** (or the smallest frequency) of transverse vibration of a simply supported beam carrying a uniformly distributed load. Fundamental frequency is also called the **first mode** or **first harmonic**. By using the other values of $m^* \left(= \dfrac{2\pi}{L}, \dfrac{3\pi}{L} \text{etc.}\right)$, we can obtain the next higher frequencies at which the beam can vibrate. These are called second harmonic, third harmonic etc (or second mode, third mode etc)

These frequencies are 4, 9, 16, ……. etc. times the fundamental frequency.

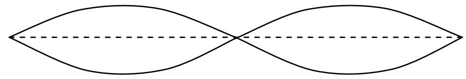
$$f = n^2 \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}} = n^2 \frac{\pi}{2}\sqrt{\frac{5}{384}\frac{g}{\delta}} = \frac{0.5615\, n^2}{\sqrt{\delta}}$$

where f is the frequency of free undamped transverse vibration of uniformly loaded simply supported beam in the n^{th} mode and $n = 1, 2, 3, 4,$ …… etc.

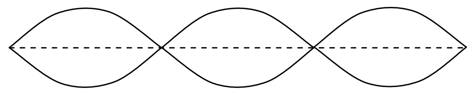




First Mode ($n = 1$)



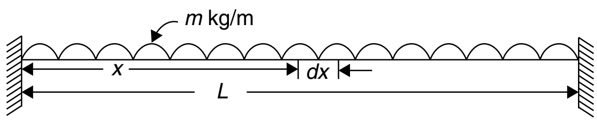
Second Mode ($n = 2$)



Third Mode ($n = 3$)

In all cases, the supports are nodes (or points of zero displacement). In the fundamental (or first) mode, there are no nodes in between the ends and one anti-node (point of maximum displacement)in between the ends. In the second mode ($n = 2$), there is one node ($= n - 1$) and two anti-nodes ($= n$) in between the ends. So, in the nth mode, there are ($n - 1$) nodes and n anti-nodes in between the ends.

Case 2 Encastre beam (Beam with fixed ends)



As seen in Case-1 (for simply supported beam), we have

$(m^*)^4 = \dfrac{m\omega^2}{EI}$ (equation 1) and

$$y = A \cos m^*x + B \sin m^*x + C \cosh m^*x + D \sinh m^*x \quad \text{(equation 2)}$$

The end conditions for fixed beam are

(i) when $x = 0$ and $x = L$, $y = 0$
($\because$ No deflection at fixed ends)

(ii) when $x = 0$ and $x = L$, $\dfrac{dy}{dx} = 0$
($\because$ slope at fixed ends is zero)

The static deflection at the centre, $\delta = \dfrac{mgL^4}{384EI}$

Using the above conditions, we can show that the fundamental frequency (smallest frequency) of natural free undamped transverse vibrations of a uniformly loaded beam with fixed ends is

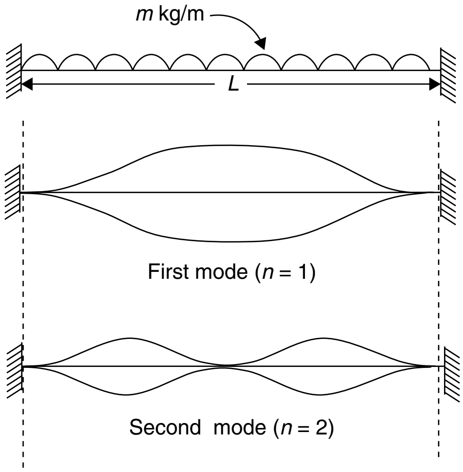
$$(f_n)_1 = 3.562\sqrt{\frac{EI}{mL^4}}$$

For the higher modes of vibration ($n = 2, 3, 4,$ etc.), the natural frequency is given by

$$f = \frac{\pi}{2}\left(n + \frac{1}{2}\right)^2 \sqrt{\frac{EI}{mL^4}}, \text{ where } n = 2, 3, 4 \text{ etc}$$

NOTE

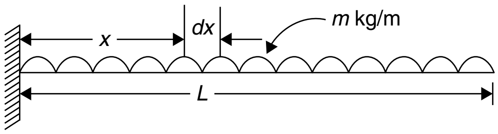
The **smallest frequency** of oscillation (fundamental frequency) **almost** corresponds to $n = 1$.



First mode ($n = 1$)

Second mode ($n = 2$)

Case 3 Cantilever beam



We have $\left(m^*\right)^4 = \dfrac{m\omega^2}{EI}$ (1)

$y = A \cos m^*x + B \sin m^*x + C \cosh m^*x + D \sinh m^*x$ (2)

The end conditions for cantilever beam are

(i) when $x = 0$, $y = 0$ ($\because$ no deflection at the fixed end)

(ii) when $x = 0$, $\dfrac{dy}{dx} = 0$ ($\because$ zero slope at fixed end)

(iii) when $x = L$, $\dfrac{d^2y}{dx^2} = 0$ ($\because$ Bending moment is zero at the free end)

(iv) when $x = L$, $\dfrac{d^3y}{dx^3} = 0$ ($\because$ shear force is zero at the free end)

The static deflection **at the free end** of the cantilever beam is $\delta = \dfrac{mgL^4}{8EI}$

Using the above conditions, we can show that the fundamental (smallest) frequency of free undamped transverse vibrations of a cantilever beam carrying a uniformly distributed load is

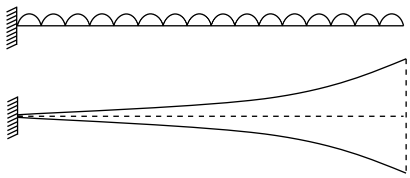
$$(f_n)_1 = 0.565\sqrt{\frac{EI}{mL^4}}$$

For higher modes of vibration, the frequencies are given by

$$f = \frac{\pi}{2}\left(n - \frac{1}{2}\right)^2\sqrt{\frac{EI}{mL^4}}, \text{ where } n = 2, 3, 4, \ldots\text{etc.}$$

NOTE

The **smallest frequency** of oscillation (fundamental frequency) **almost** corresponds to $n = 1$.



First mode ($n = 1$)

NOTES

1. Shafts which are supported on knife-edges, needle bearings and short bearings can be treated as simply supported shafts.
2. Shafts which are supported on long bearings are treated as fixed at both ends.

Example 20: A uniform shaft is 60 mm in diameter and 10 m long and may be regarded as simply supported. The density of shaft material is 7850 kg/m^3 and Young's modulus $E = 210$ GPa. The natural frequencies of first, second and third mode of undamped free transverse vibrations of the shaft are (in Hz) respectively

(A) 3.16, 12.64, 28.44 (B) 1.22, 4.88, 10.98
(C) 2.41, 9.64, 21.69 (D) 12.2, 48.8, 109.8

Solution:

Diameter of shaft, $d = 60$ mm $= 0.06$ m

$\therefore$ Second moment of area,

$$I = \frac{\pi}{64}d^4 = \frac{\pi}{64}\times(0.06)^4$$

$$= 6.3617 \times 10^{-7}\ \text{m}^4$$

$E = 210 \times 10^9$ N/m^2

$\therefore$ Flexural rigidity,

$$EI = 210 \times 10^9 \times 6.3617 \times 10^{-7}$$
$$= 133{,}595.7\ \text{Nm}^2$$

Mass per unit length,

$$m = \text{Density} \times \text{cross sectional area}$$
$$= 7850 \times \frac{\pi}{4}\times(0.06)^2$$
$$= 22.195\ \text{kg/m}$$

Length, $L = 10$ m

Fundamental frequency,

$$(f_n)_1 = \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}}$$

$$= \frac{\pi}{2}\sqrt{\frac{133595.7}{22.195\times10^4}}$$

$$= 1.22\ \text{Hz.}$$

Second harmonic $(f_n)_2 = 2^2\,(f_n)_1$

$$= 4\times1.22 = 4.88\ \text{Hz}$$

Third harmonic $(f_n)_3 = 3^2\,(f_n)_1$

$$= 9 \times 1.22$$
$$= 10.98 \text{ Hz}$$

Hence, the frequencies of first three modes of transverse undamped vibrations are 1.22 Hz., 4.88 Hz and 10.98 Hz, respectively.

Example 21: A thin rod 4 mm in diameter is held between two chucks 0.9 m apart. The rod weights 1.52 N/m and flexural stiffness is 6.236 Nm^2. The natural frequencies (in Hz) of the first three modes of undamped free transverse vibrations of the rod are respectively.

(A) 27.90, 76.90, 150.73
(B) 27.90, 111.60, 251.10
(C) 2.79, 7.69, 15.07
(D) 12.31, 33.93, 66.50

Solution:

Mass per unit length,

$$m = \frac{\text{weight/unit length}}{g}$$
$$= \frac{1.52}{9.81}$$
$$= 0.1549 \text{ kg/m}$$

Flexural stiffness, $EI = 6.236$ Nm^2

Length, $L = 0.9$ m

$$\Rightarrow \sqrt{\frac{EI}{mL^4}} = 7.8333$$

Shaft fixed at both ends

∴ Fundamental frequency,

$$(f_n)_1 = 3.562\sqrt{\frac{EI}{mL^4}}$$
$$= 3.562 \times \sqrt{\frac{6.236}{0.1549 \times 0.9^4}} = 27.90 \text{ Hz.}$$

Second harmonic

$$(f_n)_2 = \frac{\pi}{2}\left(2 + \frac{1}{2}\right)^2 \sqrt{\frac{EI}{mL^4}}$$
$$= \frac{\pi}{2} \times \frac{25}{4} \times 7.8333$$
$$= 76.90 \text{ Hz}$$

Third harmonic

$$(f_n)_3 = \frac{\pi}{2}\left(3 + \frac{1}{2}\right)^2 \sqrt{\frac{EI}{mL^4}}$$
$$= \frac{\pi}{2} \times \frac{49}{4} \times 7.8333$$
$$= 150.73 \text{ Hz.}$$

Example 22: An aluminium rod is held in a chuck with the other end unsupported. It is 10 mm diameter and 500 mm long. The density of aluminium is 2725 kg/m^3 and the modulus of elasticity E is 72 GPa. The natural frequencies of first three modes of undamped free transverse vibrations of the rod (in Hz) are respectively

(A) 31.17, 124.68, 280.53
(B) 13.15, 82.25, 228.48
(C) 7.34, 20.23, 39.65
(D) 29.05, 181.70, 504.73

Solution:

This is a cantilever beam

Diameter of rod, $d = 10$ mm $= 0.01$ m

Second moment of area,

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times (0.01)^4$$
$$= 4.91 \times 10^{-10} \text{ m}^4$$
$$E = 72 \times 10^9 \text{ N/m}^2$$

∴ Flexural rigidity,

$$EI = 72 \times 10^9 \times 4.91 \times 10^{-10}$$
$$= 35.352 \text{ Nm}^2$$

Length, $L = 0.5$ m

Mass per unit length,

$$m = \text{Density} \times \text{cross sectional area}$$
$$= 2725 \times \frac{\pi}{4} \times (0.01)^2$$
$$= 0.214 \text{ kg/m}$$

Fundamental frequency of transverse vibration,

$$(f_n)_1 = 0.565\sqrt{\frac{EI}{mL^4}}$$
$$= 0.565 \times \sqrt{\frac{35.352}{0.214 \times 0.5^4}}$$
$$= 29.05 \text{ Hz}$$

Second harmonic

$$(f_n)_2 = \frac{\pi}{2}\left(n - \frac{1}{2}\right)^2 \sqrt{\frac{EI}{mL^4}} \quad (n = 2)$$
$$= \frac{\pi}{2}\left(2 - \frac{1}{2}\right)^2 \sqrt{\frac{35.352}{0.214 \times 0.5^4}}$$
$$= \frac{\pi}{2} \times \frac{9}{4} \times 51.4115$$
$$= 181.70 \text{ Hz}$$

Third harmonic

$$(f_n)_3 = \frac{\pi}{2}\left(3 - \frac{1}{2}\right)^2 \sqrt{\frac{35.352}{0.214 \times 0.5^2}}$$
$$= \frac{\pi}{2} \times \frac{25}{4} \times 51.4115$$
$$= 504.73 \text{ Hz.}$$

Determination of Natural Frequency of Transverse Vibration of a Uniform Shaft Carrying a Combination of Distributed and Point Loads

When a shaft (or a beam) has one or more concentrated masses as well as uniformly distributed mass, the natural frequency of transverse vibration can be obtained by

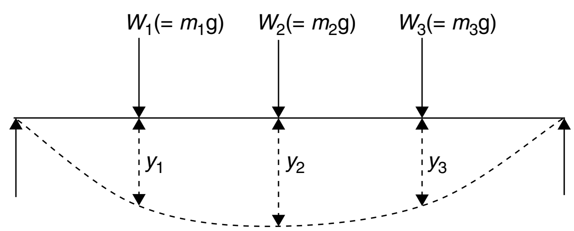
1. Energy method (or Rayleigh's method) or
2. Dunkerley's method

In general, Rayleigh's method overestimates and Dunkerley's method underestimates the natural frequency. These are explained below.

Energy method (or Rayleigh's method)

This method gives accurate results. In this method, the mass of shaft (or beam) is neglected. In fact, the mass of shaft is considered as additional point loads so that there are only concentrated loads on the system and no uniformly distributed load. This method takes into consideration that for a vibrating body, its kinetic energy in the equilibrium position is equal to its potential energy in the extreme positions.

$$KE_{\max} = PE_{\max}.$$



Consider a uniform shaft of negligible mass, carrying point loads $W_1(= m_1g)$, $W_2(= m_2g)$ and $W_3(= m_3g)$ respectively. The system is vibrating with a circular frequency of ω rad/s. The amplitudes of displacements (from equilibrium position) for points of application of W_1, W_2 and W_3 are y_1, y_2 and y_3, respectively.

For the extreme positions of shaft, maximum potential energy,

$$PE_{\max} = \Sigma\text{Mean load} \times \text{defection below load}$$

$$= \frac{1}{2}W_1y_1 + \frac{1}{2}W_2y_2 + \frac{1}{2}W_3y_3$$

$$= \frac{1}{2}(m_1gy_1) + \frac{1}{2}(m_2gy_2) + \frac{1}{2}(m_3gy_3)$$

$$= \frac{g}{2}\left[m_1y_1 + m_2y_2 + m_3y_3\right]$$

$$= \frac{g}{2}\Sigma my$$

For the equilibrium position of the shaft, maximum kinetic energy,

$$KE_{\max} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2$$

$$= \frac{1}{2}m_1(y_1\omega)^2 + \frac{1}{2}m_2(y_2\omega)^2 + \frac{1}{2}m_3(y_3\omega)^2$$

$$= \frac{1}{2}\omega^2\left[m_1y_1^2 + m_2y_2^2 + m_3y_3^2\right]$$

$$= \frac{1}{2}\omega^2\Sigma my^2$$

But $PE_{\max} = KE_{\max}$ (for undamped oscillation)

$$\Rightarrow \frac{g}{2}\Sigma my = \frac{1}{2}\omega^2\Sigma my^2$$

$$\Rightarrow \omega^2 = \frac{g\Sigma my}{\Sigma my^2}$$

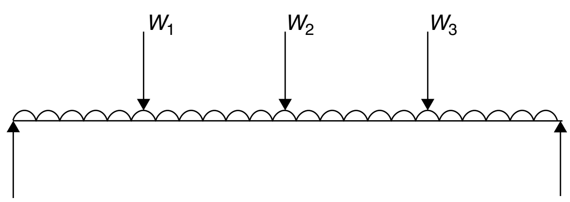
$$\omega = \sqrt{\frac{g\Sigma my}{\Sigma my^2}}$$

∴ Natural frequency of transverse vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g\Sigma my}{\Sigma my^2}}$$

Dunkerley's method

This method is used when the diameter of shaft is uniform and this method takes the weight of shaft also into consideration. It is a semi-empirical method which gives approximate results.



Consider a simply supported, uniform shaft of mass m_s and weight $W_s = m_sg$. The **maximum static deflection** of the shaft **under its own weight** is δ_s.

The concentrated loads on the shaft at various locations are W_1, W_2, W_3, etc due to masses m_1, m_2, m_3 etc ($\because W_1 = m_1g$, $W_2 = m_2g$, $W_3 = m_3g$, etc)

The **static deflection** of the shaft **under the load** W_1, W_2, W_3, etc, **when each load is acting separately**, is δ_1, δ_2, δ_3, etc

The natural frequency of transverse vibration of the shaft **with each load acting separately**, is $f_{n_1}, f_{n_2}, f_{n_3}$, etc. The natural frequency of transverse vibration of the shaft under its own uniformly distributed weight is f_{n_s} . If f_n = natural frequency of transverse vibration of the simply supported system (carrying uniformly distributed load as well as concentrated loads), then as per Dunkerley's empirical formula,

$$\frac{1}{f_n^2} = \frac{1}{f_{n_1}^2} + \frac{1}{f_{n_2}^2} + \frac{1}{f_{n_2}^2} + \frac{1}{f_{n_3}^2} + \ldots.. + \frac{1}{f_{n_s}^2}, \text{ where}$$

$$f_{n_1} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_1}} = \frac{0.4985}{\sqrt{\delta_1}},$$

$$f_{n_2} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_2}} = \frac{0.4985}{\sqrt{\delta_2}}$$

$$f_{n_3} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_3}} = \frac{0.4985}{\sqrt{\delta_3}} \text{ and}$$

$$f_{n_s} = \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}} = \frac{\pi}{2}\sqrt{\frac{5g}{384\delta_s}} = \frac{0.5615}{\sqrt{\delta_s}}.$$

∴ Dunkerley's empirical formula becomes

$$\frac{1}{f_n^2} = \frac{1}{\left(\frac{0.4985}{\sqrt{\delta_1}}\right)^2} + \frac{1}{\left(\frac{0.4985}{\sqrt{\delta_2}}\right)^2} + \frac{1}{\left(\frac{0.4985}{\sqrt{\delta_3}}\right)^2} + ..$$

$$\cdots\cdots + \frac{1}{\left(\frac{0.5615}{\sqrt{\delta_s}}\right)^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \ldots + \frac{\delta_s}{(0.5615)^2}$$

$$= \frac{1}{(0.4985)^2} \times \left[\delta_1 + \delta_2 + \delta_3 + \ldots. + \frac{\delta_s}{\left(\frac{0.5615}{0.4985}\right)^2}\right]$$

$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \ldots. + \frac{\delta_s}{1.27}\right]$$

$$\Rightarrow f_n = \frac{0.4985}{\sqrt{\left[\delta_1 + \delta_2 + \delta_3 + \ldots.. + \frac{\delta_s}{1.27}\right]}}$$

NOTES

1. The formula $\frac{1}{f_n^2} = \frac{1}{f_{n_1}^2} + \frac{1}{f_{n_2}^2} + \ldots. + \frac{1}{f_{n_s}^2}$ holds good for all types of beams but $f_{n_1}, f_{n_2}, \ldots.., f_{n_s}$ will have different values for simply supported beams, fixed beams and cantilever beams.
2. If the mass of shaft is negligible, then $\delta_s = 0$.

Example 23: A cantilever beam is 1 m long and has a cross section of 50 mm wide and 30 mm deep. The density and modulus of elasticity of the material of the beam are 2700 kg/m^3 and 80 GPa respectively. There is a mass of 4 kg attached at the free end of the beam. The natural frequency (in Hz) of the free undamped transverse vibrations of the beam in the fundamental mode due to its own weight, due to only the point load and due to the combined loads are respectively

(A) 6.7, 19.2, 9.3 (B) 13.4, 6.2, 9.4
(C) 26.4, 13.08, 11.72 (D) 33.7, 19.8, 24.3

Solution:
Length, $L = 1$ m,
Area of cross section,

$$A = 50 \times 30 = 1500 \text{ mm}^2 = 1.5 \times 10^{-3} \text{ m}^2$$

Density $\rho = 2700$ kg/m^3

m = Mass per unit length = ρA
$= 2700 \times 1.5 \times 10^{-3}$
$= 4.05$ kg/m

Second moment of area,

$$I = \frac{bd^3}{12} = \frac{0.05 \times (0.03)^3}{12}$$

$= 1.125 \times 10^{-7}$ m^4

$E = 80 \times 10^9$ N/m^2

f_{n_s} = natural frequency of transverse vibration in the fundamental mode due to weight of beam only (uniformly distributed load)

$$= \left(0.56\sqrt{\frac{EI}{mL^4}}\right) \text{(for cantilever beam)}$$

$$= 0.56\sqrt{\frac{80 \times 10^9 \times 1.125 \times 10^{-7}}{4.05 \times 1^4}}$$

$= 26.4$ Hz

δ_1 = Deflection of free end of shaft due to point load $W = Mg = 4 \times 9.81$
$= 39.24$ N

$$= \frac{WL^3}{3EI} \text{ (for cantilever beam, } W \text{ at free end)}$$

$$= \frac{39.24 \times 1^3}{3 \times 80 \times 10^9 \times 1.125 \times 10^{-7}}$$

$= 1.4533 \times 10^{-3}$ m

∴ f_{n_1} = natural frequency of transverse vibration in the fundamental mode due to point load at end of cantilever beam

$$= \frac{1}{2\pi}\sqrt{\frac{g}{\delta_1}}$$

$$= \frac{0.4985}{\sqrt{\delta_1}}$$

$$= \frac{0.4985}{\sqrt{1.4533 \times 10^{-3}}}$$

$= 13.08$ Hz.

If f_n = natural frequency of transverse vibration under combined loading, as per Dunkerley's method,

$$\frac{1}{f_n^2} = \frac{1}{f_{n_1}^2} + \frac{1}{f_{n_s}^2} \Rightarrow \frac{1}{f_n^2} = \frac{f_{n_s}^2 + f_{n_1}^2}{f_{n_1}^2\ f_{n_s}^2}$$

$$\Rightarrow f_n = \frac{f_{n_1}\ f_{n_s}}{\sqrt{f_{n_s}^2 + f_{n_1}^2}} = \frac{13.08 \times 26.4}{\sqrt{26.4^2 + 13.08^2}}$$

$$= \frac{345.312}{\sqrt{868.05}}$$

$$= 11.72 \text{ Hz.}$$

$\therefore$ The natural frequencies required are 26.4 Hz, 13.08 Hz and 11.72 Hz. respectively.

Example 24: A uniform shaft having a uniformly distributed weight of 50 N/m is supported on self-aligning bearings which are 1.5 m apart. The flexural stiffness of the shaft is 5000 Nm^2 and the end conditions may be treated as simply supported. There is a heavy pulley at the centre of the shaft with the centre of gravity of pulley coinciding with the centre line of shaft. The static deflection of the shaft due to the weight of pulley alone is 0.4 mm. The fundamental frequency of free undamped transverse vibration of the shaft under combined loading is

(A) 24.93 Hz (B) 21.87 Hz
(C) 33.71 Hz (D) 16.44 Hz.

Solution:

δ_1 = Deflection of shaft due to weight of pulley

= 4 mm

= 0.4×10^{-3} m

f_{n_1} = frequency of transverse vibration due to weight of pulley alone

$$= \frac{1}{2\pi}\sqrt{\frac{g}{\delta_1}} = \frac{0.4985}{\sqrt{\delta_1}} = \frac{0.4985}{\sqrt{0.4 \times 10^{-3}}}$$

= 24.93 Hz.

f_{n_s} = frequency of transverse vibration due to distributed weight of shaft

$$= \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}} = \frac{\pi}{2}\sqrt{\frac{gEI}{wL^4}} \qquad [\because w = mg]$$

$$= \frac{\pi}{2}\sqrt{\frac{9.81 \times 5000}{50 \times (1.5)^4}} \qquad \left\{\begin{array}{l} \because EI = 5000\ Nm^2 \\ w = 50\ N/m \\ L = 1.5 \text{ m} \end{array}\right\}$$

= 21.87 Hz.

If f_n = frequency of transverse vibration under combined loading, as per Dunkerley's formula,

$$\frac{1}{f_n^2} = \frac{1}{f_{n_1}^2} + \frac{1}{f_{n_s}^2}$$

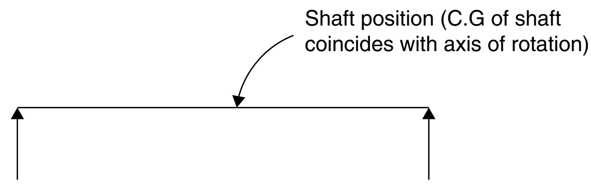
$$\Rightarrow f_n = \frac{f_{n_1}\ f_{n_s}}{\sqrt{f_{n_1}^2 + f_{n_s}^2}}$$

$$\therefore f_n = \frac{24.93 \times 21.87}{\sqrt{24.93^2 + 21.87^2}}$$

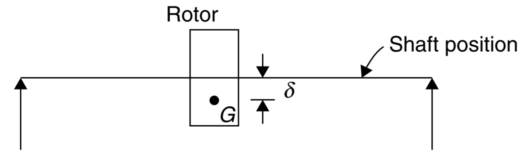
= 16.44 Hz.

Whirling Speeds (or Critical Speeds) of Shafts

Consider a horizontal shaft, whose centre of gravity coincides with the axis of rotation as shown in below Figure.



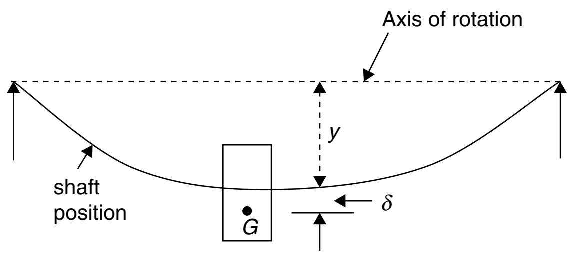
If this shaft is now loaded with a rotor of mass m (so that its weight, $W = mg$), the centre of gravity of the loaded shaft (G) is displaced from the axis of rotation by 'δ' as shown in below Figure.



Here, $mg = s\delta$, where s = stiffness of shaft and

δ = static deflection

When the shaft begins to rotate, a radially outwards force, which is proportional to δ, acts through G, the centre of gravity of the loaded shaft. This radially outward force bends the shaft in the direction of initial displacement δ so that the displacement of centre of gravity G of the loaded shaft from the axis of rotation now becomes $(y + \delta)$ as shown in below Figure.



If ω = angular velocity of shaft and

s = stiffness of shaft (i.e. force needed to produce unit deflection of shaft) at the deflected position of the shaft, the force resisting the deflection of shaft = centrifugal force through G, radially outwards

$$\Rightarrow sy = m(y + \delta)\omega^2$$

$$= m\omega^2 y + m\omega^2\delta$$

$$\Rightarrow sy - m\omega^2 y = m\omega^2\delta$$

$$(s - m\omega^2)y = m\omega^2\delta$$

$$\Rightarrow y = \frac{m\omega^2\delta}{\left(s - m\omega^2\right)} = \frac{\delta}{\left[\frac{s}{m\omega^2} - 1\right]}$$

$$\therefore y = \frac{\delta}{\left[\frac{s}{m\omega^2} - 1\right]}$$

But $\frac{s}{m} = \omega_n^2$ where

ω_n = circular frequency of transverse vibration

$$\therefore y = \frac{\delta}{\frac{\omega_n^2}{\omega^2} - 1} \text{ where } \delta = \text{static deflection}$$

Case 1

When $\omega = \omega_n$ (i.e. the angular speed of rotation of shaft is equal to the natural circular frequency of vibration of system), denominator $\left(\frac{\omega_n^2}{\omega^2} - 1\right)$ becomes zero and hence the deflection y becomes infinite. This condition is called **resonant condition**.

The speed at which the shaft runs so that the deflection of the shaft from the axis of rotation becomes infinite, is called **whirling speed** or **critical speed**. i.e. **if $\omega = \omega_n$, ω is called the critical speed, denoted as** ω_c. At the critical speed, the shaft tends to vibrate violently in the transverse direction

$$\therefore \omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}}, \text{ where}$$

δ = static deflection under load $W = mg$

If N_c = critical speed in rpm,

$$\omega_c = \frac{2\pi N_c}{60} = \sqrt{\frac{g}{\delta}}$$

$$\Rightarrow N_c = \frac{60}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{60 \times 0.4985}{\sqrt{\delta}} \text{(rpm)}$$

i.e. N_c = 60 times f_n, where N_c is in rpm

Case 2

If $\omega > \omega_n$, denominator $\left(\frac{\omega_n{}^2}{\omega^2} - 1\right)$ becomes negative, so $y < 0$ (i.e. y is in opposite direction), that is, the shaft will vibrate in the opposite direction.

Case 3

If $y = -\delta$, then G coincides with the axis of rotation, and hence the shaft will immediately stops vibrating.

NOTE

The **critical speed in rps** (revolutions per second) of a shaft which carries point loads or uniformly distributed loads or combination of both, is equal to the natural frequency of transverse vibration.

Example 25: The rotor shaft of a large electric motor supported between short bearings at both ends shows a deflection of 1.8 mm in the middle of the rotor. Assuming the rotor to be perfectly balanced and supported at knife edges at both ends, the likely critical speed (in rpm) of the shaft is
(A) 350 (B) 705 (C) 2810 (D) 4430

Solution:

$$\text{Nc(rpm)} = 60 \times \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = 60 \times \frac{0.4985}{\sqrt{\delta}}$$

$$= \frac{60 \times 0.4985}{\sqrt{1.8 \times 10^{-3}}}$$

$$= 704.98 \text{ rpm}$$

$$= 705 \text{ rpm}$$

NOTE

At this speed, the shaft will be vibrating with no nodes in between the supports. If the speed becomes

$2^2 \times 705 = 2820$ rpm, the shaft will be vibrating in the second mode and there will be one node in between the supports. Hence, 2820 rpm is the second critical speed. If the shaft rotates at $3^2 \times 705 = 6345$ rpm, it will be in the third mode and there will be two nodes in between the supports. Hence, 6345 rpm is the third critical speed.

Example 26: If two nodes are observed at a frequency of 1800 rpm during whirling of a simply supported long slender rotating shaft, the first critical speed of the shaft in rpm is
(A) 200 (B) 450 (C) 600 (D) 900

Solution:

Two nodes are observed ⇒ between the supports, there are two nodes ⇒ shaft is vibrating in the third mode (i.e. $n = 3$)

As $\left(f_n\right)_n = n^2\left(f_n\right)_1$ for simply supported system, $n = 3$ and $\left(f_n\right)_3 = 1800$, we get

$$1800 = 3^2\left(f_n\right)_1$$

$$\Rightarrow \left(f_n\right)_1 = \frac{1800}{3^2} = 200 \text{ rpm}$$

Hence, the first critical speed of shaft is 200 rpm.

NOTE

The second critical speed is $2^2 \times 200 = 800$ rpm

Example 27: A flexible rotor–shaft system comprises of a 10 kg rotor disc placed in the middle of a massless shaft of diameter 30 mm and length 500 mm between bearings (shaft is being taken massless as the equivalent mass of shaft is included in rotor mass) mounted at the ends. The bearings are assumed to simulate simply supported boundary conditions. The shaft is made of steel for which the value of E is 2.1×10^{11} Pa. What is the critical speed of rotation of the shaft?
(A) 60 Hz (B) 90 Hz
(C) 135 Hz (D) 180 Hz

Solution:

d = diameter of shaft = 30 mm

= 0.03 m

L = length of shaft = 500 mm = 0.5 m

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times (0.03)^4 = 3.976 \times 10^{-8}\,\text{m}^4$$

$E = 2.1 \times 10^{11}$ Pa

$M = 10$ kg

Static deflection at centre, $\delta = \frac{WL^3}{48EI}$ (for simply supported beam, load at centre)

$$= \frac{MgL^3}{48EI}$$

$$= \frac{10 \times 9.81 \times 0.5^3}{48 \times 2.1 \times 10^{11} \times 3.976 \times 10^{-8}}\,\text{m} = 3.06 \times 10^{-5}\,\text{m}$$

Critical speed N_c (rps) $= f_n$

$$= \frac{0.4985}{\sqrt{\delta}}\,\text{Hz}$$

$$= \frac{0.4985}{\sqrt{3.06 \times 10^{-5}}}$$

= 90.11 Hz

≃ 90 Hz.

Damped Free Vibrations

When an elastic body is set in vibratory motion, in all practical oscillations, the oscillation energy of the system gradually gets dissipated as heat in overcoming the internal molecular friction of the mass of the body and friction of the medium in which it vibrates. As the energy of oscillation is proportional to the square of amplitude of oscillation, the amplitude of oscillation keeps on decreasing with the passage of time and the vibrations die out after some time. The diminishing of vibrations with time is called **damping**.

The extent of damping can be increased by the use of dashpots or dampers. In the discussions in this chapter, it is assumed that the damping force (F_d) is proportional to the **relative velocity of the vibrating body with respect to the damper** (for slow speeds). Usually at higher speeds, damping force is proportional to square of velocity of vibration but this is not in our scope of discussion. Hence, if dashpot (or damper) is fixed and the displacement of vibrating mass from its equilibrium position is x, the velocity of vibrating mass $v = \frac{dx}{dt} = x$, which gives damping force, $F_d \propto v \propto \dot{x}$

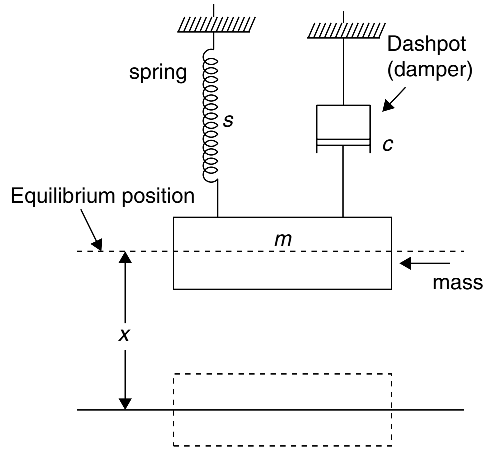
$\Rightarrow F_d = c\dot{x}$ where c = **damping coefficient** (or damping force per unit velocity). The SI unit of c is Ns/m (newton second per metre). If dashpot is not fixed but has a displacement x_1 at the instant when the vibrating body is displaced by x_2 from equilibrium position, then the relative velocity of the oscillating body with respect to damper is $(\dot{x}_2 - \dot{x}_1)$

$\therefore$ Damping force, $F_d \propto (\dot{x}_2 - \dot{x}_1)$

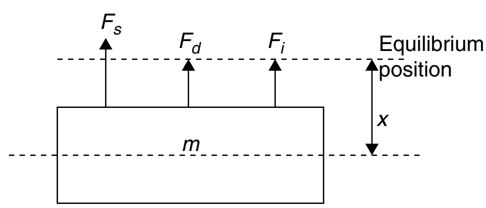
$\Rightarrow F_d = c(\dot{x}_2 - \dot{x}_1)$

The damping force opposes the relative motion of the vibrating body with respect to the damper. The value of c depends upon the dashpot type (like the size of restriction inside the dashpot, nature of fluid used, etc.)

The mathematical model of a damped free vibrating system consists of three elements (i) **Inertia element** which is represented by lumped mass for rectilinear motion and by lumped moment of inertia for angular motion (ii) **Restoring element** represented by massless linear springs for rectilinear motion and massless torsional springs for torsional motion, respectively (iii) **Damping element**, is usually represented by massless and rigid dashpots for energy dissipation. Such a model, vibrating in the vertical plane (linear vibrations) is shown below Figure.



The **mass m** is suspended from one end (lower end) of light **spring of stiffness *s***, with the **dashpot of damping coefficient *c*** between the mass and fixed support. The mass is disturbed from its equilibrium position and set into free vibrations. When the mass is displaced by a distance x below the equilibrium position, the various forces acting on the mass are shown in below Figure.



F_i = Inertial force $= m\frac{d^2x}{dt^2} = m\ddot{x}$ (upwards)

F_d = damping force $= c\frac{dx}{dt} = c\dot{x}$ (upwards)

F_s = spring force $= sx$ (upwards)

As per D'Alembert's principle,

$F_i + F_d + F_s = 0$

$\Rightarrow m\ddot{x} + c\dot{x} + sx = 0$

or $\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{s}{m}\right)x = 0$

i.e. $\frac{d^2x}{dt^2}+\left(\frac{c}{m}\right)\frac{dx}{dt}+\left(\frac{s}{m}\right)x=0$

is the second order **differential equation** for free damped vibration. We have seem earlier that $\left(\frac{s}{m}\right)=\omega_n^2$ for free vibration, where

ω_n = natural circular frequency of free vibration.

The solution to this second order differential equation is of the form $x=A\,e^{\alpha_1 t}+Be^{\alpha_2 t}$, where A and B are some constants that are evaluated by initial boundary conditions and α_1 and α_2 are the roots of the auxiliary equation

$$\alpha^2+\left(\frac{c}{m}\right)\alpha+\left(\frac{s}{m}\right)=0$$

$$\Rightarrow \alpha_{1,2}=\frac{-\frac{c}{m}\pm\sqrt{\left(\frac{c}{m}\right)^2-4\left(\frac{s}{m}\right)}}{2}$$

$$=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\frac{s}{m}}$$

$$\therefore\ \alpha_1=-\frac{c}{2m}+\sqrt{\left(\frac{c}{2m}\right)^2-\frac{s}{m}}\ \text{ and}$$

$$\alpha_2=-\frac{c}{2m}-\sqrt{\left(\frac{c}{2m}\right)^2-\frac{s}{m}}$$

We now define **damping coefficient** (ξ) as the ratio of square root of $\left(\frac{c}{2m}\right)^2$ *to* $\left(\frac{s}{m}\right)$, which **represents the degree of damping** provided in the system.

$$\therefore\ \xi=\sqrt{\frac{\left(\frac{c}{2m}\right)^2}{\left(\frac{s}{m}\right)}}\ ;\text{Also, }\sqrt{\frac{s}{m}}=\omega_n$$

$$\Rightarrow\ \xi=\frac{\left(\frac{c}{2m}\right)}{\sqrt{\frac{s}{m}}}=\frac{c}{2m\omega_n}=\frac{c}{2m\omega_n}=\frac{c}{c_c}\text{, where}$$

c_c = **critical damping coefficient**

$\Rightarrow \boldsymbol{c_c=2m\omega n}$

$\therefore$ Damping factor (ξ)

$$=\frac{\text{Damping coefficient(c)}}{\text{Critical damping coefficient}(\text{c}_\text{c})}=\frac{c}{2m\omega_n}$$

$$\therefore\ \frac{c}{2m}=\xi\omega_n\ \text{ and }\ \frac{s}{m}=\omega_n{}^2$$

Using the above values of $\frac{c}{2m}$ and $\frac{s}{m}$, equation for α_1 and α_2 become

$$\alpha_{1,2}=-\ \xi\omega_n\pm\sqrt{\xi^2-1}\ \omega_n$$

i.e. $\alpha_{1,2}=\omega_n\left[-\xi\pm\sqrt{\xi^2-1}\right]$

Depending upon the value of ξ, the roots α_1 and α_2 may be real, imaginary or complex. **The imaginary part of the root represents the extent of vibration present** in the system (i.e. **the circular frequency** of actual vibration), while the **real part of the root represents the extent of damping present** in the system.

When $\xi = 0$ (undamped system)

The roots $\alpha_{1,2}=\pm\,\mathrm{i}\omega_n$ implies **there is no real part of root** (i.e. no damping in the system) which is an **undampd system**. The imaginary part indicates that the system oscillates with a circular frequency of ω_n. The displacement equation is given by $x=A\cos\omega_n t$ which is a pure SHM, when $\xi=0$. The damping coefficient c = zero for the system

When $\xi = 1$ (critically damped system)

The roots are real and equal.

i.e. $\alpha_1=\alpha_2=-\omega_n$

The system is said to be **critically damped**.

As there is no imaginary component to the root, the system **will not vibrate** when released from disturbed position but will move to the equilibrium position and come to rest. The **time taken to come to rest will be the shortest** for critically damped system.

The displacement equation is given by $x=Ae^{-\omega_n t}$, which is an exponentially decaying curve.

The damping coefficient $c=c_c$ (critical damping coefficient) for this system.

When $\xi > 1$ (overdamped system)

The roots are $\alpha_{1,2}=-\xi\omega_n\pm\,\omega_n\sqrt{\xi^2-1}$. The roots are real and unequal. As there is no imaginary component for the root, **the system will not vibrate** when released from disturbed position but will move towards equilibrium position and come to rest there. **The time taken to come to rest** (from disturbed position to equilibrium position) **will be more** than that taken by a critically damped system (i.e. $\xi=1$ system). The displacement equation for this system is

$x=\mathrm{A}\,e^{\omega_n\left(-\xi+\sqrt{\xi^2-1}\right)t}+Be^{\omega_n\left(-\xi-\sqrt{\xi^2-1}\right)t}$ which represents a non-periodic motion. A and B are constants which depend upon initial conditions. The damping coefficient c is greater than c_c, i.e. $c>c_c$ for this system.

When $0 < \xi < 1$ (under damped system)

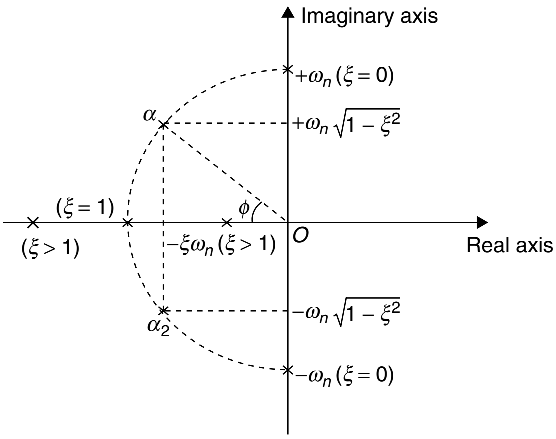
The roots are $\alpha_1=-\xi\omega_n+\mathrm{i}\omega_n\sqrt{1-\xi^2}$ and

$$\alpha_2=-\xi\omega_n-\mathrm{i}\omega_n\sqrt{1-\xi^2}$$

That is, the roots are complex conjugates. As there is an imaginary component, there will be some vibrations of decaying amplitude. The **system now vibrates** not at its natural frequency ω_n but at another frequency called as the **circular frequency of damped oscillation** (ω_d), where

$$\omega_d=\omega_n\sqrt{1-\xi^2}$$

This is the circular frequency of under damped vibrations.



In the above figure shows how α_1 and α_2 move in the complex plane as ξ is increased from zero. If ϕ is the angle made by the position vector of roots (line connecting origin O and the root α) with imaginary axis,

$$\cos\phi = \frac{\xi\omega_n}{\omega_n} = \xi \text{ or } \phi = \cos^{-1}(\xi)$$

$$\text{Also, } \tan\phi = \frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n} = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\therefore\ \phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

The displacement equation for under damped system is given by $x = \left(X_0 e^{-\xi\omega_n t}\right)\sin(\omega_d t + \phi)$, where

X_0 = amplitude of oscillation at time $t = 0$

$\omega_d = \omega_n\sqrt{1-\xi^2}$, is the circular frequency of damped oscillation

$$\phi = \cos^{-1}(\xi) \text{ or } \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

$X_0 e^{-\xi\omega_n t}$ = amplitude of oscillation at time t

Clearly, the amplitude is decaying exponentially with time but the system is also vibrating with decreasing amplitude.

T_d = time period of damped vibration

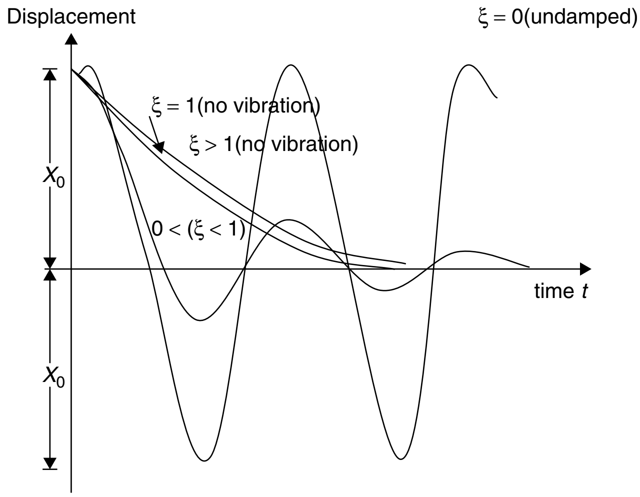
$$\Rightarrow \boldsymbol{T_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

The linear frequency of damped vibration,

$$\boldsymbol{f_d} = \frac{1}{T_d} = \frac{\omega_n\sqrt{1-\xi^2}}{2\pi}$$

For under damped system, $c < c_c$

The following figure shows the variation of displacement with time as damping factor (ξ) is increased from zero.



NOTES

The following formulae are useful in solving numerical problems.

1. Differential equation for free damped vibration is $\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{s}{m}\right)x = 0$ which can also be written as
$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$
$\left(\because \frac{c}{m} = 2\xi\omega_n \ \frac{s}{m} = \omega_n^2\right)$
2. Critical damping coefficient,
$c_c = 2m\omega_n = 2\sqrt{ms}$
3. Damping factor $\xi = \frac{c}{c_c} = \sqrt{\frac{\left(\frac{c}{2m}\right)^2}{\left(\frac{s}{m}\right)}} = \frac{c}{2\sqrt{ms}} = \frac{c}{2m\omega_n}$

Amplitude Reduction Factor

In under damped free vibrations, the ratio of the amplitude of two successive oscillations on the same side of mean position is called **amplitude reduction factor**.

Let x_1 be the amplitude of under damped free oscillation at time t and x_m be the amplitude m cycles later. If $T_d = \frac{2\pi}{\omega_d}$ is the time period of damped oscillations, the time taken for m cycles will be $t' = mT_d = \frac{2\pi m}{\omega_d} = \frac{2\pi m}{\omega_n\sqrt{1-\xi^2}}$

If X_0 is the amplitude at time $t = 0$, ξ is the damping factor and ω_n = natural frequency of undamped oscillations, we have

$$x_1 = X_0\, e^{-\xi\omega_n t} \text{ and}$$

$$x_m = X_0\, e^{-\xi\omega_n(t+t')} = X_0\ e^{-\xi\omega_n(t+mT_d)}$$

$$\therefore\ \frac{x_1}{x_m} = \frac{X_0 e^{-\xi\omega_n t}}{X_0 e^{-\xi\omega_n(t+mT_d)}} = e^{\xi\omega_n m T_d}$$

$$= e^{\xi\omega_n m.\frac{2\pi}{\omega_d}} \quad \left(\because T_d = \frac{2\pi}{\omega_d}\right)$$

$$= e^{\xi\omega_n m \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}} \quad \left(\because \omega_d = \omega_n\sqrt{1-\xi^2}\right)$$

$$= e^{\frac{2\pi\xi m}{\sqrt{1-\xi^2}}}$$

i.e. $\dfrac{x_1}{x_m} = e^{\frac{2\pi\xi m}{\sqrt{1-\xi^2}}}$

For two successive oscillations, $m = 1$

$\Rightarrow$ **Amplitude reduction factor** $= \dfrac{x_1}{x_2}$

$$= e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$

Logarithmic Decrement

The natural logarithm of the ratio of any two successive amplitudes on the same side of the mean position in an under damped system is called **logarithmic decrement**. For an under damped system, it is always a constant and is denoted as δ.

i.e. **Natural logarithm of amplitude reduction factor is called logarithmic decrement**.

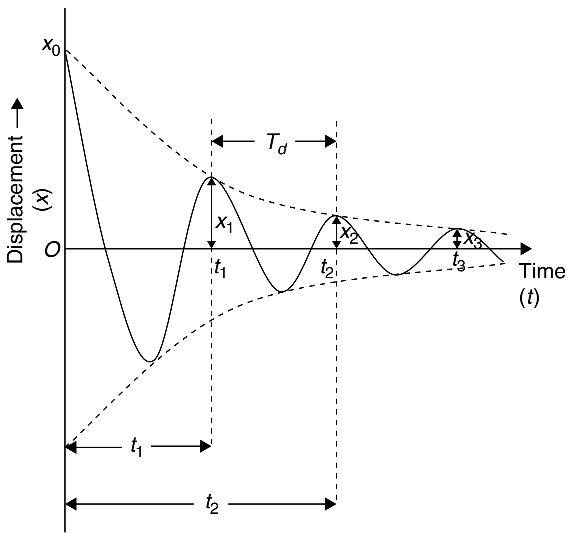
$$\therefore\ \delta = \ell n\left(\frac{x_1}{x_2}\right) = \ell n\left(\frac{2\pi\xi}{e^{\sqrt{1-\xi^2}}}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \xi\omega_n T_d$$

We can also write

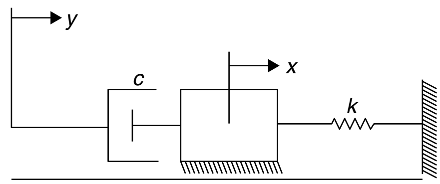
$$\ell n\left(\frac{x_1}{x_m}\right) = \ell n\left(\frac{2\pi\xi m}{e^{\sqrt{1-\xi^2}}}\right) = \frac{2\pi\xi m}{\sqrt{1-\xi^2}} = m\delta$$

If $\xi << 1$, $\delta = \mathbf{2\pi\xi}$ $\left(\because \sqrt{1-\xi^2} \approx 1\right)$

The displacement *Vs* time graph of an under damped free vibration with amplitudes at various instants is shown in below figure.



Example 28: The differential equation governing the vibrating system is



(A) $m\ddot{x} + c\ddot{x} + k(x - y) = 0$
(B) $m(\ddot{x} - \ddot{y}) + c(\dot{x} - \dot{y}) + kx = 0$
(C) $m\ddot{x} + c(\dot{x} - \dot{y}) + kx = 0$
(D) $m(\ddot{x} - \ddot{y}) + c(\dot{x} - \dot{y}) + k(x - y) = 0$

Solution:
The forces on the mass are
F_i = inertial force
$= m\ddot{x}$ ($\because$ Acceleration of mass is $\ddot{x}$)
F_s = restoring force of spring
$= kx$ (depends only on extension or compression of spring)
F_d = damping force
$= c \times$ relative velocity of mass with respect to dashpot
$= c\ (\dot{x} - \dot{y})$
As per D'Alembert's, Principle,
$F_i + F_d + F_s = 0$ for free damped vibration
$\Rightarrow m\ddot{x} + c(\dot{x} - \dot{y}) + kx = 0$.

Example 29: Critical damping is the
(A) largest amount of damping for which no oscillation occurs in free vibration.
(B) smallest amount of damping for which no oscillation occurs in free vibration.
(C) largest amount of damping for which the motion is simple harmonic in free vibration.
(D) smallest amount of damping for which the motion is simple harmonic in free vibration.

Solution:
Damping factor $\xi \geq 1$, no vibration occurs. $\xi = 1$ is called critical damping, which is the smallest amount of damping for which no oscillation occurs in free vibration.

Example 30: The damping ratio of a single degree of freedom spring-mass-damper system with mass of 1 kg, stiffness 100 N/m and viscous damping coefficient of 25 Ns/m is

Solution: Given $c = 25\ \dfrac{\text{N s}}{\text{m}}$

$s = 100$ N/m
$m = 1$ kg
$c_c = 2m\omega_n = 2\sqrt{ms} = 2\sqrt{1 \times 100}$

$$= 20\ \frac{\text{N s}}{\text{m}}$$

Damping factor $\xi = \dfrac{c}{c_c} = \dfrac{25}{20} = 1.25$

Example 31: A vehicle suspension system consists of a spring and a damper. The stiffness of the spring is 3.6 kN/m and the damping constant of the damper is 400 Ns/m. If the mass is 50 kg, then the damping factor (ξ) and damped natural frequency (f_d) respectively are

(A) 0.471 and 1.19 Hz (B) 0.471 and 7.48 Hz
(C) 0.666 and 1.35 Hz (D) 0.666 and 8.50 Hz

Solution:

Given m = 50 kg

$$s = 3.6 \times 10^3 \text{ N/m}$$
$$c = 400 \text{ Ns/m}$$
$$c_c = 2m\omega_n = 2\sqrt{ms}$$
$$= 2\sqrt{50 \times 3.6 \times 10^3}$$
$$= 848.53 \text{ Ns/m}$$

$$\therefore \; \xi = \frac{c}{c_c} = \frac{400}{848.53} = 0.471$$

$$f_d = \frac{\omega_d}{2\pi} = \frac{\omega_n\sqrt{1-\xi^2}}{2\pi}$$
$$= \frac{1}{2\pi}\sqrt{\frac{s}{m} \times \left(1-\xi^2\right)}$$
$$= \frac{1}{2\pi}\sqrt{\frac{3.6 \times 10^3}{50} \times \left(1-0.471^2\right)}$$
$$= 1.19 \text{ Hz.}$$

Hence, the damping factor is 0.471 and damped natural frequency of vibration is 1.19 Hz.

Example 32: In a spring-mass system, the mass is 0.1 kg and the stiffness of the spring is 1 kN/m. By introducing a damper, the frequency of oscillation is found to be 90% of the original value. What is the damping coefficient of the damper?

(A) 1.2 Ns/m (B) 3.4 Ns/m
(C) 8.7 Ns/m (D) 12.0 Ns/m

Solution:

Given $m = 0.1$ kg

$s = 1 \times 10^3$ N/m

$\omega_d = 0.9\ \omega_n$

We have $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\Rightarrow 0.9\ \omega_n = \omega_n\sqrt{1-\xi^2}$

$\Rightarrow (0.9)^2 = 1 - \xi^2$

$\Rightarrow \xi^2 = 1 - 0.9^2 = 0.19$

$\therefore\ \xi = \sqrt{0.19} = 0.4359$

$c_c = 2\sqrt{ms} = 2\sqrt{0.1 \times 1 \times 10^3}$

$= 20$ Ns/m

$\xi = \dfrac{c}{c_c}$

$\Rightarrow c = \xi c_c$

$= 0.4359 \times 20$

$= 8.718$ Ns/m

$\therefore\ c = 8.7$ Ns/m

Example 33: The suspension system of a two-wheeler can be equated to a single spring-mass system with a viscous damper connected in series. For a mass m = 50 kg and a spring with a stiffness of 35 kN/m, the damping coefficient of the damper provides critical damping. The damping force for a plunger velocity of 0.05 m/s (expressed in newton) is ________

Solution: $m = 50$ kg

$s = 35 \times 10^3$ N/m

$c_c = 2\sqrt{ms} = 2\sqrt{50 \times 35 \times 10^3}$

$= 2645.75$ Ns/m

Critically damped $\rightarrow \xi = 1$

$\Rightarrow c = c_c$

$v = 0.05$ m/s

$\therefore\ F_d = cv = c_c v = 2645.75 \times 0.05$

$= 132.29$

$= 132.3$ N.

Example 34: The equation of motion of a harmonic oscillator is given by $\dfrac{d^2x}{dt^2} + 2\xi\omega_n\dfrac{dx}{dt} + \omega_n{}^2 x = 0$ and the initial conditions at $t = 0$ are $x(0) = X$, $\dfrac{dx}{dt}(0) = 0$. The amplitude of $x(t)$ after n complete cycles is

(A) $Xe^{-2n\pi\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)}$ (B) $Xe^{2\pi n\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)}$
(C) $Xe^{-2n\pi\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}$ (D) X

Solution:

It is an under damped oscillator

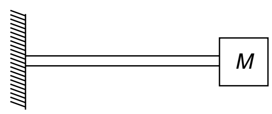
$\rightarrow$ amplitude decreases

$\rightarrow$ (B) and (D) are not answers

We have $\dfrac{X}{x_n} = e^{\frac{2\pi n\xi}{\sqrt{1-\xi^2}}}$

$$\Rightarrow x_n = \frac{X}{e^{\left(2\pi n\xi/\sqrt{1-\xi^2}\right)}}$$
$$= Xe^{-2n\pi\left(\xi/\sqrt{1-\xi^2}\right)}$$

Example 35:



A mass M, of 20 kg, is attached to the free end of a steel cantilever beam of length 1000 mm having a cross section of 25 mm × 25 mm. Assume the mass of cantilever to be negligible and E_{steel} = 200 GPa. If the lateral vibration of the system is critically damped using a viscous damper, the damping constant of the damper is

(A) 1250 Ns/m (B) 625 Ns/m
(C) 312.50 Ns/m (D) 156.25 Ns/m

Solution:
The static deflection of a cantilever beam with concentrated load $W = Mg$ at free end is given by

$$\delta = \frac{MgL^3}{3EI}$$

$$\Rightarrow \left(\frac{3EI}{L^3}\right)\delta = Mg$$

$\therefore$ Lateral stiffness, $s = \frac{3EI}{L^3}\frac{1}{2}$

Here, $E = 200 \times 10^9\ Pa$

$$I = \frac{a^4}{12} = \frac{\left(25\times10^{-3}\right)^4}{12}$$
$$= 3.2552 \times 10^{-8}\ \text{m}^4$$

$L = 1$ m

$$\therefore\ s = \frac{3EI}{L^3}$$
$$= \frac{3\times200\times10^9\times3.2552\times10^{-8}}{1^3}$$
$$= 19{,}531.2\ \text{N/m}$$

$M = 20$ kg

$$c_c = 2\sqrt{Ms} = 2\sqrt{20\times19531.2}$$
$$= 1249.99$$
$$= 1250\ \text{Ns/m}$$

$\xi = 1$ ($\because$ critically damped) $= \frac{c}{c_c}$

$\therefore\ c = c_c = 1250$ Ns/m

Direction for questions (Examples 36 and 37): A vibratory system consists of a mass 12.5 kg, a spring of stiffness 1000 N/m and a dashpot with damping coefficient of 15 Ns/m

Example 36: The value of critical damping coefficient of the system is

(A) 0.223 Ns/m (B) 17.88 Ns/m
(C) 71.4 Ns/m (D) 223.6 Ns/m

Solution:
$m = 12.5$ kg
$s = 1000$ N/s

$$c_c = 2\sqrt{ms} = 2\times\sqrt{12.5\times1000}$$
$$= 223.6\ \text{Ns/m}$$

Example 37: The value of logarithmic decrement is

(A) 1.35 (B) 0.42 (C) 0.68 (D) 0.66

Solution:
$c = 15$ Ns/m

$$\therefore\ \xi = \frac{c}{C_c} = \frac{15}{223.6} = 0.067$$

$$\delta = \ell n\left(\frac{x_1}{x_2}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$= \frac{2\pi\times0.067}{\sqrt{1-(0.067)^2}} = 0.421$$

Example 38: Two consecutive oscillations of an under damped vibrating system were found to have amplitudes of 3 mm and 0.5 mm, respectively. The logarithmic decrement (δ) and damping factor (ξ) are respectively

(A) 0.322, 1.792 (B) 3.561, 0.644
(C) 2.79, 0.61 (D) 1.792, 0.274

Solution:
$x_1 = 3$ mm
$x_2 = 0.5$ mm

$$\therefore\ \delta = \ell n\left(\frac{x_1}{x_2}\right) = \ell n\left(\frac{3}{0.5}\right)$$
$$= \ell n 6 = 1.792$$

We have $\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$

$$\Rightarrow 1.792 = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\Rightarrow (1.792)^2 = \frac{4\pi^2\xi^2}{\left(1-\xi^2\right)}$$

$$\Rightarrow 1 - \xi^2 = 4\pi^2\xi^2/(1.792)^2 = 12.294\ \xi^2$$

$$\Rightarrow 1 = 13.294\ \xi^2$$

$$\Rightarrow \xi = \sqrt{\frac{1}{13.294}} = 0.274$$

$\therefore\ \delta = 1.792$ and $\xi = 0.274$.

Example 39: A mass of 10 kg is suspended on a massless spring and set into vertical oscillations. It is observed that the amplitude reduces to 10% of its initial value after 4 oscillations. It takes 0.8 second to do that. Calculate the following.

(i) The actual frequency (in Hz) of the system.
(ii) The damping factor (ξ).
(iii) The natural frequency of oscillation (in Hz).
(iv) The spring stiffness (in kN/m)
(v) The critical damping coefficient c_c (in Ns/m)
(vi) The actual damping coefficient c (in Ns/m)

Solution:
(i) Number of vibrations in 0.8 s = 4

$$\therefore\ \text{Actual frequency } f_d = \frac{4}{0.8} = 5\ \text{Hz}$$

(ii) Amplitude after 4 oscillations
= 10% of amplitude at beginning

$$\Rightarrow x_4 = 0.1\ x_1$$

$$\Rightarrow \frac{x_1}{x_4} = \frac{1}{0.1} = 10$$

$$\therefore\ \ell n\left(\frac{x_1}{x_4}\right) = \ell n 10 = 2.3026$$

But $\ell n\left(\frac{x_1}{x_4}\right) = \frac{2\pi\xi\times4}{\sqrt{1-\xi^2}}$

$$\left(\because \ell n\left(\frac{x_1}{x_m}\right) = \frac{2\pi\xi m}{\sqrt{1-\xi^2}}\right)$$

$$\Rightarrow 2.3026 = \frac{8\pi\xi}{\sqrt{1-\xi^2}}$$

$$\Rightarrow (2.3026)^2 = \frac{64\pi^2\xi^2}{\left(1-\xi^2\right)}$$

$$\Rightarrow 5.302 - 5.302\,\xi^2 = 631.65\,\xi^2$$

$$\Rightarrow 636.952\,\xi^2 = 5.302$$

$$\Rightarrow \xi = \sqrt{\frac{5.302}{636.952}} = 0.0912$$

$\therefore$ Damping factor (ξ) = 0.0912

(iii) We have $f_d = f_n\sqrt{1-\xi^2}$

$$\Rightarrow f_n = \frac{f_d}{\sqrt{1-\xi^2}} = \frac{5}{\sqrt{1-0.0912^2}}$$

$= 5.02$ Hz

Hence, natural frequency of system, $f_n = 5.02$ Hz.

(iv) We have $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$

$\Rightarrow k = 4\pi^2 f_n^2 m$ ($m = 10$ kg)

$\Rightarrow k = 4\pi^2 \times (5.02)^2 \times 10$

$= 9949$ N/m

$= 9.95$ kN/m

$\therefore$ Stiffness of springs,

$k = 9.95$ kN/m

(v) Critical damping coefficient,

$c_c = 2m\omega_n\left(\text{or } 2\sqrt{mk}\right)$

$= 2 \times 10 \times 5.02 \times 2\pi$

$= 630.83$ Ns/m

(vi) Actual damping coefficient,

$$c = \xi c_c \quad \left(\because \xi = \frac{c}{c_c}\right)$$

$= 0.0912 \times 630.83$

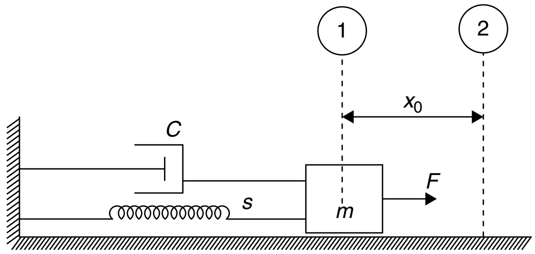
$= 57.5317$ Ns/m.

Forced Vibrations

When the vibration of system is maintained by an external excitation, it is called **forced vibration**. When an external force is applied on the system, the system will respond to the force. The nature of response depends on the type of applied force. There are usually two types of forces that can be applied on a system.

1. Step-input force (or constant force)
2. Harmonic force. We will look at the response of single degree freedom system when these forces are applied.

Step-input force (i.e. force of constant magnitude and constant direction)



Smooth floor

Consider a block of mass m on a smooth horizontal floor, connected to a light spring of stiffness s and a damper of damping coefficient c. In position 1, there is no extension of spring and no velocity for mass, so there is no damping force or spring force in the horizontal direction. Hence, position 1 is the initial equilibrium position. Now, a constant horizontal force F applied to the right on block and slowly the mass moves to position 2, where it comes to rest. Hence damper force is zero but spring force = sx_0 to the left, where x_0 = displacement of block from position 1 to position 2. In this position,

$$sx_0 = F \tag{1}$$

Hence, position 2 represents the shifted equilibrium position of the system. If a disturbing force F_0 is now applied to the right and removed, the system will execute damped vibrations under the force F. At any distance x to the right from position 2 (which is the new equilibrium position), the forces on the block in the horizontal direction are spring force,

$F_s = s(x_0 + x)$, to the left

Damping force, $F_d = c\dfrac{dx}{dt} = c\dot{x}$ to the left

Inertia force, $F_i = m\dfrac{d^2x}{dt^2} = m\ddot{x}$ to the left,

Step-input force F to the right

For dynamic equilibrium, $F_i + F_d + F_s = F$

$$\Rightarrow m\ddot{x} + c\dot{x} + s\left(x_0 + x\right) = F \tag{2}$$

Using equation 1 in equation 2, we get

$$m\ddot{x} + c\dot{x} + sx = 0$$

$[\because sx_0 = F \text{ from equation (1)}]$

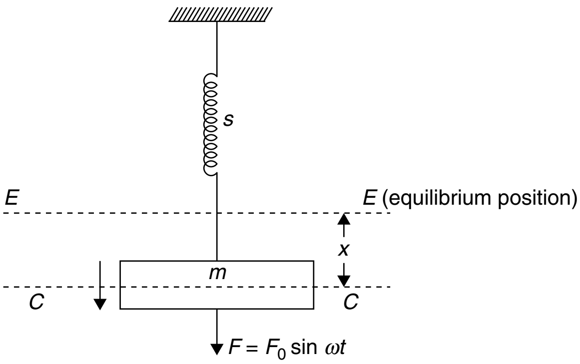
which is the same as the differential equation for **free-damped oscillation** for the same spring mass system. Hence, we can conclude that **a step-input force (or an external force of constant magnitude and direction)only shifts the equilibrium position and results in free-damped oscillations**.

NOTE

When a spring-mass-damper system is executing vertical oscillations, a step-input force equal to the weight of the block ($W = mg$) is acting vertically downwards on the system, which determines the equilibrium position of the system only. Other characteristics of the oscillating system are not affected by this step-input force.

Harmonic excitation (or an external force $F = F_0 \sin\omega t$) with no damping

Consider a block of mass m, suspended from a massless spring of stiffness s, subjected to a harmonic force $F = F_0 \sin\omega t$. At time t, let this block be at a distance x below the equilibrium position E-E. This is as shown in below Figure.



At the equilibrium position, $t = 0 \Rightarrow F = 0$, so $sx_0 = mg$. The forces on the block are,

Inertial force $F_1 = m\dfrac{d^2x}{dt^2} = m\ddot{x}$, acting upwards, spring force $F_s = sx$ ($\because sx_0$ cancels weight mg), acting upwards and

Applied force $F = F_0 \sin\omega t$, acting downwards

$\therefore$ For dynamic equilibrium, $F_i + F_s = F$

$\Rightarrow m\ddot{x} + sx = F_0 \sin\omega t$ is the differential equation for undamped forced vibration under harmonic excitation.

The solution of this differential equation will be of the form

$$x = X_0 \sin(\omega_n t + \phi) + \frac{\left(F_0/s\right)}{1-\left(\dfrac{\omega}{\omega_n}\right)^2} \sin\omega t \text{, where}$$

X_0 = amplitude of free vibration of system

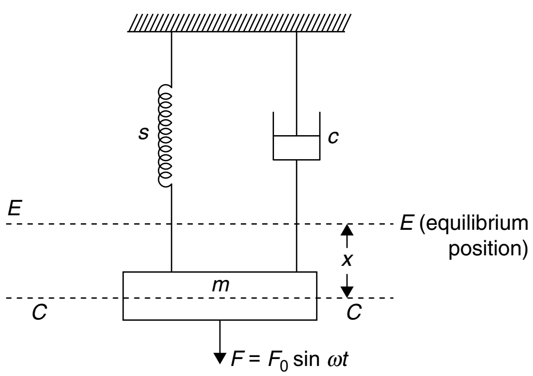
$\omega_n = \sqrt{\dfrac{s}{m}}$

= circular natural frequency of system

ω = circular frequency of exciting force (applied force)

ϕ = phase difference between applied force and resulting free vibration.

Harmonic excitation with damper (External force, $F = F_0 \sin\omega t$)



Consider a massless spring of stiffness s and a dashpot of damping coefficient c, suspended from a **fixed support**. Their free ends are connected to a block of mass m. The block is subjected to a harmonic force $F = F_0 \sin\omega t$. The mass is constrained to move only up or down (i.e. single degree of freedom). At time t, the block is displaced downwards by x from its mean position (EE). For dynamic equilibrium,

$\overline{F_i} + \overline{F_d} + \overline{F_s} + \overline{F} = 0$, where

$\overline{F_i}$ = inertial force on block

$= m\dfrac{d^2x}{dt^2} = m\ddot{x}$, acting upwards

$\overline{F_d}$ = damper force on block $= c\dfrac{dx}{dt} = c\dot{x}$, acting upwards

$\overline{F_s}$ = spring force = sx, acting upwards

$\overline{F}$ = $F_0 \sin\omega t$, acting downwards

Hence, their magnitudes are related as

$F_i + F_d + F_s = F$

$\Rightarrow m\ddot{x} + c\dot{x} + sx = F_0 \sin\omega t$, is the differential equation for **forced damped oscillation**. This is a second degree differential equation, the solution of which is of the form $x = x_c + x_p$, where

x_c = complementary solution, which represents the transient part of the forced vibrations, given by the solution of the equation $m\ddot{x} + c\dot{x} + sx = 0$, already dealt with in free-damped oscillation

$$\therefore x_c = X_0 e^{-\xi\omega_n t} \sin(\omega_d t + \varphi)$$

and x_p = particular solution, which represents the steady state part of the forced vibrations.

It can be shown that

$$x_p = \frac{F_0(\sin\omega t - \varphi)}{\sqrt{\left(s - m\omega^2\right)^2 + (c\omega)^2}}$$

Hence, in the steady state, the frequency of forced vibrations is same as the frequency of external applied force (i.e. excitor frequency).

The amplitude of forced vibration in the steady state,

$$A = x_{\max} = \frac{F_0}{\sqrt{\left(s - m\omega^2\right)^2 + (c\omega)^2}}$$

$$\Rightarrow A = \frac{(F_0/s)}{\sqrt{\left[1-\left(\dfrac{m}{s}\right)\omega^2\right]^2 + \left(\dfrac{c}{s}\omega\right)^2}}$$

$$\left[\because \frac{m}{s} = \frac{1}{\omega_n^2}\right]$$

$$= \frac{(F_0/s)}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(\dfrac{c}{s}\omega\right)^2}} \quad \left[\because \frac{c}{s} = \frac{2\xi}{\omega_n}\right]$$

$$= \frac{(F_0/s)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$[\ F_0/s]\ \because$ static deflection of spring, $\delta = [\ F_0/s]$

$$\Rightarrow A = \frac{\delta}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}},$$

where $\delta = \frac{F_0}{s}$, is static deflection of spring

ω = circular frequency of external force (excitor)

ω_n = natural frequency of undamped oscillation of the system

ξ = damping factor = $\frac{c}{c_c}$ of the system

This equation can be used only for steady state vibrations. Also, ϕ = phase difference between the applied force and the displacement of system in steady state such that

$$\Rightarrow \tan\phi = \frac{c\omega}{(s-m\omega^2)}$$

$$\text{Also, } \tan\phi = \left(\frac{c}{s}\right)\frac{\omega}{\left(1-\frac{m}{s}\omega^2\right)} = \frac{2\xi}{\omega_n}\cdot\frac{\omega}{\left(1-\frac{\omega^2}{\omega_n^2}\right)}$$

$$\left[\because \frac{m}{s} = \frac{1}{\omega_n^2}\right]$$

$$\Rightarrow \tan\phi = \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

The above expressions are for the **amplitude (A) and phase (ϕ) of steady state forced-damped vibrations.**

The ratio of maximum displacement (i.e. amplitude of steady state vibration, A) of the steady state forced vibration to the static deflection (δ) is known as **magnification factor (MF)**.

$$\therefore MF = \frac{A}{\delta} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

Hence, amplitude of steady state forced vibrations = static deflection × magnification factor

$$\Rightarrow A = \delta(MF), \text{ where } \delta = \frac{F_0}{S}$$

NOTES

1. When there is no damping (i.e. $\xi = 0$), magnification factor

$$\text{MF} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} = \frac{\omega_n^2}{(\omega_n^2-\omega^2)}$$

2. When there is no damping (i.e. $\xi = 0$) and the frequency of excitor is equal to the natural frequency of oscillator (i.e. $\omega = \omega_n$), this condition is called **resonance. The magnification factor at undamped resonance is infinite (∞).**

3. **When there is damping (i.e. $\xi \neq 0$), maximum magnification factor does not occur at $\omega = \omega_n$. At $\xi \neq 0$ and $\omega = \omega_n$,** $MF = \frac{1}{2\xi}$

4. When there is damping (i.e. $\xi \neq 0$), the maximum amplitude of steady state forced vibration occurs at an excitor frequency $\omega = \omega_r$, where $\omega_r = \omega_n\sqrt{1-2\xi^2}$, where $0 < \xi < \frac{1}{\sqrt{2}}$

   This can be easily proved as follows. As

$$MF = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}},$$

   MF will be maximum, when

   $\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2$ is minimum . Put $\frac{\omega}{\omega_n} = r$

   $\therefore MF$ is maximum, when

   $[1-r^2]^2 + (2\xi r)^2$ is minimum.

   Differentiating this function with respect to r and equating it to zero (condition for maxima or minima), we get

$$\frac{d\{(1-r^2)^2+(2\xi r)^2\}}{dr}$$

$$= 2(1-r^2)(-2r) + 8\xi^2 r = 0$$

$$\Rightarrow r = \sqrt{1-2\xi^2}$$

$$\text{i.e. } \frac{\omega}{\omega_n} = \sqrt{1-2\xi^2}$$

$$\Rightarrow \omega = \omega_n\sqrt{1-2\xi^2}$$

   $\therefore \omega_r < \omega_n$ for $MF_{\max}$, when $\xi \neq 0$ and $\xi < \frac{1}{\sqrt{2}}$

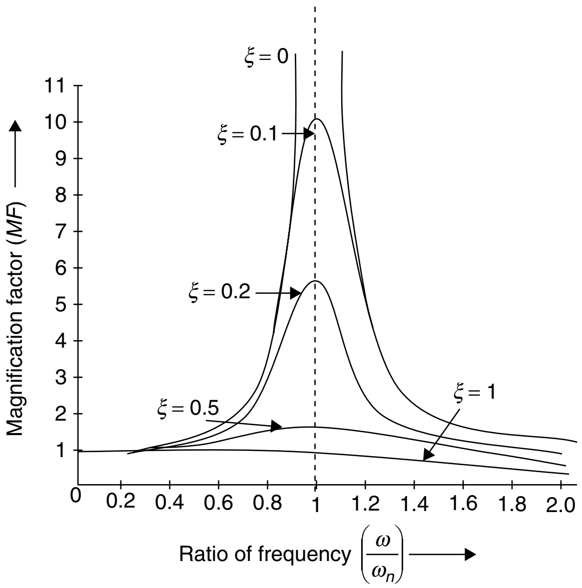
5. If $\xi > \frac{1}{\sqrt{2}}$, there is no solution for ω_r. By substituting $\omega = \omega_n\sqrt{1-2\xi^2}$, we can show that $MF_{\max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$, when $\xi \neq 0$ and

$\omega = \omega_n\sqrt{1-2\xi^2}$ and $0 < \xi < \dfrac{1}{\sqrt{2}}$.

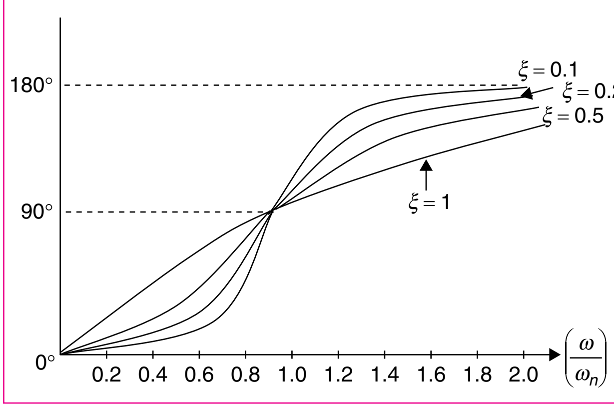
At $\xi = 0$ (undamped), $MF = \dfrac{1}{1-\left(\dfrac{\omega}{\omega_n}\right)^2}$. As ξ is incre-ased, MF keeps on decreasing.

At $\xi = 1$, $MF = \dfrac{1}{1+\left(\dfrac{\omega}{\omega_n}\right)^2}$

The variation of magnification factor (MF) with ratio of frequencies $\left(\dfrac{\omega}{\omega_n}\right)$ for various values of ξ are shown in below Figure.



The relation between phase angle (ϕ) in the steady state of forced vibrations to the angular frequency ratio $\left(\dfrac{\omega}{\omega_n}\right)$ for various values of damping factor (ξ) is shown



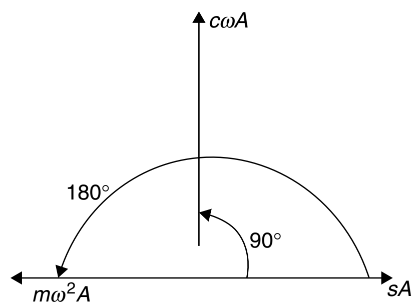
Graphical Method of Determining Maximum Amplitude (A) and Phase (ϕ) in Steady State of Forced-Damped-Vibration

The applied force at time t is $F = F_0 \sin\omega t$, hence the maximum force applied is of magnitude F_0 at an excitor frequency ω. Let us assume the displacement in steady state at that instant is $x = A\sin(\omega t - \phi)$, where A = amplitude of displacement, ϕ is the **phase lag** of the displacement with the applied force.

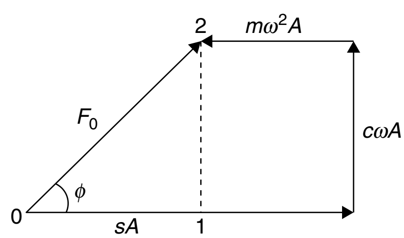
F_s = spring force $= -sx = -sA\sin(\omega t - \phi)$, with maximum spring force $= sA$ in magnitude

F_d = Damper force $= -cv = -s\omega A(\cos\omega t - \phi)$, with maximum damper force $= s\omega A$ in magnitude and leading the spring force by $\dfrac{\pi}{2}$ rad (i.e. 90° lead)

F_i = inertial force $= -m\dfrac{d^2x}{dt^2} = +m\omega^2 A\sin(\omega t - \phi)$ with maximum value of $m\omega^2 A$, leading the spring force by π rad (i.e. 180° lead) The maximum values of spring force, damper force and inertial force (in magnitude and directions) on the block of mass m are as shown below.



The maximum value of applied force (F_0) is equal and opposite to the sum of F_i, F_d and F_s $\left(\because \overline{F_i} + \overline{F_d} + \overline{F_s} = \overline{F_0}\right)$. This is shown in the vector diagram below.



From the right angled triangle 012, we have

$$F_0^2 = (sA - m\omega^2 A)^2 + (c\omega A)^2$$

$$= A^2\left\{\left[s - m\omega^2\right]^2 + (c\omega)^2\right\}$$

$$\Rightarrow F_0 = A\sqrt{\left(s - m\omega^2\right)^2 + (c\omega)^2}$$

$$\Rightarrow A = \frac{F_0}{\sqrt{\left(s - m\omega^2\right)^2 + (c\omega)^2}}$$

, which is the same expression for amplitude obtained earlier by the differential equation method.

$$\Rightarrow A = \frac{F_0}{\sqrt{\left(s - m\omega^2\right)^2 + (c\omega)^2}}$$

$$= \frac{(F_0/s)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)^2\right]}}, \text{ where}$$

$F_0/s = \delta$, the static deflection of the spring.
From triangle 012, we have

$$\tan\phi = \frac{c\omega A}{\left(sA - m\omega^2 A\right)} = \frac{c\omega}{\left(s - m\omega^2\right)}$$

$$= \left(\frac{c}{s}\right)\frac{\omega}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$\Rightarrow \tan\phi = \frac{2\xi\left(\omega/\omega_n\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]} \text{ same as calculated earlier.}$$

NOTES

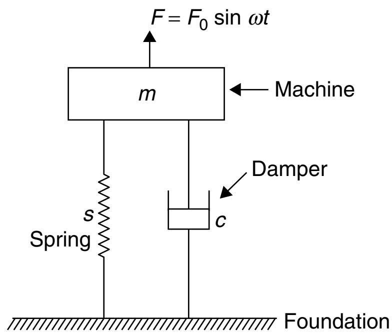
1. When $\omega = \omega_n$ in force-damped oscillation (steady state),
   $m\omega^2 A = m\omega_n^2 A = m\,\frac{s}{m}\,A = sA$
   $\therefore\ \tan\phi = \frac{c\omega A}{(sA - sA)} = \infty$
   $\Rightarrow \phi = 90^\circ\left(or\ \frac{\pi}{2}\text{rad}\right)$
   i.e. **When $\omega = \omega_n$ in force-damped oscillation, the displacement** of the vibrating body **lags** the **applied force** by $\frac{\pi}{2}$ radian.
2. If $\xi = 0$ (i.e. undamped), $\phi = 0^\circ \Rightarrow$ displacement of vibrating body and applied force are always in phase.
3. If $\omega > \omega_n$, $\phi > \frac{\pi}{2}$ rad, if $\omega < \omega_n$, $\phi < \frac{\pi}{2}$ rad when $\xi \neq 0$ (i.e. system has some damping).
4. If $\omega = \omega_n\left(\sqrt{1-2\xi^2}\right)$, i.e. when magnification factor is maximum in forced-damped vibrations $\left(0 < \xi < \frac{1}{\sqrt{2}}\right)$, then
   $$\tan\phi = \frac{\sqrt{1-2\xi^2}}{\xi} \Rightarrow \phi < 90^\circ.$$

Vibration Isolation

Whenever machines having unbalanced masses are operated, vibrations will be produced in such machines. These vibrations get transmitted to the foundation/ support/ structure on which such machines are installed. However, if springs and dampers are used in between the machines and their foundations/supports/structures, the vibrations will be transmitted through the springs and dampers. Also, other vibration isolation materials (like anti-vibration pads, etc.) can also be used between machines and their supports/foundations. The process of reducing the vibration transmitted from machines to foundations/supports/structures is called **vibration isolation**.

Transmissibility (ε)

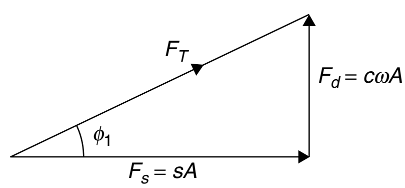
The ratio of the magnitude of the maximum force transmitted to the foundation to the maximum exciting force applied on the machine is called as **transmissibility**. It is usually denoted by the symbol ε.



Consider the system shown in the above Figure. When an exciting force $F = F_0 sin\omega t$ acts on the machine of mass m, a force is transmitted to the foundation through the spring (of stiffness s) and the damper (of damping coefficient c). If A is amplitude of vibration of the body and ω is the circular frequency of exciter (i.e. the frequency of external force on the machine), the maximum spring force $F_s = sA$ and maximum damper force, $F_d = c\omega A$. Then two forces are having a phase difference of $\frac{\pi}{2}$ rad (i.e. 90°) with each other with F_d leading F_s. The resultant force transmitted to the foundation (F_T) is the vector sum of F_s and F_d.

i.e. $\bar{F}_T = \bar{F}_s + \bar{F}_d$

Graphically, this can be represented as shown below.



$\therefore$ Resultant force transmitted to foundation,

$$F_T = \sqrt{F_s^2 + F_d^2}]$$

$$\Rightarrow F_T = \sqrt{(sA)^2 + (cA\omega)^2}$$

$$\Rightarrow F_T = A\sqrt{s^2 + c^2\omega^2}$$

But we knew $A = \frac{F_0}{\sqrt{\left(s - m\omega^2\right)^2 + (c\omega)^2}}$ from earlier discussions. Hence,

$$F_T = \frac{F_0}{\sqrt{\left(s - m\omega^2\right)^2 + (c\omega)^2}} \times \sqrt{s^2 + c^2\omega^2}$$

$$= \frac{F_0\sqrt{1 + \left(\frac{c}{s}\omega\right)^2}}{\sqrt{\left(1 - \frac{m}{s}\omega^2\right)^2 + \left(\frac{c}{s}\omega\right)^2}}$$

$$\left[\because \frac{m}{s} = \frac{1}{\omega_n^2} \text{ and } \frac{c}{s} = \frac{2\xi}{\omega_n}\right]$$

i.e. $$F_T = \frac{F_0\sqrt{1 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Transmissibility, $$\varepsilon = \frac{F_T}{F_0} = \frac{\sqrt{1 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

If no damper is used (i.e. $\xi = 0$), then

$$\varepsilon = \frac{1}{\pm\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

If damper is used $\left(\text{i.e. } 0 < \xi < \frac{1}{\sqrt{2}}\right)$ and $\omega = \omega_n$ (which is not resonance for this force damped vibrations) then transmissibility,

$$\varepsilon = \frac{\sqrt{1 + (2\xi)^2}}{2\xi} = \frac{\sqrt{1 + 4\xi^2}}{2\xi}$$

(For forced-damped oscillations with $0 < \xi < \frac{1}{\sqrt{2}}$, resonance occurs at $\omega_r = \omega = \omega_n\sqrt{1 - 2\xi^2}$). **The following points are to be noted about transmissibility**.

1. If $\left(\frac{\omega}{\omega_n}\right) < \sqrt{2}$, then the transmissibility (ε) is greater than 1 (i.e. $\varepsilon > 1$) for all values of damping factor (ξ).
2. If $\left(\frac{\omega}{\omega_n}\right) > \sqrt{2}$, then the transmissibility (ε) is less than 1 (i.e. $\varepsilon < 1$) for all values of damping factor (ξ).
3. If $\frac{\omega}{\omega_n} = \sqrt{2}$, then the transmissibility (ε) is equal to 1 (i.e. $\varepsilon = 1$) for all values of damping factor (ξ).
4. If $\frac{\omega}{\omega_n} = 1$, then transmissibility (ε) is infinite (i.e. $\varepsilon = \infty$), if no damper is used (i.e. $\xi = 0$).
5. If $\frac{\omega}{\omega_n} = 1$, then transmissibility (ε) decreases as damping factor (ξ) is increased.
6. If $\frac{\omega}{\omega_n} > \sqrt{2}$, then transmissibility (ε) increases as damping factor (ξ) is increased.
7. For a given value of damping factor (ξ), the transmissibility (ε) starts at 1.0 corresponding to $\frac{\omega}{\omega_n} = 0$ and keeps on increasing to reach a maximum value at $\frac{\omega}{\omega_n} = 1$ and then keeps on decreasing as $\frac{\omega}{\omega_n}$ becomes greater than 1. At $\frac{\omega}{\omega_n} = \sqrt{2}$, then transmissibility (ε) again becomes equal to one and becomes less than 1 for all values of $\frac{\omega}{\omega_n} > \sqrt{2}$.

The phase angle (ϕ_T) between the maximum resultant transmitted force (F_T) and the amplitude of the applied force F_0 is given by the expression

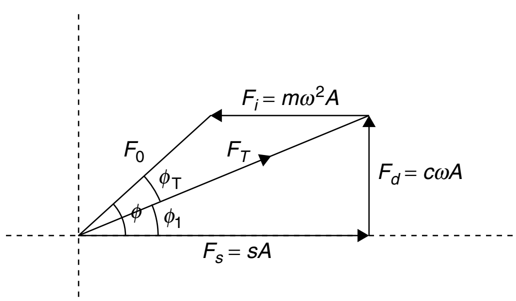
$$\phi_T = \phi - \tan^{-1}\left(\frac{F_d}{F_s}\right) = \phi - \tan^{-1}\left(\frac{c\omega A}{sA}\right)$$

$$= \phi - \tan^{-1}\left(\frac{c\omega}{s}\right)$$

$$\left(\because \frac{c}{s} = \frac{2\xi}{\omega_n}\right)$$

i.e. $\phi_T = \phi - \tan^{-1}\left(\frac{2\xi\omega}{\omega_n}\right)$, where

$$\phi = \tan^{-1}\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$ which is the phase difference between F_0 and A. This is graphically shown below.



Example 40: For an under damped harmonic oscillator, resonance

(A) occurs when excitation frequency is greater than undamped natural frequency.
(B) occurs when excitation frequency is less than undamped natural frequency.
(C) occurs when excitation frequency is equal to undamped natural frequency.
(D) never occurs.

Solution:
Resonance occurs when the frequency of excitation is such that the amplitude of forced vibration is maximum. In the case of undamped forced vibration, resonance occurs when the excitation frequency ω = natural frequency of vibration ω_n. But **in the case of under damped forced vibrations,** the maximum amplitude of forced vibration occurs when $\omega = \omega_n\sqrt{1-2\xi^2}$, which is less than ω_n, the undamped natural frequency $\left(when\ 0<\xi<\frac{1}{\sqrt{2}}\right)$.

Example 41: An automotive engine weighing 240 kg is supported on four springs with linear characteristics. Each of the front two springs have a stiffness of 16 MN/m. The engine speed (in rpm), at which resonance is likely to occur is

(A) 6040 (B) 3020 (C) 1424 (D) 955

Solution:
Mass of system $m = 240$ kg
All springs are connected in parallel.
∴ Effective stiffness of spring, s

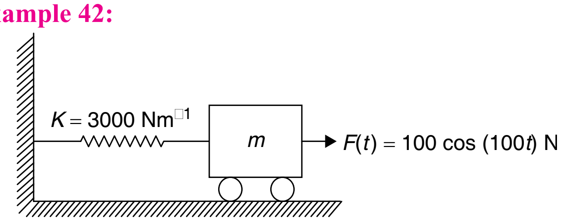
$$= (2\times s_1 + 2\times s_2)$$
$$= (2\times 16\times 10^6) + (2\times 32\times 10^6)$$
$$= 96\times 10^6 \text{ N/m}$$

Natural frequency of vibration

$$f_n = \frac{1}{2\pi}.\omega_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{96\times 10^6}{240}}\text{ Hz}$$
$$= 100.66 \text{ Hz}$$
$$= 100.66\times 60 \text{ rpm}$$
$$= 6039.6 \text{ rpm}$$

No damper is used.
∴ $f = f_n$ for resonance
⇒ Engine speed at resonance
= 6039.6 rpm
= 6040 rpm

Example 42:



A mass m attached to a spring is subjected to a harmonic force as shown in figure. The amplitude of the forced motion is observed to be 50 mm. The value of m (in kg) is

(A) 0.1 (B) 1.0 (C) 0.3 (D) 0.5

Solution:
F_0 = maximum exciter force = 100 N
No damper is used A = 50 mm
= 0.05 m

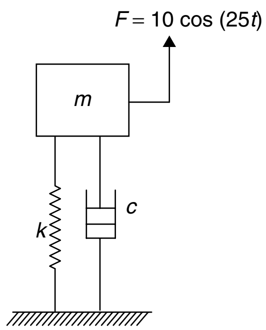
$$\therefore F_0 = \left[sA - m\omega^2 A\right]$$
$$\left\{\because F_0 = \sqrt{\left(sA-m\omega^2A\right)^2+\left(c\omega A\right)^2},\quad \text{here } c=0\right\}$$
$$\Rightarrow m\omega^2 A = sA - F_0$$
$$= (3000\times 0.05) - 100$$
$$= 50 \text{ N}$$

ω = 100 rad/s (= Exciter frequency from data)

$$\therefore m = \frac{50}{\omega^2 A} = \frac{50}{(100)^2\times 0.05} = 0.1 \text{ kg}$$

Hence, mass of system is 0.1 kg.

Example 43:



A mass-spring-dashpot system with mass m = 10 kg, spring constant k = 6250 N/m is excited by a harmonic excitation of 10 cos(25t) N. At the steady state, the vibration amplitude of the mass is 40 mm. The damping coefficient c (in Ns/m) of the dashpot is ______.

Solution: For steady state forced-damped vibrations, we have

$$A = \frac{F_0}{\sqrt{\left[s-m\omega^2\right]^2+(c\omega)^2}}$$

Here, $A = 0.04\ m$, $F_0 = 10\ N$,
s = 6250 N/m
m = 10 kg, ω = 25 rad/s ; c = ?

$$\Rightarrow (s - m\omega^2)^2 + (c\omega)^2$$
$$= \left(\frac{F_0}{A}\right)^2 = \left(\frac{10}{0.04}\right)^2 = 250^2$$
$$\therefore (c\omega)^2 = 250^2 - (s - m\omega^2)^2$$
$$= 250^2 - (6250 - 10\times 25^2)^2$$
$$= 62{,}500 - 0 = 62{,}500$$
$$\therefore c\omega = \sqrt{62500} = 250$$

$$\Rightarrow c = \frac{250}{\omega} = \frac{250}{25}$$

$$= 10\frac{\text{N s}}{\text{m}}.$$

Hence, the damping coefficient of the dashpot is $10\dfrac{\text{N s}}{\text{m}}$

Example 44: A single degree of freedom system has a mass of 2 kg, stiffness 8 N/m and viscous damping ratio 0.02. The dynamic magnification factor at an excitation frequency of 1.5 rad/s is __________ .

Solution: Given $m = 2$ kg, $s = 8$ N/m, $\xi = 0.02$, $\omega = 1.5$ rad/s

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{8}{2}} = 2 \text{ rad/s}$$

Magnification factor,

$$\text{MF} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{1}{\sqrt{\left[1-\left(\frac{1.5}{2}\right)^2\right]^2 + \left(2\times 0.02\times\frac{1.5}{2}\right)^2}}$$

$$= \frac{1}{\sqrt{(1-0.5625)^2 + (0.03)^2}}$$

$$= \frac{1}{\sqrt{(0.4375)^2 + 0.0009}}$$

$$= \frac{1}{\sqrt{0.1923}} = \frac{1}{0.4385} = 2.28$$

$\therefore$ The dynamic magnification factor is 2.28.

Example 45: A machine of 250 kg mass is supported on springs of total stiffness 100 kN/m. Machine has an unbalanced rotating force of 350 N at speed of 3600 rpm. Assuming a damping factor of 0.15, the value of transmissibility ratio is

(A) 0.0531 (B) 0.9922
(C) 0.0162 (D) 0.0028

Solution:

Mass $m = 250$ kg

Stiffness of spring, $s = 100 \times 10^3$ N/m

$= 10^5$ N/m

$F_0 = 350$ N

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 3600}{60} = 120\pi$$

$$= 376.99 \text{ rad/s}$$

$$\xi = 0.15$$

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10^5}{250}}$$

$$= 20 \text{ rad/s}$$

Transmissibility,

$$\varepsilon = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\xi\omega}{\omega_n}\right)^2}}$$

$$= \frac{\sqrt{1+\left(\frac{2\times 0.15\times 376.99}{20}\right)^2}}{\sqrt{\left[1-\left(\frac{376.99}{20}\right)^2\right]^2 + \left(\frac{2\times 0.15\times 376.99}{20}\right)^2}}$$

$$= \frac{\sqrt{1+31.977}}{\sqrt{[1-355.304]^2 + 31.977}}$$

$$= \frac{\sqrt{32.977}}{\sqrt{125531.324 + 31.977}}$$

$$= \frac{5.7426}{354.349} = 0.0162$$

Example 46: Consider a single degree of freedom system with viscous damping, excited by a harmonic force. When the frequency of exciter is equal to the natural frequency of oscillation of the system, the phase angle (in degree) of the displacement with respect to the exciting force is

(A) 0 (B) 45 (C) 90 (D) 135

Solution:

We have $\tan\phi$

$$= \frac{c\omega A}{(sA - m\omega^2 A)} = \frac{c\omega}{(s - m\omega^2)}$$

$$= \frac{\left(\frac{c}{s}\right)\omega}{\left[1-\left(\frac{m}{s}\right)\omega^2\right]} = \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$$= \frac{2\xi}{(1-1)} \quad (\because \omega = \omega_n)$$

$$= \infty$$

$\therefore \phi = \tan^{-1}(\infty) = 90°$

NOTE

$\omega = \omega_n$ is **NOT resonance** for underdamped-forced-vibrating system. $\omega_r = \omega = \omega_n\sqrt{1-2\xi^2}$ gives the maximum amplitude for that case, when $0 < \xi < \dfrac{1}{\sqrt{2}}$.

Example 47: A vibrating machine is isolated from the floor using springs. If the ratio of the excitation frequency of vibration of machine to the natural frequency of the isolation system is equal to 0.5, the transmissibility ratio of isolation is

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$

(C) $\frac{4}{3}$ (D) 2

Solution: Given $\frac{\omega}{\omega_n} = 0.5; \xi = 0$

$(\because \text{no damper is used})$

Transmissibility ε

$$= \frac{\sqrt{1+\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$= \frac{\sqrt{1+0}}{\sqrt{\left[1-(0.5)^2+0\right]}} = \frac{1}{1-(0.5)^2}$$

$$= \frac{1}{1-0.25} = \frac{1}{0.75} = \frac{4}{3}$$

Example 48: In vibration isolation, which one of the following statements is **not** correct regarding transmissibility (T)?

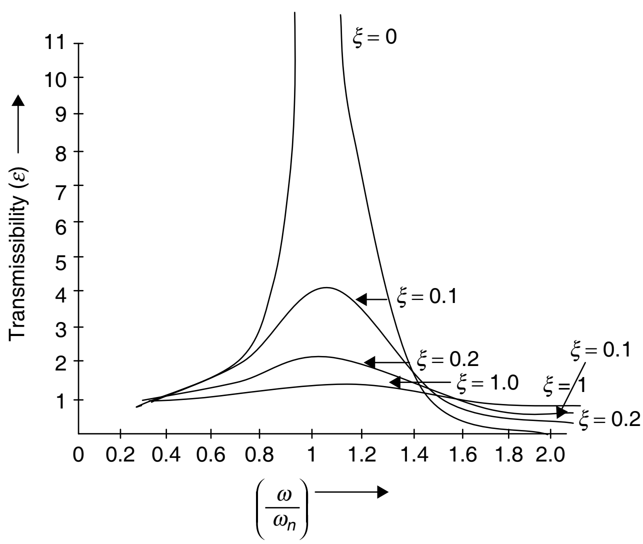
(A) T is nearly unity at small excitation frequencies.

(B) T can always be reduced by using higher damping at any excitation frequency.

(C) T is unity at the frequency ratio of $\sqrt{2}$

(D) T is infinity at resonance for undamped systems.

Solution: The variation of transmissibility (ε) with ratio of frequencies $\left(\frac{\omega}{\omega_n}\right)$ for various values of damping factor (ξ) is shown in figure below.



It can be seen that for small excitation frequency $\left(\frac{\omega}{\omega_n} < 0.3\right)$,

$\varepsilon \approx 1 \rightarrow$ statement (A) is correct.

We have $\varepsilon = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$

when $\frac{\omega}{\omega_n} = \sqrt{2}$, $\varepsilon = 1 \Rightarrow$ (C) is correct

If $\xi = 0$ (undamped), when $\frac{\omega}{\omega_n} = 1$ (i.e. resonance), ε is ∞.

$\Rightarrow$ (D) is correct.

If $\frac{\omega}{\omega_n} > \sqrt{2}$, then transmissibility (ε) increases as damping factor (ξ) is increased. For $0.3 < \frac{\omega}{\omega_n} < \sqrt{2}$ only, transmissibility decreases as damping factor (ξ) is increased

$\Rightarrow$ statement (B) is wrong. Choice (B)

Example 49: There are four samples P, Q, R and S with natural frequencies 64, 96, 128 and 256 Hz respectively. They are mounted on test setups for conducting vibration experiments. If a loud pure note of frequency 144 Hz is produced by some instrument, which of the samples will show the most perceptible induced vibration?

(A) P (B) Q (C) R (D) S

Solution: The most perceptible induced vibration will have the largest magnification factor (MF) among the given frequencies.

$$MF = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

As no data is given on ξ, we assume it as zero.

$$\therefore MF = \frac{1}{\pm\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

Here, $\omega = 144$ Hz

For $\omega_n = 64$ Hz,

$$MF_1 = \frac{1}{-\left(1-\left(\frac{144}{64}\right)^2\right)} = \frac{1}{-[1-5.0625]} = \frac{1}{4.0625} = 0.2462$$

For $\omega_n = 96$ Hz., $MF_2 = \frac{1}{-\left[1-\left(\frac{144}{96}\right)^2\right]}$

$$= \frac{1}{-[1-2.25]} = \frac{1}{1.25} = 0.8$$

For $\omega_n = 128$ Hz.,

$$MF_3 = \frac{1}{-\left[1-\left(\frac{144}{128}\right)^2\right]} = \frac{1}{-[1-1.2656]}$$

$$= \frac{1}{0.2656} = 3.765$$

For $\omega_n = 256$ Hz., $MF_4 = \dfrac{1}{\left[1-\left(\frac{144}{256}\right)^2\right]}$

$$= \frac{1}{(1-0.3164)}$$

$$= \frac{1}{0.6836}$$

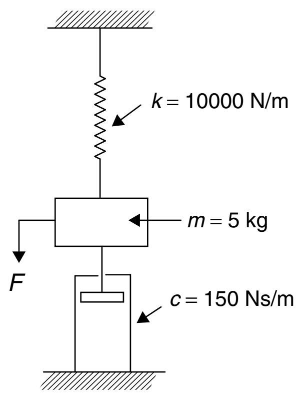
$$= 1.463$$

MF is maximum for $\omega_n = 128$ Hz

⇒ Most perceptible induced vibration will be produced for natural frequency of 128 Hz. (i.e. *R*).

Hence the option is (C)

Example 50:



The figure shows a spring-mass-dashpot system. The mass has a harmonic disturbing force applied to it given by $F = 400 \sin(30t)$ N. The amplitude of displacement of induced vibration of mass (in mm) and the maximum phase angle (in degree) of the displacement with the applied force are respectively ____________ and ____________.

Solution: $m = 5$ kg; $k = s = 10000$ N/m;
$c = 150$ Ns/m
$F_0 = 400\ N$; $\omega = 30$ rad/s

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10000}{5}} = 44.72 \text{ rad/s}$$

∴ Static deflection, $\delta = \dfrac{F_0}{s} = \dfrac{400}{10000}$

$$= 0.04 \text{ m}$$

$$c_c = 2m\omega_n = 2\sqrt{ms} = 2\times\sqrt{5\times 10000}$$

$$= 447.21 \text{ rad/s}$$

$$\xi = \frac{c}{c_c} = \frac{150}{447.21} = 0.3354$$

Magnification factor, $MF = \dfrac{A}{\delta}$

$$= \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{1}{\sqrt{\left[1-\left(\frac{30}{44.72}\right)^2\right]^2 + \left(\frac{2\times 0.3354\times 30}{44.72}\right)^2}}$$

$$= \frac{1}{\sqrt{0.3025+0.2025}}$$

$$= \frac{1}{\sqrt{0.505}}$$

$$= 1.407$$

∴ Amplitude of oscillation,

$$A = 1.407\times\delta$$
$$= 1.407\times 0.04$$
$$= 0.05628 \text{ m}$$
$$= 56.28 \text{ mm}$$

If ϕ is the phase difference between F_0 and A,

$$\tan\phi = \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$= \frac{2\times 0.3354\times\left(\frac{30}{44.72}\right)}{\left[1-\left(\frac{30}{44.72}\right)^2\right]}$$

$$= \frac{0.45}{0.55}$$

$$= 0.8182$$

$$\therefore\ \phi = \tan^{-1}(0.8182)$$

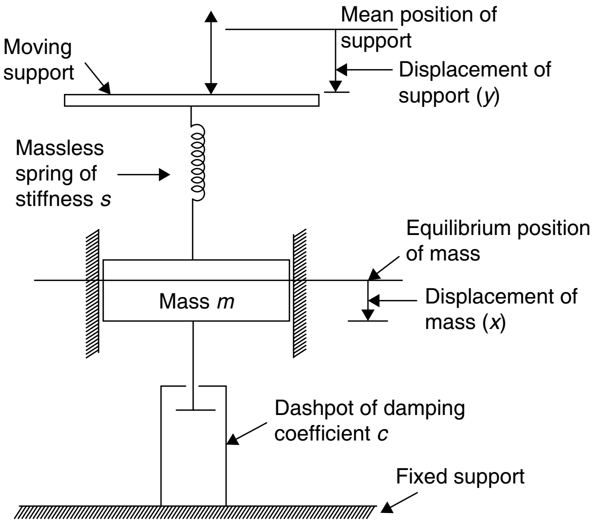
$$= 39.29°$$

Hence, the maximum amplitude of displacement of the induced vibration is 56.28 mm and the phase difference of displacement amplitude with the applied force is 39.29°.

Forced-Damped Vibrations Involving Harmonic Movement of the Support

So far we had discussed the forced-damped vibrations in which the support is fixed. Let us now consider forced vibrations in which the support is also subjected to harmonic movement. There are two cases; (i) when the damper is located between the mass and a fixed support, spring connected to a moving support (ii) when the damper and spring are located between the mass and moving support. We will discuss each of these cases.

Case 1: Damper between mass and fixed support



The arrangement of spring-mass-damper system is as shown in the above figure. The mass m is constrained to move vertically up and down only. The movable support moves vertically up and down at a circular frequency of ω (rad/s) and with a displacement amplitude of a. This is usually achieved using a cam arrangement. At time $t = 0$, the movable support passes through its mean position. The displacement of the movable support at any time t (measured from its mean position) is y, so that the equation of simple harmonic motion of the support is $y = a \sin\omega t$

The displacement of the mass m, measured from its mean position is denoted as x and the amplitude of this displacement is A. It is not necessary that the oscillation of mass m is in phase with the oscillation of support. If the oscillation of mass m has a phase difference of ϕ with the oscillation of the support, then we can write the displacement of mass m as given by

$$x = A\sin(\omega t + \phi)$$

Hence, the velocity of damper piston is $v = \dfrac{dx}{dt}$

At any given time t, the spring is stretched or compressed by an amount $(x - y)$ so that spring force $F_s = s(x - y)$

The damper force, $F_d = c\left(\dfrac{dx}{dt}\right)$

The inertial force, $F_i = m\dfrac{d^2x}{dt^2}$

There is no applied force on the mass

Hence, the dynamic balance gives

$$0 = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + s(x - y)$$

$$\Rightarrow sy = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + sx \quad [\text{But } y = a\sin\omega t]$$

$$\text{i.e. } s\,a\sin\omega t = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + sx$$

The above equation is of the form

$F_0 \sin\omega t = m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + sx$, which is the equation for forced-damped vibration. Hence, in the equation of forced-damped vibration, replace F_0 with sa (i.e. $F_0 = sa$) to obtain the steady state solution for this vibration

$$\therefore A = \frac{sa}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$

$$= \frac{a}{\sqrt{\left[1 - \left(\frac{m}{s}\right)\omega^2\right]^2 + \left(\frac{c}{s}\omega\right)^2}}$$

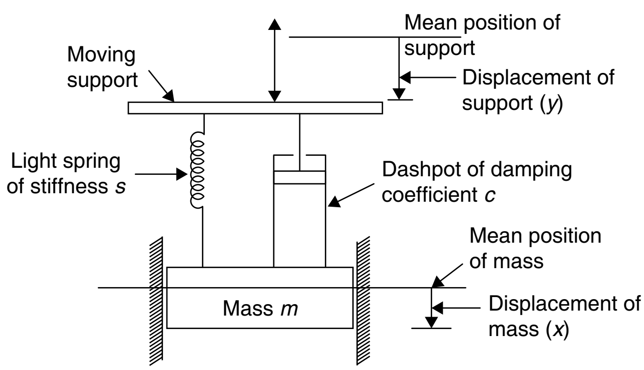
$$\Rightarrow A = \frac{a}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}, \text{ where}$$

$$\omega_n = \sqrt{\frac{s}{m}} \text{ and}$$

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{ms}}$$

$$\text{Also, phase } \phi = \tan^{-1}\left\{\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}\right\}$$

Case 2: Damper between the mass and moving support



The displacement of support from mean position is $y = a \sin \omega t$. The support passes through its mean position at time $t = 0$

The change in length of spring is $(x - y)$, hence the spring force is $F_s = s(x - y)$

As the damper is now between the moving support and the mass, velocity of mass with respect to damper is now

$$v = \frac{d}{dt}(x - y) = \frac{dx}{dt} - \frac{dy}{dt}$$

$\therefore$ Damping force, $F_d = c\left(\frac{dx}{dt} - \frac{dy}{dt}\right)$

Inertial force on mass, $F_i = m\,\frac{d^2x}{dt^2}$

As $y = a \sin\omega t$ [i.e. displacement of support]

$$\frac{dy}{dt} = a\omega \cos \omega t$$

As there are no other forces on m, we have

$0 = F_i + F_d + F_s$

$\Rightarrow 0 = m\,\frac{d^2x}{dt^2} + c\left(\frac{dx}{dt} - \frac{dy}{dt}\right) + s(x - y)$

$\Rightarrow c\,\frac{dy}{dt} + sy = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + sx$

$\Rightarrow c(a\omega \cos\omega t) + s(a \sin\omega t)$

$= m\,\frac{d^2x}{dt^2} + c\frac{dx}{dt} + sx$

LHS is the sum of two SHMs having a phase difference of $\frac{\pi}{2}$ rad

$\Rightarrow \sqrt{(ca\omega)^2 + (sa)^2}\,\sin(\omega t + \alpha)$

$= m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + sx$

where $\alpha = \tan^{-1}\left(\frac{c\omega}{s}\right) = 2\xi\left(\frac{\omega}{\omega_n}\right)$

Compare this equation with

$F_0 \sin\omega t = m\,\frac{d^2x}{dt^2} + c\frac{dx}{dt} + sx$. The steady state solution for the vibration of mass m is

$x = A\sin\,[(\omega t + \alpha) + \phi]$, where

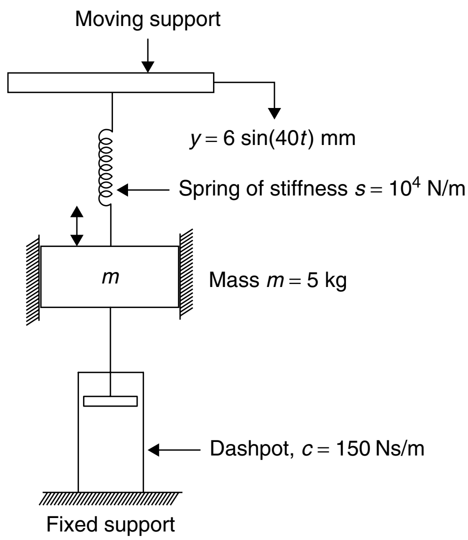
$$A = \frac{sa}{\sqrt{(s - m\omega^2)^2 + (c\omega)^2}}$$

$\Rightarrow A = \dfrac{a}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\,\frac{\omega}{\omega_n}\right)^2}}$, where

$$\omega_n = \sqrt{\frac{s}{m}} \text{ and } \xi = \frac{c}{c_c} = \frac{c}{2\sqrt{ms}}$$

The phase difference $\phi = \tan^{-1}\dfrac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$

Example 51:



The figure shows a spring-mass-dashpot system. The dashpot is in between the mass m = 5 kg and the fixed support. The damping coefficient is 150 Ns/m. The spring of stiffness 10^4 N/m is connected to a moving support which is moved vertically as per the relation $y = 6 \sin(40t)$ mm, where y represents the vertical displacement of the support from its mean position at time t. The amplitude of displacement of mass m (in mm) and the phase difference (in degree) of the displacement of mass m with the displacement of support are respectively ___________ and _________ .

Solution: Given $s = 10^4$ N/m; $c = 150$ Ns/m,

$m = 5$ kg

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10^4}{5}} = 44.72 \text{ rad/s}$$

$$c_c = 2\sqrt{ms} = 2\sqrt{5 \times 10^4}$$

$= 447.21$ Ns/m

$$\xi = \frac{c}{c_c} = \frac{150}{447.21} = 0.335$$

From the equation of motion of support, $a = 6 \times 10^{-3}$ m and $\omega = 40$ rad/s

$$\therefore\ A = \frac{a}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\,\frac{\omega}{\omega_n}\right)^2}}$$

$$= \frac{6\times10^{-3}}{\sqrt{\left[1-\left(\frac{40}{44.72}\right)^2\right]^2+\left(\frac{2\times0.335\times40}{44.72}\right)^2}}$$

$$= \frac{6\times10^{-3}}{\sqrt{0.04+0.359}} = \frac{6\times10^{-3}}{\sqrt{0.399}}$$

$= 9.499 \times 10^{-3}$ m

$= 9.5$ mm

$$\tan\phi = \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$= \frac{2\times0.335\times\left(\frac{40}{44.72}\right)}{\left[1-\left(\frac{40}{44.72}\right)^2\right]}$$

$$= \frac{0.5993}{[1-0.8]} = \frac{0.5993}{0.2} = 2.9965$$

$\Rightarrow \phi = \tan^{-1}2.9965 = 71.54°$

Hence, the maximum phase difference between displacement of mass and displacement of support is 71.54°.

Torsional Oscillations

When a shaft, fixed at one end, is subjected to a disturbing torque at its free end and about the axis of the shaft, the shaft gets twisted in the direction of the disturbing torque. Owing to the elasticity of the shaft (more specifically due to the torsional stiffness of the shaft), a torque which opposes this twist, called restoring torque is setup in the shaft. When the disturbing torque is removed, this restoring torque untwists the shaft. In this process, the strain energy in the shaft gets converted to rotational kinetic energy. So the shaft, on reaching the original position, overshoots and gets twisted in the opposite sense. Again restoring torques are set up which untwists the shaft and this process goes on. This process is known as **torsional vibration**.

We know from the lessons on torsion of shafts (in Strength of Materials) that $\frac{T}{J} = \frac{G\theta}{L}$, where

T = torque on shaft

J = polar second moment of area of cross section of shaft

G = modulus of rigidity of material of shaft

L = length of shaft and

θ = angle of twist of shaft

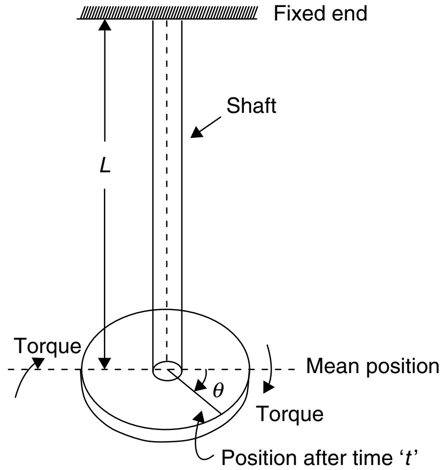
The **torsional stiffness** of shaft (q) is defined as the restoring torque per unit twist.

$$\therefore q = \frac{T}{\theta} = \frac{GJ}{L}$$

For a circular solid shaft of diameter d,

$$J = \frac{\pi d^4}{32}$$

Natural Frequency of Free Torsional Vibrations (Single Rotor)



Consider a uniform shaft of length L, whose upper end is fixed and the lower end carries a heavy uniform disc of mass m. The plane of disc is parallel to the cross sectional area of the shaft. The moment of inertia of the shaft about its axis is negligible.

Let the disc be subjected to a disturbing torque about the axis of the shaft. On releasing this torque, the shaft executes torsional vibrations. If θ is the angular displacement of shaft from its mean position after time t, the angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$ for the shaft.

m = mass of disc

k = radius of gyration of disc about axis of shaft

I = mass moment of inertia of disc about axis of shaft $= mk^2$

q = torsional stiffness of shaft

$= \frac{GJ}{L}$ where L = length of shaft

J = polar second moment of area

G = modulus of rigidity of shaft material

In the free-torsional vibration of the shaft, at any instant t, the torques acting on the disc are

(i) τ_i = Inertial torque $= I\alpha$

$= I\frac{d^2\theta}{dt^2}$ (acting opposite to direction of α)

(ii) τ_r = Restoring torque

$= q\theta$ (opposite direction of θ)

As per D'Alembert's, principle, $\tau_i + \tau_r = 0$

$$\Rightarrow I\frac{d^2\theta}{dt^2} + q\theta = 0$$

$\Rightarrow \dfrac{d^2\theta}{dt^2} + \left(\dfrac{q}{I}\right)\theta = 0$ which is the differential equation for free torsional vibration, which is SHM.

$\therefore \omega_n^2 =$ coefficient of $\theta = \dfrac{q}{I}$

$\Rightarrow \omega_n = \sqrt{\dfrac{q}{I}}$, where ω_n = natural circular frequency of free - torsional oscillations.

Time period, $T_n = \dfrac{2\pi}{\omega_n} = 2\pi\sqrt{\dfrac{I}{q}}$ and natural frequency

$f_n = \dfrac{1}{T_n} = \dfrac{1}{2\pi}\sqrt{\dfrac{q}{I}}$ for free-torsional vibrations.

If the mass moment of inertia of the shaft about its axis is not negligible, then it can be shown that the effective mass moment of inertia of system will be equal to mass moment of inertia of disc plus one-third of the mass moment of inertia of the shaft.

i.e. $I_{eff} = I + \dfrac{I_1}{3}$, where

I = mass moment of inertia of disc and

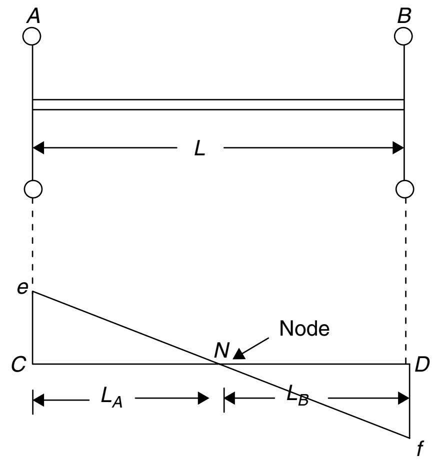
I_1 = mass moment of inertia of shaft

In that case, $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{q}{I_{eff}}} = \dfrac{1}{2\pi}\sqrt{\dfrac{q}{\left(I + \dfrac{I_1}{3}\right)}}$

At the fixed end of the shaft, the **amplitude of torsional vibrations is zero** for a single rotor system and it forms a **node**. **The amplitude of torsional vibration is maximum at the free end** (at the location of the rotor). In torsional vibrations, the shaft remains undisturbed by the vibrations at the **node**, which is the section of the shaft where the amplitude of vibration is zero.

Free Torsional Vibrations (Two Rotor System)

In a two-rotor system, there are two discs (or rotors), one at each free end of the shaft. For torsional vibrations, the **discs are twisted in opposite directions** and released (if the discs are twisted in same direction, the shaft will only rotate and will not get twisted). Consequently certain length of the shaft **twists in one direction**, while the remaining length twists **in opposite direction**. There is a section of the shaft which does not get twisted and remains unaffected by the vibrations. This section is called **node**. Hence, the section of the shaft at the node can be considered fixed. The section of shaft from the node to one rotor and the section of shaft from the node to the other rotor vibrate with **same frequency** but in **opposite directions**. The **amplitudes of oscillations at the rotors may or may not be equal**. These are explained in the following Figure.



L is the length of the shaft. Two rotors A and B are fixed at the free ends. The rotors are twisted in opposite directions and released. The section at point N of the shaft is not affected by twisting and it is the node. L_A is the length of the shaft from node N to rotor A and L_B is the length of shift from node N to rotor B. **The length L_A and L_B get twisted in opposite directions but vibrate with same frequency**. Ce and Df are the amplitudes of oscillations at rotor A and B respectively. The line ef, known as the **elastic line** of the shaft, passes through the node N.

We have $L_A + L_B = L$, length of shaft

I_A = mass moment of inertia of rotor A

I_B = mass moment of inertia of rotor B

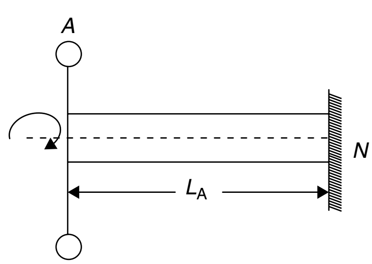
q_A = torsional stiffness of length L_A

$= \dfrac{GJ}{L_A}$

q_B = torsional stiffness of length L_B

$= \dfrac{GJ}{L_B}$

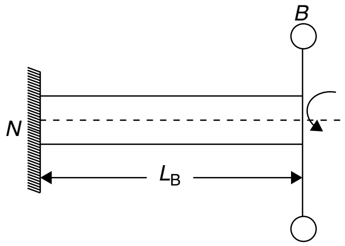
The length L_A can be considered as a single rotor system, fixed at node N.



$\therefore f_{nA}$ = natural frequency of torsional vibration of rotor A

$= \dfrac{1}{2\pi}\sqrt{\dfrac{q_A}{I_A}} = \dfrac{1}{2\pi}\sqrt{\dfrac{GJ}{L_A I_A}}$

Similarly, the length L_B can be considered to be a single rotor system, fixed at N.



f_{nB} = natural frequency of torsional vibration of rotor B

$$= \frac{1}{2\pi}\sqrt{\frac{q_B}{I_B}} = \frac{1}{2\pi}\sqrt{\frac{GJ}{L_B I_B}}$$

As the natural frequency of L_A and L_B are same,

$$f_{nA} = f_{nB}$$

$$\Rightarrow \frac{1}{2\pi}\sqrt{\frac{GJ}{L_A I_A}} = \frac{1}{2\pi}\sqrt{\frac{GJ}{L_B I_B}}$$

$$\Rightarrow L_A I_A = L_B I_B$$

$$\text{or } \frac{L_A}{L_B} = \frac{I_B}{I_A}$$

$$\Rightarrow \frac{(L_A + L_B)}{L_B} = \frac{(I_B + I_A)}{I_A}$$

$$\Rightarrow \frac{L}{L_B} = \frac{I_B + I_A}{I_A}$$

$$\text{or } L_B = \frac{I_A L}{(I_A + I_B)} \text{ and } L_A = \frac{I_B L}{(I_A + I_B)}$$

If I_A, I_B and L are known, the location of node N can be determined using the expressions for L_A and L_B. A two rotor system has one natural frequency of torsional vibration. **The node in this case is located between the rotors and nearer to the rotor of larger mass moment of inertia**.

Free Torsional Vibration of Three Rotor System

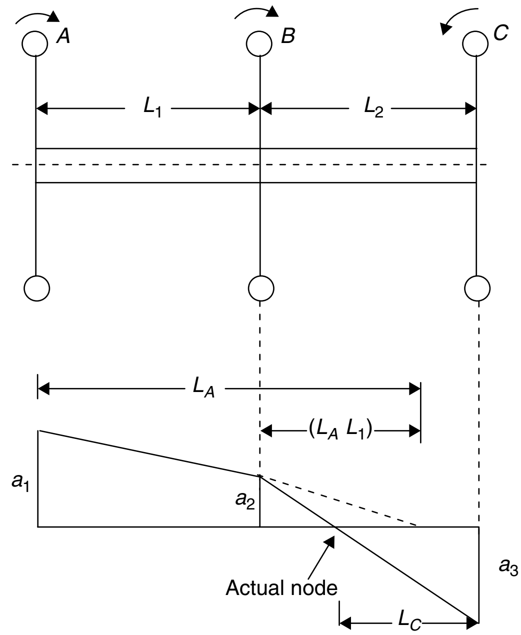
In this case, three rotors A, B and C are mounted on a shaft. There are two possible nodes of torsional vibrations of this shaft and hence there are two natural frequencies of torsional vibration of a three rotor system. In one mode there is only one node and in the other mode, there are two nodes.

Three-Rotors Single Node System

When rotors A and B are twisted in same sense and rotor C is twisted in opposite sense, a single node is formed between rotors B and C.

When rotors B and C are twisted in one sense and rotor A is twisted in opposite sense, a single node is formed between rotors A and B.

The amplitudes of vibration of rotor A, rotor B and rotor C are a_1, a_2 and a_3, respectively.

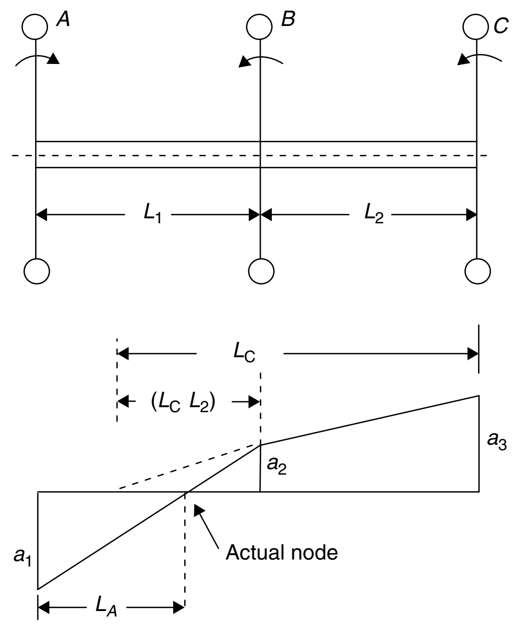


Rotors A and B rotate in same sense, rotor C rotates in opposite sense. An actual node is formed between B and C. The length $L_A > L_1$ but L_A **does not** give the actual node point. The actual node point in this case is given by L_c.

$$\text{We have } \frac{a_1}{L_A} = \frac{a_2}{(L_A - L_1)}$$

$$\text{Also, } \frac{a_3}{L_c} = \frac{a_2}{(L_2 - L_c)}$$

Now let us look at the case when rotors B and C rotate in same sense and rotor A rotates in opposite sense.



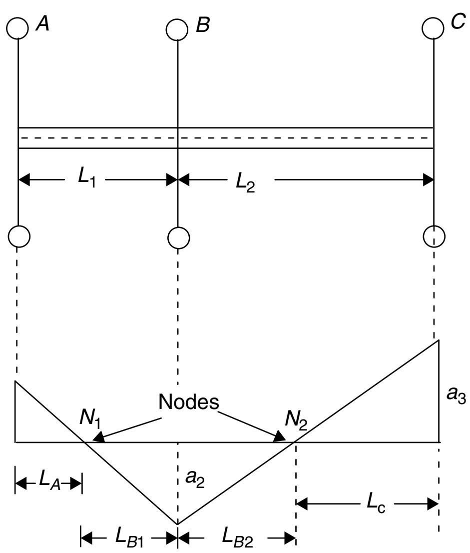
Actual node in this case is obtained in between A and B and L_A gives the location of actual node. L_c does not give location of actual node. $L_c > L_2$

Also, $\frac{a_3}{L_c} = \frac{a_2}{(L_c - L_2)}$ and

$$\frac{a_1}{L_A} = \frac{a_2}{(L_1 - L_A)}$$

Three - Rotors Double Node System

When the rotors at the free ends (A and C) rotate in the same sense and rotor in between them (B) rotates in the opposite sense, two nodes are produced, one between A and B and then other between B and C. The shaft can be assumed to be fixed at the nodes.



Let I_A, I_B and I_C = mass moment of inertia of rotors A, B and C respectively

L_1 = distance between rotors A and B

L_2 = distance between rotors B and C

L_A = distance of rotor A from node N_1

L_{B_1} = distance of rotor B from node N_1

L_{B_2} = distance of rotor B from node N_2

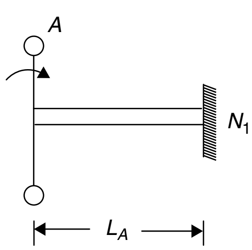
L_C = distance of rotor C from node N_2

G = modulus of rigidity of shaft material

d = diameter of shaft

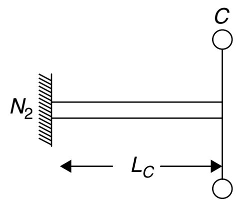
J = polar moment of inertia of shaft cross section

$$= \frac{\pi}{32} d^4.$$



The natural frequency of torsional vibration of rotor A,

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{q_A}{I_A}} = \frac{1}{2\pi}\sqrt{\frac{GJ}{L_A I_A}}$$



The Natural Frequency of Torsional Vibration of Rotor C

$$f_{nC} = \frac{1}{2\pi}\sqrt{\frac{q_c}{I_c}} = \frac{1}{2\pi}\sqrt{\frac{GJ}{L_c I_c}}$$

q_{B_1} = Torque required to produced unit twist on length L_{B_1}

$$= \frac{GJ}{L_{B_1}}$$

q_{B_2} = Torque required to produce unit twist on length L_{B_2}

$$= \frac{GJ}{L_{B_2}}$$

$\therefore q_B$ = Torque required to produce unit twist on rotor B

$$= q_{B_1} + q_{B_2} = GJ\left[\frac{1}{L_{B_1}} + \frac{1}{L_{B_2}}\right]$$

$$= GJ\left[\frac{1}{(L_1 - L_A)} + \frac{1}{(L_2 - L_1)}\right]$$

Hence, natural frequency of vibration of rotor B,

$$f_{nB} = \frac{1}{2\pi}\sqrt{\frac{q_B}{I_B}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{GJ}{I_B}\left[\frac{1}{(L_1 - L_A)} + \frac{1}{(L_2 - L_1)}\right]}$$

The natural frequencies of torsional vibration of rotor A, rotor B, and rotor C are all equal.

$$\therefore f_{nA} = f_{nB} = f_{nC}$$

$$f_{nA} = f_{nC} \Rightarrow \frac{1}{2\pi}\sqrt{\frac{GJ}{I_A L_A}} = \frac{1}{2\pi}\sqrt{\frac{GJ}{I_C L_C}}$$

$$\Rightarrow L_A I_A = L_C I_C$$

or $\frac{L_A}{L_C} = \frac{I_C}{I_A} \Rightarrow L_A = \left(\frac{I_C}{I_A}\right) L_C$

Also, $f_{nC} = f_{nB}$

$$\Rightarrow \frac{1}{L_c I_c} = \frac{1}{I_B}\left[\frac{1}{(L_1 - L_A)} + \frac{1}{(L_2 - L_c)}\right]$$

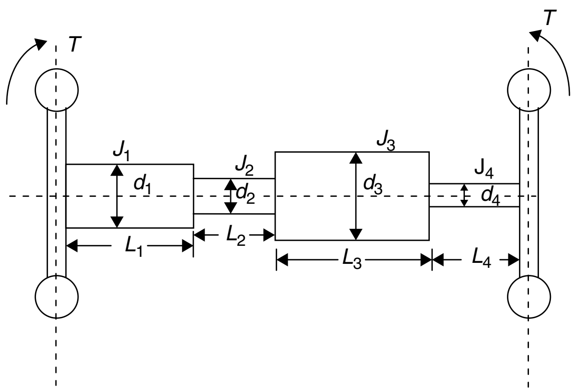
By eliminating L_A from the above equation, we obtain a quadratic equation in L_C, which gives two values of L_C and corresponding two values of L_A. One value of L_A and corresponding value of L_C gives the position of nodes. The fundamental frequency determined using the two nodes positions is called the **two-node frequency**.

NOTE

In a multi rotor torsional vibration system, the number of nodes that can occur is equal to number of rotors minus one.

Torsionally Equivalent Shaft

When a compound shaft (made of different diameters and different lengths) is subjected to equal and opposite torques, the shaft twists through some angle. **The torsionally equivalent shaft** of this compound shaft **is a shaft of uniform diameter and certain length** which **undergoes the same amount of angular twist** as the compound shaft, **when subjected to same amount of equal and opposite torques**.



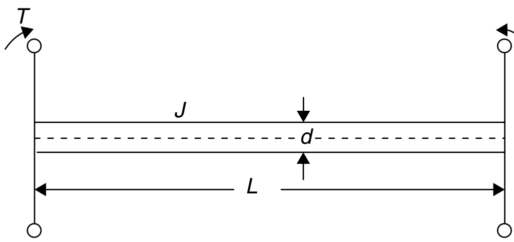


Figure shows a compound shaft subjected to equal and opposite torque T and Figure shows its torsionally equivalent shaft of diameter d and length L.

The torsion equation for a shaft is $\dfrac{T}{J} = \dfrac{G\theta}{L}$

$$\Rightarrow \theta = \frac{TL}{GJ}, \text{ where } J = \frac{\pi}{32}d^4$$

For shaft of diameter d_1 and length L_1 angle of twist θ_1 is given by $\theta_1 = \dfrac{TL_1}{J_1 G} = \dfrac{TL_1}{\left(\dfrac{\pi}{32}d_1^{\,4}G\right)}$

Similarly, $\theta_2 = \dfrac{TL_2}{\left(\dfrac{\pi}{32}d_2^{\,4}G\right)}$

$$\theta_3 = \frac{TL_3}{\left(\dfrac{\pi}{32}d_3^{\,4}G\right)} \text{ and}$$

$$\theta_4 = \frac{TL_4}{\left(\dfrac{\pi}{32}d_4^{\,4}G\right)}$$

For torsionally equivalent shaft,

$$\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4$$

$$\Rightarrow \frac{TL}{\left(\dfrac{\pi}{32}d^4G\right)} = \frac{TL_1}{\left(\dfrac{\pi}{32}d_1^{\,4}G\right)} + \frac{TL_2}{\left(\dfrac{\pi}{32}d_2^{\,4}G\right)} + \frac{TL_3}{\left(\dfrac{\pi}{32}d_3^{\,4}G\right)} + \frac{TL_4}{\left(\dfrac{\pi}{32}d_4^{\,4}G\right)}$$

$$\Rightarrow \frac{L}{d^4} = \frac{L_1}{d_1^{\,4}} + \frac{L_2}{d_2^{\,4}} + \frac{L_3}{d_3^{\,4}} + \frac{L_4}{d_4^{\,4}}$$

$$\text{or } L = L_1\left(\frac{d}{d_1}\right)^4 + L_2\left(\frac{d}{d_2}\right)^4 + L_3\left(\frac{d}{d_3}\right)^4 + L_4\left(\frac{d}{d_4}\right)^4$$

NOTE

Here, we assumed that all shafts are made of same material i.e. G is same for all shafts.

Example 52: A flywheel of moment of inertia 30 kg m^2 is fitted at the lower end of a uniform vertical shaft of diameter 20 mm and length 800 mm, made of material of modulus of rigidity 80 GPa. The upper end of the shaft is rigidly fixed. The natural frequency of torsional vibration of the system (in Hz) is

(A) 2.31 (B) 1.15
(C) 4.76 (D) 9.31

Solution:

Polar moment of inertia, $J = \dfrac{\pi}{32}d^4$

$$= \frac{\pi}{32}\times(0.02)^4 = 1.57\times10^{-8}\,m^4$$

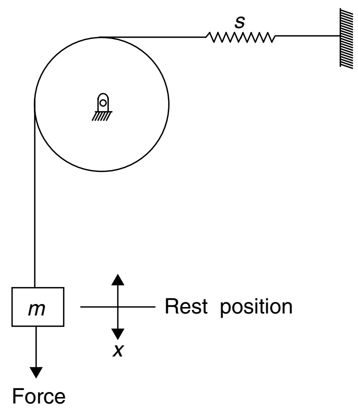
Torsional stiffness,

$$q = \frac{GJ}{L} = \frac{80\times10^9\times1.57\times10^{-8}}{0.8}$$

$$= 1570 \text{ Nm/rad}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{1570}{30}} = 1.15 \text{ Hz.}$$

Direction for questions (Examples 53 and 54):



A large drum of radius 0.5 m is mounted on a horizontal shaft. A belt runs over it as shown in figure with a mass of 400 kg on one end. The other end is restrained with a spring of stiffness 300 kN/m. The drum has a moment of inertia 60 kg m^2 about its axis. The belt does not slip on the drum. The maximum displacement of the mass is 5 mm from its mean position, when it is pulled down and released, resulting in its vertical free undamped oscillations.

Example 53: The natural frequency of vibration of the system (in Hz) is
(A) 1.73 (B) 5.94 (C) 3.45 (D) 7.36

Solution:
At time t, the extension of spring is x
If α is the angular acceleration of the drum, the torque (τ) required to accelerate the drum is $\tau = I\alpha$, where I = moment of inertia of drum
But torque = Force × radius $\Rightarrow \tau = FR$

$$\Rightarrow F = \frac{\tau}{R} = \frac{I\alpha}{R}$$

Hence, inertia force required to accelerate the drum, $F_{i1} = \dfrac{I\alpha}{R}$

If a is the linear acceleration of the mass, inertia force required to accelerate the mass,

$$F_{i2} = ma$$

For no slip of belt, the linear acceleration a is related to the angular acceleration α as

$$a = R\alpha \Rightarrow \alpha = \frac{a}{R}$$

$$\therefore F_{i1} = \frac{I\alpha}{R} = \frac{Ia}{R^2}$$

Spring force $F_s = sx$ (s = stiffness of spring)
For free vibrations, for dynamic equilibrium,

$$F_{i1} + F_{i2} + F_s = 0$$

$$\Rightarrow \frac{Ia}{R^2} + ma + kx = 0$$

$$\Rightarrow \left(\frac{I}{R^2} + m\right)a = -kx$$

$$\therefore a = -\left[\frac{k}{\left(\frac{I}{R^2} + m\right)}\right]x \Rightarrow \text{SHM}$$

$$\therefore \omega_n^2 = \frac{k}{\left(\frac{I}{R^2} + m\right)}$$

$$\Rightarrow \omega_n = \sqrt{\frac{k}{\left(\frac{I}{R^2} + m\right)}}$$

$$= \sqrt{\frac{300 \times 10^3}{\left(\frac{60}{0.5^2} + 400\right)}}$$

$$= 21.65 \text{ rad/s}$$

$$\therefore f_n = \frac{\omega_n}{2\pi} = \frac{21.65}{2\pi} = 3.45 \text{ Hz.}$$

Example 54: The magnitudes of the maximum and minimum forces in the belt are respectively
(A) 5424 N, 2424 N (B) 4862 N, 2986 N
(C) 4862 N, 2424 N (D) 5424 N, 2986 N

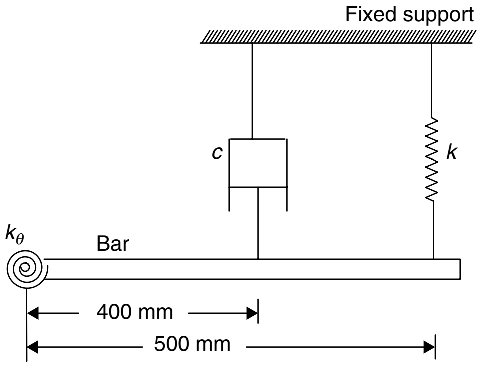
Solution:
Consider the portion of the belt connected to the spring. The static force on belt (when at rest)
= weight of block
= $mg = 400 \times 9.81 = 3924$ N
When the oscillation changes the length of spring by 5 mm, the tension is reduced or increased by
$kx = 300 \times 10^3 \times 5 \times 10^{-3} = 1500$ N
Hence, maximum force on this part of belt
= $mg + kx = 3924 + 1500 = 5424$ N
Minimum force on this part of belt
= $mg - kx$
= $3924 - 1500 = 2424$ N
Now, let us consider the portion of belt connected to the mass
The maximum acceleration occurs when the mass is about to reverse direction and this is when amplitude A = 5 mm.
In SHM, we have
$a_{max} = \omega^2 A = (21.65)^2 \times 5 \times 10^{-3}$
= 2.344 m/s^2
Force in the belt to accelerate mass
= ma_{max}
= 400×2.344
= 937.6 N
≈ 938 N
∴ Force in belt when maximum acceleration of mass is upwards
= $mg + ma_{max}$
= $3924 + 938 = 4862$ N

Force in belt when maximum acceleration of mass is downwards

$$= mg - ma_{max}$$
$$= 3924 - 938 = 2986 \text{ N}$$

Hence, the maximum tension in belt is 5424 N and minimum tension in belt is 2424 N.

Direction for questions (Examples 55 and 56):



A uniform rigid slender bar of mass 10 kg, hinged at the left end is suspended horizontally with the help of spring and damper arrangement as shown in the figure, where k = 2 kN/m, c = 500 Ns/m and the stiffness of the torsional spring k_θ is 1 kNm/rad. Ignore the hinge dimensions.

Example 55: The undamped natural frequency of oscillations of the bar about the hinge point is

(A) 42.43 rad/s (B) 30 rad/s
(C) 17.32 rad/s (D) 14.14 rad/s

Solution:

Mass of rod $m = 10$ kg

Length of rod, $L = 0.5$ m

Mass moment of inertia of rod about the hinge,

$$I = \frac{mL^2}{3} = \frac{10 \times 0.5^2}{3}$$
$$= 0.833 \text{ kg m}^2$$

Consider the instant when the rod has undergone a small angular displacement θ from its equilibrium position. The angular velocity of rod, $\omega = \dfrac{d\theta}{dt}$ and angular acceleration of rod, $\alpha = \dfrac{d^2\theta}{dt^2}$ at that instant.

Extension of spring $x_1 = L\theta = 0.5\theta$

Spring force $F_s = kx_1 = 2 \times 10^3 \times 0.5\theta$

$$= 1000\ \theta$$

Restoring torque due to spring force, $\tau_s = F_s L$

$$= 1000\ \theta \times 0.5$$
$$= 500\ \theta \text{ (Nm)}$$

Damping force, $F_d = cv = cr\omega$

$$= 500 \times 0.4 \times \frac{d\theta}{dt}$$
$$= 200 \frac{d\theta}{dt}$$

Restoring torque due to damper, $\tau_D = F_d r$

$$= 200 \frac{d\theta}{dt} \times 0.4$$
$$= 80 \frac{d\theta}{dt} \text{ (Nm)}$$

Restoring torque due to torsional spring, $\tau_\theta = k_\theta \theta$

$$= 1000\ \theta \text{ (Nm)}$$

Inertial torque on rod, $\tau_i = I\alpha$

$$= 0.833 \times \frac{d^2\theta}{dt^2}$$

For dynamic equilibrium for free vibration, $\tau_i + \tau_D + \tau_s + \tau_\theta = 0$

$$\Rightarrow 0.833 \frac{d^2\theta}{dt^2} + 80 \frac{d\theta}{dt} + 500\theta + 1000\theta = 0$$

$$\Rightarrow 0.833 \frac{d^2\theta}{dt^2} + 80 \frac{d\theta}{dt} + 1500\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{80}{0.833}\right)\frac{d\theta}{dt} + \left(\frac{1500}{0.833}\right)\theta = 0,$$

which is the differential equation for transverse vibration of the rod. This is of the form $\dfrac{d^2\theta}{dt^2} + 2\xi\omega_n\left(\dfrac{d\theta}{dt}\right) + \omega_n^2\theta = 0$

Comparing the coefficients of θ,

we get

$$\omega_n^2 = \frac{1500}{0.833} = 1800.72$$

$\Rightarrow \omega_n = \sqrt{1800.72} = 42.43$ rad/s.

Example 56: The damping coefficient in the vibration equation is given by

(A) 500 Nms/rad (B) 500 Ns/m
(C) 80 Nms/rad (D) 80 Ns/m

Solution:

The differential equation for the vibration is

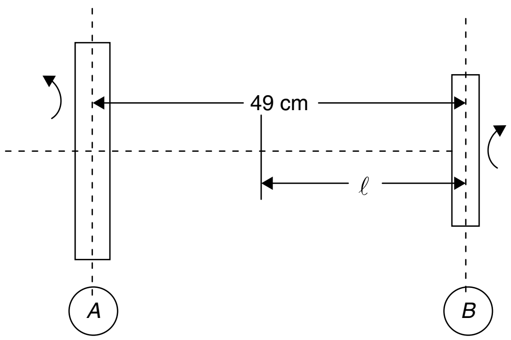
$0.833 \dfrac{d^2\theta}{dt^2} + 80 \dfrac{d\theta}{dt} + 1500\ \theta = 0$ as established earlier.

This is of the form $I \dfrac{d^2\theta}{dt^2} + c_{eq} \dfrac{d\theta}{dt} + k_{eq}\theta = 0$ where

k_{eq} = equivalent torsional spring constant and c_{eq} = equivalent torsional damping coefficient (in Nms/rad)

$\Rightarrow c_{eq} = 80$ Nms/rad.

Example 57:



The above figure shows two rotors A and B connected by an elastic shaft undergoing torsional vibration. The rotor A has a mass of 6 kg and a radius of gyration of 80 cm, while the rotor B has a mass of 4 kg and a radius of gyration of 40 cm. The distance ℓ (from rotor B) at which the node of torsional vibration occurs is

(A) 7 cm (B) 33 cm
(C) 36 cm (D) 42 cm

Solution:

$$I_A = m_A k_A^2 = 6 \times 0.8^2 = 3.84 \text{ kg m}^2$$

$$I_B = m_B k_B^2 = 4 \times 0.4^2 = 0.64 \text{ kg m}^2$$

We have $I_A L_A = I_B L_B$, where L_A = distance from rotor A to node and L_B = distance from rotor B to node = ℓ

Also, $L_A = (L - L_B) = (49 - L_B)$

$$\therefore L_B = \frac{I_A}{I_B} L_A = \frac{I_A}{I_B}(L - L_B)$$

$$= \frac{3.89}{0.64}(49 - L_B) = 6(49 - L_B)$$

$$\Rightarrow 7\, L_B = 6 \times 49$$

$$\Rightarrow L_B = \frac{6 \times 49}{7} = 42 \text{ cm}$$

Example 58: Two heavy rotating masses are connected by shafts of lengths L_1, L_2 and L_3 and the corresponding diameters are d_1, d_2 and d_3. The system is reduced to a torsionally equivalent system having uniform diameter d_1 of the shaft. The equivalent length of the shaft is

(A) $\dfrac{L_1 + L_2 + L_3}{3}$

(B) $L_1 + L_2\left(\dfrac{d_1}{d_2}\right)^3 + L_3\left(\dfrac{d_1}{d_3}\right)^3$

(C) $L_1 + L_2\left(\dfrac{d_1}{d_2}\right)^4 + L_3\left(\dfrac{d_1}{d_3}\right)^4$

(D) $L_1 + L_2 + L_3$

Solution:

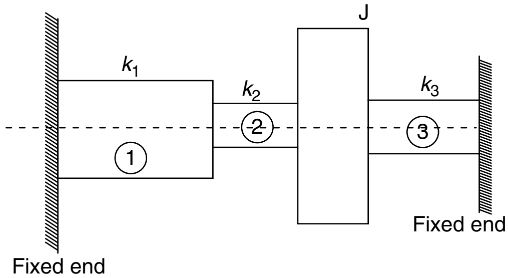
If d and L are the diameter and length of torsionally equivalent shaft, then

$$L = L_1\left(\frac{d}{d_1}\right)^4 + L_2\left(\frac{d}{d_2}\right)^4 + L_3\left(\frac{d}{d_3}\right)^{\prime}$$

Here, $d = d_1$

$$\Rightarrow L = L_1 + L_2\left(\frac{d_1}{d_2}\right)^4 + L_3\left(\frac{d_1}{d_3}\right)^4$$

Example 59:



Consider the arrangement shown in the figure where J is the **combined polar mass moment of inertia** of the disc and the shafts, k_1, k_2 and k_3 are the **torsional stiffnesses** of shafts 1, 2 and 3, respectively. The natural circular frequency of torsional oscillation of the disc is given by

(A) $\sqrt{\dfrac{k_1 + k_2 + k_3}{J}}$

(B) $\dfrac{k_1k_2 + k_2k_3 + k_3k_1}{J(k_1 + k_2)}$

(C) $\sqrt{\dfrac{k_1k_2k_3}{J(k_1k_2 + k_2k_3 + k_3k_1)}}$

(D) $\sqrt{\dfrac{k_1k_2 + k_2k_3 + k_3k_1}{J(k_2 + k_3)}}$

Solution:

$\omega_n = \sqrt{\dfrac{q_{\text{eff}}}{I_{\text{eff}}}}$; Given $I_{\text{eff}} = J$

q_{eff} = effective torsional stiffness of system

T_1 = Torque required to produce a twist of 1 radian on shaft 1
= torsional stiffness of shaft 1
= k_1 (data)

T_2 = Torque required to produce a twist of 1 radian on shaft 2
= torsional stiffness of shaft 2
= k_2 (data)

T_3 = torque required to produce a twist of 1 radian on shaft 3
= torsional stiffness of shaft 3
= k_3 (data)

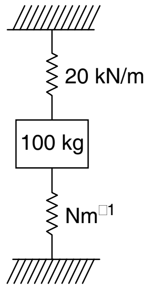
$\therefore$ T = torque required to rotate the rotor through 1 radian
$= T_1 + T_2 + T_3$
$= k_1 + k_2 + k_3$
= effective torsional stiffness of system

$\Rightarrow q_{\text{eff}} = k_1 + k_2 + k_3$

$$\therefore \omega_n = \sqrt{\frac{q_{\text{eff}}}{I_{\text{eff}}}} = \sqrt{\frac{k_1 + k_2 + k_3}{J}}$$

Exercises

Practice Problems I

1. 

A mass of 100 kg is held between two massless springs, each of spring constant 20 kN/m. The natural frequency of vibration of the system in cycles/second is

(A) $\frac{1}{2\pi}$ (B) $\frac{5}{\pi}$
(C) $\frac{10}{\pi}$ (D) $\frac{20}{\pi}$

2. The equation of the vibration of a system is
$\ddot{x} + 36\pi^2 x = 0$. Its natural frequency is
(A) 46 Hz (B) 3π Hz
(C) 3 Hz (D) 6π Hz

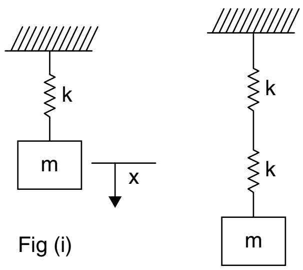
3. 

Fig (i)

Fig (ii)

For the spring mass system shown in
Fig. (i), the frequency of vibration is N. When one more identical spring is added in series (as in Fig. (ii)), the frequency of vibration becomes(springs are massless)

(A) $\frac{N}{2}$ (B) $\frac{N}{\sqrt{2}}$
(C) $\sqrt{2}\ N$ (D) 2N

4. If a block of mass m oscillates on a spring having a mass m_s and stiffness k, then the natural frequency of the system is given by

(A) $\sqrt{\frac{k}{\left(m + \frac{m_s}{3}\right)}}$ (B) $\sqrt{\frac{k}{\left(\frac{m}{3} + m_s\right)}}$

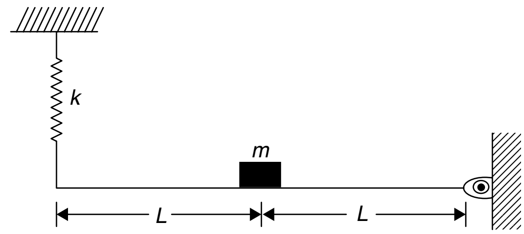
(C) $\sqrt{\frac{3k}{(m + m_s)}}$ (D) $\sqrt{\frac{k}{(m + m_s)}}$

5. A rod of uniform diameter is suspended from one of its ends in vertical plane. The mass of the rod is m and length ℓ. The natural frequency of oscillation of this rod (in Hz) for small amplitude angular oscillation is

(A) $\frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$ (B) $\frac{1}{2\pi}\sqrt{\frac{g}{3\ell}}$

(C) $\frac{1}{2\pi}\sqrt{\frac{2g}{3\ell}}$ (D) $\frac{1}{2\pi}\sqrt{\frac{3g}{2\ell}}$

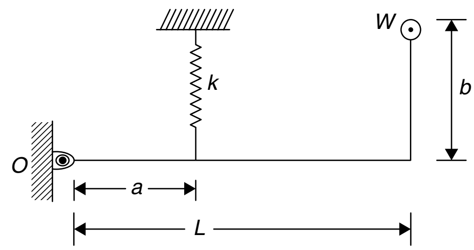
6. A uniform rigid rod of mass $m = 1$ kg and length $L = 1$ m is hinged at its centre and laterally supported at one end by a spring of spring constant $k = 300$ N/m. The natural frequency ω_n in rad/s is
(A) 10 (B) 20 (C) 30 (D) 40

7. 

A concentrated mass m is attached at the centre of a uniform rod of length $2L$ as shown in figure. The mass of rod is negligible and it is kept in a horizontal position by a spring of stiffness k. For a very small amplitude of vibration, neglecting the weights of the rod and spring, the undamped natural frequency of the system is

(A) $\sqrt{\frac{k}{m}}$ (B) $\sqrt{\frac{2k}{m}}$

(c) $\sqrt{\frac{k}{2m}}$ (D) $\sqrt{\frac{4k}{m}}$

8. 

For the system shown in figure, the moment of inertia of the weight W and the bar about the pivot point is I_0. The system will execute transverse vibrations in the vertical plane, when

(A) $b < \frac{ka^2}{W}$ (B) $b = \frac{ka^2}{W}$

(C) $b > \frac{ka^2}{W}$ (D) $a = 0$

9. A shaft has two heavy rotors mounted on it. The transverse natural frequencies, considering each of the rotor

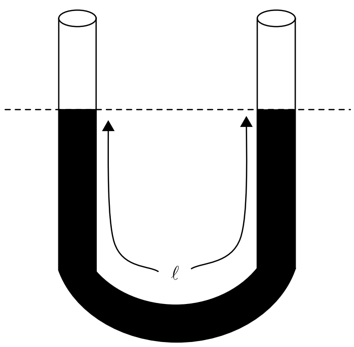
separately, are 100 cycles/second and 200 cycles/second, respectively. The lowest critical speed of the shaft is
(A) 5367 rpm (B) 6000 rpm
(C) 9360 rpm (D) 12,000 rpm

10. The critical speed of a rotating shaft depends upon
(A) mass.
(B) stiffness.
(C) mass and stiffness.
(D) mass, stiffness and eccentricity.

11. The critical speed of a uniform shaft with a rotor at the centre of the span can be reduced by
(A) reducing the shaft length.
(B) reducing the rotor mass.
(C) increasing the rotor mass.
(D) increasing the shaft diameter.

12. The static deflection of a shaft under a flywheel is 4 mm. Take $g = 10$ m/s^2. The critical speed of the shaft in rad/s is
(A) 50 (B) 20 (C) 10 (D) 5

13.



A U-tube of cross sectional area 'a', contains a liquid of density σ. The total length of the liquid column is ℓ (see figure). With a light tight-fitting plunger, the liquid column in one limb is pushed down by a small distance x and released. The natural frequency of subsequent free oscillations of the liquid column is

(A) $\sqrt{\frac{2g}{\ell}}$ (B) $\sqrt{\frac{g}{\ell}}$ (C) $\sqrt{\frac{g}{2\ell}}$ (D) $\sqrt{\frac{3g}{2\ell}}$

14. When a uniform solid shaft, supported horizontally on needle bearings (treat as simply supported), rotates at a speed of 1233 rpm, two nodes were observed between the two bearings. The speed at which the shaft should rotate so that only one node is observed in between the two bearings is
(A) 822 rpm (B) 548 rpm
(C) 616.5 rpm (D) 137 rpm

Direction for questions 15 to 17: A rolor has a mass of 12 kg and mounted midway on a 30 mm diameter shaft supported at the ends on short bearings. The bearings are 1 m apart. The centre of mass of rolor is 0.11 mm away from the axis of the shaft. The shaft rotates at 2500 rpm.

Take $E = 2 \times 10^{11}$ N/m^2 for material of shaft. Mass of shaft is negligible.

15. The fundamental frequency (in rad s^{-1}) of the system is
(A) 123.74 (B) 207.46
(C) 193.25 (D) 178.35

16. The amplitude of steady state vibration (in mm) is
(A) 0.3392 (B) 0.1895
(C) 0.2053 (D) 0.4133

17. The dynamic force on the bearing (in N) is
(A) 127.36 (B) 94.15
(C) 168.85 (D) 78.36

18. The equation of motion for a single degree freedom system with viscous damping is $16\ddot{x} + 5\dot{x} + 4x = 0$. The damping ratio ($\xi$) of the system is

(A) $\frac{5}{64}$ (B) $\frac{5}{16}$ (C) $\frac{5}{4\sqrt{2}}$ (D) $\frac{2}{5}$

19. In a vibrating system, the spring has stiffness 32 N/m and the mass 2 kg. The system is having a damper whose coefficient of viscous damping is 18 N s/m. The system is
(A) over damped. (B) under damped.
(C) critically damped. (D) undamped.

Direction for questions 20 and 21: A vibrating system consists of a mass 12.5 kg, a spring of stiffness 1000 N/m and a dashpot with damping coefficient of 15 Ns/m.

20. The value of critical damping coefficient of the system is (in Ns/m)
(A) 0.223 (B) 17.88 (C) 71.4 (D) 223.6

21. The value of logarithmic decrement is
(A) 1.35 (B) 0.42 (C) 0.68 (D) 0.66

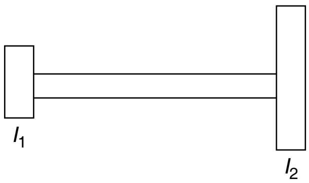
22. A free-damped vibrating system has a mass of 200 kg, a spring of stiffness 40 N/mm and a damping factor (ξ) of 0.22. The time (in second) in which the mass would settle down to deflection equal to $\frac{1}{50}^{\text{th}}$ of its initial deflection and the number of oscillations completed to reach this value of deflection are respectively
(A) 1.73, 3.54 (B) 0.9, 2.8
(C) 1.26, 2.76 (D) 2.2, 4.6

23. Large field guns which come to initial position after firing in shortest possible time are
(A) under damped.
(B) critically damped.
(C) over damped.
(D) undamped.

24. A vibrating system has mass 3 kg, stiffness 21 N/m and damper having damping coefficient 10 Ns/m. An exciting force of magnitude $27\sin 2t$ N is acting on the system. The time period of oscillation in the transient state is
(A) 3.14 s (B) 3.06 s (C) 3.27 s (D) 2.95 s

25. Transmissibility ratio will be equal to unity for all values of damping factor, if $\frac{\omega}{\omega_n}$ is equal to ________.

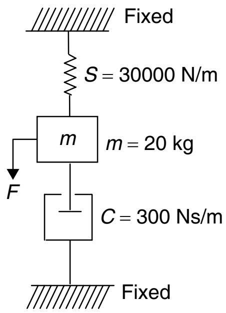
26.



In the two rotor system shown in the figure, ($I_1 < I_2$), a node of torsional vibration is situated

(A) between I_1 and I_2 but nearer to I_1.
(B) between I_1 and I_2 but nearer to I_2.
(C) exactly at the middle of the shaft.
(D) nearer to I_1 but outside.

Direction for questions 27 and 28:



A harmonic force $F = 800 \sin(30t)$ N is applied to a spring- mass- damper system shown in figure.

27. The amplitude of forced vibrations (steady state) will be (in mm)

(A) 26.7 (B) 53.4 (C) 48.06 (D) 65.42

28. The phase angle between the displacement and applied force in the steady state is

(A) 23.71° (B) 47.35° (C) 53.64° (D) 36.92°

Direction for questions 29 and 30: A machine of mass 700 kg is supported on four identical springs, connected in parallel, each spring having a stiffness of 350 kN/m. There is an unbalanced rotating element in the machine, which results in a disturbing force of 700 N at a speed of 4500 rpm. Assume a damping factor (ξ) of 0.25.

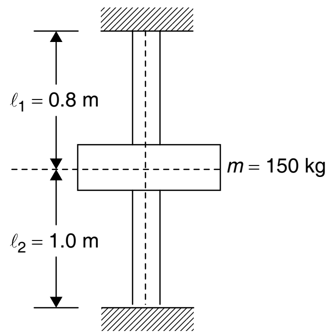
29. The transmissibility of the system is

(A) 0.049 (B) 0.09 (C) 0.13 (D) 0.19

30. The magnitude of the maximum force transmitted to the foundation is

(A) 133 N (B) 34.3 N
(C) 63 N (D) 91 N

31.

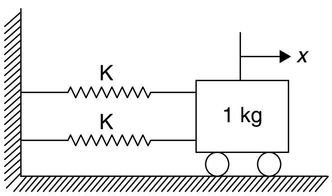


A disc of mass $m = 150$ kg and radius of gyration 0.5 m is mounted on a vertical shaft of diameter 50 mm as shown in figure. The modulus of rigidity of shaft material is 80 GN/m^2. The distance of disc from fixed supports are $\ell_1 = 0.8$ m and $\ell_2 = 1.0$ m respectively. The frequency of torsional vibration of the disc (in Hz) is

(A) 5.19 (B) 19.67
(C) 8.64 (D) 13.44

Practice Problems 2

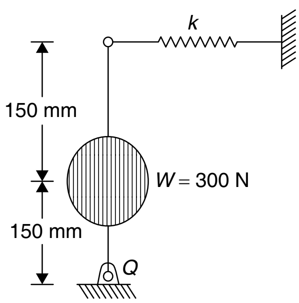
1.



A mass of 1 kg is attached to two identical springs, each with stiffness $K = 20$ kN/m as shown in the figure. Under frictionless conditions, the natural frequency of the system (in Hz) is close to

(A) 32 (B) 23 (C) 16 (D) 1

2.



A uniform stiff rod of length 300 mm and having a weight of 300 N is pivoted at one end and connected to a spring at the other end. For keeping the rod vertical in a stable equilibrium position, the minimum value of spring constant k needed is

(A) 300 N/m (B) 400 N/m
(C) 500 N/m (D) 1000 N/m

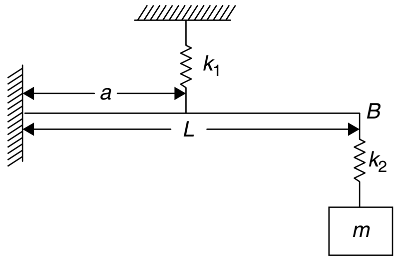
3. Under logarithmic decrement, the amplitudes of successive vibrations are

(A) constant.
(B) in arithmetic progression.
(C) in geometric progression.
(D) in logarithmic progression.

4. A simple spring mass vibrating system has a natural frequency of N. If the spring stiffness is halved and the mass is doubled, then the natural frequency will become

(A) $\frac{N}{2}$ (B) 2N (C) 4N (D) 8N

5. A mass of 1 kg is attached to the end of a spring with a stiffness of 0.7 N/mm. The critical damping coefficient of this system in (N s/m) is ________.

6. 

In the system shown in figure, bar AB is assumed to be rigid and weightless. The natural frequency of vibration of the system is given by

(A) $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1 k_2 \left(\dfrac{a}{L}\right)^2}{m\left[k_2 + \left(\dfrac{a}{L}\right)^2 k_1\right]}}$

(B) $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1 k_2 a}{mL(k_1 + k_2)}}$

(C) $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_1 L}{m k_2 a}}$

(D) $f_n = \dfrac{1}{2\pi}\left(\dfrac{L}{a}\right)\sqrt{\dfrac{k_1 k_2}{(k_1 + k_2)m}}$

7. The value of the natural frequency obtained by Rayleigh's method
   (A) is always greater than the actual fundamental frequency.
   (B) is always less than the actual fundamental frequency.
   (C) depends upon the initial deflection curve chosen and may be greater than or less than the actual fundamental frequency.
   (D) is independent of the initial deflection curve chosen.

8. The mass moment of inertia of the two rotors in a two-rotor system are 100 kg m^2 and 10 kg m^2 respectively. The length of shaft of uniform diameter between the rotors is 110 cm. The distance of the node from the rotor of lower moment of inertia is
   (A) 80 cm (B) 90 cm
   (C) 100 cm (D) 110 cm

9. If a spring- mass dashpot system is subjected to excitation by a constant amplitude harmonic force, then at resonance ($\omega = \omega_n$), its amplitude of vibration will be
   (A) infinity.
   (B) inversely proportional to damping factor (ξ).
   (C) directly proportional to damping factor (ξ).
   (D) decreases exponentially with time.

10. The natural frequency (in rad/s) of transverse vibrations of a massless simply supported beam of length L, having a mass m attached at its mid span is given by
   (A) $\left(\dfrac{mL^3}{48\,EI}\right)^{1/2}$ (B) $\left(\dfrac{48\,mL^3}{EI}\right)^{1/2}$
   (C) $\left(\dfrac{48\,EI}{mL^3}\right)^{1/2}$ (D) $\left(\dfrac{3\,EI}{mL^3}\right)^{1/2}$

11. During torsional vibration of a shaft, the node is characterised by the
   (A) maximum angular velocity.
   (B) maximum angular displacement.
   (C) maximum angular acceleration.
   (D) zero angular displacement.

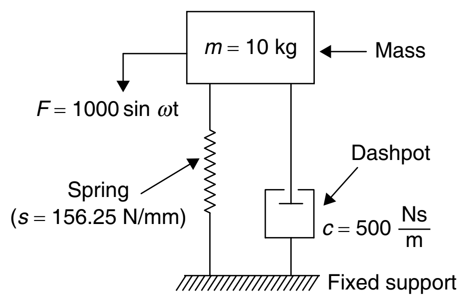
12. A shaft of 50 mm diameter and 1 m length carries a disc which has mass eccentricity equal to 190 micron. The displacement of the shaft at a speed which is 90% of initial speed, expressed in micron, is
   (A) 810 (B) 900 (C) 800 (D) 820

13. The differential equation of a vibrating system is given as $4\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 15x = 20\cos 4t$. In the steady state, the maximum amplitude of vibration of the system is (x is in metre, t in second)
   (A) 52.1 cm (B) 45.7 cm
   (C) 34.2 cm (D) 25.3 cm

14. A vibrating system consists of a mass of 250 kg a spring of stiffness 80 N/mm and a damper with damping coefficient $1280\ \dfrac{\text{Ns}}{\text{m}}$. The frequency of vibration of this system is
   (A) 2.82 Hz (B) 4.37 Hz
   (C) 7.93 Hz (D) 12.31 Hz

15. The differential equation for a vibrating system is given as $12\dfrac{d^2x}{dt^2} + 1000x = 0$. The time period of vibration of the system is
   (A) 3.14 *s* (B) 1.57 *s* (C) 0.32 *s* (D) 0.69 *s*

Direction for questions 16 to 20:



A spring-mass-dashpot system has a mass of 10 kg, connected by a massless spring of stiffness 156.25 N/mm and a dashpot of damping coefficient $500\ \dfrac{\text{Ns}}{\text{m}}$ as shown in figure. A periodic force F = 1000 *sinωt* N acts on the mass such that it executes 1-*D* force- damped oscillations in the vertical plane. The value of ω (in rad/s) is such that the amplitude of oscillation of the system in the steady state is the maximum possible value of amplitude for this system.

16. The frequency of applied force, ω (in rad/s) is
(A) 125 (B) 119.9 (C) 122.5 (D) 130.3

17. The magnification factor (*MF*) is equal to
(A) 2.50 (B) 2.55 (C) 2.46 (D) 2.73

18. The phase difference between the applied force and the displacement of the mass is
(A) 90° (B) 50.33°
(C) 78.22° (D) 45°

19. The transmissibility of the system is
(A) 2.96 (B) 2.50 (C) 2.55 (D) 2.73

20. The magnitude of the maximum force transmitted to the support (in N) is
(A) 2730 (B) 2960
(C) 1000 (D) 2500

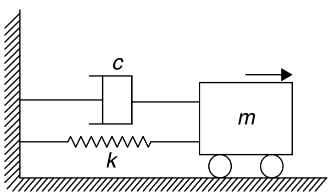
Direction for questions 21 to 23: A machine is mounted on three springs connected in parallel and also fitted with a dashpot. The other ends of springs and dashpot are rigidly fixed. The mass of the machine is 100 kg and the stiffness of springs are 12 N/mm, 14 N/mm and 16 N/mm, respectively. The amplitude of vibration decreases from 60 mm to 12 mm in 2 oscillations.

21. The damping factor (ξ) of the system is
(A) 0.0985 (B) 0.1265
(C) 0.1539 (D) 0.1743

22. The damping coefficient of the damper $\left(\text{in } \dfrac{\text{Ns}}{\text{m}}\right)$ is
(A) 225.7 (B) 369.8
(C) 469.3 (D) 518.5

23. The time period of damped vibration (in s) is
(A) 0.152 (B) 0.247
(C) 0.309 (D) 0.336

24.



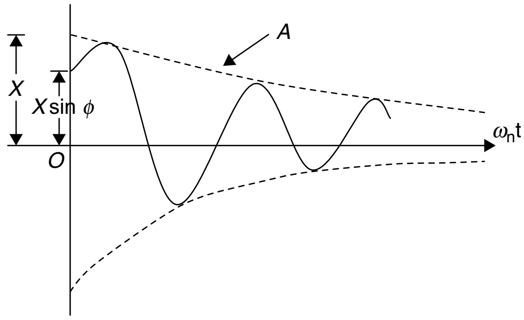
Smooth floor

In the system shown in figure, the stiffness of the spring is 100 N/mm and the damping coefficient of the dashpot is 1000 Ns/m. The system with the damper is known to be a critically damped system. If the dashpot is now removed and the system is set into 1 *D* longitudinal vibrations, the frequency of free vibrations will be (in Hz) nearly
(A) 300 Hz (B) 200 Hz
(C) 50 Hz (D) 32 Hz

25. The logarithmic decrement of a damped single degree of freedom system is δ. If the stiffness of the spring is doubled and the mass is made half, then the logarithmic decrement of the new system will be equal to
(A) $\dfrac{\delta}{4}$ (B) $\dfrac{\delta}{2}$
(C) δ (D) 2δ

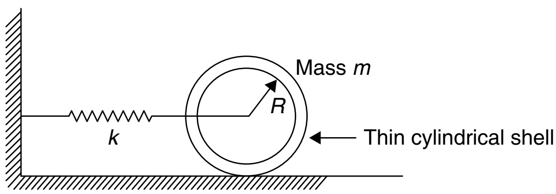
26. In a multi-rotor system of torsional vibrations, maximum number of nodes that can occur is
(A) Two.
(B) equal to number of rotors.
(C) equal to number of rotors plus one.
(D) equal to number of rotors minus one.

27.



A damped free vibration is expressed by the general equation $x = Xe^{(-\xi\omega_n t)}\sin\left(\sqrt{1-\xi^2}\,\omega_n t+\phi\right)$, which is shown graphically above. The envelope *A* has the equation.
(A) Xe^{-t} (B) $X\sin\left(\sqrt{1-\xi^2}\right)\omega_n t$
(C) $e^{-\xi\omega_n t}$ (D) $Xe^{-\xi\omega_n t}$

28.



A thin cylindrical shell of radius *R* and mass *m* is connected to a light, horizontal spring of stiffness *k* shown in figure. If the thin cylindrical shell is free to roll on horizontal surface without slipping, its natural frequency (in rad/s) is
(A) $\sqrt{\dfrac{2k}{3m}}$ (B) $\sqrt{\dfrac{k}{2m}}$
(C) $\sqrt{\dfrac{k}{m}}$ (D) $\sqrt{\dfrac{2k}{m}}$

29. Rayleigh's method computing the fundamental natural frequency is based on
(A) conservation of energy.
(B) conservation of momentum.
(C) conservation of masses.
(D) laws of statics.

30. Consider the following statements. Transmissibility of vibrations

(i) is more than 1, when $\frac{\omega}{\omega_n} < \sqrt{2}$

(ii) is less than 1, when $\frac{\omega}{\omega_n} > \sqrt{2}$

(iii) increases as damping is increased.

The correct statements are

(A) (i), (ii) and (iii)
(B) (i) and (ii) only
(C) (ii) and (iii) only
(D) (i) and (iii) only

31. Which of the following type of viscous damping will give periodic motion to the vibrating body?

(i) under-damping
(ii) critical-damping
(iii) over-damping

(A) (i) only (B) (ii) only
(C) (iii) only (D) (i) and (ii) only

32. In case of free vibrations with viscous damping, the damping force is proportional to

(A) the displacement.
(B) the velocity.
(C) the acceleration.
(D) the natural frequency.

33. A reciprocating engine, running at 80 rad/*s*, is supported on springs. The static deflection of the spring is 1 mm. Take $g = 10$ m/s^2. When the engine runs, what will be the frequency of vibration of the system?

(A) 80 rad/s (B) 90 rad/s
(C) 100 rad/s (D) 160 rad/s

34. A uniform vertical bar, fixed at upper end, carries a heavy concentrated mass at the other end. The system is executing longitudinal vibrations. The inertia of the bar may be taken into account by which one of the following portions of the mass of the bar at the free end?

(A) $\frac{5}{384}$ (B) $\frac{1}{48}$
(C) $\frac{33}{140}$ (D) $\frac{1}{3}$

35. The equation of motion of a damped viscous vibration is $3\ddot{x} + 9\dot{x} + 27x = 0$. The logarithmic decrement is

(A) 1.57 (B) 2.63 (C) 3.63 (D) 5.31

36. Critical speed of a shaft with a disc supported in between is equal to the natural frequency of the system in

(A) Transverse vibrations.
(B) Torsional vibrations.
(C) Longitudinal vibrations.
(D) Longitudinal vibrations provided the shaft is vertical.

37. A shaft has two heavy rotors mounted on it. The transverse natural frequencies, considering each of the rotor separately, are 150 cycles/second and 250 cycles/second, respectively. The lowest critical speed is (the shaft is weightless).

(A) 11357 rpm (B) 5367 rpm
(C) 9367 rpm (D) 7717 rpm

38. A shaft carries a weight *W* at the centre. The *CG* of the weight is displaced by an amount *e* from the axis of rotation. If *y* is the additional displacement of the *CG* from the axis of rotation due to the centrifugal force, Then, the ratio of *y* to *e* (where ω_c = critical speed of shaft and ω = angular speed of shaft) is given by

(A) $\dfrac{1}{\left(\dfrac{\omega_c}{\omega}\right)^2 + 1}$ (B) $\dfrac{1}{\left(\dfrac{\omega_c}{\omega}\right)^2 - 1}$

(C) $\left(\dfrac{\omega_c}{\omega}\right)^2 + 1$ (D) $\dfrac{\omega}{\left(\dfrac{\omega_c}{\omega}\right)^2 - 1}$

39. Consider a harmonic motion $x = 1.25 \sin\left(5t - \frac{\pi}{6}\right)$ cm.

Match List-I with List-II and select correct answer using the codes gives below the lists.

List-I	List-II
P. Amplitude (cm)	**1.** $\frac{5}{2\pi}$
Q. Frequency (cycle/s)	**2.** 1.25
R. Initial phase angle (rad)	**3.** $\frac{2\pi}{5}$
S. Time period (s)	**4.** $\frac{\pi}{6}$

Codes:

	P	Q	R	S
(A)	4	1	2	3
(B)	2	3	4	1
(C)	4	3	2	1
(D)	2	1	4	3

40. A vertical shaft 90 mm in diameter and 1200 mm long has its upper end fixed to the ceiling. The lower end carries a disc of weight 4000 N, having a radius of gyration of 300 mm. The modulus of rigidity of the material of the shaft is 0.8×10^5 N/mm^2. The frequency of torsional vibrations of the shaft is

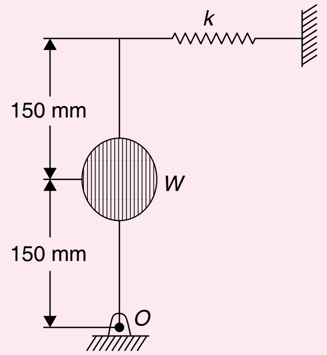
(A) 25.25 Hz
(B) 17.22 Hz
(C) 37.63 Hz
(D) 10.18 Hz

Previous Years' Questions

1. A vibrating machine is isolated from the floor using springs. If the ratio of excitation frequency of vibration of machine to the natural frequency of the isolation system is equal to 0.5, the transmissibility of ratio of isolation is **[2004]**

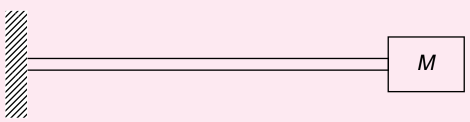
   (A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) $\frac{4}{3}$ (D) 2

2. A uniform stiff rod of length 300 mm and having a weight of 300 N is pivoted at one end and connected to a spring at the other end. For keeping the rod vertical in a stable position the minimum value of spring constant K needed is **[2004]**



   (A) 300 N/m (B) 400 N/m
   (C) 500 N/m (D) 1000 N/m

3. A mass M, of 20 kg is attached to the free end of a steel cantilever beam of length 1000 mm having a cross-section of 25×25 mm. Assume the mass of the cantilever to be negligible and $E_{steel} = 200$ GPa. If the lateral vibration of this system is critically damped using a viscous damper, the damping constant of the damper is **[2004]**



   (A) 1250 Ns/m (B) 625 Ns/m
   (C) 312.50 Ns/m (D) 156.25 Ns/m

4. There are four samples P, Q, R and S, with natural frequencies 64, 96, 128 and 256 Hz respectively. These are mounted on test setups for conducting vibration experiments. If a loud pure note of frequency 144 Hz is produced by some instrument, which of the samples will show the most perceptible induced vibration? **[2005]**

   (A) P (B) Q (C) R (D) S

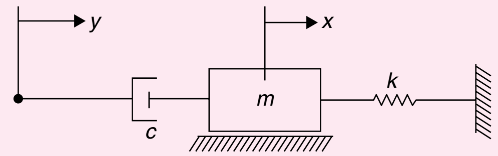
5. In a spring-mass system, the mass is 0.1 kg and the stiffness of the spring is 1 kN/m. By introducing a damper, the frequency of oscillation is found to be 90% of the original value. What is the damping coefficient of the damper? **[2005]**

   (A) 1.2 N.s/m (B) 3.4 N.s/m
   (C) 8.7 N.s/m (D) 120.N.s/m

6. A weighing machine consists of a 2 kg pan resting on a spring. In this condition, the pan resting on the spring, the length of the spring is 200 mm. When a mass of 20 kg is placed on the pan, the length of the spring becomes 100 mm. For the spring, the undeformed length I_0 and the spring constant k (stiffness) are **[2006]**

   (A) $I_0 = 220$ mm, $k = 1862$ N/m
   (B) $I_0 = 210$ mm, $k = 1960$ N/m
   (C) $I_0 = 200$ mm, $k = 1960$ N/m
   (D) $I_0 = 200$ mm, $k = 2156$ N/m

7. The differential equation governing the vibrating system is: **[2006]**



   (A) $m\ddot{x} + c\dot{x} + k(x–y) = 0$

   (B) $m(\dot{x} - \dot{y}) + c(\dot{x} - \dot{y}) + kx = 0$

   (C) $m\ddot{x} + c\,\dot{x} - \dot{y} + kx = 0$

   (D) $m(\ddot{x} - \ddot{y}) + c(\dot{x} - \dot{y}) + k(x - y) = 0$

8. A machine of 250 kg mass is supported on springs of total stiffness 100 kN/m. Machine has an unbalanced rotating force of 350 N at speed of 3600 rpm. Assuming a damping factor of 0.15, the value of transmissibility ratio is: **[2006]**

   (A) 0.0531 (B) 0.9922
   (C) 0.0162 (D) 0.0028

Direction for questions 9 and 10: A vibratory system consists of a mass 12.5 kg, a spring of stiffness 1000 N/m, and a dashpot with damping coefficient of 15 Ns/m.

9. The value of critical damping of the system is: **[2006]**

   (A) 0.223 Ns/m (B) 17.88 Ns/m
   (C) 71.4 Ns/m (D) 223.6 Ns/m

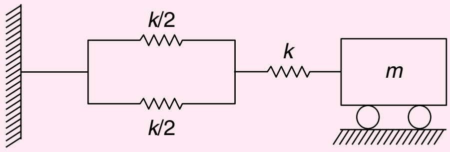
10. The value of logarithmic decrement is: **[2006]**

   (A) 1.35 (B) 1.32 (C) 0.68 (D) 0.66

11. For an under-damped harmonic oscillator, resonance **[2007]**

   (A) occurs when excitation frequency is greater than undamped natural frequency
   (B) occurs when excitation frequency is less than undamped natural frequency
   (C) occurs when excitation frequency is equal to undamped natural frequency
   (D) never occurs

12. The natural frequency of the system shown below is **[2007]**



(A) $\sqrt{\dfrac{k}{2m}}$ (B) $\sqrt{\dfrac{k}{m}}$

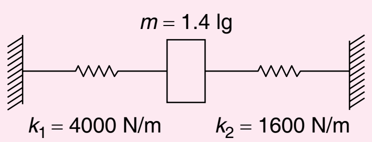
(C) $\sqrt{\dfrac{2k}{m}}$ (D) $\sqrt{\dfrac{3k}{m}}$

13. The equation of motion of a harmonic oscillator is given by $\dfrac{d^2x}{dt^2}+2\zeta\omega_n\dfrac{dx}{dt}+\omega_n^2x=0$, and the initial conditions at $t=0$ are $x(0)=X, \dfrac{dx}{dt}(0)$. The amplitude of $x(t)$ after n complete cycles is **[2007]**

(A) $Xe^{-2n\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}$ (B) $Xe^{2n\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}$

(C) $Xe^{-2n\pi\frac{\sqrt{1-\zeta^2}}{\zeta}}$ (D) X

14. The natural frequency of the spring mass system shown in the figure is closest to **[2008]**



(A) 8 Hz (B) 10 Hz
(C) 12 Hz (D) 14 Hz

15. A uniform rigid rod of mass m = 1 kg and length L = 1 m is hinged at its center and laterally supported at one end by a spring of spring constant k = 300 N/m. The natural frequency ω_n in rad/s is **[2006]**

(A) 10 (B) 20 (C) 30 (D) 40

16. The rotor shaft of a large electric motor supported between short bearings at both ends shows a deflection of 1.8 mm in the middle of the rotor. Assuming the rotor to be perfectly balanced and supported at knife edges at both the ends, the likely critical speed (in rpm) of the shaft is **[2009]**

(A) 350 (B) 705 (C) 2810 (D) 4430

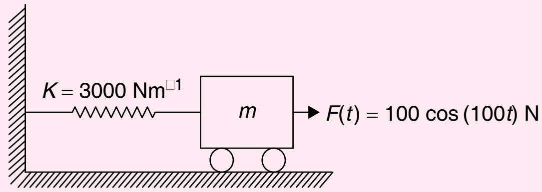
17. A vehicle suspension system consists of a spring and a damper. The stiffness of the spring is 3.6 kN/m and the damping constant of the damper is 400 Ns/m. If the mass is 50 kg, then the damping factor (d) and damped natural frequency (f_n), respectively, are **[2009]**

(A) 0.471 and 1.19 Hz
(B) 0.471 and 7.48 Hz
(C) 0.666 and 1.35 Hz
(D) 0.666 and 8.50 Hz

18. The natural frequency of a spring-mass system on earth is ω_n. The natural frequency of this system on the moon ($g_{moon} = g_{earth}/6$) is **[2010]**

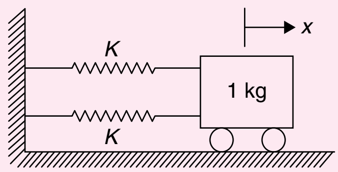
(A) ω_n (B) $0.408\omega_n$
(C) $0.204\omega_n$ (D) $0.167\omega_n$

19. A mass m attached to a spring is subjected to a harmonic force as shown in figure. The amplitude of the forced motion is observed to be 50 mm. The value of m (in kg) is **[2010]**



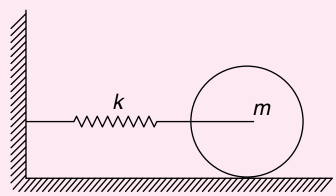
(A) 0.1 (B) 1.0 (C) 0.3 (D) 0.5

20. A mass of 1 kg is attached to two identical springs each with stiffness k = 20 kN/m as shown in the figure. Under frictionless condition, the natural frequency of the system in Hz is close to **[2011]**



(A) 32 (B) 23 (C) 16 (D) 11

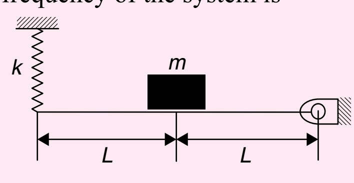
21. A disc of mass m is attached to a spring of stiffness k as shown in the figure. The disc rolls without slipping on a horizontal surface. The natural frequency of vibration of the system is. **[2011]**



(A) $\dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$ (B) $\dfrac{1}{2\pi}\sqrt{\dfrac{2k}{m}}$

(C) $\dfrac{1}{2\pi}\sqrt{\dfrac{2k}{3m}}$ (D) $\dfrac{1}{2\pi}\sqrt{\dfrac{3k}{2m}}$

22. A concentrated mass m is attached at the centre of a rod of length $2L$ as shown in the figure. The rod is kept in horizontal equilibrium position by a spring of stiffness k. For very small amplitude of vibration, neglecting the weights of the rod and spring, the undamped natural frequency of the system is **[2011]**



(A) $\sqrt{\frac{k}{m}}$ (B) $\sqrt{\frac{2k}{m}}$

(C) $\sqrt{\frac{k}{2m}}$ (D) $\sqrt{\frac{4k}{m}}$

23. If two nodes are observed at a frequency of 1800 rpm during whirling of a simply supported long slender rotating shaft, the first critical speed of the shaft in rpm is **[2013]**

(A) 200 (B) 450

(C) 600 (D) 900

24. A single degree of freedom system having mass 1 kg and stiffness 10 kN/m initially at rest is subjected to an impulse force of magnitude 5 kN for 10^{-4} seconds. The amplitude in mm of the resulting free vibration is **[2013]**

(A) 0.5 (B) 1.0

(C) 5.0 (D) 10.0

25. Critical damping is the **[2014]**

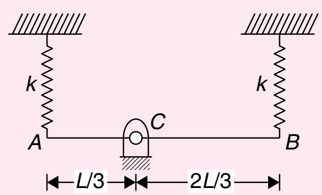
(A) largest amount of damping for which no oscillation occurs in free vibration.

(B) smallest amount of damping for which no oscillation occurs in free vibration.

(C) largest amount of damping for which the motion is simple harmonic in free vibration.

(D) smallest amount of damping for which the motion is simple harmonic in free vibration.

26. A rigid uniform rod *AB* of length *L* and mass m is hinged at *C* such that $AC = L/3$, $CB = 2L/3$. Ends *A* and *B* are supported by springs of spring constant *k*. The natural frequency of the system is given by **[2014]**



(A) $\sqrt{\frac{k}{2m}}$ (B) $\sqrt{\frac{k}{m}}$

(C) $\sqrt{\frac{2k}{m}}$ (D) $\sqrt{\frac{5k}{m}}$

27. In vibration isolation, which one of the following statements is **NOT** correct regarding Transmissibility (*T*)? **[2014]**

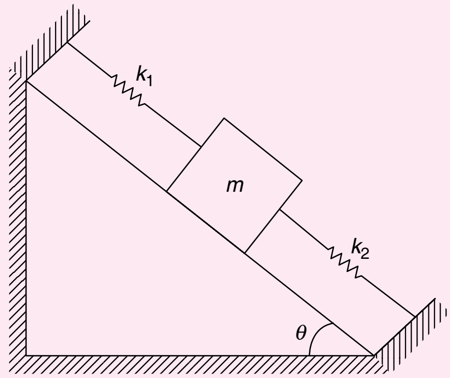
(A) *T* is nearly unity at small excitation frequencies.

(B) *T* can be always reduced by using higher damping at any excitation frequency.

(C) *T* is unity at the frequency ratio of $\sqrt{2}$.

(D) *T* is infinity at resonance for undamped systems.

28. What is the natural frequency of the spring mass system shown below? The contact between the block and the inclined plane is frictionless. The mass of the block is denoted by *m* and the spring constants are denoted by k_1 and k_2 as shown below. **[2014]**



(A) $\sqrt{\frac{k_1+k_2}{2m}}$ (B) $\sqrt{\frac{k_1+k_2}{4m}}$

(C) $\sqrt{\frac{k_1-k_2}{m}}$ (D) $\sqrt{\frac{k_1+k_2}{m}}$

29. Consider a single degree-of-freedom system with viscous damping excited by a harmonic force. At resonance, the phase angle (in degree) of the displacement with respect to the exciting force is **[2014]**

(A) 0 (B) 45

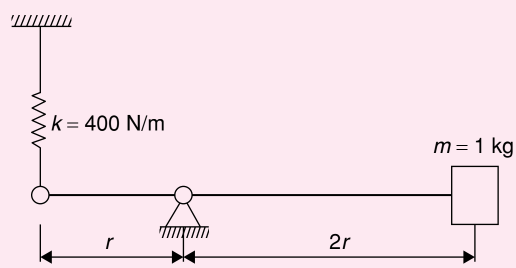
(C) 90 (D) 135

30. The damping ratio of a single degree of freedom spring-mass-damper system with mass of 1 kg, stiffness 100 N/m and viscous damping coefficient of 25 N.s/m is _______ **[2014]**

31. A point mass is executing simple harmonic motion with an amplitude of 10 mm and frequency of 4 Hz. The maximum acceleration (m/s^2) of the mass is _____ **[2014]**

32. A single degree of freedom system has a mass of 2 kg, stiffness 8 N/m and viscous damping ratio 0.02. The dynamic magnification factor at an excitation frequency of 1.5 rad/s is _____ **[2014]**

33. Considering massless rigid rod and small oscillations, the natural frequency (in rad/s) of vibration of the system shown in the figure is **[2015]**



(A) $\sqrt{\frac{400}{1}}$ (B) $\sqrt{\frac{400}{2}}$

(C) $\sqrt{\frac{400}{3}}$ (D) $\sqrt{\frac{400}{4}}$

34. A precision instrument package ($m = 1$ kg) needs to be mounted on a surface vibrating at 60 Hz. It is desired that only 5% of the base surface vibration amplitude be transmitted to the instrument. Assume that the isolation is designed with its natural frequency significantly lesser than 60 Hz, so that the effect of damping may be ignored. The stiffness (in N/m) of the required mounting pad is ______. **[2015]**

35. In a spring-mass system, the mass is m and the spring constant is k. The critical damping coefficient of the system is 0.1 kg/s. In another spring-mass system, the mass is $2m$ and the spring constant is $8k$. The critical damping coefficient (in kg/s) of this system is ______. **[2015]**

36. A slider-degree-freedom spring-mass system is subjected to a sinusoidal force of 10 N amplitude and frequency ω along the axis of the spring. The stiffness of the spring is 150 N/m, damping factor is 0.2 and the undamped natural frequency is 10ω. At steady state, the amplitude of vibration (in m) is approximately: **[2015]**

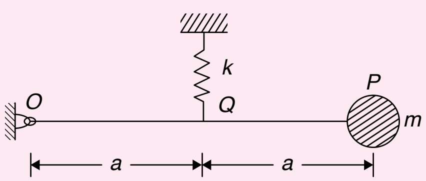
(A) 0.05 (B) 0.07
(C) 0.70 (D) 0.90

37. Which of the following statements are TRUE for damped vibrations? **[2015]**

(P) For a system having critical damping, the value of damping ratio is unity and system does not undergo a vibratory motion.
(Q) Logarithmic decrement method is used to determine the amount of damping in a physical system.
(R) In case of damping due to dry friction between moving surfaces resisting forces of constant magnitude acts opposite to the relative motion.
(S) For the case of viscous damping, drag force is directly proportional to the square of relative velocity.

(A) P and Q only (B) P and S only
(C) P, Q and R only (D) Q and S only

38. Figure shows a single degree of freedom system. The system consists of a massless rigid bar OP hinged at O and mass m at end P. The natural frequency of vibration of the system is: **[2015]**

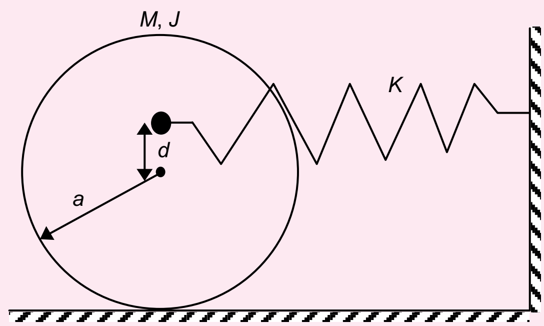


(A) $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{4m}}$ (B) $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{2m}}$

(C) $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (D) $f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

39. A single degree of freedom spring mass system with viscous damping has a spring constant of 10 kN/m. The system is excited by a sinusoidal force of amplitude 100 N. If the damping factor (ratio) is 0.25, the amplitude of steady state oscillation at resonance is ______ mm. **[2016]**

40. A solid disc with radius a is connected to a spring at a point d above the center of the disc. The other end of the spring is fixed to the vertical wall. The disc is free to roll without slipping on the ground. The mass of the disc is M and the spring constants is K. The polar moment of inertia for the disc about its center is $J = Ma^2/2$.



The natural frequency of this system is rad/s is given by: **[2016]**

(A) $\sqrt{\frac{2K(a+d)^2}{3Ma^2}}$ (B) $\sqrt{\frac{2K}{3M}}$

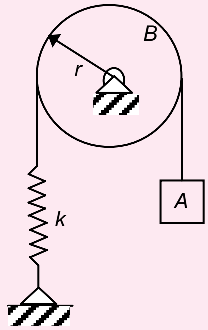
(C) $\sqrt{\frac{2K(a+d)^2}{Ma^2}}$ (D) $\sqrt{\frac{K(a+d)^2}{Ma^2}}$

41. A single degree of freedom mass-spring-viscous damper system with mass m, spring constant k and viscous damping coefficient q is critically damped. The correct relation among m, k and q is: **[2016]**

(A) $q = \sqrt{2km}$ (B) $q = 2\sqrt{km}$

(C) $q = \sqrt{\frac{2k}{m}}$ (D) $q = 2\sqrt{\frac{k}{m}}$

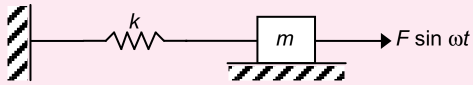
42. The system shown in the figure consists of block A of mass 5 kg connected to a spring through a mass less rope passing over pulley B of radius r and mass 20 kg. The spring constant k is 1500 N/m. If there is no slipping of the rope over the pulley, the natural frequency of the system is _____ rad/s. **[2016]**



43. The static deflection of a spring under gravity, when a mass of 1 kg is suspended from it, is 1 mm. Assume the acceleration due to gravity $g = 10\ \text{m/s}^2$. The natural frequency of this spring-mass system (in rad/s) is ______.

[2016]

44. A single degree of freedom spring-mass is subjected to a harmonic force of constant amplitude. For an excitation frequency of $\sqrt{\frac{3k}{m}}$, the ratio of the amplitude of steady state response to the static deflection of the spring is __________. **[2016]**



Answer Keys

Exercises

Practice Problems 1

1. C	**2.** C	**3.** B	**4.** A	**5.** D	**6.** C	**7.** D	**8.** A	**9.** A	**10.** C
11. C	**12.** A	**13.** A	**14.** B	**15.** D	**16.** C	**17.** D	**18.** B	**19.** A	**20.** D
21. B	**22.** C	**23.** B	**24.** B	**25.** $\sqrt{2}$	**26.** B	**27.** B	**28.** D	**29.** A	**30.** B
31. C									

Practice Problems 2

1. A	**2.** C	**3.** C	**4.** A	**5.** 52.92	**6.** A	**7.** A	**8.** C	**9.** B	**10.** C
11. D	**12.** A	**13.** C	**14.** A	**15.** D	**16.** B	**17.** B	**18.** C	**19.** D	**20.** A
21. B	**22.** D	**23.** C	**24.** D	**25.** C	**26.** D	**27.** D	**28.** B	**29.** A	**30.** B
31. A	**32.** B	**33.** A	**34.** D	**35.** C	**36.** A	**37.** D	**38.** B	**39.** D	**40.** B

Previous Years' Questions

1. C	**2.** C	**3.** A	**4.** C	**5.** C	**6.** B	**7.** C	**8.** C	**9.** D	
10. None	**11.** C	**12.** A	**13.** A	**14.** B	**15.** C	**16.** B	**17.** A	**18.** A	**19.** A
20. A	**21.** C	**22.** D	**23.** D	**24.** C	**25.** B	**26.** D	**27.** B	**28.** D	**29.** C
30. 1.24 to 1.26		**31.** 6.3 to 6.4		**32.** 2 to 2.4	**33.** D	**34.** 6750 to 7150		**35.** 0.38 to 0.42	
36. B	**37.** C	**38.** A	**39.** 20	**40.** A	**41.** B	**42.** 10	**43.** 100	**44.** 0.5	