



# BINOMIAL THEOREM

## INTRODUCTION

In previous classes, you have learnt the squares and cubes of binomial expressions like  $a + b$ ,  $a - b$  and used these to find the values of numbers like  $(103)^2$ ,  $(998)^3$  by expressing these as  $(103)^2 = (100 + 3)^2$ ,  $(998)^3 = (1000 - 2)^3$  etc. However, for higher powers like  $(103)^7$ ,  $(998)^9$ , the calculations become difficult by repeated multiplication. This problem of evaluation of such numbers was overcome by using a result called **Binomial theorem**. The general form of the binomial expression is  $a + b$  and the expansion of  $(a + b)^n$ ,  $n \in \mathbf{N}$ , is called the **binomial theorem** for positive integral index. **The binomial theorem enables us to expand any power of a binomial expression.** It was first given by Sir Isaac Newton.

## Development of Binomial Theorem

We know that

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \text{ etc.}$$

(Assume  $a + b \neq 0$ )

From the above expansions, we observe that :

- (i) The total number of terms in each expansion is one more than the index. For example, in the expansion of  $(a + b)^3$ , the number of terms is 4 whereas the index of  $(a + b)^3$  is 3.
- (ii) The powers (indices) of the first quantity ' $a$ ' goes on decreasing by 1 whereas the powers of the second quantity ' $b$ ' goes on increasing by 1, in successive terms.
- (iii) In each of the expansion, the sum of indices of  $a$  and  $b$  is the same and is equal to the index of  $(a + b)$ . For example, in each term of the expansion of  $(a + b)^3$ , the sum of indices of  $a$  and  $b$  is 3.

The coefficients of the terms in the above expansions can be written in the form of a table as :

| INDEX OF BINOMIAL | COEFFICIENTS OF VARIOUS TERMS |   |   |   |   |
|-------------------|-------------------------------|---|---|---|---|
| 0                 | 1                             |   |   |   |   |
| 1                 | 1                             |   | 1 |   |   |
| 2                 | 1                             |   | 2 | 1 |   |
| 3                 | 1                             | 3 | 3 | 1 |   |
| 4                 | 1                             | 4 | 6 | 4 | 1 |

We observe that the coefficients form a certain pattern.

We notice that :

- (i) each row starts with 1 and ends with 1.
- (ii) leaving first two rows i.e. from third row onwards, each coefficient (except the first and the last) in a row is the sum of two coefficients in the preceding row, one just before it and the other just after it.

The above pattern (arrangement of numbers) is known as **Pascal's Triangle**.

In this pattern, the numbers involved in addition and the results can be indicated as shown in the table below. The table can be extended by writing a few more rows :

| INDEX OF BINOMIAL | COEFFICIENTS OF VARIOUS TERMS                                                                                                                                                                                                                                                                                                                                                                                                            |
|-------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 0                 | 1                                                                                                                                                                                                                                                                                                                                                                                                                                        |
| 1                 | 1  1                                                                                                                                                                                                                                                                                                                                                    |
| 2                 | 1  2  1                                                                                                                                                                                                                                                                |
| 3                 | 1  3  3  1                                                                                                                                                                            |
| 4                 | 1  4  6  4  1                                                                                        |
| 5                 | 1  5  10  10  5  1 |
| 6                 | 1    6    15    20    15    6    1                                                                                                                                                                                                                                                                                                                                                                                                       |
| ...               | ... ..                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| ...               | ... ..                                                                                                                                                                                                                                                                                                                                                                                                                                   |

The above table can be continued till any index we like. Expansions for the higher powers of Binomial can be written by using Pascal's triangle. For example, let us expand  $(a + b)^6$  by using Pascal's triangle. The row for index 6 is

1        6        15        20        15        6        1

Using this row for coefficients and the observations (i), (ii) and (iii), we get

$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

By making use of the concept of combinations i.e.  ${}^nC_r = \frac{{}^n P_r}{r!}, 0 \leq r \leq n, n$  a non-negative integer, also  ${}^nC_n = 1 = {}^nC_0$ , the binomial expansions can be written as

$(a + b)^1 = a + b$   
 $\qquad\qquad\qquad = {}^1C_0 a^1 + {}^1C_1 b^1$   
 $(a + b)^2 = a^2 + 2ab + b^2$   
 $\qquad\qquad\qquad = {}^2C_0 a^2 + {}^2C_1 a^{2-1} b^1 + {}^2C_2 b^2$   
 $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$   
 $\qquad\qquad\qquad = {}^3C_0 a^3 + {}^3C_1 a^{3-1} b^1 + {}^3C_2 a^{3-2} b^2 + {}^3C_3 b^3$  etc.

By looking at the above expansions, we can easily guess the general formula for the expansion of  $(a + b)^n, n \in \mathbb{N}$ .

**8.1 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX**

If  $n$  is a natural number,  $a$  and  $b$  are any numbers, then

$(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n.$

**Proof.** We shall prove the theorem by using the principle of mathematical induction.

Let  $P(n)$  be the statement :

$$(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_{n-1}ab^{n-1} + {}^nC_n b^n.$$

Here  $P(1)$  means

$$(a + b)^1 = {}^1C_0 a^1 + {}^1C_1 b^1$$

i.e.  $a + b = 1 \times a + 1 \times b$ , which is true

$\Rightarrow P(1)$  is true.

Let  $P(m)$  be true

$$\text{i.e. } (a + b)^m = {}^mC_0 a^m + {}^mC_1 a^{m-1}b + {}^mC_2 a^{m-2}b^2 + \dots + {}^mC_{m-1}ab^{m-1} + {}^mC_m b^m \quad \dots(i)$$

For  $P(m + 1)$  :

$$\begin{aligned} (a + b)^{m+1} &= (a + b)^m (a + b) \\ &= ({}^mC_0 a^m + {}^mC_1 a^{m-1}b + {}^mC_2 a^{m-2}b^2 + \dots + {}^mC_{m-1}ab^{m-1} + {}^mC_m b^m) (a + b) \quad (\text{using (i)}) \\ &= {}^mC_0 a^{m+1} + {}^mC_1 a^m b + {}^mC_2 a^{m-1}b^2 + \dots + {}^mC_{m-1}a^2 b^{m-1} + {}^mC_m ab^m \\ &\quad + {}^mC_0 a^m b + {}^mC_1 a^{m-1}b^2 + {}^mC_2 a^{m-2}b^3 + \dots + {}^mC_{m-1}ab^m + {}^mC_m b^{m+1} \\ &\quad \quad \quad (\text{by actual multiplication}) \\ &= {}^mC_0 a^{m+1} + ({}^mC_1 + {}^mC_0) a^m b + ({}^mC_2 + {}^mC_1) a^{m-1} b^2 + \dots \\ &\quad \quad \quad + ({}^mC_m + {}^mC_{m-1}) ab^m + {}^mC_m b^{m+1} \quad (\text{grouping like terms}) \\ &= {}^{m+1}C_0 a^{m+1} + {}^{m+1}C_1 a^m b + {}^{m+1}C_2 a^{m-1} b^2 + \dots + {}^{m+1}C_m ab^m + {}^{m+1}C_{m+1} b^{m+1} \end{aligned}$$

(Because we know that  ${}^mC_0 = 1 = {}^{m+1}C_0$ ,  ${}^mC_m = 1 = {}^{m+1}C_{m+1}$ )

and  ${}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$ ,  $r = 1, 2, 3, \dots, m$

$$\Rightarrow {}^mC_1 + {}^mC_0 = {}^{m+1}C_1, {}^mC_2 + {}^mC_1 = {}^{m+1}C_2, \dots, {}^mC_m + {}^mC_{m-1} = {}^{m+1}C_m)$$

$\Rightarrow P(m + 1)$  is true.

Hence, by principle of mathematical induction,  $P(n)$  is true for all  $n \in \mathbf{N}$ .

The notation  $\sum_{r=0}^n {}^nC_r a^{n-r} b^r$  stands for

$${}^nC_0 a^n b^0 + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \dots + {}^nC_r a^{n-r}b^r + \dots + {}^nC_n a^{n-n}b^n$$

(Note that  $b^0 = 1$  and  $a^{n-n} = a^0 = 1$ )

Hence, the **binomial theorem** can be written as

$$(a + b)^n = \sum_{r=0}^n {}^nC_r a^{n-r} b^r.$$

### 8.1.1 Some important observations

1. The total number of terms in the expansion of  $(a + b)^n$  is  $(n + 1)$  i.e. one more than the index  $n$ .
2. The sum of indices of  $a$  and  $b$  in each term is  $n$ . In the first term of the expansion of  $(a + b)^n$ , the index of  $a$  starts with  $n$ , goes on decreasing by 1 in every successive term and ends with 0, whereas the index of  $b$  starts with zero, goes on increasing by 1 in every successive term and ends with  $n$ .
3. The coefficients  ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$  are called **binomial coefficients**.
4. Since  ${}^nC_r = {}^nC_{n-r}$ ,  $r = 0, 1, 2, \dots, n$   
 $\Rightarrow {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, {}^nC_2 = {}^nC_{n-2}, \dots$

Therefore, the coefficients of terms equidistant from the beginning and end are equal.

### 8.1.2 Some special cases

1. Replacing 'b' by '-b' in the binomial expansion of  $(a + b)^n$ , we get

$$\begin{aligned}
 (a - b)^n &= {}^nC_0 a^n + {}^nC_1 a^{n-1}(-b) + {}^nC_2 a^{n-2}(-b)^2 + \dots \\
 &\quad + {}^nC_r a^{n-r}(-b)^r + \dots + {}^nC_n (-b)^n \\
 &= {}^nC_0 a^n - {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots \\
 &\quad + (-1)^r {}^nC_r a^{n-r} b^r + \dots + (-1)^n {}^nC_n b^n \\
 &= \sum_{r=0}^n (-1)^r {}^nC_r a^{n-r} b^r
 \end{aligned}$$

Thus, the terms in the expansion of  $(a - b)^n$  are alternatively positive and negative. The last term is positive or negative according as  $n$  is even or odd.

2. Putting  $a = 1$  and  $b = x$  in the binomial expansion of  $(a + b)^n$ , we get

$$\begin{aligned}
 (1 + x)^n &= {}^nC_0 1^n + {}^nC_1 1^{n-1} x + {}^nC_2 1^{n-2} x^2 + \dots + {}^nC_r 1^{n-r} x^r + \dots + {}^nC_n x^n \\
 &= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n \\
 &= \sum_{r=0}^n {}^nC_r x^r.
 \end{aligned}$$

3. Putting  $a = 1$  and  $b = -x$  in the binomial expansion of  $(a + b)^n$ , we get

$$\begin{aligned}
 (1 - x)^n &= {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + \dots + (-1)^r {}^nC_r x^r + \dots + (-1)^n {}^nC_n x^n \\
 &= \sum_{r=0}^n (-1)^r {}^nC_r x^r.
 \end{aligned}$$

4. In the expansion of  $(1 + x)^n$ ,  $n \in \mathbf{N}$

- (i)  ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_r + \dots + {}^nC_n = 2^n$ .  
(ii)  ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^n {}^nC_n = 0$ .  
(iii)  ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{n-1}$ .

**Proof.** We know that

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n \quad \dots(1)$$

- (i) On putting  $x = 1$  in (1), we get

$$\begin{aligned}
 (1 + 1)^n &= {}^nC_0 + {}^nC_1 \cdot 1 + {}^nC_2 \cdot 1^2 + \dots + {}^nC_r \cdot 1^r + \dots + {}^nC_n \cdot 1^n \\
 \Rightarrow {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_r + \dots + {}^nC_n &= 2^n.
 \end{aligned}$$

Thus, the sum of the binomial coefficients in the expansion of  $(1 + x)^n$ ,  $n \in \mathbf{N}$ , is  $2^n$ .

- (ii) On putting  $x = -1$  in (1), we get

$$\begin{aligned}
 (1 - 1)^n &= {}^nC_0 + {}^nC_1 (-1) + {}^nC_2 (-1)^2 + {}^nC_3 (-1)^3 + \dots + {}^nC_n (-1)^n \\
 \Rightarrow {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^n {}^nC_n &= 0.
 \end{aligned}$$

- (iii) From part (ii), we get

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots$$

$$\therefore \text{The sum of each} = \frac{1}{2} (\text{sum of the coefficients of all terms})$$

$$= \frac{1}{2} \cdot 2^n \quad \text{(using part (i))}$$

$$= 2^{n-1}$$

$$\Rightarrow {}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{n-1}.$$

Thus, the sum of the coefficients of odd terms in the expansion of  $(1 + x)^n$ ,  $n \in \mathbf{N}$ , is equal to the sum of the coefficients of even terms and each is equal to  $2^{n-1}$ .

**REMARKS**

1. If  $n$  is a positive odd integer, then

$(a + b)^n + (a - b)^n$  and  $(a + b)^n - (a - b)^n$  both have same number of terms equal to  $\frac{n+1}{2}$ .

2. If  $n$  is a positive even integer, then

(i)  $(a + b)^n + (a - b)^n$  has  $\left(\frac{n}{2} + 1\right)$  terms and

(ii)  $(a + b)^n - (a - b)^n$  has  $\frac{n}{2}$  terms.

**ILLUSTRATIVE EXAMPLES**

**Example 1.** Find the number of terms in the expansions of the following :

$$(i) (7x + 2y)^9 \quad (ii) \left(2x - \frac{3}{x^3}\right)^{10} \quad (iii) (1 + 2x + x^2)^{11}$$

$$(iv) (x + 2y - 3z)^n, n \in \mathbb{N}.$$

**Solution.** (i) As the number of terms in the expansion of  $(x + a)^n$  is  $(n + 1)$ , therefore, the number of terms in the expansion of  $(7x + 2y)^9 = 9 + 1 = 10$ .

(ii) The number of terms in the given expansion =  $10 + 1 = 11$ .

(iii) Given expansion =  $(1 + 2x + x^2)^{11} = ((1 + x)^2)^{11} = (1 + x)^{22}$ ,

$\therefore$  the number of terms in the given expansion =  $22 + 1 = 23$ .

(iv) Given expansion =  $(x + 2y - 3z)^n = (x + (2y - 3z))^n$   
 $= {}^nC_0 x^n + {}^nC_1 x^{n-1} (2y - 3z)^1 + {}^nC_2 x^{n-2} (2y - 3z)^2 + \dots$   
 $+ {}^nC_{n-1} (x)^1 (2y - 3z)^{n-1} + {}^nC_n (2y - 3z)^n.$

Clearly, the first term in the above expansion gives one term, second term gives 2 terms, third term gives 3 terms and so on, the last term gives  $(n + 1)$  terms.

$\therefore$  The total number of terms in the given expansion

$$= 1 + 2 + 3 + \dots + (n + 1) = \frac{(n + 1)(n + 2)}{2}.$$

**Example 2.** Expand the following :

$$(i) (3x - 2y)^4 \quad (ii) \left(x^2 + \frac{3}{x}\right)^4, x \neq 0. \quad (\text{NCERT})$$

**Solution.** (i)  $(3x - 2y)^4 = (3x + (-2y))^4$   
 $= {}^4C_0 (3x)^4 + {}^4C_1 (3x)^3 (-2y) + {}^4C_2 (3x)^2 (-2y)^2$   
 $+ {}^4C_3 (3x)^1 (-2y)^3 + {}^4C_4 (-2y)^4$   
 $= 1.81x^4 + 4.27x^3(-2y) + 6.9x^2.4y^2 + 4.3x(-8y^3) + 1.16y^4$   
 $= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4.$

(ii)  $\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3 \left(\frac{3}{x}\right) + {}^4C_2 (x^2)^2 \left(\frac{3}{x}\right)^2 + {}^4C_3 x^2 \left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$   
 $= x^8 + 4 \cdot x^6 \cdot \frac{3}{x} + 6 \cdot x^4 \cdot \frac{9}{x^2} + 4 \cdot x^2 \cdot \frac{27}{x^3} + \frac{81}{x^4}$   
 $= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$

**Example 3.** Expand the following :

$$(i) (2x^2 + 3y)^5 \qquad (ii) \left( \frac{2x^2}{3} - \frac{3}{2x} \right)^4.$$

$$\begin{aligned} \text{Solution. (i)} \quad (2x^2 + 3y)^5 &= {}^5C_0 (2x^2)^5 + {}^5C_1 (2x^2)^4 (3y) + {}^5C_2 (2x^2)^3 (3y)^2 \\ &\quad + {}^5C_3 (2x^2)^2 (3y)^3 + {}^5C_4 (2x^2)^1 (3y)^4 + {}^5C_5 (3y)^5 \\ &= 2^5 x^{10} + 5 \cdot 2^4 \cdot 3 \cdot x^8 y + 10 \cdot 2^3 \cdot 3^2 \cdot x^6 y^2 + 10 \cdot 2^2 \cdot 3^3 \cdot x^4 y^3 + 5 \cdot 2 \cdot 3^4 \cdot x^2 y^4 + 3^5 y^5 \\ &= 32x^{10} + 240x^8y + 720x^6y^2 + 1080x^4y^3 + 810x^2y^4 + 243y^5. \end{aligned}$$

$$\begin{aligned} (ii) \quad \left( \frac{2x^2}{3} - \frac{3}{2x} \right)^4 &= {}^4C_0 \left( \frac{2x^2}{3} \right)^4 + {}^4C_1 \left( \frac{2x^2}{3} \right)^3 \left( \frac{-3}{2x} \right) + {}^4C_2 \left( \frac{2x^2}{3} \right)^2 \left( \frac{-3}{2x} \right)^2 \\ &\quad + {}^4C_3 \left( \frac{2x^2}{3} \right)^1 \left( \frac{-3}{2x} \right)^3 + {}^4C_4 \left( \frac{-3}{2x} \right)^4 \\ &= \left( \frac{2}{3} \right)^4 x^8 - 4 \cdot \left( \frac{2}{3} \right)^3 \cdot x^6 \cdot \frac{3}{2x} + 6 \cdot \left( \frac{2}{3} \right)^2 \cdot x^4 \cdot \left( \frac{3}{2} \right)^2 \cdot \frac{1}{x^2} \\ &\quad - 4 \cdot \left( \frac{2}{3} \right)^1 \cdot x^2 \cdot \left( \frac{3}{2} \right)^3 \cdot \frac{1}{x^3} + 1 \cdot \left( \frac{3}{2} \right)^4 \cdot \frac{1}{x^4} \\ &= \frac{16}{81} x^8 - \frac{16}{9} x^5 + 6x^2 - \frac{9}{x} + \frac{81}{16x^4}. \end{aligned}$$

**Example 4.** Expand the following :

$$(i) (3x^2 - 2ax + 3a^2)^3$$

(NCERT)

$$(ii) (1 - x + x^2)^4.$$

(NCERT Exemplar Problems)

$$\begin{aligned} \text{Solution. (i)} \quad (3x^2 - 2ax + 3a^2)^3 &= (3(x^2 + a^2) - 2ax)^3 \\ &= {}^3C_0 (3(x^2 + a^2))^3 - {}^3C_1 (3(x^2 + a^2))^2 \cdot 2ax + {}^3C_2 3(x^2 + a^2) \cdot (2ax)^2 - {}^3C_3 (2ax)^3 \\ &= 27(x^2 + a^2)^3 - 3 \cdot 9(x^2 + a^2)^2 \cdot 2ax + 3 \cdot 3(x^2 + a^2) \cdot 4a^2x^2 - 8a^3x^3 \\ &= 27(x^6 + 3x^4a^2 + 3x^2a^4 + a^6) - 54ax(x^4 + 2a^2x^2 + a^4) + 36a^2x^2(x^2 + a^2) - 8a^3x^3 \\ &= 27x^6 + 81a^2x^4 + 81a^4x^2 + 27a^6 - 54ax^5 - 108a^3x^3 - 54a^5x + 36a^2x^4 + 36a^4x^2 - 8a^3x^3 \\ &= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6. \end{aligned}$$

$$\begin{aligned} (ii) \quad (1 - x + x^2)^4 &= ((1 - x) + x^2)^4 \\ &= {}^4C_0 (1 - x)^4 + {}^4C_1 (1 - x)^3 \cdot (x^2)^1 + {}^4C_2 (1 - x)^2 \cdot (x^2)^2 + {}^4C_3 (1 - x)^1 \cdot (x^2)^3 + {}^4C_4 (x^2)^4 \\ &= 1 \cdot (1 - 4x + 6x^2 - 4x^3 + x^4) + 4(1 - 3x + 3x^2 - x^3)x^2 \\ &\quad + 6(1 - 2x + x^2)x^4 + 4(1 - x)x^6 + 1 \cdot x^8 \\ &= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8. \end{aligned}$$

**Example 5.** Using binomial theorem, find the values of

$$(i) (99)^4$$

$$(ii) (98)^5$$

(NCERT)

$$(iii) (1.02)^6 \text{ correct to 5 decimal places.}$$

$$\text{Solution. (i)} \quad (99)^4 = (100 - 1)^4 = (10^2 - 1)^4$$

$$\begin{aligned} &= {}^4C_0 (10^2)^4 - {}^4C_1 (10^2)^3 \cdot 1 + {}^4C_2 (10^2)^2 \cdot 1^2 - {}^4C_3 (10^2)^1 \cdot 1^3 + {}^4C_4 1^4 \\ &= 1 \cdot 10^8 - 4 \cdot 10^6 + 6 \cdot 10^4 - 4 \cdot 10^2 + 1 \\ &= 100000000 - 4000000 + 60000 - 400 + 1 \\ &= 96059601. \end{aligned}$$

$$(ii) \quad (98)^5 = (100 - 2)^5 = (10^2 - 2)^5$$

$$\begin{aligned} &= {}^5C_0 (10^2)^5 - {}^5C_1 (10^2)^4 \cdot 2 + {}^5C_2 (10^2)^3 \cdot 2^2 - {}^5C_3 (10^2)^2 \cdot 2^3 + {}^5C_4 (10^2)^1 \cdot 2^4 - {}^5C_5 2^5 \\ &= 1 \times 10^{10} - 5 \times 10^8 \times 2 + 10 \times 10^6 \times 4 - 10 \times 10^4 \times 8 + 5 \times 10^2 \times 16 - 1 \times 32 \end{aligned}$$

$$\begin{aligned}
 &= 10000000000 - 1000000000 + 40000000 - 800000 + 8000 - 32 \\
 &= 10040008000 - 1000800032 \\
 &= 9039207968.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad (1.02)^6 &= (1 + .02)^6 && |(1+x)^n \\
 &= {}^6C_0 + {}^6C_1 (.02) + {}^6C_2 (.02)^2 + {}^6C_3 (.02)^3 + {}^6C_4 (.02)^4 + {}^6C_5 (.02)^5 + {}^6C_6 (.02)^6 \\
 &= 1 + 6(.02) + 15(.0004) + 20(.000008) + 15(.00000016) \\
 &\quad + 6(.0000000032) + 1(.000000000064) \\
 &= 1 + .12 + .006 + .00016 + .0000024 + \dots \\
 &= 1.12616 \text{ correct to 5 decimal places.}
 \end{aligned}$$

**Example 6.** Which number is larger :  $(1.2)^{4000}$  or 800?

$$\begin{aligned}
 \text{Solution. } (1.2)^{4000} &= (1 + 0.2)^{4000} && |(1+x)^n \\
 &= {}^{4000}C_0 + {}^{4000}C_1(0.2) + \text{other positive terms} \\
 &= 1 + 4000(0.2) + \text{other positive terms} \\
 &= 1 + 800 + \text{other positive terms} \\
 &> 800.
 \end{aligned}$$

Hence,  $(1.2)^{4000} > 800$ .

**Example 7.** Find the value of  $(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4$ . (NCERT)

**Solution.** Let  $\sqrt{a^2 - 1} = b$ .

$$\begin{aligned}
 \therefore (a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4 &= (a^2 + b)^4 + (a^2 - b)^4 \\
 &= ({}^4C_0 (a^2)^4 + {}^4C_1 (a^2)^3b + {}^4C_2 (a^2)^2b^2 + {}^4C_3 a^2b^3 + {}^4C_4 b^4) \\
 &\quad + ({}^4C_0 (a^2)^4 - {}^4C_1 (a^2)^3b + {}^4C_2 (a^2)^2b^2 - {}^4C_3 a^2b^3 + {}^4C_4 b^4) \\
 &= 2({}^4C_0 a^8 + {}^4C_2 a^4b^2 + {}^4C_4 b^4) \\
 &= 2(a^8 + 6a^4(a^2 - 1) + (a^2 - 1)^2) && \text{(putting the value of } b) \\
 &= 2(a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1) \\
 &= 2(a^8 + 6a^6 - 5a^4 - 2a^2 + 1).
 \end{aligned}$$

**Example 8.** Expand  $(a + b)^6 - (a - b)^6$ . Hence find the value of  $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$ .

(NCERT)

$$\begin{aligned}
 \text{Solution. } (a + b)^6 - (a - b)^6 &= ({}^6C_0 a^6 + {}^6C_1 a^5b + {}^6C_2 a^4b^2 + {}^6C_3 a^3b^3 + {}^6C_4 a^2b^4 + {}^6C_5 ab^5 + {}^6C_6 b^6) \\
 &\quad - ({}^6C_0 a^6 - {}^6C_1 a^5b + {}^6C_2 a^4b^2 - {}^6C_3 a^3b^3 + {}^6C_4 a^2b^4 - {}^6C_5 ab^5 + {}^6C_6 b^6) \\
 &= 2({}^6C_1 a^5b + {}^6C_3 a^3b^3 + {}^6C_5 ab^5) \\
 &= 2(6a^5b + 20a^3b^3 + 6ab^5) \\
 &= 4ab(3a^4 + 10a^2b^2 + 3b^4).
 \end{aligned}$$

Putting  $a = \sqrt{3}$  and  $b = \sqrt{2}$ , we get

$$\begin{aligned}
 (\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 &= 4\sqrt{3} \sqrt{2} [3(\sqrt{3})^4 + 10(\sqrt{3})^2 (\sqrt{2})^2 + 3(\sqrt{2})^4] \\
 &= 4\sqrt{6} (3 \times 9 + 10 \times 3 \times 2 + 3 \times 4) \\
 &= 4\sqrt{6} (27 + 60 + 12) = 396\sqrt{6}.
 \end{aligned}$$

**Example 9.** Using binomial theorem, evaluate  $(\sqrt{3} + 1)^5 - (\sqrt{3} - 1)^5$ . Hence show that the value of  $(\sqrt{3} + 1)^5$  lies between 152 and 153.

**Solution.**  $(\sqrt{3} + 1)^5 - (\sqrt{3} - 1)^5$

$$\begin{aligned} &= \left( {}^5C_0(\sqrt{3})^5 + {}^5C_1(\sqrt{3})^4 + {}^5C_2(\sqrt{3})^3 + {}^5C_3(\sqrt{3})^2 + {}^5C_4(\sqrt{3})^1 + {}^5C_5 \right) \\ &\quad - \left( {}^5C_0(\sqrt{3})^5 - {}^5C_1(\sqrt{3})^4 + {}^5C_2(\sqrt{3})^3 - {}^5C_3(\sqrt{3})^2 + {}^5C_4(\sqrt{3})^1 - {}^5C_5 \right) \\ &= 2({}^5C_1(\sqrt{3})^4 + {}^5C_3(\sqrt{3})^2 + {}^5C_5) \\ &= 2(5.9 + 10.3 + 1) = 2(45 + 30 + 1) = 152 \end{aligned}$$

$$\Rightarrow (\sqrt{3} + 1)^5 = 152 + (\sqrt{3} - 1)^5 \quad \dots(i)$$

But we know that

$$\sqrt{3} = 1.732 \Rightarrow 0 < \sqrt{3} - 1 < 1$$

$$\Rightarrow 0 < (\sqrt{3} - 1)^5 < 1 \quad (\because 0 < a < 1 \Rightarrow 0 < a^n < 1 \text{ for all } n \in \mathbf{N})$$

$$\begin{aligned} \therefore \text{ From (i), } (\sqrt{3} + 1)^5 &= 152 + (\sqrt{3} - 1)^5 \\ &= 152 + \text{a positive real number less than 1} \end{aligned}$$

$$\Rightarrow (\sqrt{3} + 1)^5 \text{ lies between 152 and 153.}$$

**Example 10.** If  $P$  be the sum of odd terms and  $Q$  be the sum of even terms in the expansion of  $(x + a)^n$ , prove that

$$(i) \quad P^2 - Q^2 = (x^2 - a^2)^n$$

$$(ii) \quad 2(P^2 + Q^2) = (x + a)^{2n} + (x - a)^{2n}$$

$$(iii) \quad 4PQ = (x + a)^{2n} - (x - a)^{2n}.$$

(NCERT Exemplar Problems)

**Solution.**  $(x + a)^n$

$$\begin{aligned} &= {}^nC_0 x^n + {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 + {}^nC_3 x^{n-3}a^3 + \dots + {}^nC_n a^n \\ &= ({}^nC_0 x^n + {}^nC_2 x^{n-2}a^2 + \dots) + ({}^nC_1 x^{n-1}a + {}^nC_3 x^{n-3}a^3 + \dots) \\ &= P + Q \end{aligned} \quad \dots(1)$$

$$\begin{aligned} (x - a)^n &= {}^nC_0 x^n - {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 - {}^nC_3 x^{n-3}a^3 + \dots + {}^nC_n (-1)^n a^n \\ &= ({}^nC_0 x^n + {}^nC_2 x^{n-2}a^2 + \dots) - ({}^nC_1 x^{n-1}a + {}^nC_3 x^{n-3}a^3 + \dots) \\ &= P - Q \end{aligned} \quad \dots(2)$$

$$(i) \quad \text{L.H.S.} = P^2 - Q^2 = (P + Q)(P - Q)$$

$$\begin{aligned} &= (x + a)^n (x - a)^n \quad (\text{using (1) and (2)}) \\ &= ((x + a)(x - a))^n \\ &= (x^2 - a^2)^n = \text{R.H.S.} \end{aligned}$$

$$(ii) \quad \text{L.H.S.} = 2(P^2 + Q^2) = (P + Q)^2 + (P - Q)^2$$

$$\begin{aligned} &= ((x + a)^n)^2 + ((x - a)^n)^2 \quad (\text{using (1) and (2)}) \\ &= (x + a)^{2n} + (x - a)^{2n}. \end{aligned}$$

$$(iii) \quad \text{L.H.S.} = 4PQ = (P + Q)^2 - (P - Q)^2$$

$$\begin{aligned} &= ((x + a)^n)^2 - ((x - a)^n)^2 \quad (\text{using (1) and (2)}) \\ &= (x + a)^{2n} - (x - a)^{2n}. \end{aligned}$$

**Example 11.** Write the binomial expansion of  $(1 + x)^{n+1}$ , when  $x = 8$ . Deduce that  $9^{n+1} - 8n - 9$  is divisible by 64 for all  $n \in \mathbf{N}$ . (NCERT)

**Solution.** By binomial theorem,

$$(1 + x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+1}C_3 x^3 + \dots + {}^{n+1}C_{n+1} x^{n+1}.$$

Putting  $x = 8$ , we get

$$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \dots + {}^{n+1}C_{n+1} 8^{n+1}, \text{ which is the required binomial expansion of } (1 + x)^{n+1} \text{ when } x = 8$$



$$\begin{aligned} \Rightarrow 9^{n+1} &= 1 + (n+1)8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \dots + {}^{n+1}C_{n+1} 8^{n+1} \\ &\quad (\because {}^{n+1}C_0 = 1 \text{ and } {}^{n+1}C_1 = n+1) \\ \Rightarrow 9^{n+1} - 8n - 9 &= 64({}^{n+1}C_2 + {}^{n+1}C_3 8 + \dots + {}^{n+1}C_{n+1} 8^{n-1}) \\ &= 64\lambda, \text{ where } \lambda \text{ is some integer} \\ \Rightarrow 9^{n+1} - 8n - 9 &\text{ is divisible by 64 for all } n \in \mathbf{N}. \end{aligned}$$

**Example 12.** By using Binomial theorem, prove that  $3^{2n+2} - 8n - 9$  is divisible by 64 for all natural numbers  $n$ .

**Solution.** We have,  $3^{2n+2} - 8n - 9 = (3^2)^{n+1} - 8n - 9$

$$\begin{aligned} &= (1+8)^{n+1} - 8n - 9 \\ &= (1 + {}^{n+1}C_1 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \dots + {}^{n+1}C_{n+1} 8^{n+1}) - 8n - 9 \\ &= 1 + (n+1) \times 8 + 8^2({}^{n+1}C_2 + {}^{n+1}C_3 8 + \dots + {}^{n+1}C_{n+1} 8^{n-1}) - 8n - 9 \\ &= 9 + 8n + 64({}^{n+1}C_2 + {}^{n+1}C_3 8 + \dots + {}^{n+1}C_{n+1} 8^{n-1}) - 8n - 9 \\ &= 64\lambda, \text{ where } \lambda \text{ is some integer} \\ \Rightarrow 3^{2n+2} - 8n - 9 &\text{ is divisible by 64 for all natural numbers } n. \end{aligned}$$

**Example 13.** Using binomial theorem, prove that  $6^n - 5n$  always leaves the remainder 1 when divided by 25, for all  $n \in \mathbf{N}$ . (NCERT)

**Solution.** We have,  $6^n - 5n = (1+5)^n - 5n$   $|(1+x)^n$

$$\begin{aligned} &= ({}^nC_0 + {}^nC_1 \times 5 + {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \dots + {}^nC_n \times 5^n) - 5n \\ &= 1 + 5n + {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \dots + {}^nC_n \times 5^n - 5n \\ &= 1 + 5^2({}^nC_2 + {}^nC_3 \times 5 + \dots + {}^nC_n \times 5^{n-2}) \\ &= 1 + 25\lambda, \text{ where } \lambda = {}^nC_2 + {}^nC_3 \times 5 + \dots + {}^nC_n \times 5^{n-2} \text{ is some integer.} \end{aligned}$$

Thus,  $6^n - 5n = 25\lambda + 1$ , where  $\lambda$  is some integer

$\Rightarrow 6^n - 5n$  leaves the remainder 1 when divided by 25.

## EXERCISE 8.1

**Very short answer type questions (1 to 4) :**

1. Find the number of terms in the expansions of the following :

$$(i) \left(3x - \frac{7}{y^2}\right)^8 \quad (ii) (1 + 2x + x^2)^7 \quad (iii) (x^2 - 6x + 9)^{10}.$$

2. Find the number of terms in the expansion of the following :

$$(i) (1 + 3x + 3x^2 + x^3)^5 \quad (ii) (a - b + c)^6.$$

3. Find the number of terms in the expansions of the following :

$$(i) (2x + 3y)^{49} + (2x - 3y)^{49} \quad (ii) (\sqrt{3} + 5x^2)^{93} - (\sqrt{3} - 5x^2)^{93}.$$

4. Find the number of terms in the expansions of the following :

$$(i) (4x^2 + 5\sqrt{3}y)^{100} + (4x^2 - 5\sqrt{3}y)^{100}$$

$$(ii) (1 + 7\sqrt{2}x)^{50} - (1 - 7\sqrt{2}x)^{50}.$$

5. By using binomial theorem, expand the following :

$$(i) (2x + 3y)^5 \quad (ii) (1 - 2x)^5 \quad (\text{NCERT}) \quad (iii) \left(\frac{2x}{3} - \frac{3}{2x}\right)^4$$

$$(iv) \left(\frac{2}{x} - \frac{x}{2}\right)^5 \quad (\text{NCERT}) \quad (v) \left(\frac{x}{3} + \frac{1}{x}\right)^5 \quad (\text{NCERT}) \quad (vi) (2x - 3)^6 \quad (\text{NCERT})$$

$$(vii) \left(x + \frac{1}{x}\right)^6 \quad (NCERT) \quad (viii) (1 + x + x^2)^3.$$

6. Using binomial theorem, find the values of :

$$(i) (96)^3 \quad (NCERT) \quad (ii) (101)^4 \quad (NCERT) \quad (iii) (102)^5 \quad (NCERT) \\ (iv) (99)^5 \quad (NCERT) \quad (v) (999)^3 \quad (vi) (10.1)^4.$$

7. (i) Find an approximation of  $(0.99)^5$  using the first three terms of its expansion.  
(ii) Find the value of  $(1.01)^5$  correct to 5 decimal places.

8. Which number is larger

$$(i) (1.1)^{10000} \text{ or } 1000 ? \quad (NCERT) \quad (ii) (1.01)^{1000000} \text{ or } 10000 ? \quad (NCERT)$$

9. Simplify the following :

$$(i) (x^2 - \sqrt{1-x^2})^4 + (x^2 + \sqrt{1-x^2})^4 \quad (NCERT Exemplar Problems) \\ (ii) (x + \sqrt{x-1})^6 + (x - \sqrt{x-1})^6$$

10. Using binomial theorem, evaluate the following :

$$(i) (\sqrt{3} + \sqrt{2})^3 + (\sqrt{3} - \sqrt{2})^3 \quad (ii) (\sqrt{3} + 1)^5 - (\sqrt{3} - 1)^5 \\ (iii) (3 + \sqrt{2})^5 - (3 - \sqrt{2})^5 \quad (iv) (2 + \sqrt{3})^7 + (2 - \sqrt{3})^7.$$

11. Find  $(a+b)^4 - (a-b)^4$ . Hence evaluate  $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$ . (NCERT)

12. Using Binomial theorem, expand  $(x+y)^5 + (x-y)^5$ . Hence find the value of

$$(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5.$$

13. Using Binomial theorem, find  $(x+1)^6 + (x-1)^6$ . Hence or otherwise evaluate

$$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6. \quad (NCERT)$$

14. Using Binomial theorem, expand  $(a+b)^4$ . Hence or otherwise prove that

$$(1-x)^8 = (1-2x)^4 + 4x^2(1-2x)^3 + 6x^4(1-2x)^2 + 4x^6(1-2x) + x^8.$$

**Hint.**  $(1-x)^8 = ((1-x)^2)^4 = ((1-2x) + x^2)^4$ .

15. In the Binomial expansion of  $(\sqrt[3]{3} + \sqrt{2})^5$ , find the term which does not contain irrational expression.

16. Prove that  $\sum_{r=0}^n 3^r {}^nC_r = 4^n$ . (NCERT)

**Hint.**  $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$ . Put  $x = 3$ .

17. By using binomial theorem, prove that :

$$(i) 2^{3n} - 7n - 1 \text{ is divisible by } 49, \text{ for all } n \in \mathbf{N}.$$

$$(ii) 3^{3n} - 26n - 1 \text{ is divisible by } 676, \text{ for all } n \in \mathbf{N}.$$

**Hint.** (i)  $2^{3n} - 7n - 1 = (2^3)^n - 7n - 1 = 8^n - 7n - 1 = (1+7)^n - 7n - 1$ .

$$(ii) 3^{3n} - 26n - 1 = (3^3)^n - 26n - 1 = (1+26)^n - 26n - 1.$$

## 8.2 GENERAL AND MIDDLE TERMS

### 8.2.1 General term

If  $n$  is any natural number and  $a, b$  are any numbers, then

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_r a^{n-r} b^r + \dots + {}^nC_n b^n.$$

In the binomial expansion of  $(a + b)^n$ , we find that the first term is  ${}^nC_0 a^n$ , second term is  ${}^nC_1 a^{n-1} b$ , the third term is  ${}^nC_2 a^{n-2} b^2$  and so on. On looking at the pattern of successive terms, we find that  $(r + 1)$ th term is  ${}^nC_r a^{n-r} b^r$ . It is denoted by  $T_{r+1}$ .

Thus,  $T_{r+1} = {}^nC_r a^{n-r} b^r$ . This is called the **general term**.

Hence, **general term** =  $T_{r+1} = {}^nC_r a^{n-r} b^r$ .

### Particular cases

- (i) In the expansion of  $(a - b)^n$ ,  $T_{r+1} = (-1)^r {}^nC_r a^{n-r} b^r$ .
- (ii) In the expansion of  $(1 + x)^n$ ,  $T_{r+1} = {}^nC_r x^r$ .
- (iii) In the expansion of  $(1 - x)^n$ ,  $T_{r+1} = (-1)^r {}^nC_r x^r$ .

### REMARKS

1. Coefficient of  $x^r$  in the expansion of  $(1 + x)^n$  is  ${}^nC_r$ .
2.  $r$ th term from the end in the expansion of  $(a + b)^n$  is  $r$ th term from the beginning in the expansion of  $(b + a)^n$ .

*Alternatively*, as the total number of terms in the expansion of  $(a + b)^n$  is  $n + 1$ ,

$\therefore$   $r$ th term from end has  $((n + 1) - r)$  i.e.  $(n - r + 1)$  terms before it, therefore, it is  $(n - r + 2)$ th term from beginning.

### 8.2.2 Middle term or terms

Since the binomial expansion of  $(a + b)^n$  contains  $(n + 1)$  terms, therefore

- (i) if  $n$  is even then the number of terms in the expansion is odd, so there is only one middle term and  $\left(\frac{n}{2} + 1\right)$ th term i.e.  $T_{\frac{n}{2}+1}$  is the middle term.
- (ii) if  $n$  is odd then the number of terms in the expansion is even, so there are two middle terms and  $\left(\frac{n+1}{2}\right)$ th,  $\left(\frac{n+1}{2} + 1\right)$ th i.e.  $T_{\frac{n+1}{2}}$ ,  $T_{\frac{n+3}{2}}$  are the two middle terms.

### ILLUSTRATIVE EXAMPLES

**Example 1.** Find the 7th term in the expansion of  $\left(2x^3 - \frac{3}{2x}\right)^{10}$ .

**Solution.** We know that in the expansion of  $(a + b)^n$ ,  $T_{r+1} = {}^nC_r a^{n-r} b^r$ .

$\therefore$  In the expansion of  $\left(2x^3 - \frac{3}{2x}\right)^{10}$ ,

$$\begin{aligned} T_7 = T_{6+1} &= {}^{10}C_6 (2x^3)^{10-6} \left(-\frac{3}{2x}\right)^6 = {}^{10}C_4 (2x^3)^4 \left(\frac{3}{2x}\right)^6 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^4 \cdot x^{12} \cdot \frac{3^6}{2^6 x^6} = \frac{76545}{2} x^6. \end{aligned}$$

**Example 2.** Find the 13th term in the expansion of  $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ ,  $x > 0$ . (NCERT)

**Solution.** In the expansion of  $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ ,

$$\begin{aligned} T_{13} = T_{12+1} &= {}^{18}C_{12} (9x)^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12} \\ &= {}^{18}C_6 (9x)^6 \times \frac{1}{3^{12} x^6} \quad (\because {}^nC_r = {}^nC_{n-r}) \end{aligned}$$

$$\begin{aligned}
 &= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13}{1 \times 2 \times 3 \times 4 \times 5 \times 6} \times (3^2)^6 x^6 \times \frac{1}{3^{12} x^6} \\
 &= 18564.
 \end{aligned}$$

**Example 3.** Find the fourth term from the end in the expansion of  $\left(\frac{3}{x^2} - \frac{x^3}{3}\right)^9$ .

**Solution.** The fourth term from the end in the expansion of  $\left(\frac{3}{x^2} - \frac{x^3}{3}\right)^9$

$$= \text{the fourth term from the beginning in the expansion of } \left(-\frac{x^3}{3} + \frac{3}{x^2}\right)^9.$$

(Interchanging  $a$  and  $b$  in  $(a + b)^n$ )

$$\begin{aligned}
 \therefore T_4 = T_{3+1} &= {}^9C_3 \left(-\frac{x^3}{3}\right)^{9-3} \left(\frac{3}{x^2}\right)^3 = \frac{9.8.7}{1.2.3} \cdot \left(-\frac{x^3}{3}\right)^6 \cdot \left(\frac{3}{x^2}\right)^3 \\
 &= 84 \cdot \frac{x^{18}}{3^6} \cdot \frac{3^3}{x^6} = 84 \cdot \frac{x^{12}}{3^3} = \frac{28}{9} x^{12}.
 \end{aligned}$$

*Alternatively*

4th term from the end =  $(9 - 4 + 2)$ th i.e. 7th term from the beginning of the given expansion.

$$\begin{aligned}
 \therefore \text{4th term from the end} &= {}^9C_6 \left(\frac{3}{x^2}\right)^{9-6} \left(-\frac{x^3}{3}\right)^6 = {}^9C_3 \left(\frac{3}{x^2}\right)^3 \left(\frac{x^3}{3}\right)^6 \\
 &= \frac{9.8.7}{1.2.3} \cdot \frac{3^3}{x^6} \cdot \frac{x^{18}}{3^6} = \frac{28}{9} x^{12}.
 \end{aligned}$$

**Example 4.** Find the  $r$ th term from the end in the expansion of  $(x + a)^n$ ,  $n \in \mathbb{N}$ . (NCERT)

**Solution.** The  $r$ th term from the end in the expansion of  $(x + a)^n$

= the  $r$ th term from the beginning in the expansion of  $(a + x)^n$

$$= T_r = T_{(r-1)+1} = {}^nC_{r-1} a^{n-(r-1)} x^{r-1}$$

$$= {}^nC_{r-1} x^{r-1} a^{n-r+1}.$$

**Example 5.** Find  $x$  if the 17th and 18th terms of the expansion  $(2 + x)^{50}$  are equal. (NCERT)

**Solution.** In the expansion of  $(2 + x)^{50}$ ,

$$T_{17} = T_{16+1} = {}^{50}C_{16} 2^{50-16} x^{16} \text{ and}$$

$$T_{18} = T_{17+1} = {}^{50}C_{17} 2^{50-17} x^{17}.$$

$$\text{Given } T_{17} = T_{18} \Rightarrow {}^{50}C_{16} 2^{34} x^{16} = {}^{50}C_{17} 2^{33} x^{17}$$

$$\Rightarrow \frac{\underline{50}}{\underline{34} \underline{16}} \times 2 = \frac{\underline{50}}{\underline{33} \underline{17}} \times x$$

$$\Rightarrow \frac{2}{34 \times \underline{33} \times \underline{16}} = \frac{x}{\underline{33} \times 17 \times \underline{16}} \Rightarrow \frac{2}{34} = \frac{x}{17}$$

$$\Rightarrow x = 1.$$

**Example 6.** Find the middle term in  $\left(\frac{2x^2}{3} - \frac{3}{2x}\right)^{12}$ .

**Solution.** Total number of terms in the given expansion =  $12 + 1 = 13$  (odd).

$\therefore$  There is only one middle term given by  $T_{\frac{12}{2}+1}$  i.e.  $T_7$ .

## ANSWERS

## EXERCISE 8.1

1. (i) 9                      (ii) 15                      (iii) 21                      2. (i) 16                      (ii) 28
3. (i) 25                      (ii) 47                      4. (i) 51                      (ii) 25
5. (i)  $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$
- (ii)  $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$                       (iii)  $\frac{16}{81}x^4 - \frac{16}{9}x^2 + 6 - \frac{9}{x^2} + \frac{81}{16x^4}$
- (iv)  $\frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^3}{8} - \frac{x^5}{32}$                       (v)  $\frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$
- (vi)  $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$
- (vii)  $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$
- (viii)  $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$
6. (i) 884736                      (ii) 104060401                      (iii) 11040808032
- (iv) 9509900499                      (v) 997002999                      (vi) 10406.0401
7. (i) 0.951                      (ii) 1.05101                      8. (i)  $(1.1)^{10000}$                       (ii)  $(1.01)^{1000000}$
9. (i)  $2x^8 - 12x^6 + 14x^4 - 4x^2 + 2$                       (ii)  $2(x^6 + 15x^5 - 29x^3 + 12x^2 + 3x - 1)$
10. (i)  $18\sqrt{3}$                       (ii) 152                      (iii)  $1178\sqrt{2}$                       (iv) 10084                      11.  $8ab(a^2 + b^2); 40\sqrt{6}$
12.  $2(x^5 + 10x^3y^2 + 5xy^4); 58\sqrt{2}$                       13.  $2(x^6 + 15x^4 + 15x^2 + 1); 198$
14.  $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$                       15. Third term = 60

## EXERCISE 8.2

1. (i)  $(-1)^r {}^6C_r x^{12-2r} y^r$                       (ii)  $(-1)^r {}^{12}C_r x^{24-r} y^r$                       (iii)  $(-1)^r {}^{12}C_r x^{24-3r}$
2.  $112.36x^{10}$                       3.  $-1760x^9y^3$                       4.  $\frac{1760}{x^3}$                       5.  $\frac{672}{x^3}$                       6.  ${}^{10}C_5$
7. (i)  $-{}^{10}C_5$                       (ii)  $-20x^3$                       8.  ${}^{2n}C_n x^n$                       9. 2, -2                      10. -3
11. -4                      12. 4                      13. 8th                      14. 7th                      15.  $2^{50}$
16.  $2^{49}$                       17. 63                      18. (i)  $-\frac{|25|}{|15||10|} \cdot \frac{2^{10}}{x^{20}}$                       (ii)  $\frac{|3n|}{|n||2n|} \cdot \frac{1}{x^n}$
19.  $\frac{7}{8}$
20. (i)  $61236x^5y^5$                       (ii)  $-\frac{105}{8}x^9, \frac{35}{48}x^{12}$                       (iii)  $\frac{59136a^6b^6}{x^6}$                       (iv)  $\frac{189}{8}x^{17}, -\frac{21}{16}x^{19}$
22. (i) 1512                      (ii) -252                      (iii)  $55.2^8.3^5$                       (iv) 0                      (v) -25344
23.  $-9720; -\frac{40}{27}$
24. (i) 672                      (ii) 924                      (iii) 0                      25. -438                      26. 990
27. (i) -3432                      (ii) 495                      (iii)  $\frac{5}{12}$                       (iv)  $-3003 \times 3^{10} \times 2^5$                       28. 4                      30.  $\frac{9}{7}$
32. 6                      33. 1, 14                      34. 15                      35.  $3003y^4x^{10}$
36. 30th and 31st terms                      37. 8                      38. 11;  $462x^7$
39.  $a = 2, n = 4$                       40.  $n = 11, x = 2$                       41.  $(1 + 2)^5$                       42.  $(3 + 5)^6$                       43. 12
44.  $n = 7, r = 3$                       45. 7