

14

Complex Numbers

14.1 COMPLEX NUMBERS

A **complex number** z is defined as an ordered pair of real numbers and is written as $z = (a, b)$ or $z = a + ib$, where a, b are real numbers and $i = \sqrt{-1}$.

Thus, a number of the form $a + ib$ where a, b are real numbers and $i = \sqrt{-1}$ is called a complex number.

For example, $3 + 5i, -2 + 3i, -2 + i\sqrt{5}, 7 + i\left(-\frac{2}{3}\right)$ are all complex numbers.

The system of numbers $\mathbf{C} = \{z; z = a + ib; a, b \in \mathbf{R}\}$ is called the set of **complex numbers**.

Real and imaginary parts of a complex number

If $z = a + ib$ ($a, b \in \mathbf{R}$) is a complex number, then a is called the **real part**, denoted by $\text{Re}(z)$ and b is called **imaginary part**, denoted by $\text{Im}(z)$.

For example :

- (i) If $z = 2 + 3i$, then $\text{Re}(z) = 2$ and $\text{Im}(z) = 3$.
- (ii) If $z = -3 + \sqrt{5}i$, then $\text{Re}(z) = -3$ and $\text{Im}(z) = \sqrt{5}$.
- (iii) If $z = 7$, then $z = 7 + 0i$, so that $\text{Re}(z) = 7$ and $\text{Im}(z) = 0$.
- (iv) If $z = -5i$, then $z = 0 + (-5)i$, so that $\text{Re}(z) = 0$ and $\text{Im}(z) = -5$.

Note that imaginary part is a real number.

In $z = a + ib$ ($a, b \in \mathbf{R}$), if $b = 0$ then $z = a$, which is a **real number**. If $a = 0$ and $b \neq 0$, then $z = ib$, which is called **purely imaginary number**. If $b \neq 0$, then $z = a + ib$ is **non-real** complex number. Since every real number a can be written as $a + 0i$, we see that $\mathbf{R} \subset \mathbf{C}$ i.e. the set of real numbers $\mathbf{R}$ is a **proper subset** of $\mathbf{C}$, the set of complex numbers.

Note that $\sqrt{3}, 0, 2, \pi$ are real numbers; $3 + 2i, 3 - 2i$ etc. are non-real complex numbers; $2i, -\sqrt{2}i$ etc. are purely imaginary numbers.

Equality of two complex numbers

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are called **equal**, written as $z_1 = z_2$, if and only if $a = c$ and $b = d$.

For example, if the complex numbers $z_1 = a + ib$ and $z_2 = -3 + 5i$ are equal, then $a = -3$ and $b = 5$.

14.2 ALGEBRA OF COMPLEX NUMBERS

In this section, we shall define the usual mathematical operations—addition, subtraction, multiplication, division, square, power etc. on complex numbers.

14.2.1 Addition of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their sum $z_1 + z_2$ is defined as

$$z_1 + z_2 = (a + c) + i(b + d).$$

For example, let $z_1 = 2 + 3i$ and $z_2 = -5 + 4i$, then

$$z_1 + z_2 = (2 + (-5)) + (3 + 4)i = -3 + 7i.$$

If z_1, z_2, z_3 are any complex numbers, then it is easy to see that

$$(i) \quad z_1 + z_2 = z_2 + z_1 \quad (\text{Commutative law})$$

$$(ii) \quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad (\text{Associative law})$$

$$(iii) \quad z + 0 = z = 0 + z, \text{ so } 0 \text{ acts as the *additive identity* for the set of complex numbers.}$$

Negative of a complex number

For a complex number, $z = a + ib$, the negative is defined as $-z = (-a) + i(-b) = -a - ib$.

Note that $z + (-z) = (a - a) + i(b - b) = 0 + i0 = 0$.

Thus, $-z$ acts as the *additive inverse* of z .

14.2.2 Subtraction of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then the difference of z_2 from z_1 is defined as

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (a + ib) + (-c - id) \\ &= (a - c) + i(b - d). \end{aligned}$$

For example, let $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, then

$$\begin{aligned} z_1 - z_2 &= (2 + 3i) - (-1 + 4i) \\ &= (2 + 3i) + (1 - 4i) \\ &= (2 + 1) + (3 - 4)i = 3 - i \end{aligned}$$

$$\begin{aligned} \text{and } z_2 - z_1 &= (-1 + 4i) - (2 + 3i) \\ &= (-1 + 4i) + (-2 - 3i) \\ &= (-1 - 2) + (4 - 3)i = -3 + i. \end{aligned}$$

14.2.3 Multiplication of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their product $z_1 z_2$ is defined as

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

Note that intuitively,

$$(a + ib)(c + id) = ac + ibc + iad + i^2 bd; \text{ now put } i^2 = -1.$$

For example, let $z_1 = 3 + 7i$ and $z_2 = -2 + 5i$, then

$$\begin{aligned} z_1 z_2 &= (3 + 7i)(-2 + 5i) \\ &= (3 \times (-2) - 7 \times 5) + i(3 \times 5 + 7 \times (-2)) \\ &= -41 + i. \end{aligned}$$

If z_1, z_2, z_3 are any complex numbers, then it is easy to see that

$$(i) \quad z_1 z_2 = z_2 z_1 \quad (\text{Commutative law})$$

$$(ii) \quad (z_1 z_2) z_3 = z_1 (z_2 z_3) \quad (\text{Associative law})$$

(iii) $z.1 = z = 1.z$, so 1 acts as the *multiplicative identity* for the set of complex numbers.

(iv) **Multiplicative inverse of a non-zero complex number**

For every non-zero complex number $z = a + ib$, we have the complex number

$$\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \text{ (denoted by } z^{-1} \text{ or } \frac{1}{z}) \text{ such that}$$

$$z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z \quad (\text{check it})$$

$\frac{1}{z}$ is called the multiplicative inverse of z .

$$\text{Note that intuitively, } \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$$

(v) **Multiplication of complex numbers is distributive over addition of complex numbers**

If z_1, z_2 and z_3 are any three complex numbers, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

$$\text{and } (z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

These results are known as *distributive laws*.

14.2.4 Division of complex numbers

Division of a complex number $z_1 = a + ib$ by $z_2 = c + id \neq 0$ is defined as

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 \cdot z_2^{-1} = (a+ib) \cdot \frac{c-id}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}.$$

Note that intuitively,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}.$$

For example, if $z_1 = 3 + 4i$ and $z_2 = 5 - 6i$, then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{3+4i}{5-6i} = \frac{3+4i}{5-6i} \times \frac{5+6i}{5+6i} = \frac{(3 \times 5 - 4 \times 6) + (3 \times 6 + 4 \times 5)i}{5^2 - 6^2 \times i^2} \\ &= \frac{-9 + 38i}{25 + 36} = -\frac{9}{61} + \frac{38}{61}i. \end{aligned}$$

14.2.5 Integral powers of a complex number

If z is any complex number, then positive integral powers of z are defined as

$$z^1 = z, z^2 = z.z, z^3 = z^2.z, z^4 = z^3.z \text{ and so on.}$$

If z is any non-zero complex number, then negative integral powers of z are defined as :

$$z^{-1} = \frac{1}{z}, z^{-2} = \frac{1}{z^2}, z^{-3} = \frac{1}{z^3} \text{ etc.}$$

If $z \neq 0$, then $z^0 = 1$.

14.2.6 Powers of i

Integral power of i are defined as :

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = i^2.i = (-1).i = -i,$$

$$i^4 = (i^2)^2 = (-1)^2 = 1, i^5 = i^4.i = 1.i = i,$$

$$i^6 = i^4.i^2 = 1.(-1) = -1, \text{ and so on.}$$

$$i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$$

Remember that $\frac{1}{i} = -i$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1, \quad i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i,$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, \text{ and so on.}$$

Note that $i^4 = 1$ and $i^{-4} = 1$. It follows that for any integer k ,

$$i^{4k} = 1, \quad i^{4k+1} = i, \quad i^{4k+2} = i^2 = -1, \quad i^{4k+3} = i^3 = -i.$$

14.2.7 Modulus of a complex number

Modulus of a complex number $z = a + ib$, denoted by $\text{mod}(z)$ or $|z|$, is defined as

$$|z| = \sqrt{a^2 + b^2}, \text{ where } a = \text{Re}(z), \quad b = \text{Im}(z).$$

Sometimes, $|z|$ is called **absolute value** of z . Note that $|z| \geq 0$.

For example,

$$(i) \text{ If } z = -3 + 5i, \text{ then } |z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}.$$

$$(ii) \text{ If } z = 3 - \sqrt{7}i, \text{ then } |z| = \sqrt{3^2 + (-\sqrt{7})^2} = \sqrt{9 + 7} = 4.$$

Properties of modulus of a complex number

If z, z_1 and z_2 are complex numbers, then

$$(i) \quad |-z| = |z|$$

$$(ii) \quad |z| = 0 \text{ if and only if } z = 0$$

$$(iii) \quad |z_1 z_2| = |z_1| |z_2|$$

$$(iv) \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \text{ provided } z_2 \neq 0.$$

Proof. (i) Let $z = a + ib$, then $-z = -a - ib$.

$$\therefore \quad |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

$$(ii) \text{ Let } z = a + ib, \text{ then } |z| = \sqrt{a^2 + b^2}.$$

$$\text{Now } |z| = 0 \text{ iff } \sqrt{a^2 + b^2} = 0$$

$$\text{i.e. iff } a^2 + b^2 = 0 \text{ i.e. iff } a^2 = 0 \text{ and } b^2 = 0$$

$$\text{i.e. iff } a = 0 \text{ and } b = 0 \text{ i.e. iff } z = 0 + i0$$

$$\text{i.e. iff } z = 0.$$

$$(iii) \text{ Let } z_1 = a + ib, \text{ and } z_2 = c + id, \text{ then}$$

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

$$\begin{aligned} \therefore |z_1 z_2| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2 c^2 + b^2 d^2 - 2abcd + a^2 d^2 + b^2 c^2 + 2abcd} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \quad (\because a^2 + b^2 \geq 0, c^2 + d^2 \geq 0) \\ &= |z_1| |z_2|. \end{aligned}$$

$$(iv) \text{ Here } z_2 \neq 0 \Rightarrow |z_2| \neq 0.$$

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow |z_1| = |z_2 z_3|$$

$$\Rightarrow |z_1| = |z_2| |z_3|$$

(using part (iii))

$$\Rightarrow \frac{|z_1|}{|z_2|} = |z_3| \Rightarrow \frac{|z_1|}{|z_2|} = \left| \frac{z_1}{z_2} \right| \quad \left(\because z_3 = \frac{z_1}{z_2} \right)$$

Remark. From (iii), on replacing both z_1 and z_2 by z , we get

$$|z z| = |z| |z| \text{ i.e. } |z^2| = |z|^2.$$

Similarly, $|z^3| = |z^2 z| = |z^2| |z| = |z|^2 |z| = |z|^3$ etc.

14.2.8 Conjugate of a complex number

Conjugate of a complex number $z = a + ib$, denoted by $\bar{z}$, is defined as

$$\bar{z} = a - ib \text{ i.e. } \overline{a + ib} = a - ib.$$

For example,

$$(i) \overline{2 + 5i} = 2 - 5i, \quad \overline{2 - 5i} = 2 + 5i$$

$$(ii) \overline{-3 - 7i} = -3 + 7i, \quad \overline{-3 + 7i} = -3 - 7i.$$

Properties of conjugate of a complex number

If z , z_1 and z_2 are complex numbers, then

$$(i) \overline{(\bar{z})} = z$$

$$(ii) \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$(iii) \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$(iv) \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$(v) \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \text{ provided } z_2 \neq 0$$

$$(vi) |\bar{z}| = |z|$$

$$(vii) z \bar{z} = |z|^2$$

$$(viii) z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

Proof. (i) Let $z = a + ib$, where $a, b \in \mathbf{R}$, so that $\bar{z} = a - ib$.

$$\therefore \overline{(\bar{z})} = \overline{a - ib} = a + ib = z.$$

(ii) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} \\ &= (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2. \end{aligned}$$

(iii) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} \overline{z_1 - z_2} &= \overline{(a + ib) - (c + id)} = \overline{(a - c) + i(b - d)} \\ &= (a - c) - i(b - d) = (a - ib) - (c - id) = \bar{z}_1 - \bar{z}_2. \end{aligned}$$

(iv) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(ad + bc)} \\ &= (ac - bd) - i(ad + bc). \end{aligned}$$

$$\text{Also } \overline{\bar{z}_1 \bar{z}_2} = \overline{(a - ib)(c - id)} = \overline{(ac - bd) - i(ad + bc)}.$$

$$\text{Hence } \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

(v) Here $z_2 \neq 0 \Rightarrow \bar{z}_2 \neq 0$.

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow \bar{z}_1 = \overline{z_2 z_3}$$

$$\Rightarrow \bar{z}_1 = \bar{z}_2 \bar{z}_3$$

(using part (iv))

$$\Rightarrow \frac{\bar{z}_1}{\bar{z}_2} = \bar{z}_3 \Rightarrow \frac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\frac{z_1}{z_2}\right)}$$

$$\left(\because z_3 = \frac{z_1}{z_2} \right)$$

(vi) Let $z = a + ib$, then $\bar{z} = a - ib$.

$$\therefore |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(vii) Let $z = a + ib$, then $\bar{z} = a - ib$.

$$\begin{aligned}\therefore z \bar{z} &= (a + ib)(a - ib) \\ &= (aa - b(-b)) + i(a(-b) + ba) \quad (\text{Def. of multiplication}) \\ &= (a^2 + b^2) + i \cdot 0 \\ &= a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2.\end{aligned}$$

Remember that $(a + ib)(a - ib) = a^2 + b^2$.

(viii) Let $z = a + ib \neq 0$, then $|z| \neq 0$.

$$\begin{aligned}\therefore z \bar{z} &= (a + ib)(a - ib) = a^2 + b^2 = |z|^2 \\ \Rightarrow \frac{z \bar{z}}{|z|^2} &= 1 \Rightarrow \frac{\bar{z}}{|z|^2} = \frac{1}{z} = z^{-1}\end{aligned}$$

$$\text{Thus, } z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

Remark From (iv), on replacing both z_1 and z_2 by z , we get

$$\overline{z \bar{z}} = \bar{z} \bar{\bar{z}} \text{ i.e. } \overline{z^2} = (\bar{z})^2.$$

$$\text{Similarly, } \overline{(z^3)} = \overline{(z^2 z)} = (\overline{z^2}) \bar{z} = (\bar{z})^2 \bar{z} = (\bar{z})^3 \text{ etc.}$$

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate the following :

$$(i) i^{50}$$

$$(ii) i^{-57}$$

$$(iii) (-\sqrt{-1})^{31}$$

$$(iv) i + i^2 + i^3 + i^4$$

$$(v) \frac{i + i^2 + i^4}{1 + i^2 + i^4}$$

$$(vi) \frac{i^2 + i^4 + i^6 + i^7}{1 + i^2 + i^3}.$$

Solution. (i) $i^{50} = i^{4 \cdot 12 + 2} = i^2 = -1$.

$$(ii) i^{-57} = i^{4 \cdot (-15) + 3} = i^3 = -i.$$

$$(iii) (-\sqrt{-1})^{31} = (-i)^{31} = (-1)^{31} \cdot i^{31} = -i^{4 \cdot 7 + 3} = -i^3 = -(-i) = i.$$

$$(iv) i + i^2 + i^3 + i^4 = i + (-1) + (-i) + (1) = 0.$$

$$(v) \frac{i + i^2 + i^4}{1 + i^2 + i^4} = \frac{i + (-1) + (1)}{1 + (-1) + (1)} = \frac{i}{1} = i.$$

$$(vi) \frac{i^2 + i^4 + i^6 + i^7}{1 + i^2 + i^3} = \frac{(-1) + (1) + (-1) + (-i)}{1 + (-1) + (-i)} = \frac{-1 - i}{-i} = \frac{1 - i}{i} + 1 = -i + 1 = 1 - i.$$

Example 2. (i) Show that $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$ for all $n \in \mathbb{N}$.

$$(ii) \text{ If } z = \frac{3}{2} - 2i, \text{ find } \operatorname{Re}(z), \operatorname{Im}(z), \bar{z}, |z|, z + \bar{z}, z - \bar{z}.$$

$$(iii) \text{ If } \operatorname{Re}(z) = \sqrt{2} \text{ and } \operatorname{Im}(z) = -\sqrt{3}, \text{ find } z, \bar{z}, |z|, |\bar{z}|.$$

$$(iv) \text{ Find values of } x \text{ and } y \text{ if } (3x - 1) + (\sqrt{3} + 2y)i = 5.$$

Solution. (i) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n (1 + i + i^2 + i^3)$
 $= i^n (1 + i - 1 - i) = i^n \times 0 = 0.$

$$(ii) \text{ Given } z = \frac{3}{2} - 2i \Rightarrow \operatorname{Re}(z) = \frac{3}{2}, \operatorname{Im}(z) = -2,$$

$$\therefore \bar{z} = \frac{3}{2} + 2i,$$

$$|z| = \sqrt{\left(\frac{3}{2}\right)^2 + (-2)^2} = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{9+16}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2},$$

$$z + \bar{z} = \left(\frac{3}{2} - 2i\right) + \left(\frac{3}{2} + 2i\right) = \frac{3}{2} + \frac{3}{2} = 3 \text{ and}$$

$$z - \bar{z} = \left(\frac{3}{2} - 2i\right) - \left(\frac{3}{2} + 2i\right) = -4i.$$

$$(iii) \quad z = \operatorname{Re}(z) + i \cdot \operatorname{Im}(z) = \sqrt{2} - \sqrt{3}i,$$

$$\bar{z} = \sqrt{2} + \sqrt{3}i,$$

$$|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{3})^2} = \sqrt{2+3} = \sqrt{5} \text{ and}$$

$$|\bar{z}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2} = \sqrt{2+3} = \sqrt{5}.$$

(iv) Equating real and imaginary parts on both sides, we get

$$3x - 1 = 5, \quad \sqrt{3} + 2y = 0 \Rightarrow x = 2, \quad y = -\frac{\sqrt{3}}{2}.$$

Example 3. (i) Simplify $(-2 + 3i) + 3\left(-\frac{1}{2}i + 1\right) - (2i)$.

(ii) Simplify $(-3 + \sqrt{-5})(3 + 4\sqrt{-5})$.

(iii) Find the conjugate and modulus of i^7 .

Solution. (i) Given expression $= (-2 + 3) + \left(3 - \frac{3}{2} - 2\right)i$
 $= 1 - \frac{1}{2}i.$

$$\begin{aligned} (ii) \quad (-3 + \sqrt{-5})(3 + 4\sqrt{-5}) &= (-3 + \sqrt{5}i)(3 + 4\sqrt{5}i) \\ &= -9 - 12\sqrt{5}i + 3\sqrt{5}i + \sqrt{5} \cdot 4\sqrt{5}i^2 \\ &= -9 - 9\sqrt{5}i + 20(-1) \\ &= -29 - 9\sqrt{5}i. \end{aligned}$$

$$(iii) \quad \text{Let } z = i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i,$$

$$\text{so } \bar{z} = \overline{-i} = i \text{ and } |z| = |-i| = \sqrt{0^2 + (-1)^2} = 1.$$

Example 4. If $i = \sqrt{-1}$, prove the following :

$$(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i) = x^4 + 4. \quad (\text{I.S.C. 2004})$$

$$\begin{aligned} \text{Solution. L.H.S.} &= [(x+1+i)(x+1-i)][(x-1+i)(x-1-i)] \\ &= [(x+1)^2 - i^2][(x-1)^2 - i^2] \\ &= [x^2 + 2x + 1 - (-1)][x^2 - 2x + 1 - (-1)] \\ &= (x^2 + 2 + 2x)(x^2 + 2 - 2x) \\ &= (x^2 + 2)^2 - (2x)^2 \\ &= x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4 = \text{R.H.S.} \end{aligned}$$

Example 5. Express each of the following in the **standard form** $a + ib$.

$$(i) \quad \frac{i}{1+i}$$

$$(ii) \quad (-1 + \sqrt{3}i)^{-1}$$

$$(iii) \quad (2i - i^2)^2 + (1 - 3i)^3 \quad (iv) \quad \frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i}.$$

$$\text{Solution. (i)} \quad \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i-i^2}{1-i^2} = \frac{i-(-1)}{1-(-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i.$$

$$\begin{aligned} (ii) \quad (-1 + \sqrt{3}i)^{-1} &= \frac{1}{-1 + \sqrt{3}i} = \frac{1}{-1 + \sqrt{3}i} \cdot \frac{-1 - \sqrt{3}i}{-1 - \sqrt{3}i} \\ &= \frac{-1 - \sqrt{3}i}{(-1)^2 - (\sqrt{3}i)^2} = \frac{-1 - \sqrt{3}i}{1 - (-3)} = -\frac{1}{4} - \frac{\sqrt{3}}{4}i. \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad (2i - i^2)^2 + (1 - 3i)^3 &= (2i + 1)^2 + (1 - 3i)^3 \\
 &= (4i^2 + 4i + 1) + (1 - 9i + 27i^2 - 27i^3) \\
 &= -4 + 4i + 1 + 1 - 9i - 27 + 27i = -29 + 22i.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i} &= \frac{(1+i)(3+i)(3+i) - (1-i)(3-i)(3-i)}{(3-i)(3+i)} \\
 &= \frac{(1+i)(8+6i) - (1-i)(8-6i)}{9-i^2} = \frac{(2+14i) - (2-14i)}{9+1} = \frac{28i}{10} = 0 + \frac{14}{5}i.
 \end{aligned}$$

Example 6. (i) Find the conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

(ii) Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

(iii) Find the modulus of $\frac{(2-3i)^2}{-1+5i}$.

Solution. (i) Let $z = \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{6+9i-4i+6}{2-i+4i+2}$

$$\begin{aligned}
 &= \frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} \\
 &= \frac{48-36i+20i+15}{4^2-(3i)^2} = \frac{63-16i}{16-9(-1)} \\
 &= \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i.
 \end{aligned}$$

$$\therefore \text{Conjugate of } z = \frac{63}{25} + \frac{16}{25}i.$$

$$\begin{aligned}
 \text{(ii)} \quad z &= \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} \\
 &= \frac{(1+i^2+2i) - (1+i^2-2i)}{1^2-i^2} = \frac{4i}{1+1} = 2i \\
 &= 0 + 2i.
 \end{aligned}$$

$$\therefore \text{Modulus of } z = \sqrt{0^2 + 2^2} = \sqrt{4} = 2.$$

(iii) Let $z = \frac{(2-3i)^2}{-1+5i}$, then

$$\begin{aligned}
 |z| &= \left| \frac{(2-3i)^2}{-1+5i} \right| = \frac{|(2-3i)^2|}{|-1+5i|} & \left(\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right) \\
 &= \frac{|2-3i|^2}{|-1+5i|} & (\because |z^2| = |z|^2) \\
 &= \frac{(\sqrt{(2)^2 + (-3)^2})^2}{\sqrt{(-1)^2 + 5^2}} = \frac{(\sqrt{13})^2}{\sqrt{26}} = \frac{13}{\sqrt{26}} = \sqrt{\frac{13}{2}}.
 \end{aligned}$$

Example 7. If $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = a + ib$, find the real numbers a and b . With these values of a and b , also find the modulus of $a + ib$. (I.S.C. 2009)

Solution. Given $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = a + ib$

$$\Rightarrow (-2 + \sqrt{3}i)(-3 + 2\sqrt{3}i) = a + ib$$

$$\Rightarrow 6 - 4\sqrt{3}i - 3\sqrt{3}i + \sqrt{3} \cdot 2\sqrt{3}i^2 = a + ib$$

$$\Rightarrow 6 - 7\sqrt{3}i + 6(-1) = a + ib$$

$$\Rightarrow 0 - 7\sqrt{3}i = a + ib.$$

Equating real and imaginary parts on both sides, we get

$$a = 0 \text{ and } b = -7\sqrt{3}.$$

$$\begin{aligned} |a + ib| &= \sqrt{a^2 + b^2} = \sqrt{0^2 + (-7\sqrt{3})^2} = \sqrt{0 + 147} \\ &= \sqrt{147} = 7\sqrt{3}. \end{aligned}$$

Example 8. Find the values of x and y given that $(x + iy)(2 - 3i) = 4 + i$. (I.S.C. 2008)

Solution. Given $(x + iy)(2 - 3i) = 4 + i$

$$\Rightarrow x + iy = \frac{4 + i}{2 - 3i} = \frac{4 + i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$$

$$\Rightarrow x + iy = \frac{8 + 12i + 2i + 3i^2}{2^2 - 9i^2} = \frac{8 + 14i + 3(-1)}{4 - 9(-1)}$$

$$\Rightarrow x + iy = \frac{5}{13} + \frac{14}{13}i.$$

Equating real and imaginary parts on both sides, we get

$$x = \frac{5}{13} \text{ and } y = \frac{14}{13}.$$

Example 9. Given that $\frac{2\sqrt{3} \cos 30^\circ - 2i \sin 30^\circ}{\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)} = A + iB$, find the values of A and B .

(I.S.C. 2002)

Solution. Given $\frac{2\sqrt{3} \cos 30^\circ - 2i \sin 30^\circ}{\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)} = A + iB$

$$\begin{aligned} \Rightarrow A + iB &= \frac{2\sqrt{3} \cdot \frac{\sqrt{3}}{2} - 2i \cdot \frac{1}{2}}{\sqrt{2}\left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)} = \frac{3 - i}{1 + i} \\ &= \frac{3 - i}{1 + i} \times \frac{1 - i}{1 - i} = \frac{3 - 3i - i + i^2}{1^2 - i^2} \\ &= \frac{3 + (-1) - 4i}{1 - (-1)} = \frac{2 - 4i}{2} = 1 - 2i. \end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$A = 1 \text{ and } B = -2.$$

Example 10. Find the real values of x and y satisfying the equality

$$\frac{x - 2 + (y - 3)i}{1 + i} = 1 - 3i.$$

(I.S.C. 2003)

Solution. Given $\frac{x - 2 + (y - 3)i}{1 + i} = 1 - 3i$

$$\Rightarrow (x - 2) + (y - 3)i = (1 - 3i)(1 + i)$$

$$\Rightarrow (x - 2) + (y - 3)i = (1 + 3) + (-3 + 1)i$$

$$\Rightarrow (x - 2) + (y - 3)i = 4 - 2i.$$

Equating real and imaginary parts on both sides, we get

$$x - 2 = 4 \text{ and } y - 3 = -2$$

$$\Rightarrow x = 6 \text{ and } y = 1.$$

Example 11. Find the real values of x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

Solution. According to given, $(x - iy)(3 + 5i) = \overline{-6 - 24i}$

$$\Rightarrow 3x + 5xi - 3yi - 5y(-1) = -6 + 24i$$

$$\Rightarrow (3x + 5y) + (5x - 3y)i = -6 + 24i.$$

Equating real and imaginary parts on both sides, we get

$$3x + 5y = -6 \text{ and } 5x - 3y = 24.$$

Solving these equations simultaneously, we get

$$x = 3 \text{ and } y = -3.$$

Example 12. For what real values of x and y are the following numbers equal

(i) $(1 + i)y^2 + (6 + i)$ and $(2 + i)x$

(ii) $x^2 - 7x + 9yi$ and $y^2i + 20i - 12$?

Solution. (i) $(1 + i)y^2 + (6 + i) = (2 + i)x$

$$\Rightarrow (y^2 + 6) + i(y^2 + 1) = 2x + ix$$

$$\Rightarrow y^2 + 6 = 2x \text{ and } y^2 + 1 = x$$

$$\Rightarrow x = 5 \text{ and } y^2 = 4 \Rightarrow x = 5 \text{ and } y = \pm 2.$$

$$\text{Hence, } x = 5, y = 2; x = 5, y = -2.$$

(ii) $x^2 - 7x + 9yi = y^2i + 20i - 12$

$$\Rightarrow (x^2 - 7x) + i(9y) = (-12) + i(y^2 + 20)$$

$$\Rightarrow x^2 - 7x = -12 \text{ and } 9y = y^2 + 20$$

$$\Rightarrow x^2 - 7x + 12 = 0 \text{ and } y^2 - 9y + 20 = 0$$

$$\Rightarrow (x - 4)(x - 3) = 0 \text{ and } (y - 5)(y - 4) = 0$$

$$\Rightarrow x = 4, 3 \text{ and } y = 5, 4.$$

Hence, $x = 4, y = 5; x = 4, y = 4; x = 3, y = 5; x = 3, y = 4.$

Example 13. If z is a complex number and $iz^3 + z^2 - z + i = 0$, then prove that $|z| = 1$.

Solution. Given $iz^3 + z^2 - z + i = 0$

$$\Rightarrow iz^3 + z^2 + i^2z + i = 0$$

$$(\because i^2 = -1)$$

$$\Rightarrow z^2(iz + 1) + i(iz + 1) = 0$$

$$\Rightarrow (z^2 + i)(iz + 1) = 0$$

$$\Rightarrow z^2 = -i \text{ or } iz = -1$$

$$\Rightarrow z^2 = -i \text{ or } z = -\frac{1}{i} = -(-i) = i.$$

$$\text{Now } z = i \Rightarrow |z| = |i| = \sqrt{0^2 + 1^2} = 1$$

$$\text{and } z^2 = -i \Rightarrow |z^2| = |-i| \Rightarrow |z|^2 = \sqrt{0^2 + (-1)^2} = 1 \Rightarrow |z| = 1.$$

Hence, $|z| = 1$ in both cases.

Example 14. Solve the equation $2z = |z| + 2i$.

Solution. Let $z = x + iy$ where $x, y \in \mathbf{R}$ so that $|z| = \sqrt{x^2 + y^2}$.

$$\text{Given } 2z = |z| + 2i \Rightarrow 2(x + iy) = \sqrt{x^2 + y^2} + 2i$$

$$\Rightarrow 2x = \sqrt{x^2 + y^2} \text{ and } 2y = 2$$

$$\Rightarrow 4x^2 = x^2 + y^2 \text{ and } y = 1$$

$$\Rightarrow 3x^2 = 1 \text{ and } y = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \text{ and } y = 1.$$

$$\begin{aligned}
 &= \frac{|\beta - \alpha| |\bar{\beta}|}{|\beta - \alpha|} \quad (\because |z_1 z_2| = |z_1| |z_2| \text{ and } \bar{z}_1 - \bar{z}_2 = \overline{z_1 - z_2}) \\
 &= \frac{|\beta - \alpha| |\beta|}{|\beta - \alpha|} \quad (\because |\bar{z}| = |z|) \\
 &= |\beta| = 1 \quad (\because |\beta| = 1 \text{ given})
 \end{aligned}$$

Hence, $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = 1.$

Example 26. If z_1, z_2 are complex numbers such that $\left| \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right| = 1$ and $|z_2| \neq 1$, then find $|z_1|$.

Solution. Given $\left| \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right| = 1 \Rightarrow \left| \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right|^2 = 1$

$$\begin{aligned}
 &\Rightarrow \left(\frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right) \left(\overline{\frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2}} \right) = 1 \quad (\because |z|^2 = z \bar{z}) \\
 &\Rightarrow \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \cdot \frac{\bar{z}_1 - 3\bar{z}_2}{3 - \bar{z}_1 z_2} = 1 \\
 &\Rightarrow (z_1 - 3z_2)(\bar{z}_1 - 3\bar{z}_2) = (3 - z_1 \bar{z}_2)(3 - \bar{z}_1 z_2) \\
 &\Rightarrow z_1 \bar{z}_1 - 3z_1 \bar{z}_2 - 3z_2 \bar{z}_1 + 9z_2 \bar{z}_2 = 9 - 3\bar{z}_1 z_2 - 3z_1 \bar{z}_2 + z_1 \bar{z}_1 z_2 \bar{z}_2 \\
 &\Rightarrow |z_1|^2 + 9|z_2|^2 = 9 + |z_1|^2 |z_2|^2 \\
 &\Rightarrow |z_1|^2 |z_2|^2 - |z_1|^2 - 9|z_2|^2 + 9 = 0 \\
 &\Rightarrow |z_1|^2 (|z_2|^2 - 1) - 9(|z_2|^2 - 1) = 0 \\
 &\Rightarrow (|z_2|^2 - 1)(|z_1|^2 - 9) = 0 \\
 &\Rightarrow |z_1|^2 = 9 \quad (\because |z_2| \neq 1 \text{ given}) \\
 &\Rightarrow |z_1| = 3 \quad (\because |z_1| \geq 0)
 \end{aligned}$$

Example 27. If $|z_1| = |z_2| = \dots = |z_n| = 1$, prove that

$$|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|.$$

Solution. Given $|z_1| = |z_2| = \dots = |z_n| = 1$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \dots = |z_n|^2 = 1$$

$$\Rightarrow z_1 \bar{z}_1 = 1, z_2 \bar{z}_2 = 1, \dots, z_n \bar{z}_n = 1$$

$$\Rightarrow \frac{1}{z_1} = \bar{z}_1, \frac{1}{z_2} = \bar{z}_2, \dots, \frac{1}{z_n} = \bar{z}_n.$$

$$\begin{aligned}
 \therefore \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right| &= |\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n| \\
 &= |\overline{z_1 + z_2 + \dots + z_n}| = |z_1 + z_2 + \dots + z_n|. \quad (\because |\bar{z}| = |z|)
 \end{aligned}$$

EXERCISE 14.1

1. Evaluate

(i) $\sqrt{-9} \sqrt{-4}$

(ii) $\sqrt{(-9)(-4)}$

(iii) $i + i^3 + i^5 + \dots + i^{15}$

(iv) $\frac{1}{1 + i + i^2}$

(v) $(-\sqrt{-4})^3$

(vi) $\sum_{k=0}^{100} i^k.$

2. If $z = -3 - i$, find $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, $|z|$, z^{-1} .

3. Express each of the following in the standard form $a + ib$:

$$(i) \frac{1}{3-4i} \quad (ii) \frac{1}{-2-\sqrt{-3}} \quad (iii) \frac{13i}{2-3i} \quad (\text{I.S.C. 2000})$$

$$(iv) \frac{(1+i)^2}{3-i} \quad (v) \frac{3-4i}{(4-2i)(1+i)} \quad (vi) \frac{1-2i}{2+i} + \frac{3+i}{2-i}. \quad (\text{I.S.C. 2006})$$

4. Find x and y if

$$(i) 2y + (3x - y)i = 5 - 2i$$

$$(ii) 3 + ix^2y \text{ and } x^2 + y + 4i \text{ are conjugate complexes of each other.}$$

$$(iii) (x - yi)(2 + 3i) = \frac{x + 2i}{1 - i}$$

$$(iv) (x + iy)(2 - 3i) \text{ is the conjugate of } 4 - i.$$

5. If $z_1 = 1 - i$, $z_2 = -2 + 4i$, calculate the values of a and b if $a + bi = \frac{z_1 z_2}{z_1}$.

6. Solve the equation :

$$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i, \quad x, y \in \mathbf{R}, \quad i = \sqrt{-1}.$$

7. (i) If $|z_1| = |z_2|$, does it follow that $z_1 = z_2$?

(ii) Show that $z = -\bar{z}$ iff z is either zero or purely imaginary.

8. If $z_1, z_2, z_3, \dots, z_n$ are any complex numbers, prove that

$$(i) |z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$$

$$(ii) \overline{z_1 + z_2 + z_3 + \dots + z_n} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \dots + \bar{z}_n$$

$$(iii) \overline{z_1 z_2 \dots z_n} = \bar{z}_1 \bar{z}_2 \bar{z}_3 \dots \bar{z}_n.$$

Hint. Use induction.

9. (i) Simplify $(3 + 2i)^3$. Also find its conjugate.

(ii) Simplify $(1 - 2i)^{-3}$. Also find its conjugate.

(iii) Separate into real and imaginary parts $\frac{3 + \sqrt{-1}}{2 - \sqrt{-1}}$. Also find its modulus.

$$(iv) \text{Simplify } (\sqrt{5} + 7i)(\sqrt{5} - 7i)^2 + (-2 + 7i)^2.$$

(v) Express the square of $\frac{i}{1+i}$ in the form $x + iy$.

(vi) Show that $\frac{\sqrt{7} + \sqrt{3}i}{\sqrt{7} - \sqrt{3}i} + \frac{\sqrt{7} - \sqrt{3}i}{\sqrt{7} + \sqrt{3}i}$ is real.

(vii) If $z = 1 + i$, evaluate $z^3 - 2z^2 + 3z - 4$

(viii) If $z = 3 + 5i$, evaluate $z^3 + \bar{z} + 198$.

10. Find the modulus of the following numbers :

$$(i) \frac{(1+3i)(2-5i)}{(2-\sqrt{6}i)(-3+2\sqrt{5}i)} \quad (ii) \frac{(3+4i)(4-5i)}{(4-3i)(6+7i)}.$$

11. If z is a complex number and $|2z - 1| = |z - 2|$, show that $|z| = 1$.

12. Let $z = x + iy$ and $w = \frac{1-iz}{z-i}$, show that if $|w| = 1$, then z is purely real.

13. If x is real and $\frac{1-ix}{1+ix} = a - ib$, prove that $a^2 + b^2 = 1$.

Hint. $\left| \frac{1-ix}{1+ix} \right| = |a - ib|$.

14. Using $z_1 = 2 - 3i$, $z_2 = 4 + 5i$, $z_3 = -1 - i$, verify that complex number multiplication is left as well as right distributive over addition.

15. If $\left(\frac{1-i}{1+i} \right)^{500} = a + ib$, find the values of a and b .

Hint. First simplify $\frac{1-i}{1+i} = -i$.

16. Find the least positive integral value of n such that $\left(\frac{1+i}{1-i} \right)^n$ is real.

Hint. First simplify $\frac{1+i}{1-i} = i$.

17. Solve the equation $|1 - i|^x = 2^x$.

18. Solve the equation $|z| + z = 2 + i$.

19. Show that if $\left| \frac{z-5i}{z+5i} \right| = 1$, then z is a real number.

20. If $(1+i)(1+2i)(1+3i) \dots (1+ni) = x + iy$, show that $2.5.10 \dots (1+n^2) = x^2 + y^2$.

Hint. Take modulus of both sides and use the fact that

$$|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|.$$

21. If $(x + iy)^{1/3} = a + ib$, prove that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$.

Hint. Cube both sides and find x, y in terms of a and b .

22. If $f(z) = z^4 + 9z^3 + 35z^2 - z + 4$, find $f(-5 + 2\sqrt{-4})$.

23. Find the complex numbers satisfying the equations

$$\left| \frac{z-4}{z-8} \right| = 1 \text{ and } \left| \frac{z-12}{z-8i} \right| = \frac{5}{3}.$$

24. Prove that

(i) if $x + iy = \frac{a+ib}{a-ib}$, then $x^2 + y^2 = 1$

(ii) if $x + iy = \sqrt{\frac{1+i}{1-i}}$, then $x^2 + y^2 = 1$

(iii) if $x - iy = \sqrt{\frac{a-ib}{c-id}}$, then $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$.

25. If $a + ib = \frac{c+i}{c-i}$, where c is real, then prove that

$$a^2 + b^2 = 1 \text{ and } \frac{b}{a} = \frac{2c}{c^2 - 1}.$$

26. If z_1, z_2 are complex numbers such that $\left| \frac{z_1 - 2z_2}{2 - \bar{z}_1 z_2} \right| = 1$ and $|z_2| \neq 1$, then find $|z_1|$.

14.2 REPRESENTATIONS OF COMPLEX NUMBERS

A complex number can be represented geometrically or trigonometrically.

14.2.1 Geometric representation

We know that corresponding to every real number there exists a unique real number on the number line (called real axis) and conversely, corresponding to every point on the line there exists a unique real number *i.e.* there is a one-one correspondence between the set $\mathbf{R}$ of real numbers and the points on the real axis.

In a similar way, corresponding to every ordered pair (x, y) of real numbers there exists a unique point P in the co-ordinate plane with x as *abscissa* and y as *ordinate* of the point P and conversely, corresponding to every point P in the plane there exists a unique ordered pair of real numbers. Thus, there is a one-one correspondence between the set of ordered pairs $\{(x, y); x, y \in \mathbf{R}\}$ and the points in the co-ordinate plane.

The point P with co-ordinates (x, y) is said to represent the complex number $z = x + iy$.

It follows that the complex number $z = x + iy$ can be uniquely represented by the point $P(x, y)$ in the co-ordinate plane and conversely, corresponding to the point $P(x, y)$ in the plane there exists a unique complex number $z = x + iy$. The plane is called the **complex plane** and the representation of complex numbers as points in the plane is called **Argand diagram**.

Notice that length $OP = \sqrt{x^2 + y^2} = |z|$.

Figure 14.2 represents the complex numbers $0, 2, -2, 3i, -3i, 2 + 3i, -2 + 3i, -2 - 3i, 2 - 3i$.

Note that every real number $x = x + 0i$ is represented by point $(x, 0)$ lying on x -axis, and every purely imaginary number iy is represented by point $(0, y)$ lying on y -axis. Consequently, x -axis is called the **real axis** and y -axis is called the **imaginary axis**.

Since any complex number $z = x + iy$ is represented by point $P(x, y)$ in complex plane, many geometric ideas arise from this — distance between two points ; equation of line ; equation of a circle ; describing some region in complex plane *e.g.* all points lying inside a circle, and so on.

1. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then the distance between two corresponding points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1 P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2| = |z_2 - z_1|.$$

2. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ are two complex numbers and $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are two corresponding points, then the point Q dividing $[P_1 P_2]$ *i.e.* the segment $P_1 P_2$ in the ratio $m : n$ is given by

$$Q\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

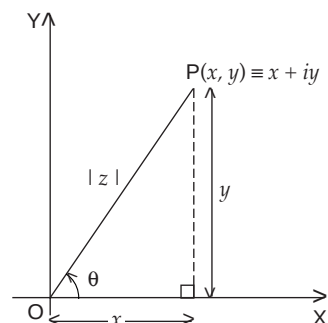


Fig. 14.1.

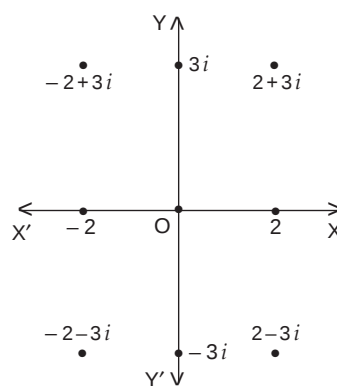


Fig. 14.2.

So corresponding complex number is

$$\begin{aligned} z_3 &= \frac{mx_2 + nx_1}{m+n} + i \frac{my_2 + ny_1}{m+n} \\ &= \frac{m(x_2 + i y_2) + n(x_1 + i y_1)}{m+n} \\ &= \frac{mz_2 + nz_1}{m+n}. \end{aligned}$$

In particular, putting $m = n = 1$, the **mid-point** of $[P_1P_2]$ is given by $\frac{z_1 + z_2}{2}$.

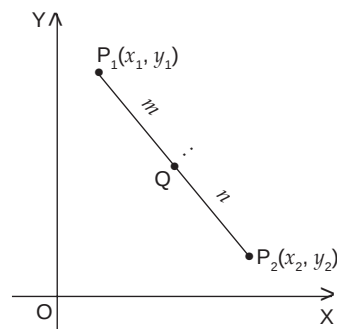


Fig. 14.3.

3. If $z = x + iy$, $x, y \in \mathbf{R}$, is represented by the point $P(x, y)$ in the complex plane, then the complex numbers $-z$, $\bar{z}$, $-\bar{z}$ are represented by the points $P'(-x, -y)$, $Q(x, -y)$ and $Q'(-x, y)$ respectively in the complex plane (see fig. 14.4.).

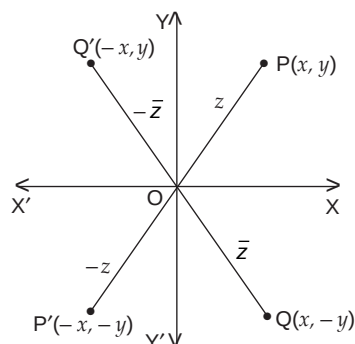


Fig. 14.4.

4. **Geometric representation of sum of two complex numbers.**

Let the complex numbers z_1 and z_2 be represented by the points P and Q respectively in the complex plane. Complete the parallelogram OPRQ and let its diagonals meet at M. Since diagonals of a parallelogram bisect each other, M is mid-point of $[PQ]$ and so it

represents the complex number $\frac{z_1 + z_2}{2}$.

Let R represent the complex number z . As M is also mid-point of $[OR]$, it will represent the complex number $\frac{0 + z}{2}$

$$\Rightarrow \frac{z}{2} = \frac{z_1 + z_2}{2} \Rightarrow z = z_1 + z_2.$$

Hence, the point R represents the complex number $z_1 + z_2$.

This is known as **parallelogram law**.

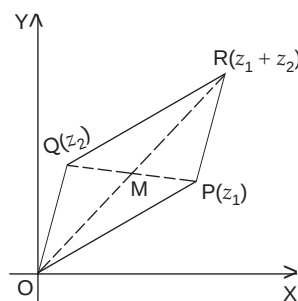


Fig. 14.5.

5. **Geometric representation of difference of two complex numbers.**

Let the complex numbers z_1 and z_2 be represented by the points P and Q respectively in the complex plane. Join QO and produce it to Q' such that O is mid-point of $[QQ']$ then Q' represents the complex number $-z_2$. Complete the parallelogram OPRQ' then the point R represents the complex number $z_1 + (-z_2)$ i.e. $z_1 - z_2$.

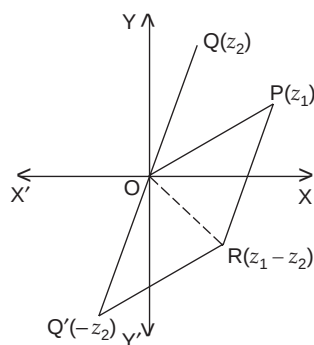


Fig. 14.6.

14.2.2 Trigonometric (polar) representation

We have already seen that a complex number $x + iy$ is uniquely represented by a point $P(x, y)$ in the complex plane. The angle XOP is called **amplitude** or **argument** of z and is written as $\text{amp}(z)$ or $\arg(z)$.

Let $OP = |z| = r > 0$, and $\text{amp}(z) = \theta$.

From fig. 14.7, we see that

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

$$\therefore z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta).$$

This form of z is called **trigonometric form** or **polar form**. The number $(\cos \theta + i \sin \theta)$ is usually abbreviated as $\text{cis } \theta$. Thus, if modulus of z is r and $\text{amp}(z) = \theta$, then

$$z = r(\cos \theta + i \sin \theta) = r \text{cis } \theta.$$

The unique value of θ such that $-\pi < \theta \leq \pi$ is called **principal value of amplitude** or **argument**.

Note that the number zero cannot be put into the form $r(\cos \theta + i \sin \theta)$. Therefore, if $z = 0$ then the $\arg(z)$ is not defined.

For example :

- (i) Let $z = 1 + i$, then $r = \sqrt{1^2 + 1^2} = 2$

$$\text{and } r \cos \theta = 1, r \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}.$$

So the principal value of amplitude is $\frac{\pi}{4}$.

$$\therefore z = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

Conversely, if $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

$$\begin{aligned} \text{then } z &= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ &= \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = 1 + i. \end{aligned}$$

- (ii) Let $z = 1 - i$, then $r = \sqrt{1^2 + (-1)^2} = 2$

$$\text{and } r \cos \theta = 1, r \sin \theta = -1$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = -\frac{\pi}{4}.$$

So the principal value of amplitude is $-\frac{\pi}{4}$.

$$\therefore z = \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right).$$

Conversely, if $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$

$$\begin{aligned} \text{then } z &= \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right) \\ &= \sqrt{2} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) = 1 - i. \end{aligned}$$

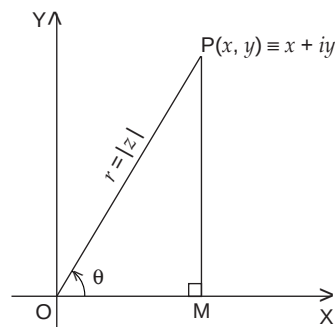


Fig. 14.7.

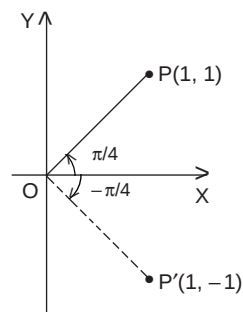


Fig. 14.8.

26. If complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle at C, then show that

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2).$$

Hint. Take C as origin. Let $BC = CA = k$, then

$$z_1 = k + i0 = k, z_2 = 0 + ik = ik \text{ and } z_3 = 0 + i0 = 0.$$

27. If $\omega (\neq 1)$ is a cube root of unity, then find the conjugate of $2\omega - 3i$.
 28. If z is a complex number, prove that $(z - 1)(\bar{z} - 1) = |z - 1|^2$.

ANSWERS

EXERCISE 14.1

1. (i) -6 (ii) 6 (iii) 0 (iv) $-i$ (v) $8i$ (vi) 1 .
2. $-3, -1, -3 + i, \sqrt{10}, -\frac{3}{10} + \frac{1}{10}i$.
3. (i) $\frac{3}{25} + \frac{4}{25}i$ (ii) $-\frac{2}{7} + \frac{\sqrt{3}}{7}i$ (iii) $-3 + 2i$
 (iv) $-\frac{1}{5} + \frac{3}{5}i$ (v) $\frac{1}{4} - \frac{3}{4}i$ (vi) $1 + 0i$.
4. (i) $x = \frac{1}{6}, y = \frac{5}{2}$ (ii) $x = \pm 2, y = -1$ (iii) $x = \frac{2}{21}, y = -\frac{8}{21}$
 (iv) $x = \frac{5}{13}, y = \frac{14}{13}$.
5. $a = 4, b = 2$. 6. $x = 3, y = -1$.
7. (ii) No; e.g. let $z_1 = 1 + i, z_2 = 1 - i$.
9. (i) $-9 + 46i; -9 - 46i$ (ii) $-\frac{11}{125} - \frac{2}{125}i; -\frac{11}{125} + \frac{2}{125}i$ (iii) $1 + i, \sqrt{2}$
 (iv) $(54\sqrt{5} - 45) - 406i$ (v) $0 + \frac{1}{2}i$ (vii) $-3 + i$ (viii) $3 + 5i$.
10. (i) 1 (ii) $\sqrt{\frac{41}{85}}$. 15. $a = 1, b = 0$. 16. $n = 2$.
17. $x = 0$ is the only solution. 18. $z = \frac{3}{4} + i$ is the only solution.
22. -160 . 23. Solutions are $6 + 8i, 6 + 17i$. 26. 2 .

EXERCISE 14.2

1. (i) $2 \operatorname{cis} \frac{\pi}{6}$ (ii) $\sqrt{2} \operatorname{cis} \left(-\frac{3\pi}{4}\right)$ (iii) $\pi \operatorname{cis} 0$ (iv) $\operatorname{cis} 0$
 (v) $\sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4}\right)$ (vi) $2 \operatorname{cis} \frac{\pi}{3}$ (vii) $\sqrt{2} \operatorname{cis} \frac{\pi}{4}$.
2. (i) $\sqrt{2}; \frac{3\pi}{4}$ (ii) $1; \pi - \tan^{-1} \frac{4}{3}$ (iii) $\sqrt{2}; -\frac{3\pi}{4}$. 3. $\frac{\pi}{10}$.
4. (i) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (ii) $1 - i$. 14. Maximum value $= \sqrt{2} + 1$, minimum value $= \sqrt{2} - 1$.

EXERCISE 14.3

1. (i) $|z| = 2$ (ii) $|z + 1 + i| > \sqrt{2}$
 (iii) $|z - 1 - i| = \sqrt{2}$ (iv) $\operatorname{Re}(z) > 0$
 (v) $\operatorname{Re}(z) = 0$ (vi) $\operatorname{Re}(z) > 0$ and $\operatorname{Im}(z) > 0$.