Theorem 6.7 : If $\bar{x}, \bar{y} \in R^3$, $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$ and $(\bar{x} \hat{,} \bar{y}) = \alpha$, then

(1) $\bar{x} \cdot \bar{y} = |\bar{x}|\,|\bar{y}|\, cos\alpha$

(2) $|\bar{x} \times \bar{y}| = |\bar{x}|\,|\bar{y}|\, sin\alpha$

(3) $\bar{x} \perp (\bar{x} \times \bar{y})$, $\bar{y} \perp (\bar{x} \times \bar{y})$

Proof : (1) By definition of the measure of the angle between two vectors, $\alpha = cos^{-1} \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

$\therefore \quad cos\alpha = \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

$\therefore \quad \bar{x} \cdot \bar{y} = |\bar{x}|\,|\bar{y}|\, cos\alpha$

(2) By Lagrange's identity,

$|\bar{x} \times \bar{y}|^2 + |\bar{x} \cdot \bar{y}|^2 = |\bar{x}|^2\,|\bar{y}|^2$

$$\begin{aligned} \therefore \quad |\bar{x} \times \bar{y}|^2 &= |\bar{x}|^2\,|\bar{y}|^2 - |\bar{x} \cdot \bar{y}|^2 \\ &= |\bar{x}|^2\,|\bar{y}|^2 - |\bar{x}|^2\,|\bar{y}|^2\, cos^2\alpha \\ &= |\bar{x}|^2\,|\bar{y}|^2\,(1 - cos^2\alpha) \\ &= |\bar{x}|^2\,|\bar{y}|^2\, sin^2\alpha \end{aligned}$$

$\therefore \quad |\bar{x} \times \bar{y}| = |\bar{x}|\,|\bar{y}|\, sin\alpha$ **($sin\alpha \geq 0$ as $0 \leq \alpha \leq \pi$)**

(3) Let $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$

Now, $\bar{x} \cdot (\bar{x} \times \bar{y}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$

$\therefore \quad \bar{x} \perp (\bar{x} \times \bar{y})$

Similarly, $\bar{y} \cdot (\bar{x} \times \bar{y}) = 0$. So $\bar{y} \perp (\bar{x} \times \bar{y})$.

Thus, $(\bar{x} \times \bar{y})$ is a vector orthogonal to both $\bar{x}$ and $\bar{y}$. And so $\pm \dfrac{\bar{x} \times \bar{y}}{|\bar{x} \times \bar{y}|}$ are unit vectors orthogonal to both $\bar{x}$ and $\bar{y}$.

Geometrical Interpretation of $\bar{x} \times \bar{y}$:

When the positive X-axis is rotated in anticlockwise direction to the positive Y-axis, a right handed screw would advance in positive direction of Z-axis as shown in figure 6.15.

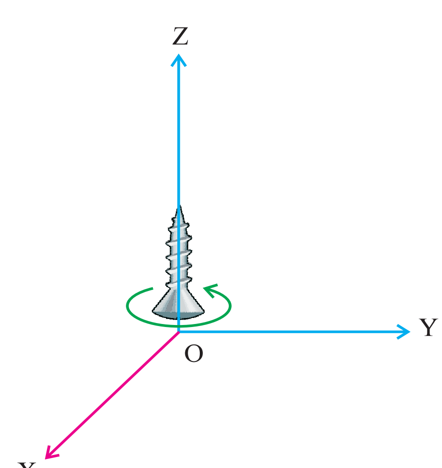


Figure 6.15

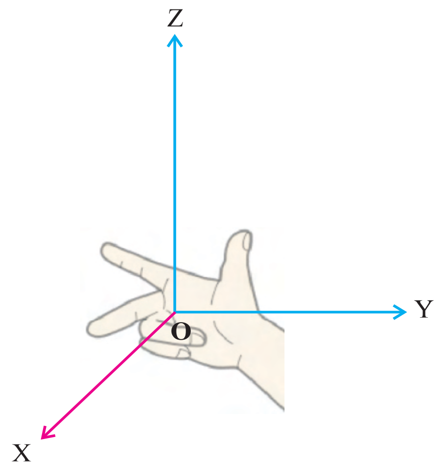


Figure 6.16

As $|\bar{x} \times \bar{y}| = |\bar{x}|\,|\bar{y}|\,sin\theta$, $\theta = (\bar{x}, \hat{\ } \bar{y})$

So, $\bar{x} \times \bar{y} = |\bar{x}|\,|\bar{y}|\,sin\theta\ \hat{n}$, where $\hat{n}$ is the unit vector in the direction of $\bar{x} \times \bar{y}$.

Direction of $\bar{x} \times \bar{y}$ can be determined by using right hand thumb rule i.e. if we keep fingers of our right hand in the direction of $\bar{x}$ and turning the fingers towards $\bar{y}$, then the direction shown by the thumb of the right hand is the direction of $\bar{x} \times \bar{y}$.

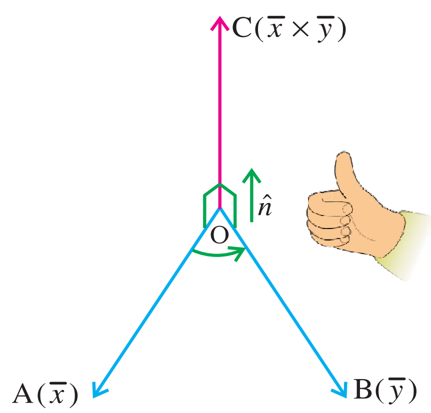


Figure 6.17

Example 14 : Find the measure of the angle between the vectors (1, −1, 2) and (2, −1, 1).

Solution : Let, $\bar{x} = (1, -1, 2)$ and $\bar{y} = (2, -1, 1)$

Now, $cos(\bar{x}, \hat{\ } \bar{y}) = \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

$$= \frac{(1, -1, 2) \cdot (2, -1, 1)}{\sqrt{1+1+4}\,\sqrt{4+1+1}} = \frac{2+1+2}{\sqrt{6}\sqrt{6}}$$

$$= \frac{5}{6}$$

$\therefore \quad (\bar{x}, \hat{\ } \bar{y}) = cos^{-1}\frac{5}{6}$

Example 15 : If the measure of the angle between the vectors $\sqrt{3}\hat{i} + \hat{j}$ and $a\hat{i} + \sqrt{3}\hat{j}$ is $\frac{\pi}{3}$, find a.

Solution : Let, $\bar{x} = \sqrt{3}\hat{i} + \hat{j} = (\sqrt{3}, 1)$ and $\bar{y} = a\hat{i} + \sqrt{3}\hat{j} = (a, \sqrt{3})$

It is given that $(\overline{x}, \hat{\ } \overline{y}) = \frac{\pi}{3}$

$\therefore \quad cos(\overline{x}, \hat{\ } \overline{y}) = cos\frac{\pi}{3}$

$\therefore \quad \frac{\overline{x} \cdot \overline{y}}{|\overline{x}||\overline{y}|} = \frac{1}{2}$ **(i)**

Now $\overline{x} \cdot \overline{y} = (\sqrt{3}, 1) \cdot (a, \sqrt{3}) = \sqrt{3}a + \sqrt{3}$, $|\overline{x}| = \sqrt{3+1} = 2$, $|\overline{y}| = \sqrt{a^2+3}$

$\therefore \quad \frac{\sqrt{3}a + \sqrt{3}}{2\sqrt{a^2+3}} = \frac{1}{2}$ **(using (i))**

$\therefore \quad \sqrt{3}(a + 1) = \sqrt{a^2+3}$ **(ii)**

$\therefore \quad 3(a^2 + 2a + 1) = a^2 + 3$

$\therefore \quad 2a^2 + 6a = 0$

$\therefore \quad 2a(a + 3) = 0$

$\therefore \quad a = 0 \quad$ or $\quad a = -3$

$a = -3$ does not satisfy (ii) as $\sqrt{3}(-2) \neq \sqrt{12} = 2\sqrt{3}$

For $a = 0$, $\sqrt{3}(a + 1) = \sqrt{3}$, $\sqrt{a^2+3} = \sqrt{3}$. Hence $a = 0$.

Example 16 : If $|\overline{x}| = |\overline{y}| = 1$ and $(\overline{x}, \hat{\ } \overline{y}) = \theta$, then prove that $|\overline{x} - \overline{y}cos\theta| = sin\theta$

Solution : $|\overline{x} - \overline{y}cos\theta|^2 = |\overline{x}|^2 - 2\overline{x} \cdot \overline{y}cos\theta + |\overline{y}cos\theta|^2$

$= 1 - 2\,cos\theta \cdot cos\theta + |\overline{y}|^2 cos^2\theta$ **($|\overline{x}| = 1$)**

$\left(cos\theta = \frac{\overline{x} \cdot \overline{y}}{|\overline{x}||\overline{y}|} \Rightarrow cos\theta = \frac{\overline{x} \cdot \overline{y}}{1}\right)$

$= 1 - 2\,cos^2\theta + cos^2\theta$ **($|\overline{y}| = 1$)**

$= 1 - cos^2\theta$

$= sin^2\theta$

$\therefore \quad |\overline{x} - \overline{y}cos\theta| = sin\theta$ **($0 \leq \theta \leq \pi$)**

Example 17 : If $\overline{x} = \hat{i} + a\hat{j} + 3\hat{k}$ and $\overline{y} = 2\hat{i} - \hat{j} + 5\hat{k}$ are orthogonal, find a.

Solution : Here $\overline{x} = (1, a, 3)$, $\overline{y} = (2, -1, 5)$

$\overline{x} \perp \overline{y} \iff \overline{x} \cdot \overline{y} = 0$

$\iff 2 - a + 15 = 0$

$\iff a = 17$

$\therefore \quad a = 17$

Example 18 : Find unit vectors orthogonal to both (1, 2, 3) and (2, −1, 4).

Solution : $\overline{x} = (1, 2, 3)$,

$\overline{y} = (2, -1, 4)$

$\therefore \quad \overline{x} \times \overline{y} = (11, 2, -5)$ and $|\overline{x} \times \overline{y}| = \sqrt{121+4+25} = \sqrt{150} = 5\sqrt{6}$

$\therefore$ Unit vectors orthogonal to the given vectors are $\pm \frac{\overline{x} \times \overline{y}}{|\overline{x} \times \overline{y}|} = \pm\left(\frac{11}{5\sqrt{6}}, \frac{2}{5\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

6.10 Projection of a Vector

If $\bar{a}$ and $\bar{b}$ are non-zero vectors and they are not orthogonal to each other, then the projection of $\bar{a}$ on $\bar{b}$ is defined as the vector $\left(\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\right)\bar{b}$ and is denoted by $\text{Proj}_{\bar{b}}\,\bar{a}$.

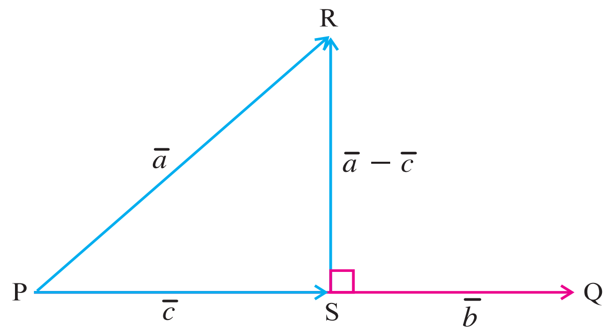


Figure 6.18

Let $\overrightarrow{PR} = \bar{a}$ and $\overrightarrow{PQ} = \bar{b}$ have the same initial point P. Also S is the foot of perpendicular from R to $\overleftrightarrow{PQ}$. Then we assert that $\overrightarrow{PS} = \text{Proj}_{\bar{b}}\,\bar{a}$. (as shown in figure 6.18)

Let $\bar{c} = \overrightarrow{PS}$, $\bar{c} \neq \bar{0}$ **(Why ?)**

Then $\overrightarrow{SR} = \bar{a} - \bar{c}$ since $\overrightarrow{PS} + \overrightarrow{SR} = \overrightarrow{PR} = \bar{a}$

$\bar{c}$ and $\bar{b}$ are in the same or in the opposite directions.

$\therefore \quad \bar{c} = k\bar{b},\ k \in R - \{0\}$

$\therefore \quad \bar{c}\cdot\bar{b} = k\bar{b}\cdot\bar{b} = k|\bar{b}|^2$

$\therefore \quad k = \frac{\bar{c}\cdot\bar{b}}{|\bar{b}|^2}$

As $\overleftrightarrow{RS} \perp \overleftrightarrow{PS}$, $(\bar{a} - \bar{c}) \perp \bar{b}$

$\therefore \quad (\bar{a} - \bar{c})\cdot\bar{b} = 0$

$\therefore \quad \bar{a}\cdot\bar{b} = \bar{c}\cdot\bar{b}$

$\therefore \quad k = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}$, since $k = \frac{\bar{c}\cdot\bar{b}}{|\bar{b}|^2}$

$\therefore \quad \overrightarrow{PS} = \bar{c} = \left(\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\right)\bar{b} = \text{Proj}_{\bar{b}}\,\bar{a}$.

Magnitude of projection vector is PS = $\frac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|^2}|\bar{b}| = \frac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|}$.

$\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$ is called the component of $\bar{a}$ along $\bar{b}$ and is denoted by $\text{Comp}_{\bar{b}}\,\bar{a}$.

Note : If two vectors of R^3 are given, then we can think as above by taking corresponding two bound vectors.

If $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are two vectors in R^3, then if we take equal vector as $\overrightarrow{AC}$ with initial point A, then we have the same result. Projection of $\overrightarrow{AB}$ on $\overrightarrow{PQ}$ is the vector $\overrightarrow{AC}$.

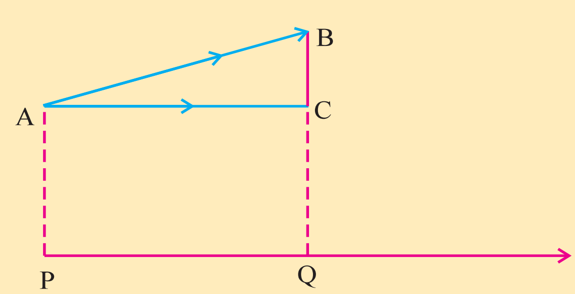


Figure 6.19

Area of a Triangle :

In ΔABC $\overrightarrow{AB} = \bar{c}$, $\overrightarrow{BC} = \bar{a}$, $\overrightarrow{CA} = \bar{b}$.

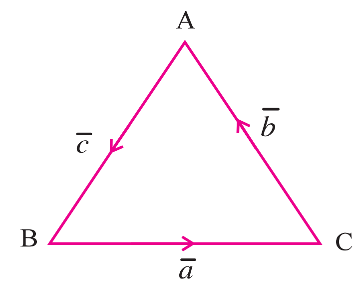


Figure 6.20

$$\text{Area of } \Delta ABC = \frac{1}{2} bc \, sinA$$

$$= \frac{1}{2} |\bar{b}| \, |\bar{c}| \, sinA$$

$$= \frac{1}{2} |\bar{b} \times \bar{c}|$$

$$(\bar{b} \, \hat{,} \, \bar{c}) = \pi - A \text{ and } sin(\pi - A) = sinA$$

Thus, **area of ΔABC** $= \frac{1}{2} |\bar{b} \times \bar{c}| = \frac{1}{2} |\bar{a} \times \bar{b}| = \frac{1}{2} |\bar{c} \times \bar{a}|$

Note : This formula can be applied in R^3 only.

Area of ΔABC is also given by

$$\Delta = \frac{1}{2} bc \sqrt{1 - cos^2 A}$$

$$= \frac{1}{2} |\bar{b}| \, |\bar{c}| \cdot \sqrt{1 - \left(\frac{\bar{b} \cdot \bar{c}}{|\bar{b}| |\bar{c}|} \right)^2}$$

$$\therefore \quad \Delta = \frac{1}{2} \sqrt{|\bar{b}|^2 |\bar{c}|^2 - |\bar{b} \cdot \bar{c}|^2}$$

Note : This formula can be applied in R^2 as well as in R^3.

Area of a Parallelogram :

$\square$OACB is a parallelogram with

$\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$.

$\overline{BM} \perp \overline{OA}$.

$\therefore$ BM = OB$sin\alpha$ = $|\bar{b}| \, sin\alpha$

$\therefore$ Area of $\square^m$ OACB = OA $\cdot$ BM

$= |\bar{a}| \, |\bar{b}| \, sin\alpha$

$\therefore$ **Area of $\square^m$ OACB** $= |\bar{a} \times \bar{b}|$

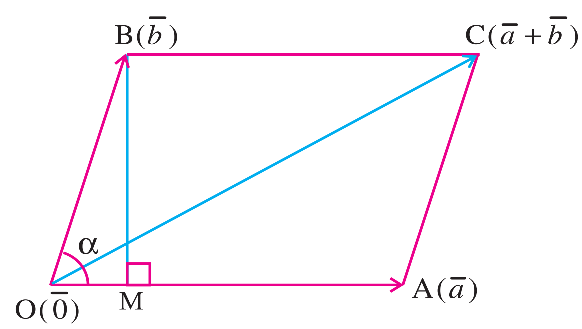


Figure 6.21

Note : **Area of $\square^m$ ABCD** $= \frac{1}{2} |\bar{x} \times \bar{y}|$

if $\overrightarrow{AC} = \bar{x}$, $\overrightarrow{BD} = \bar{y}$.

Let M be the point of intersection of the diagonals, then

$\overrightarrow{AM} = \frac{1}{2} \bar{x}$ and $\overrightarrow{BM} = \frac{1}{2} \bar{y}$

Area of $\square^m$ ABCD = 4(Area of ΔABM)

$= 4(\frac{1}{2} |\overrightarrow{AM} \times \overrightarrow{BM}|)$

Area of $\square^m$ ABCD $= 2 \left| \frac{1}{2} \bar{x} \times \frac{1}{2} \bar{y} \right| = \frac{1}{2} |\bar{x} \times \bar{y}|$

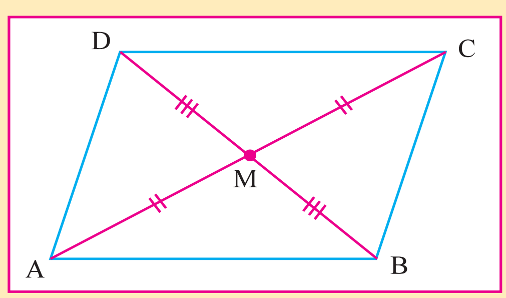


Figure 6.22

Example 19 : Find projection, component and magnitude of projection of $2\hat{i} + \hat{j} + \hat{k}$ on $-4\hat{i} - 2\hat{j} + 4\hat{k}$.

Solution : Here $\bar{a} = (2, 1, 1)$, $\bar{b} = (-4, -2, 4)$

$\therefore \quad \bar{a} \cdot \bar{b} = -8 - 2 + 4 = -6$ and $|b| = \sqrt{16+4+16} = 6$

$$\therefore \quad \text{Proj}_{\bar{b}}\, \bar{a} = \left(\frac{\bar{a} \cdot \bar{b}}{|\bar{b}|^2}\right) \bar{b} = \frac{-6}{36}(-4, -2, 4) = \frac{1}{6}(4, 2, -4) = \frac{1}{3}(2, 1, -2)$$

$$\therefore \quad \text{Comp}_{\bar{b}}\, \bar{a} = \frac{\bar{a} \cdot \bar{b}}{|\bar{b}|} = \frac{-6}{6} = -1$$

$$\text{Magnitude of Proj}_{\bar{b}}\, \bar{a} = \frac{|\bar{a} \cdot \bar{b}|}{|\bar{b}|} = \frac{|-6|}{6} = 1.$$

Volume of a Parallelopiped :

A parallelopiped is a solid consisting of six faces which are parallelograms.

Suppose $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors in R^3,

$\therefore \quad (\bar{a} \times \bar{b}) \cdot \bar{c} \neq 0$

Let the position vector of O be $\bar{0}$. $\overrightarrow{OA} = \bar{a}$, $\overrightarrow{OC} = \bar{b}$ represent vectors $\bar{a}$ and $\bar{b}$ respectively.

Here, □ OABC is a parallelogram.

$\therefore \quad$ Area of $\square^m$OABC $= |\bar{a} \times \bar{b}|$

Also $\bar{a} \times \bar{b}$ (i.e. $\overrightarrow{OM}$) is perpendicular to $\bar{a}$ and $\bar{b}$ both.

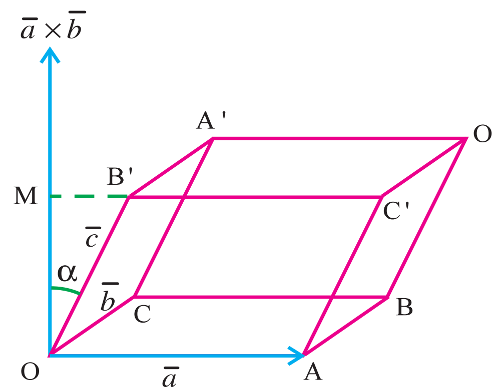


Figure 6.23

$\therefore \quad$ Height of parallelopiped OABC − B'C'O'A' = Magnitude of projection of $\bar{c}$ on $\bar{a} \times \bar{b}$ (i.e. OM)

$$= \frac{|\bar{c} \cdot (\bar{a} \times \bar{b})|}{|\bar{a} \times \bar{b}|}$$

Volume of parallelopiped = Area of base × height

$$= |\bar{a} \times \bar{b}| \frac{|\bar{c} \cdot (\bar{a} \times \bar{b})|}{|\bar{a} \times \bar{b}|}$$

$$= |\bar{c} \cdot (\bar{a} \times \bar{b})|$$

$\therefore \quad$ **Volume of parallelopiped** $= |[\bar{c}\ \bar{a}\ \bar{b}]| = |[\bar{a}\ \bar{b}\ \bar{c}]|$

Note : Let us note that $\bar{a}$, $\bar{b}$, $\bar{c}$ are the vectors denoting three consecutive edges of the parallelopiped.

Example 20 : Find the volume of the parallelopiped three of whose edges are $\overrightarrow{OA} = (2, 1, 1)$, $\overrightarrow{OB} = (3, -1, 1)$, $\overrightarrow{OC} = (-1, 1, -1)$.

Solution : Here, $\bar{a} = (2, 1, 1)$, $\bar{b} = (3, -1, 1)$, $\bar{c} = (-1, 1, -1)$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = 2(0) - 1(-2) + 1(2) = 4$$

Volume of parallelopiped $= \left|\,[\bar{a}\ \bar{b}\ \bar{c}]\,\right| = |4| = 4$

6.11 Direction *cosines*, Direction Angles and Direction Ratios of a Vector

We know that $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$ are unit vectors of R^3 in the positive directions of X-axis, Y-axis and Z-axis respectively. If $\bar{x} = (x_1, x_2, x_3)$ is a non-zero vector of R^3 and makes angles of measures α, β and γ with the positive directions of X-axis, Y-axis and Z-axis respectively, then α, β and γ are called the direction angles of $\bar{x}$ and $cos\alpha$, $cos\beta$, $cos\gamma$ are called the direction *cosines* of $\bar{x}$.

As α is the measure of the angle between $\bar{x}$ and $\hat{i}$, we have,

$$cos\alpha = \frac{\bar{x} \cdot \hat{i}}{|\bar{x}||\hat{i}|} = \frac{(x_1, x_2, x_3) \cdot (1, 0, 0)}{\sqrt{x_1^2 + x_2^2 + x_3^2} \cdot 1} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

Similarly, $cos\beta = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$ **and** $cos\gamma = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$.

If we take $l = cos\alpha$, $m = cos\beta$, $n = cos\gamma$

then $(l, m, n) = (cos\alpha, cos\beta, cos\gamma) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right)$

$$= \left(\frac{x_1}{|\bar{x}|}, \frac{x_2}{|\bar{x}|}, \frac{x_3}{|\bar{x}|}\right)$$

$$= \frac{1}{|\bar{x}|}(x_1, x_2, x_3) = \frac{\bar{x}}{|\bar{x}|} = \hat{x}$$

Now, $l^2 + m^2 + n^2 = cos^2\alpha + cos^2\beta + cos^2\gamma = \frac{x_1^2 + x_2^2 + x_3^2}{x_1^2 + x_2^2 + x_3^2} = 1$

Also $(cos\alpha, cos\beta, cos\gamma) = \frac{\bar{x}}{|\bar{x}|} = \hat{x}$

$\therefore$ $(cos\alpha, cos\beta, cos\gamma)$ is the unit vector in direction of $\bar{x}$ as $\frac{\bar{x}}{|\bar{x}|} = k\bar{x}$, where $k = \frac{1}{|\bar{x}|} > 0$.

If $\bar{x} = (x_1, x_2, x_3)$, $\bar{x} \neq \bar{0}$ and $m \neq 0$, then let $m\bar{x} = (mx_1, mx_2, mx_3)$. The components of $m\bar{x}$, namely, mx_1, mx_2 and mx_3 are called direction ratios (or direction numbers) of $\bar{x}$. Direction ratios of $k\bar{x}$ are $m(kx_1)$, $m(kx_2)$, $m(kx_3)$ $(m \neq 0, k \neq 0)$. Direction numbers of $\bar{x}$ and $m\bar{x}$ are same. For $m > 0$, $\bar{x}$, $m\bar{x}$ have same direction *cosines*. For $m < 0$, direction *cosines* of $\bar{x}$ and $m\bar{x}$ are additive inverses. Also, the direction angles of $\bar{x}$ are $\alpha = cos^{-1}\frac{x_1}{|\bar{x}|}$, $\beta = cos^{-1}\frac{x_2}{|\bar{x}|}$, $\gamma = cos^{-1}\frac{x_3}{|\bar{x}|}$.

$\frac{m\bar{x}}{|m\bar{x}|} = \frac{m\bar{x}}{|m||\bar{x}|} = \frac{m\bar{x}}{m|\bar{x}|} = \frac{\bar{x}}{|\bar{x}|}$, $m > 0$

$\therefore$ If $m > 0$, direction *cosines* of $\bar{x}$ and $m\bar{x}$ are same.

And if $m < 0$, $|m| = -m$. Hence direction *cosines* of $\bar{x}$ and $m\bar{x}$ are additive inverses.

Example 21 : Find direction *cosines* and direction angles of $\sqrt{2}\,\hat{i} - \hat{j} + \hat{k}$.

Solution : Since $\bar{x} = (\sqrt{2}, -1, 1)$, $|\bar{x}| = \sqrt{2+1+1} = 2$

If α, β and γ are the direction angles of $\bar{x}$, then $cos\alpha = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, $cos\beta = -\frac{1}{2}$, $cos\gamma = \frac{1}{2}$

$\therefore \quad \alpha = \frac{\pi}{4}$, $\beta = \pi - cos^{-1}\frac{1}{2} = \frac{2\pi}{3}$ and $\gamma = \frac{\pi}{3}$

$\therefore$ Direction *cosines* of $\bar{x}$ are $\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}$ and direction angles are $\frac{\pi}{4}, \frac{2\pi}{3}$ and $\frac{\pi}{3}$.

Example 22 : If a vector $\bar{x}$ makes angles with measure $\frac{\pi}{3}, \frac{2\pi}{3}$ with X-axis and Y-axis respectively, then find the measure of the angle made by $\bar{x}$ with Z-axis.

Solution : Let $\bar{x}$ make angles with measures α, β and γ with X-axis, Y-axis and Z-axis respectively. Then $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$. Here $\alpha = \frac{\pi}{3}$, $\beta = \frac{2\pi}{3}$

$\therefore \quad cos^2\frac{\pi}{3} + cos^2\frac{2\pi}{3} + cos^2\gamma = 1$

$\therefore \quad \frac{1}{4} + \frac{1}{4} + cos^2\gamma = 1$

$\therefore \quad cos^2\gamma = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \quad cos\gamma = \pm\frac{1}{\sqrt{2}}$

$\therefore \quad \gamma = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

Miscellaneous Examples

Example 23 : If $|\bar{x}| = 2$, $|\bar{y}| = 4$, $|\bar{z}| = 1$ and $\bar{x} + \bar{y} + \bar{z} = \bar{0}$, find $\bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{z} + \bar{z}\cdot\bar{x}$.

Solution : $|\bar{x} + \bar{y} + \bar{z}|^2 = |\bar{x}|^2 + |\bar{y}|^2 + |\bar{z}|^2 + 2\bar{x}\cdot\bar{y} + 2\bar{y}\cdot\bar{z} + 2\bar{z}\cdot\bar{x}$.

$\therefore \quad 0 = 4 + 16 + 1 + 2(\bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{z} + \bar{z}\cdot\bar{x})$

$\therefore \quad \bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{z} + \bar{z}\cdot\bar{x} = -\frac{21}{2}$.

Example 24 : If A(1, 1, 1), B(0, 2, 5), C(−3, 3, 2) and D(−1, 1, −6) are four points in R^3, find the measure of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. What can you conclude about $\overrightarrow{AB}$ and $\overrightarrow{CD}$?

Solution : $\overrightarrow{AB} = (0, 2, 5) - (1, 1, 1) = (-1, 1, 4)$ and $\overrightarrow{CD} = (-1, 1, -6) - (-3, 3, 2) = (2, -2, -8)$

$|\overrightarrow{AB}| = \sqrt{1+1+16} = 3\sqrt{2}$ and $|\overrightarrow{CD}| = \sqrt{4+4+64} = 6\sqrt{2}$

$$cos(\overrightarrow{AB}\,\hat{,}\,\overrightarrow{CD}) = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\,|\overrightarrow{CD}|} = \frac{-2-2-32}{3\sqrt{2}\times 6\sqrt{2}} = \frac{-36}{36} = -1$$

$\therefore \quad (\overrightarrow{AB}\,\hat{,}\,\overrightarrow{CD}) = \pi$

As the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ has measure π, they are in opposite directions.

Also, $\overrightarrow{AB} \times \overrightarrow{CD} = \bar{0}$, so $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Note : $\overrightarrow{CD} = -2\overrightarrow{AB}$. Hence $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear and in opposite directions.

Example 25 : Express $\bar{x} = 3\hat{i} - \hat{j} + 2\hat{k}$ as a sum of two vectors $\bar{a}$ and $\bar{b}$ such that $\bar{a}$ is parallel to $\bar{y}$ and $\bar{b}$ is perpendicular to vector $\bar{y}$, where $\bar{y} = 2\hat{i} - \hat{k}$.

Solution : $\bar{a}$ is parallel to $\bar{y}$.

So $\bar{a} = m\bar{y}$, $m \in R - \{0\}$

$\therefore \quad \bar{a} = 2m\hat{i} - m\hat{k} = (2m, 0, -m)$

Now, $\bar{x} = \bar{a} + \bar{b}$

$\therefore \quad \bar{b} = \bar{x} - \bar{a} = (3, -1, 2) - (2m, 0, -m) = (3 - 2m, -1, 2 + m)$

Again, $\bar{b} \perp \bar{y}$.

$\therefore \quad \bar{b} \cdot \bar{y} = 0$

$\therefore \quad (3 - 2m, -1, 2 + m) \cdot (2, 0, -1) = 0$

$\therefore \quad 6 - 4m - 2 - m = 0$

$\therefore \quad m = \frac{4}{5}$

$\therefore \quad \bar{a} = \frac{8}{5}\hat{i} - \frac{4}{5}\hat{k}$ and $\bar{b} = \left(3 - 2\left(\frac{4}{5}\right)\right)\hat{i} - \hat{j} + \left(2 + \frac{4}{5}\right)\hat{k} = \frac{7}{5}\hat{i} - \hat{j} + \frac{14}{5}\hat{k}$

$\therefore$ For the above $\bar{a}$ and $\bar{b}$, we have $\bar{x} = \bar{a} + \bar{b}$.

Example 26 : Prove that $\bar{a} \times [\bar{a} \times (\bar{a} \times \bar{b})] = |\bar{a}|^2(\bar{b} \times \bar{a})$

Solution : $\bar{a} \times [\bar{a} \times (\bar{a} \times \bar{b})] = [\bar{a} \cdot (\bar{a} \times \bar{b})]\,\bar{a} - (\bar{a} \cdot \bar{a})(\bar{a} \times \bar{b})$

$= [\bar{a}\ \bar{a}\ \bar{b}]\,\bar{a} - |\bar{a}|^2(-(\bar{b} \times \bar{a}))$

$= \bar{0} + |\bar{a}|^2(\bar{b} \times \bar{a})$

$= |\bar{a}|^2(\bar{b} \times \bar{a})$

Example 27 : For non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$, if $\bar{a} \times \bar{b} = \bar{c}$, $\bar{b} \times \bar{c} = \bar{a}$, then prove that $|\bar{b}| = 1$.

Solution : $\bar{b} \times \bar{c} = \bar{a}$

$\therefore \quad (\bar{b} \times \bar{c}) \cdot \bar{b} = \bar{a} \cdot \bar{b}$

$\therefore \quad [\bar{b}\ \bar{c}\ \bar{b}] = \bar{a} \cdot \bar{b}$

$\therefore \quad \bar{a} \cdot \bar{b} = 0$ **(i)**

Now, $\bar{b} \times \bar{c} = \bar{a}$

$\therefore \quad \bar{b} \times (\bar{a} \times \bar{b}) = \bar{a}$ **($\bar{c} = \bar{a} \times \bar{b}$)**

$\therefore \quad (\bar{b} \cdot \bar{b})\bar{a} - (\bar{b} \cdot \bar{a})\bar{b} = \bar{a}$

$\therefore \quad |\bar{b}|^2\bar{a} = \bar{a}$ **(using (i))**

$\therefore \quad (|\bar{b}|^2 - 1)\bar{a} = \bar{0}$

$\therefore$ Since $\bar{a} \neq \bar{0}$, $|\bar{b}|^2 = 1$ **($\alpha\bar{x} = \bar{0} \Rightarrow \alpha = 0$ or $\bar{x} = \bar{0}$)**

$\therefore$ $|\bar{b}| = 1$

Example 28 : A(1, 1, 2), B(2, 3, 5), C(1, 3, 4) and D(0, 1, 1) are the vertices of a parallelogram ABCD. Find its area.

Solution : Method 1 : Adjacent sides of $\square^m$ABCD are

$\overrightarrow{AB} = (2, 3, 5) - (1, 1, 2) = (1, 2, 3)$ and

$\overrightarrow{BC} = (1, 3, 4) - (2, 3, 5) = (-1, 0, -1)$

Area $= |\overrightarrow{AB} \times \overrightarrow{BC}| = |(-2 - 0, -(-1 + 3), 0 + 2)|$

$= |(-2, -2, 2)|$

$= \sqrt{4+4+4}$

$= 2\sqrt{3}$

Method 2 : Vector along the diagonal $\overline{AC}$ is $\overrightarrow{AC} = (0, 2, 2)$ and

Vector along the diagonal $\overline{BD}$ is $\overrightarrow{BD} = (-2, -2, -4)$.

$\therefore$ $\overrightarrow{AC} \times \overrightarrow{BD} = (-8 + 4, -(0 + 4), 0 + 4)$

$= (-4, -4, 4)$

$\therefore$ Area $= \frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{BD}|$

$= \frac{1}{2} |(-4, -4, 4)|$

$= \frac{1}{2} \sqrt{16+16+16}$

$= 2\sqrt{3}$

Example 29 : If α, β, γ are the direction angles of $\bar{x}$, prove that $sin^2\alpha + sin^2\beta + sin^2\gamma = 2$. Also find the value of $cos2\alpha + cos2\beta + cos2\gamma$.

Solution : α, β, γ are the direction angles of $\bar{x}$.

$\therefore$ $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

$\therefore$ $1 - sin^2\alpha + 1 - sin^2\beta + 1 - sin^2\gamma = 1$

$\therefore$ $sin^2\alpha + sin^2\beta + sin^2\gamma = 2$

Again, $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

$\therefore$ $\frac{1+cos2\alpha}{2} + \frac{1+cos\,2\beta}{2} + \frac{1+cos2\gamma}{2} = 1$

$\therefore$ $3 + cos2\alpha + cos2\beta + cos2\gamma = 2$

$\therefore$ $cos2\alpha + cos2\beta + cos2\gamma = -1$

Example 30 : Find a unit vector in XY-plane perpendicular to $4\hat{i} - 3\hat{j} + 2\hat{k}$.

Solution : Let the required vector in XY-plane be $(a, b, 0)$ and it is perpendicular to $(4, -3, 2)$.

$\therefore \quad (a, b, 0) \cdot (4, -3, 2) = 0$

$\therefore \quad 4a - 3b = 0$

$\therefore \quad a = \frac{3b}{4}$

Now, $(a, b, 0)$ is a unit vector.

$\therefore \quad a^2 + b^2 = 1$

$\therefore \quad \frac{9b^2}{16} + b^2 = 1$

$\therefore \quad 25b^2 = 16$

$\therefore \quad b = \pm\frac{4}{5}, \; a = \pm\frac{3}{5}$

$\therefore$ Required vector is $\pm\frac{1}{5}(3, 4, 0)$.

Example 31 : $\bar{a}$ is a unit vector and $\bar{b} = (3, 0, -4)$. The measure of the angle between them is $\frac{\pi}{6}$. If the diagonals of the parallelogram are $(3\bar{a} + \bar{b})$ and $(\bar{a} + 3\bar{b})$, then obtain the area of the parallelogram.

Solution : Area of parallelogram $= \frac{1}{2}\,|(3\bar{a} + \bar{b}) \times (\bar{a} + 3\bar{b})|$

$$= \frac{1}{2}\,|3(\bar{a} \times \bar{a}) + \bar{b} \times \bar{a} + 9(\bar{a} \times \bar{b}) + 3(\bar{b} \times \bar{b})|$$

$$= \frac{1}{2}\,|-(\bar{a} \times \bar{b}) + 9(\bar{a} \times \bar{b})| = 4\,|\bar{a} \times \bar{b}|$$

Now, $|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}|\,sin(\bar{a}, \hat{\ } \bar{b})$

$$= (1)\,(\sqrt{9+16})\,\left(sin\frac{\pi}{6}\right)$$

$$= (5)\left(\frac{1}{2}\right)$$

$$= \frac{5}{2}$$

$\therefore$ Area $= 4 \times \frac{5}{2} = 10$

Exercise 6

1. If $\bar{x} = (-1, 2, 3)$, $\bar{y} = (2, -1, 3)$ and $\bar{z} = (3, 2, 1)$, show that $\bar{x} \times (\bar{y} \times \bar{z}) \neq (\bar{x} \times \bar{y}) \times \bar{z}$.

2. Prove that $[\bar{x} + \bar{y} \quad \bar{y} + \bar{z} \quad \bar{z} + \bar{x}] = 2\,[\bar{x} \quad \bar{y} \quad \bar{z}]$.

3. Does $\bar{x} \cdot \bar{y} = \bar{x} \cdot \bar{z}$ imply $\bar{y} = \bar{z}$? Why ?

4. Does $\bar{x} \times \bar{y} = \bar{x} \times \bar{z}$ imply $\bar{y} = \bar{z}$? Why ?

5. If $\bar{x} \cdot \bar{y} = \bar{x} \cdot \bar{z}$ and $\bar{x} \times \bar{y} = \bar{x} \times \bar{z}$ and $\bar{x} \neq \bar{0}$, then prove that $\bar{y} = \bar{z}$.

6. Find a, b, c if $a(1, 3, 2) + b(1, -5, 6) + c(2, 1, -2) = (4, 10, -8)$.

7. If $m\bar{a} = n\bar{b}$, $m, n \in$ N, then prove that $\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}|$. If $m, n \in$ Z $-$ {0}, what can be said ?

8. Prove that $\bar{x} \times (\bar{y} \times \bar{z}) + \bar{y} \times (\bar{z} \times \bar{x}) + \bar{z} \times (\bar{x} \times \bar{y}) = \bar{0}$.

9. Find direction angles and direction *cosines* of the following vectors :

(1) (1, 0, −1) (2) $\hat{j} + \hat{k}$ (3) $5\hat{i} + 12\hat{j} + 84\hat{k}$.

10. If $(\widehat{\bar{x}, \bar{y}}) = \alpha$, then prove that $\sin\frac{\alpha}{2} = \frac{1}{2}|\bar{x} - \bar{y}|$, where $\bar{x}$ and $\bar{y}$ are unit vectors.

11. Find unit vectors in R^2 orthogonal to (5, −12).

12. If $\bar{x}, \bar{y}, \bar{z}$ are non-coplanar, then prove that $\bar{x} + \bar{y}$, $\bar{y} + \bar{z}$ and $\bar{z} + \bar{x}$ are non-coplanar.

13. Prove that $(\bar{a} - \text{Proj}_{\bar{b}}\bar{a})$ is orthogonal to $\bar{b}$.

14. Prove that (1, 2, 3) and (2, 1, 3) are not collinear.

15. Prove that (1, 2, 3), (2, 3, 5) and (5, 8, 13) are coplanar.

16. If the angle between (*a*, 2) and (*a*, −2) has measure $\frac{\pi}{3}$, find *a*.

17. Prove that $a\hat{i} + 3\hat{j} + 2\hat{k}$ cannot be orthogonal to $-a\hat{i} + \hat{j} - 2\hat{k}$.

18. Find $|\bar{a} \times \bar{b}|$, if $|\bar{a}| = 4$, $|\bar{b}| = 5$ and $(\bar{a} \cdot \bar{b}) = -6$.

19. If (*a*, 1, 1), (1, *b*, 1) and (1, 1, *c*) are coplanar, prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$.

20. $\bar{a} \times \bar{b} = \bar{a} \times \bar{c}$, $\bar{a} \neq \bar{0}$, $\bar{b} \neq \bar{c}$, then show that $\bar{b} = \bar{c} + k\bar{a}$, $k \in R$

21. If $\bar{a}$ is orthogonal to both $\bar{b}$ and $\bar{c}$ and $\bar{a}, \bar{b}, \bar{c}$ are unit vectors and $(\widehat{\bar{b}, \bar{c}}) = \frac{\pi}{6}$, show that $\bar{a} = \pm 2(\bar{b} \times \bar{c})$.

22. Prove that $[(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})] \cdot \bar{d} = (\bar{a} \cdot \bar{d})[\bar{a} \;\; \bar{b} \;\; \bar{c}]$.

23. Prove by using vectors that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$.

24. Find the area of the triangle whose verticies are (4, −3, 1), (2, −4, 5), (1, −1, 0).

25. Find the projection of $4\hat{i} + \hat{j} + 3\hat{k}$ on $\hat{i} - \hat{j} + \hat{k}$ and its magnitude.

26. Find the projection of (*a, b, c*) on Y-axis and its magnitude.

27. If A(3, 2, −4), B(4, 3, −4), C(3, 3, 3) and D(4, 2, −3), find projection of $\overrightarrow{AD}$ on $\overrightarrow{AB} \times \overrightarrow{AC}$.

28. Use vectors to prove $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ for ΔABC.

29. Obtain *cosine* formula for a triangle by using vectors.

30. Express $2\hat{i} + 3\hat{j} + \hat{k}$ as a sum of two vectors out of which one vector is perpendicular to $2\hat{i} - 4\hat{j} + \hat{k}$ and another is parallel to $2\hat{i} - 4\hat{j} + \hat{k}$.

31. Find unit vector in R^3 which makes an angle of measure $\frac{\pi}{4}$ with $\hat{i}$ and perpendicular to $\hat{k}$.

32. If the sum of two unit vectors is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.

33. If $\bar{a} = (1, 1, 1)$ and $\bar{c} = (0, 1, -1)$ are two given vectors, find $\bar{b}$ such that $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$.

34. Find the volume of parallelopiped whose edges are $\overrightarrow{OA} = (3, 1, 4)$, $\overrightarrow{OB} = (1, 2, 3)$, $\overrightarrow{OC} = (2, 1, 5)$.

35. Prove that if $\bar{x} \times \bar{y} = \bar{0}$, then $\bar{x} = k\bar{y}$, $k \in R - \{0\}$, $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$

36. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

(1) If $\bar{x} = (-2, 1, -2)$, then a unit vector in the direction of $\bar{x}$ is ...... ☐

(a) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ (c) $\left(-\frac{2}{9}, \frac{1}{9}, -\frac{2}{9}\right)$ (d) $\left(\frac{2}{9}, -\frac{1}{9}, \frac{2}{9}\right)$

(2) ...... is not a unit vector. $(\alpha \neq \frac{n\pi}{2}, n \in Z)$

(a) $(cos\alpha, sin\alpha)$ (b) $(-cos\alpha, -sin\alpha)$ (c) $(-cos2\alpha, sin2\alpha)$ (d) $(cos2\alpha, sin\alpha)$

(3) $\bar{x} \times \bar{y} = (7, 2, -3)$, then $\bar{y} \times \bar{x} =$ ......

(a) $(7, 2, -3)$ (b) $(-3, 2, 7)$ (c) $(-7, -2, 3)$ (d) $(3, -2, -7)$

(4) $|\bar{x}| = |\bar{y}| = 1$, $\bar{x} \perp \bar{y}$, $|\bar{x} + \bar{y}| =$ ......

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 1 (d) 0

(5) If $\bar{x} = 3\bar{y}$, then $\bar{x} \times \bar{y} =$ ......

(a) $3|\bar{y}|^2$ (b) $3|\bar{x}|^2$ (c) $\bar{0}$ (d) $\frac{1}{3}|\bar{y}|^2$

(6) $\bar{x} = (2, 3)$, $\bar{y} = (5, -2)$ are ...... vectors.

(a) collinear (b) non-collinear (c) same directional (d) of opposite direction

(7) If $\bar{x} = (a, 4, 2a)$ and $\bar{y} = (2a, -1, a)$ are perpendicular to each other, then $a =$ ....

(a) 2 (b) 1 (c) 4 (d) any real number

(8) $(a, 1, -2)$, $(1, 1, 3)$, $(8, 5, 0)$ are coplanar then $a =$ ....

(a) -5 (b) 5 (c) -2 (d) 2

(9) If $\bar{x} = (3, 1, 0)$, $\bar{y} = (2, 2, 3)$, $\bar{z} = (-1, 2, 1)$ and $\bar{x} \perp (\bar{y} + k\bar{z})$, then $k =$ ......

(a) 8 (b) 4 (c) $\frac{1}{8}$ (d) $\frac{1}{4}$

(10) If $\bar{x} = (1, 2, 4)$, $\bar{y} = (-1, -2, k)$, $k \neq -4$, then $|\bar{x} \cdot \bar{y}|$ ...... $|\bar{x}||\bar{y}|$.

(a) $<$ (b) $>$ (c) $=$ (d) $\geq$

(11) $\bar{x} = (-1, 4, -2)$, $\bar{y} = (-4, 16, -8)$, then $|\bar{x} + \bar{y}|$ ...... $|\bar{x}| + |\bar{y}|$.

(a) $=$ (b) $>$ (c) $\geq$ (d) $\leq$

(12) $(3, 6, -9)$ and ...... have same direction ratios.

(a) $(1, 2, 3)$ (b) $(\pi, 2\pi, 3\pi)$ (c) $(-1, -2, 3)$ (d) $(1, 2, 0)$

(13) If $\bar{a} = (-3, 1, 0)$ and $\bar{b} = (1, -1, -1)$, then $\text{Comp}_{\bar{a}}\bar{b} =$ ....

(a) $\frac{4}{\sqrt{10}}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{-4}{\sqrt{10}}$ (d) $-\frac{\sqrt{3}}{4}$

(14) The area of the parallelogram whose diagonals are $\hat{j} + \hat{k}$ and $\hat{i} + \hat{k}$ is ....

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{3}{2}$ (c) 3 (d) $\sqrt{3}$

(15) Magnitude of the projection of $(-1, 2, -1)$ on $\hat{i}$ is ....

(a) $\frac{1}{\sqrt{6}}$ (b) $-\frac{1}{\sqrt{6}}$ (c) 1 (d) -1

(16) $\bar{a}$ is a non-zero vector, then number of unit vectors collinear with $\bar{a}$ is ....

(a) 1 (b) 2 (c) 3 (d) infinitely many.

(17) The area of the parallelogram whose adjacent sides are $\hat{i} + \hat{k}$ and $\hat{i} + \hat{j}$ is ....

(a) 3 (b) $\sqrt{3}$ (c) $\frac{3}{2}$ (d) $\frac{\sqrt{3}}{2}$

(18) If $\bar{x}$ and $\bar{y}$ are non-collinear, non-zero vectors, then number of unit vectors orthogonal to both $\bar{x}$ and $\bar{y}$ is ....

(a) 2 (b) 4 (c) none (d) infinitely many.

(19) If θ is the measure of the angle between vectors $\bar{x}$ and $\bar{y}$ such that $\bar{x} \cdot \bar{y} \geq 0$, then ...... ☐

(a) $0 \leq \theta \leq \pi$ (b) $\frac{\pi}{2} \leq \theta \leq \pi$ (c) $0 \leq \theta \leq \frac{\pi}{2}$ (d) $0 < \theta < \frac{\pi}{2}$

(20) The unit vector in the direction of sum of the vectors (1, 1, 1), (2, −1, −1) and (0, 2, 6) is .... ☐

(a) $-\frac{1}{7}(3, 2, 6)$ (b) $\frac{1}{49}(3, 2, 6)$ (c) $\frac{1}{7}(3, -2, 6)$ (d) $\frac{1}{7}(3, 2, 6)$

(21) The expression ...... is meaningless. ☐

(a) $\bar{a} \cdot (\bar{b} \times \bar{c})$ (b) $(\bar{a} \cdot \bar{b})\, \bar{c}$ (c) $\bar{a} \times (\bar{b} \cdot \bar{c})$ (d) $\bar{a} \times (\bar{b} \times \bar{c})$

(22) If $\bar{x} = \hat{i} - \hat{j} + \hat{k}$, $\bar{y} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\bar{z} = \hat{i} + a\hat{j} + b\hat{k}$ are coplanar and $|\bar{z}| = \sqrt{3}$, then ...... ☐

(a) $a = 1, b = -1$ (b) $a = 1, b = \pm 1$ (c) $a = -1, b = \pm 1$ (d) $a = \pm 1, b = 1$

(23) If A(3, −1), B(2, 3) and C(5, 1), then $m\angle A$ = ... ☐

(a) $\cos^{-1} \frac{3}{\sqrt{34}}$ (b) $\pi - \cos^{-1} \frac{3}{\sqrt{34}}$ (c) $\sin^{-1} \frac{5}{\sqrt{34}}$ (d) $\frac{\pi}{2}$

(24) If $|\bar{x} \cdot \bar{y}| = \cos\alpha$, then $|\bar{x} \times \bar{y}|$ = ... ☐

(a) $\pm \sin\alpha$ (b) $\sin\alpha$ (c) $-\sin\alpha$ (d) $\sin^2\alpha$

(25) If $\bar{x} \cdot \bar{y} = 0$, then $\bar{x} \times (\bar{x} \times \bar{y})$ = ..., where $|\bar{x}| = 1$. ☐

(a) $\bar{x} \times \bar{y}$ (b) $\bar{x}$ (c) $-\bar{y}$ (d) $\bar{y} \times \bar{x}$

•

Summary

We have studied the following points in this chapter :

1. $R^2 = \{(x, y) \mid x \in R, y \in R\}$ and $R^3 = \{(x, y, z) \mid x \in R, y \in R, z \in R\}$ are vector spaces over R.

2. Properties of vector space were listed.

3. Magnitude of a Vector : If $\bar{x} = (x_1, x_2, x_3)$, then magnitude of $\bar{x}$ is $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

$\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$ are unit vectors in the positive direction of X-axis, Y-axis and Z-axis respectively. If $\bar{x} = (x_1, x_2)$, then $|\bar{x}| = \sqrt{x_1^2 + x_2^2}$. In R^2, $\hat{i} = (1, 0)$, $\hat{j} = (0, 1)$.

4. Direction of vectors : Let $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$

If **(i)** $\bar{x} = k\bar{y}$, $k > 0$, then $\bar{x}$ and $\bar{y}$ are vectors having same direction.

(ii) $\bar{x} = k\bar{y}$, $k < 0$, then $\bar{x}$ and $\bar{y}$ are vectors having opposite directions.

(iii) $\bar{x} \neq k\bar{y}$, for any $k \in R$, then $\bar{x}$ and $\bar{y}$ are vectors having different directions.

5. Non-zero vectors $\bar{x}$ and $\bar{y}$ are equal if and only if $|\bar{x}| = |\bar{y}|$ and $\bar{x}$ and $\bar{y}$ have the same direction.

6. If $\bar{x} \neq \bar{0}$, then $\frac{1}{|\bar{x}|}\bar{x}$ is a unit vector in the direction of $\bar{x}$ and it is denoted by $\hat{x}$.

7. If $A(x_1, x_2, x_3)$ and $B(y_1, y_2, y_3)$ are two distinct points in R^3, then

$\overrightarrow{AB} = (y_1 - x_1, y_2 - x_2, y_3 - x_3)$

8. $P(x_1, x_2, x_3) \in R^3$, then

(i) Distance of P from XY-plane $= |x_3|$. from YZ-plane $= |x_1|$ and from ZX-plane $= |x_2|$.

(ii) Distance of P from X-axis $= \sqrt{x_2^2 + x_3^2}$.

(iii) Distance of P from origin $= \sqrt{x_1^2 + x_2^2 + x_3^2}$.

9. **Triangle law of vector addition :** If A, B and C are non-collinear points, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

10. **Inner Product :** If $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, then inner product of $\bar{x}$ and $\bar{y}$ is $\bar{x} \cdot \bar{y} = x_1y_1 + x_2y_2 + x_3y_3$. If $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$, then $\bar{x} \cdot \bar{y} = x_1y_1 + x_2y_2$.

Properties of inner product were studied.

11. **Outer Product :** If $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, then outer product of $\bar{x}$ and $\bar{y}$ is $\bar{x} \times \bar{y} = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$.

Properties of outer product were studied.

12. **Box Product :** If $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$ and $\bar{z} = (z_1, z_2, z_3)$, then box product of $\bar{x}$, $\bar{y}$ and $\bar{z}$ is

$$\bar{x} \cdot (\bar{y} \times \bar{z}) = [\bar{x} \quad \bar{y} \quad \bar{z}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Properties of box product were studied.

13. **Vector Triple Product :** If $\bar{x}, \bar{y}, \bar{z} \in R^3$, then vector triple product of $\bar{x}$, $\bar{y}$ and $\bar{z}$ is $\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$.

14. **Lagrange's Identity :** $(\bar{x} \cdot \bar{y})^2 + |\bar{x} \times \bar{y}|^2 = |\bar{x}|^2 |\bar{y}|^2$

15. **Cauchy-Schwartz Inequality :** $|\bar{x} \cdot \bar{y}| \leq |\bar{x}| |\bar{y}|$

16. **Triangle Inequality :** $|\bar{x} + \bar{y}| \leq |\bar{x}| + |\bar{y}|$.

17. **Measure of the angle between two non-zero vecotrs :** $(\bar{x} \wedge \bar{y}) = cos^{-1} \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

18. If $\bar{x} \cdot \bar{y} = 0 \Leftrightarrow \bar{x} \perp \bar{y}$

19. **Projection of a Vector :** If $\bar{a}$ and $\bar{b}$ are non-zero vectors and they are not orthogonal, then the projection of $\bar{a}$ on $\bar{b}$ is $Proj_{\bar{b}}\,\bar{a} = \left(\frac{\bar{a} \cdot \bar{b}}{|\bar{b}|^2}\right)\bar{b}$.

Component of $\bar{a}$ on $\bar{b}$ is $\text{Comp}_{\bar{b}}\,\bar{a} = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$.

Magnitude of $\text{Proj}_{\bar{b}}\,\bar{a} = \dfrac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|}$.

20. Area of ΔABC : If $\bar{a} = \overrightarrow{BC}$, $\bar{b} = \overrightarrow{CA}$, $\bar{c} = \overrightarrow{AB}$, then

area of $\Delta ABC = \frac{1}{2}|\bar{b} \times \bar{c}|$

$= \frac{1}{2}\sqrt{|\bar{b}|^2|\bar{c}|^2 - |\bar{b}\cdot\bar{c}|^2}$

21. Area of a Parallelogram : Area of $\square^m$ ABCD $= |\overrightarrow{AB} \times \overrightarrow{BC}|$

$= \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$

22. Volume of a Parallelopiped : If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are the edges of a parallelopiped, then volume of parallelopiped $= |[\bar{a}\ \bar{b}\ \bar{c}]|$.

23. Collinear Vectors : Non-zero vectors $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ are collinear if and only if $x_1y_2 - x_2y_1 = 0$.

Non-zero vectors $\bar{x}$ and $\bar{y}$ of R^3 are collinear if and only if $\bar{x} \times \bar{y} = \bar{0}$.

24. Coplanar Vectors : If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the vectors of R^3 and we can find $\alpha, \beta, \gamma \in R$ with at least one of them non-zero, such that $\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \bar{0}$, then $\bar{x}$, $\bar{y}$ and $\bar{z}$ are said to be coplanar vectors.

The vectors which are not coplanar are said to be non-coplanar or linearly independent vectors.

25. Distinct non-zero vectors $\bar{x}$, $\bar{y}$, $\bar{z}$ of R^3 are coplanar if and only if $[\bar{x}\ \bar{y}\ \bar{z}] = 0$.

26. Direction *cosines*, Direction Angles and Direction Ratios of a Vector : If $\bar{x} = (x_1, x_2, x_3)$ is a non-zero vector of R^3 and makes angles of measures α, β and γ with the positive directions of X-axis, Y-axis and Z-axis respectively, then α, β and γ are called the **direction angles** of $\bar{x}$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the **direction *cosines*** of $\bar{x}$.

Here, $\cos\alpha = \dfrac{\bar{x}\cdot\hat{i}}{|\bar{x}||\hat{i}|} = \dfrac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$, $\cos\beta = \dfrac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$ and $\cos\gamma = \dfrac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$.

For $m \neq 0$, mx_1, mx_2, mx_3 are called direction ratios of $\bar{x}$.

Three Dimensional Geometry

7

To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.

– Pierre de Fermat

7.1 Introduction

We have studied plane geometry in standard IX and X and studied the same concepts in the light of coordinate geometry in standard X and XI. Now in the semester II, we studied about the vector space which was explained with the concept of three dimensional coordinate system in R^3 and vectors in R^3. Now, question arises whether we can study a line, a plane, a square, a triangle, a sphere,... in R^3? The answer is yes. Vectors can help us to study such concepts. In this chapter, we shall study about the equations of a line and a plane in space.

Before we study lines in space, let us be clear about some differences in plane geometry and three dimensional geometry. Given two lines in a plane, there are three possibilities : (1) lines are parallel, (2) lines are coincident and (3) lines intersect in unique point. These can be very easily seen by drawing lines on a paper, but when we think of two lines in R^3, basically there are two possibilities : They are in the same plane or there is no plane containing these two lines. If they are in the same plane, they are called coplanar and for them, there are three possibilities as discussed above. If two lines are not in the same plane, they are called non-coplanar or skew.

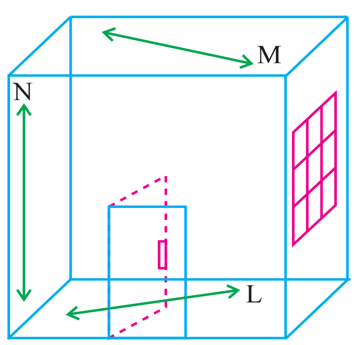


Figure 7.1

In figure 7.1, we see that line L is in the plane of floor and line M is in the plane of ceiling. These lines L and M are in different parallel planes and there is no plane containing them. Hence these lines are skew lines or non-coplanar lines. Such a possibility cannot be observed in plane geometry. Observing carefully one can imagine that $L \perp N$ and $M \perp N$ but L and N as well as M and N are not intersecting each other. This is not observed in the plane geometry.

Figure 7.2 is a picture of three mutually perpendicular lines in space. This is not possible in plane geometry.

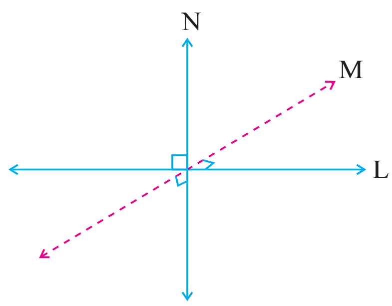


Figure 7.2

7.2 Direction of a line

We know about the direction of a vector. If A and B are two distinct points of a line L in R^3, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have opposite directions. If direction of $\overrightarrow{AB}$ is $\overline{l}$, then direction of $\overrightarrow{BA}$ is $-\overline{l}$. Both $\pm\overline{l}$ are called directions of $\overleftrightarrow{AB}$. (i.e. line L)

Thus, when we talk about $\overline{l}$ as the direction of a line L, we mean to say that direction of any non-zero vector on L can be $\overline{l}$ or $-\overline{l}$.

Note : **(1)** Lines in space will be denoted by letters L, M, N,...

(2) A line in space can uniquely be determined if

(i) it passes through a given point and has the direction of a non-zero vector $\bar{l}$. (or $-\bar{l}$) written briefly as 'direction $\bar{l}$'.

(ii) it passes through two distinct points.

7.3 Equation of a line passing through A($\bar{a}$) and having the same direction as a non-zero vector $\bar{l}$

Let L be the line passing through A($\bar{a}$) and having direction $\bar{l}$.

Let P($\bar{r}$) be any point on the line L and P ≠ A.

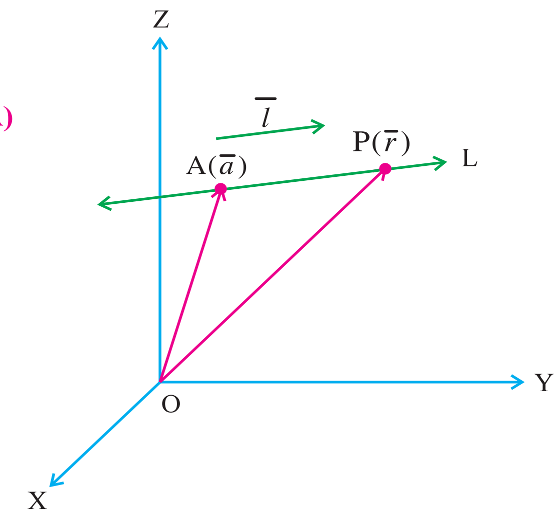


Figure 7.3

∴ Direction of $\overrightarrow{AP}$ is $\bar{l}$ or $-\bar{l}$.

∴ $\overrightarrow{AP} = k\bar{l}$, $k \in R - \{0\}$. **($k \neq 0$ as P ≠ A)**

∴ $\bar{r} - \bar{a} = k\bar{l}$

∴ $\bar{r} = \bar{a} + k\bar{l}$

Also, if $k = 0$, then $\bar{r} = \bar{a}$

i.e. P = A and A is also on L.

∴ For every point P($\bar{r}$) on L,

$\bar{r} = \bar{a} + k\bar{l}$, $k \in R$.

Conversely, if P($\bar{r}$) is any point in space such that $\bar{r} = \bar{a} + k\bar{l}$ for some $k \in R$,

then (i) if $k = 0$ then $\bar{r} = \bar{a}$ or P = A.

and (ii) if $k \neq 0$, then $\bar{r} \neq \bar{a}$ and $\bar{r} - \bar{a} = k\bar{l}$, where $k \neq 0$

∴ $\overrightarrow{AP} = k\bar{l}$

∴ $\overrightarrow{AP}$ has the same direction as $\bar{l}$ or direction opposite to that of $\bar{l}$. But A ∈ L (given). So P ∈ L.

Thus, P($\bar{r}$) ∈ L ⇔ $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$

∴ **The vector equation of line L is $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$**

Vector equation of a line gives the position vector of any point on the line.

The equation does not depend upon the choice of $\bar{a}$. If $\bar{b} \in L$, let $\bar{b} = \bar{a} + k_1\bar{l}$

Then $\bar{b} + k\bar{l} = \bar{a} + k_1\bar{l} + k\bar{l}$

$= \bar{a} + (k_1 + k)\bar{l}$

$= \bar{a} + t\bar{l}$, $t \in R$

∴ $\{\bar{b} + k\bar{l} \mid k \in R\} = \{\bar{a} + k\bar{l} \mid k \in R\}$

Parametric Equations of a Line :

Suppose a line L has direction $\bar{l} = (l_1, l_2, l_3)$ and passes through $\bar{a} = (x_1, y_1, z_1)$. Let P($\bar{r}$) ∈ L.

Suppose $\bar{r} = (x, y, z)$. Also $\bar{a} = (x_1, y_1, z_1)$ and $\bar{l} = (l_1, l_2, l_3)$.

$\therefore \quad \bar{r} = \bar{a} + k\bar{l}, \quad k \in R$

$\therefore \quad (x, y, z) = (x_1, y_1, z_1) + k(l_1, l_2, l_3), \quad k \in R$

$\therefore \quad (x - x_1, y - y_1, z - z_1) = (kl_1, kl_2, kl_3)$

$\therefore \quad x - x_1 = kl_1, \quad y - y_1 = kl_2, z - z_1 = kl_3$ **(i)**

$$\therefore \quad \left.\begin{aligned} x &= x_1 + kl_1 \\ y &= y_1 + kl_2 \\ z &= z_1 + kl_3 \end{aligned}\right\} \quad k \in R$$

These equations are called the parametric equations of line L passing through (x_1, y_1, z_1) and having direction (l_1, l_2, l_3) and k is the parameter.

Cartesian Equation (Symmetric Form) :

If we eliminate the parameter k from above equations, we get

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3} (= k) \text{ provided } l_1 \neq 0, l_2 \neq 0, l_3 \neq 0 \qquad \textbf{(using (i)) (ii)}$$

This is called the symmetric form of the Cartesian equations of line L.

If $l_1 = 0$ and $l_2 \neq 0$, $l_3 \neq 0$, then (i) gives

$$x = x_1, \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$$

[Here actually $x - x_1 = kl_1$ and as $l_1 = 0$, so $x - x_1 = 0$, i.e. $x = x_1$.]

This can also be written as $\frac{x - x_1}{0} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3} (= k)$

[Here, $\frac{x - x_1}{0}$ does not mean that denominator is zero. This is only a symbolic form.]

It simply means $x = x_1 + 0k$, $y = y_1 + kl_2$, $z = z_1 + kl_3$

$\therefore \quad x = x_1, y = y_1 + kl_2, z = z_1 + kl_3.$

Similarly, we can write the equation if any of l_1, l_2, l_3 is zero (of course not for $l_1 = l_2 = l_3 = 0$).

If $l_1 = l_2 = 0$ in equation (i) then $x = x_1$, $y = y_1$ and z is arbitrary.

This can be written symbolically as $\frac{x - x_1}{0} = \frac{y - y_1}{0} = \frac{z - z_1}{l_3} = k$ $(l_3 \neq 0$ as $\bar{l} \neq \bar{0})$

Agian 0 in denominator does not mean division by zero. It simply means $x - x_1 = 0$ or $x = x_1$ and $y = y_1$.

Note : If l_1, l_2, l_3 are direction *cosines* of a line L passing through $A(x_1, y_1, z_1)$, then the equation of L is $\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$, where $l_1^2 + l_2^2 + l_3^2 = 1$.

Example 1 : Find the equation of the line passing through A(2, 1, −4) and having direction (1, −1, 2), in the vector form and also in the symmetric form.

Solution : Here, $\bar{a} = (2, 1, -4)$ and $\bar{l} = (1, -1, 2)$.

$\therefore$ The vector equation of the line L, $\bar{r} = \bar{a} + k\bar{l}, \quad k \in R$ gives,

$\bar{r} = (2, 1, -4) + k(1, -1, 2), \quad k \in R$

This is the vector equation of the line.

Symmetric Form : Symmetric form of the equation of line is $\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

$\therefore \quad \frac{x - 2}{1} = \frac{y - 1}{-1} = \frac{z + 4}{2}$ is the equation of the line in symmetric form.

7.4 Equation of a line passing through two distinct points

Suppose a line L passes through $A(\bar{a})$ and $B(\bar{b})$, $A \neq B$.

Let $P(\bar{r})$ be a point on $\overleftrightarrow{AB}$ and $P \neq A$.

$P(\bar{r}) \in \overleftrightarrow{AB} \Leftrightarrow$ directions of $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are same or opposite.

$\Leftrightarrow \overrightarrow{AP} = k\overrightarrow{AB}, \; k \in R - \{0\}$

($k \neq 0$ as $P \neq A$)

$\Leftrightarrow \bar{r} - \bar{a} = k(\bar{b} - \bar{a})$

$\Leftrightarrow \bar{r} = \bar{a} + k(\bar{b} - \bar{a})$

$\Leftrightarrow \bar{r} = (1 - k)\bar{a} + k\bar{b}, \; k \in R - \{0\}$

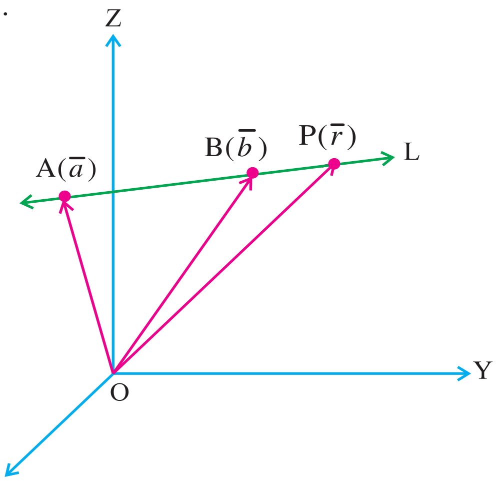


Figure 7.4

Also, $k = 0 \Leftrightarrow \bar{r} = \bar{a}$ and $A(\bar{a}) \in \overleftrightarrow{AB}$

$\therefore$ **The vector equation of $\overleftrightarrow{AB}$ is $\bar{r} = (1 - k)\bar{a} + k\bar{b}, \; k \in R$**

or $\bar{r} = \bar{a} + k(\bar{b} - \bar{a}), \; k \in R$

Taking $k = 1 - t$, $\bar{r} = (1 - (1 - t))\bar{a} + (1 - t)\bar{b}, \; t \in R$

$= t\bar{a} + (1 - t)\bar{b} = \bar{b} + t(\bar{a} - \bar{b})$.

[**Compare :** In R^2, $x = tx_2 + (1 - t)x_1$, $y = ty_2 + (1 - t)y_1$]

Thus roles of $\bar{a}$ and $\bar{b}$ can be interchanged or you can choose any pair of distinct points of L, to get its equation.

Parametric Form :

Suppose $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{r} = (x, y, z)$.

$\therefore \quad \bar{r} = \bar{a} + k(\bar{b} - \bar{a}), \; k \in R$ gives,

$(x, y, z) = (x_1, y_1, z_1) + k(x_2 - x_1, y_2 - y_1, z_2 - z_1), \quad k \in R$

$\therefore \quad x - x_1 = k(x_2 - x_1), \quad y - y_1 = k(y_2 - y_1), \; z - z_1 = k(z_2 - z_1)$ **(i)**

$$\therefore \left.\begin{aligned} x &= x_1 + k(x_2 - x_1) \\ y &= y_1 + k(y_2 - y_1) \\ z &= z_1 + k(z_2 - z_1) \end{aligned}\right\} \quad k \in R$$

are the parametric equations of $\overleftrightarrow{AB}$, k is a parameter.

Symmetric Form :

Eliminating parameter k from above equations, we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

[**Compare :** In R^2 $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$]

This is the symmetric form of the Cartesian equation of $\overleftrightarrow{AB}$.

Here, also if $x_1 = x_2$, then we get

$$\frac{x - x_1}{0} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

which can be understood as $x = x_1$, $\frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$.

[Here denominator of $x - x_1$ is not zero, it only means $x = x_1$. The form is only symbolic.]

Example 2 : Write vector form of the line $\frac{3 - x}{3} = \frac{2y - 3}{5} = \frac{z}{2}$.

Solution : Line is $\frac{x - 3}{-3} = \frac{y - \frac{3}{2}}{\frac{5}{2}} = \frac{z - 0}{2}$.

Here, $\bar{a} = \left(3, \frac{3}{2}, 0\right)$ and $\bar{l} = \langle -3, \frac{5}{2}, 2 \rangle = \langle -6, 5, 4 \rangle$

$\therefore$ The vector form of the equation of the line is $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$

$\therefore$ $\bar{r} = \left(3, \frac{3}{2}, 0\right) + k(-6, 5, 4)$, $k \in R$

Example 3 : Convert the equation of the line $\bar{r} = (5, -2, 4) + k(0, -4, 3)$, $k \in R$ in the Cartesian form.

Solution : Here, $\bar{a} = (5, -2, 4) = (x_1, y_1, z_1)$ and $\bar{l} = (0, -4, 3) = (l_1, l_2, l_3)$

Cartesian form of the equation of line is $\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

$\therefore$ $x - 5 = 0$, $\frac{y + 2}{-4} = \frac{z - 4}{3}$ **($l_1 = 0$)**

Example 4 : Find the equation of the line passing through the points (2, 2, −3) and (1, 3, 5).

Solution : The equation of the line pasing through $\bar{a}$ and $\bar{b}$ is $\bar{r} = \bar{a} + k(\bar{b} - \bar{a})$, $k \in R$

Here $\bar{a} = (2, 2, -3)$ and $\bar{b} = (1, 3, 5)$, $\bar{b} - \bar{a} = (-1, 1, 8)$.

$\therefore$ $\bar{r} = (2, 2, -3) + k(-1, 1, 8)$, $k \in R$

Cartesian form of the equation of the line L is $\frac{x - 2}{-1} = \frac{y - 2}{1} = \frac{z + 3}{8}$.

7.5 Collinear Points

Let $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ be distinct points in R^3.

A, B, C are collinear $\Leftrightarrow C \in \overleftrightarrow{AB}$

$\Leftrightarrow \bar{c} = \bar{a} + k(\bar{b} - \bar{a})$, for some $k \in R$,

($\overleftrightarrow{AB}$ has equation $\bar{r} = \bar{a} + k(\bar{b} - \bar{a})$, $k \in R$)

$\Leftrightarrow \bar{c} - \bar{a} = k(\bar{b} - \bar{a})$

$\therefore$ A, B, C are collinear $\Leftrightarrow (\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$

Thus, $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$ is necessary and sufficient condition for $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ to be collinear.

There is a theorem also stating the necessary and sufficient condition for collinearity. This theorem is stated below and we accept it without proof.

Theorem 7.1 : If A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are three distinct points in space, then a necessary and sufficient condition for A, B, C to be collinear is that there exist three non-zero real numbers *l, m, n* such that $l + m + n = 0$ and $l\bar{a} + m\bar{b} + n\bar{c} = \bar{0}$.

We obtain a necessary condition for collinearity of three points.

A, B, C are collinear $\Rightarrow (\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$

$$\Rightarrow (\bar{c} \times \bar{b}) - (\bar{a} \times \bar{b}) - (\bar{c} \times \bar{a}) + (\bar{a} \times \bar{a}) = \bar{0}$$

Also $\bar{a} \times \bar{a} = \bar{0}$ and $\bar{c} \times \bar{b} = -\bar{b} \times \bar{c}$

$$\Rightarrow (\bar{a} \times \bar{b}) + (\bar{b} \times \bar{c}) + (\bar{c} \times \bar{a}) = \bar{0}$$

$$\Rightarrow (\bar{a} \times \bar{b}) \cdot \bar{c} + (\bar{b} \times \bar{c}) \cdot \bar{c} + (\bar{c} \times \bar{a}) \cdot \bar{c} = 0$$

$$\Rightarrow [\bar{a}\ \bar{b}\ \bar{c}] = 0$$

$[\bar{a}\ \bar{b}\ \bar{c}] = 0$ is a necessary condition for A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) to be collinear. However as a following examples show that it is not a sufficient condition.

We also note that $[\bar{a}\ \bar{b}\ \bar{c}] \neq 0 \Rightarrow$ A, B, C are non-collinear as contrapositive of above statement, but $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ does not guarantee any conclusion. Following examples will clear this.

For example : Consider A(1, 2, 0), B(–4, 1, 9) and C(2, 4, 0).

Let $\bar{a} = (1, 2, 0)$, $\bar{b} = (-4, 1, 9)$ and $\bar{c} = (2, 4, 0)$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 1 & 2 & 0 \\ -4 & 1 & 9 \\ 2 & 4 & 0 \end{vmatrix} = 1(-36) - 2(-18) + 0 = 0$$

Now, $\bar{c} - \bar{a} = (1, 2, 0)$

$\bar{b} - \bar{a} = (-5, -1, 9)$

$(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = (18, -9, 9) \neq \bar{0}$

$\therefore$ A, B, C are non-collinear, though $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

We shall take one simple example, let $\bar{a} = (0, 0, 0)$, $\bar{b} = (1, 2, 3)$, $\bar{c} = (2, 3, 4)$.

Then $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

But $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{c} \times \bar{b} \neq \bar{0}$

$\therefore$ $\bar{a}, \bar{b}, \bar{c}$ are not collinear.

Example 5 : Prove that (–1, 2, 5), (–2, 4, 2) and (1, –2, 11) are collinear.

Solution : Method 1 : $\bar{a} = (-1, 2, 5)$, $\bar{b} = (-2, 4, 2)$, $\bar{c} = (1, -2, 11)$

$\therefore$ $\bar{c} - \bar{a} = (2, -4, 6)$ and

$\bar{b} - \bar{a} = (-1, 2, -3)$

$\therefore$ $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = (0, 0, 0) = \bar{0}$

$\therefore$ The given points are collinear.

Method 2 : First of all, we shall find the equation of the line passing through two points $A(\bar{a}) = (-1, 2, 5)$ and $B(\bar{b}) = (-2, 4, 2)$.

Equation of $\overleftrightarrow{AB}$ is $\bar{r} = \bar{a} + k(\bar{b} - \bar{a}),\ k \in R$

$\therefore \quad \bar{r} = (-1, 2, 5) + k(-1, 2, -3),\ k \in R$

Now we shall prove that the third point $C(\bar{c}) = (1, -2, 11)$ is on this line.

Let, if possible $\bar{r} = \bar{c} = (1, -2, 11)$ lie on $\overleftrightarrow{AB}$.

We must have $(1, -2, 11) = (-1 - k, 2 + 2k, 5 - 3k)$ for some $k \in R$.

$\therefore$ We must have $1 = -1 - k,\ -2 = 2 + 2k,\ 11 = 5 - 3k$ for some $k \in R$.

$\therefore$ $k = -2$ satisfies all the three equations. So $C(\bar{c})$ lie on $\overleftrightarrow{AB}$.

$\therefore$ A, B, C are collinear.

7.6 The Measure of the Angle Between Two Lines in Space

Suppose $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ are two lines in space.

(i) If $\bar{l} = \bar{m}$ or $\bar{l} = -\bar{m}$ then $\bar{l} \times \bar{m} = \bar{0}$. Then the measure of the angle between the lines is defined to be zero. Since direction of lines are same, they are coincident or parallel.

(ii) If $\bar{l} \perp \bar{m}$ i.e. $\bar{l} \cdot \bar{m} = 0$, then the lines are mutually perpendicular. Then the measure of angle between the lines is defined to be $\frac{\pi}{2}$.

(iii) If $\bar{l} \neq \pm\bar{m}$ and $\bar{l} \cdot \bar{m} \neq 0$ i.e. lines are neither perpendicular nor parallel or coincident. We define the measure of the acute angle between $\bar{l}$ and $\bar{m}$ as the measure of the angle between the lines.

If α is the measure of angle between the lines, then

$$\cos\alpha = \frac{|\bar{l} \cdot \bar{m}|}{|\bar{l}||\bar{m}|},\ 0 < \alpha < \frac{\pi}{2}$$

which also holds good for $\alpha = 0$ and $\frac{\pi}{2}$

Note : For $\alpha = 0$, $|\bar{l} \cdot \bar{m}| = |\bar{l}||\bar{m}|$.

$\therefore \quad \bar{l} \times \bar{m} = \bar{0}$

Thus, $\cos\alpha = \frac{|\bar{l} \cdot \bar{m}|}{|\bar{l}||\bar{m}|},\ 0 \le \alpha \le \frac{\pi}{2}$

7.7 Condition for intersection of two distinct lines

Theorem 7.2 : If two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ intersect in a point, then $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Proof : Suppose two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ intersect at $C(\bar{c})$.

$\therefore \quad \bar{c} = \bar{a} + k_1\bar{l} = \bar{b} + k_2\bar{m}$, for some $k_1, k_2 \in R$

$\therefore \quad \overline{a} - \overline{b} = k_2\overline{m} - k_1\overline{l}$

$\therefore \quad (\overline{a} - \overline{b})\cdot(\overline{l} \times \overline{m}) = (k_2\overline{m} - k_1\overline{l})\cdot(\overline{l} \times \overline{m}) = k_2\overline{m}\cdot(\overline{l} \times \overline{m}) - k_1\overline{l}\cdot(\overline{l} \times \overline{m})$

$= 0 - 0 = 0$

$\therefore \quad (\overline{a} - \overline{b})\cdot(\overline{l} \times \overline{m}) = 0$

$\therefore$ If the lines intersect in a point, then $(\overline{a} - \overline{b})\cdot(\overline{l} \times \overline{m}) = 0$

Note : This condition is necessary but it is not sufficient. Why ?

If $\overline{a} = (x_1, y_1, z_1)$, $\overline{b} = (x_2, y_2, z_2)$, $\overline{l} = (l_1, l_2, l_3)$, $\overline{m} = (m_1, m_2, m_3)$, then the condition $(\overline{a} - \overline{b})\cdot(\overline{l} \times \overline{m}) = 0$ is transformed into

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

This is the Cartesian form of the condition, when two lines intersect.

Example 6 : Find the measure of the angle between the lines $\frac{x-2}{2} = \frac{y-1}{2} = \frac{z+3}{1}$ and $\frac{x+2}{4} = \frac{y-4}{1} = \frac{z-3}{8}$.

Solution : Line L has equations $\frac{x-2}{2} = \frac{y-1}{2} = \frac{z+3}{1}$ and M has equations $\frac{x+2}{4} = \frac{y-4}{1} = \frac{z-3}{8}$.

$\therefore \quad \overline{l} = (2, 2, 1)$ and $\overline{m} = (4, 1, 8)$

If α is measure of the angle between the given lines, then $\left(0 \le \alpha \le \frac{\pi}{2}\right)$

$$cos\alpha = \frac{|\overline{l}\cdot\overline{m}|}{|\overline{l}||\overline{m}|} = \frac{|8+2+8|}{\sqrt{9}\cdot\sqrt{81}} = \frac{18}{3\cdot 9} = \frac{2}{3}$$

$\therefore \quad \alpha = cos^{-1}\frac{2}{3}$

Example 7 : If the lines $\frac{x-5}{7} = \frac{y-5}{k} = \frac{z-2}{1}$ and $\frac{x}{1} = \frac{y-3}{2} = \frac{z+1}{3}$ are perpendicular to each other, find k.

Solution : Here, $\overline{l} = (7, k, 1)$ and $\overline{m} = (1, 2, 3)$

As the lines are perpendicular, $\overline{l}\cdot\overline{m} = 0$

$\therefore \quad 7 + 2k + 3 = 0$

$\therefore \quad 2k = -10$

$\therefore \quad k = -5$

Example 8 : Find the Cartesian equation of the line which passes through the point (2, -4, 5) and is parallel to the line $\overline{r} = (-3, 4, 8) + k(3, 5, 6)$, $k \in R$.

Solution : Here lines are parallel, so the direction of both the lines should be same.

$\therefore$ Direction of required line is $\overline{l} = (3, 5, 6) = (l_1, l_2, l_3)$ and it passes through the point $(2, -4, 5) = (x_1, y_1, z_1)$.

(2, −4, 5) does not lie on $\bar{r} = (-3, 4, 8) + k(3, 5, 6), k \in R$

as (2, −4, 5) = (−3, 4, 8) + k(3, 5, 6) for some $k \in R$

$\Rightarrow (5, -8, -3) = k(3, 5, 6)$

But $5 = 3k, -8 = 5k, -3 = 6k$ is not true for any $k \in R$.

∴ The equation of the line parallel to the given line and passing through (x_1, y_1, z_1) is

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$$

∴ $\frac{x - 2}{3} = \frac{y + 4}{5} = \frac{z - 5}{6}$ is the equation of the line passing through (2, −4, 5) and parallel to given line.

Condition for coplanar and non-coplanar lines :

Theorem 7.3 : A necessary condition for lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$, to be coplanar is that $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Proof : If the two distinct lines L and M are coplanar, then either they intersect or they are parallel.

If they intersect, then by theorem 7.2, $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

If they are parallel, then $\bar{l} \times \bar{m} = \bar{0}$. So $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Thus, if the lines are coplanar, then $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Is this condition sufficient also ?

Non-coplanar or skew lines : If there is no plane that contains both the lines L and M, then L and M are called non-coplanar or skew lines.

From theorem 7.3, it is clear that $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) \neq 0 \Rightarrow$ lines $\bar{r} = \bar{a} + k\bar{l}$ and $\bar{r} = \bar{b} + k\bar{m}$ are skew lines.

Example 9 : Examine whether the lines L : $\frac{x - 3}{4} = \frac{y + 2}{-1} = \frac{z + 1}{-1}$ and M : $\frac{x}{2} = \frac{z + 3}{3}$, $y = -1$ are coplanar or not.

Solution : M can be taken as $\frac{x}{2} = \frac{y + 1}{0} = \frac{z + 3}{3}$

Here $\bar{a} = (3, -2, -1)$, $\bar{l} = (4, -1, -1)$ and $\bar{b} = (0, -1, -3)$, $\bar{m} = (2, 0, 3)$

∴ $\bar{a} - \bar{b} = (3, -1, 2)$

$$(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = \begin{vmatrix} 3 & -1 & 2 \\ 4 & -1 & -1 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= 3(-3) + 1(14) + 2(2)$$

$$= -9 + 14 + 4 = 9$$

Hence L and M are non-coplanar or skew.

7.8 Perpendicular distance of a point from a line

Suppose $\bar{r} = \bar{a} + k\bar{l}$ is the equation of a line L passing through $A(\bar{a})$ and having direction $\bar{l}$ and $P(\bar{p})$ is any point in R^3.

If $P \in L$, then perpendicular distance between P and L is zero.

If $P \notin L$, P and L determine unique plane π.

Let M be the foot of perpendicular in the plane π from P to line L and $(\bar{l}, \widehat{\overrightarrow{AP}}) = \alpha$, let $M \neq A$.

where $0 < \alpha < \frac{\pi}{2}$.

$\therefore$ PM = Perpendicular distance from P to L.

$= AP \sin\alpha$

$= \frac{|\overrightarrow{AP}||\bar{l}|\sin\alpha}{|\bar{l}|}$ $\quad (\bar{l} \neq \bar{0})$

$= \frac{|\overrightarrow{AP} \times \bar{l}|}{|\bar{l}|}$ $\quad (\alpha = (\overrightarrow{AP}, \widehat{\bar{l}}))$

$= \frac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$

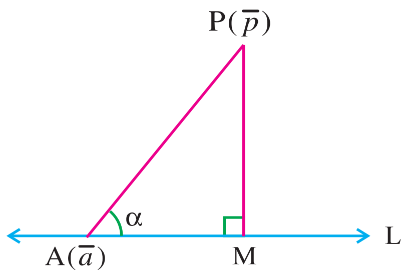


Figure 7.5

Thus, PM $= \frac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$ or $|(\bar{p} - \bar{a}) \times \hat{l}|$ $\quad \left(\hat{l} = \frac{\bar{l}}{|\bar{l}|}\right)$

Second proof :

$AM = |\text{Proj}_{\bar{l}}\overrightarrow{AP}| = \frac{|\overrightarrow{AP} \cdot \bar{l}|}{|\bar{l}|}$

Now, $PM^2 = AP^2 - AM^2$

$= AP^2 - \frac{|\overrightarrow{AP} \cdot \bar{l}|^2}{|\bar{l}|^2}$

$= \frac{|\overrightarrow{AP}|^2 |\bar{l}|^2 - |\overrightarrow{AP} \cdot \bar{l}|^2}{|\bar{l}|^2}$

$\therefore \quad PM^2 = \frac{|\overrightarrow{AP} \times \bar{l}|^2}{|\bar{l}|^2}$ **(Lagrange's identity)**

$\therefore \quad \mathbf{PM} = \frac{|\overrightarrow{AP} \times \bar{l}|}{|\bar{l}|} = \frac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|} = |(\bar{p} - \bar{a}) \times \hat{l}|$

Note : If P lies on perpendicular to A, both the proofs fail, but the result is true.

Example 10 : Find the perpendicular distance of the point (1, 2, −4) from the line $\frac{x-3}{2} = \frac{y-3}{3} = \frac{z+5}{6}$.

Solution : Here, point P(1, 2, −4) and A($\bar{a}$) = (3, 3, −5), $\bar{l}$ = (2, 3, 6)

$\overrightarrow{AP} = (1 - 3, 2 - 3, -4 + 5) = (-2, -1, 1)$ and

$\bar{l} = (2, 3, 6)$

$\overrightarrow{AP} \times \bar{l} = (-9, 14, -4)$

$|\bar{l}| = \sqrt{4 + 9 + 36} = 7$

Perpendicular distance of P from the given line $= \frac{|\overrightarrow{AP} \times \bar{l}|}{|\bar{l}|} = \frac{|(-9, 14, -4)|}{7}$

$= \frac{\sqrt{81 + 196 + 16}}{7} = \frac{\sqrt{293}}{7}$

Perpendicular distance between two parallel lines :

Let L : $\bar{r} = \bar{a} + k\bar{l}$, $k \in$ R and M : $\bar{r} = \bar{b} + k\bar{l}$, $k \in$ R be two parallel lines in R^3.

Since L || M, they determine unique plane.

The distance between L and M is the perpendicular distance between A($\bar{a}$) and M (or between B($\bar{b}$) and L).

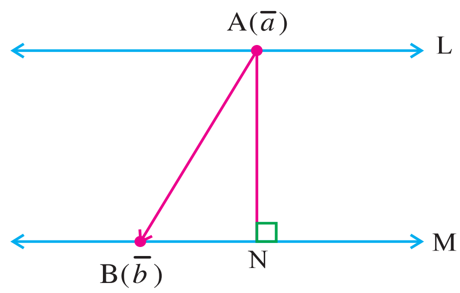


Figure 7.6

∴ **Distance between L and M is**

$$\frac{|\overrightarrow{AB} \times \bar{l}|}{|\bar{l}|} = \frac{|(\bar{b} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$$

Example 11 : Find the distance between the lines L : $\frac{x-4}{3} = \frac{y+1}{-2} = \frac{z-2}{6}$ and

M : $\bar{r} = (2, 3, -1) + k(-3, 2, -6)$, $k \in$ R

Solution : Here, $\bar{a} = (4, -1, 2)$; $\bar{l} = (3, -2, 6)$, $\bar{b} = (2, 3, -1)$; $\bar{m} = (-3, 2, -6)$

If possible, let A($\bar{a}$) $\in$ M.

Then $(4, -1, 2) = (2, 3, -1) + k(-3, 2, -6)$ for some $k \in$ R

∴ $(2, -4, 3) = k(-3, 2, -6)$ for some $k \in$ R

∴ $2 = -3k$, $-4 = 2k$, $3 = -6k$

This is not possible for any $k \in$ R as first equation gives $k = -\frac{2}{3}$ and this k does not satisfy other two equations.

∴ A($\bar{a}$) $\notin$ M

Also $\bar{l} = -\bar{m}$

∴ $\bar{l} \times \bar{m} = -\bar{m} \times \bar{m} = \bar{0}$

Now $\bar{l} \times \bar{m} = \bar{0}$ and A($\bar{a}$) $\notin$ M

∴ Given lines are parallel.

$\bar{a} - \bar{b} = (2, -4, 3)$ and

$\bar{l} = (3, -2, 6)$

∴ $(\bar{a} - \bar{b}) \times \bar{l} = (-18, -3, 8)$, $|\bar{l}| = \sqrt{9+4+36} = 7$

Perpendicular distance between given lines $= \frac{|(\bar{a} - \bar{b}) \times \bar{l}|}{|\bar{l}|}$

$$= \frac{\sqrt{324+9+64}}{7} = \frac{\sqrt{397}}{7}$$

Perpendicular distance between two skew lines :

Let L : $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$ and M : $\overline{r} = \overline{b} + k\overline{m}$, $k \in R$ be skew lines of R^3.

As L and M are skew lines, $(\overline{a} - \overline{b}) \cdot (\overline{l} \times \overline{m}) \neq 0$ **(Theorem 7.3)**

We shall assume that for skew lines L and M, there exist points P ∈ L and Q ∈ M such that $\overleftrightarrow{PQ} \perp L$ and $\overleftrightarrow{PQ} \perp M$.

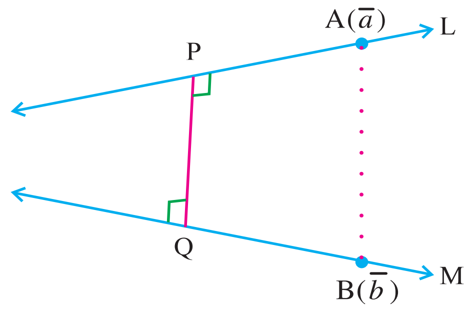


Figure 7.7

$\therefore$ $\overrightarrow{PQ} \cdot \overline{l} = 0$, $\overrightarrow{PQ} \cdot \overline{m} = 0$

$\therefore$ Direction *cosines* of $\overrightarrow{PQ}$ and $\overline{l} \times \overline{m}$ are same.

Now, $\overrightarrow{PQ}$ = projection of $\overrightarrow{AB}$ on $\overrightarrow{PQ}$.

$$\therefore \quad \overrightarrow{PQ} = \left[\frac{|(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m})|}{|\overline{l} \times \overline{m}|}\right]\left[\frac{\overline{l} \times \overline{m}}{|\overline{l} \times \overline{m}|}\right]$$

$$\therefore \quad PQ = \frac{|(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m})|}{|\overline{l} \times \overline{m}|}$$

$$\text{Also, } PQ = \frac{|\overline{b} - \overline{a}||\overline{l} \times \overline{m}| \, cos\,\alpha}{|\overline{l} \times \overline{m}|}$$

$= |\overline{b} - \overline{a}| \, |cos\alpha|$ where $\alpha = ((\overline{b} - \overline{a}), \widehat{\ } (\overline{l} \times \overline{m}))$

$\therefore$ $PQ \leq |\overline{b} - \overline{a}|$ **($|cos\alpha| \leq 1$)**

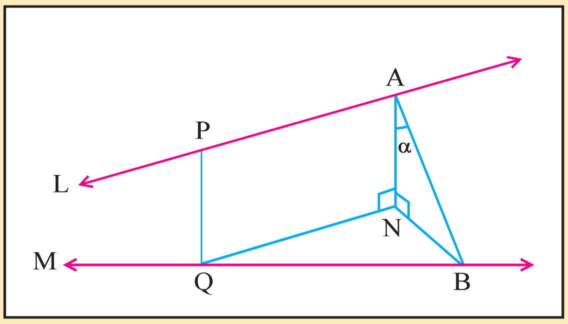
$\therefore$ Distance PQ is less than or equal to the distance between any pair of points on L and M.

$\therefore$ PQ is the shortest distance between L and M.

Thus, $PQ = \dfrac{|(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m})|}{|\overline{l} \times \overline{m}|}$ is the perpendicular distance or the shortest distance between L and M.

$\overleftrightarrow{PQ}$ and L are intersecting lines, so there is a plane π containing them. □ PANQ is a rectangle in the plane π.

$\overleftrightarrow{AN}$ and $\overleftrightarrow{PQ}$ are parallel lines.



If the measure of the angle between $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ is α, then the measure of the angle between $\overleftrightarrow{AB}$ and $\overleftrightarrow{AN}$ is α. Now, in the plane containing $\overleftrightarrow{AN}$ and $\overleftrightarrow{AB}$,

AN = AB*cos*α, because in ΔANB,

$m\angle ANB = \frac{\pi}{2}$

($\because$ $\overline{AN} \perp \overline{QN}$ and $\overline{AN} \perp \overline{QB}$, so $\overline{AN}$ is perpendicular to the plane containing $\overline{QN}$ and $\overline{QB}$.)

$\therefore \quad PQ = AN = |\,AB\cos\alpha\,|$

$$= \frac{|\,\overrightarrow{AB}\cdot\bar{l}\times\bar{m}\,|}{|\,\bar{l}\times\bar{m}\,|} = \frac{|\,(\bar{b}-\bar{a})\cdot(\bar{l}\times\bar{m})\,|}{|\,\bar{l}\times\bar{m}\,|}$$

Example 12 : Find the shortest distance between the lines $\bar{r} = (1, 1, 0) + k(2, -1, 1)$, $k \in R$ and $\bar{r} = (2, 1, -1) + k(3, -5, 2)$, $k \in R$.

Solution : Here, $\bar{a} = (1, 1, 0)$; $\bar{l} = (2, -1, 1)$ and $\bar{b} = (2, 1, -1)$; $\bar{m} = (3, -5, 2)$

$\bar{b} - \bar{a} = (1, 0, -1)$

$$(\bar{b}-\bar{a})\cdot(\bar{l}\times\bar{m}) = \begin{vmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= 1(3) - 1(-7) = 10 \neq 0$$

$\therefore$ Given lines are skew lines.

$\bar{l} = (2, -1, 1)$,

$\bar{m} = (3, -5, 2)$

$\therefore \quad \bar{l} \times \bar{m} = (3, -1, -7)$

$\therefore \quad |\,\bar{l}\times\bar{m}\,| = \sqrt{9+1+49} = \sqrt{59}$, $(\bar{b}-\bar{a})\cdot(\bar{l}\times\bar{m}) = 3 + 0 + 7 = 10$

$\therefore$ The shortest distance between given lines $= \dfrac{|\,(\bar{b}-\bar{a})\cdot(\bar{l}\times\bar{m})\,|}{|\,\bar{l}\times\bar{m}\,|} = \dfrac{10}{\sqrt{59}}$

7.9 To determine the nature of pair of lines of R^3

Let $L : \bar{r} = \bar{a} + k\bar{l}$, $k \in R$

$M : \bar{r} = \bar{b} + k\bar{m}$, $k \in R$ be two lines

If $\bar{l} \times \bar{m} = \bar{0}$, then L and M are parallel or coincident.

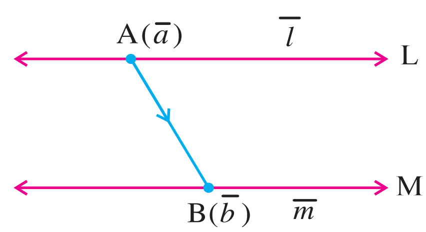


Figure 7.8

Suppose L || M

Here, $\overrightarrow{AB}$ and $\bar{l}$ are non-collinear vectors.

$\therefore \quad \overrightarrow{AB} \times \bar{l} = (\bar{b}-\bar{a}) \times \bar{l} \neq \bar{0}$

Conversely if $\overrightarrow{AB} \times \bar{l} = (\bar{b}-\bar{a}) \times \bar{l} \neq \bar{0}$, then $\overrightarrow{AB}$ and $\bar{l}$ are non-collinear.

$\therefore$ L || M, if L and M have same directions and $(\bar{b}-\bar{a}) \times \bar{l} \neq \bar{0}$.

But, if $(\bar{b}-\bar{a}) \times \bar{l} = \bar{0}$, then L is not parallel to M, so L and M are coincident.

Hence if $\bar{l} \times \bar{m} = \bar{0}$, $(\bar{b}-\bar{a}) \times \bar{l} = \bar{0}$, lines are coincident.

If $\bar{l} \times \bar{m} = \bar{0}$, $(\bar{b}-\bar{a}) \times \bar{l} \neq \bar{0}$, lines are parallel.

If two lines of R^3 are given, then we want to determine whether they are parallel or intersecting or coincident or skew. We can decide by the following flow-chart, based on the entire previous discussion.

$$L : \bar{r} = \bar{a} + k\bar{l}, \ k \in R$$
$$M : \bar{r} = \bar{b} + k\bar{m}, \ k \in R$$
Find $\bar{l} \times \bar{m}$

- $\bar{l} \times \bar{m} = \bar{0}$
  - $(\bar{b} - \bar{a}) \times \bar{l} \neq \bar{0}$: L and M are parallel
  - $(\bar{b} - \bar{a}) \times \bar{l} = \bar{0}$: L and M are coincident
- $\bar{l} \times \bar{m} \neq \bar{0}$
  - $(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) \neq 0$: L and M are skew
  - $(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$: L and M are intersecting

Example 13 : Identify the nature (i.e. skew, parallel, coincident and intersecting) of the following lines :

(1) $\bar{r} = (2, -5, 1) + k(3, 2, 6), k \in R$ and $\frac{x-7}{1} = \frac{y}{2} = \frac{z+6}{2}$

(2) $\frac{2x-4}{1} = \frac{3-y}{3} = \frac{z}{1}$ and $\bar{r} = (1, 1, -1) + k(1, -6, 2), k \in R$

(3) $\bar{r} = (1, -2, -3) + k(-1, 1, -2), k \in R$ and $\bar{r} = (4, -2, -1) + k(1, 2, -2), k \in R$

(4) $\bar{r} = (3 + t)\hat{i} + (1 - t)\hat{j} + (-2 - 2t)\hat{k}, t \in R$ and $x = 4 + k, y = -k, z = -4 - 2k, k \in R$

Solution : (1) Here, $\bar{a} = (2, -5, 1), \bar{l} = (3, 2, 6)$

$\bar{b} = (7, 0, -6); \bar{m} = (1, 2, 2)$

$\bar{b} - \bar{a} = (5, 5, -7)$

$\bar{l} \times \bar{m} = (-8, 0, 4) \neq \bar{0}$ and $(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = (5, 5, -7) \cdot (-8, 0, 4)$

$= -40 - 28 = -68 \neq 0$

$\therefore$ The given lines are skew lines.

(2) The equation of the first line is $\frac{x-2}{\frac{1}{2}} = \frac{y-3}{-3} = \frac{z}{1}$

$\therefore \bar{a} = (2, 3, 0); \bar{l} = <\frac{1}{2}, -3, 1> = <1, -6, 2>$

$\bar{b} = (1, 1, -1); \bar{m} = (1, -6, 2)$

$(\bar{b} - \bar{a}) = (-1, -2, -1)$

Now $\bar{l} \times \bar{m} = (0, 0, 0) = \bar{0}$ and $(\bar{b} - \bar{a}) \times \bar{m} = (-1, -2, -1) \times (1, -6, 2) = (-10, 1, 8) \neq \bar{0}$

$\therefore$ Lines are parallel.

(3) $\bar{a} = (1, -2, -3); \bar{l} = (-1, 1, -2)$

$\bar{b} = (4, -2, -1); \bar{m} = (1, 2, -2)$

$(\bar{b} - \bar{a}) = (3, 0, 2)$

Now $\bar{l} \times \bar{m} = (2, -4, -3) \neq \bar{0}$ and $(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = (3, 0, 2) \cdot (2, -4, -3)$

$= 6 + 0 - 6 = 0$

$\therefore$ The lines are intersecting.

(4) $\bar{a} = (3, 1, -2); \bar{l} = (1, -1, -2)$

$\bar{b} = (4, 0, -4); \bar{m} = (1, -1, -2)$

$(\bar{b} - \bar{a}) = (1, -1, -2)$

Now $\bar{l} \times \bar{m} = (0, 0, 0) = \bar{0}$ and $(\bar{b} - \bar{a}) \times \bar{l} = (1, -1, -2) \times (1, -1, -2) = \bar{0}$

$\therefore$ The lines are coincident.

Exercise 7.1

1. Find the vector and Cartesian equation of the line passing through (2, −1, 3) and having direction $2\hat{i} - 3\hat{j} + 4\hat{k}$.

2. Find the equation of the line passing through the points (2, 3, −9) and (4, 3, −5) in symmetric and in vector form.

3. Are the points (0, 1, 1), (0, 4, 4) and (2, 0, 1) collinear ? Why ?

4. Find the direction *cosines* of the line $x = 4z + 3$, $y = 2 - 3z$.

5. Find the vector and Cartesian equation of the line passing through (1, −2, 1) and perpendicular to the lines $x + 3 = 2y = -12z$ and $\frac{x}{2} = \frac{y+6}{2} = \frac{3z-9}{1}$.

6. Prove that the lines L : $\frac{x+2}{3} = \frac{y-2}{-1}$, $z + 1 = 0$ and M : $\{(4 + 2k, 0, -1 + 3k) \mid k \in R\}$ intersect each other. Also find the point of their intersection.

7. Find the measure of the angle between the lines $\overline{r} = (1, 2, 1) + k(2, 3, -1)$, $k \in R$ and $\frac{x-1}{4} = \frac{y-2}{3}$, $z = 3$.

8. Show that the line through the points (2, 1, −1) and (−2, 3, 4) is perpendicular to the line through the points (9, 7, 8) and (11, 6, 10).

9. Identify whether the following lines are parallel, intersecting, skew or coincident :

(1) $\overline{r} = (1, 2, -3) + k(3, -2, 1)$, $k \in R$ and $\frac{x-1}{2} = \frac{3-y}{2} = \frac{z-5}{-1}$.

(2) $\frac{x-5}{-2} = \frac{y-3}{-2} = \frac{z+2}{4}$ and $\frac{x-2}{1} = \frac{3-y}{-1} = \frac{z+2}{-2}$.

(3) $x = \frac{y-1}{1} = \frac{z+1}{3}$ and $\{(2, 1 + 3k, 2 + k) \mid k \in R\}$.

(4) $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2}$ and $x = 1 + 2t$, $y = t$, $z = 4 + 5t$, $t \in R$.

(5) $\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z-1}{3}$ and $\frac{x-1}{-2} = \frac{y+2}{4} = \frac{z-2}{-6}$.

10. Show that $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$ and $\frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$ are skew lines. Find the shortest distance between them.

11. Find the perpendicular distance of (−5, 3, 4) from the line $\frac{x+2}{-4} = \frac{y-6}{5} = \frac{z-5}{3}$.

12. Find the perpendicular distance between the lines $x = 3 - 2k$, $y = k$, $z = 3 - k$, $k \in R$ and $x = 2k - 3$, $y = 2 - k$, $z = 7 + k$, $k \in R$.

*

7.10 Plane

Let us recall the postulates of plane we studied in earlier class.

(1) Three distinct non-collinear points determine unique plane.

(2) There is a unique plane containing two parallel lines.

(3) There is a unique plane containing two intersecting lines.

Plane passing through three distinct non-collinear points :

Suppose $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$ are three distinct non-collinear points of R^3.

∴ A, B, C determine unique plane π.

Let $P(\overline{r})$ be any point of the plane π and let P ≠ A.

$\therefore$ $\overrightarrow{AP}$, $\overrightarrow{AB}$, $\overrightarrow{AC}$ are coplanar.

$\therefore$ $\overrightarrow{AP}$ is a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\therefore$ $\overrightarrow{AP} = m\overrightarrow{AB} + n\overrightarrow{AC}$, where $m, n \in R$ and $m^2 + n^2 \neq 0$.

$\therefore$ $\overline{r} - \overline{a} = m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a})$

If $\overline{r} = \overline{a}$ and $A(\overline{a}) \in \pi$, then $m = n = 0$.

$\therefore$ **$\overline{r} = \overline{a} + m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a}), m, n \in R$** **(i)**

Conversely, if $P(\overline{r})$ satisfies

$\overline{r} - \overline{a} = m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a}), m, n \in R, m^2 + n^2 \neq 0$

then $\overrightarrow{AP} = m(\overrightarrow{AB}) + n(\overrightarrow{AC})$

i.e. $\overrightarrow{AP}$ is in the plane of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\therefore$ A is in π. So $P \in \pi$.

If $m = n = 0$, then $\overline{r} = \overline{a}$ i.e. $P = A \in \pi$.

Thus, $P(\overline{r}) \in \pi$ if and only if $\overline{r}$ satisfies (i).

$\therefore$ The plane π determined by $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$ has equation

$\overline{r} = \overline{a} + m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a})$ $m, n \in R$

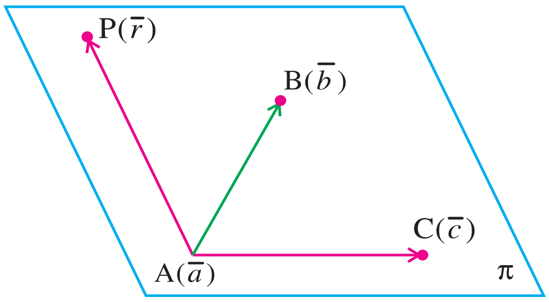


Figure 7.9

Now, $\overline{r} = (1 - m - n)\overline{a} + m\overline{b} + n\overline{c}$ if $P(\overline{r}) \in \pi$.

Let $1 - m - n = l$ i.e. $l + m + n = 1$

$\therefore$ **$\overline{r} = l\overline{a} + m\overline{b} + n\overline{c}$, where $l, m, n \in R$ and $l + m + n = 1$.**

This is the vector equation of the plane containing three distinct non-collinear points $A(\overline{a})$, $B(\overline{b})$ and $C(\overline{c})$.

Parametric equations of a plane :

Let $P(x, y, z)$, be any point of the plane passing through non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

$\therefore$ $\overline{r} = l\overline{a} + m\overline{b} + n\overline{c}$,

$\therefore$ $(x, y, z) = l(x_1, y_1, z_1) + m(x_2, y_2, z_2) + n(x_3, y_3, z_3)$

$\therefore$ **$x = lx_1 + mx_2 + nx_3$**

$y = ly_1 + my_2 + ny_3$

$z = lz_1 + mz_2 + nz_3$ where $l, m, n \in R$ and $l + m + n = 1$

are the parametric equations of the plane through A, B, C and l, m, n are the parameters.

Other forms :

If $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$ are three non-collinear distinct points, they determine a unique plane π.

$P(\overline{r}) \in \pi \Leftrightarrow \overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ are coplanar **(P ≠ A)**

$\Leftrightarrow (\overline{r} - \overline{a}), (\overline{b} - \overline{a}), (\overline{c} - \overline{a})$ are coplanar **(P ≠ A)**

$\Leftrightarrow (\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0$ **(ii)**

Also, if $\overline{r} = \overline{a}$, then $\overline{r} - \overline{a} = \overline{0}$.

$\therefore$ $(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0$, $\forall P(\overline{r}) \in \pi$

Thus, the vector equation of the plane through distinct non-collinear points A, B, C is $(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0$

Cartesian form (Scalar form) :

Let $\overline{r} = (x, y, z)$, $\overline{a} = (x_1, y_1, z_1)$, $\overline{b} = (x_2, y_2, z_2)$, $\overline{c} = (x_3, y_3, z_3)$

$\therefore$ **The equation $(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0$ becomes,**

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

This is the Cartesian equation or scalar form of the equation of the plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Condition for four distinct points of R^3 to be coplanar :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ be points of R^3.

A, B, C, D are coplanar $\Leftrightarrow$ D lies on the plane determined by A, B, C

$$\Leftrightarrow D(x_4, y_4, z_4) \text{ satisfies } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Thus $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

Example 14 : Find the equation of the plane passing through A(–6, 0, 7), B(1, 2, 2) and C(3, –5, –4), if possible.

Solution : Let us examine if A, B, C are collinear or not.

$$\begin{vmatrix} -6 & 0 & 7 \\ 1 & 2 & 2 \\ 3 & -5 & -4 \end{vmatrix} = -6(2) + 7(-11) = -89 \neq 0$$

$\therefore$ A, B, C are non-collinear.

$\therefore$ There is a unique plane passing through A, B, C.

Cartesian equation of the plane passing through A, B, C is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x + 6 & y - 0 & z - 7 \\ 1 + 6 & 2 - 0 & 2 - 7 \\ 3 + 6 & -5 - 0 & -4 - 7 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x + 6 & y & z - 7 \\ 7 & 2 & -5 \\ 9 & -5 & -11 \end{vmatrix} = 0$$

$\therefore \quad (x + 6)(-47) - y(-32) + (z - 7)(-53) = 0$

$\therefore \quad -47x - 282 + 32y - 53z + 371 = 0$

$\therefore \quad -47x + 32y - 53z + 89 = 0$

$\therefore \quad 47x - 32y + 53z - 89 = 0$ is the equation of the plane passing through A, B and C.

Example 15 : Does a unique plane pass through A(4, −2, −1), B(5, 0, −3) and C(3, −4, 1)? If so, find its equation.

Solution : Let us examine if A, B, C are collinear or not.

$$\begin{vmatrix} 4 & -2 & -1 \\ 5 & 0 & -3 \\ 3 & -4 & 1 \end{vmatrix} = 4(-12) + 2(14) - 1(-20)$$

$$= -48 + 28 + 20 = 0$$

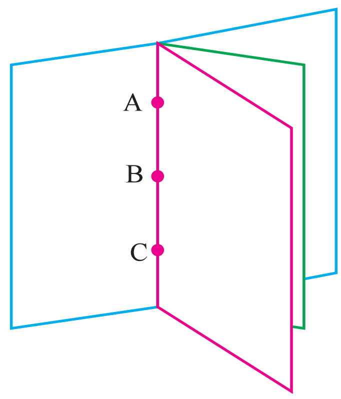


Figure 7.10

This is not enough to confirm that given points are collinear. So let us verify using the condition whether

$(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$ is true or not.

$\therefore \quad \bar{c} - \bar{a} = (-1, -2, 2)$

$\bar{b} - \bar{a} = (1, 2, -2) = -(\bar{c} - \bar{a})$

$\therefore \quad (\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$

$\therefore$ A, B, C are collinear.

$\therefore$ A, B, C do not determine a unique plane.

Example 16 : Show that the points A(1, 0, 2), B(−1, 2, 0), C(2, 3, 11) and D(1, −3, −4) are coplanar.

Solution : $$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = \begin{vmatrix} -2 & 2 & -2 \\ 1 & 3 & 9 \\ 0 & -3 & -6 \end{vmatrix} = -2(9) - 2(-6) - 2(-3)$$

$$= -18 + 12 + 6 = 0$$

$\therefore$ A, B, C, D are coplanar points.

7.11 Intercepts of a Plane

If a plane π intersects three coordinate axes at points A(a, 0, 0), B(0, b, 0) and C(0, 0, c), then a, b, c are called the X-intercept, the Y-intercept and the Z-intercept of the plane π respectively.

If the plane π does not intersect X-axis, then X-intercept of the plane π is said to be undefined and similarly for intersection of the plane with Y-axis or Z-axis also.

Equation of a plane making intercepts *a, b, c* on the coordinate axes :

Suppose intercepts made by a the plane π on X-axis, Y-axis and Z-axis are respectively a, b and c. (where $a \neq 0$, $b \neq 0$, $c \neq 0$).

$\therefore$ A(a, 0, 0), B(0, b, 0) and C(0, 0, c) are points on the plane π.

Obviously, A, B, C are non-collinear. **(Why ?)**

$\therefore$ Parametric equations of the plane π through A, B, C are

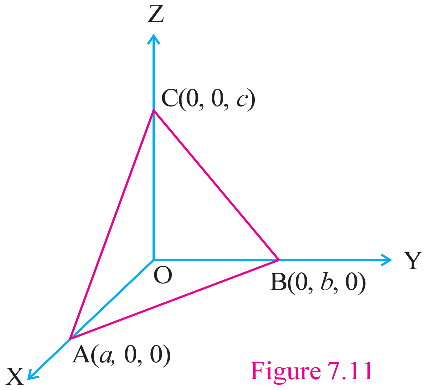


Figure 7.11

$$\therefore \left.\begin{aligned} x &= lx_1 + mx_2 + nx_3 = la \\ y &= ly_1 + my_2 + ny_3 = mb \\ z &= lz_1 + mz_2 + nz_3 = nc \end{aligned}\right\} \text{ where } l, m, n \in \mathbb{R}, l + m + n = 1$$

$$\therefore \quad l = \frac{x}{a}, m = \frac{y}{b}, n = \frac{z}{c}$$

Since $l + m + n = 1$, **$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is the equation of the plane having intercepts a, b and c.** **($abc \neq 0$)**

Another Method :

Using cartesian form of the equation of the plane passing through A(a, 0, 0), B(0, b, 0), C(0, 0, c), we get $\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$ as the equation of the plane through A, B and C.

$$\therefore \quad \begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$\therefore \quad (x - a)bc - y(-ac) + z(ab) = 0$

$\therefore \quad bcx - abc + acy + abz = 0$

$\therefore \quad bcx + acy + abz = abc$

$\therefore \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is the equation of the plane having intercepts a, b, c. **($abc \neq 0$)**

Example 17 : Find the equation of the plane making X-intercept 4, Y-intercept -6 and Z-intercept 3.

Solution : Here $a = 4$, $b = -6$, $c = 3$ is given.

$\therefore$ The equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\therefore \quad \frac{x}{4} + \frac{y}{-6} + \frac{z}{3} = 1$

$\therefore \quad 3x - 2y + 4z = 12$ is the equation of the plane having X-intercept 4, Y-intercept -6 and Z-intercept 3.

Example 18 : Find the intercepts made by the plane $2x - 3y + 5z = 15$ on the coordinate axes.

Solution : The equation of the given plane is $2x - 3y + 5z = 15$

$\therefore \quad \frac{x}{\frac{15}{2}} + \frac{y}{-5} + \frac{z}{3} = 1$ **(dividing both the sides by 15)**

$\therefore$ Comparing with the equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, X-intercept $= \frac{15}{2}$, Y-intercept $= -5$, Z-intercept $= 3$.

Example 19 : Find the intercepts made by the plane $3y + 2z = 12$ on the coordinate axes.

Solution : The equation of plane is $3y + 2z = 12$

$\therefore \quad \frac{y}{4} + \frac{z}{6} = 1$

$\therefore$ Comparing with $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, X-intercept is undefined, Y-intercept $= 4$ and Z-intercept $= 6$.

Another Method :

The equation of the plane is $3y + 2z = 12$.

It intersects X-axis where $y = 0 = z$.

$\therefore$ But then $0 + 0 = 12$

This is not true.

$\therefore$ $3y + 2z = 12$ does not intersect X-axis.

$\therefore$ It has no X-intercept.

To find Y-intercept, let $x = 0 = z$.

$\therefore$ $3y = 12$

$\therefore$ $y = 4$

$\therefore$ Y-intercept is 4.

To find Z-intercept, let $x = y = 0$.

$\therefore$ $2z = 12$

$\therefore$ $z = 6$

$\therefore$ Z-intercept is 6.

7.12 Normal to a plane

There exists a line which is perpendicular to every line in the plane. It is called a normal to the plane. Usually, normal is denoted by $\bar{n}$ or $\bar{n}_1$, $\bar{n}_2$, $\bar{n}_3$,...

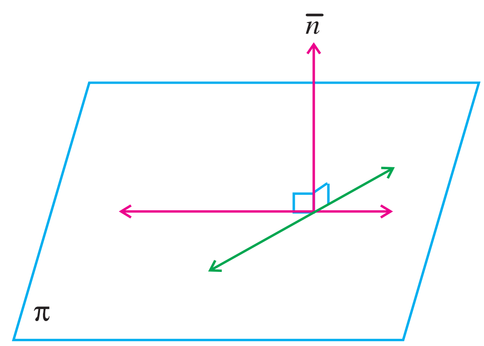


Figure 7.12

Vector equation of the plane passing through A($\bar{a}$) and having normal $\bar{n}$:

Let the plane passing through A($\bar{a}$) and having normal $\bar{n}$ be π.

Let P($\bar{r}$) be any point in the plane π.

$\therefore$ P($\bar{r}$) $\in$ π, P $\neq$ A $\Rightarrow \overrightarrow{AP} \in \pi$

$\Rightarrow \overrightarrow{AP} \perp \bar{n}$

$\Rightarrow \overrightarrow{AP} \cdot \bar{n} = 0$

$\Rightarrow (\bar{r} - \bar{a}) \cdot \bar{n} = 0$

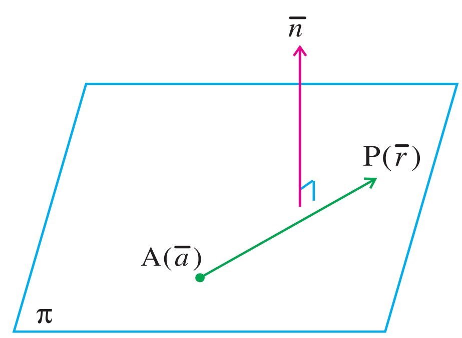


Figure 7.13

If P = A, then $\bar{r} = \bar{a}$ so $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$ holds good.

$\therefore$ $\forall$ P($\bar{r}$) $\in$ π, $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$

Conversely, if P($\bar{r}$) is any point in space such that $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$, then $\overrightarrow{AP} \perp \bar{n}$.

As A $\in$ π, P $\in$ π.

Thus, P($\bar{r}$) $\in$ π $\Leftrightarrow (\bar{r} - \bar{a}) \cdot \bar{n} = 0$

$\Leftrightarrow \bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$

$\therefore$ $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ is the vector equation of the plane passing through A($\bar{a}$) and having normal $\bar{n}$.

Let $\bar{a} \cdot \bar{n} = d$

$\therefore$ $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ becomes $\bar{r} \cdot \bar{n} = d$

Cartesian form :

Let $\overline{r} = (x, y, z)$, $\overline{n} = (a, b, c)$ and $\overline{a} = (x_1, y_1, z_1)$

$\therefore$ $\overline{r} \cdot \overline{n} = d$ becomes $(x, y, z) \cdot (a, b, c) = d$ where $d = \overline{a} \cdot \overline{n} = ax_1 + by_1 + cz_1$.

$\therefore$ **$ax + by + cz = d$, $a^2 + b^2 + c^2 \neq 0$ as $\overline{n} \neq \overline{0}$ is the equation of the plane having normal $\overline{n} = (a, b, c)$**

Note : The ordered triplet formed by the coefficient of x, y, z in the equation of a plane represents the normal $\overline{n}$ of the plane.

Example 20 : Find the equation of the plane passing through $(4, 5, -1)$ having normal $3\hat{i} - \hat{j} + \hat{k}$.

Solution : Here $\overline{a} = (4, 5, -1)$, $\overline{n} = (3, -1, 1)$

$\therefore$ The equation of the plane $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$ gives $(x, y, z) \cdot (3, -1, 1) = (4, 5, -1) \cdot (3, -1, 1)$

$\therefore$ $3x - y + z = 12 - 5 - 1 = 6$

$\therefore$ $3x - y + z = 6$ is the equation of the plane passing through $(4, 5, -1)$ and having normal $3\hat{i} - \hat{j} + \hat{k}$.

Example 21 : Find the normal and the vector equation of the plane $2x - z + 1 = 0$.

Solution : Cartesian equation of plane is $2x - z + 1 = 0$.

$\therefore$ Normal $\overline{n} = (2, 0, -1)$ **(see note)**

$\therefore$ Vector equation $\overline{r} \cdot \overline{n} = d$ is $2x - z + 1 = (2, 0, -1) \cdot (x, y, z) + 1 = 0$

$\therefore$ The vector equation is $\overline{r} \cdot (2, 0, -1) + 1 = 0$

7.13 Equation of the plane using normal through the origin

Let $N(\overline{n})$ be the foot of perpendicular from origin to the plane π.

Let $ON = p$

$\therefore$ $|\overline{n}| = p$.

Let α, β, γ be the direction angles of $\overline{n}$.

$\therefore$ Direction *cosines* of $\overline{n}$ are $cos\alpha$, $cos\beta$, $cos\gamma$.

$\therefore$ Unit vector in the direction of $\overline{n}$ (i.e. $\hat{n}$) is

$$\hat{n} = \frac{\overline{n}}{|\overline{n}|} = \frac{\overline{n}}{p} = (cos\alpha, cos\beta, cos\gamma)$$

$\therefore$ $\overline{n} = (pcos\alpha, pcos\beta, pcos\gamma)$

Let $P(\overline{r})$ be any point of the plane π.

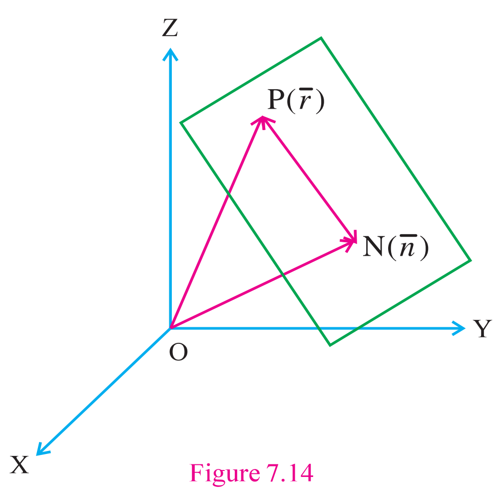


Figure 7.14

Here $\overline{a} = \overline{n} = (pcos\alpha, pcos\beta, pcos\gamma)$

The equation of the plane $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$ becomes

$(x, y, z) \cdot (pcos\alpha, pcos\beta, pcos\gamma) = p^2$ **(as $\overline{a} \cdot \overline{n} = \overline{n} \cdot \overline{n} = |\overline{n}|^2 = p^2$)**

$\therefore$ $xcos\alpha + ycos\beta + zcos\gamma = p$ is the equation of a plane having α, β, γ as the direction angles of the normal and p, the length of the normal.

Note : If the equation of the plane is $ax + by + cz = d$, then to convert such an equation into the form of $xcos\alpha + ycos\beta + zcos\gamma = p$, we divide the given equation by $|\overline{n}|$. That is

$$\frac{a}{|\overline{n}|}x + \frac{b}{|\overline{n}|}y + \frac{c}{|\overline{n}|}z = \frac{d}{|\overline{n}|}$$

If $d > 0$, then let $\overline{n} = (a, b, c)$ so that $\frac{d}{|\overline{n}|} = p$ is positive.

$\therefore \quad \frac{\overline{n}}{|\overline{n}|} = \left(\frac{a}{|\overline{n}|}, \frac{b}{|\overline{n}|}, \frac{c}{|\overline{n}|}\right) = \hat{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ and $\frac{d}{|\overline{n}|} = p$

If $d < 0$, then let $\overline{n} = (-a, -b, -c)$ so that $\frac{-d}{|\overline{n}|} = p$ is positive.

$\therefore \quad -ax - by - cz = -d$

$\therefore \quad \frac{\overline{n}}{|\overline{n}|} = \left(\frac{-a}{|\overline{n}|}, \frac{-b}{|\overline{n}|}, \frac{-c}{|\overline{n}|}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$ and $\frac{-d}{|\overline{n}|} = p$.

Example 22 : Find the direction *cosines* and the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 14 = 0$.

Solution : The plane π has the equation $2x - 3y + 6z = -14$ (given) **(i)**

We shall represent the equation in the form $\frac{a}{|\overline{n}|}x + \frac{b}{|\overline{n}|}y + \frac{c}{|\overline{n}|}z = \frac{d}{|\overline{n}|}$.

Here $d = -14 < 0$.

The equation can be written as $-2x + 3y - 6z = 14$, so that $d > 0$.

Let $\overline{n} = (-2, 3, -6)$, $|\overline{n}| = \sqrt{4+9+36} = 7$.

$\therefore \quad p = \frac{-d}{|\overline{n}|} = \frac{14}{7} = 2$, $(\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{-2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$

Thus, the length of perpendicular from origin is 2 and direction *cosines* of the normal are $\frac{-2}{7}, \frac{3}{7}, \frac{-6}{7}$.

Intersection of a Line and a plane :

Let the equation $\overline{r} = \overline{a} + k\overline{l}$, $k \in$ R represent a line and the equation $\overline{r} \cdot \overline{n} = d$ represents a plane. $(\overline{n} \neq \overline{0})$

Consider the intersection of the line $\overline{r} = \overline{a} + k\overline{l}$ and the plane $\overline{r} \cdot \overline{n} = d$. $(\overline{l} \neq \overline{0}, \overline{n} \neq \overline{0})$

Suppose $\overline{l} = (l_1, l_2, l_3)$, $\overline{n} = (a, b, c)$, $\overline{a} = (x_1, y_1, z_1)$.

If the point $\overline{r}_1 = \overline{a} + k_1\overline{l}$ for some $k_1 \in$ R of the line is also on the plane, then

$(\overline{a} + k_1\overline{l}) \cdot \overline{n} = d$.

$\therefore \quad k_1(\overline{l} \cdot \overline{n}) = d - \overline{a} \cdot \overline{n}$ **(i)**

Now,

(1) If $\overline{l} \cdot \overline{n} = 0$ and $d - \overline{a} \cdot \overline{n} \neq 0$, then (i) is impossible.

$\therefore$ If $\overline{l} \cdot \overline{n} = 0$ and $ax_1 + by_1 + cz_1 \neq d$, then the line and the plane do not intersect. We say that the line is parallel to the plane.

(2) If $\overline{l} \cdot \overline{n} = 0$ and also $d - \overline{a} \cdot \overline{n} = 0$, then (i) is satisfied for every $k_1 \in$ R.

In this case, every point of the line is in the plane.

Thus, if $ax_1 + by_1 + cz_1 = d$ and $\overline{l} \cdot \overline{n} = 0$ then the line lies in the plane.

(3) If $\overline{l} \cdot \overline{n} \neq 0$, then we get a unique value of k_1 by $k_1 = \frac{d - \overline{a} \cdot \overline{n}}{\overline{l} \cdot \overline{n}}$. So in this case, exactly one point of the line is on the plane. i.e. the line intersects the plane in exactly one point.

7.14 Measure of the Angle between two planes

The measure of the angle of between two planes is defined to be the measure of the angle between their normals.

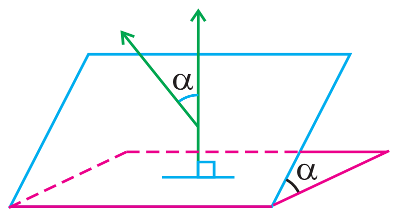


Figure 7.15

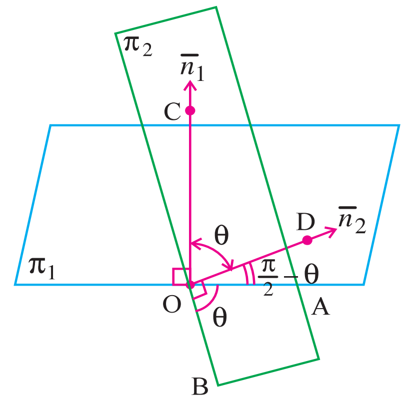


Figure 7.16

Since we take angle between two lines (normals) to be the acute angle between the lines, the angle between the planes is an acute angle.

Figure 7.16 shows that the measure of the angle between two normals $\overline{n}_1$ and $\overline{n}_2$ is θ, i.e. $(\overline{n}_1, \widehat{\ } \overline{n}_2) = \theta = m\angle COD$,

but $m\angle COA = \frac{\pi}{2}$, so $m\angle DOA = \frac{\pi}{2} - \theta$.

Again, $\overline{n}_2$ is a normal of π_2, so $m\angle BOD = \frac{\pi}{2}$

$\therefore$ $m\angle AOB = \theta$, the angle between two planes.

Let $\pi_1 : \overline{r} \cdot \overline{n}_1 = d_1$ and

$\pi_2 : \overline{r} \cdot \overline{n}_2 = d_2$ be the equations of given planes.

(1) $\pi_1 \perp \pi_2 \Leftrightarrow \overline{n}_1 \perp \overline{n}_2 \Leftrightarrow \overline{n}_1 \cdot \overline{n}_2 = 0$

$\therefore$ **The measure of the angle between the planes π_1 and π_2 is $\frac{\pi}{2} \Leftrightarrow \overline{n}_1 \cdot \overline{n}_2 = 0$.**

(2) For distinct planes π_1 and π_2 we define π_1 is parallel to π_2 if they do not intersect. In this case

$\overline{n}_1 = \overline{n}_2 = \overline{n}$.

$\therefore$ **$\pi_1 \parallel \pi_2 \Leftrightarrow \overline{n}_1 \times \overline{n}_2 = \overline{0}$**

$\therefore$ **The measure of the angle between π_1 and π_2 is zero $\Leftrightarrow \overline{n}_1 \times \overline{n}_2 = \overline{0}$.**

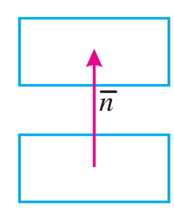


Figure 7.17

(3) Let θ be the measure of the angle between the planes π_1 and π_2, so that $0 < \theta < \frac{\pi}{2}$.

$$\therefore \quad \cos\theta = \frac{|\overline{n}_1 \cdot \overline{n}_2|}{|\overline{n}_1||\overline{n}_2|}$$

$$\therefore \quad \theta = \cos^{-1} \frac{|\overline{n}_1 \cdot \overline{n}_2|}{|\overline{n}_1||\overline{n}_2|}$$

which also holds true for $\theta = 0$ and $\frac{\pi}{2}$. **(Verify !)**

If $\pi_1 : a_1x + b_1y + c_1z = d_1$ and $\pi_2 : a_2x + b_2y + c_2z = d_2$ are given planes, then $\overline{n}_1 = (a_1, b_1, c_1)$ and $\overline{n}_2 = (a_2, b_2, c_2)$.

$$\theta = \cos^{-1} \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Example 23 : Find the measure of the angle between the planes $2x - y + z + 6 = 0$ and $x + y + 2z - 3 = 0$.

Solution : $\pi_1 : 2x - y + z + 6 = 0$. So $\overline{n}_1 = (2, -1, 1)$

$\pi_2 : x + y + 2z - 3 = 0$. So $\overline{n}_2 = (1, 1, 2)$

Now, $\bar{n}_1 \cdot \bar{n}_2 = 2(1) + (-1)1 + 1(2) = 3$

$$|\bar{n}_1| = \sqrt{4+1+1} = \sqrt{6}, \quad |\bar{n}_2| = \sqrt{1+1+4} = \sqrt{6}$$

$$\therefore \quad \theta = cos^{-1} \frac{|\bar{n}_1 \cdot \bar{n}_2|}{|\bar{n}_1||\bar{n}_2|} = cos^{-1} \frac{|3|}{\sqrt{6}\sqrt{6}} = cos^{-1} \frac{1}{2} = \frac{\pi}{3}$$

$\therefore$ The measure of the angle between given planes is $\frac{\pi}{3}$.

7.15 Equation of the plane passing through two parallel lines

Let $\bar{r} = \bar{a} + k\bar{l}$, $k \in$ R and

$\bar{r} = \bar{b} + k\bar{l}$, $k \in$ R be two parallel lines.

$\therefore$ They determine unique plane π.

Also, $\bar{b} \notin \{\bar{r} \mid \bar{r} = \bar{a} + k\bar{l}, k \in \text{R}\}$

$\therefore$ $\bar{b} \neq \bar{a} + k\bar{l}$, for any $k \in$ R

$\therefore$ $\bar{b} - \bar{a} \neq k\bar{l}$, for any $k \in$ R

$\therefore$ $(\bar{b} - \bar{a}) \times \bar{l} \neq \bar{0}$

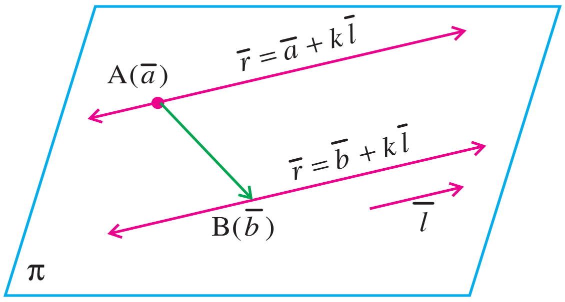


Figure 7.18

So let $\bar{n} = (\bar{b} - \bar{a}) \times \bar{l}$. Then $\bar{n} \neq \bar{0}$

We assert that the equation of the required plane π is $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$.

i.e. $(\bar{r} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] = 0$

Now, we shall show that this plane π contains both of the given lines.

For $\bar{r} = \bar{a} + k\bar{l}$

$$(\bar{r} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] = (k\bar{l}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] = 0$$

$\therefore$ Every point of line $\bar{r} = \bar{a} + k\bar{l}$ is in the plane $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$.

$\therefore$ The plane π contains the line $\bar{r} = \bar{a} + k\bar{l}$, $k \in$ R.

Similarly, for $\bar{r} = \bar{b} + k\bar{l}$

$$\begin{aligned}(\bar{r} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] &= (\bar{b} + k\bar{l} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] \\ &= (\bar{b} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] + k\bar{l} \cdot [(\bar{b} - \bar{a}) \times \bar{l}] \\ &= 0\end{aligned}$$

$\therefore$ The line $\bar{r} = \bar{b} + k\bar{l}$ is a subset of the plane $(\bar{r} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] = 0$.

Hence, **$(\bar{r} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] = 0$ is the equation of plane containing given parallel lines.**

Cartesian form :

Let $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$ and $\bar{l} = (l_1, l_2, l_3)$.

The Cartesian form of the equation of the plane containing two parallel lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & l_2 & l_3 \end{vmatrix} = 0$$

Example 24 : Show that lines L : $\frac{x-3}{3} = \frac{y-3}{-4} = \frac{z-5}{2}$ and M : $\frac{x}{6} = \frac{y-5}{-8} = \frac{z-2}{4}$ are parallel and find the equation of the plane containing them.

Solution : Here, $\overline{l} = (3, -4, 2)$, $\overline{m} = (6, -8, 4)$. So, $\overline{l} \times \overline{m} = \overline{0}$.

$\therefore$ L = M or L $\parallel$ M

Also, for (3, 3, 5) and $\frac{3}{6} = \frac{3-5}{-8} = \frac{5-2}{4}$ is not true. So $(3, 3, 5) \notin$ M.

$\therefore$ $(3, 3, 5) \in$ L, $(3, 3, 5) \notin$ M

$\therefore$ L $\neq$ M

Hence L $\parallel$ M

Now, $\overline{a} = (3, 3, 5)$, $\overline{b} = (0, 5, 2)$ and $\overline{l} = (3, -4, 2)$.

$\therefore$ The equation of the plane containing L and M is $\begin{vmatrix} x-3 & y-3 & z-5 \\ 0-3 & 5-3 & 2-5 \\ 3 & -4 & 2 \end{vmatrix} = 0$

$\therefore$ $\begin{vmatrix} x-3 & y-3 & z-5 \\ -3 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix} = 0$

$\therefore$ $(x-3)(-8) - (y-3)(3) + (z-5)(6) = 0$

$\therefore$ $-8x + 24 - 3y + 9 + 6z - 30 = 0$

$\therefore$ $8x + 3y - 6z = 3$ is the equation of the plane passing through given parallel lines.

7.16 Equation of the plane containing two intersecting lines

Let $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$ and

$\overline{r} = \overline{b} + k\overline{m}$, $k \in R$ be two intersecting lines.

$\therefore$ They determine unique plane π.

Also, $\overline{l} \times \overline{m} \neq \overline{0}$ and $(\overline{a} - \overline{b}) \cdot (\overline{l} \times \overline{m}) = 0$.

(Why ?)

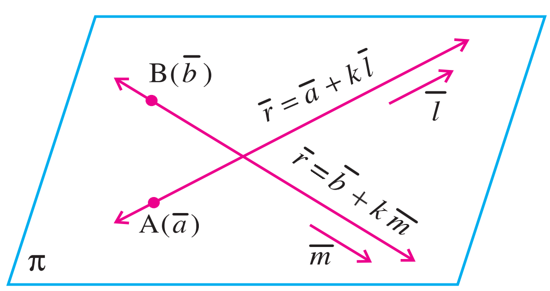


Figure 7.19

Taking $\overline{n} = \overline{l} \times \overline{m}$, we have $\overline{n} \neq \overline{0}$.

$(\overline{r} - \overline{a}) \cdot \overline{n} = 0$ represents a plane π.

i.e. $(\overline{r} - \overline{a}) \cdot (\overline{l} \times \overline{m}) = 0$ is the equation of a plane π. **(as $\overline{n} \neq \overline{0}$)**

Now, we shall show that plane π contains given lines.

For $\overline{r} = \overline{a} + k\overline{l}$,

$(\overline{r} - \overline{a}) \cdot (\overline{l} \times \overline{m}) = (k\overline{l}) \cdot (\overline{l} \times \overline{m}) = 0$

$\therefore$ Every point of $\overline{r} = \overline{a} + k\overline{l}$ is in the plane $(\overline{r} - \overline{a}) \cdot (\overline{l} \times \overline{m}) = 0$.

Similarly, for $\bar{r} = \bar{b} + k\bar{m}$,

$(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = (\bar{b} + k\bar{m} - \bar{a}) \cdot (\bar{l} \times \bar{m})$

$= (\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) + (k\bar{m}) \cdot (\bar{l} \times \bar{m})$ $\quad$ **$((\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0)$**

$= 0$

$\therefore$ Every point of $\bar{r} = \bar{b} + k\bar{m}$ is in the plane $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$.

Hence, $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$ is the equation of a plane containing given intersecting lines.

Cartesian form :

Let $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$, $\bar{l} = (l_1, l_2, l_3)$ and $\bar{m} = (m_1, m_2, m_3)$.

The Cartesian form of the equation of the plane containing two intersecting lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

Note : (1) In the formula $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$, we can also use $\bar{b}$ in place of $\bar{a}$ i.e. $(\bar{r} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$ is also the equation of plane containing two intersecting lines.

(2) To get the equation of the plane we need three non-collinear points. So A($\bar{a}$) and B($\bar{b}$) are two given points of the plane. The third point C can be any point of the given lines (which can be obtained by taking $k \in R - \{0\}$ in any of the given equations.)

Example 25 : Prove that L : $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and M : $\frac{x-4}{5} = \frac{y-1}{2} = z$ are coplanar and find the equation of the plane containing them.

Solution : Here, $\bar{a} = (1, 2, 3)$, $\bar{l} = (2, 3, 4)$ and

$\bar{b} = (4, 1, 0)$, $\bar{m} = (5, 2, 1)$.

$\bar{l} \times \bar{m} = (-5, 18, -11) \neq \bar{0}$ and $\bar{b} - \bar{a} = (3, -1, -3)$

$(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = (3, -1, -3) \cdot (-5, 18, -11) = -15 - 18 + 33 = 0$

$\therefore$ L and M are intersecting lines and so coplanar.

The equation of the plane containing L and M is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x - 1 & y - 2 & z - 3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

$\therefore$ $(x - 1)(-5) - (y - 2)(-18) + (z - 3)(-11) = 0$

$\therefore$ $-5x + 5 + 18y - 36 - 11z + 33 = 0$

$\therefore$ $5x - 18y + 11z - 2 = 0$ is the equation of the required plane.

Another Method : A(1, 2, 3), B(4, 1, 0) are given.

Taking $k = 1$ in the equation $\bar{r} = (1, 2, 3) + k(2, 3, 4)$, $k \in R$ of line L, we get C(3, 5, 7) as a point on line L.

Obviously, A, B, C are not collinear as $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 3 & 5 & 7 \end{vmatrix} = 7 - 56 + 51 \neq 0$.

So the equation of the required plane through A, B, C is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 4-1 & 1-2 & 0-3 \\ 3-1 & 5-2 & 7-3 \end{vmatrix} = 0$

$\therefore \begin{vmatrix} x-1 & y-2 & z-3 \\ 3 & -1 & -3 \\ 2 & 3 & 4 \end{vmatrix} = 0$

$\therefore \quad (x - 1)(5) - (y - 2)(18) + (z - 3)(11) = 0$

$\therefore \quad 5x - 5 - 18y + 36 + 11z - 33 = 0$

$\therefore \quad 5x - 18y + 11z - 2 = 0$

Note : Similar approach can also be taken for finding the equation of the plane containing two parallel lines.

7.17 Perpendicular distance from a point outside a plane to the plane

Let $\pi : \bar{r} \cdot \bar{n} = d$ be the equation of a given plane and $P(\bar{p})$ be a given point, $P \notin \pi$.

If $M(\bar{m})$ is the foot of the perpendicular from $P(\bar{p})$ to the plane π, then we need to find the distance PM.

$\therefore$ Direction of $\overrightarrow{MP}$ and $\bar{n}$ are same.

$\therefore$ The equation of $\overleftrightarrow{MP}$ is $\bar{r} = \bar{p} + k\bar{n}$, $k \in R$

As $M(\bar{m}) \in \overleftrightarrow{MP}$ so $\bar{m} = \bar{p} + k_1\bar{n}$,

for some $k_1 \in R - \{0\}$

Also, $M(\bar{m}) \in \pi$. So $\bar{m} \cdot \bar{n} = d$

$\therefore \quad (\bar{p} + k_1\bar{n}) \cdot \bar{n} = d$

$\therefore \quad k_1 \, |\bar{n}|^2 = d - \bar{p} \cdot \bar{n}$

$\therefore \quad k_1 = \dfrac{d - \bar{p} \cdot \bar{n}}{|\bar{n}|^2}$ $\qquad (\bar{n} \neq \bar{0})$ **(i)**

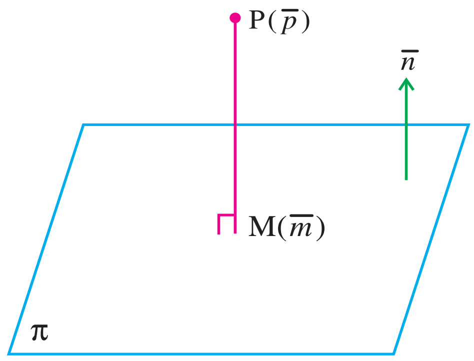


Figure 7.20

Now, $PM = |\overrightarrow{PM}| = |\bar{m} - \bar{p}|$ $\qquad (\because \bar{m} = \bar{p} + k_1\bar{n})$

$= |k_1\bar{n}| = |k_1| \, |\bar{n}|$

$\therefore \qquad \mathbf{PM} = \dfrac{|d - \bar{p} \cdot \bar{n}|}{|\bar{n}|^2} \times |\bar{n}| = \dfrac{|\bar{p} \cdot \bar{n} - d|}{|\bar{n}|}$

Cartesian form :

Let $P(x_1, y_1, z_1)$ be the given point and $ax + by + cz = d$ be the given plane.

$\therefore \quad \bar{p} = (x_1, y_1, z_1)$, $\bar{n} = (a, b, c)$

$\therefore$ Perpendicular distance from P to $\pi = \dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$

Also, if the equation of the plane is taken as $ax + by + cz + d = 0$, the perpendicular distance

$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad \textbf{(replacing } d \textbf{ by } -d \textbf{ in } \overline{r} \cdot \overline{n} = d\textbf{)}$$

Note : (1) The foot of perpendicular from the point $P(\overline{p})$ to the plane $\overline{r} \cdot \overline{n} = d$ is $M(\overline{m})$ where $\overline{m} = \overline{p} + k_1\overline{n}$, $k_1 = \frac{d - \overline{p} \cdot \overline{n}}{|\overline{n}|^2}$.

(2) Compare with $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$, the perpendicular distance of (x_1, y_1) from $ax + by + c = 0$.

Example 26 : Find the perpendicular distance from point $(-1, 2, -2)$ to the plane $3x - 4y + 2z + 44 = 0$.

Solution : $\overline{p} = (-1, 2, -2)$ and $\pi : 3x - 4y + 2z = -44$ are given. So $d = -44$.

$\therefore$ Perpendicular distance from P to plane $\pi = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$

$$= \frac{|3(-1) - 4(2) + 2(-2) + 44|}{\sqrt{3^2 + (-4)^2 + 2^2}} = \frac{29}{\sqrt{29}} = \sqrt{29}$$

Distance between two parallel planes :

Suppose $\pi_1 : \overline{r} \cdot \overline{n} = d_1$ and $\pi_2 : \overline{r} \cdot \overline{n} = d_2$ are two parallel planes.

The perpendicular distance of any point $A(\overline{a})$ in π_1 to the plane π_2 is the distance between two parallel planes.

$A(\overline{a}) \in \pi$. Hence $\overline{a} \cdot \overline{n} = d_1$

$\therefore$ Perpendicular distance of $A(\overline{a})$ from

$\overline{r} \cdot \overline{n} = d_2$ is $\frac{|\overline{a} \cdot \overline{n} - d_2|}{|\overline{n}|} = \frac{|d_1 - d_2|}{|\overline{n}|}$

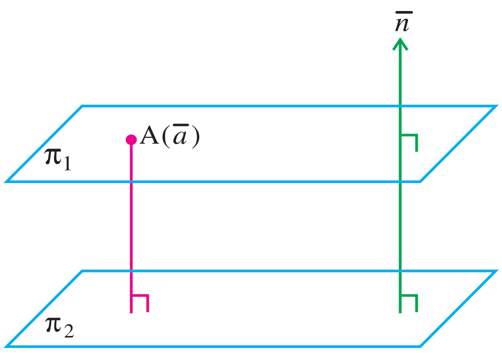


Figure 7.21

Example 27 : Find the distance between the planes $2x - 2y - z + 4 = 0$ and $4y + 2z - 4x + 1 = 0$.

Solution : $\left.\begin{array}{l}\pi_1 : 2x - 2y - z + 4 = 0 \\ \pi_2 : 4y + 2z - 4x + 1 = 0\end{array}\right\} \Rightarrow \begin{array}{l}\pi_1 : 4x - 4y - 2z = -8 \\ \pi_2 : 4x - 4y - 2z = 1\end{array}$

$\therefore$ $\overline{n} = (4, -4, -2)$, $d_1 = -8$, $d_2 = 1$

$\therefore$ Perpendicular distance between the given planes $= \frac{|d_1 - d_2|}{|\overline{n}|}$

$$= \frac{|-8 - 1|}{\sqrt{4^2 + (-4)^2 + (-2)^2}}$$

$$= \frac{9}{6} = \frac{3}{2}$$

By using above formula, we can obtain the formula for the shortest distance between two skew lines.

Let $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$ and $\overline{r} = \overline{b} + k\overline{m}$, $k \in R$ be two skew lines. So $(\overline{a} - \overline{b}) \cdot (\overline{l} \times \overline{m}) \neq 0$

First of all, let P $(\bar{a}+k_2\bar{l}\,)$ for some $k_2 \in$ R be any point on L and Q $(\bar{b}+k_1\bar{m}\,)$ for some $k_1 \in$ R be any point on M.

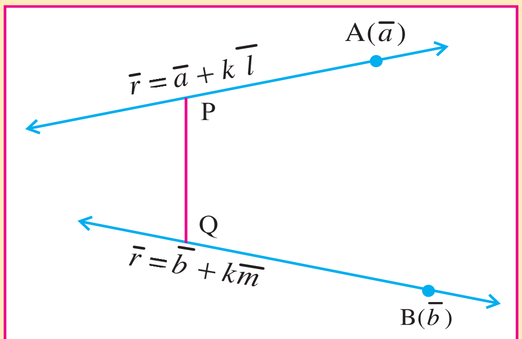


Figure 7.22

$\therefore$ $\overrightarrow{PQ} = \bar{b} - \bar{a} + k_1\bar{m} - k_2\bar{l}$

Now, if $\overleftrightarrow{PQ}$ is perpendicular to both L and M, then

$(\bar{b} - \bar{a} + k_1\bar{m} - k_2\bar{l}\,)\cdot\bar{l} = 0$

and $(\bar{b} - \bar{a} + k_1\bar{m} - k_2\bar{l}\,)\cdot\bar{m} = 0$

$\therefore$ $(\bar{l}\cdot\bar{m})\,k_1 - |\bar{l}|^2\,k_2 = (\bar{a} - \bar{b})\cdot\bar{l}$

$|\bar{m}|^2\,k_1 - (\bar{l}\cdot\bar{m})\,k_2 = (\bar{a} - \bar{b})\cdot\bar{m}$

As, lines are skew lines, so

$$(\bar{l}\cdot\bar{m})(\bar{l}\cdot\bar{m}) - |\,l\,|^2\,|\,m\,|^2 = |\,\bar{l}\cdot\bar{m}\,|^2 - |\bar{l}|^2|\bar{m}|^2$$

$$= -\,|\,\bar{l}\times\bar{m}\,|^2 \neq 0$$

$\therefore$ There exist unique $k_1 \in$ R and $k_2 \in$ R, such that $\overleftrightarrow{PQ} \perp$ L and $\overleftrightarrow{PQ} \perp$ M

But directions of L and M are $\bar{l}$ and $\bar{m}$ respectively.

$\therefore$ Direction of $\overleftrightarrow{PQ}$ is $\bar{l} \times \bar{m}$.

The plane $(\bar{r} - \bar{a})\cdot(\bar{l} \times \bar{m}) = 0$ passes through L. Since $(\bar{a} + k\bar{l} - \bar{a})\cdot(\bar{l} \times \bar{m}) = 0$

Similarly $(\bar{r} - \bar{b})\cdot(\bar{l} \times \bar{m}) = 0$ passes through M.

.Direction of $\overleftrightarrow{PQ}$ is $\bar{l}\times\bar{m}$ and it is perpendicular to both the planes.

$$\therefore \quad PQ = \frac{|d_1 - d_2|}{|\bar{l}\times\bar{m}|}$$

$$= \frac{|\bar{a}.(\bar{l}\times\bar{m}) - \bar{b}.(\bar{l}\times\bar{m})|}{|\bar{l}\times\bar{m}|}$$

$$= \frac{|(\bar{a}-\bar{b})(\bar{l}\times\bar{m})|}{|\bar{l}\times\bar{m}|}$$

7.18 Angle between a line and a plane

Suppose $\bar{r} = \bar{a} + k\,\bar{l}$ is the equation of a given line and $\bar{r}\cdot\bar{n} = d$ is the equation of a given plane. Suppose the line intersects the plane at P and is not perpendicular to the plane. M is the foot of the perpendicular from A($\bar{a}$) on the plane. Then $\angle$APM is called the angle between the given line and the given plane.

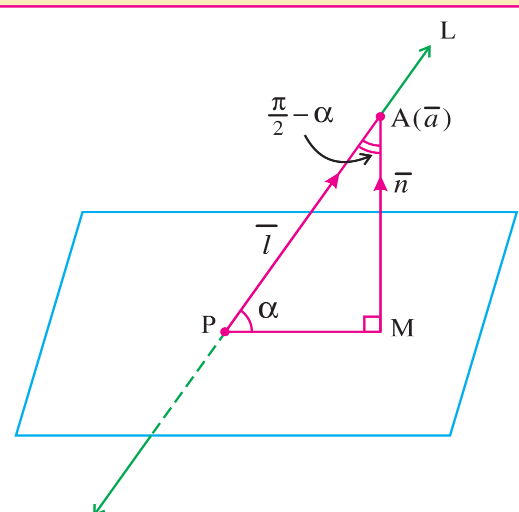


Figure 7.23

Let $m\angle$APM = α, $0 < \alpha < \frac{\pi}{2}$

$\therefore$ $\frac{\pi}{2} - \alpha = (\bar{l}\,\hat{,}\,\bar{n})$

$\therefore$ $cos\left(\frac{\pi}{2} - \alpha\right) = \frac{|\bar{l}\cdot\bar{n}|}{|\bar{l}\,||\bar{n}|}$

$\therefore \quad sin\alpha = \frac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$

$\therefore \quad \alpha = sin^{-1} \frac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$ **is the measure of the angle between the line and the plane.**

Example 28 : Find the measure of the angle between the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z+1}{1}$ and the plane $\bar{r} \cdot (-2, 2, -1) = 1$.

Solution : Here $\bar{l} = (2, 2, 1)$, $\bar{n} = (-2, 2, -1)$

$\bar{l} \cdot \bar{n} = 2(-2) + 2(2) + 1(-1) = -1$

$|\bar{l}| = \sqrt{2^2 + 2^2 + 1^2} = 3, \quad |\bar{n}| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = 3$

$\therefore$ The measure of the angle between the given line and the plane $= sin^{-1} \frac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$

$= sin^{-1} \frac{|-1|}{3(3)} = sin^{-1} \frac{1}{9}$

7.19 Intersection of two planes

Let $\pi_1 : \bar{r} \cdot \bar{n}_1 = d_1$ and $\pi_2 : \bar{r} \cdot \bar{n}_2 = d_2$ be two intersecting planes.

$\therefore \quad \bar{n}_1 \times \bar{n}_2 \neq \bar{0}$

Let $\bar{n} = \bar{n}_1 \times \bar{n}_2$.

Suppose $A(\bar{a})$ is a point of intersection of π_1 and π_2.

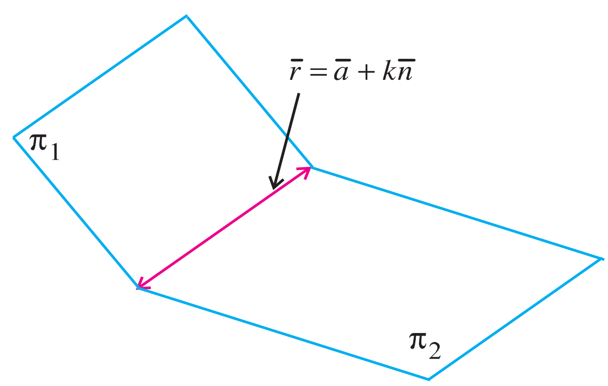


Figure 7.24

$\therefore \quad A(\bar{a}) \in \pi_1$ and $A(\bar{a}) \in \pi_2$.

$\therefore \quad \bar{a} \cdot \bar{n}_1 = d_1$ and $\bar{a} \cdot \bar{n}_2 = d_2$

$\therefore$ The equations of π_1 and π_2 are

$\bar{r} \cdot \bar{n}_1 = d_1 = \bar{a} \cdot \bar{n}_1$

$\therefore \quad (\bar{r} - \bar{a}) \cdot \bar{n}_1 = 0$

Similarly $(\bar{r} - \bar{a}) \cdot \bar{n}_2 = 0$

$\therefore$ If $P(\bar{r})$ is on both π_1 and π_2, then $(\bar{r} - \bar{a}) \perp \bar{n}_1$ and $(\bar{r} - \bar{a}) \perp \bar{n}_2$, $P \neq A$.

$\therefore \quad \bar{r} - \bar{a} = k(\bar{n}_1 \times \bar{n}_2), k \in R - \{0\}$

$\therefore \quad \bar{r} - \bar{a} = k\bar{n}, k \in R - \{0\}$ $\quad (\bar{n} = \bar{n}_1 \times \bar{n}_2)$

If $k = 0$, then P = A. So $\bar{r} = \bar{a} + k\bar{n}, k \in R$.

Thus, if $P(\bar{r}) \in \pi_1 \cap \pi_2$, then $\bar{r} = \bar{a} + k\bar{n}, k \in R$.

This is the equation of a line.

$\therefore$ Every point of $\pi_1 \cap \pi_2$ is on the line $\bar{r} = \bar{a} + k\bar{n}, k \in R$.

Conversely, if $P(\bar{r})$ is on the line $\bar{r} = \bar{a} + k\bar{n}, k \in R$, then

$(\bar{r} - \bar{a}) \cdot \bar{n}_1 = k\bar{n} \cdot \bar{n}_1 = k(\bar{n}_1 \times \bar{n}_2) \cdot \bar{n}_1 = 0$

and $(\bar{r} - \bar{a}) \cdot \bar{n}_2 = k\bar{n} \cdot \bar{n}_2 = k(\bar{n}_1 \times \bar{n}_2) \cdot \bar{n}_2 = 0$

Thus, $P(\bar{r}) \in \pi_1 \cap \pi_2$.

Hence, $\pi_1 \cap \pi_2$ is the line given by the equation $\overline{r} = \overline{a} + k\overline{n}$, $k \in R$ where $\overline{n} = \overline{n}_1 \times \overline{n}_2$.

Thus two planes $\overline{r} \cdot \overline{n}_1 = d_1$ and $\overline{r} \cdot \overline{n}_2 = d_2$ intersect in the line $\overline{r} = \overline{a} + k(\overline{n}_1 \times \overline{n}_2)$ $k \in R$ provided $\overline{n}_1 \times \overline{n}_2 \neq \overline{0}$.

Equation of a plane passing through the intersection of two planes :

Suppose $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are two intersecting planes.

The equation of any plane passing through their line of intersection is

$$l(a_1x + b_1y + c_1z + d_1) + m(a_2x + b_2y + c_2z + d_2) = 0,\ l^2 + m^2 \neq 0$$

Conversely, any plane whose equation can be expressed in the form,

$l(a_1x + b_1y + c_1z + d_1) + m(a_2x + b_2y + c_2z + d_2) = 0$, $l^2 + m^2 \neq 0$ will certainly contain the line of intersection of the two given planes.

We shall assume both these statements without proof.

Here $l^2 + m^2 \neq 0$ means atleast one of l, m is non-zero.

If $l = 0$, then $m \neq 0$ and hence the required plane is $a_2x + b_2y + c_2z + d_2 = 0$.

If $l \neq 0$, then the required plane is not $a_2x + b_2y + c_2z + d_2 = 0$.

$\therefore$ $l(a_1x + b_1y + c_1z + d_1) + m(a_2x + b_2y + c_2z + d_2) = 0$ becomes

$$a_1x + b_1y + c_1z + d_1 + \frac{m}{l}(a_2x + b_2y + c_2z + d_2) = 0$$

Let $\frac{m}{l} = \lambda$

If $a_2x + b_2y + c_2z + d_2 = 0$ is not the required plane, then

the equation of the required plane is

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0,\ \lambda \in R$$

Example 29 : Find the equation of the plane passing through the intersection of the planes $2x + 3y + z - 1 = 0$ and $x + y - z - 7 = 0$ and also passing through the point (1, 2, 3). Also obtain the equation of the line of intersection of these planes.

Solution : For (1, 2, 3), $x + y - z - 7 = 1 + 2 - 3 - 7 = -7 \neq 0$

$\therefore$ (1, 2, 3) is not in the plane $x + y - z - 7 = 0$.

$\therefore$ $x + y - z - 7 = 0$ is not the required plane.

Suppose the required plane has equation $2x + 3y + z - 1 + \lambda(x + y - z - 7) = 0$ **(i)**

It passes through (1, 2, 3)

$\therefore$ $2 + 6 + 3 - 1 + \lambda(1 + 2 - 3 - 7) = 0$

$\therefore$ $-7\lambda = -10$

$\therefore$ $\lambda = \frac{10}{7}$. Substitute $\lambda = \frac{10}{7}$ in (i).

$$2x + 3y + z - 1 + \frac{10}{7}(x + y - z - 7) = 0$$

$\therefore$ $14x + 21y + 7z - 7 + 10x + 10y - 10z - 70 = 0$

$\therefore$ $24x + 31y - 3z - 77 = 0$

The direction of the line of intersection is $\overline{n} = \overline{n}_1 \times \overline{n}_2 = (2, 3, 1) \times (1, 1, -1) = (-4, 3, -1)$.

Let us take $z = 0$ in both the equations of planes.

$\therefore$ We get $2x + 3y = 1$ and $x + y = 7$.

Solving these equations we get $x = 20$, $y = -13$.

$\therefore$ A point of intersection is A(20, -13, 0)

$\therefore$ The equation of the required line $\overline{r} = \overline{a} + k\overline{n}$, $k \in$ R gives,

$\overline{r} = (20, -13, 0) + k(-4, 3, -1)$, $k \in$ R

Note : To find a common point of two planes, we can take any one of x, y, z as known number so that the other two can be uniquely determined.

Exercise 7.2

1. Find the unit normal to the plane $4x - 2y + z - 7 = 0$.

2. If possible, find the vector and Cartesian equation of the plane passing through (1, 1, -1), (2, -1, -3) and (3, 0, 1).

3. Find the equation of the plane parallel to $2x - 3y - 5z + 1 = 0$ and passing through (1, 2, -3).

4. Find the equation of the plane passing through (5, -1, 2) and perpendicular to the line which passes through (-2, 1, 1) and (0, 5, 1). Also find the intercepts made by this plane on the co-ordinate axes.

5. Find the equation of the plane passing through (2, 0, 1) and containing the line

$\overline{r} = (1, 4, -1) + k(2, -3, 3)$, $k \in$ R.

6. Show that the points (2, 7, 3), (-10, -10, 2), (-3, 3, 2) and (0, -2, 4) are coplanar. Also find the equation of the plane passing through them.

7. Obtain the equation of the plane which passes through (3, 4, -5) and (1, 2, 3) and parallel to Z-axis.

8. Find the measure of the angle between the planes $2x + y - z - 1 = 0$ and $x - y - 2z + 7 = 0$.

9. Find the measure of the angle between the line $\frac{x-2}{2} = \frac{y-2}{-3} = \frac{z-1}{2}$ and the plane $2x + y - 3z + 4 = 0$.

10. Find the perpendicular distance to the plane $3x + 2y - 5z - 13 = 0$ from the point (5, 3, 4).

11. Find the perpendicular distance between the planes $12x - 6y + 4z - 21 = 0$ and $6x - 3y + 2z - 1 = 0$.

12. Find the equation of the plane passing through A(1, 3, 5) and perpendicular to $\overline{AP}$, where P is (3, -2, 1)

13. Find the equation of the plane passing through the point (1, 1, -1) and containing the line

$\overline{r} = (2, -4, -6) + k(1, 8, -3)$, $k \in$ R.

14. Find the equation of the plane passing through the intersecting lines $\frac{x+1}{1} = \frac{3-y}{1} = \frac{z+5}{2}$ and $\frac{x+1}{3} = \frac{y-3}{1} = \frac{z+5}{2}$.

*

Miscellaneous Examples

Example 30 : If a line makes angles of measures α, β, γ, δ with the four diagonals of a cube, prove that $cos2\alpha + cos2\beta + cos2\gamma + cos2\delta = -\frac{4}{3}$.

Solution : Assume that each side of the cube is of unit length. Then the vertices can be taken as shown in the figure 7.25.

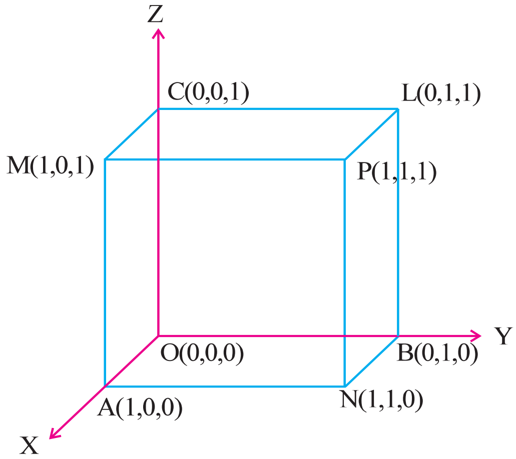


Figure 7.25

The four diagonals of the cube are $\overrightarrow{OP} = (1, 1, 1)$, $\overrightarrow{AL} = (-1, 1, 1)$, $\overrightarrow{BM} = (1, -1, 1)$, $\overrightarrow{CN} = (1, 1, -1)$.

Suppose the line has direction *cosines* l, m, n. So $l^2 + m^2 + n^2 = 1$.

If α, β, γ and δ are the measure of the angles made by the line with the diagonals $\overrightarrow{OP}$, $\overrightarrow{AL}$, $\overrightarrow{BM}$ and $\overrightarrow{CN}$ respectively, then

$$cos\alpha = \frac{|l + m + n|}{\sqrt{3}},\ cos\beta = \frac{|-l + m + n|}{\sqrt{3}},\ cos\gamma = \frac{|l - m + n|}{\sqrt{3}} \text{ and } cos\delta = \frac{|l + m - n|}{\sqrt{3}}.$$

Now,
$$\begin{aligned}
cos2\alpha + cos2\beta + cos2\gamma + cos2\delta &= 2cos^2\alpha - 1 + 2cos^2\beta - 1 + 2cos^2\gamma - 1 + 2cos^2\delta - 1 \\
&= 2(cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta) - 4 \\
&= \frac{2}{3}[(l + m + n)^2 + (-l + m + n)^2 + \\
&\qquad (l - m + n)^2 + (l + m - n)^2] - 4 \\
&= \frac{2}{3}[4(l^2 + m^2 + n^2)] - 4 \\
&= \frac{8}{3} - 4 \qquad\qquad (l^2 + m^2 + n^2 = 1) \\
&= -\frac{4}{3}
\end{aligned}$$

Image of a point in the line (plane) : If M is the foot of perpendicular from A to a line (plane) and B is the point such that M is the mid-point of $\overline{AB}$, then B is called the image of A in the line (plane).

Example 31 : Find the image of A(1, 2, 3) in the line L : $\frac{x-6}{3} = \frac{y-7}{2} = \frac{7-z}{2}$.

Solution : The line has equation $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$.

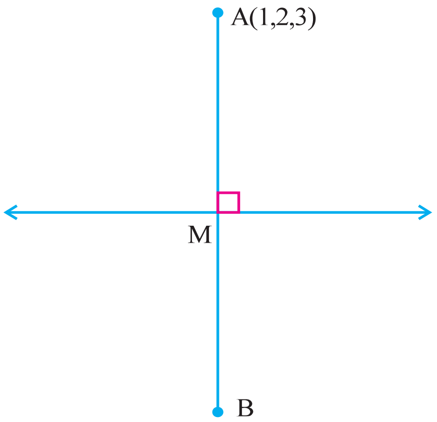


Figure 7.26

Here $\bar{a} = (6, 7, 7)$, $\bar{l} = (3, 2, -2)$. Let M be the foot of perpendicular from A(1, 2, 3) to L.

M $\in$ L. So M is $(6 + 3k, 7 + 2k, 7 - 2k)$ for some $k \in$ R.

$$\begin{aligned}
\overrightarrow{AM} &= (6 + 3k, 7 + 2k, 7 - 2k) - (1, 2, 3) \\
&= (5 + 3k, 5 + 2k, 4 - 2k)
\end{aligned}$$

$\overrightarrow{AM} \perp$ L

$\therefore\ \overrightarrow{AM} \cdot \bar{l} = 0$

$\therefore\ (5 + 3k, 5 + 2k, 4 - 2k) \cdot (3, 2, -2) = 0$

$\therefore\ 15 + 9k + 10 + 4k - 8 + 4k = 0$

$\therefore \quad 17k + 17 = 0$

$\therefore \quad k = -1$

$\therefore$ The foot of perpendicular is $M(6 + 3k, 7 + 2k, 7 - 2k) = M(3, 5, 9)$.

If $B(x, y, z)$ is the image of A in the given line, then M is the mid-point of $\overline{AB}$.

$\therefore \quad (3, 5, 9) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z+3}{2}\right)$

$\therefore \quad x = 5, y = 8, z = 15$

$\therefore$ The image of A is B(5, 8, 15).

Example 32 : The direction numbers l, m, n of two lines satisfy $l + m + n = 0$ and $l^2 - m^2 + n^2 = 0$. Find the measure of the angle between the lines.

Solution : Here $l + m + n = 0$

$\therefore \quad m = -l - n$

Also $l^2 - m^2 + n^2 = 0$

$\therefore \quad l^2 - (-l - n)^2 + n^2 = 0$

$\therefore \quad l^2 - l^2 - 2ln - n^2 + n^2 = 0$

$\therefore \quad ln = 0$

$\therefore \quad l = 0$ or $n = 0$

As l, m, n are the direction numbers, $(l, m, n) \neq (0, 0, 0)$

If $l = 0$, then $n = -m$

$\therefore$ Direction numbers are $(0, m, -m)$

If $n = 0$, then $l = -m$

$\therefore$ Direction numbers are $(-m, m, 0)$

If α is the measure of the angle between the two lines, then

$$\cos\alpha = \frac{|(0, m, -m) \cdot (-m, m, 0)|}{\sqrt{2m^2} \cdot \sqrt{2m^2}}$$

$$= \frac{|m^2|}{2|m^2|} = \frac{1}{2}$$

$\therefore \quad \alpha = \frac{\pi}{3}$

Example 33 : Find the point of intersection of the line $\frac{x-4}{2} = \frac{y-5}{2} = \frac{z-3}{1}$ and the plane $x + y + z - 2 = 0$. Also find the distance between this point and the point (8, 9, 5).

Solution : Here $\bar{a} = (4, 5, 3)$, $\bar{l} = (2, 2, 1)$.

Let P be the point of intersection. So P is on the given line.

$\therefore$ P is $(5 + 2k, 3 + k, 4 + 2k)$ for some $k \in$ R, P is also on the plane $x + y + z - 2 = 0$.

$\therefore$ $4 + 2k + 5 + 2k + 3 + k - 2 = 0$

$\therefore$ $5k = -10$

$\therefore$ $k = -2$

$\therefore$ The point of intersection is $(4 + 2(-2), 5 + 2(-2), 3 + (-2)) = (0, 1, 1)$

The distance between P(0, 1, 1) and Q(8, 9, 5) is given by

$$PQ = \sqrt{(8-0)^2 + (9-1)^2 + (5-1)^2} = \sqrt{64 + 64 + 16} = \sqrt{144} = 12$$

Example 34 : Find the equation of the plane passing through $(2, 2, -2)$ and $(-2, -2, 2)$ and perpendicular to the plane $2x - 3y + z - 7 = 0$.

Solution : Let the equation of the required plane be $ax + by + cz + d = 0$.

If $\overline{n}$ is normal to this plane, then $\overline{n} = (a, b, c)$.

Since this plane is perpendicular to $2x - 3y + z - 7 = 0$.

$\therefore$ $\overline{n} \cdot (2, -3, 1) = 0$ **(i)**

Also A(2, 2, −2) and B(−2, −2, 2) lie in the plane.

$\therefore$ $\overrightarrow{AB}$ lies in the plane. $\overrightarrow{AB} = (-4, -4, 4)$

$\therefore$ $\overline{n} \cdot (-4, -4, 4) = 0$

$\therefore$ $\overline{n} \cdot (-1, -1, 1) = 0$ **(ii)**

From (i) and (ii), $\overline{n} = (2, -3, 1) \times (-1, -1, 1)$

$\therefore$ $\overline{n} = (-2, -3, -5)$ or $\overline{n} = (2, 3, 5)$

Since the plane passes through $(2, 2, -2)$ its equation is given by $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$.

$\therefore$ $\overline{r} \cdot (2, 3\ 5) = (2, 2, -2) \cdot (2, 3\ 5)$

$\therefore$ $2x + 3y + 5z = 4 + 6 - 10 = 0$

$\therefore$ The equation of the required plane is $2x + 3y + 5z = 0$.

Example 35 : Find the foot of the perpendicular from P(9, 6, −2) to the plane passing through the point A(4, 5, 2), B(2, 3, −1) and C(6, −1, −1). Also find the perpendicular distance from P to this plane.

Solution : The equation of the plane is

$$\begin{vmatrix} x-4 & y-5 & z-2 \\ 2-4 & 3-5 & -1-2 \\ 6-4 & -1-5 & -1-2 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x-4 & y-5 & z-2 \\ -2 & -2 & -3 \\ 2 & -6 & -3 \end{vmatrix} = 0$$

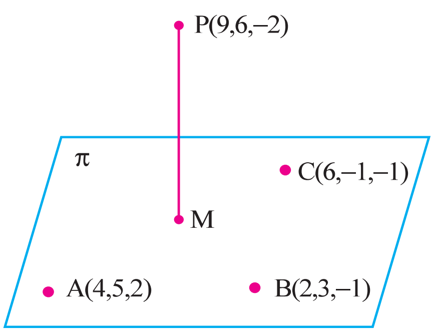


Figure 7.27

$\therefore$ $(x - 4)(-12) - (y - 5)(12) + (z - 2)(16) = 0$

$\therefore$ $3(x - 4) + 3(y - 5) - 4(z - 2) = 0$

$\therefore$ $3x + 3y - 4z - 19 = 0$ is the equation of plane through A, B and C.

Let M be the foot of perpendicular from the P(9, 6, −2) to the plane π : $3x + 3y - 4z - 19 = 0$.

Here, $\bar{n} = (3, 3, -4)$

Equation of $\overleftrightarrow{PM}$ is $\bar{r} = \bar{p} + k\bar{n}$, $k \in R$

$\therefore \quad \bar{r} = (9, 6, -2) + k(3, 3, -4)$, $k \in R$

$\therefore$ M is $(9 + 3k, 6 + 3k, -2 - 4k)$ for some $k \in R$

Now, $M \in \pi$

$\therefore \quad 3(9 + 3k) + 3(6 + 3k), -4(-2 - 4k) - 19 = 0$

$\therefore \quad 27 + 9k + 18 + 9k + 8 + 16k - 19 = 0$

$\therefore \quad 34k = -34$

$\therefore \quad k = -1$

$\therefore$ The foot of the perpendicular is $M(9 + 3(-1), 6 + 3(-1), -2 - 4(-1))$

$\therefore$ M is (6, 3, 2)

Perpendicular distance PM $= \sqrt{(9-6)^2 + (6-3)^2 + (-2-2)^2}$

$= \sqrt{9+9+16}$

$= \sqrt{34}$

Example 36 : Show that (i) The line $\bar{r} = (1, 2, -3) + k(4, -3, 2)$, $k \in R$ is parallel to the plane $3x + 2y - 3z = 5$. (ii) The plane $2x - 3y + 4z = 0$ contains the line $\bar{r} = (1, -2, -2) + k(1, 2, 1)$, $k \in R$

Solution : (i) Here, the equation of the line L is $\bar{r} = (1, 2, -3) + k(4, -3, 2)$, $k \in R$ and the plane π has equation $3x + 2y - 3z = 5$.

$\therefore \quad A(\bar{a}) = (1, 2, -3)$, $\bar{l} = (4, -3, 2)$ and $\bar{n} = (3, 2, -3)$

Now, $\bar{l} \cdot \bar{n} = 4(3) - 3(2) + 2(-3) = 12 - 6 - 6 = 0$

$\therefore \quad \bar{l} \perp \bar{n}$. So L is parallel to π or π contains L.

Also $\bar{a} \cdot \bar{n} = (1, 2, -3) \cdot (3, 2, -3) = 3 + 4 + 9 = 16 \neq 0$

$\therefore$ The line is parallel to the plane.

(ii) Here, the equation of the line L is $\bar{r} = (1, -2, -2) + k(1, 2, 1)$, $k \in R$ and the equation of the plane π is $2x - 3y + 4z = 0$.

$\therefore \quad A(\bar{a}) = (1, -2, -2)$, $\bar{l} = (1, 2, 1)$ and $\bar{n} = (2, -3, 4)$

Now, $\bar{l} \cdot \bar{n} = 1(2) + 2(-3) + 1(4) = 2 - 6 + 4 = 0$

$\therefore \quad \bar{l} \perp \bar{n}$. So L is parallel to π or π contains L.

$\bar{a} \cdot \bar{n} = (1, -2, -2) \cdot (2, -3, 4) = 2 + 6 - 8 = 0$

$\therefore$ The plane π contains the line L.

Exercise 7

1. Find the foot of perpendicular from P(1, 0, 3) to the line passing through the points A(4, 7, 1) and B(5, 9, −1). Also find the equation of perpendicular line $\overleftrightarrow{AB}$ through P and perpendicular distance from P to $\overleftrightarrow{AB}$.

2. Find the measure of the angle between two lines, if their direction *cosines l, m, n* satisfy $l + m + n = 0$ and $m^2 + n^2 = l^2$.

3. Prove that the lines $x = 2, \frac{y-1}{3} = \frac{z-2}{1}$ and $x = \frac{y-1}{1} = \frac{z+1}{3}$ are skew. Find the shortest distance between them.

4. Find the point of intersection of the lines $\frac{x+3}{2} = \frac{5-y}{1} = \frac{1-z}{1}$ and $\frac{x+3}{2} = \frac{y-5}{3} = \frac{z-1}{1}$. Also find the measure of the angle between them.

5. Find the equation of the line passing through (1, 2, 3) and perpendicular to both the lines $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-1}$ and $\frac{x-5}{-3} = \frac{y+8}{1} = \frac{z-5}{5}$.

6. Find the equation of the line equally inclined to the co-ordinate axes and passing through (3, −2, −4).

7. Find the point of intersection of the line $\frac{x-1}{2} = \frac{2-y}{3} = \frac{z+3}{4}$ and the plane $2x + 4y - z = 1$. Also find the measure of the angle between them.

8. Find the equation of the plane parallel to X-axis and whose Y and Z-intercepts are 2 and 3 respectively.

9. Find the image of (1, 5, 1) in the plane $x - 2y + z + 5 = 0$.

10. Find the foot of perpendicular from (0, 2, −2) to the plane $2x - 3y + 4z - 44 = 0$, the equation of this perpendicular and the perpendicular distance between the point and the plane.

11. Find the equation of the plane through the line of intersection of the planes $2x + 3y - z - 4 = 0$ and $x + y + z - 2 = 0$ and through the point (1, 2, 2). Also find the equation of the line of intersection of these planes.

12. If the centroid of the triangle formed by the intersection of a plane with the coordinate axes is (2, 1, −1), find the equation of this plane.

13. Prove that the lines $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z+4}{4}$ and $\frac{x-7}{5} = \frac{y+6}{1} = \frac{z+8}{2}$ intersect each other. Find the equation of the plane containing them.

14. Find the equation of the plane whose intercepts are equal to half of the intercepts of the plane $3x + 4y - 6z = 12$.

15. Find the equation of the perpendicular bisector plane of the line-segment joining the points (1, 2, −3) and (−3, 6, 4).

16. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The equation of the line passing through origin with direction angles $\frac{2\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3}$ is ...... ☐

(a) $x = \frac{y}{-\sqrt{2}} = z$ (b) $\frac{x}{-1} = \frac{y}{-\sqrt{2}} = z$ (c) $x = \frac{y}{-\sqrt{2}} = -z$ (d) $x = \frac{y}{\sqrt{2}} = z$

(2) Line passing through (3, 4, 5) and (4, 5, 6) has direction *cosines* ...... ☐

(a) 1, 1, 1 (b) $\sqrt{3}, \sqrt{3}, \sqrt{3}$ (c) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (d) 7, 9, 11

(3) Lines $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$ and $\frac{x-2}{2} = \frac{y+1}{1} = \frac{3-z}{-3}$ are ...... lines. ☐

(a) parallel (b) perpendicular

(c) coincident (d) intersecting in an acute angle

(4) Line through origin and parallel to Y-axis is ...... ☐

(a) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ (b) $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$ (c) $\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$ (d) $\frac{x}{1} = \frac{y}{1} = \frac{z}{0}$

(5) The measure of the angle between the lines $x = k + 1$, $y = 2k - 1$, $z = 2k + 3$, $k \in$ R and $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-1}{-2}$ is ...... ☐

(a) $sin^{-1}\frac{4}{3}$ (b) $cos^{-1}\frac{4}{9}$ (c) $sin^{-1}\frac{\sqrt{5}}{3}$ (d) $\frac{\pi}{2}$

(6) A normal to the plane $x = 2$ is... ☐

(a) (0, 1, 1) (b) (2, 0, 2) (c) (1, 0, 0) (d) (0, 1, 0)

(7) Direction of the line perpendicular to the plane $3x - 4y + 7z = 2$ and passing through (−1, 2, 4) is ..... . ☐

(a) (3, 4, 7) (b) (4, −6, 3) (c) (−3, 4, −7) (d) (−1, 2, 4)

(8) Perpendicular distance of origin from the plane $\bar{r} \cdot (12, -4, 3) = 65$ is ...... ☐

(a) 65 (b) 5 (c) −5 (d) $\frac{5}{13}$

(9) Plane $2x + 3y + 6z - 15 = 0$ makes angle of measure ...... with X-axis. ☐

(a) $cos^{-1}\frac{3\sqrt{5}}{7}$ (b) $sin^{-1}\frac{3}{7}$ (c) $sin^{-1}\frac{2}{\sqrt{7}}$ (d) $tan^{-1}\frac{2}{7}$

(10) Perpendicular distance between the planes $2x - y + 2z = 1$ and $4x - 2y + 4z = 1$ is .... ☐

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{1}{6}$ (d) 6

(11) The plane passing through the points (1, 1, 1), (1, −1, 1) and (−1, 3, −5) will pass through (2, k , 4) for .... ☐

(a) no value of k (b) two values of k

(c) any value of k (d) unique k

(12) If the foot of the perpendicular from the origin to a plane is (a, b, c), the equation of the plane is .... ☐

(a) $ax + by + cz = a + b + c$ (b) $ax + by + cz = abc$

(c) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (d) $ax + by + cz = a^2 + b^2 + c^2$

(13) Equation of the line L passing through A(−2, 2, 3) and perpendicular to $\overleftrightarrow{AB}$ is ...... where coordinates of B are (13, −3, 13). ☐

(a) $\frac{x-2}{3} = \frac{y+2}{13} = \frac{z+3}{2}$ (b) $\frac{x+2}{3} = \frac{y-2}{13} = \frac{z-3}{2}$

(c) $\frac{x+2}{15} = \frac{y-2}{-5} = \frac{z-3}{10}$ (d) $\frac{x-2}{15} = \frac{y+2}{-5} = \frac{z+3}{10}$

(14) If $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, then k = ... ☐

(a) 7 (b) 6 (c) −7 (d) any value of k

(15) Perpendicular distance of (2, −3, 6) from $3x - 6y + 2z + 10 = 0$ is .... ☐

(a) $\frac{13}{7}$ (b) $\frac{46}{7}$ (c) 7 (d) $\frac{10}{7}$

(16) Line passing through (2, −3, 1) and (3, −4, −5) intersects ZX-plane at .... ☐

(a) (−1, 0, 13) (b) (−1, 0, 19) (c) $\left(\frac{13}{6}, 0, \frac{-19}{6}\right)$ (d) (0, −1, 13)

(17) If lines $\bar{r} = (2, -3, 7) + k(2, a, 5)$, $k \in$ R and $\bar{r} = (1, 2, 3) + k(3, -a, a)$, $k \in$ R are perpendicular to each other, then a .... ☐

(a) 2 (b) −6 (c) 1 (d) −1

●

Summary

We have studied the following points in this chapter :

1. Vector equation of the line pasing through A($\bar{a}$) and having the direction of a non-zero vector $\bar{l}$ is $\bar{r} = \bar{a} + k\bar{l}$, $k \in$ R

Parametric equations :

$$\left.\begin{aligned} x &= x_1 + kl_1 \\ y &= y_1 + kl_2 \\ z &= z_1 + kl_3 \end{aligned}\right\} \quad k \in \text{R}$$

Cartesian equations (symmetric form) : $\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

If $l_1 = 0$ and $l_2 \neq 0$ and $l_3 \neq 0$, then equation is $x = x_1$, $\frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$,

we can write it as $\frac{x - x_1}{0} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

2. Equation of a line passing through two distinct points A($\bar{a}$) and B($\bar{b}$) :

Vector equation : $\bar{r} = \bar{a} + k(\bar{b} - \bar{a}),\ k \in R$

Parametric equations :

$$\left.\begin{aligned} x &= x_1 + k(x_2 - x_1) \\ y &= y_1 + k(y_2 - y_1) \\ z &= z_1 + k(z_2 - z_1) \end{aligned}\right\}\ k \in R$$

Symmetric Form :

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

3. Collinear points : A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are collinear if and only if $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$.

4. If A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are collinear, $[\bar{a}\ \bar{b}\ \bar{c}] = 0$. But $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ does not assure that points are collinear.

5. The measure of the angle between two lines : $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ are two distinct lines. If α is the measure of angle between the lines,

then $cos\alpha = \dfrac{|\bar{l} \cdot \bar{m}|}{|\bar{l}||\bar{m}|}$; $0 \le \alpha \le \frac{\pi}{2}$

Lines are perpendicular if and only if $\bar{l} \cdot \bar{m} = 0$.

6. If two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ intersect in a point, then $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0,\ \bar{l} \times \bar{m} \ne \bar{0}$.

It can also be stated as $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$,

where $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{l} = (l_1, l_2, l_3)$ and $\bar{m} = (m_1, m_2, m_3)$.

7. Non-coplanar or skew lines : For two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$, if $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) \ne 0$, then they are non-coplanar or skew lines.

8. Perpendicular distance of a point P($\bar{p}$) from a line $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ is $\dfrac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$.

9. Perpendicular distance between two parallel lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{l},\ k \in R$ is $\dfrac{|(\bar{b} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$.

10. Perpendicular (shortest) distance between two skew lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ is $\dfrac{|(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m})|}{|\bar{l} \times \bar{m}|}$.

11. Vector equation of the plane passing through three distinct non-collinear points A($\bar{a}$), B($\bar{b}$) and C($\bar{c}$) is $\bar{r} = l\bar{a} + m\bar{b} + n\bar{c}$, where $l, m, n \in R$ and $l + m + n = 1$.

Parametric Form :

$x = lx_1 + mx_2 + nx_3$

$y = ly_1 + my_2 + ny_3$

$z = lz_1 + mz_2 + nz_3$ where $l, m, n \in$ R and $l + m + n = 1$ and the points are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Cartesian Form : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

12. Four distinct points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $C(x_4, y_4, z_4)$ are coplanar if and only if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0.$

13. Equation of the plane making intercepts a, b and c on X-axis, Y-axis and Z-axis respectively is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ $(abc \neq 0)$.

14. Equation of the plane passing through $A(\overline{a})$ and having normal $\overline{n}$:

Vector equation : $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$

Cartesian form : If $\overline{r} = (x, y, z)$, $\overline{n} = (a, b, c)$, then the equation is $ax + by + cz = d$, $(d = \overline{a} \cdot \overline{n})$

15. Equation of the plane using normal through the origin : Let $N(\overline{n})$ be the foot of perpendicular from the origin and $|\overline{n}| = p$. Then the equation of the plane is $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction *cosines* of $\overline{n}$.

16. Measure of the angle between the planes $\overline{r} \cdot \overline{n}_1 = d_1$ and $\overline{r} \cdot \overline{n}_2 = d_2$: If θ is the measure of the angle between them, then $\cos\theta = \frac{|\overline{n}_1 \cdot \overline{n}_2|}{|\overline{n}_1||\overline{n}_2|}$; $0 \leq \theta \leq \frac{\pi}{2}$.

Planes are perpendicular if and only if $\overline{n}_1 \cdot \overline{n}_2 = 0$.

17. Equation of the plane passing through two parallel lines $\overline{r} = \overline{a} + k\overline{l}$, $k \in$ R and $\overline{r} = \overline{b} + k\overline{l}$, $k \in$ R is $(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] = 0$.

Cartesian form :

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & l_2 & l_3 \end{vmatrix} = 0$$

where $\overline{a} = (x_1, y_1, z_1)$, $\overline{b} = (x_2, y_2, z_2)$ and $\overline{l} = (l_1, l_2, l_3)$.

18. Equation of the plane passing through two intersecting lines $\overline{r} = \overline{a} + k\overline{l}$, $k \in$ R and $\overline{r} = \overline{b} + k\overline{m}$, $k \in$ R is $(\overline{r} - \overline{a}) \cdot (\overline{l} \times \overline{m}) = 0$.

Cartesian form :

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0,$$ where $\overline{a} = (x_1, y_1, z_1)$, $\overline{l} = (l_1, l_2, l_3)$ and $\overline{m} = (m_1, m_2, m_3)$.

19. Perpendicular distance from a point P($\bar{p}$) to the plane $\bar{r} \cdot \bar{n} = d$ is $\frac{|\bar{p} \cdot \bar{n} - d|}{|\bar{n}|}$.

Cartesian form :

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

where equation of the plane is $ax + by + cz = d$ and point P is (x_1, y_1, z_1).

20. Perpendicular distance between two parallel planes $\pi_1 : \bar{r} \cdot \bar{n} = d_1$ and $\pi_2 : \bar{r} \cdot \bar{n} = d_2$ is $\frac{|d_1 - d_2|}{|\bar{n}|}$.

21. If the measure of the angle between the line $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and the plane $\bar{r} \cdot \bar{n} = d$ is α, then $\alpha = sin^{-1} \frac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$; $0 < \alpha < \frac{\pi}{2}$.

22. Intersection of two planes $\pi_1 : \bar{r} \cdot \bar{n}_1 = d_1$ and $\pi_2 : \bar{r} \cdot \bar{n}_2 = d_2$ is a line given by the equation $\bar{r} = \bar{a} + k\bar{n}$, $k \in R$ where $\bar{n} = \bar{n}_1 \times \bar{n}_2$.

23. Equation of a plane passing through the intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$.

Mahavira

Mahavira was a 9th-century Indian mathematician from Gulbarga who asserted that the square root of a negative number did not exist. He gave the sum of a series whose terms are squares of an arithmetical progression and empirical rules for area and perimeter of an ellipse. He was patronised by the great Rashtrakuta king Amoghavarsha. Mahavira was the author of Ganit Saar Sangraha. He separated Astrology from Mathematics. He expounded on the same subjects on which Aryabhata and Brahmagupta contended, but he expressed them more clearly. He is highly respected among Indian Mathematicians, because of his establishment of terminology for concepts such as equilateral, and isosceles triangle; rhombus; circle and semicircle. Mahavira's eminence spread in all South India and his books proved inspirational to other Mathematicians in Southern India.

ANSWERS

Exercise 1.1

1. 15 cm^3/sec **2.** $\frac{2}{3}\pi rh$ **3.** $\frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}$ **4.** 4 cm^2/sec **5.** 3 cm^2/sec

6. **(1)** 27π cm^3/sec **(2)** 36π cm^2/sec **7.** 80π cm^2/sec

8. **(1)** 1 cm^2/sec **(2)** 1 cm/sec **(3)** 0.5 cm/sec **9.** 4 cm/sec **10.** $\frac{1}{8\pi}$ cm/sec **11.** ₹ 21.42

12. ₹ 615 **13.** 2 m/min **14.** 0.1 cm/sec **15.** 0.25 m^2/sec **16.** $\frac{3}{20}\sqrt{\frac{3}{7}}$ m/sec

17. 12π cm^2/sec **18.** -36 $units/sec$ **19.** $(1, 1), (-1, -1)$ **20.** $(1, 2)$

Exercise 1.2

7. **(1)** Increasing on R **(2)** Decreasing on R **(3)** Increasing on $(1, \infty)$, Decreasing on $(-\infty, 1)$

(4) Increasing on $\left(-\infty, \frac{3}{2}\right)$, Decreasing on $\left(\frac{3}{2}, \infty\right)$ **(5)** Increasing on R

(6) Decreasing on $(-\infty, -1)$ and $(0, 2)$, Increasing on $(-1, 0)$ and $(2, \infty)$

(7) Increasing on $\left(0, \frac{\pi}{4}\right)$ and Decreasing on $\left(\frac{\pi}{4}, \pi\right)$

(8) Decreasing on $(-\infty, -2)$ and $(-1, \infty)$, Increasing on $(-2, -1)$

(9) Strictly increasing on $(1, 3)$, $(3, \infty)$; Strictly decreasing on $(-\infty, -1)$, $(-1, 1)$

(10) Decreasing **(11)** Increasing **(12)** Decreasing

11. Decreasing in $(-\infty, -2)$ and $(1, 3)$; Increasing in $(-2, 1)$ and in $(3, \infty)$

12. Increasing in $\left(2k\pi, (4k+1)\frac{\pi}{2}\right)$ and $\left((4k+3)\frac{\pi}{2}, (2k+2)\pi\right)$, $k \in Z$

Decreasing in $\left((4k+1)\frac{\pi}{2}, (2k+1)\pi\right)$ and $\left((2k+1)\pi, (4k+3)\frac{\pi}{2}\right)$, $k \in Z$

14. Decreasing in $\left(0, \frac{\pi}{4}\right)$, Increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

15. $a < -2$ **16.** $a \in \left[0, \frac{1}{3}\right)$

21. **(1)** Increasing in $(-\infty, -2)$ and $(6, \infty)$; Decreasing in $(-2, 6)$

(2) Increasing in $(1, \infty)$, Decreasing in $(-\infty, 1)$

(3) Increasing in $\left(-\infty, \frac{4}{3}\right)$, $(2, \infty)$ and Decreasing in $\left(\frac{4}{3}, 2\right)$

(4) Increasing in $(-\infty, 1)$ and $(3, \infty)$ Decreasing in $(1, 3)$

(5) Increasing on R^+ **(6)** Increasing on R^+

(7) Increasing in $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, Decreasing in $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$

(8) Increasing in $((2k-1)\pi, 2k\pi)$, Decreasing in $(2k\pi, (2k+1)\pi)$, $k \in Z$

(9) Increasing in $(0, \infty)$, Decreasing in $(-\infty, 0)$

(10) Decreasing in $(-\infty, -2)$, Increasing in $(-2, \infty)$

(11) Increasing in $(-1, \infty)$, Decreasing in $(-\infty, -1)$

(12) Increasing in $(-\infty, -2)$ and $(0, \infty)$, Decreasing in $(-2, 0)$

(13) Increasing in $(0, e^2)$, Decreasing in (e^2, ∞)

(14) Increasing in $\left(\frac{1}{e}, \infty\right)$, Decreasing in $\left(0, \frac{1}{e}\right)$

Exercise 1.3

1. $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ **2.** $yy_1 = 2a(x + x_1)$ **3.** 17 **4.** -1 **5.** $y = 4x + 1$ **6.** $2x + 4y = 9$

11. **(1)** $y = 1$ **(2)** $\left(2n\pi + \frac{\pi}{2}, 1\right)$, $n \in Z - \{0\}$ **12.** $x + y = \sqrt{2}$

13. **(1)** $y = 0$ at $(0, 0)$ **(2)** $y = 2x$ at $(1, 2)$ and $(-1, -2)$ **14.** $a = 2$, $b = -7$

15. $a = 5$, $b = -4$ **16.** $x\cos\frac{\theta}{2} - y\sin\frac{\theta}{2} = a\,\theta\cos\frac{\theta}{2} - 2a\sin\frac{\theta}{2}$ **18.** $(-1, -2)$

19. $a = 2$, $b = -1$ **20.** $x + y = 6$, horizontal at $(0, 0)$, $\left(2^{\frac{4}{3}}, 2^{\frac{5}{3}}\right)$, vertical at $(0, 0)$, $\left(2^{\frac{5}{3}}, 2^{\frac{4}{3}}\right)$

22. **(1)** $5x + 4y + 16 = 0$ **(2)** $x - \sqrt{2}y + 9 = 0$ **(3)** $x - y = 0$

(4) $9x - 2y - 5 = 0$ **(5)** $9x + 13y - 40 = 0$

23. $(1, 1)$, $(-1, -1)$ **24.** **(1)** $\tan^{-1}\frac{4}{3}$ **(2)** $\tan^{-1} 2$ at $(2, 1)$ and $(2, -1)$

25. $x + 2y = (4k + 1)\frac{\pi}{2}$, $k \in Z$ **26.** $x + y = 3$, $x + y + 1 = 0$ **28.** $a = -\frac{1}{2}$, $b = -\frac{3}{4}$, $c = 3$

Exercise 1.4

1. $\frac{73}{120}$ or 0.6083 **2.** 0.9999 **3.** $\frac{323}{108}$ or 2.9907 **4.** $\frac{1023}{256}$ or 3.9961 **5.** 19.975

6. 2.00125 **7.** $\frac{\sqrt{3}}{2} + \frac{\pi}{360}$ **8.** $\frac{\sqrt{3}}{2} + \frac{\pi}{360}$ **9.** $\frac{1}{\sqrt{3}} + \frac{\pi}{135}$ **10.** 4.6062

11. 1.0004343 **12.** 2.003125 **13.** $\frac{\pi}{2}\ cm^3$ **14.** $4\pi r^2\,\Delta r$ **15.** 0.5 %

16. $\frac{\sqrt{3}\pi x}{6}$ % **17.** 1.12 **18.** 4.05 **19.** $60\ cm^3$ **20.** $5.184\pi\ cm^2$

21. $\frac{1}{2} + \frac{\sqrt{3}\pi}{72}$

Exercise 1.5

1. Local minimum at $x = \frac{1}{3}$, $f\left(\frac{1}{3}\right) = \frac{122}{27}$; Local maximum at $x = 3$, $f(3) = 14$

2. Local minimum at $x = -\sqrt{3}$, $f(\sqrt{3}) = f(-\sqrt{3}) = -9$

Local minimum at $x = \sqrt{3}$

Local maximum at $x = 0$, $f(0) = 0$

3. No extreme value. f is increasing on R^+

4. Local minimum at $x = (2n + 1)\pi$, $f((2n + 1)\pi) = -2$

Local maximum at $x = 2n\pi$, $f(2n\pi) = 2$

5. Local and global minimum at $x = 0$, $f(0) = 0$

6. Local and global maximum at $x = 1$, $f(1) = \frac{1}{e}$

Global minimum at $x = 0$, $f(0) = 0$

7. Local and global maximum at $x = e$, $f(e) = \frac{1}{e}$

Global minimum at $x = 1$, $f(1) = 0$

8. Local and global maximum at $x = 0$, $f(0) = 4$

Global minimum at $x = \pm 4$, $f(\pm 4) = 0$

9. Global minimum $f(1) = \frac{1}{2}$; Global maximum $f(2) = \frac{2}{3}$; f is $\uparrow$. No local minimum or local maximum.

10. Local and global maximum at $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$

Local and global minimum at $x = \frac{5\pi}{4}$, $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

11. Local and global maximum at $x = \frac{11\pi}{6}$, $f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

Local and global minimum at $x = \frac{7\pi}{6}$, $f\left(\frac{7\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

12. Local maximum at $x = \frac{2}{3}$, $f\left(\frac{2}{3}\right) = \frac{2}{3^{\frac{3}{2}}}$

13. Local and global minimum at $x = 2$, $f(2) = 61$

Global maximum at $x = 0$, $f(0) = 125$

14. Local and global maximum at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 1$

Local and global minimum at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$, $f\left(\frac{3\pi}{4}\right) = f\left(\frac{7\pi}{4}\right) = -1$

15. Local and global minimum at $x = 2$, $f(2) = 75$

Global maximum at $x = 3$, $f(3) = 89$

16. Length $(l) = \frac{20}{\pi + 4}$ *m*, Breadth $(b) = \frac{10}{\pi + 4}$ *m* **18.** 8, 8 **19.** $x = 10$, $y = 25$

22. Minimum distance 10 for P(4, 3) and Maximum distance 20 for Q(–4, –3)

24. Length = Breadth = 2 *m*, Height = 1 *m*, Minimum surface = 12 m^2

25. $a = 0$, $b = -1$, $c = 2$ **26.** 25 cm^2

Exercise 1

1. $\frac{5}{2\sqrt{3\pi}}$ *cm/sec* **2.** 15 *m/sec* **3.** –3 *cm/min*

4. Increasing in $(-\infty, -2)$ and $(3, \infty)$, Decreasing in $(-2, 3)$

5. **(1)** $(1, 3)$, $(3, \infty)$ **(2)** $(-\infty, -1)$, $(-1, 1)$ **7.** Increasing in $(-2, \infty)$, Decreasing in $(-\infty, -2)$

8. Decreasing in $(-\infty, 0)$ and $(2, \infty)$, Increasing in $(0, 2)$

10. $\frac{x}{a} + \frac{y}{b} = 1$ **11.** $\frac{\pi}{2}$ at $(0, 0)$, $tan^{-1}\frac{3}{4}$ at $(4a, 4a)$ **13.** $(-1, 2)$, $(1, -2)$

14. Local and global minimum at $x = \frac{\pi}{3}, f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$

Local and global maximum at $x = \frac{5\pi}{3}, f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$

15. Global minimum at $x = 0,\ f(0) = 0$

16. Local minimum at $x = 1, f(1) = 3$

17. Increasing in $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, Decreasing in $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ and $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$.

Local minimum $f\left(\frac{-\pi}{3}\right) = \frac{-4\pi}{3} + \sqrt{3}$, Local maximum $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$

18. Increasing in $\left(0, \frac{3}{4}\right)$, Decreasing in $\left(\frac{3}{4}, 1\right)$.

Local maximum at $x = \frac{3}{4},\ f\left(\frac{3}{4}\right) = \frac{5}{4}$

19. Critical numbers 0, 4 and 6. Increasing in (0, 4) and Decreasing in (4, 6).

Local and global maximum at $x = 4, f(4) = 2^{\frac{5}{3}}$, Global minimum at $x = 0$ and $6, f(0) = f(6) = 0$

20. Local and global minimum at $x = \frac{\pi}{4}, f\left(\frac{\pi}{4}\right) = \frac{1}{2}$

Global maximum at $x = 0, \frac{\pi}{2}, f(0) = f\left(\frac{\pi}{2}\right) = 1$

27. **(1)** $\tan^{-1}\frac{3}{11}$ **(2)** $\tan^{-1}\frac{9}{2}$ **(3)** $\tan^{-1}\frac{1}{2}$ **(4)** $\tan^{-1}\left(\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}\right)$ at $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$ and $\frac{\pi}{2}$ at (0, 0)

(5) $\tan^{-1}\frac{9}{13}$ **(6)** $\tan^{-1}\frac{1}{2}$ at (1, 1) and (1, −1) and touch each other at (0, 0)

29. **(1)** (b) **(2)** (d) **(3)** (a) **(4)** (b) **(5)** (d) **(6)** (a) **(7)** (c) **(8)** (b) **(9)** (d) **(10)** (b)
(11) (a) **(12)** (c) **(13)** (b) **(14)** (d) **(15)** (d) **(16)** (b) **(17)** (b) **(18)** (a) **(19)** (c) **(20)** (d)
(21) (d) **(22)** (c) **(23)** (a) **(24)** (c) **(25)** (d) **(26)** (a) **(27)** (c) **(28)** (a) **(29)** (b) **(30)** (c)
(31) (b) **(32)** (b) **(33)** (b) **(34)** (d) **(35)** (d) **(36)** (c) **(37)** (a) **(38)** (a) **(39)** (b) **(40)** (c)
(41) (d) **(42)** (b) **(43)** (b) **(44)** (a) **(45)** (a) **(46)** (a) **(47)** (a) **(48)** (b) **(49)** (a) **(50)** (b)
(51) (b) **(52)** (a) **(53)** (b) **(54)** (a) **(55)** (b)

Exercise 2.1

1. $\frac{x^3}{3}\log x - \frac{1}{9}x^3 + c$

2. $\left(\frac{3+5x}{7}\right)\sin 7x + \frac{5}{49}\cos 7x + c$

3. $x\cos^{-1}x - \sqrt{1-x^2} + c$

4. $e^{3x}\left[\frac{x^2}{3} - \frac{2}{9}x + \frac{2}{27}\right] + c$

5. $\frac{x^3}{3}\tan^{-1}x - \frac{1}{6}[x^2 - \log(1+x^2)] + c$

6. $x\,\mathrm{cosec}^{-1}x + \log\left|x + \sqrt{x^2-1}\right| + c$

7. $\frac{x}{2}[\sin(\log x) - \cos(\log x)] + c$

8. $\frac{1}{2}\left[\sec x\tan x + \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|\right] + c$

9. $-x\cot\frac{x}{2} + 2\log\left|\sin\frac{x}{2}\right| + c$

10. $-\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2 + c$

11. $2x\tan^{-1}x - \log(1+x^2) + c$

12. $-\frac{1}{2}(x\,\mathrm{cosec}^2x + \cot x) + c$

13. $\frac{3}{4}(x\sin x + \cos x) + \frac{1}{12}\left(x\sin 3x + \frac{1}{3}\cos 3x\right) + c$

14. $\frac{1}{n}(x^n\sin x^n + \cos x^n) + c$

15. $\left(x-\frac{x^3}{3}\right)\log x - x + \frac{x^3}{9} + c$

16. $-\frac{\log x}{x+1} + \log\left(\frac{x}{x+1}\right) + c$

17. $-\frac{\sin^{-1}x}{x} + \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c$

18. $2(\sqrt{x} - \sqrt{1-x}\ \sin^{-1}\sqrt{x}) + c$

Exercise 2.2

1. $\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$

2. $\sqrt{2}\left[\frac{x}{2}\sqrt{x^2+5} + \frac{5}{2}\log\left|x+\sqrt{x^2+5}\right|\right] + c$

3. $\frac{x}{2}\sqrt{5x^2-3} - \frac{3}{2\sqrt{5}}\log\left|\sqrt{5}x + \sqrt{5x^2-3}\right| + c$

4. $\frac{4x+3}{8}\sqrt{4-3x-2x^2} + \frac{41}{16\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$

5. $\frac{1}{2}\left[\frac{2x+1}{2}\sqrt{4x^2+4x-15} - 8\log\left|2x+1+\sqrt{4x^2+4x-15}\right|\right] + c$

6. $\frac{1}{3}\left[\frac{x^3}{2}\sqrt{8-x^6} + 4\sin^{-1}\frac{x^3}{2\sqrt{2}}\right] + c$

7. $\frac{\sin x}{2}\sqrt{4-\sin^2 x} + 2\sin^{-1}\left(\frac{\sin x}{2}\right) + c$

8. $e^x \log \sin x + c$

9. $-e^x \cot\frac{x}{2} + c$

10. $\frac{e^{2x}}{2}\tan x + c$

11. $e^x\left(\frac{x-2}{x+2}\right) + c$

12. $\frac{e^x}{\sqrt{x^2+1}} + c$

13. $\frac{e^x}{1+x^2} + c$

14. $-\frac{1}{3}(1+x-x^2)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$

15. $(x^2+x+1)^{\frac{3}{2}} - \frac{7(2x+1)}{8}\sqrt{x^2+x+1} - \frac{21}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + c$

16. $-\frac{2}{3}(2+3x-x^2)^{\frac{3}{2}} - \frac{2x-3}{2}\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$

17. $\frac{e^{2x}}{10}(\sin 4x - 2\cos 4x) + c$

18. $-e^{-\frac{x}{2}} + \frac{e^{-\frac{x}{2}}}{17}(-\cos 2x + 4\sin 2x) + c$

19. $\frac{3^x}{2\log 3} - \frac{3^x}{2(4+(\log 3)^2)}((\log 3)\cos 2x + 2\sin 2x) + c$

20. $\frac{e^{2x}}{8}(\sin 2x + \cos 2x) - \frac{e^{2x}}{20}(\cos 4x + 2\sin 4x) + c$

Exercise 2.3

1. $\log\left|\frac{x(x-1)^2}{(x+1)^2}\right| + c$

2. $\frac{5}{2}\log|x-1| - 8\log|x-2| + \frac{11}{2}\log|x-3| + c$

3. $\frac{x^2}{2} - x - 2\log|x-2| + \log|x-3| + c$

4. $\frac{1}{3\sqrt{2}}\ \tan^{-1}(\sqrt{2}\,x) + \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + c$

5. $\frac{1}{3\sqrt{2}}\ \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{3\sqrt{2}}\ \tan^{-1}(\sqrt{2}\,x) + c$

6. $\frac{5}{6}\log(x^2+5) - \frac{1}{3}\log(x^2+2) + c$

7. $-2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + c$

8. $-\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\left(\frac{x}{3}\right) + c$

9. $x + 2\log|2e^x+1| - 3\log|3e^x+1| + c$

10. $\frac{1}{2}\log\left|\frac{\tan\theta-3}{\tan\theta-1}\right| + c$

11. $\frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log(x^2+1) + c$

12. $-\frac{1}{8}\log|x+1| + \frac{1}{8}\log|x-1| - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2} + c$

13. $-\frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + c$

14. $\frac{1}{10}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + \frac{2}{5}\log|3+2\cos x| + c$

Exercise 2

1. $\frac{x^3}{3}\sin^{-1}x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{\frac{3}{2}} + c$

2. $\frac{x}{2}\cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + c$

3. $-x\cot\frac{x}{2} + c$

4. $\frac{1}{2}\log\left|\frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}\right| - \tan^{-1}(\sqrt{\sin x}) + c$

5. $x\log\left|x+\sqrt{x^2+a^2}\right| - \sqrt{x^2+a^2} + c$

6. $(x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$

7. $2\sqrt{x} - 2\sqrt{1-x}\,\sin^{-1}\sqrt{x} + c$

8. $\frac{1}{2}e^x\sec x + c$

9. $\frac{x}{\log x} + c$

10. $x\log(\log x) + c$

11. $-\frac{1}{3}(2ax-x^2)^{\frac{3}{2}} + \frac{a(x-a)}{2}\sqrt{2ax-x^2} + \frac{a^3}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + c$

12. $\frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{11}{8}(2x+1)\sqrt{x^2+x} + \frac{11}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$

13. $\frac{1}{2}\log\left|\frac{\sin x-1}{\sin x+1}\right| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right| + c$

14. $\frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|2\cos x+1| + c$

15. $\frac{1}{8}\log\left|\frac{\sin x-1}{\sin x+1}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right| + c$

16. $x\tan^{-1}x + x\tan^{-1}(1-x) + \frac{1}{2}\log|x^2-2x+2| + \tan^{-1}(x-1) - \frac{1}{2}\log(x^2+1) + c$

17. $\frac{2}{\sqrt{\cos x}} - \frac{1}{2}\log\left|\frac{\sqrt{\cos x}+1}{\sqrt{\cos x}-1}\right| + \tan^{-1}(\sqrt{\cos x}) + c$

18. $\frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{2(1+\sin x)} + c$

19. $\frac{1}{2}\log\left|\tan\frac{x}{2}\right| + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + c$

20. (1) (a) (2) (b) (3) (c) (4) (a) (5) (c) (6) (c) (7) (a) (8) (d) (9) (b) (10) (b) (11) (a)

Exercise 3.1

1. 8
2. 10
3. $\frac{94}{3}$
4. $\frac{38}{3}$
5. $e - e^{-1}$
6. $\frac{1}{3}(e^2 - e^{-1})$
7. $6\log_3 e$
8. 3
9. $e^2 - 3$
10. $2\log_a e$
11. 26
12. $\sin b - \sin a$
13. 2
14. 1
15. 20

Exercise 3.2

1. $\frac{1}{3}\cdot 2^{\frac{5}{2}}$
2. $\left(1-\frac{\pi}{4}\right)$
3. $\frac{\pi}{4}$
4. $\frac{1}{2}\log 2$
5. $\sqrt{2}$
6. $\sqrt{2} - 1$
7. $\frac{\pi}{2}$
8. $\frac{1}{5}\log 6$
9. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$
10. $2 - \frac{\pi}{2}$
11. $\frac{\pi}{3\sqrt{3}}$
12. $6 - 4\log 2$
13. $\frac{\pi}{6}$
14. $\tan^{-1}e - \frac{\pi}{4}$
15. $\frac{\pi}{4} - \frac{1}{2}\log 2$
16. $\frac{\pi}{3}$
17. $-\frac{\pi}{4}$
18. $\frac{1}{2} - \frac{\sqrt{3}\pi}{12}$
19. $\tan^{-1}\frac{1}{3}$
20. $\frac{1}{\sqrt{10}}\tan^{-1}\sqrt{\frac{2}{5}}$
21. $\frac{\pi}{2} - 1$
22. $\frac{\pi}{2} - 1$
23. $\frac{1}{2}\log\left(\frac{32}{27}\right)$
24. $\frac{1}{2}(\sqrt{2} - 1) + \frac{1}{2}\log(\sqrt{2} + 1)$
25. $\frac{1}{2}\log\frac{8}{5}$
26. $\frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}$
27. $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$
28. $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$
29. 4
30. 47
31. $e^4 + 5 - \frac{\pi}{2}$
32. $\frac{13}{10}$
33. 4
34. 0
35. 0
36. 2
37. $\frac{1}{2}$
38. $\frac{9}{2}$

Exercise 3.3

1. (1) 0 (2) 0 (3) 0 (4) 0 (5) $\frac{\pi}{2}$ (6) 2

2. (1) 0 (2) 0

Exercise 3

3. **(1)** $\frac{\pi^2}{4}$ **(2)** π **5.** -64 **8.** $\frac{1}{2}(1 - \log 2)$ **9.** $\frac{1}{2ab} \log \left|\frac{a+b}{a-b}\right|$

10. $\frac{1}{\sqrt{2}} \tan^{-1}\frac{3}{2\sqrt{2}}$ **11.** 0 **12.** $\frac{\pi}{6\sqrt{3}} - \frac{1}{2} \log \frac{3}{2}$ **13.** $\frac{\pi}{4} - \frac{1}{2} \log 2$ **14.** $\frac{\pi}{8} \log 2$ **15.** $\frac{2}{3} + \log\left(\frac{2}{3}\right)$

16. $\frac{\pi^2}{4}$ **17.** $2(\sqrt{2} - 1)$ **18.** $\frac{38}{3}$ **19.** $\frac{15 + e^8}{2}$

22. **(1)** (c) **(2)** (a) **(3)** (a) **(4)** (c) **(5)** (a) **(6)** (b) **(7)** (c) **(8)** (a) **(9)** (b) **(10)** (b)
(11) (a) **(12)** (a) **(13)** (b) **(14)** (b) **(15)** (d) **(16)** (b) **(17)** (d) **(18)** (a) **(19)** (b) **(20)** (c)

Exercise 4.1

1. $\frac{13}{3}$ **2.** 9 **3.** 3 **4.** $\frac{136}{3}$ **5.** $\frac{32}{3}$ **6.** 36 **7.** πa^2 **8.** $\frac{32}{3}$

Exercise 4.2

1. 27 **2.** $\frac{9}{2}$ **3.** $\frac{4}{\pi}$ **4.** $\frac{64}{3}$ **5.** $\frac{5}{6}$ **6.** $\frac{32}{3}$ **7.** $\frac{19}{6}$ **8.** $\frac{64}{3}$ **9.** 8 **10.** $\frac{15}{2}$

11. 4π **12.** $\frac{20}{3}(\sqrt{5} - 2)$

Exercise 4

1. $\frac{125}{6}$ **2.** $\frac{2}{3}$ **3.** $\frac{1}{6}$ **4.** $\frac{\pi}{4}$ **5.** $\frac{8}{3}$ **6.** $\frac{9}{8}$ **7.** $\frac{4}{3}(8 + 3\pi)$ **8.** $\frac{13}{3}$ **10.** $\frac{23}{6}$

11. $\frac{8\pi}{3} - 2\sqrt{3}$ **12.** $\frac{32}{3}$ **13.** 2 **14.** $\frac{5\pi}{4} - \frac{1}{2}$ **15.** $\frac{9}{2}$ **16.** $\frac{4}{3}$

17. **(1)** (c) **(2)** (d) **(3)** (c) **(4)** (b) **(5)** (c) **(6)** (c) **(7)** (b) **(8)** (d) **(9)** (c) **(10)** (b)
(11) (a) **(12)** (b) **(13)** (d) **(14)** (a) **(15)** (b) **(16)** (d) **(17)** (d) **(18)** (c) **(19)** (a) **(20)** (b)

Exercise 5.1

1.

Sr. No.	Order	Degree
1	2	1
2	1	4
3	2	Undefined
4	1	1
5	3	2
6	2	2
7	1	2
8	3	2
9	2	1
10	2	3

Exercise 5.2

1. $(x-y)^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=\left(x+y\frac{dy}{dx}\right)^2$

2. $\frac{d^2y}{dx^2}=0$ **3.** $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$ **4.** $(x^2-y^2)\frac{dy}{dx}=2xy$

9. (1) $\frac{d^2y}{dx^2}=0$ (2) $x\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)=y\frac{dy}{dx}$ (3) $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$

(4) $x^2y_2+xy_1-y=0$ (5) $x\frac{dy}{dx}=3y$ (6) $y_2=4(y_1-y)$ (7) $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\frac{dy}{dx}$

Exercise 5.3

1. (1) $2y^3+3y^2=3x^2+6\log|x|+c$ (2) $e^y=(y+1)(e^x+1)\,c$

(3) $\sin y=c\cos x$ (4) $\log|y|=\log|\sec x|-\tan x+c$

(5) $(e^y+1)\sin x=c$ (6) $\tan^{-1}y=x+\frac{x^3}{3}+c$

(7) $x=c\log y$ (8) $\frac{1}{y}=2x^2+1$

(9) $y=x^2+\log x$ (10) $8e^y=x(y+2)^2$

(11) $4e^x+\frac{1}{y^2}=8$ (12) $x\sec y=2$

(13) $y=(x+1)\log(x+1)-x+3$ (14) $y=\sin^{-1}a\cdot x+1$

(15) $y=\sec x$ (16) $y=\frac{e^x}{x+1}+c$

2. (1) $\tan(x+y)-\sec(x+y)=x+c$ (2) $c(x-y+2)=e^{2y-x}$

(3) $(x+y+2)\,c=e^{y+1}$ (4) $c(e^{x+y}+1)=e^y$

(5) $y-a\tan^{-1}\left(\frac{x+y}{a}\right)=c$

Exercise 5.4

1. (1) $(x-y)^2=cx\,e^{-\frac{y}{x}}$ (2) $\sec\frac{y}{x}=xyc$

(3) $x\tan\frac{y}{2x}=c$ (4) $e^{\frac{x}{y}}=y+c$

(5) $-\cos\frac{y}{x}=\log x+c$ (6) $2e^{\frac{x}{y}}=\log\frac{c}{y}$

(7) $\frac{\sqrt{2}y+x}{\sqrt{2}y-x}=cx^{2\sqrt{2}}$ (8) $ye^{\frac{x}{y}}+x=c$

(9) $e^{\frac{y}{x}}=xc$ (10) $y\left(\log\frac{y}{x}-1\right)=c$

(11) $-e^{-\frac{y}{x}}=\log xc$ (12) $yx^2=c(y+2x)$

(13) $\sin\frac{y}{x}=xc$

2. **(1)** $x^2(x^2 + 2y^2) = 3$

(2) $e^{-\frac{y}{x}} = \log x$

(3) $e^{\cos\frac{y}{x} - 1} = x$

(4) $xe^{\frac{y^2}{x^2}} = e$

(5) $x = e^{1 - \frac{2x}{y}}$

(6) $e^{\frac{y}{x+y}} = x$

Exercise 5.5

1. $y = \frac{1}{5}[2\sin x - \cos x] + ce^{-2x}$

2. $y = -e^{-x} + cx$

3. $\frac{y}{x} = \log x + c$

4. $\frac{y}{1+x^2} = x + c$

5. $y + x + 1 = ce^x$

6. $yx^2 = e^x(x^2 - 2x + 2) + c$

7. $y = -\frac{5}{4}e^{-3x} + ce^{-2x}$

8. $(1 + x^2)y = \frac{4x^3}{3} + c$

9. $xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

10. $y\log x = -\frac{2}{x}(1 + \log x) + c$

11. $y = (\cot x + 1) + ce^{\cot x}$

12. $\frac{x}{y} = 2y + c$

Exercise 5.6

1. $y = ce^{-\frac{x}{4y}}$

2. 16 times, 3000

3. $x^2 = -\frac{9}{4}y$

4. 14 years, 6.9 %

5. $m_0 = 125\ g$

6. $y^2 = 2kx$, (k is arbitrary constant)

7. $y^2 - x^2 = 3$

Exercise 5

5. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx}$

6. (1) $1 + \tan\left(\frac{x+y}{2}\right) = ce^x$

(2) $y(x^2 + 1)^2 = \tan^{-1}x + c$

(3) $2e^{\frac{x}{y}} = \log\left|\frac{c}{y}\right|, y \neq 0$

(4) $x^2(x^2 - 2y^2) = c$

(5) $x^2 + y^2 = 2x$

(6) $y = \tan x - 1 + ce^{-\tan x}$

7. **(1)** (b) **(2)** (a) **(3)** (b) **(4)** (c) **(5)** (b) **(6)** (c) **(7)** (c) **(8)** (b) **(9)** (d) **(10)** (a) **(11)** (b) **(12)** (c) **(13)** (d) **(14)** (a) **(15)** (a) **(16)** (b)

Exercise 6.1

1. **(1)** 4 **(2)** 5 **(3)** $3\sqrt{2}$

2. $\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$

3. $(6, -4, -4)$

4. $12\hat{i} - 8\sqrt{3}\hat{j} + 8\hat{k}$

5. $\frac{7}{\sqrt{110}}\hat{i} + \frac{6}{\sqrt{110}}\hat{j} - \frac{5}{\sqrt{110}}\hat{k}$

6. Scalar components are 3, −6, 7; Vector components are $3\hat{i}, -6\hat{j}, 7\hat{k}$

7. **(i)** 5 **(ii)** 5 **(iii)** $5\sqrt{2}$

Exercise 6.2

1. 5 **2.** $(-7, 3, 5)$ **3.** $(-6, -24, 6)$ **4.** $8\sqrt{3}$

5. 3 **6.** $(-5, 5, 0)$ **7.** -2 **8.** 0

9. $(11, -11, 11)$ **10.** $7\sqrt{6}$

Exercise 6

6. $a = 1, b = -1, c = 2$

9. **(1)** $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}; \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$ **(2)** $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}; 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

(3) $cos^{-1}\frac{1}{17};\ cos^{-1}\frac{12}{85};\ cos^{-1}\frac{84}{85};\ \frac{1}{17}, \frac{12}{85}; \frac{84}{85}$

11. $\left(\frac{12}{13}, \frac{5}{13}\right)$ or $\left(-\frac{12}{13}, -\frac{5}{13}\right)$ **16.** $\pm 2\sqrt{3}$ **18.** $2\sqrt{91}$ **24.** $\frac{7\sqrt{6}}{2}$

25. $(2, -2, 2), 2\sqrt{3}$ **26.** $b\hat{j}; |b|$ **27.** $\left(\frac{56}{99}, \frac{-56}{99}, \frac{8}{99}\right)$

30. $\left(\frac{8}{3}, \frac{5}{3}, \frac{4}{3}\right) + \left(\frac{-2}{3}, \frac{4}{3}, \frac{-1}{3}\right) = (2, 3, 1)$ **31.** $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$ **33.** $\left(\frac{5}{3}, \frac{2}{3}, \frac{2}{3}\right)$ **34.** 10

36. **(1)** (b) **(2)** (d) **(3)** (c) **(4)** (b) **(5)** (c) **(6)** (b) **(7)** (b) **(8)** (d) **(9)** (a) **(10)** (a)
(11) (a) **(12)** (c) **(13)** (c) **(14)** (a) **(15)** (c) **(16)** (b) **(17)** (b) **(18)** (a) **(19)** (c) **(20)** (d)
(21) (c) **(22)** (d) **(23)** (a) **(24)** (b) **(25)** (c)

Exercise 7.1

1. $\vec{r} = (2, -1, 3) + k(2, -3, 4), k \in R;\ \frac{x-2}{2} = \frac{y+1}{-3} = \frac{z-3}{4}$

2. $\frac{x-2}{2} = \frac{z+9}{4}, y - 3 = 0;\ \vec{r} = (2, 3, -9) + k(2, 0, 4), k \in R$

3. Non-collinear **4.** $\frac{4}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}}$

5. $\vec{r} = (1, -2, 1) + k\left(\frac{1}{3}, \frac{-1}{2}, 1\right), k \in R;\ \frac{3(x-1)}{1} = \frac{2(y+2)}{-1} = \frac{z-1}{1}$

6. (4, 0, −1) **7.** $\cos^{-1}\dfrac{17}{5\sqrt{14}}$ **9.** **(1)** Skew **(2)** Parallel **(3)** Skew **(4)** Intersecting **(5)** Parallel

10. $\dfrac{107}{\sqrt{1038}}$ **11.** $\dfrac{\sqrt{457}}{5}$ **12.** $\sqrt{\dfrac{118}{3}}$

Exercise 7.2

1. $\frac{1}{\sqrt{21}}(4, -2, 1)$ **2.** $\bar{r} \cdot (2, 2, -1) = 5;\ 2x + 2y - z = 5$ **3.** $2x - 3y - 5z = 11$

4. $x + 2y - 3 = 0;\ 3, \frac{3}{2}$, not defined **5.** $6x - y - 5z = 7$ **6.** $13x - 7y - 37z + 134 = 0$

7. $x - y + 1 = 0$ **8.** $\frac{\pi}{3}$ **9.** $\sin^{-1}\left(\frac{5}{\sqrt{238}}\right)$ **10.** $\frac{12}{\sqrt{38}}$ **11.** $\frac{19}{14}$

12. $2x - 5y - 4z + 33 = 0$ **13.** $55x - 2y + 13z = 40$ **14.** $x - y - z - 1 = 0$

Exercise 7

1. $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$, $\bar{r} = (1, 0, 3) + k(2, 7, 8),\ k \in R;\ \sqrt{13}$ **2.** $\frac{\pi}{3}$ **3.** $\frac{7}{\sqrt{74}}$ **4.** $(-3, 5, 1), \frac{\pi}{2}$

5. $\frac{x-1}{11} = \frac{y-2}{-2} = \frac{z-3}{7}$ **6.** $\frac{x-3}{1} = \frac{y+2}{1} = \frac{z+4}{1}$ **7.** $(3, -1, 1);\ \sin^{-1}\frac{12}{\sqrt{609}}$

8. $\frac{y}{2} + \frac{z}{3} = 1$ **9.** (2, 3, 2) **10.** $(4, -4, 6);\ \frac{x}{2} = \frac{y-2}{-3} = \frac{z+2}{4};\ 2\sqrt{29}$

11. $4x + 7y - 5z - 8 = 0;\ \frac{x-2}{4} = \frac{y}{-3} = \frac{z}{-1}$ **12.** $x + 2y - 2z = 6$ **13.** $2x + 16y - 13z - 22 = 0$

14. $3x + 4y - 6z = 6$ **15.** $8x - 8y - 14z = -47$

16. **(1)** (c) **(2)** (c) **(3)** (a) **(4)** (b) **(5)** (d) **(6)** (c) **(7)** (c) **(8)** (b) **(9)** (a) **(10)** (c) **(11)** (c) **(12)** (d) **(13)** (b) **(14)** (a) **(15)** (b) **(16)** (b) **(17)** (d)

TERMINOLOGY

(In Gujarati)

Approximate Value	આસન્ન મૂલ્ય
Box Product	પેટીગુણન
Coincident	સંપાતી
Collinear Vectors	સમરેખ સદિશો
Component	ઘટક
Coplanar Vectors	સમતલીય સદિશો
Coplanar	સમતલીય
Definite Integration	નિયત સંકલન
Degree	પરિમાણ
Dependent Variable	અવલંબી ચલ
Differential Equation	વિકલ સમીકરણ
Direction Angles	દિક્ખૂણા
Direction Cosines	દિક્કોસાઈન
Direction of Line	રેખાની દિશા
Direction Ratios	દિક્ગુણોત્તર (દિક્ સંખ્યાઓ)
Error	ત્રુટિ
Free Vector	મુક્ત સદિશ
Global	વૈશ્વિક
Having same Direction	સમદિશ
Homogeneous	સમપરિમાણ
Improper Rational Function	અનુચિત સંમેય વિધેય
Independent Variable	સ્વતંત્ર ચલ
Initial Condition	પ્રારંભિક શરત
Inner Product	અંતઃ ગુણન
Integrating Factor (I.F.)	સંકલ્યકારક અવયવ
Integration by Parts	ખંડશઃ સંકલન
Linear Combination	સુરેખ સંયોજન
Linear Differential Equation	સુરેખ વિકલ સમીકરણ
Lower Limit	અધઃસીમા
Monotonic	એકસૂત્રી

Normal	અભિલંબ
Opposite Direction	વિરુદ્ધ દિશા
Order	કક્ષા
Outer Product of Vectors	સદિશોનું બહિર્ગુણન
Parallelopiped	સમાંતર ફલક
Particular Solution	વિશિષ્ટ ઉકેલ
Perpendicular Bisector Plane	લંબદ્વિભાજક સમતલ
Projection Vector	પ્રક્ષેપ સદિશ
Proper Rational Function	ઉચિત સંમેય વિધેય
Rate	દર
Scalar Product	અદિશ ગુણાકાર
Singular Solution	અસામાન્ય ઉકેલ
Skew Lines	વિષમતલીય રેખાઓ
Strictly Decreasing Function	ચુસ્ત ઘટતું વિધેય
Strictly Increasing Function	ચુસ્ત વધતું વિધેય
Subnormal	અવાભિલંબ
Subtangent	અવસ્પર્શક
Symmetric Form	સંમિત સ્વરૂપ
Tangent	સ્પર્શક
Triangle Inequality	ત્રિકોણીય અસમતા
Upper Limit	ઉર્ધ્વસીમા
Variable Separable	વિયોજનીય ચલ
Vector Product	સદિશ ગુણાકાર
Vector Triple Produc t	સદિશનું ત્રિગુણન
Vector	સદિશ

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