2.1 Functions

If A and B are two non-empty sets, then a rule f which associated to each $x \in A,$ a unique number $y \in B,$ is called a function from A to B and we write, $f: A \to B$.

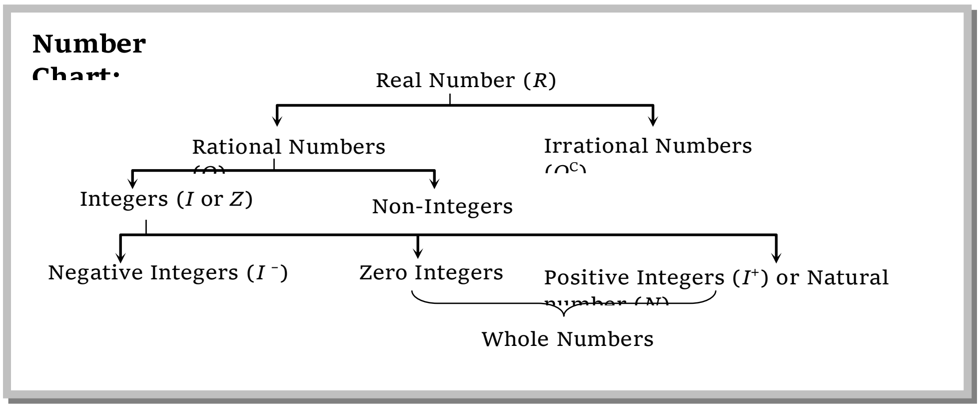
2.1.1 Some Important Definitions

(1) **Real numbers :** Real numbers are those which are either rational or irrational. The set of real numbers is denoted by R.

(i) **Rational numbers :** All numbers of the form p/q where p and q are integers and $q \neq 0$, are called rational numbers and their set is denoted by Q. *e.g.* $\frac{2}{3}, -\frac{5}{2}, 4\left(\text{as} \quad 4 = \frac{4}{1}\right)$ are rational numbers.

(ii) **Irrational numbers :** Those are numbers which can not be expressed in form of p/q are called irrational numbers and their set is denoted by Q^c (*i.e.,* complementary set of Q) *e.g.* $\sqrt{2}, \ 1-\sqrt{3}, \ \pi$ are irrational numbers.

(iii) **Integers :** The numbers ....– 3, – 2, – 1, 0, 1, 2, 3, ... are called integers. The set of integers is denoted by I or Z. Thus, I or Z ={.......,– 3, – 2, –1, 0, 1, 2, 3,......}



Note :
- ❒ Set of positive integers I^+ = {1, 2, 3, ...}
- ❒ Set of negative integers I^- = {– 1, – 2, – 3, ...}.
- ❒ Set of non negative integers ={0, 1, 2, 3, ..}
- ❒ Set of non positive integers = {0, – 1, – 2, – 3,.....}
- ❒ Positive real numbers: $R^+ = (0, \infty)$
- ❒ Negative real numbers: $R^- = (-\infty, 0)$
- ❒ R_0 : all real numbers except 0 (Zero)
- ❒ Imaginary numbers: $C = \{i, \omega, ...\}$

- Even numbers: $E = \{0, 2, 4, 6, ...\}$
- Odd numbers: $0 = \{1, 3, 5, 7, ...\}$
- Prime numbers : The natural numbers greater than 1 which is divisible by 1 and itself only, called prime numbers.
- In rational numbers the digits are repeated after decimal
- 0 (zero) is a rational number
- In irrational numbers, digits are not repeated after decimal
- π and e are called special irrational quantities
- ∞ is neither a rational number nor an irrational number

(2) **Related quantities :** When two quantities are such that the change in one is accompanied by the change in other, *i.e.,* if the value of one quantity depends upon the other, then they are called related quantities. *e.g.* the area of a circle $(A = \pi r^2)$ depends upon its radius (r) as soon as the radius of the circle increases (or decreases), its area also increases (or decreases). In the given example, A and r are related quantities.

(3) **Variable:** A variable is a symbol which can assume any value out of a given set of values. The quantities, like height, weight, time, temperature, profit, sales etc, are examples of variables. The variables are usually denoted by x, y, z, u, v, w, t etc. There are two types of variables mainly:

(i) **Independent variable :** A variable which can take any arbitrary value, is called independent variable.

(ii) **Dependent variable :** A variable whose value depends upon the independent variable is called dependent variable. *e.g.* $y = x^2$, if $x = 2$ then $y = 4 \Rightarrow$ so value of y depends on x. y is dependent and x is independent variable here.

(4) **Constant :** A constant is a symbol which does not change its value, *i.e.*, retains the same value throughout a set of mathematical operation. These are generally denoted by a, b, c etc. There are two types of constant.

(i) **Absolute constant :** A constant which remains the same throughout a set of mathematical operation is known as absolute constant. All numerical numbers are absolute constants, *i.e.* $2, \sqrt{3}, \pi$ etc. are absolute constants.

(ii) **Arbitrary constant :** A constant which remains same in a particular operation, but changes with the change of reference, is called arbitrary constant *e.g.* $y = mx + c$ represents a line. Here m and c are constants, but they are different for different lines. Therefore, m and c are arbitrary constants.

(5) **Absolute value :** The absolute value of a number x, denoted by $|x|$, is a number that satisfies the conditions

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}.$$ We also define |*x*| as follows, |*x*|= maximum {*x*, – *x*} or $|x| = \sqrt{x^2}$

The properties of absolute value are

(i) The inequality $|x| \le a$ means $-a \le x \le a$

(ii) The inequality $|x| \ge a$ means $x \ge a$ or $x \le -a$

(iii) $|x \pm y| \le |x| + |y|$ and $|x \pm y| \ge |x| - |y|$

(iv) $|xy| = |x||y|$

(v) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \ne 0$

(6) **Greatest integer:** Let $x \in R$. Then [*x*] denotes the greatest integer less than or equal to *x*; *e.g.* [1.34]=1, [– 4.57]= – 5, [0.69] = 0 etc.

(7) **Fractional part :** We know that $x \ge [x]$. The difference between the number '*x*' and it's integral value '[*x*]' is called the fractional part of *x* and is symbolically denoted as {*x*}. Thus, $\{x\} = x - [x]$

e.g., if *x* = 4.92 then [*x*] = 4 and {*x*}= 0.92.

Note : ❒ Fractional part of any number is always non-negative and less than one.

2.1.2 Intervals

If a variable *x* assumes any real value between two given numbers, say *a* and *b* (*a*<*b*) as its value, then *x* is called a continuous variable. The set of real numbers which lie between two specific numbers, is called the interval.

There are four types of interval:

(1) **Open interval :** Let *a* and *b* be two real numbers such that *a*<*b*, then the set of all real numbers lying strictly between *a* and *b* is called an open interval and is denoted by]*a*, *b*[or (*a*, *b*). Thus,]*a*, *b*[or (*a*, *b*) = $\{x \in R : a < x < b\}$ *a*<*x*<*b* (*a*) *b* Open	(2) **Closed interval :** Let *a* and *b* be two real numbers such that *a*<*b*, then the set of all real numbers lying between *a* and *b* including *a* and *b* is called a closed interval and is denoted by [*a*, *b*]. Thus, [*a*, *b*] = $\{x \in R : a \le x \le b\}$ $a \le x \le b$ [*a*] *b* Closed
(3) **Open-Closed interval :** It is denoted by]*a*, *b*] or (*a*, *b*] and]*a*, *b*] or (*a*, *b*] = $\{x \in R : a < x \le b\}$ $a < x \le b$ (*a*] *b* Open closed	(4) **Closed-Open interval :** It is denoted by [*a*, *b*[or [*a*, *b*) and [*a*, *b*[or [*a*, *b*) = $\{x \in R : a \le x < b\}$ $a \le x < b$ [*a*) *b* Closed open

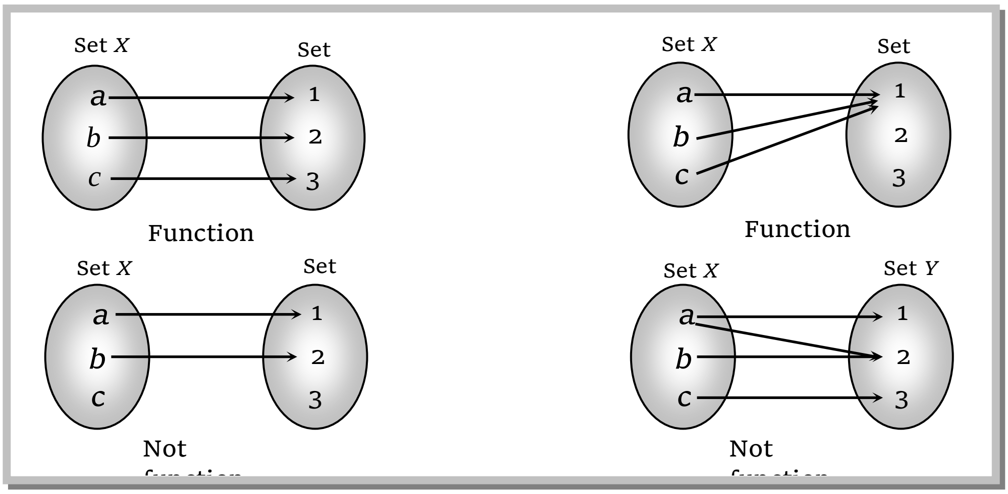
2.1.3 Definition of Function

(1) Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f: X \to Y$ where $y = f(x), x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre image of y).

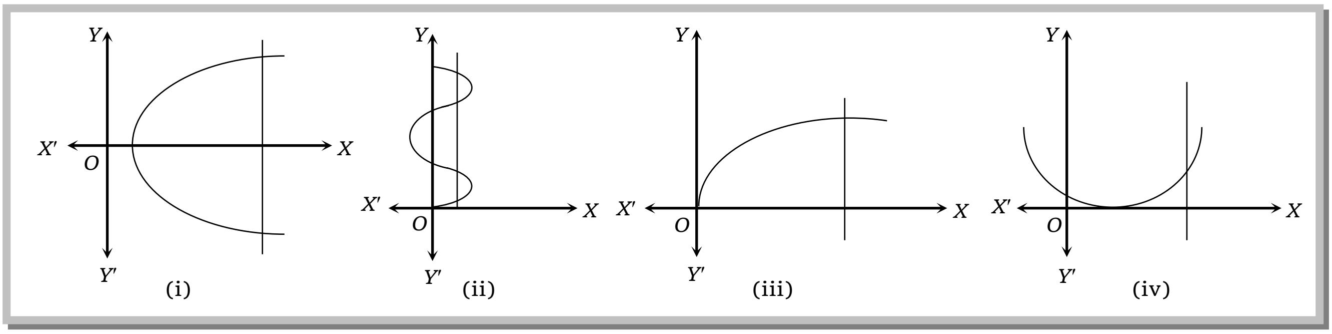
Two things should always be kept in mind:

(i) A mapping $f: X \to Y$ is said to be a function if each element in the set X has it's image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.

(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can not be multi-valued (A mapping that is multi-valued is called a relation from X and Y) *e.g.*



(2) **Testing for a function by vertical line test :** A relation $f: A \to B$ is a function or not it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more than one point then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.



(3) **Number of functions :** Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

(4) **Value of the function :** If $y = f(x)$ is a function then to find its values at some value of x, say $x = a$, we directly substitute $x = a$ in its given rule $f(x)$ and it is denoted by $f(a)$.

e.g. If $f(x) = x^2 + 1$, then $f(1) = 1^2 + 1 = 2$, $f(2) = 2^2 + 1 = 5$, $f(0) = 0^2 + 1 = 1$ *etc.*

Example: 1 If A contains 10 elements then total number of functions defined from A to A is

(a) 10 (b) 2^{10} (c) 10^{10} (d) $2^{10} - 1$

Solution: (c) According to formula, total number of functions = n^n

Here, n = 10. So, total number of functions = 10^{10}.

Example: 2 If $f(x) = \frac{x - |x|}{|x|}$, then $f(-1) =$ **[SCRA 1996]**

(a) 1 (b) – 2 (c) 0 (d) 2

Solution: (b) $f(-1) = \frac{-1 - |-1|}{|-1|} = \frac{-1-1}{1} = -2$.

Example: 3 If $f(y) = \log y$, then $f(y) + f\left(\frac{1}{y}\right)$ is equal to **[Rajasthan PET 1996]**

(a) 2 (b) 1 (c) 0 (d) – 1

Solution: (c) Given $f(y) = \log y \Rightarrow f(1/y) = \log(1/y)$, then $f(y) + f\left(\frac{1}{y}\right) = \log y + \log(1/y) = \log 1 = 0$.

Example: 4 If $f(x) = \log\left[\frac{1+x}{1-x}\right]$, then $f\left[\frac{2x}{1+x^2}\right]$ is equal to **[MP PET 1999; Rajasthan PET 1999; UPSEAT 2003]**

(a) $[f(x)]^2$ (b) $[f(x)]^3$ (c) $2f(x)$ (d) $3f(x)$

Solution: (c) $f(x) = \log\left(\frac{1+x}{1-x}\right)$

$$\therefore f\left(\frac{2x}{1+x^2}\right) = \log\left[\frac{1 + \frac{2x}{1+x^2}}{1 - \frac{2x}{1+x^2}}\right] = \log\left[\frac{x^2+1+2x}{x^2+1-2x}\right] = \log\left[\frac{1+x}{1-x}\right]^2 = 2\log\left[\frac{1+x}{1-x}\right] = 2f(x)$$

Example: 5 If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, then **[Orissa JEE 2002]**

(a) $f\left(\frac{\pi}{4}\right) = 2$ (b) $f(-\pi) = 2$ (c) $f(\pi) = 1$ (d) $f\left(\frac{\pi}{2}\right) = -1$

Solution: (d) $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$$f(x) = \cos(9x) + \cos(-10x) = \cos(9x) + \cos(10x) = 2\cos\left(\frac{19x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{19\pi}{4}\right)\cos\left(\frac{\pi}{4}\right); \; f\left(\frac{\pi}{2}\right) = 2 \times \frac{-1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = -1 .$$

Example: 6 If $f : R \to R$ satisfies $f(x+y) = f(x) + f(y)$, for all $x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is

(a) $\frac{7n}{2}$ (b) $\frac{7(n+1)}{2}$ (c) $7n(n+1)$ (d) $\frac{7n(n+1)}{2}$

Solution: (d) $f(x+y) = f(x) + f(y)$

put $x = 1, y = 0 \Rightarrow f(1) = f(1) + f(0) = 7$

put $x = 1, y = 1 \Rightarrow f(2) = 2.f(1) = 2.7$; similarly $f(3) = 3.7$ and so on

$$\therefore \sum_{r=1}^{n} f(r) = 7(1 + 2 + 3 + ... + n) = \frac{7n(n+1)}{2}.$$

Example: 7 If $f(x) = \frac{1}{\sqrt{x + 2\sqrt{2x-4}}} + \frac{1}{\sqrt{x - 2\sqrt{2x-4}}}$ for $x > 2$, then $f(11) =$ **[EAMCET 2003]**

(a) $\frac{7}{6}$ (b) $\frac{5}{6}$ (c) $\frac{6}{7}$ (d) $\frac{5}{7}$

Solution: (c) $f(x) = \frac{1}{\sqrt{x + 2\sqrt{2x-4}}} + \frac{1}{\sqrt{x - 2\sqrt{2x-4}}}$

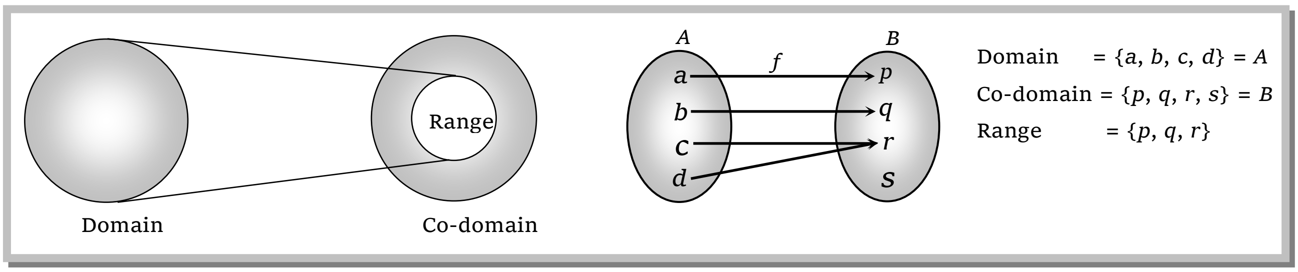
$$f(11) = \frac{1}{\sqrt{11 + 2\sqrt{18}}} + \frac{1}{\sqrt{11 - 2\sqrt{18}}} = \frac{1}{3+\sqrt{2}} + \frac{1}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{7} + \frac{3+\sqrt{2}}{7} = \frac{6}{7}.$$

2.1.4 Domain, Co-domain and Range of Function

If a function *f* is defined from a set of *A* to set *B* then for $f : A \to B$ set *A* is called the domain of function *f* and set *B* is called the co-domain of function *f*. The set of all *f*-images of the elements of *A* is called the range of function *f*.

In other words, we can say Domain = All possible values of *x* for which *f*(*x*) exists.

Range = For all values of *x*, all possible values of *f*(*x*).



(1) **Methods for finding domain and range of function**

(i) **Domain**

(a) Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0

(b) Denominator $\neq 0$.

(c) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively then the domain of $f(x) \pm g(x)$ or $f(x).g(x)$ is $D_1 \cap D_2$.

(d) While domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x) = 0\}$.

(e) Domain of $\left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$

(ii) **Range :** Range of $y = f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain.

(a) If domain $\in$ finite number of points $\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

(b) If domain $\in$ *R* or *R* – [some finite points]. Then express *x* in terms of *y*. From this find *y* for *x* to be defined (*i.e.*, find the values of *y* for which *x* exists).

(c) If domain $\in$ a finite interval, find the least and greatest value for range using monotonicity.

Important Tips

☞ *If f(x) is a given function of x and if a is in its domain of definition, then by f(a) it means the number obtained by replacing x by a in f(x) or the value assumed by f(x) when x = a.*

☞ *Range is always a subset of co-domain.*

Example: 8 Domain of the function $\frac{1}{\sqrt{x^2 - 1}}$ is **[Roorkee 1987; Rajasthan PET 2000]**

(a) $(-\infty, -1) \cup (1, \infty)$ (b) $(-\infty, -1] \cup (1, \infty)$ (c) $(-\infty, -1) \cup [1, \infty)$ (d) None of these

Solution: (a) For domain, $x^2 - 1 > 0 \Rightarrow (x-1)(x+1) > 0$

$\Rightarrow x < -1$ or $x > 1 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$.

Example: 9 The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is **[Roorkee 1998]**

(a) R^+ (b) R^- (c) R_0 (d) R

Solution: (b) For domain, $|x| - x > 0 \Rightarrow |x| > x$. This is possible, only when $x \in R^-$.

Example: 10 Find the domain of definition of $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$ **[IIT 2001; UPSEAT 2001]**

(a) $(-3, \infty)$ (b) $\{-1, -2\}$ (c) $(-3, \infty) - \{-1, -2\}$ (d) $(-\infty, \infty)$

Solution: (c) Here $f(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2} = \frac{\log_2(x+3)}{(x+1)(x+2)}$ exists if,

Numerator $x + 3 > 0 \Rightarrow x > -3$ ...... (i)

and denominator $(x+1)(x+2) \neq 0 \Rightarrow x \neq -1, -2$ ...... (ii)

Thus, from (i) and (ii); we have domain of $f(x)$ is $(-3, \infty) - \{-1, -2\}$.

Example: 11 The domain of the function $f(x)=\sqrt{(2-2x-x^2)}$ is **[BIT Ranchi 1992]**

(a) $-3 \le x \le \sqrt{3}$ (b) $-1-\sqrt{3} \le x \le -1+\sqrt{3}$

(c) $-2 \le x \le 2$ (d) None of these

Solution: (b) The quantity square root is positive, when $-1-\sqrt{3} \le x \le -1+\sqrt{3}$.

Example: 12 If the domain of function $f(x)=x^2-6x+7$ is $(-\infty,\infty)$, then the range of function is

(a) $(-\infty,\infty)$ (b) $[-2,\infty)$ (c) $(-2,3)$ (d) $(-\infty,-2)$

Solution: (b) $x^2-6x+7=(x-3)^2-2$ Obviously, minimum value is –2 and maximum ∞.

Example: 13 The domain of the function $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$ is **[AMU 1999]**

(a) $[-4,\infty)$ (b) $[-4,4]$ (c) $[0,4]$ (d) $[0,1]$

Solution: (d) $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$

clearly $f(x)$ is defined if

$4+x \ge 0 \Rightarrow x \ge -4$

$4-x \ge 0 \Rightarrow x \le 4$

$x(1-x) \ge 0 \Rightarrow x \ge 0$ and $x \le 1$

$\therefore$ Domain of $f=(-\infty,4]\cap[-4,\infty)\cap[0,1]=[0,1]$.

Example: 14 The domain of the function $\sqrt{\log(x^2-6x+6)}$ is **[Roorkee 1999; MP PET 2002]**

(a) $(-\infty,\infty)$ (b) $(-\infty,3-\sqrt{3})\cup(3+\sqrt{3},\infty)$

(c) $(-\infty,1]\cup[5,\infty)$ (d) $[0,\infty)$

Solution: (c) The function $f(x)=\sqrt{\log(x^2-6x+6)}$ is defined when $\log(x^2-6x+6)\ge 0$

$\Rightarrow x^2-6x+6 \ge 1 \Rightarrow (x-5)(x-1)\ge 0$

This inequality hold if $x \le 1$ or $x \ge 5$. Hence, the domain of the function will be $(-\infty,1]\cup[5,\infty)$.

Example: 15 The domain of definition of the function $y(x)$ given by $2^x+2^y=2$ is **[IIT Screening 2000; DCE 2001]**

(a) (0, 1] (b) [0, 1] (c) $(-\infty,0]$ (d) $(-\infty,1)$

Solution: (d) $2^y=2-2^x$

y is real if $2-2^x \ge 0 \Rightarrow 2>2^x \Rightarrow 1>x$

$\Rightarrow \quad x\in(-\infty,1)$

Example: 16 The domain of the function $f(x)=\sin^{-1}[\log_2(x/2)]$ is
[AIEEE 2002; Rajasthan PET 2002]

(a) [1, 4] (b) [– 4, 1] (c) [– 1, 4] (d) None of these

Solution: (a) $f(x)=\sin^{-1}[\log_2(x/2)]$

Domain of $\sin^{-1}x$ is $x\in[-1,1]$

$\Rightarrow \quad -1 \le \log_2(x/2) \le 1 \Rightarrow \frac{1}{2} \le \frac{x}{2} \le 2 \Rightarrow 1 \le x \le 4$

$\therefore x\in[1,4]$.

Example: 17 The domain of the derivative of the function $f(x)=\begin{cases}\tan^{-1}x & , |x|\le 1\\ \frac{1}{2}(|x|-1) & , |x|>1\end{cases}$ is **[IIT Screening 2002]**

(a) $R-\{0\}$ (b) $R-\{1\}$ (c) $R-\{-1\}$ (d) $R-\{-1, 1\}$

Solution: (c) $f(x) = \begin{cases} \frac{1}{2}(-x-1), x < -1 \\ \tan^{-1} x, -1 \le x \le 1 \\ \frac{1}{2}(x+1), x > 1 \end{cases} \Rightarrow \quad f'(x) = \begin{cases} -\frac{1}{2}, & x < -1 \\ \frac{1}{1+x^2}, & -1 < x < 1 \\ \frac{1}{2}, & x > 1 \end{cases}$

$f'(-1-0) = -\frac{1}{2}$; $f'(-1+0) = \frac{1}{1+(-1+0)^2} = \frac{1}{2}$

$f'(1-0) = \frac{1}{1+(1-0)^2} = \frac{1}{2}$; $f'(1+0) = \frac{1}{2}$

$\therefore$ $f'(-1)$ does not exist.

$\therefore$ domain of $f'(x) = R - \{-1\}$.

Example: 18 Domain of definition of the function $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$, is **[AIEEE 2003]**

(a) (1, 2) (b) (–1, 0) ∪ (1, 2) (c) (1, 2) ∪ (2, ∞) (d) (–1, 0) ∪ (1, 2) ∪ (2, ∞)

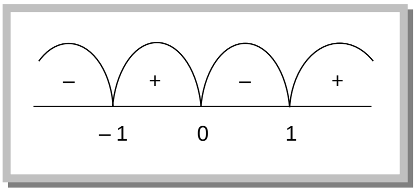
Solution: (d) $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$

So, $4 - x^2 \ne 0 \Rightarrow x \ne \pm\sqrt{4} \Rightarrow x \ne \pm 2$

and $x^3 - x > 0 \Rightarrow x(x^2 - 1) > 0 \Rightarrow x > 0, |x| > 1$

$\therefore$ $D = (-1, 0) \cup (1, \infty) - \{2\}$

$D = (-1, 0) \cup (1, 2) \cup (2, \infty)$.



Example: 19 The domain of the function $f(x) = \log_{3+x}(x^2 - 1)$ is **[Orissa JEE 2003]**

(a) $(-3, -1) \cup (1, \infty)$ (b) $[-3, -1) \cup [1, \infty)$

(c) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$ (d) $[-3, -2) \cup (-2, -1) \cup [1, \infty]$

Solution: (c) $f(x)$ is to be defined when $x^2 - 1 > 0$

$\Rightarrow x^2 > 1, \Rightarrow x < -1$ or $x > 1$ and $3 + x > 0$

$\therefore$ $x > -3$ and $x \ne -2$

$\therefore$ $D_r = (-3, -2) \cup (-2, -1) \cup (1, \infty)$.

Example: 20 Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$, for real value x, is

(a) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (c) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (d) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

Solution: (a) $-\frac{\pi}{6} \le \sin^{-1}(2x) \le \frac{\pi}{2} \Rightarrow -\frac{1}{2} \le 2x \le 1 \Rightarrow x \in \left[-\frac{1}{4}, \frac{1}{2}\right]$.

Example: 21 The range of $f(x) = \cos x - \sin x$, is **[MP PET 1995]**

(a) $(-1, 1)$ (b) $[-1, 1)$ (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[-\sqrt{2}, \sqrt{2}]$

Solution: (d) Let, $f(x) = \cos x - \sin x \Rightarrow f(x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) \Rightarrow f(x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

Now since, $-1 \le \cos\left(x + \frac{\pi}{4}\right) \le 1 \Rightarrow -\sqrt{2} \le f(x) \le \sqrt{2} \Rightarrow f(x) \in [-\sqrt{2}, \sqrt{2}]$

Trick : $\because$ Maximum value of $\cos x - \sin x$ is $\sqrt{2}$ and minimum value of $\cos x - \sin x$ is $-\sqrt{2}$.

Hence, range of $f(x) = [-\sqrt{2}, \sqrt{2}]$.

Example: 22 The range of $\frac{1+x^2}{x^2}$ is **[Karnataka CET 1989]**

(a) $(0, 1)$ (b) $(1, \infty)$ (c) [0, 1] (d) $[1, \infty)$

Solution: (b) Let $y = \frac{1+x^2}{x^2}$ $\Rightarrow x^2 y = 1 + x^2 \Rightarrow x^2(y-1) = 1 \Rightarrow x^2 = \frac{1}{y-1}$

Now since, $x^2 > 0 \Rightarrow \frac{1}{y-1} > 0 \Rightarrow (y-1) > 0 \Rightarrow y > 1 \Rightarrow y \in (1, \infty)$

Trick : $y = \frac{1+x^2}{x^2} = 1 + \frac{1}{x^2}$. Now since, $\frac{1}{x^2}$ is always > 0 $\Rightarrow y > 1 \Rightarrow y \in (1, \infty)$.

Example: 23 For real values of x, range of the function $y = \frac{1}{2 - \sin 3x}$ is

(a) $\frac{1}{3} \le y \le 1$ (b) $-\frac{1}{3} \le y < 1$ (c) $-\frac{1}{3} > y > -1$ (d) $\frac{1}{3} > y > 1$

Solution: (a) $\because y = \frac{1}{2 - \sin 3x}$, $\therefore 2 - \sin 3x = \frac{1}{y} \Rightarrow \sin 3x = 2 - \frac{1}{y}$

Now since,

$-1 \le \sin 3x \le 1 \Rightarrow -1 \le 2 - \frac{1}{y} \le 1 \Rightarrow -3 \le -\frac{1}{y} \le -1 \Rightarrow 1 \le \frac{1}{y} \le 3 \Rightarrow \frac{1}{3} \le y \le 1$.

Example 24 If $f(x) = a\cos(bx + c) + d$, then range of $f(x)$ is **[UPSEAT 2001]**

(a) $[d + a, d + 2a]$ (b) $[a - d, a + d]$ (c) $[d + a, a - d]$ (d) $[d - a, d + a]$

Solution: (d) $f(x) = a\cos(bx + c) + d$ ..... (i)

For minimum $\cos(bx + c) = -1$

from (i), $f(x) = -a + d = (d - a)$,

for maximum $\cos(bx + c) = 1$

from (i), $f(x) = a + d = (d + a)$

$\therefore$ Range of $f(x) = [d - a, d + a]$.

Example: 25 The range of the function $f(x) = \frac{x+2}{|x+2|}$ is **[Rajasthan PET 2002]**

(a) {0, 1} (b) {– 1, 1} (c) R (d) $R - \{-2\}$

Solution: (b) $f(x) = \frac{x+2}{|x+2|} = \begin{cases} -1, & x < -2 \\ 1, & x > -2 \end{cases}$

$\therefore$ Range of $f(x)$ is $\{-1, 1\}$.

Example: 26 The range of $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right), -\infty < x < \infty$ is **[Orissa JEE 2002]**

(a) $[1, \sqrt{2}]$ (b) $[1, \infty)$ (c) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$ (d) $(-\infty, -1] \cup [1, \infty)$

Solution: (a) $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right)$

We know that, $0 \le \cos^2 x \le 1$ at $\cos x = 0$, $f(x) = 1$ and at $\cos x = 1$, $f(x) = \sqrt{2}$

$\therefore\ 1 \le x \le \sqrt{2} \Rightarrow x \in [1, \sqrt{2}]$.

Example: 27 Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}; x \in R$ is **[IIT Screening 2003]**

(a) $(1, \infty)$ (b) $(1, 11/7)$ (c) $(1, 7/3]$ (d) $(1, 7/5]$

Solution: (c) $f(x) = 1 + \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \Rightarrow$ Range $= (1, 7/3]$.

2.1.5 Algebra of Functions

Let $f(x)$ and $g(x)$ be two real and single-valued functions, with domains X_f, X_g and ranges Y_f and Y_g respectively. Let $X = X_f \cap X_g \ne \phi$. Then, the following operations are defined.

(1) **Scalar multiplication of a function :** $(c\,f)(x) = c\,f(x)$, where c is a scalar. The new function $c\,f(x)$ has the domain X_f.

(2) **Addition/subtraction of functions :** $(f \pm g)(x) = f(x) \pm g(x)$. The new function has the domain X.

(3) **Multiplication of functions :** $(fg)(x) = (g\,f)(x) = f(x)g(x)$. The product function has the domain X.

(4) **Division of functions :**

(i) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. The new function has the domain X, except for the values of x for which $g(x) = 0$.

(ii) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$. The new function has the domain X, except for the values of x for which $f(x) = 0$.

(5) **Equal functions :** Two function f and g are said to be equal functions, if and only if

(i) Domain of f = domain of g

(ii) Co-domain of f = co-domain of g

(iii) $f(x) = g(x) \forall x \in$ their common domain

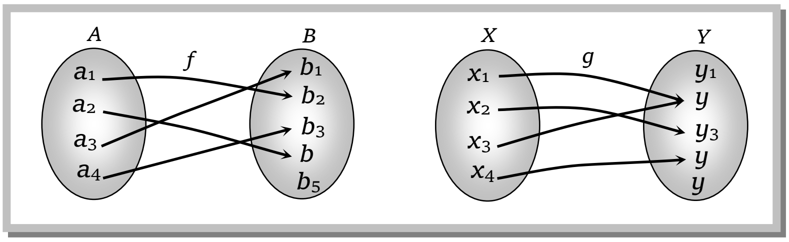
(6) **Real valued function :** If R, be the set of real numbers and A, B are subsets of R, then the function $f: A \to B$ is called a real function or real –valued function.

2.1.6 Kinds of Function

(1) **One-one function (injection) :** A function $f: A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B. Thus, $f: A \to B$ is one-one.

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$ $\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

e.g. Let $f: A \to B$ and $g: X \to Y$ be two functions represented by the following diagrams.



Clearly, $f: A \to B$ is a one-one function. But $g: X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function g.

(i) **Method to check the injectivity of a function**

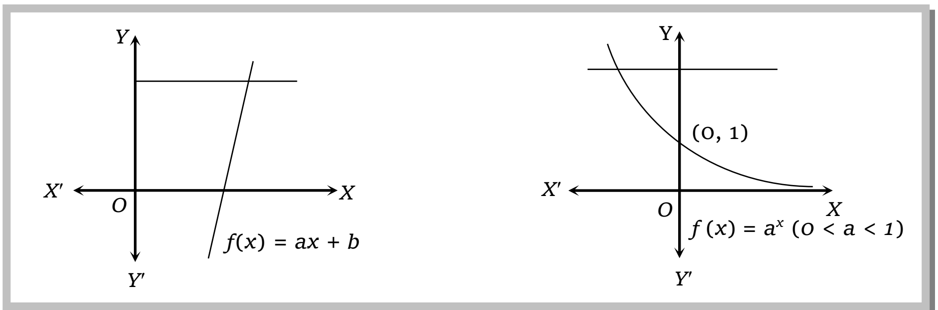
Step I : Take two arbitrary elements x, y (say) in the domain of f.

Step II : Put $f(x) = f(y)$.

Step III : Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives $x = y$ only, then $f: A \to B$ is a one-one function (or an injection). Otherwise not.

Note : ❒ If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

❒ If the graph of the function $y = f(x)$ is given and each line parallel to x-axis cuts the given curve at maximum one point then function is one-one. *e.g.*

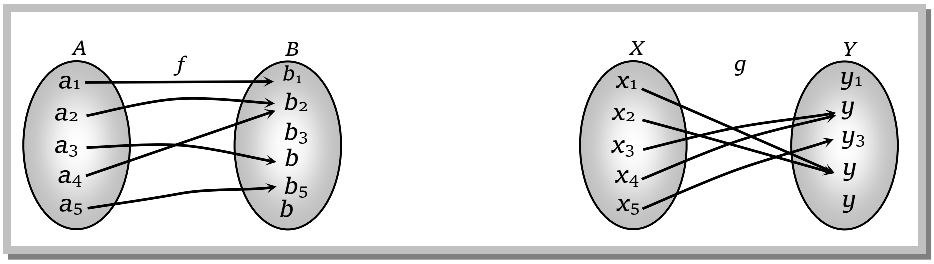


(ii) **Number of one-one functions (injections) :** If A and B are finite sets having m and n elements respectively, then number of one-one functions from A to B $= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(2) **Many-one function :** A function $f: A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

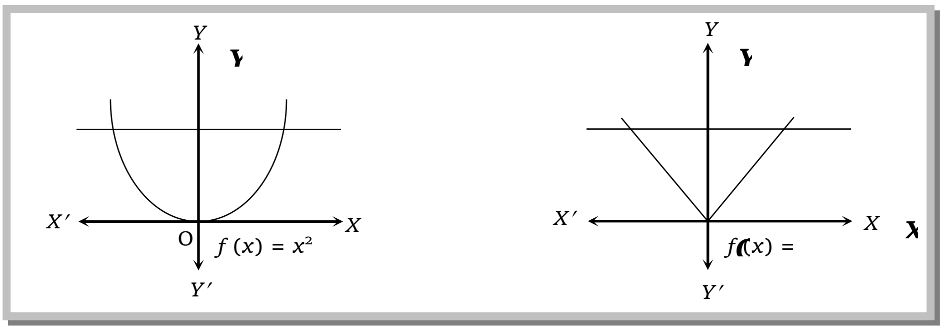
Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

In other words, $f : A \to B$ is a many-one function if it is not a one-one function.



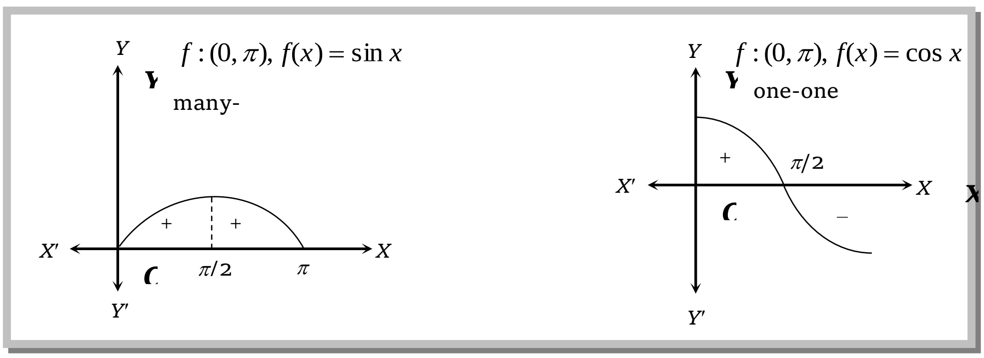
Note : ❐ If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many-one.

❐ If the graph of $y = f(x)$ is given and the line parallel to x-axis cuts the curve at more than one point then function is many-one.



❐ If the domain of the function is in one quadrant then the trigonometrical functions are always one-one.

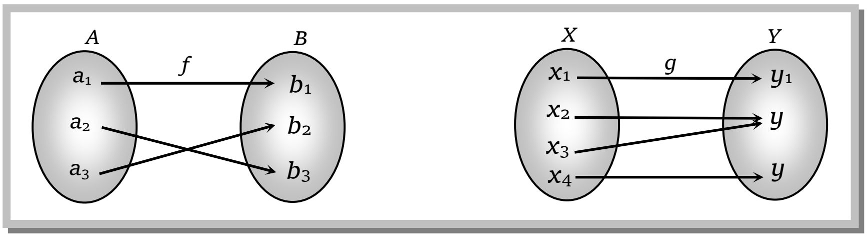
❐ If trigonometrical function changes its sign in two consecutive quadrants then it is one-one but if it does not change the sign then it is many-one.



❐ In three consecutive quadrants trigonometrical functions are always many-one.

(3) **Onto function (surjection) :** A function $f : A \to B$ is onto if each element of B has its pre-image in A. Therefore, if $f^{-1}(y) \in A, \forall y \in B$ then function is onto. In other words, Range of f = Co-domain of f.

e.g. The following arrow-diagram shows onto function.

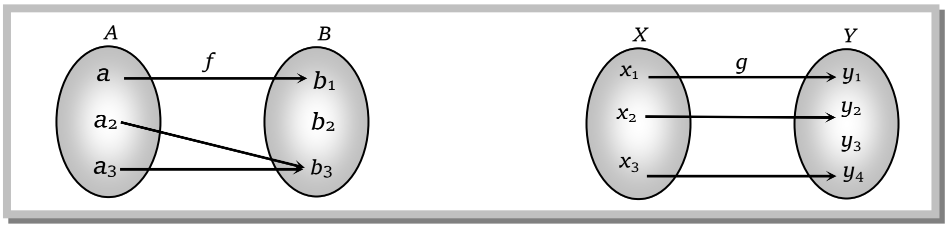


(i) **Number of onto function (surjection) :** If A and B are two sets having m and n elements respectively such that $1 \le n \le m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r}\ {}^nC_r r^m$.

(4) **Into function :** A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words, $f : A \to B$ is an into function if it is not an onto function.

e.g. The following arrow-diagram shows into function.



(i) **Method to find onto or into function**

(a) If range = co-domain, then $f(x)$ is onto and if range is a proper subset of the co-domain, then $f(x)$ is into.

(b) Solve $f(x) = y$ by taking x as a function of y *i.e.*, $g(y)$ (say).

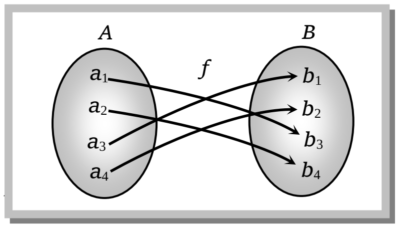
(c) Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain for $y \in$ co-domain, then $f(x)$ is onto and if any one of the above requirements is not fulfilled, then $f(x)$ is into.

(5) **One-one onto function (bijection) :** A function $f : A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f : A \to B$ is a bijection if

(i) It is one-one i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) It is onto i.e., for all $y \in B$, there exists $x \in$



Clearly, f is a bijection since it is both injective as well as surjective.

Number of one-one onto function (bijection) : If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijection from A to B is the total number of arrangements of n items taken all at a time *i.e.* $n!$.

(6) **Algebraic functions :** Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations +, −, × and ÷ are called algebraic functions.

e.g., (i) $x^{\frac{3}{2}}+5x$ (ii) $\frac{\sqrt{x+1}}{x-1}, x \neq 1$ (iii) $3x^4-5x+7$

The algebraic functions can be classified as follows:

(i) **Polynomial or integral function :** It is a function of the form $a_0x^n+a_1x^{n-1}+....+a_{n-1}x+a_n$, where $a_0 \neq 0$ and $a_0, a_1, ..., a_n$ are constants and $n \in N$ is called a polynomial function of degree n

e.g. $f(x)=x^3-2x^2+x+3$ is a polynomial function.

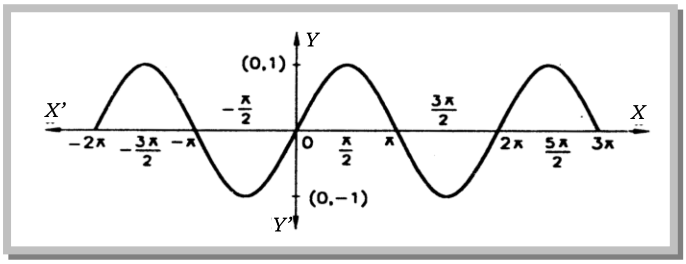
Note : ❐ The polynomial of first degree is called a linear function and polynomial of zero degree is called a constant function.

(ii) **Rational function :** The quotient of two polynomial functions is called the rational function. *e.g.* $f(x)=\frac{x^2-1}{2x^3+x^2+1}$ is a rational function.

(iii) **Irrational function :** An algebraic function which is not rational is called an irrational function. *e.g.* $f(x)=x+\sqrt{x}+6,\ g(x)=\frac{x^3-\sqrt{x}}{1+x^{1/4}}$ are irrational functions.

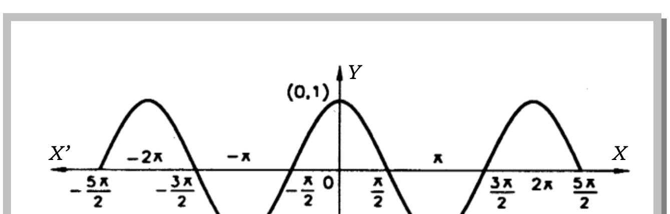
(7) **Transcendental function :** A function which is not algebraic is called a transcendental function. *e.g.*, trigonometric; inverse trigonometric , exponential and logarithmic functions are all transcendental functions.

(i) **Trigonometric functions :** A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, cosecant) of variable angles.



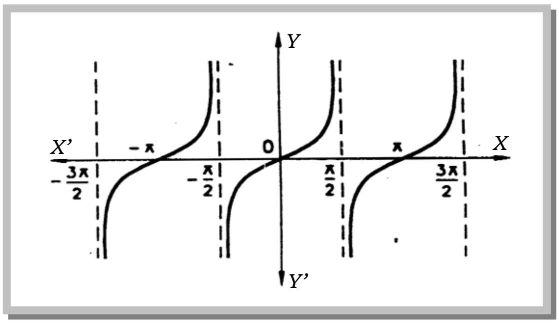
(a) **Sine function :** The function that associates to each real numbers x to $\sin x$ is called the sine function. Here x is the radian measure of the angle. The domain of the sine function is R and the range is [−1, 1].

(b) **Cosine function:** The function that associates to each real number x to $\cos x$ is called the cosine function. Here x is the radian measure of the angle. The domain of the cosine function is R and the range is [− 1, 1].



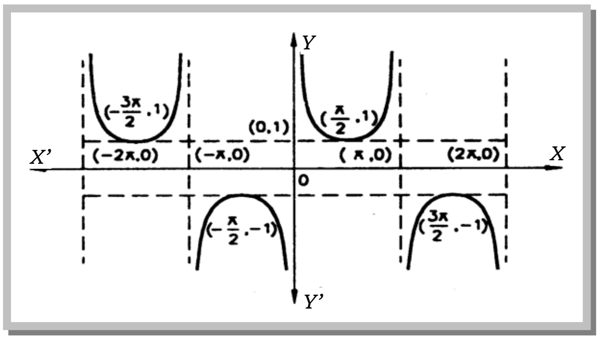
(c) **Tangent function :** The function that associates a real number x to tan x is called the tangent function.

Clearly, the tangent function is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.,* $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc. So, the domain of the tangent function is $R-\{(2n+1)\frac{\pi}{2} | \; n \in I\}$. Since it takes every value between $-\infty$ and ∞. So, the range is R. Graph of $f(x) = \tan x$ is shown in figure.



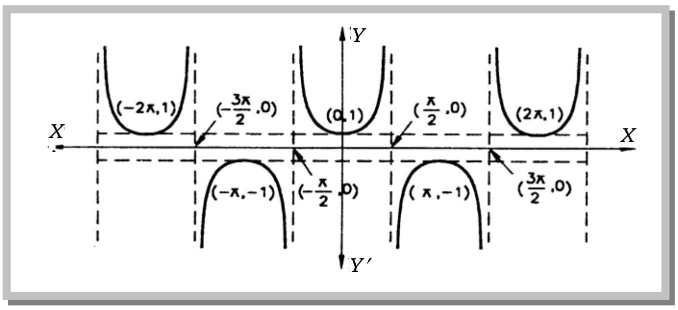
(d) **Cosecant function :** The function that associates a real number x to cosec$x$ is called the cosecant function.

Clearly, cosec x is not defined at $x = n\pi, n \in I$. *i.e.,* $0, \pm\pi, \pm 2\pi, \pm 3\pi$ etc. So, its domain is $R-\{n\pi | \; n \in I\}$. Since $\operatorname{cosec} x \geq 1$ or $\operatorname{cosec} x \leq -1$. Therefore, range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \operatorname{cosec} x$ is shown in figure.

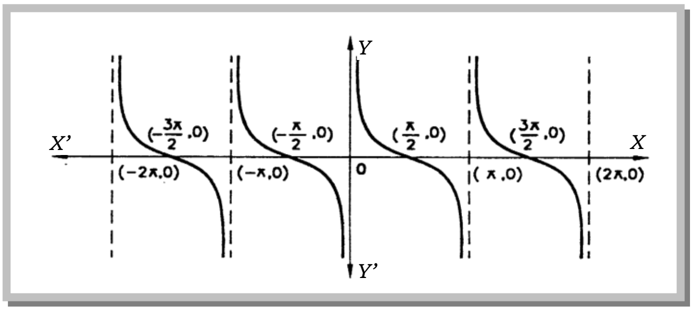


(e) **Secant function :** The function that associates a real number x to sec x is called the secant function.

Clearly, sec x is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.,* $(2\pi+1)\frac{\pi}{2}$, where $n \in I$. So, its domain is $R-\{(2n+1)\frac{\pi}{2} | \; n \in I\}$. Also, $|\sec x| \geq 1$, therefore its range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \sec x$ is shown in figure.



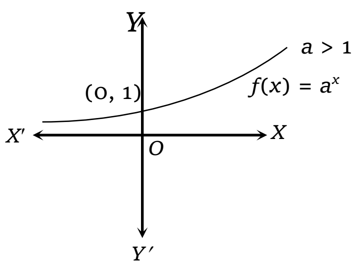
(f) **Cotangent function :** The function that associates a real number x to cot x is called the cotangent function. Clearly, cot x is not defined at $x = n\pi, n \in I$ i.e., at $n = 0, \pm\pi, \pm 2\pi$ etc. So, domain of $\cot x$ is $R - \{n\pi \mid n \in I\}$. The range of $f(x) = \cot x$ is R as is evident from its graph in figure.



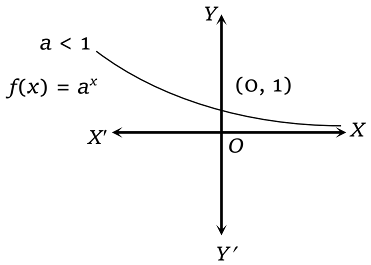
(ii) **Inverse trigonometric functions**

Function	**Domain**	**Range**	**Definition of the function**
$\sin^{-1} x$	$[-1,1]$	$[-\pi/2,\ \pi/2]$	$y = \sin^{-1} x \Leftrightarrow x = \sin y$
$\cos^{-1} x$	$[-1,\ 1]$	$[0,\ \pi]$	$y = \cos^{-1} x \Leftrightarrow x = \cos y$
$\tan^{-1} x$	$(-\infty,\ \infty)$ or R	$(-\pi/2,\ \pi/2)$	$y = \tan^{-1} x \Leftrightarrow x = \tan y$
$\cot^{-1} x$	$(-\infty,\ \infty)$ or R	$(0,\ \pi)$	$y = \cot^{-1} x \Leftrightarrow x = \cot y$
$\operatorname{cosec}^{-1} x$	$R - (-1,\ 1)$	$[-\pi/2,\ \pi/2] - \{0\}$	$y = \operatorname{cosec}^{-1} x \Leftrightarrow x = \operatorname{cosec} y$
$\sec^{-1} x$	$R - (-1,\ 1)$	$[0,\ \pi] - [\pi/2]$	$y = \sec^{-1} x \Leftrightarrow x = \sec y$

(iii) **Exponential function :** Let $a \neq 1$ be a positive real number. Then $f : R \to (0, \infty)$ defined by $f(x) = a^x$ is called exponential function. Its domain is R and range is $(0, \infty)$.

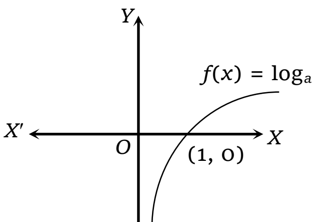


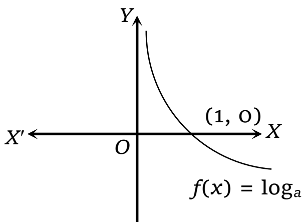
graph of $f(x) = a^x$, when $a > 1$



graph of $f(x) = a^x$, when $a < 1$

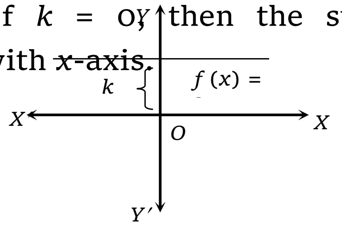
(iv) **Logarithmic function :** Let $a \neq 1$ be a positive real number. Then $f : (0, \infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R.





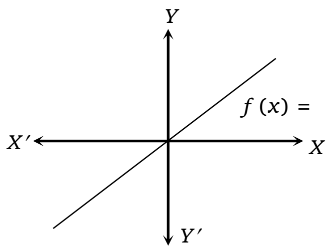
(8) **Explicit and implicit functions :** A function is said to be explicit if it can be expressed directly in terms of the independent variable. If the function can not be expressed directly in terms of the independent variable or variables, then the function is said to be implicit. *e.g.* $y = \sin^{-1} x + \log x$ is explicit function, while $x^2 + y^2 = xy$ and $x^3 y^2 = (a-x)^2 (b-y)^2$ are implicit functions.

(9) **Constant function :** Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to x-axis as shown in figure and it is above or below the x-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with x-axis.

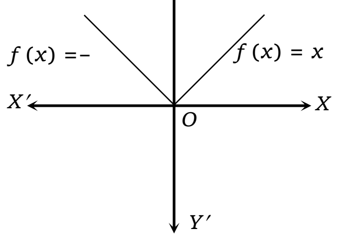


(10) **Identity function :** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on R. Clearly, the domain and range of the identity function is R.

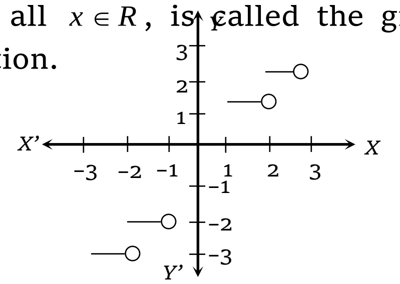
The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of x-axis.



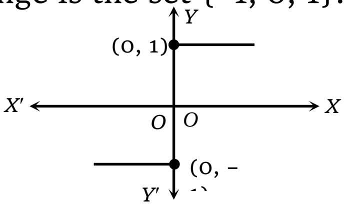
(11) **Modulus function :** The function defined by $f(x)=|x|=\begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function. The domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers.



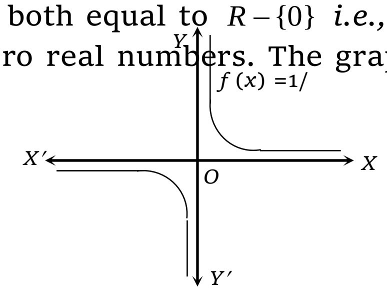
(12) **Greatest integer function:** Let $f(x)=[x]$, where $[x]$ denotes the greatest integer less than or equal to x. The domain is R and the range is I. *e.g.* [1.1] = 1, [2.2] = 2, [– 0.9] = –1, [– 2.1] = – 3 etc. The function f defined by $f(x)=[x]$ for all $x \in R$, is called the greatest integer function.



(13) **Signum function :** The function defined by $f(x)=\begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ or $f(x)=\begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ is called the signum function. The domain is R and the range is the set {–1, 0, 1}.



(14) **Reciprocal function:** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R-\{0\}$ *i.e.,* the set of all non-zero real numbers. The graph is as shown.



Domain and Range of Some Standard Functions

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	$\{K\}$
Reciprocal function $\frac{1}{x}$	R_0	R_0
$x^2, \|x\|$	R	$R^+ \cup \{0\}$
$x^3, x\|x\|$	R	R
Signum function	R	$\{-1, 0, 1\}$
$x+\|x\|$	R	$R^+ \cup \{0\}$
$x-\|x\|$	R	$R^- \cup \{0\}$

$[x]$	R	I
$x-[x]$	R	$[0, 1)$
$\sqrt{x}$	$[0, \infty)$	R
a^x	R	R^+
$\log x$	R^+	R
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...\right\}$	R
$\cot x$	$R-\{0, \pm\pi, \pm 2\pi, ...\}$	R
$\sec x$	$R-\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...\right\}$	$R-(-1, 1)$
$\text{cosec } x$	$R-\{0, \pm\pi, \pm 2\pi, ...\}$	$R-(-1, 1)$
$\sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$R-(-1,1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	$R-(-1,1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

Important Tips

- ☞ *Any function, which is entirely increasing or decreasing in the whole of a domain, is one-one.*
- ☞ *Any continuous function $f(x)$, which has at least one local maximum or local minimum, is many-one.*
- ☞ *If any line parallel to the x-axis cuts the graph of the function at most at one point, then the function is one-one and if there exists a line which is parallel to the x-axis and cuts the graph of the function in at least two points, then the function is many-one.*
- ☞ *Any polynomial function $f: R \to R$ is onto if degree of f is odd and into if degree of f is even.*
- ☞ *An into function can be made onto by redefining the co-domain as the range of the original function.*

Example: 28 Function $f: N \to N, f(x) = 2x+3$ is **[IIT 1973; UPSEAT 1983]**

(a) One-one onto (b) One-one into (c) Many-one onto (d) Many –one into

Solution: (b) f is one-one because $f(x_1) = f(x_2) \Rightarrow 2x_1+3 = 2x_2+3 \Rightarrow x_1 = x_2$

Further $f^{-1}(x) = \frac{x-3}{2} \notin N$ (domain) when x = 1, 2, 3 etc.

$\therefore$ f is into which shows that f is one-one into.

Example: 29 The function $f: R \to R$ defined by $f(x) = (x-1)(x-2)(x-3)$ is **[Roorkee 1999]**

(a) One-one but not onto (b) Onto but not one-one

(c) Both one-one and onto (d) Neither one-one nor onto

Solution: (b) We have $f(x) = (x-1)(x-2)(x-3) \Rightarrow f(1) = f(2) = f(3) = 0 \Rightarrow f(x)$ is not one-one

For each $y \in R$, there exists $x \in R$ such that $f(x) = y$. Therefore f is onto.

Hence, $f: R \to R$ is onto but not one-one.

Example: 30 Find number of surjection from A to B where $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$ **[IIT Screening 2001]**

(a) 13 (b) 14 (c) 15 (d) 16

Solution: (b) Number of surjection from A to B $= \sum_{r=1}^{2} (-1)^{2-r}\ {}^2C_r (r)^4$

$= (-1)^{2-1}\ {}^2C_1 (1)^4 + (-1)^{2-2}\ {}^2C_2 (2)^4 = -2 + 16 = 14$

Therefore, number of surjection from A to B = 14.

Trick : Total number of functions from A to B is 2^4 of which two function $f(x) = a$ for all $x \in A$ and $g(x) = b$ for all $x \in A$ are not surjective. Thus, total number of surjection from A to B $= 2^4 - 2 = 14$.

Example: 31 If $A = \{a, b, c\}$, then total number of one-one onto functions which can be defined from A to A is

(a) 3 (b) 4 (c) 9 (d) 6

Solution: (d) Total number of one-one onto functions = $3!$

Example: 32 If $f: R \to R$, then $f(x) = |x|$ is **[Rajasthan PET 2000]**

(a) One-one but not onto (b) Onto but not one-one

(c) One-one and onto (d) None of these

Solution: (d) $f(-1) = f(1) = 1$ $\therefore$ function is many-one function.

Obviously, f is not onto so f is neither one-one nor onto.

Example: 33 Let $f: R \to R$ be a function defined by $f(x) = \frac{x-m}{x-n}$, where $m \neq n$. Then **[UPSEAT 2001]**

(a) f is one-one onto (b) f is one-one into (c) f is many one onto (d) f is many one into

Solution: (b) For any $x, y \in R$, we have

$$f(x) = f(y) \Rightarrow \frac{x-m}{x-n} = \frac{y-m}{y-n} \Rightarrow x = y$$

$\therefore f$ is one-one

Let $\alpha \in R$ such that $f(x) = \alpha \Rightarrow \frac{x-m}{x-n} = \alpha \Rightarrow x = \frac{m - n\alpha}{1-\alpha}$

Clearly $x \notin R$ for $\alpha = 1$. So, f is not onto.

Example: 34 The function $f: R \to R$ defined by $f(x) = e^x$ is **[Karnataka CET 2002; UPSEAT 2002]**

(a) Onto (b) Many-one (c) One-one and into (d) Many one and onto

Solution: (c) Function $f: R \to R$ is defined by $f(x) = e^x$. Let $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$ or $e^{x_1} = e^{x_2}$ or $x_1 = x_2$. Therefore f is one-one. Let $f(x) = e^x = y$. Taking log on both sides, we get $x = \log y$. We know that negative real numbers have no pre-image or the function is not onto and zero is not the image of any real number. Therefore function f is into.

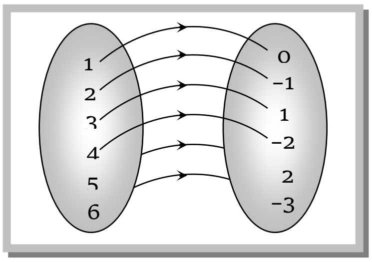
Example: 35 A function f from the set of natural numbers to integers defined by $f(n) = \begin{cases} \frac{n-1}{2}, \text{ when } n \text{ is odd} \\ -\frac{n}{2}, \text{ when } n \text{ is even} \end{cases}$, is **[AIEEE 2003]**

(a) One-one but not onto (b) Onto but not one-one

(c) One-one and onto both (d) Neither one-one nor onto

Solution: (c) $f: N \to I$

$f(1) = 0, f(2) = -1, f(3) = 1, f(4) = -2, f(5) = 2$ and $f(6) = -3$ so on.



In this type of function every element of set A has unique image in set B and there is no element left in set B. Hence f is one-one and onto function.

2.1.7 Even and Odd function

(1) **Even function :** If we put $(-x)$ in place of x in the given function and if $f(-x) = f(x)$, $\forall x \in$ domain then function $f(x)$ is called even function. *e.g.* $f(x) = e^x + e^{-x}$, $f(x) = x^2$, $f(x) = x \sin x$, $f(x) = \cos x$, $f(x) = x^2 \cos x$ all are even function.

(2) **Odd function :** If we put $(-x)$ in place of x in the given function and if $f(-x) = -f(x)$, $\forall x \in$ domain then $f(x)$ is called odd function. *e.g.* $f(x) = e^x - e^{-x}$, $f(x) = \sin x$, $f(x) = x^3$, $f(x) = x \cos x$, $f(x) = x^2 \sin x$ all are odd function.

Important Tips

☞ *The graph of even function is always symmetric with respect to y-axis.*

☞ *The graph of odd function is always symmetric with respect to origin.*

☞ *The product of two even functions is an even function.*

☞ *The sum and difference of two even functions is an even function.*

☞ *The sum and difference of two odd functions is an odd function.*

☞ *The product of two odd functions is an even function.*

☞ *The product of an even and an odd function is an odd function*

☞ *It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function. e.g.* $f(x) = x^2 + x^3$, $f(x) = \log_e x$, $f(x) = e^x$.

☞ *The sum of even and odd function is neither even nor odd function.*

☞ *Zero function* $f(x) = 0$ *is the only function which is even and odd both.*

Example: 36 Which of the following is an even function **[UPSEAT 1998]**

(a) $x\left(\frac{a^x - 1}{a^x + 1}\right)$ (b) $\tan x$ (c) $\frac{a^x - a^{-x}}{2}$ (d) $\frac{a^x + 1}{a^x - 1}$

Solution: (a) We have : $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

$$f(-x) = -x\left(\frac{a^{-x} - 1}{a^{-x} + 1}\right) = -x\left(\frac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1}\right) = -x\left(\frac{1 - a^x}{1 + a^x}\right) = x\left(\frac{a^x - 1}{a^x + 1}\right) = f(x)$$

So, $f(x)$ is an even function.

Example: 37 Let $f(x) = \sqrt{x^4 + 15},$ then the graph of the function $y = f(x)$ is symmetrical about

(a) The x-axis (b) The y-axis (c) The origin (d) The line $x = y$

Solution: (b) $f(x) = \sqrt{x^4 + 15} \Rightarrow f(-x) = \sqrt{(-x)^4 + 15} = \sqrt{x^4 + 15} = f(x)$

$\Rightarrow f(-x) = f(x) \Rightarrow f(x)$ is an even function $\Rightarrow f(x)$ is symmetric about y-axis.

Example: 38 The function $f(x) = \log(x + \sqrt{x^2 + 1})$ is

(a) An even function (b) An odd function (c) Periodic function (d) None of these

Solution: (b) $f(x) = \log(x + \sqrt{x^2 + 1})$ and $f(-x) = -\log(x + \sqrt{x^2 + 1}) = -f(x)$, so $f(x)$ is an odd function.

Example: 39 Which of the following is an even function **[Rajasthan PET 2000]**

(a) $f(x) = \frac{a^x + 1}{a^x - 1}$ (b) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$ (c) $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$ (d) $f(x) = \sin x$

Solution: (b) In option (a), $f(-x) = \frac{a^{-x} + 1}{a^{-x} - 1} = \frac{1 + a^x}{1 - a^x} = -\frac{a^x + 1}{a^x - 1} = -f(x)$ So, It is an odd function.

In option (b), $f(-x) = (-x)\frac{a^{-x} - 1}{a^{-x} + 1} = -x\frac{(1 - a^x)}{1 + a^x} = x\frac{(a^x - 1)}{(a^x + 1)} = f(x)$ So, It is an even function.

In option (c), $f(-x) = \frac{a^{-x} - a^x}{a^{-x} + a^x} = -f(x)$ So, It is an odd function.

In option (d), $f(-x)=\sin(-x)=-\sin x=-f(x)$ So, It is an odd function.

Example: 40 The function $f(x)=\sin\left(\log(x+\sqrt{x^2+1})\right)$ is **[Orissa JEE 2002]**

(a) Even function (b) Odd function (c) Neither even nor odd (d) Periodic function

Solution: (b) $f(x)=\sin\left(\log(x+\sqrt{1+x^2})\right)$

$\Rightarrow f(-x)=\sin[\log(-x+\sqrt{1+x^2})]$ $\Rightarrow f(-x)=\sin\log\left((\sqrt{1+x^2}-x)\dfrac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}\right)$

$\Rightarrow f(-x)=\sin\log\left[\dfrac{1}{(x+\sqrt{1+x^2})}\right]$ $\Rightarrow f(-x)=\sin\left[\log(x+\sqrt{1+x^2})^{-1}\right]$

$\Rightarrow f(-x)=\sin\left[-\log(x+\sqrt{1+x^2})\right]$ $\Rightarrow f(-x)=-\sin\left[\log(x+\sqrt{1+x^2})\right]\Rightarrow f(-x)=-f(x)$

$\therefore f(x)$ is odd function.

2.1.8 Periodic Function

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x+T)=f(x), \forall x \in$ domain. Here the least positive value of T is called the period of the function. Clearly $f(x)=f(x+T)=f(x+2T)=f(x+3T)=......$ *e.g.* $\sin x, \cos x, \tan x$ are periodic functions with period $2\pi, 2\pi$ and π respectively.

Some standard results on periodic functions

Functions	Periods
(1) $\sin^n x,\ \cos^n x,\ \sec^n x,\ cosec^n x$	$\begin{cases}\pi; \text{ if } n \text{ is even}\\ 2\pi; \text{ if } n \text{ is odd or fraction}\end{cases}$
(2) $\tan^n x,\ \cot^n x$	$\pi; n$ is even or odd.
(3) $\lvert\sin x\rvert, \lvert\cos x\rvert, \lvert\tan x\rvert,$ $\lvert\cot x\rvert, \lvert\sec x\rvert, \lvert cosec\, x\rvert$	π
(4) $x-[x]$	1
(5) Algebraic functions e.g., $\sqrt{x}, x^2, x^3+5, ...$ etc	Period does not exist

Important Tips

- *If $f(x)$ is periodic with period T, then $c.f(x)$ is periodic with period T, $f(x + c)$ is periodic with period T and $f(x) \pm c$ is periodic with period T. where c is any constant.*
- *If a function $f(x)$ has a period T, then the function $f(ax+b)$ will have a period $\frac{T}{|a|}$.*
- *If $f(x)$ is periodic with period T then $\frac{1}{f(x)}$ is also periodic with same period T.*
- *If $f(x)$ is periodic with period T, $\sqrt{f(x)}$ is also periodic with same period T.*
- *If $f(x)$ is periodic with period T, then $a f(x) + b$, where $a, b \in R (a \neq 0)$ is also a periodic function with period T.*
- *If $f_1(x), f_2(x), f_3(x)$ are periodic functions with periods T_1, T_2, T_3 respectively then; we have*

  $h(x) = af_1(x) \pm bf_2(x) \pm cf_3(x)$, *has period as,*

$$= \begin{cases} \text{L.C.M.of } \{T_1, T_2, T_3\}; & \text{if } h(x) \text{ is not an even function} \\ \frac{1}{2}\text{L.C.M.of } \{T_1, T_2, T_3\}; & \text{if } h(x) \text{ is an even function} \end{cases}$$

Example: 41 The period of the function $f(x) = 2\cos\frac{1}{3}(x - \pi)$ is **[DCE 1998]**

(a) 6π (b) 4π (c) 2π (d) π

Solution: (a) $f(x) = 2\cos\frac{1}{3}(x - \pi) = 2\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$

Now, since $\cos x$ has period $2\pi \Rightarrow \cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$ has period $\frac{2\pi}{\frac{1}{3}} = 6\pi$

$\Rightarrow 2\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$ has period $= 6\pi$.

Example: 42 The function $f(x) = \sin\frac{\pi x}{2} + 2\cos\frac{\pi x}{3} - \tan\frac{\pi x}{4}$ is periodic with period **[EAMCET 1992]**

(a) 6 (b) 3 (c) 4 (d) 12

Solution: (d) $\because$ $\sin x$ has period $= 2\pi \Rightarrow \sin\frac{\pi x}{2}$ has period $= \frac{2\pi}{\frac{\pi}{2}} = 4$

$\because$ $\cos x$ has period $= 2\pi \Rightarrow \cos\frac{\pi x}{3}$ has period $= \frac{2\pi}{\frac{\pi}{3}} = 6 \Rightarrow 2\cos\frac{\pi x}{3}$ has period = 6

$\because$ $\tan x$ has period $= \pi \Rightarrow \tan\frac{\pi x}{4}$ has period $= \frac{\pi}{\frac{\pi}{4}} = 4$.

L.C.M. of 4, 6 and 4 =12, period of $f(x)$ = 12.

Example: 43 The period of $|\sin 2x|$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

Solution: (b) Here $|\sin 2x| = \sqrt{\sin^2 2x} = \sqrt{\frac{(1-\cos 4x)}{2}}$

Period of $\cos 4x$ is $\frac{\pi}{2}$. Hence, period of $|\sin 2x|$ will be $\frac{\pi}{2}$

Trick : $\because$ $\sin x$ has period $= 2\pi \Rightarrow$ $\sin 2x$ has period $= \frac{2\pi}{2} = \pi$

Now, if $f(x)$ has period p then $|f(x)|$ has period $\frac{p}{2} \Rightarrow |\sin 2x|$ has period $= \frac{\pi}{2}$.

Example: 44 If $f(x)$ is an odd periodic function with period 2, then $f(4)$ equals **[IIT 1991]**

(a) 0 (b) 2 (c) 4 (d) – 4

Solution: (a) Given, $f(x)$ is an odd periodic function. We can take $\sin x$, which is odd and periodic.

Now since, $\sin x$ has period = 2 and $f(x)$ has period = 2.

So, $f(x) = \sin(\pi x) \Rightarrow f(4) = \sin(4\pi) = 0$.

Example: 45 The period of the function $f(x) = \sin^2 x$ is **[UPSEAT 1991, 2002; AIEEE 2002]**

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) None of these

Solution: (b) $\sin^2 x = \frac{1-\cos 2x}{2} \Rightarrow$ Period $= \frac{2\pi}{2} = \pi$.

Example: 46 The period of $f(x) = x - [x]$, if it is periodic, is **[AMU 2000]**

(a) $f(x)$ is not periodic (b) $\frac{1}{2}$ (c) 1 (d) 2

Solution: (c) Let $f(x)$ be periodic with period T. Then,

$f(x+T) = f(x)$ for all $x \in R \Rightarrow x + T - [x+T] = x - [x]$ for all $x \in R \Rightarrow x + T - x = [x+T] - [x]$

$\Rightarrow [x+T] - [x] = T$ for all $x \in R \Rightarrow T = 1, 2, 3, 4, ...$

The smallest value of T satisfying,

$f(x+T) = f(x)$ for all $x \in R$ is 1.

Hence $f(x) = x - [x]$ has period 1.

Example: 47 The period of $f(x) = \sin\left(\frac{\pi x}{n-1}\right) + \cos\left(\frac{\pi x}{n}\right)$, $n \in Z$, $n > 2$ is

(a) $2\pi n(n-1)$ (b) $4n(n-1)$ (c) $2n(n-1)$ (d) None of these

Solution: (c) $f(x) = \sin\left(\frac{\pi x}{n-1}\right) + \cos\left(\frac{\pi x}{n}\right)$

Period of $\sin\left(\frac{\pi x}{n-1}\right) = \frac{2\pi}{\left(\frac{\pi}{n-1}\right)} = 2(n-1)$ and period of $\cos\left(\frac{\pi x}{n}\right) = \frac{2\pi}{\left(\frac{\pi}{n}\right)} = 2n$

Hence period of $f(x)$ is LCM of $2n$ and $2(n-1) \Rightarrow 2n(n-1)$.

Example: 48 If a, b be two fixed positive integers such that $f(a+x) = b + [b^3 + 1 - 3b^2 f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{\frac{1}{3}}$ for all real x, then $f(x)$ is a periodic function with period

(a) a (b) $2a$ (c) b (d) $2b$

Solution: (b) $f(a+x) = b + (1 + \{b - f(x)\}^3)^{1/3} \Rightarrow f(a+x) - b = \{1 - \{f(x) - b\}^3\}^{1/3}$

$\Rightarrow \phi(a+x) = \{1 - \{\phi(x)\}^3\}^{1/3}$ $[\ \phi(x) = f(x) - b\ ] \Rightarrow \phi(x+2a) = \{1 - \{\phi(x+a)\}^3\}^{1/3} = \phi(x)$

$\Rightarrow f(x+2a) - b = f(x) - b \Rightarrow f(x+2a) = f(x)$

$\therefore$ $f(x)$ is periodic with period $2a$.

2.1.9 Composite Function

If $f : A \to B$ and $g : B \to C$ are two function then the composite function of *f* and *g*, $gof\, A \to C$ will be defined as $gof(x) = g[f(x)], \forall x \in A$

(1) **Properties of composition of function :**

(i) *f* is even, *g* is even $\Rightarrow fog$ even function.

(ii) *f* is odd, *g* is odd $\Rightarrow fog$ is odd function.

(iii) *f* is even, *g* is odd $\Rightarrow fog$ is even function.

(iv) *f* is odd, *g* is even $\Rightarrow fog$ is even function.

(v) Composite of functions is not commutative *i.e.*, $fog \neq gof$

(vi) Composite of functions is associative *i.e.*, $(fog)oh = fo(goh)$

(vii) If $f : A \to B$ is bijection and $g : B \to A$ is inverse of *f*. Then $fog = I_B$ and $gof = I_A$.

where, I_A and I_B are identity functions on the sets *A* and *B* respectively.

(viii) If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is bijection and $(gof)^{-1} = (f^{-1} og^{-1})$.

(ix) $fog \neq gof$ but if , $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also, $(fog)(x) = (gof)(x) = (x)$.

Important Tips

- *gof(x) is simply the g-image of f(x), where f(x) is f-image of elements $x \in A$.*
- *Function gof will exist only when range of f is the subset of domain of g.*
- *fog does not exist if range of g is not a subset of domain of f.*
- *fog and gof may not be always defined.*
- *If both f and g are one-one, then fog and gof are also one-one.*
- *If both f and g are onto, then gof is onto.*

Example: 49 If $f : R \to R, f(x) = 2x - 1$ and $g : R \to R,\ g(x) = x^2$ then $(gof)(x)$ equals **[Rajasthan PET 1987]**

(a) $2x^2 - 1$ (b) $(2x-1)^2$ (c) $4x^2 - 2x + 1$ (d) $x^2 + 2x - 1$

Solution: (b) $gof(x) = g\{f(x)\} = g(2x-1) = (2x-1)^2$.

Example: 50 If $f : R \to R, f(x) = (x+1)^2$ and $g : R \to R, g(x) = x^2 + 1$, then $(fog)(-3)$ is equal to **[Rajasthan PET 1999]**

(a) 121 (b) 144 (c) 112 (d) 11

Solution: (a) $fog(x) = f\{g(x)\} = f(x^2+1) = (x^2+1+1)^2 = (x^2+2)^2 \Rightarrow fog(-3) = (9+2)^2 = 121$.

Example: 51 $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then $(gof)(x)$ is equal to **[IIT 1996]**

(a) 1 (b) –1 (c) 2 (d) – 2

Solution: (a) $f(x) = \sin^2 x + \sin^2(x + \pi/3) + \cos x \cos(x + \pi/3) = \frac{1 - \cos 2x}{2} + \frac{1 - \cos(2x + 2\pi/3)}{2} + \frac{1}{2}\{2\cos x \cos(x + \pi/3)\}$

$= \frac{1}{2}[1 - \cos 2x + 1 - \cos(2x + 2\pi/3) + \cos(2x + \pi/3) + \cos \pi/3]$

$= \frac{1}{2}\left[\frac{5}{2} - \{\cos 2x + \cos\left(2x + \frac{2\pi}{3}\right)\} + \cos\left(2x + \frac{\pi}{3}\right)\right] = \frac{1}{2}\left[\frac{5}{2} - 2\cos\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2x + \frac{\pi}{3}\right)\right] = 5/4$ for all x.

$\therefore \quad gof(x) = g(f(x)) = g(5/4) = 1 \quad [\because g(5/4) = 1$ (given)]

Hence, $gof(x) = 1,$ for all x.

Example: 52 If $g(x) = x^2 + x - 2$ and $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2,$ then $f(x)$ is equal to **[Roorkee 1998; MP PET 2002]**

(a) $2x - 3$ (b) $2x + 3$ (c) $2x^2 + 3x + 1$ (d) $2x^2 - 3x - 1$

Solution: (a) $g(x) = x^2 + x - 2 \Rightarrow (gof)(x) = g[f(x)] = [f(x)]^2 + f(x) - 2$

Given, $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2 \qquad \therefore \quad \frac{1}{2}[f(x)]^2 + \frac{1}{2}f(x) - 1 = 2x^2 - 5x + 2$

$\Rightarrow [f(x)]^2 + f(x) = 4x^2 - 10x + 6 \Rightarrow f(x)[f(x) + 1] = (2x - 3)[(2x - 3) + 1] \Rightarrow f(x) = 2x - 3$.

Example: 53 If $f(y) = \frac{y}{\sqrt{1 - y^2}}, g(y) = \frac{y}{\sqrt{1 + y^2}},$ then $(fog)(y)$ is equal to

(a) $\frac{y}{\sqrt{1 - y^2}}$ (b) $\frac{y}{\sqrt{1 + y^2}}$ (c) y (d)

Solution: (c) $f[g(y)] = \frac{y/\sqrt{1 + y^2}}{\sqrt{1 - \left(\frac{y}{\sqrt{1 + y^2}}\right)^2}} = \frac{y}{\sqrt{1 + y^2}} \times \frac{\sqrt{1 + y^2}}{\sqrt{1 + y^2 - y^2}} = y$

Example: 54 If $f(x) = \frac{2x - 3}{x - 2},$ then $[f\{f(x)\}]$ equals **[Rajasthan PET 1996]**

(a) x (b) $-x$ (c) $\frac{x}{2}$ (d) $-\frac{1}{x}$

Solution: (a) $f[f(x)] = \frac{2\left(\frac{2x - 3}{x - 2}\right) - 3}{\left(\frac{2x - 3}{x - 2}\right) - 2} = x$

Example: 55 Suppose that $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x,$ then $f(x)$ is **[MP PET 2000; Karnataka CET 2002]**

(a) $1 + 2x^2$ (b) $2 + x^2$ (c) $1 + x$ (d)

Solution: (b) $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$ ..... (i)

$\Rightarrow \quad f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$

Put $1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2$

then, $f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2$

therefore, $f(x) = 2 + x^2$.

Example: 56 Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, then for all x, $f(g(x))$ is equal to **[IIT Screening 2001; UPSEAT 2001]**

(a) x (b) 1 (c) $f(x)$ (d)

Solution: (b) Here $g(x) = 1 + n - n = 1, x = n \in Z$

$1 + n + k - n = 1 + k, \; x = n + k \quad$ (where $n \in Z, \; 0 < k < 1$)

Now $f(g(x)) = \begin{cases} -1, & g(x) < 0 \\ 0, & g(x) = 0 \\ 1, & g(x) > 0 \end{cases}$

Clearly, $g(x) > 0$ for all x. So, $f(g(x)) = 1$ for all x.

Example: 57 If $f(x) = \frac{2x+1}{3x-2}$, then $(fof)(2)$ is equal to **[Kerala (Engg.) 2002]**

(a) 1 (b) 3 (c) 4 (d) 2

Solution: (d) Here $f(2) = \frac{5}{4}$

Hence $(fof)(2) = f(f(2)) = f\left(\frac{5}{4}\right) = \frac{2 \times \frac{5}{4} + 1}{3 \times \frac{5}{4} - 2} = 2$.

Example: 58 If $f : R \to R$ and $g : R \to R$ are given by $f(x) = |x|$ and $g(x) = [x]$ for each $x \in R$, then $\{x \in R : g(f(x)) \le f(g(x))\} =$ **[EAMCET 2003]**

(a) $Z \cup (-\infty, 0)$ (b) $(-\infty, 0)$ (c) Z (d)

Solution: (d) $g(f(x)) \le f(g(x)) \Rightarrow g(|x|) \le f[x] \Rightarrow [|x|] \le |[x]|$. This is true for $x \in R$.

2.1.10 Inverse Function

If $f : A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each element $b \in B$ with element $a \in A$, such that $f(a) = b$, is called the inverse function of the function $f : A \to B$

$$f^{-1} : B \to A,\ f^{-1}(b) = a \Rightarrow f(a) = b$$

In terms of ordered pairs inverse function is defined as $f^{-1} = (b, a)$ if $(a, b) \in f$.

Note : ❒ For the existence of inverse function, it should be one-one and onto.

Important Tips

- *Inverse of a bijection is also a bijection function.*
- *Inverse of a bijection is unique.*
- $(f^{-1})^{-1} = f$
- *If f and g are two bijections such that (gof) exists then* $(gof)^{-1} = f^{-1}og^{-1}$.
- *If* $f : A \to B$ *is a bijection then* $f^{-1} : B \to A$ *is an inverse function of f.* $f^{-1}of = I_A$ *and* $fof^{-1} = I_B$. *Here* I_A, *is an identity function on set A, and* I_B, *is an identity function on set B.*

Example: 59 If $f : R \to R$ is given by $f(x) = 3x - 5$, then $f^{-1}(x)$ **[IIT Screening 1998]**

(a) Is given by $\frac{1}{3x-5}$ (b) Is given by $\frac{x+5}{3}$

(c) Does not exist because *f* is not one-one (d) Does not exist because *f* is not onto

Solution: (b) Clearly, $f : R \to R$ is a one-one onto function. So, it is invertible.

Let $f(x)=y$. then, $3x-5=y \Rightarrow x=\frac{y+5}{3} \Rightarrow f^{-1}(y)=\frac{y+5}{3}$. Hence, $f^{-1}(x)=\frac{x+5}{3}$.

Example: 60 Let $f: R \to R$ be defined by $f(x)=3x-4$, then $f^{-1}(x)$ is

(a) $3x+4$ (b) $\frac{1}{3}x-4$ (c) $\frac{1}{3}(x+4)$ (d) $\frac{1}{3}(x-4)$

Solution: (c) $f(x)=3x-4=y \Rightarrow y=3x-4 \Rightarrow x=\frac{y+4}{3} \Rightarrow f^{-1}(y)=\frac{y+4}{3} \Rightarrow f^{-1}(x)=\frac{x+4}{3}$.

Example: 61 If the function $f: R \to R$ be such that $f(x)=x-[x]$, where $[y]$ denotes the greatest integer less than or equal to y, then $f^{-1}(x)$ is

(a) $\frac{1}{x-[x]}$ (b) $[x]-x$ (c) Not defined (d) None of these

Solution: (c) $f(x)=x-[x]$ Since, for $x=0 \Rightarrow f(x)=0$

For $x=1 \Rightarrow f(x)=0$.

For every integer value of x, $f(x)=0$

$\Rightarrow$ $f(x)$ is not one-one $\Rightarrow$ So $f^{-1}(x)$ is not defined.

Example: 62 If $f:[1,\infty) \to [1,\infty)$ is defined as $f(x)=2^{x(x-1)}$ then $f^{-1}(x)$ is equal to **[IIT Screening 1999]**

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$

(c) $\frac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$ (d) Not defined

Solution: (b) Given $f(x)=2^{x(x-1)} \Rightarrow x(x-1)=\log_2 f(x)$

$\Rightarrow x^2-x-\log_2 f(x)=0 \Rightarrow x=\frac{1\pm\sqrt{1+4\log_2 f(x)}}{2}$

Only $x=\frac{1+\sqrt{1+4\log_2 f(x)}}{2}$ lies in the domain

$\therefore$ $f^{-1}(x)=\frac{1}{2}[1+\sqrt{1+4\log_2 x}]$

Example: 63 Which of the following function is invertible **[AMU 2001]**

(a) $f(x)=2^x$ (b) $f(x)=x^3-x$ (c) $f(x)=x^2$ (d) None of these

Solution: (a) A function is invertible if it is one-one and onto.

Example: 64 If $f(x)=x^2+1$, then $f^{-1}(17)$ and $f^{-1}(-3)$ will be **[UPSEAT 2003]**

(a) 4, 1 (b) 4, 0 (c) 3, 2 (d) None of these

Solution: (d) Let $y=x^2+1 \Rightarrow x=\pm\sqrt{y-1}$

$\Rightarrow f^{-1}(y)=\pm\sqrt{y-1} \Rightarrow f^{-1}(x)=\pm\sqrt{x-1}$

$\Rightarrow f^{-1}(17)=\pm\sqrt{17-1}=\pm 4$

and $f^{-1}(-3)=\pm\sqrt{-3-1}=\pm\sqrt{-4}$, which is not possible.