2

Relations and Functions

2.01 Introduction

Relation is a common word used in daily life, we come across different types of relation in our daily life. For example :

(i) Delhi is the capital of India.
(ii) Shyam is son of Sohan.
(iii) 5 is the divisor of 15.
(iv) Triangle ABC is similar to triangle DEF.
(v) Set B, is a subset of set A.

In all these, we notice that a relation involves pair of object elements, places or persons in certain order. In this Chapter, we will learn how to link pair of objects from two sets and then introduce relations between the two objects in the pair.

Statement : A statement is a valid sentence which is either true or false

For Example -

(i) The sunrises in the east.
(ii) London is the capital of America.
(iii) 49 is the square of 7.
(iv) 90° is called a right angle.

All the above are statements where as (i), (iii) and (iv) are true and (ii) is false

2.02 Open Sentence

Those Sentences which do not give any additional information to be cosnidered as true or false are known as open sentences.

Example :

(i) $x+5=20$
(ii) $-5<x<3$
(iii) x, is a city in India
(iv) $x^2+y^2=10$
(v) $x>2y+3$

Above sentences are all open sentences. In example (i), (ii) and (iii) only one variable is used whereas in (iv) and (v) two variables x and y are used. Such open sentences with variable x is represented by $P(x)$ and with two variables x, y is represented by $P(x, y)$. sentence having more than two variables are also possible.

The variable selected from the set of open sentence are called replacement set and for those values for which the open sentence is cosnidered as true are called as solution set.

2.03 Ordered Pair

Generally there is no importance of order of elements in sets. For example $A=\{a, b, c, d\}$ and $B=\{d, a, c, b\}$ then there is no difference between A and B. Hence A = B, thus, it is clear that by changing the order of elements of set no change occur in set.

But if in any set the order of elements has importance so that sets are called as ordered set. For example we know that $235 \neq 523$ whereas, in both numbers 2, 3 and 5 digits are used. Here the order of digit is important order set of 2 digits is called as ordered pair. It is denoted from (a, b), (x, y) etc. elearly $(a, b) \neq (b, a)$ and $(a, b)=(c, d) \Leftrightarrow a=c, b=d$. In ordered pair (a, b) a is called as first element and b is called second element. Both elements of ordered pair can be same or can be different. For example (5, 7), (x, y), (3, 3), (a, a) all denotes ordered pair.

If the number of elements in any ordered set is n, then this type of set is called as ordered n-tupple and it represented as $(a_1, a_2, a_3 ... a_n)$.

2.04 Cartesian Product of Two Sets

Given two non-empty sets A and B the cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B, i.e.,

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Example : If $A = \{a, b, c\}$ and $B = \{x, y\}$ then $A \times B = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\}$

Remakrs:

(i) If $A = \phi$ or $B = \phi$ then $A \times B = \phi$.

(ii) If there are m elements in A and n elements in B, then there will be mn elements in $A \times B$.

(iii) If A and B are non empty sets and one or both are infinite set then number of elements in A × B will be infinite i.e. A × B will be infinit set.

2.05 Relation

Let A and B are two non empty sets. A relation R from a set A to set B is defined as $P(x, y)$, where $x \in A,\ y \in B$ i.e. $R = \{(x, y) : x \in A,\ y \in B,\ P(x, y)\}$ for any value of x, y if

(i) If $P(a, b)$ is true if element a of set A is related to element b of set B then we say, $a\,R\,b$ or $(a, b) \in R$.

(ii) If $P(a, b)$ is false if element a of set A is not related to element b of set B then we say $a \not R\, b$ or $(a, b) \notin R$.

Example 1. If $A = \{1, 2, 3, 5, 7\}$, $B = \{1, 4, 6, 9\}$ and $\{P(x, y) : y$ is twice of $x\}$ then

$R = \{(x, y) : x \in A,\ y \in B,\ P(x, y)\}$ A is related to B then 2R4, 3R6 but $1 \not R 4$, $3 \not R 9$ etc.

this can be shown as $(2, 4) \in R$, $(3, 6) \in R$ but $(1, 4) \notin R$, $(3, 9) \notin R$ etc.

Example 2. If N is a set of natural numbers and $P(x, y) : x$, is a divisor of y, then

$R = \{(x, y) : x,\ y \in N,\ P(x, y)\}$ R is a relation in which $2R2$, $2R4$, $5R10$, $2 \not R 3$, $7 \not R 4$ etc.

or $(2, 2) \in R, (2, 4) \in R, (5, 10) \in R$ but $(2, 3) \notin R, (7, 4) \notin R$ etc.

Example 3. If $A = \{1, 2, 3, 4, 5\}$, $B : \{2, 3, 4, 5, 6\}$ and $P(x, y) : x$ is greater then y, then

$R = \{(x, y) : x \in A,\ y \in B,\ P(x, y)\}$ A is related to B then $3R2$, $4R3$, $4R2$, $5R3$, $5R4$ but $2 \not R 4$, $3 \not R 5$ etc. this can be shown as $(3, 2) \in R, (5, 3) \in R$ but $(2, 4) \notin R, (3, 5) \notin R$ etc.

Remarks : It is clear from the above examples

(i) It is not necessary that every element of set A is related to some element of set B. It means A can have the elements which are not releated to any element of B.

(ii) Any element of A can be related to more than one elements of B.

(iii) More than one element of A can be related to only one element of B

2.06 Relation as a Set of Ordered Pairs

With the concept of open statement we see that if in $P(a,b)$ where $a \in A,\ b \in B$ is true then $(a,b) \in R$ i.e. all the elements of the relationwill belong to $A \times B$. Thus $R \subseteq A \times B$.

Converse : Any subset of type (a, b) of A × B will be set of ordered pair, Hence, define a relation from A to B. Thus from set A to set B any relation can be defiend as follows.

Definition : Any defined relation R, from a set A to a set B, is a subset of A × B i.e. $R \subseteq A \times B$.

Remakrs: If number of elements in A and B are m and n respectively. Then A × B contains $m \times n$ elements. Hence number of non-empty subsets will be $2^{mn} - 1$. Also, non-empty relation defined from A to B will be $2^{mn}-1$.

2.07 Domain and Range of a Relation

The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

The set of all second elements of the ordered pains in a relation R from a set A to a set B is called the *range* of the relation R. If means domain of R is $\{a_1(a, b) \in R\}$ and range of R, is $\{b_1(a, b) \in R\}$. It is clear that domain of R is subset of A and range of R is subset of B.

Example 1. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$ Let $R = \{(a, b) \mid a \in A, b \in B$ a is divisor of $b\}$. If this is a relation from A to B, then

$$R = \{(1, 2)\ (1, 4)\ (1, 6)\ (1, 8)\ (1, 10)\ (2, 2)\ (2, 4)\ (2, 6)\ (2, 8)\ (2, 10)\ (3, 6)\ (4, 4)\ (4, 8)\ (5, 10)\}$$

Hence domain of $R = \{1, 2, 3, 4, 5\}$ and range of $R = \{2, 4, 6, 8, 10\} = B$

Example 2. Define A relation is defined as

$$R = \left\{(x, y) \middle| x, y \in z,\ x^2 + y^2 \le 4\right\}$$

Domain of $R = \{-2, -1, 0, 1, 2\}$ and Range $= \{-2, -1, 0, 1, 2\}$

2.08 Inverse Relation

Let R be a relation from set A to set B. Then the inverse of R will be R^{-1}, defined from set B to set A as

$$R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$$

i.e. $\quad (a,b) \in R \quad \Leftrightarrow \quad (b,a) \in R^{-1}$

or $\quad aRb \quad \Leftrightarrow \quad bR^{-1}a$

Thus domain of R^{-1} = Range of R and Range of R^{-1} = Domain of R

Example 1. If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 8\}$ and a relation defined from A to B is $R = \{(1,4), (2,5), (3,6)\}$ then $R^{-1} = \{(4,1), (5,2), (6, 3)\}$

again domain of $R^{-1} = \{4,5,6\}$ range of R and Range of $R^{-1} = \{1,2,3\}$ domain of R.

Example 2. If a relation in N is defiend as 'x' is less than 'y' then $R = \{(x, y) | x, y \in N,\ x < y\}$ so its inverse relation $R^{-1} = \{(x, y) | x, y \in N,\ x > y\}$ is defined by x is greater then y.

2.09 Identity Relation

In a set A iffany element of set A is related itself set A then the relation is known as identity relation. It is represented as I_A thus : $I_A = \{(a,a) | a \in A\}$.

Example : If $A = \{x, y, z\}$ then $I_A = \{(x,x), (y,y), (z,z)\}$.

Illustrative Examples

Example 1. If $A = \{a,b,c,d\}$, $B = \{p,q,r,s\}$ explain with reason which amongst the following is relattion from A to B

(i) $R_1 = \{(a,q),(b,s),(c,r),(c,s)\}$ (ii) $R_2 = \{(b,q),(b,r),(b,s)\}$

(iii) $R_3 = \{(a,p),(b,q),(r,a),(d,s),(p,a)\}$ (iv) $R_4 = \{(d,p),(a,p),(b,s),(s,a)\}$

Solution :

(i) Clearly $R_1 \subseteq A \times B$ $\therefore$ R_1 is a relation from A to B.

(ii) Clearly $R_2 \subseteq A \times B$ $\therefore$ R_2 is a relation from A to B.

(iii) Clearly $(r,a) \in R_3$ but $(r,a) \not\subseteq A \times B$ and $(p,a) \in R_3$ but $(p,a) \not\subseteq A \times B$.

$\therefore$ $R_3 \not\subseteq A \times B$ $\therefore$ R_3 is not a relation from A to B

(iv) Clearly $(s,a) \in R_4$ but $(s,a) \not\subseteq A \times B$ $\therefore$ R_4 also is not a relation from A to B

Example 2. Set C is a set of complex numbers. Relation R is defined as { $x\,R\,y \Leftrightarrow$ x is a conjugate of y} Explain with reason which of the following statements are true or false-

(i) $2R2$ (ii) i R i (iii) $-3R3$ (iv) $(1-i)R(1-i)$

(v) $(1-i)R(1+i)$ (vi) $(-1+i)R(1+i)$

Solution : (i) $2 = 2 + i \cdot 0$, conjugate of $2 = 2 - i \cdot 0 = 2$ $\therefore$ $2R2$ is true

(ii) $i = 0 + i \cdot 1$, therefore its conjugate of $i = 0 - i \cdot 1 = -i$ $\therefore$ $i \not R i$ $\therefore$ $i R i$ is false.

(iii) $-3 = -3 + i \cdot 0$ therefore its conjugate is $= -3 - i \cdot 0 = -3$. $\therefore$ $-3 \not R 3$ $\therefore$ $-3R3$ is false.

(iv) Conjugate of $(1-i)$ is $(1+i)$ $\therefore$ $(1-i) \not R (1-i)$ $\therefore$ $(1-i)R(1-i)$ is false.

(v) Conjugate of $(1-i)$ is $(1+i)$ $\therefore$ $(1-i)R(1+i)$ is true

(vi) Conjugate of $(-1+i)$ is $(-1-i)$ $\therefore$ $(-1+i)R(1+i)$ is false.

Example 3. A is a set of first 10 natural numbers. A relation R is defined as $xRy \Leftrightarrow x+2y=10$ then

(i) Write R and R^{-1} as a set of the ordered pairs

(ii) Find the domain of R and R^{-1}

(iii) Find the range of R and R^{-1}

Solution : Here $A=\{1,2,3,4,5,6,7,8,9,10\}$ according to the definition $xRy \Leftrightarrow x+2y=10$

$$\Leftrightarrow y=\frac{10-x}{2} \Rightarrow x=1,\ y=\frac{9}{2}\notin A.$$

1 is not related to any element of A, similarly 3, 5, 7, 9 and 10 also are not related to the any element of A Again

When $x=2; y=\frac{10-2}{2}=4\in A \Rightarrow 2R4$

When $x=4; y=\frac{10-4}{2}=3\in A \Rightarrow 4R3$

When $x=6; y=\frac{10-6}{2}=2\in A \Rightarrow 6R2$

When $x=8; y=\frac{10-8}{2}=1\in A \Rightarrow 8R1$

(i) $\therefore\ R=\{(2,4),(4,3),(6,2),(8,1)\}$ and $R^{-1}=\{(4,2),(3,4),(2,6),(1,8)\}$.

(ii) Domain of $R=\{2,4,6,8\}$ and Domain of $R^{-1}=\{4,3,2,1\}$

(iii) Range of $R=\{4,3,2,1\}$ and Range of $R^{-1}=\{2,4,6,8\}$

Example 4. In set $A=\{2,\ 4,\ 5\}$ set $B=\{1,\ 2,\ 3,\ 4,\ 6,\ 8\}$ a relation R is defined as "x divides y".
Represent R as a set of ordersed pair and also find its domain and range.

Solution : As $2\in A$, we see that this divides the elements 2, 4 6 and 8 of set B.

$(2,2)\in R, (2,4)\in R, (2,6)\in R, (2,8)\in R$

Similarly $(4,4)\in R, (4,8)\in R$

But element 5 of A does not divide any of the element of B.

So, $R=\{(2,2),(2,4),(2,6),(2,8),(4,4),(4,8)\}$

Domain of $R=\{2,\ 4\}$ and Range of $R=\{2,\ 4,\ 6,\ 8\}$

Example 5. Find the Inverse relation of the following

(i) $R=\{(x,y),x,y\in N,x+2y=8\}$

(ii) R, is related from set $A=\{8.9,10,11\}$ to set $B=\{5,6,7,8\}$ as $y=x-2$

(iii) $R=\{(x,y),\ x,y\in N,\ x \text{ is a divisor of } y\}$

Solution : (i) $x+2y=8 \Rightarrow y=\dfrac{8-x}{2}$ $\because y\in N$ $\therefore$ x should be always less then 8.

By putting $x=1$, $y=7/2\notin N$ $\therefore$ 1 is not related to any natural number.

By putting $x=2$ we have $y=3\in N$ $\therefore$ $(2,3)\in R$

similarly $(4,2,)\in R$ and $(6,1)\in R$

$\therefore$ $R=\{(2,3),(4,2),(6,1)\}$ $\therefore$ $R^{-1}=\{(3,2),(2,4),(1,6)\}$

(ii) taking $x=8\in A$ we have $y=8-2=6\in B$ $\therefore$ $(8,6)\in R$

$x=9\in A$ we have $y=9-2=7\in B$ $\therefore$ $(9,7)\in R$

$x=10\in A$ we have $y=10-2=8\in B$ $\therefore$ $(10,8)\in R$

$x=11\in A$ we have $y=11-2=9\notin B$

$\therefore$ $R=\{(8,6),(9,7),(10,8)\}$ $\Rightarrow$ $R^{-1}=\{(6,8),(7,9),(8,10)\}$

(iii) *For elements of N* if x is a divisor of y then y will be the multiple of x

$\therefore$ $R^{-1}=\{(x,y)\,|\,x,y\in N,\ x \text{ is a multiple of } y\}$

Exercise 2.1

1. If $A=\{1,2,3\}$, $B=\{4,5,6\}$ then which amongst the following is related from A to B? Explain with reason.

(i) $\{(1,4),(3,5),(3,6)\}$ (ii) $\{(1,6),(2,6),(3,6)\}$

(iii) $\{(1,5),(3,4),(5,1),(3,6)\}$ (iv) $\{(2,4),(2,6),(3,6),(4,2)\}$

(v) $A\times B$

2. Express the following Relations on a set of N in set builder form.

(i) $\{(1,3),(2,5),(3,7),(4,9),\ldots\}$ (ii) $\{(2,3),(4,2),(6,1)\}$

(iii) $\{(2,1),(3,2),(4,3),(5,4),\ldots\}$

3. Define a relation R from a set $A=\{2,3,4,5\}$ to a set $B=\{3,6,7,10\}$ in such a way that $xRy \Leftrightarrow x$ is a prime divisor of y Write relation R in the form of set of ordered pair. Write down the domain and the range of R .

4. Define a relation R on set of integers Z such a way that $xRy \Leftrightarrow x^2+y^2=25$ then write R and R^{-1} in the form of set of ordered pair and also find the domain and range.

5. Define a relation $R=\phi$ on set of Complex number C to a set of real numbers R in such a way that $x\phi y \Leftrightarrow |x|=y$

Explain with reason which of the following is true or false.

(i) $(1+i)\phi 3$ (ii) $3\phi(-3)$ (iii) $(2+3i)\phi 13$ (iv) $(1+i)\phi 1$

6. Defined a realtion R "$x<y$" on a set $A=\{1,2,3,4,5\}$ to set $B=\{1,4,5\}$ then write R in form of set of ordered pair and also find R^{-1} .

7. Write the following realtion in roster form

(i) R_1, A relation defined as "$x=2y$" from a set $A=\{1,2,3,4,5,6\}$ to set $B=\{1,2,3\}$

(ii) R_2, A relation defined as "$y=x-2$" from a set $A=\{8,9,10,11\}$ to set $B=\{5,6,7,8\}$

(iii) R_3, A relation defined as $2x+3y=12$ from a set $A=\{0,1,2,\ldots,10\}$ to set?

(v) R_4, A relation defined as "x is a divisor of y' on a set $A=\{5,6,7,8\}$ to set $B=\{10,12,15,16,18\}$

8. Find the Inverse of the following relations

(i) $R=\{(2,3),(2,4),(3,3),(3,2),(4,2)\}$ (ii) $R=\{(x,y)|x,y\in N; x<y\}$

(iii) R, Define a relation $2x+3y=12$ on a set $A=\{0,1,2,\ldots,10\}$ to set?

2.10 Kinds of Relations

(i) Reflexive Relation: If a relation R is defined in set A, every element of set A is related to itself then that relation is known as Reflexive Relation i.e. R is reflexive if $(a,a)\in R,\ \forall a\in A.$

If an element belongs to in set A which is not related to itself then the given relation is not reflexive.

Remark : For a reflexive relation $(a,a)\in R$ but this does not mean that element a does not have relation except a. It means the relation of a with self also be with other element of A. But in identify relation a has relation only with a. It means every identity relation is a reflexive relation but each reflexive relation is not identity relation.

Example 1. Let $A=\{a,b,c,d\}$ and $R=\{(a,a),(a,d),(b,a),(b,b),(c,d),(c,c)(d,d)\}$ defiend on set A then R is reflexive because $(a,a)\in R,(b,b)\in R,(c,c)\in R$ and $(d,d)\in R$ but if there exist a relation R_1 defined on A such that

$$R_1=\{(a,a),(a,d),(b,c),(b,d),(c,c),(c,d),(d,b)\}$$

then R_1 is not reflexive as $b\in A$ but $(b,b)\not\in R_1$ similarly $d\in A$ but $(d,d)\not\in R_1$.

Example 2. If R is a relation defined on a set of natural numbers N such that $xRy\Leftrightarrow x\geq y$ then R is reflexive as $x\in N\Rightarrow x=x$ but if R is defiend as $xRy\ \Leftrightarrow\ x>y$ then R is not reflexive as for all $x>x$ is not true.

Example 3. If R is a relation defined on a set A of parallel lines in a plane elements of N, defined as $xRy\Leftrightarrow x$ is parallel to y will be reflexive as every line is parallel to itself but if R is defined as $xRy\Leftrightarrow x$ is perpendicular to y then R will not be reflexive as a line cannot be perpendicualr to itself.

(ii) Symmetric Relation: A relation defined on a non-empty set A defined on a realtion R, if element a is related to b then b should also relates to a, then relation R will be called as Symmetric i.e. Symmetric relation R is defined as

$$(a,b)\in R \Rightarrow (b,a)\in R\ \forall\ a,b\in A.$$

R is not Symmetric on A if for atleast two elements a, b is such that $(a,b)\in R$ but $(b,a)\notin R$.

Note: If R is Symmetric on set A then $xRy \Leftrightarrow yRx$.

i.e. $(x,y)\in R \Rightarrow (y,x)\in R \Rightarrow (x,y)\in R^{-1}$

$\therefore \quad R\subseteq R^{-1}$ (1)

similarly $(x,y)\in R^{-1} \Rightarrow (y,x)\in R \Rightarrow (x,y)\in R$ (R is Symmetric)

$\therefore \quad R^{-1}\subseteq R$ (2)

from (1) and (2) $R=R^{-1}$

Example 1. If a relation R_1 and R_2 defined on set $A=\{a,b,c,d\}$ such that

$$R_1=\{(a,b),(b,d),(b,c)(b,a),(d,b),(c,b)\}$$

and $R_2=\{(a,c),(a,d),(b,d),(c,a),(d,b),(a,b),(b,a)\}$

then, R_1 is symmetric but R_2 is not symmetric as $(a,d)\in R_2$ but $(d,a)\not\subseteq R_2$.

Example 2. The relation " is congruent to $(\cong)$" on a set A of Triangles is symmetric as $\Delta_1\cong\Delta_2\Rightarrow\Delta_2\cong\Delta_1$.

Example 3. If R is defined as $xRy\Leftrightarrow x$ is perpendicular to y on a set A of lines in a plane then R is symmetric as for all $\ell_1,\ell_2\in A\ \ \ell_2\perp\ell_1\Rightarrow\ell_1\perp\ell_2$.

(iii) Anti-symmetric Relation: If R is a realtion defined on set A is such that the realtion of element a with b and of b with a is true iff $a=b$ then R is called Anti-symmetric relation i.e.

$$(a,b)\in R \text{ and } (b,a)\in R \Rightarrow a=b,\ \forall\, a,b,\in A$$

R is not Anti-symmetric relation iff atleast two elements a, b in A are present for all $(a,b)\in R$ and $(b,a)\in R$ but $a\neq b$

Example 1. Define a relation R on the sets of set S as $ARB\Leftrightarrow A$ is a subset of B then R is Anti-symmetric relation as for any two sets A and B, $A\subseteq B$ and $B\subseteq A\Rightarrow A=B$

Example 2. Define a relation R on the set of natural numbers N such that $x\,R\,y\Leftrightarrow x$ is a divisor of y then R is Anti-symmetric relation as for every N, x is a divisor of y then y is a divisor of x is true iff $x=y$

Example 3. Define a realtion R on the set of Real numbers A such that $x\,R\,y\Leftrightarrow x\geq y$ then R is Anti-symmetric relation as $x\geq y$ and $y\geq x\Rightarrow x=y$

(iv) Transitive Relation: A relation a non-empty set A defined on a is transitive, if element a is related to b, b is related to c then a is related to c, such relations are called as Transitive Relation i.e. for all $(a,b)\in R$

and $(b,c) \in R \Rightarrow (a,c) \in R, \ \forall \ a,b,c, \in A$

R will not be Transitive iff $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

Example 1. In set $A = \{a,b,c,d\}$, $R = \{(a,b),(a,c),(a,d),(b,d),(b,c)\}$ is a transitive relation.

Example 2. Set A is defined on lines in a plane as "x is parallel to y" is a transitive relation as for all $\ell_1, \ell_2 \ell_3 \in A$ and $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3 \Rightarrow \ell_1 \parallel \ell_3$.

Example 3. If a relation R is definedon a set of natural numbers N such that $x R y \Leftrightarrow x$ and y are odd then R is Transitive as for all $x, y, z \in N$ and $x R y$ and $y R z \Rightarrow x, y$ and y, z are odd i.e. x, z both are odd so xRz

(v) Equivalence Relation: A relation on a set A is said to be an Equivalence relation iff

(i) R is Reflexive that is $(a,a) \in R \ \forall a \in A$

(ii) R is Symmetric that is $(a,b) \in R \Rightarrow (b,a) \in R, \ \forall a,b \in A$

(iii) R is Transitive if that is $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R, \ \forall a,b,c \in A$

Example 1. Define a relation R on a set N set of natural numbers such that $x R y \Leftrightarrow x = y$ then R is an Equivalence relation iff for all $a,b,c, \in N$

(i) $a = a$ $\quad \therefore a R a \ \forall a \in N$ $\quad$ R is Reflexive

(ii) $a = b \Rightarrow b = a$ $\quad \therefore a R b \Rightarrow b R a$ $\quad$ R is Symmetric

(iii) $a = b$ and $b = c \Rightarrow a = c$

$a R b$ and $b R c \Rightarrow a R c$ $\quad$ R is Transitive

Example 2. Define a relation R in a plane with set A as a set of points such that $x R y \Leftrightarrow x$ and y are equidistant from the origin then R is an Equivalence relation as

(i) $x \in A \Rightarrow x$ and y are equidistant from the origin

$\therefore x R x \ \forall x \in A \ \therefore$ R is Reflexive

(ii) Let $x, y \in A$ and $x R y$ i.e. x and y are also equidistant from the origin $\Rightarrow$ y and x are also equidistant from the origin i.e. $y R x$

$\therefore x R y \Rightarrow y R x \quad \therefore R$ is Symmetric

(iii) Let $x, y, z \in A$ and $x R y$ and $y R z$.

i.e. x and y are equidsistant from the origin and y and z are equidistant fromt he origin

$\therefore$ x and z are also equidsitant from the origin

i.e. $x R z \ \therefore x R y$ and $y R z \Rightarrow x R z \ \therefore R$ is transitive

Thus R is an Equivalence relation

(iv) Partial Order Relation: A relation R on set A is said to be a Partial Order relation if

(a) R is reflexive $\quad$ (b) R is Anti Symmetric $\quad$ (c) R is transitive

If a relation R on set A is a Partial Order relation then set A is called as Partially ordered set.

Example 1. "x is a subset of y" defiend on a set A is a Partial Order realtion

Example 2. "x is a divisor of y" defined on a set N of natural numbers is a Partial Order relation because this relation is reflexive. Anti-symmetric and transitve.

(v) Total Order Relation: A relation R on set A is said ot be a Total Order relation if

(a) R is a Partial Order relation

(b) For every $a, b \in A$ either $(a,b) \in R$ or $(b,a) \in R$ or $a = b$ is true

Example 1. On a set of natural numbers N " $x \le y$ " is a Total Order relation if for all $x, y \in N$, $x \le y$ or $y \le x$ or any one is true if $x = y$.

Illustrative Examples

Example 6. Test the Reflexivity, Symmetry and Transitivity of R and P

(i) In N : $a\,R\,b \Leftrightarrow b$ divides a

(ii) $\alpha\,P\,\beta \Leftrightarrow \alpha \perp \beta$ where α and β are the lines in a plane

Solution : (i) By the definition of R $a\,R\,b \Leftrightarrow a|b$.

Reflexivity: R is Reflexive, for every natural number a, $a \mid a$

Symmetry : R is not symmetric as for all $a, b \in N$ when a divides b but b does not divide a till $a = b$

Transitivity : R is transtiive as for all $a, b, c, \in N$, $a|b$ and $b|c \Rightarrow a|c$

(ii) By the definition of P, $\alpha\,P\,\beta \Rightarrow \alpha \perp \beta$.

Reflexivity: P is not Reflexive as for any straight line α , $\alpha \perp \alpha$ is not true

Symmetry : P is symmetric, as for any straight line α, β , $\alpha \perp \beta \Rightarrow \beta \perp \alpha$

Transitivity : P is not transitive as for straight lines α, β, γ

$$\alpha \perp \beta \text{ and } \beta \perp \gamma \Rightarrow \alpha \parallel \gamma.$$

Example 7. N is a set of natural numbers. If from $N \times N$ a realtion R is defined in such a way that $(a,b)R(c,d) \Leftrightarrow a+d=b+c$ where $a, b, c, d \in N$, then prove that R is an equivalence relation.

Solution : (i) Clearly $(a,b)\,R\,(a,b)$ as $a+b=b+a$ therefore R is reflexive

(ii) Let $(a,b),(c,d) \in N \times N$ then $(a,b)\,R\,(c,d) \Rightarrow a+d=b+c$

$$\Rightarrow c+b=d+a$$

$$\Rightarrow (c,d)\,R\,(a,b) \text{ thus } R \text{ is symmetric}$$

(iii) Let $(a,b),(c,d),(e,f) \in N \times N$ then $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$

$\Rightarrow a+d=b+c$ and $c+f=d+e$

$\Rightarrow a+d+c+f=b+c+d+e \Rightarrow a+f=b+e \Rightarrow (a,b)\,R\,(e,f)$

$\therefore$ R is transitive

$\therefore$ R is an Equivalence relation

Example 8. Let $X = \{x_1, x_2, x_3, x_4\}$. Define a relation R_1, R_2, R_3 on X such that-

(i) $R_1 = \{(x_1,x_1),(x_2,x_2),(x_3,x_3),(x_2,x_3),(x_3,x_2)\}$

(ii) $R_2 = \{(x_1,x_1),(x_2,x_2),(x_3,x_3),(x_4,x_4),(x_2,x_3),(x_3,x_4),(x_2,x_4)\}$

(iii) $R_3 = \{(x_1,x_1),(x_2,x_2),(x_3,x_3),(x_4,x_4),(x_2,x_3),(x_3,x_2),(x_3,x_4),(x_4,x_3)\}$

Discuss the Reflexivity, Symmetry and Transitivity of the above relations.

Solution:

(i) Clearly R_1 is symmetric and transitive but not reflexive

because $x_4 \in X$ but $(x_4, x_4) \not\in R_1$

(ii) Celarly R_2 is reflexive and transitive but not symmetric

because $(x_3, x_4) \in R_2$ but $(x_4, x_3) \not\in R_2$, $(x_2, x_4) \in R_2$ but $(x_4, x_2) \not\in R_2$

(iii) Clearly R_3 si reflexive and symmetric but not transitive

because $(x_2, x_3) \in R_3$ and $(x_3, x_4) \in R_3$ but $(x_2, x_4) \not\in R_3$

Note: It is clear from the above example that reflexive, symmetric and transitive relation are all independent. It measn if one property satisfied then no compulision for second property to be satisfied.

Example 9. In a set of integers I define a relation $\cong$ such that-

If $m, n \in I$ then $m \cong n$ $(|k|)$ $\Leftrightarrow$ $m - n$ is divided by k where k is a non zero integer. Prove that it is an Equivalence relation.

Solution:

(i) If $a \in I$ then $a - a = 0$ Clearly divisible by k $\therefore$ $a \cong a$ $(|k|)$ $\forall a \in I$

Here for the given relation each element of 1 relatesto itself.

Thus the relation is reflexive.

(ii) Let $m, n \in I$ and $m \cong n$ $(|k|)$

then $m - n$ is divisible by k i.e., ~ $m - n = qk$

where k is an integer $\therefore n - m = (-q)k$ where $-q$ is an integer

therefore $n \equiv m$ $(|k|)$

Thus the relation is symmetric.

(iii) Let $m, n, p \in I$ where $m \cong n$ $(|k|)$ and $n \cong p$ $(|k|)$

therefore $m - n = qk$ and $n - p = rk$ where $q, r \in I$

Thus $m - p = (m - n) + (n - p) = qk + rk = (q + r)k$ where $(q + r) \in I$ therefore $m \cong p$ $(|k|)$

Thus, the relation is transitive.

Therefore the given relation is an equivalence relation.

Note: If $m \cong n$ $(|k|)$ then it is read as "m is congruent to n modulus k"

Example 10. If R is an equivalence relation on set A then prove that its inverse relation R^{-1} is also an equivalence relation.

Solution : $\because$ R is a relation on set A

$\therefore$ $R \subseteq A \times A \Rightarrow R^{-1} \subseteq A \times A$

$\therefore$ R^{-1} is also a relation on set A. Now we will prove that R^{-1} is also an equivalence relation

(i) Reflexivity: If $a \in A$ then $a \in A \Rightarrow (a, a) \in R \Rightarrow (a, a) \in R^{-1}$

Similarly $(a, a) \in R^{-1}, \forall a \in A$ $\therefore$ R^{-1} is reflexive

(ii) Symmetry: Let $(a, b) \in R^{-1}$ then

$(a,b)\in R^{-1} \Rightarrow (b,a)\in R \Rightarrow (a,b)\in R \Rightarrow (b,a)\in R^{-1}$ $\because R$ is symmetric

similarly $(a,b)\in R^{-1} \Rightarrow (b,a)\in R^{-1}$

Thus R is symmetric

(iii) Transitivity: Let $(a,b)\in R^{-1}$ and $(b,c)\in R^{-1}$

then $(a,b)\in R^{-1}$ and $(b,c)\in R^{-1} \Rightarrow (b,a)\in R$ and $(c,b)\in R$

$\Rightarrow (c,b)\in R$ and $(b,a)\in R$

$\Rightarrow (c,a)\in R$, $\because R$ transitive $\Rightarrow (a,c)\in R^{-1}$

similarly $(a,b)\in R^{-1}$ and $(b,c)\in R^{-1} \Rightarrow (a,c)\in R^{-1}$

thus R^{-1} is transitive $\therefore$ R^{-1} is an equivalence relation.

Example 11. If two equivalencerelations R and S are defined on a set A then prove that $R\cap S$ is also an Equivalence relation.

Solution : R and S are defined on set A $\therefore$ $R\subseteq A\times A$ and $S\subseteq A\times A \Rightarrow R\cap S\subseteq A\times A$

$\therefore$ $R\cap S$ is also defined on set A

now let us examine the reflexivity symmetry and transitivity on a set A of $R\cap S$

(i) Reflexivity - let a_1 be an element on set A

then $a\in A \Rightarrow (a,a)\in R$ and $(a,a)\in S$ $\because$ R and S are reflexive

$\Rightarrow (a,a)\in R\cap S$

$\therefore$ $(a,a)\in R\cap S, \forall a\in A$ $\therefore$ $R\cap S$ is reflexive.

(ii) Symmetry: Let a and b are such that $(a,b)\in R\cap S$ then

$(a,b)\in R\cap S \Rightarrow (a,b)\in R$ and $(a,b)\in S$

$\Rightarrow (b,a)\in R$ and $(b,a)\in S$

$\Rightarrow (b,a)\in R\cap S$ $\therefore R\cap S$ is assymmetric.

(iii) Transitivity: Let a,b,c are the elements of A

$(a,b)\in R\cap S$ and $(b,c)\in R\cap S$

$\therefore$ $(a,b)\in R$ and $(a,b)\in S$ and $(b,c)\in R$ and $(b,c)\in S$

now $(a,b)\in R$ and $(b,c)\in R \Rightarrow (a,c)\in R$ $\because$ R is transitive

and $(a,b)\in S$ and $(b,c)\in S \Rightarrow (a,c)\in S$ $\because$ R is transitive

similarly $(a,c)\in R$ and $(a,c)\in S \Rightarrow (a,c)\in R\cap S$

i.e. $(a,b)\in R\cap S$ and $(b,c)\in R\cap S \Rightarrow (a,c)\in R\cap S$

$\therefore$ $R\cap S, A$ is transitive $\therefore$ $R\cap S$ is an Equivalence relation.

Example 12. A relation R is defiend on a set of integers I such that $x\,R\,y \Leftrightarrow (x-y)$ is an even integer, Prove that it is an equivalence relation.

Solution: (i) Let $x \in I$ then $x - x = 0$ which is an even integer

$\therefore$ $x\,R\,x, \forall\, x \in I.$

$\therefore$ R is reflexive.

(ii) Let $x, y \in I$ and $x\,R\,y$ then $(x-y)$ is an even integer

$\therefore$ $y - x = -(x-y)$ is also an even integer

$\therefore$ $y\,R\,x$ and $x\,R\,y \Rightarrow y\,R\,x$ $\therefore$ R is symmetric

(iii) Let $x, y, z \in I$ and $x\,R\,y$ and $y\,R\,z$

i.e. $(x-y)$ is an even integer then $(y-z)$ is also an even integer.

now $x - z = (x-y)+(y-z) =$ even integer

$\therefore$ $x\,R\,z.$ similarly $x\,R\,y$ and $y\,R\,z \Rightarrow x\,R\,z$

$\therefore$ R is transitive $\therefore$ R is an equivalence relation

Exercise 2.2

1. Examine the Reflexivity, Symmetric and Transitivity of the following relations.
   (i) $m\,R_1\,n \Leftrightarrow m$ and n are odd $, \forall\, m, n \in N$
   (ii) On set A of (Power set) $P(A)$, $A\,R_2\,B \Leftrightarrow A \subseteq B, \forall\, A, B \in P(A)$
   (iii) Set S of lines defined on (Three dimensional Space) as $L_1 R_3 L_2 \Leftrightarrow L_1$ and L_2 are coplaner $\forall\, L_1, L_2 \in S$
   (iv) $a\,R_4 b \Leftrightarrow b, a$ is divisible by a, $\forall\, a, b \in N$
2. Set P is defined on a set of non-zero Real numbers R_0 such that
   (i) $x\,P\,y \Leftrightarrow x^2 + y^2 = 1$ (ii) $x\,P\,y \Leftrightarrow x\,y = 1$
   (iii) $x\,P\,y \Leftrightarrow (x+y)$ is a rational number (iv) $x\,P\,y \Leftrightarrow x/y$ is a rational number
   Examine the Reflexivity, Symmetric and Transitivity fo the above relations.
3. A relation R_1 is defiend on a set of Real numbers R such that
   $$(a,b) \in R_1 \Leftrightarrow 1 + ab > 0, \forall\, a, b \in R$$
   Prove that R_1 is Reflexive and Symmetric but not Transitive.
4. N is a set of natural numbers. If a relation R defined on $N \times N$ be such that $(a,b)\,R\,(c,d) \Leftrightarrow ad = bc\ \forall\, (a,b), (c,d) \in N \times N$ then Prove that R is an equivalence relation.
5. In a set Q_0 of non-zero rational numbers a relation R is defined as $a\,R\,b \Leftrightarrow a = 1/b, \forall\, a, b \in Q_0$. Is R an equivalence realtion?
6. Let $X = \{(a,b) \mid a, b \in R\}$ where I is a set of intgers. A relation R_1 is defiend on x such that

$(a,b) R_1 (c,d) \Leftrightarrow b-d=a-c$

Prove that R_1 is an Equivalence relation.

7. A relation R is defiend on a set of triangles T in a plane wuch that $x R y \Leftrightarrow x, y$ is similar to y Prove that R is an Equivalence relation.

8. Let $A=\{1,2,3\} A$ relation R is defined as

$$R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(1,3),(3,1),(2,3),(3,2)\}$$

Examine the reflexivity symmetric and transitivity of R

9. A relation R is defined on a set of non-zero Complex number C_0 such that - $z_1 R z_2 \Leftrightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is Real

Prove that R is an Equivalence relation.

10. If R is define on a set X as "A and B are disjoint sets" then examine the reflexivity, symmetry and transitivity of R.

11. A relation R is defined on a set of N of natural numbers such that $a R b$ if a is a divisor of b. Prove that R is Partial order relation and not a Total order relation.

12. Show that for subsets of N the relation "x divides y" is transitive or not-

(i) $\{2,4,6,8.......\}$ (ii) $\{0,2,4,6.......\}$ (iii) $\{3,9,5,15.......\}$

(iv) $\{5,15,30\}$ (v) $\{1,2,3,4,\}$ (vi) $\{a,b,ab\}, \forall\, a,b \in R$

2.11 Functions

In this Section, we study a special type fo relation called *fucntion*. It is one of the most important concepts in mathematics. We can, visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as 'mapping' used to denote a function.

We have seen while establishing a relation from set A to set B that A can have one or more such elements which are does not related to element B. It was also possible that any element of A is related to one or more element of set B. But if in a defined relation from set A to set B such that each element of A is related to only one element of B. Then these type of relations are called as function.

Hence we can define a function as:

Definition:

A relation *f* from a set A to a set B is said to be a *function* if every element of set A has one and only one image in set B. In other words, a function *f* is a relation from a non-empty set A to a non-empty set B such that the domain of *f* is A and no two distinct ordered pairs in *f* have the same first element. If *f* is a function from A to B and $(a, b) \in f$, then $f(a)=b$, where b is called the *image* of a under *f* and a is called the *preimage* of b under *f*. The function *f* from A to B is denoted by $f : A \to B$.

Remark : *f* is a function when *f*(*x*), the image fo *x* under the function *f* or value of *x* on *f*.

2.12 Function as a Set of Ordered Pair

Function $f : A \to B$ is a special relation which can be expressed as a set of ordered pairs. Therefore a function from set A to set B can be written as a set of ordered pairs, where the first element is from set A and the second from set B. First elements of any two ordered pair are not equal and each element of A is first element of any of the ordered pair. Such that $f = \{(a,\ b) \mid b = f(x),\ a \in A,\ b \in B\}$

f is a function if

(i) No two first element of ordered pair is same.

(ii) Every element of *A* is a first element of the ordered pair.

Note: It is clear from the above defintion that a function from set A to set B is a subset of A × B in which every element of A is the first element of orderecd pair in *f*

Example 1. Let $A = \{0,1,2,3\}$ and $B = \{a,b,c,d\}$

and $$f_1 = \{(0,a),(1,a),(2,c),(3,d)\}$$

$$f_2 = \{(0,b),(1,a),(2,c),(3,d),(0,d)\}$$

$$f_3 = \{(0,a),(1,a),(3,a)\}$$

$$f_4 = \{(0,d),(1,c),(2,c),(3,b)\}$$

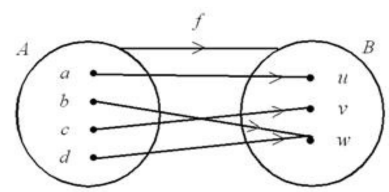
Thus we can see that f_1 and f_4 are function from A to B because in both every element of A has a relation with only one element of B. But f_2 is into function because in this element o of A is related to *b* and *d* of B. f_3 is not function because in this element 2 of A is not having relation with any element of B.

Example 2. If $A = \{a,b,c,d\}, B = \{u,v,w\}$ and *f* relates the elements of set A to be such that

$$f(a) = u, f(b) = w, f(c) = v, f(d) = w$$

then *f* is a function from A to B

$$f = \{(a,u),(b,w),(c,v),(d,w)\}$$



It can be expressed as an arrow diagram shown in figure 2.1

Fig. 2.1

Example 3. If $A = \{a_1,a_2,a_3,a_4\},\ B = \{b_1,b_2,b_3,b_4\}$ and f_1, f_2, f_3 relate the elements of A to the elements of B

It is clear that f_1 is a function whereas f_2 and f_3 is not a function because there no image of a_3 in B under f_2 and there is two images of b_1 and b_3 of b_3 under f_3.

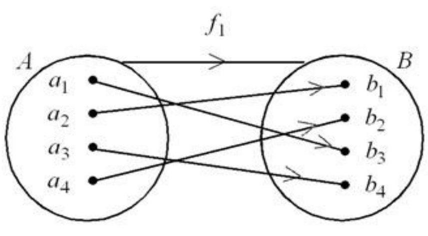


Fig. 2.2

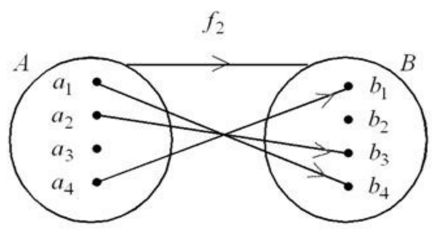


Fig. 2.3

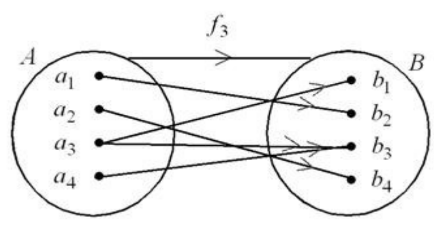


Fig. 2.4

Example 4. If $A = \{x,y,z\},\ B = \{a,b,c\}$ and $f = \{(x,b),(y,c),(z,a),(x,c)\}$ then *f* is a not a function from A to B as the first two elements of *f*, (*x*, *b*) and (*x*, *c*) are same.

Example 5. If $f: R \to R$ and $f(x) = \log x$ then *f* is not a function as $f(-3) = \log(-3)$ is not real but if $f: R^+ \to R, f(x) = \log x$, here *f* is a function.

Example 6. $f: R^+ \to R$ and $f(x) = \pm\sqrt{x}$ f is not a function.

but if *f* is defined as $f(x) = +\sqrt{x}$, or $f(x) = -\sqrt{x}$, then *f*, $R^+ \to R$ will be a function.

2.13 Domain, Co-domain and Range of a Function

In other words, a function *f* is a relation from a non-empty set A to a non-empty set B such that the domain of *f* is A and no two distinct ordered pairs in *f* have the same first element. Also the elements of set B are known as co-domain. All the elements of B which are associated to elements of A are called the range. i.e. If *f* is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the *image* of a under *f* and a is called the *preimage* of b under *f*.

$\therefore$ $f(A) = \{f(a), a \in A\}$ clearly $f(A) \subseteq B$

If a function is expressed in the form of set of ordered pairs then set of first elements of ordered pair of *f* is called domain and set of second element is called as range i.e. Domain of $f\{a \mid (a, b) \in f\}$, range of $f\{b \mid (a, b) \in f\}$.

Example 1. If $A = \{1, 2, 3, 4,\}, B = \{a, b, c, d\}$ and $f: A \to B$ is defiend as set of ordered pairs as $f = \{(1, a), (2, b), (3, b), (4, c)\}$ then domain of $f = \{1, 2, 3, 4,\} = A$

Co-domain of $f = B$

and range of $f = \{a, b, c\}$

Example 2. *f* is a function defined on a set of real numbers R such that $f(x) = x^2, \forall x \in R$ then domain of $f = R$

Co-domain $f = R$

and range of $f = R^+ \cup \{0\}$ where R^+ is a positive real number because square of any real number must be zero or any positive number.

Example 3. If $A = \{1, 2, 3, 4\}$ and $f: A \to N,\ f(x) = 3x + 2$ then

$$f(1) = 3.1 + 2 = 5$$
$$f(2) = 3.2 + 2 = 8$$
$$f(3) = 3.3 + 2 = 11$$
$$f(4) = 3.4 + 2 = 14$$

$\therefore$ range of $f = \{5, 8, 11, 14\}$

Example 4. If $f: R \to R$ and $f(x) = sin\,x$ then range of $f = \{x \in R | -1 \le x \le 1\}$ as we know that range of

$\sin x$ is $-1 \le x \le 1$.

Example 5. If $f: R \to R, f(x) = e^x$ range of *f* is a positive real number R^+ for all *x*, e^x is always a positive real number.

2.14 Constant Function

If the elements of domain set are associated to only one element of co-domain set then that function is known as constant function. Clearly there is only one element as a range.

$\therefore$ $f: A \to B$ is a constant function $\Leftrightarrow f(x) = c, x \in A$ where c is an element of co-domain B

Example 1. $f: N \to R, f(x) = \frac{2}{3}, \forall x \in N$ is a constant function as $f(N) = \left\{\frac{2}{3}\right\}$

2.15 Identify Function

If every element of set *A* is associated to itself then that function is called as Identity function. It is denoted by I_A $\therefore$ $I_A(x) = x, \forall x \in A$.

2.16 Equal Functions

Two functions f and g are called equal functions if

(i) domain of f = domain of g

(ii) co-domain of f = co-domain of g

(iii) $f(x) = g(x), \forall x$

Example 1. If $A = \{1,2\}, B = \{3,6\}$ and $f: A \to B, f(x) = x^2 + 2$, $g: A \to B, g(x) = 3x$ then the domain and co-domain of *f* and *g* are equal, here we see that

$$f(1) = 1^2 + 2 = 3 = g(1),$$

$$f(2) = 2^2 + 2 = 6 = g(2)$$

$$\therefore \quad f = g$$

Example 2. If $f(x) = x^2$, when $0 \le x \le 1$ and $g(x) = x^2$, when $2 \le x \le 8$ here *f* and *g* are not equal function as their domains are different.

Example 3. If $f: R \to R$ is defined as $f(x) = \frac{x^2 - 4}{x - 2}$ when $x \ne 2$ and $f(2) = 5$ and $g: R \to R, g(x) = x + 2, \forall x \in R$ here *f* and *g* are not equal functions as $f(2) \ne g(2)$.

2.17 Polynomial Function

Function $f: \mathbf{R} \to \mathbf{R}$ is known as a polynomial function if for every *x* in *R*

$y = f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ where *n* is a non-negative integer and

$a_0, a_1, a_2, a_2, ..., a_n \in \boldsymbol{R}$.

Example: Define the function $f : \boldsymbol{R} \to \boldsymbol{R}$ by $y = f(x) = x^2,\ x \in \boldsymbol{R}$. Complete the table given below by using this definition. What is the domain and range of this function? Draw the graph of *f*.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

Solution : The completed table is given below

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

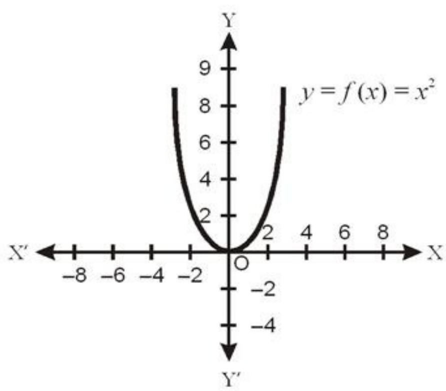


Fig. 2.05

Domain of $f = \{x : x \in R\}$, Range of $f = \{x^2 : x \in R\}$, the graph of *f* is indicated in Fig. 2.05

2.18 Rational Function

Function of the type $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial function of *x* defined in a domain, where $g(x) \neq 0$ are rational functions.

Example. Define the real valued function $f : \boldsymbol{R} - \{0\} \to \boldsymbol{R}$ defined by $f(x) = \dfrac{1}{x},\ x \in \boldsymbol{R} - \{0\}$ Complete the Table given below using this defintion. What are the domain and range of this function?

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
$y =$	...	...	...	...	...	...	...	...	...

Solution : The completed Table is given by

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
$y =$	-0.5	-0.67	-1	-2	4	2	1	0.67	0.5

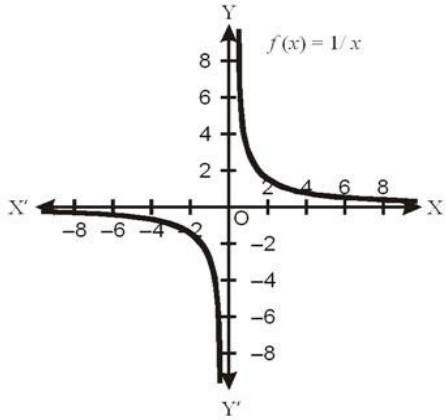


Fig. 2.06

The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of *f* is given in Fig. 2.06.

2.19 Modulus Function

The function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = |x|$ for each $x \in \boldsymbol{R}$ is called *modulus function*. For each non-negative vlaue of *x*, $f(x)$ is equal to *x*. But for negative values of *x*, the value of $f(x)$ is the negative of the value of *x*, i.e.

$$f(x) = |x| = \begin{cases} x, \ x \ge 0 \\ -x, \ x < 0 \end{cases}$$

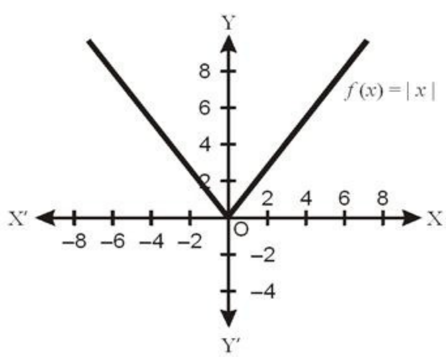


Fig. 2.07

The graph of the modulus function is given in fig. 2.07

2.20 Signum Function

The function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined for each $x \in \boldsymbol{R}$

$$f(x) = \begin{cases} 1 \ ; & ; \ x > 0 \\ 0 \ ; & ; \ x = 0 \\ -1 ; & ; \ x < 0 \end{cases}$$

is called the *signum function*. The domain of the signum function is R and the range is the set {–1, 0, 1}

The graph of the signum function is given by the figure 2.08.

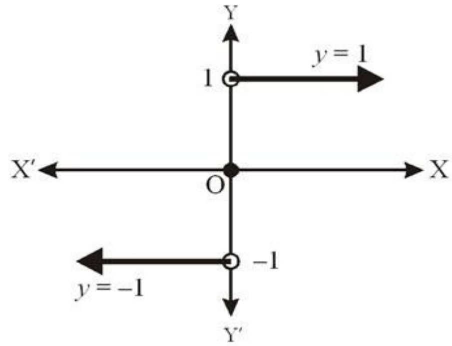


Fig. 2.08

2.21 Greatest Integer Function

The function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = [x],\ x \in \boldsymbol{R}$ assumes the value of the greatest integer, less than or equal to x. Such a function is called the *greatest integer* function. From the defintion of $[x]$, we can see that

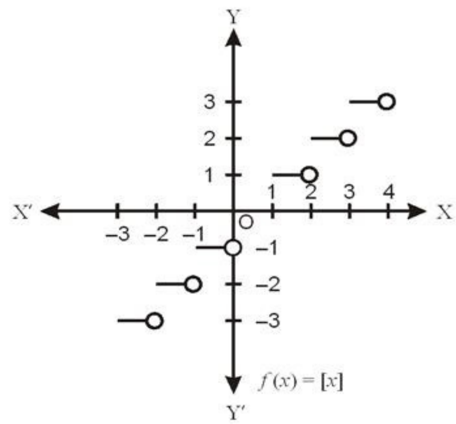
$$[x] = -1 \text{ If } -1 \le x < 0$$

$$[x] = 0 \text{ If } 0 \le x < 1$$

$$[x] = 1 \text{ If } 1 \le x < 2$$

$$[x] = 2 \text{ If } 2 \le x < 3$$

The graph of the function is shown in Fig. 2.09



$f(x) = [x] \le x$

Fig. 2.09

Illustrative Examples

Example 13. If $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$ which of the following is a function from A to B

(i) $f_1 = \{(a,1), (b,2), (c,3)\}$

(ii) $f_2 = \{(a,2), (b,3), (c,1), (b,4)\}$

(iii) $f_3 = \{(a,4), (b,4), (c,1)\}$

(iv) $f_4 = \{(a,1), (a,2), (a,3), (a,4)\}$

(v) $f_5 = \{(a,2), (b,2), (c,2)\}$

Solution : (i) f_1 is a function as every element of A is related to only one element of B

(ii) f_2 is not a function as the element b of A is related to element 3 and 4 of set B

(iii) f_3 is a function

(iv) f_4 is not a function as a element A is related to many element of B also b and c of A is not related to any element B.

(v) f_5 is a function. This is a constant function from A to B in which every element of A is related to a defined element 2 to B.

Example 14. Explain with reason which of followingone is a function, if $A = \{a, b, c\}$, $B = \{x, y, z\}$.

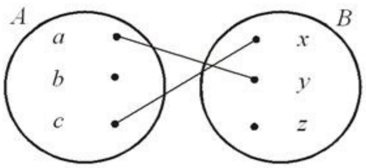


Fig. 2.10

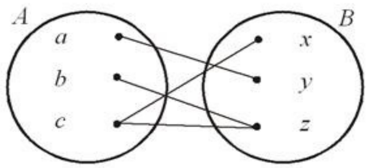


Fig. 2.11

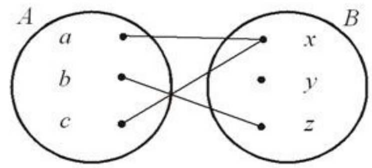


Fig. 2.12

Solution : In fig. 2.10 since there is no image of element b in set B hence it is not a function.

In Fig. 2.11 The element c of set A has two images in set B hence it is not a function

In Fig. 2.12 every element of set A has only one image in set B, hence it is a function.

Example 15. Find the domain when the function $f(x)=2x^2-1$ and $g(x)=1-3x$ are equal. Also find the domain when the functions are not equal.

Solution : For the functions f and g we have $f(x)=g(x)$

$\Rightarrow \quad 2x^2-1=1-3x \quad \Rightarrow 2x^2+3x-2=0 \quad \Rightarrow (2x-1)(x+2)=0 \quad \Rightarrow x=1/2,\ x=-2$

$\therefore$ for the domain $\left\{\frac{1}{2},-2\right\}$ functions f and g are equal

If the functions f and g are defined for the set of Real numbers then they are not equal.

Example 16. Find the range of functions defined from R to R

(i) $f(x)=1-|x-2|$ (ii) $g(x)=1+3\cos 2x$ (iii) $h(x)=\dfrac{x}{1+x^2}$

Solution : (i) $f(x)=1-|x-2|$

we know that $0\le|x-2|<\infty, \forall x\in R$ (multiply by -1)

$\Rightarrow \quad -\infty<-|x-2|\le 0, \forall x\in R$ (adding $+1$)

$\Rightarrow \quad -\infty<1-|x-2|\le 1, \forall x\in R$

$\Rightarrow \quad -\infty<f(x)\le 1, \forall x\in R \quad \therefore$ range of $f=(-\infty,1]$

(ii) $g(x)=1+3\cos 2x$

We know that $-1\le\cos 2x\le 1$, $\forall x\in R$

$\Rightarrow \quad -3\le 3\cos 2x\le 3, \forall x\in R$ (multiplying by 3)

$\Rightarrow \quad -2\le 1+3\cos 2x\le 4, \forall x\in R$ (adding $+1$)

$\Rightarrow \quad -2\le g(x)\le 4$

$\therefore$ Range of $g=[-2,4]$

(iii) $h(x)=\dfrac{x}{1+x^2}$

Let $y=h(x)$ then $y=\dfrac{x}{1+x^2}\Rightarrow x^2y-x+y=0$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{1-4y^2}}{2y}, \ y \neq 0, \ y = 0$ only when $x = 0$.

x will be real if $1-4y^2 \geq 0$, if $4y^2 - 1 \leq 0 \Rightarrow y^2 - \frac{1}{4} \leq 0$

$\Rightarrow \quad \left(y - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \leq 0 \Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2} \quad \Rightarrow \quad y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$\therefore$ Range of $h = [-1/2, 1/2]$

Example 17. Write the following as a set of ordered pairs and find which amongst these are functions

(i) $\{(x,y) | y = 3x, x \in \{1,2,3\}, y \in \{3,6,9,12\}\}$

(ii) $\{(x,y) | y = x+2, x, y \in N\}$

(iii) $\{(x,y) | y > x+1, x \in \{1,2\}, y \in \{2,4,6\}\}$

(iv) $\{(x,y) | y = x^2, x, y \in N\}$

Solution : (i) given $y = 3x \ \therefore$ putting $x = 1,2,3$, $y = 3,6,9$

realtion as a set of ordered pairs $= \{(1,3),(2,6),(3,9)\}$ clearly it is a fucntion

(ii) given $y = x+2, x, y, \in N$

putting $x = 1,2,3,...$, $y = 3,4,5...$

relation as a set of ordered pairs $= \{(1,3),(2,4),(3,5),...,(x,x+2),...\}$ clearly it is a function.

(iii) $y > x+1 \ \therefore$ for $x = 1$, $y = 4,6$ and for $x = 2$, $y = 4,6$

$\therefore$ relation as a set of ordered pairs $= \{(1,4),(1,6),(2,4),(2,6)\}$

It is not a function as element 1 and 2 has two images.

(iv) $y = x^2, x \in N \ \therefore$ taking $x = 1,2,3,...$, $y = 1,4,9,...$

$\therefore$ relation as a set of ordered pairs $= \{(1,1),(2,4),(3,9),...(x,x^2),...\}$

clearly it is a function.

Example 18. Define a function $f : \boldsymbol{R} \rightarrow \boldsymbol{R}$ for $f(x) = x^3, \ x \in \boldsymbol{R}$ draw the graph

Solution : Here $f(0) = 0, \ f(1) = 1, \ f(-1) = -1, \ f(2) = 8, \ f(-2) = -8, \ f(3) = 27, \ f(-3) = -27$,

etc. $f = \{(x, \ x^3) : \ x \in \boldsymbol{R}\}$ graph of f is shown in fig. 2.13.

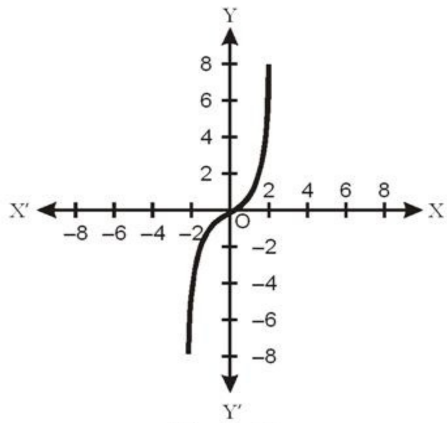


Fig. 2.13

Exercise 2.03

1. Explain with reasons which of the following relations are functions:

(a) $\{(1,2),(2,3),(3,4),(2,1)\}$ (b) $\{(a,0),(b,0),(c,1),(d,1)\}$

(c) $\{(1,a),(2,b),(1,b),(2,a)\}$ (d) $\{(a,a),(b,b),(c,c)\}$

(e) $\{(a,b)\}$ (f) $\{(4,1),(4,2),(4,3),(4,4)\}$

(g) $\{(1,4),(2,4),(3,4),(4,4)\}$ (h) $\{(x,y) | x,y \in R \wedge y^2 = x\}$

(i) $\{(x,y) | x,y \in R \wedge x^2 = y\}$ (j) $\{(x,y) | x,y \in R \wedge x = y^3\}$

(k) $\{(x,y) | x,y \in R \wedge y = x^3\}$

2. If $f : R \to R, f(x) = x^2$ then find

(i) range of f (ii) $\{x | f(x) = 4\}$ (iii) $\{y | f(y) = -1\}$

3. . . . $A = \{-2,-1,0,1,2\}$ and function defined $f : A \to R$ as $f(x) = x^2 + 1$, find the range of f.

4. Let $A = \{-2,-1,0,1,2\}$ and $f : A \to Z$, , where $f(x) = x^2 + 2x - 3$ then find

(i) range of f (ii) pre-image of 6, –3 and 5.

5. If $f : R \to R$, where $f(x) = e^x$ then find

(a) image set of R on f (b) $\{y | f(y) = 1\}$

(c) Is $f(x+y) = f(x)f(y)$ true?

6. If $f : R^+ \to R$ where $f(x) = \log x$, where R^+ is a set of positive integers then find

(a) $f(R^+)$ (b) $\{y | f(y) = -2\}$ (c) Is $f(x \cdot y) = f(x) + f(y)$ true?

7. If $f = \left\{\left(x, \frac{x^2}{1+x^2}\right), x \in R\right\}$ is a function from R to R then find the range of f

8. Is $g=\{(1,1),(2,3),(3,5),(4,7)\}$ a function?

If g is represented by the formula $g(x)=\alpha x+\beta$ then find the value of α and β

9. Find the difference between Constant function and Signum function.

2.22 Algebra of Real Functions

In this Section, we shall learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real fucntions and divide one real function by another.

(i) Addition of two real functions Let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be any two real functions, where $X \subset \boldsymbol{R}$. Then, we define $(f+g) : X \to \boldsymbol{R}$, by $(f+g)(x)=f(x)+g(x)$ for all $x \in X$.

(ii) Subraction of a real function from another Let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be any two real functions, where $X \subset \boldsymbol{R}$. Then, we define $(f-g) : X \to \boldsymbol{R}$ by $(f-g)(x)=f(x)-g(x)$, for all $x \in X$.

(iii) Multiplication by a scalar Let $f : X \to \boldsymbol{R}$ be a real valued function and α be scalar. Here by scalar, we mean a real number. Then the product α, f is a function from X to **R** defined by $(\alpha f)(x)=\alpha f(x),\ x \in X$

(iv) Multiplication of two real functions The product (or multiplication) of two real functions $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ is a function $fg : X \to \boldsymbol{R}$ defiend by $(fg)(x)=f(x)g(x),\ x \in X$. This is also called *pointwise multiplication*

(v) Quotient of two real functions Let f and g be two real functions defined from $X \to \boldsymbol{R}$ where $X \subset R$. The quotient of f by g denoted by is a function defined by, $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$] for all $x \in X$

provided $g(x) \neq 0$, .

2.23 Kinds of Functions

If f is a function from set A to B than according to function f each elements of A will relate to only one element of B. So, we can relate various element of A with various element of B. This is also possible that two or more the two element of A are related to one element of B whereas if the question of pre-image of B, this is possible that here is no pre-image of B in A. It is also possible that more than one element of B has pre-image is A. On the basis of all these possibility functions from A to B are defined as:

(i) One-one function or Injective function:

If $f : A \to B$ is a function f is called as one-one function if the images of distinct element under f are distinct. i.e.~

$f : A \to B$ is one-one if $a \neq b \Rightarrow f(a) \neq f(b), \forall a,b \in A,$ in other words

$f : A \to B$ is one-one if $f(a) = f(b) \Rightarrow a = b, \forall a,b \in A$

Example 1. Function *f* describing the captials of various countries is a one-one function, as all the countries have unique capitals.

Example 2. If $A = \{1,2,3,4\}, B = \{2,5,8,11,13\}$ and $f : A \to B, f(x) = 3x - 1$

defined as $f(1) = 2,\ f(2) = 5, f(3) = 8, f(4) = 11$ we see that the images of distinct elements under *f* are distinct. thus *f* is one-one.

Example 3. If $f : A \to B$ and $g : X \to Y$ are the two functions as shown in the arrow diagram

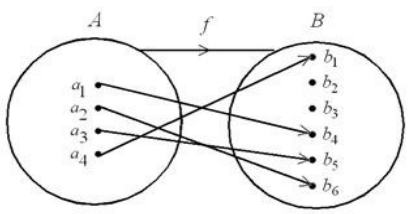


Fig. 2.14

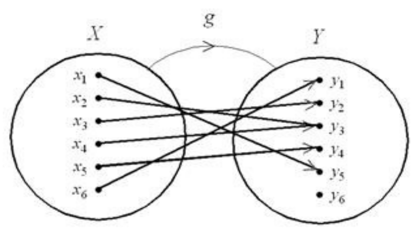


Fig. 2.15

Clearly $f : A \to B$ is one-one function but $g : X \to Y$ is not a one-one function as x_2 and x_4 of X has two image y_3 in Y.

Example 4. Function $f : N \to Z$ where $f(x) = x^2$ is a one-one functions as for $x, y \in N, x \neq y \Rightarrow x^2 \neq y^2$ $\Rightarrow f(x) \neq f(y)$ but $f : Z \to Z, f(x) = x^2$ is not a one-one function as $3, -3 \in Z$ is an element, $3 \neq -3$ but q $f(-3) = 9 = f(3)$ i.e. the image of -3 and 3 is 9.

Remark: If $f : A \to B$ is any function then $x = y \to f(x) = f(y)$, for *x*, all elements of A, true for *y* but $f(x) = f(y) \to x = y$ is true only when *f* is one-one function.

(ii) Many-one function:

If $f : A \to B$ is a function *f* is called as many-one function if the two or more then two elements of set A has a unique image in set B. This a function $f : A \to B$ is a many one function if there exist atleast two elements a and b in A such that for $a \neq b$, $f(a) = f(b)$

If the function is not one-one then certainly it is many-one.

Example 1. If $A = \{1,5,6,8\}$ and $B = \{2,3,4,7\}$ then $f : A \to B$ is defined as

$$f(1) = 3, f(5) = 4, f(6) = 3 \text{ and } f(8) = 2$$

then *f* is a many-one function as the elements 1 and 6 of A has a unique image 3 in B.

Example 2. Function $f : Z \to Z, f(x) = |x|$ is a many-one function as for $a \in Z$, $a \neq -a$ but $f(a) = f(-a)$ $[\because |a| = |-a|]$.

Example 3. $f : R \to R,\ f(x) = \sin x$ is a many one function as $\sin x$ is a **Periodic function** i.e. the value of the angles repeats after certain intervals.

Example 4. If $f : A \to B$ and $g : X \to Y$ as shown in the diagram

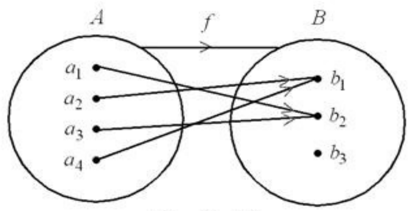


Fig. 2.16

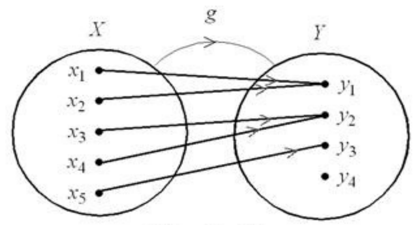


Fig. 2.17

here *f* and *g* both are many-one functions.

(iii) Onto function or Surjective function:

A function from set A to set B is said to be onto if every element of A is associated to all the elements of B i.e. all the elements of B are the images of the elements of A

$\therefore$ $f : A \to B$ is onto function if for $b \in B \Rightarrow \exists a \in A$ such that $f(a) = b$.

clearly if *f* is onto then $f(A) = B$ i.e. co-domain of *f* = range of *f*.

Note: For any function $f : A \to B$ if co-domain and range are not equal then it is not onto function.

Example 1. A function $f : Q \to Q, f(x) = 2x$ is onto function because for every element x of Q there exists a pre-image $x/2$ in the domain Q.

Example 2. If $A = \{-1, 1, -2, 2\}, B = \{1, 4\}$ and $f : A \to B, f(x) = x^2$ then *f* is onto function as $f(A) = B$

Example 3. If $f : A \to B$ and $g : X \to Y$ as shown in the diagram.

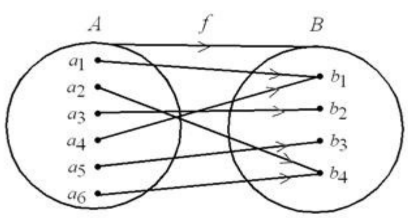


Fig. 2.18

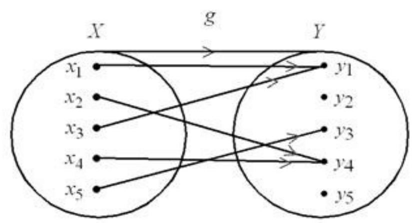


Fig. 2.19

f is onto where as *g* is not onto as for the elements y_2 and y_5 of *Y* there is no pre-image in X

Example 4. $f : R \to R,\ f(x) = x^2$ is not onto function because $f(R) = R^+ \cup (0) \neq R$.

(iv) Into function:

A function from set A to set B is said to be into if there exist atleast one element in set B which is not the image of any element of set A $\therefore$ *f* is into if $f(A) \neq B$

Note: If a function $f : A \to B$ is not onto then certainly it is into.

Example 1. $f : R \to R, f(x) = |x|$ is an into function $f(R) = R^+ \cup \{0\} \neq R$

Example 2. $f : R \to R, f(x) = e^x$ is an into function as

$$f(R) = R^+ \neq R \left[\because\ e^x > 0\ \forall x \in R\right]$$

Example 3. $f : R \to R, f(x) = \cos x$ is an into functions as

$$f(R) = \{x \in R | -1 \le x \le 1\} \ne R$$

(v) One-one onto function or Bijective function:

A function $f : A \to B$ is said to be one-one and onto if it is both one-one and onto i.e $f : A \to B$ will be

(a) *f* is one-one i..e $f(a) = f(b) \Rightarrow a = b, \forall a, b \in A$

(b) *f* is onto i.e. $\forall\, b \in B \Rightarrow \exists a \in A$ so that $f(a) = b$

Example 1. If Z_+ is a set of positive integers and E is a set of even positive integers and defined as $f : Z_+ \to E, f(x) = 2x$ then *f* is one-one onto function for all $x, y \in Z_+$ and $f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$ $\therefore$ *f* is one-one function.

also *f* is onto as $y \in E \Rightarrow \exists \frac{y}{2} \in z^+$ so that $f\left(\frac{y}{2}\right) = 2\frac{y}{2} = y$

Example 2. If $f : R \to R, f(x) = 2x + 3$ is one-one onto function then for all $x, y \in R$ if $f(x) = f(y) \Rightarrow 2x + 3 = 2y + 3 \Rightarrow x = y$ $\therefore f$ is one-one.

again $y \in R$ (co-domain) if possible let *x* be the pre-image of *y* then $f(x) = y$ i.e. $2x + 3 = y$ or $x = \frac{y-3}{2} \in R$ i.e. *f* is onto.

Example 3. If $A = \{1,2,3,4\}$ and $f : A \to A, f(x) = 5 - x$ such that $f = \{(1,4),(2,3),(3,2),(4,1)\}$ clearly the image of different elements of set A under *f* are different also $f(A) = A$ $\therefore$ *f* is a bijective function.

Example 4. If $A = \{1,2,3,4\}$ and $B = \{1,4,9,16\}$ and $f : A \to B, f(x) = x^2$ then clearly *f* is one-one onto function.

Illustrative Examples

Example 19. Let $f(x) = x^2$ and $g(x) = 2x + 1$ are two real valued functions then find $(f+g)(x)$, $(f-g)(x)$, $(f/g)(x)$

Solution : Clearly $(f+g)(x) = x^2 + 2x + 1$, $(f-g)(x) = x^2 - 2x - 1$,

$$(fg)(x) = x^2(2x+1) = 2x^3 + x^2,\ (f/g)(x) = \frac{x^2}{2x+1},\ x \ne \frac{1}{2}$$

Example 20. Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two negative real valued functions then find, $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $(f/g)(x)$.

Solution : Here we get these results:

$$(f+g)(x) = \sqrt{x} + x,\ (f-g)(x) = \sqrt{x} - x,$$

$$(fg)x = \sqrt{x}(x) = x^{3/2} \text{ and } \left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}},\ x \neq 0$$

Example 21. Define the following functions with reasons ont the basis of one-one, onto, into or many-one:

(i) $f : N \to N, f(x) = x^2$ (ii) $f : Z \to Z, f(x) = 2x + 1$

(iii) $f : R \to R, f(x) = x^3 + 3$

Solution : (i) Let $x_1, x_2 \in N$ then $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one

again $f(N) = \{1^2, 2^2, 3^2, 4^2, ...\} = \{1, 4, 9, 16, ...\} \neq N$ (co-domain)

$\therefore$ f is into

$\therefore$ f is one-one into function.

(ii) Let $x_1, x_2 \in Z$ then $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2$ $\therefore$ f is one-one. Again let $y \in Z$ (co-domain) if possible let x be the pre-image of y under f then $f(x) = y$ or $2x + 1 = y \Rightarrow x = y - 1/2$ now if $y \in Z$ then its not necessary that $(y - 1/2) \in Z$ i.e. there may be many elements of Z (co-domain) which do not have any pre-image in Z (domain) $\therefore$ f is into function.

$\therefore$ f is one-one into function.

(iii) If $x_1, x_2 \in R$ then $f(x_1) = f(x_2) \Rightarrow x_1^3 + 3 = x_2^3 + 3 \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ $\therefore$ f is one-one. again let $y \in R$ (co-domain) if possible let x be the pre-image of y under f then $f(x) = y$ i.e. $x^3 + 3 = y$ or $x = (y - 3)^{1/3}$

now $y \in R \Rightarrow (y - 3)^{1/3} \in R$

$\therefore$ pre-image of every element of R (co-domain) is present in R (domain)

$\therefore$ f is onto hence f is one-one onto function.

Example 22. If $X = \left\{x \middle| x \in R \text{ and} -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$ and $Y = \{y | y \in R \text{ and} -1 \leq x \leq 1\}$ then prove that $f : X \to Y, f(x) = \sin x$ is one-one onto function.

Solution : Let $x_1, x_2 \in X$ then $f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2 \Rightarrow x_1 = x_2$ $\left[\because -\frac{\pi}{2} \leq x_1, x_2 \leq \frac{\pi}{2}\right]$

$\therefore$ f is one-one

again let for all $y \in Y$, x be the pre-image of y under f

then $f(x) = y \Rightarrow sin x = y \Rightarrow x = sin^{-1} y$ now since $-1 \leq y \leq 1$, then the $\therefore$ $-\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$

i.e. for all $x \in X$ f is onto function

$\therefore$ f is one-one onto function.

Example 23. Illustrate the following functions as one-one, onto, into and many-one:

(i) $f: R \to R, f(x) = 1 + x^2$

(ii) $f: N \to N, f(n) = \begin{cases} (n+1)/2 \text{ ; if } n \text{ is odd} \\ n/2 \text{ ; if } n \text{ is even} \end{cases}$

(iii) $f: R \to R, f(x) = \begin{cases} 1 \text{ ; if } x \text{ is rational} \\ -1 \text{ ; if } x \text{ is irrational} \end{cases}$

Solution : (i) Let $x_1, x_2 \in R$ then

$$f(x_1) = f(x_2) \Rightarrow 1 + x_1^2 = 1 + x_2^2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$$

$\therefore$ *f* is not one-one thus it is many-one

again let for $x \in R$ if possible let *x* is the pre-image of *y* (co-domain)

then $f(x) = y \Rightarrow x^2 + 1 = y \Rightarrow x = \sqrt{y-1}$

If $y < 1$ then *x* is imaginary $\therefore$ *f* is into function

thus *f* is many-one into function.

(ii) by the defintion of the function $3, 4 \in N$ and

$$f(3) = \frac{3+1}{2} = 2 \text{ , } f(4) = \frac{4}{2} = 2$$

i.e. $f(3) = f(4)$

thus *f* is many-one

again $f(1) = 1, f(3) = 2, \ldots.. f(2n-1) = n$ etc. $\therefore$ range of $f = N$ thus *f* is onto

$\therefore$ *f* is many-one onto fucntion.

(iii) We know that real numbers are either rational or irrational, as per the question the image of all rational numbers is 1 and that of rational is -1 hence *f* is many-one, also range of $f = \{-1, 1\} \neq R$

$\therefore$ *f* is into fucntion

$\therefore$ *f* is many-one into function.

Example 24. Classify the following functions as one-one, many-one, onto and into, also find their range

(i) $f: C \to R, f(z) = |z|$

(ii) $f: R_0 \to R_0, f(x) = \frac{1}{x}$

(iii) $f: R \to R, f(x) = ax + b, a, b \in R, a \neq 0$

Solution : (i) We see that $z_1 = x + iy$ and $z_2 = x - iy (y \neq 0)$ are different elements of domain C but

$$\left. \begin{array}{l} f(z_1) = |x + iy| = \sqrt{x^2 + y^2} \\ f(z_2) = |x - iy| = \sqrt{x^2 + y^2} \end{array} \right\} \Rightarrow f(z_1) = f(z_2)$$

thus the two domain elements of *f* has the same image hence *f* is many-one

again range of : $f = \{|z| : z \in c\} = R^+ \cup \{0\} \neq R$ (co-domain)

$\therefore$ f is not onto

thus f is many-one into function.

(ii) If $x_1, x_2 \in R_0$ then $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$ thus f is one-one function

again let $y \in R_0$ (co-domain) then $1/y \in R_0$ (domain) so that $f(1/y) = y$

$\therefore$ f is onto function.

thus range of f = co-domain $= R_0$ and f = is one-one onto function.

(iii) Let $x_1, x_2 \in R$ then $f(x_1) = f(x_2) \Rightarrow ax_1 + b = ax_2 + b$

$\Rightarrow ax_1 = ax_2 \Rightarrow x_1 = x_2$ $[\because a \neq 0]$ $\therefore f$ is one-one function

again let $y \in R$ (co-domain), let x be the pre-image of y

then $f(x) = y$ or $ax + b = y$ or $x = \frac{y-b}{a} \in R$ $[\because a \neq 0]$

$\therefore f$ is one-one onto function

Example 25. If $A = R - \{2\}$ and $B = R - \{1\}$ defined as $f : A \to B, f(x) = \frac{x-1}{x-2}$ then prove that it is one-one onto function.

Solution : If $x, y \in A$ then $f(x) = f(y) \Rightarrow \frac{x-1}{x-2} = \frac{y-1}{y-2}$

$$\Rightarrow (x-1)(y-2) = (x-2)(y-1)$$

$$\Rightarrow xy - y - 2x + 2 = xy - x - 2y + 2$$

$$\Rightarrow -y - 2x = -x - 2y \Rightarrow x = y$$

$\therefore f$ is one-one fucntion.

again for $y \in B$ let x be the pre-image of y then

$$f(x) = y \Rightarrow \frac{x-1}{x-2} = y \Rightarrow x - 1 = y(x-2) \Rightarrow x = \frac{1-2y}{1-y}$$

clearly if $y \neq 1$, x is a real number also $\frac{1-2y}{1-y} \neq 2$ as by taking $\frac{1-2y}{1-y} = 2$ we get as 1 = 0 which is absurd and meaningless

every element of set B has some pre-image in A hence f is into

$\therefore f$ is one-one onto function

Exercise 2.4

1. Classify with reason the following functions as one-one, many-one, onto and into,

(i) $f : Q \to Q, f(x) = 3x + 7$ (ii) $f : C \to R, f(x + iy) = x$

(iii) $f : R \to [-1, 1], f(x) = \sin x$ (iv) $f : N \to Z, f(x) = |x|$

2. If $A=\{x|-1\le x\le 1\}=B$ define a function from A to B. Show that it is one-one, many-one, onto and into

(i) $f(x)=\frac{x}{2}$ (ii) $g(x)=|x|$ (iii) $h(x)=x^2$ (iv) $k(x)=sin\,\pi\,x$

3. If $f:C\to C, f(x+iy)=(x-iy)$ then prove that *f* is one-one onto function.

4. Give one example of each function -

(i) one-one into (ii) many-one onto

(iii) onto but not one-one (iv) one-one but not onto

(v) neither one-one nor onto (vi) one-one onto

5. Prove that the fucntion $f:R\to R, f(x)=cos\,x$ is many-one into function. Convert the domain and co-domain of *f* so that *f* becomes

(i) one-one into (ii) many-one onto (iii) one-one onto

6. If $N=\{1,2,3,4,.....\}$, $O=\{1,3,5,7,.....\}, E=\{2,4,6,8,...\}$ and f_1, f_2 are defined as:

$$f_1:N\to O,\ f_1(x)=2x-1\quad ;\qquad f_2:N\to E,\ f_2(x)=2x$$

then prove that f_1 and f_2 are one-one onto function.

7. If the function *f* is defined from set of real number to real number then classify with reason whether the functions are one-one, many-one, onto or into

(i) $f(x)=x^2$ (ii) $f(x)=x^3$ (iii) $f(x)=x^3+3$ (iv) $f(x)=x^3-x$

Miscellaneous Exercise 2

1. If $A=\{a,b,c,d\}$ and $B=\{p,q,r,s\}$ then a relation from A to B is

(A) $\{(a,p),(b,r),(c,r)\}$ (B) $\{(a,p),(b,q),(c,r),(s,d)\}$

(C) $\{(b,a),(q,b),(c,r)\}$ (D) $\{(c,s),(d,s),(r,a),(q,b)\}$

2. Define a relation R on a set of natural number N sch that $x\,R\,y \Leftrightarrow x+4y=16$, then range fo R is

(A) $\{1,2,4\}$ (B) $\{1,3,4\}$ (C) $\{1,2,3\}$ (D) $\{2,3,4\}$

3. The set builder form of relation $\{(1,2),(2,5),(3,10),(4,17),...\}$ on N is

(A) $\{(x,y)|x,y\in N,\ y=2x+1\}$ (B) $\{(x,y)|x,y\in N,\ y=x^2+1\}$

(C) $\{(x,y)|x,y\in N,\ y=3x-1\}$ (D) $\{(x,y)|x,y\in N,\ y=x+3\}$

4. If $A=\{2,3,4\}$ and $B=\{3,4,5,6,7,8\}$. Define a relation R from A to B such that "x divides y" then R^{-1} will be

(A) $\{(4,2),(6,2),(8,2),(3,3),(6,3),(4,4),(8,4)\}$ (B) $\{(2,4),(2,6),(2,8),(3,3),(3,6),(4,4),(4,8)\}$

(C) $\{(3,3),(4,4),(8,4)\}$ (D) $\{(4,2),(6,3),(8,4)\}$

5. A relation R "$x<y$" on a set of Real numbes is

(A) reflexive and transitive
(B) symmetric and transitive
(C) anti-symmetric and transitive
(D) reflexive and anti-symmetric

6. A relation R from a set of non-zero integers such that $x R y \Leftrightarrow x^y = y^x$ then R is
(A) reflexive and symmetric but not transitive
(B) reflexive and anti-symmetric but not transitive
(C) reflexive, anti-symmetric and transitive
(D) reflexive, symmetric and transitive

7. Define a relation R "x is a divisor of y" then find which among the following sets on N is ordered set
(A) $\{36, 3, 9\}$ (B) $\{7, 77, 11\}$ (C) $\{3, 6, 9, 12, 24\}$ (D) $\{1, 2, 3, 4, \ldots\}$

8. Which amongst the following is not an Equivalence relation defined on a set of integers Z
(A) $aR_1b \Leftrightarrow (a+b)$ is an even integer
(B) $aR_2b \Leftrightarrow (a-b)$ is an even integer
(C) $aR_3b \Leftrightarrow a < b$
(D) $aR_4b \Leftrightarrow a = b$

9. Define a relation R on a set $A = \{1, 2, 3\}$ as $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$ then R is
(A) reflexive but not transitive
(B) reflexive but not symmetric
(C) symmetric and transitive
(D) neither symmetric nor reflexive

10. If $A = \{a, b, c\}$ then the number of non-empty relations defined will be
(A) 511 (B) 512 (C) 8 (D) 7

11. If $A = \{1, 2, 3, 4\}$ then which amongst the following is a function in A
(A) $f_1 = \{(x, y) | y = x + 1\}$
(B) $f_2 = \{(x, y) | x + y > 4\}$
(C) $f_3 = \{(x, y) | y < x\}$
(D) $f_4 = \{(x, y) | x + y = 5\}$

12. Function $f : N \to N, f(x) = 2x + 3$ is
(A) one-one onto (B) one-one into (C) many-one onto (D) many-one into

13. Which amongst the below given functions is onto from R to R
(A) $f(x) = |x|$ (B) $f(x) = e^{-x}$ (C) $f(x) = x^3$ (D) $f(x) = \sin x$

14. Which amongst the below given functions is one-one from R to R
(A) $f(x) = |x|$ (B) $f(x) = \cos x$ (C) $f(x) = e^x$ (D) $f(x) = x^2$

15. Function $f : R \to R, f(x) = x^2 + x$
(A) one-one onto (B) one-one into (C) many-one onto (D) many-one into

16. Which amongst the below given fucntions is onto:
(A) $f : Z \to Z, f(x) = |x|$
(B) $f : N \to Z, f(x) = |x|$
(C) $f : R_0 \to R^+, f(x) = |x|$
(D) $f : C \to R, f(x) = |x|$

17. Domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is
(A) R^+ (B) R^- (C) R_0 (D) R

18. If x is Real then the range of $f(x) = \dfrac{x}{1 + x^2}$:

(A) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (B) $(-2, 2)$ (C) $(-1, 1)$ (D) $\left(-\frac{1}{2}, \frac{1}{2}\right)$

19. The range of $f(x) = \cos\frac{x}{3}$ is

(A) $(0, \infty)$ (B) $\left(-\frac{1}{3}, \frac{1}{3}\right)$ (C) $[-1, 1]$ (D) $[0, 1]$

20. Which amongst the below given fucntions defiend on R to $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ is one-one onto

(A) $f(x) = \tan x$ (B) $f(x) = \sin x$ (C) $f(x) = \cos x$ (D) $f(x) = e^x + e^{-x}$

21. Find the domain and range of R

$$R = \{(x+1, x+5) | x \in \{0, 1, 2, 3, 4, 5\}\}$$

22. If $A = \{1, 2\}$ then write the number of non-empty relations on A

23. Find the domain and range of the relations

(i) $R_1 = \{(x, y) | x, y \in N;\ x + y = 10\}$

(ii) $R_2 = \{(x, y) | y = |x - 1|, x \in Z;\ |x| \le 3\}$

24. Define a relation R_1 and R_2 on a set of real numbers R as

(i) $aR_1b \Leftrightarrow a - b > 0$ (ii) $aR_2b \Leftrightarrow |a| \le b$

Examine the reflexivity, symmetric and transitivity of R.

25. Define a relation R on a set of natural number N

$$aRb \Leftrightarrow a^2 - 4ab + 3b^2 = 0, \quad (a, b \in N)$$

Prove that R is reflexive but not symmetric and transitive

26. Defien a relation R_1 and R_2 on a set of real numbers R as

(i) $aR_1b \Leftrightarrow |a| = |b|$ (ii) $aR_2b \Leftrightarrow |a| \le |b|$

Prove that R_1 is an equivalence relation and R_2 is not an equivalence relation.

27. Define a relation R on set $A = \{1, 2, 3\}$:

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (1,3), (3,1), (2,3), (3,2)\}$$

Examine the reflexivity, symmetric and transitivity of R.

28. Find the domain of the function $1/\sqrt{(x+1)(x+2)}$.

Important Points

1. ***Ordered pair*** A pair of elements grouped together in a particular order

$$(a, b) \ne (b, a) \text{ and } (a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d$$

2. ***Cartesian product*** $A \times B$ of two sets A and B is given by

$A \times B = \{(a,b) | a \in A, b \in B\}$.

3. If any relation R from set A to set B, then A × B is subset converse each subset of A × B, defined a relation from A to B.

4. If n(A) = m and n (B) = n then the number of non-empty relations from A to B will be $2^{mn} - 1$.

5. Domain of $R = \{a | (a,b) \in R\}$, range $= \{b | (a,b) \in R\}$.

6. Let R^{-1} be the inverse relation defined from set A to set B then $R^{-1} = \{(a,b) | (a,b) \in R\}$ i.e. , $R^{-1} = (a,b) \in R \Leftrightarrow (a,b) \in R^{-1}$

7. Range of R =Domain of R^{-1} and Domain of R =Range of R^{-1}

8. **Identitiy Relation -** Every element of set A is related to itself it is denoted by I_A i.e. $I_A = \{(a,a) | a \in A \text{ and } (a,b) \not\in I_A\}$.

9. **Reflexive Relation -** Reflexive relation R in R is a relation called reflexive relation i.e. R will reflexive with $a \in R \Rightarrow (a,a) \in R$.

10. **Symmetric Relation -** ***symmetric relation R in A is a relation satisfying***

    $(a,b) \in R \Rightarrow (b,a) \in R, \forall a,b \in A$

11. **Transitive Relation -** *Transitive relation R in A is a relation satisfying*

    $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R, \forall a,b,c, \in A.$

12. **Equivalence Relation -** *Equivalence relation* R in A is a relation which is reflexive, symmetric and transitive.

13. **Anti-Symmetric Relation** - Anti-Symmetric *relation* R in A is a relation satisfying

    $(a,b) \in R$ and $(b,c) \in R \Rightarrow a = b, \forall\ a,b \in A$

14. **Partial Order Relation** - Partial Order *relation* R in A is a relation which is reflexive, anti-symmetric and transitive.

15. **Total Order Relation** - Total Order *relation* R in A is a relation if

    (i) R is Partial Order *relation*

    (ii) $a,b \in R \Rightarrow$ or $(a,b) \in R$ or $(b,a) \in R$ or $a = b$

16. **Function** - A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B.

17. **Domain and Co-domain** - The ***domain*** of R is the set of all first elements of the ordered pairs in a relation R. The co-domain is a set of all elements of set B.

18. **Image** - The ***image*** of an element x under a relation R is given by y, where $(x, y) \in R$.

19. **Range -** The ***range*** of the relation R is the set of all second elements of the ordered pairs in a relation R

20. **Constant Function :** A function whose each element of domain is related to one element of co-domain, called constant function.

21. **Identify function :** A function defined from A to A in which each element of A is related to itself only, called as identity function of A.

22. **Equal function -** Two functions *f* and *g* are said to be equal if

(i) Domain of *f* = Domain of *g* (ii) Co-Domain of *f* = Co-Domain of *g*

(iii) $f(x) = g(x) \ \forall \ x$

23. **Polynomial Function -** Functions $f: R \rightarrow R$ is a polynomial function if $y = f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ where *n* is a non-negative integer and $a_0, a_1, a_2, \ldots a_n \in R$.

24. **Rational Function :** Function of type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are in same domain, polynomial function *x*, in which $g(x) \neq 0$ are called rational function.

25. **Modulus Function :** $f(x) = |x|$ for every $x \in R$ defined function $f: R \rightarrow R$ called modulus function i.e.

$$f(x) = \begin{cases} x & : \ \forall \ x \geq 0 \\ -x & : \ \forall \ x < 0 \end{cases}$$

26. **Signum Function :** For every $x \in R$,

$$f(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

a defined function $f: R \rightarrow R$ is called signum fimctopm.

27. **Greatest Integer Function :** A defined function $f: R \rightarrow R$, from $f(x) = [x]$, $x \in R$ take the greatest integer *x*, $[x] \leq x$ is called greatest integer function.

28. **One-One function -** Function $f: A \rightarrow B$ is one-one if $f(a) = f(b) \Rightarrow a = b, \forall a, b \in A$

29. **Many-one function -** if the function is not one-one then it is many-one.

30. **Onto function -** $f: A \rightarrow B$ is onto if $b \in B \Rightarrow \exists \, a \in A$ so that $f(a) = b$ i.e. in this condition codomian of *f* = range of *f*.

31. **Into function -** If *f* is not onto then it is into. Again if any function is both one-one and onto so that function is called one-one on to function.

Answers

Exercise 2.1

1. (i), (ii) and (v) are relations

2. (i) $\{(x, y) | x, y \in N, \ y = 2x + 1\}$ (ii) $\{(x, y) | x, y \in N, \ x + 2y = 8\}$ (iii) $\{(x, y) | x, y \in N, \ y = x - 1\}$

3. $R = \{(2,3), (2,7), (3,7), (3,10), (4,3), (4,7), (5,3), (5,6), (5,7)\}$

Domain of $R = \{2, 3, 4, 5\}$, Range of $R = \{3, 6, 7, 10\}$

4. $R=\{(0,5),(0,-5),(3,4),(-3,4),(3,-4),(-3,-4),(4,3),(4,-3),(-4,3),(-4,-3),(5,0),(-5,0)\}$
$R^{-1}=\{(5,0),(-5,0),(4,3),(4,-3),(-4,3),(-4,-3),(3,4),(3,-4),(-3,4),(-3,-4),(0,5),(0.-5)\}$

domain of $R=\{0,3,-3,4,-4,5,-5\}=R^{-1}$ domain

5. (i) false (ii) flase (iii) false (iv) false

6. (i) $R=\{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\}$
(ii) $R^{-1}=\{(4,1),(5,1),(4,2),(5,2)(4,3),(5,3),(5,4)\}$

7. (i) $R_1=\{(2,1),(4,2),(6,3)\}$ (ii) $R_2=\{(8,6),(9,7),(10,8)\}$
(iii) $R_3=\{(0,4),(3,2),(6,0)\}$ (iv) $R_4=\{(5,10),(5,15),(6,12),(6,18),(8,16)\}$

8. (i) $R^{-1}=\{(3,2),(4,2),(3,3),(2,3),(2,4)\}$ (ii) $R^{-1}=\{(x,y)|x,y\in N,\ x>y\}$
(iii) $R^{-1}=\{(4,0),(2,3),(0,6)\}$

Exericse 2.2

1. (i) R_1 symmetric and transitive but not reflexive
(ii) R_2 reflexive and transitive but nto symmetric
(iii) R_3 reflexive and symmetric but not transitive
(iv) reflexive and transitive but nto symmetric
2. (i) only symmetric
(ii) only symmetric
(iii) only symmetric
(iv) reflexive, symmetric and transitive
5. No 8. reflexive, symmetric and transitive 10. only symmetric
12. (i) No (ii) No (iii) No (iv) yes (v) No (vi) No

Exericse 2.3

1. (a) No (b) Function (c) No (d) Function (e) Function
(f) No (g) Function (h) No (i) Function (j) Function
(k) Fungion

2. (i) $\{x\in R|0\le x<\infty\}$ (ii) $\{2,-2\}$ (iii) ϕ

3. Range $f=\{1,2,5\}$

4. (i) $f(A)=\{-4,-3,0,5\}$ (ii) ϕ, $\{0,2\}$, -2

5. (a) image set of f in $R=R^+$ (b) $\{0\}$ (c) True

6. (a) R (b) $\{e^{-2}\}$ (c) True

7. Range of $f=\{y=f(x)|0\le y<1\}$ 8. Yes, $\alpha=2,\ \beta=-1$

Exercise 2.4

1. (i) one-one onto (ii) Many-one onto

(iii) many-one onto (iv) one-one into

2. (i) f one-one into (ii) g many-one into
(iii) h many-one into (iv) many-one onto

4. (i) $f: N \to N,\ f(x) = 2x$ (ii) $f: R_0 \to R^+,\ f(x) = x^2$
(iii) $f: z_0 \to N,\ f(x) = |x|$ (iv) $f: Z \to Z,\ f(x) = 2x$
(v) $f: R \to R,\ f(x) = x^2$ (vi) $f: Z \to Z,\ f(x) = -x$

5. (i) $f: [0, \pi] \to R$ (ii) $f: R \to [-1, 1]$ (iii) $f: [0, \pi] \to [-1, 1]$

7. (i) many-one into (ii) one-one onto
(iii) one-one onto (iv) many-one onto

Miscellaneous Exercise 2

1. A	**2.** C	**3.** B	**4.** A	**5.** C	**6.** D
7. A	**8.** C	**9.** B	**10.** A	**11.** D	**12.** B
13. C	**14.** C	**15.** D	**16.** C	**17.** B	**18.** A
19. C	**20.** A				

21. domain $= \{0, 1, 2, 3, 4, 5, 6\}$, Range $= \{5, 6, 7, 8, 9, 10\}$

22. $\{(1,1)\}, \{(2,2)\}, \{(1,2)\}, \{(2,1)\},$
$\{(1,1),(1,2)\}, \{(1,1),(2,1)\}, \{(2,2),(1,2)\}, \{(2,2),(2,1)\}, \{(1,2),(2,1)\},$
$\{(1,1),(2,2),(1,2)\}, \{(1,1),(2,2),(2,1)\}, \{(1,1),(1,2),(2,1)\}, \{(2,2),(1,2),(2,1)\},$
$\{(1,1),(1,2),(2,1),(2,2)\}$

23. (i) domain $= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, Range $= \{9, 8, 7, 6, 5, 4, 3, 2, 1\}$
(ii) domain $= \{-3, -2, -1, 0, 1, 2, 3\}$, Range $= \{4, 3, 1, 0, 2\}$

24. (i) R_1 only transitive
(ii) R_2 only transitive

27. reflexive, symmetric and transitive

28. $(-\infty, -2) \cup (-1, \infty)$