Exercise 11.R

Answer 1CC.

(a)

If a sequence $\{a_n\}$ has Limit L then we write

$$\lim_{n\to\infty} a_n = L$$

If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges.

(b)

Given a series

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$, let s_n denote its nth partial sum

$$s_n = \sum_{i=1}^{n} a_n = a_1 + a_2 + a_3 + \ldots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number,

Then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$\sum_{n=1}^{\infty} a_n = s$$

(c)

$\lim_{n\to\infty} a_n = 3$ means the sequence $\{a_n\}$ converges to 3

(d)

$\sum_{n=1}^{\infty} a_n = 3$ means the series $\sum_{n=1}^{\infty} a_n$ is converges and its sum is 3.

Answer 1E.

For checking the convergence of the sequence $a_n = \dfrac{2+n^3}{1+2n^3}$

We find the limit as $n \to \infty$. If the limit exists then a_n is convergent other wise divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n^3}{1+2n^3}$$

$$= \lim_{n\to\infty} \frac{2/n^3+1}{1/n^3+2} \qquad \left[\text{dividing numbertor and denomiator by } n^3\right]$$

$$= \frac{0+1}{0+2}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \frac{1}{2}$$

Thus sequence a_n is $\boxed{\text{convergent}}$

Answer 1P.

We have $f(x) = \sin(x^3)$

Since $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$..... ....(1) (is Maclaurin series for $\sin x$)

Then $\sin(x^3) = x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} + \ldots\ldots$ .....(2)

This is the Maclaurin series for $\sin(x^3)$

Coefficient of x^{15} is $c_{15} = \dfrac{f^{15}(0)}{15!}$ since $c_n = \dfrac{f^n(0)}{n!}$

Comparing with (2) we have

$$\frac{f^{(15)}(0)}{15!} = \frac{1}{5!}$$

$$\Rightarrow f^{(15)}(0) = \frac{15!}{5!}$$

$$\Rightarrow f^{(15)}(0) = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

$$\Rightarrow \boxed{f^{(15)}(0) = 10897286400}$$

Answer 1TFQ.

False

Example:

$\lim_{n\to\infty} = \dfrac{1}{n} = 0$ but $\sum \dfrac{1}{n}$ is divergent

Answer 2CC.

(a)

A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \le M$ for all $n \ge 1$

It is bounded below if there is a number m such that $m \le a_n$ for all $n \ge 1$

If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

(b)

A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$

It is called increasing if $a_n > a_{n+1}$ for all $n \geq 1$

A sequence is monotonic if it is either increasing or decreasing

(c)

Every bounded, monotonic sequence is convergent.

Answer 2E.

For checking the convergence of sequence $a_n = \dfrac{9^{n+1}}{10^n}$

We find the limit as $n \to \infty$, if limit exists then a_n is convergent otherwise divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{9^{n+1}}{10^n}$$
$$= \lim_{n\to\infty} \quad 9\left(\frac{9}{10}\right)^n$$
$$= \lim_{n\to\infty} \quad 9.(0.9)^n$$

Since 0.9 is less than 1 so $(0.9)^n \to 0$ as $n \to \infty$

Then $\lim_{n\to\infty} a_n = 9\cdot 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

Thus the sequence a_n is $\boxed{\text{convergent}}$

Answer 2P.

Consider the function, $f(x) = \lim_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1}$.

If $|x| < 1$,then $0 \leq x^2 < 1$

$\lim_{n\to\infty} x^{2n} = 0$

So $f(x) = \lim_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1}$

$$f(x) = \frac{0-1}{0+1}$$
$$= -1$$

If $|x| = 1$ that implies $x = \pm 1$ and $x^2 = 1$ that implies $x^{2n} = 1$

So, $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$

$$= \frac{1-1}{1+1}$$

$$= 0$$

If $|x| > 1$ that implies $x^2 > 1$, $\lim\limits_{n \to \infty} x^{2n} = \infty$

So, $f(x) = \lim\limits_{n \to \infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$

$$= \lim_{n \to \infty} \frac{x^{2n}\left(1 - \frac{1}{x^{2n}}\right)}{x^{2n}\left(1 + \frac{1}{x^{2n}}\right)}$$

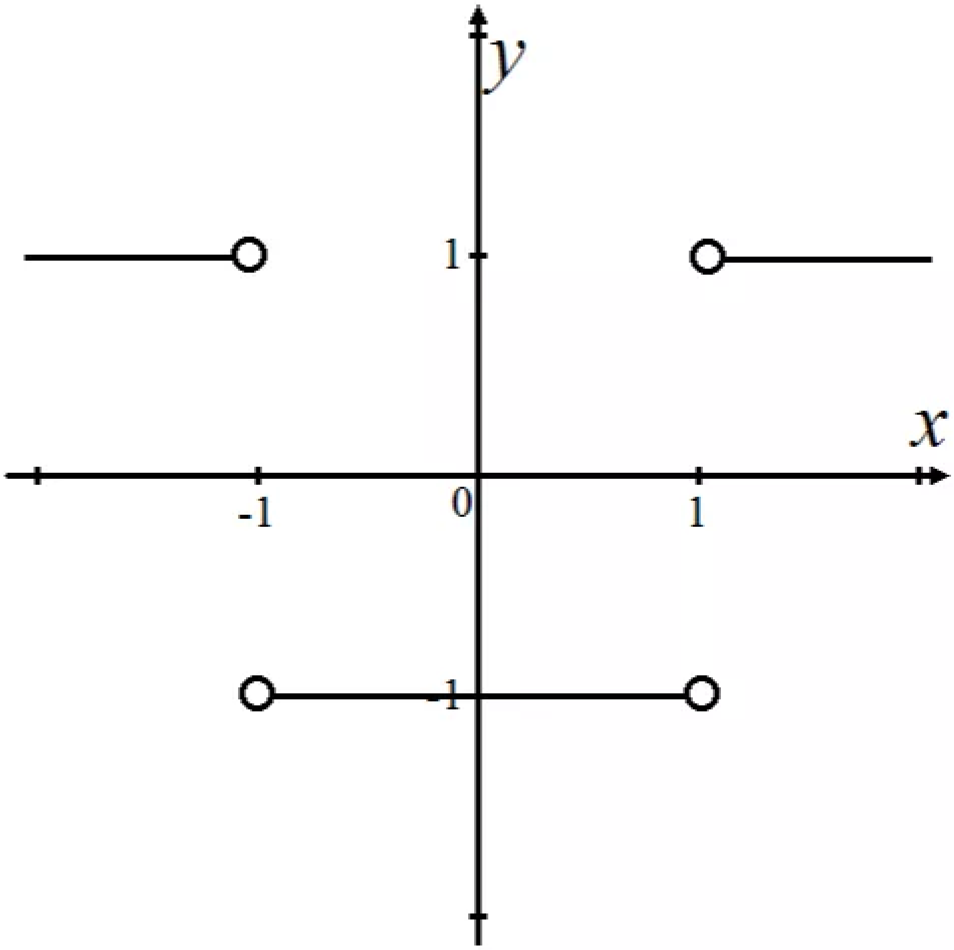
$$= \lim_{n \to \infty} \frac{\left(1 - \frac{1}{x^{2n}}\right)}{\left(1 + \frac{1}{x^{2n}}\right)}$$

$$= \frac{(1-0)}{(1+0)}$$

$$= 1$$

Therefore,

$f(x)=1$ if $x<-1$
$f(x)=0$ if $x=-1$
$f(x)=-1$ if $-1<x<1$
$f(x)=0$ if $x=1$
$f(x)=1$ if $x>1$



The above graph as shown as f is continuous everywhere except at $x=-1,1$

Answer 2TFQ.

False

Since $\sum_{n=1}^{\infty}\frac{1}{n^{\sin 1}}$ is a p-series with $p=\sin 1<1$, so the series is divergent.

The p-series $\sum_{n=1}^{\infty}\frac{1}{n^{p}}$ is convergent if $p>1$ and divergent if $p\leq 1$

Answer 3CC.

(a)

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$$

Is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent

(b)

The series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series.

It is convergent if $p>1$ and divergent if $p \leq 1$

Answer 3E.

For checking the convergence of sequence $a_n = \frac{n_3}{1+n^2}$

We find the limit as $n \to \infty$, if limit exists then a_n is convergent otherwise divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n_3}{1+n^2}$$

Dividing numerators and denominator by n^2

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\frac{1}{n^2}+1}$$

$$= \frac{\infty}{0+1}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \infty$$

So limit does not exist

Then the sequence a_n *is* $\boxed{\text{divergent}}$

Answer 3P.

(A) We have $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Then $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$

$$\Rightarrow 2\cot 2\theta = \frac{1-\tan^2\theta}{\tan\theta}$$

$$\Rightarrow 2\cot 2\theta = \frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}$$

$$\Rightarrow 2\cot 2\theta = \cot\theta - \tan\theta$$

Putting $\qquad \theta = \dfrac{x}{2}$

$$2\cot x = \cot\frac{x}{2} - \tan\frac{x}{2}$$

$$\Rightarrow \boxed{\tan\frac{x}{2} = \cot\frac{x}{2} - 2\cot x}$$

(B) $\qquad \tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}} \qquad$ from part (a)

Given series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{x}{2^n}$

n^{th} Partial sum

$$s_n = \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \frac{\tan(x/16)}{16} + \ldots\ldots$$

$$\Rightarrow s_n = \frac{1}{2}\left(\cot\frac{x}{2} - 2\cot x\right) + \frac{1}{4}\left(\cot\frac{x}{4} - 2\cot\frac{x}{2}\right) + \frac{1}{8}\left(\cot\frac{x}{6} - 2\cot\frac{x}{4}\right) + \ldots\ldots\ldots$$

$$+\ldots\ldots\ldots + \frac{1}{2^n}\left(\cot\left(x/2^n\right) - 2\cot\left(x/2^{n-1}\right)\right)$$

$$= \left(\frac{1}{2}\cot\frac{x}{2} - \cot x\right) + \left(\frac{1}{4}\cot\frac{x}{4} - \frac{1}{2}\cot\frac{x}{2}\right) + \left(\frac{1}{8}\cot\frac{x}{8} - \frac{1}{4}\cot\frac{x}{4}\right) + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\left(\frac{\cot\left(x/2^n\right)}{2^n} - \frac{\cot\left(x/2^{n-1}\right)}{2^{n-1}}\right)$$

By telescoping sum $\Rightarrow s_n = -\cot x + \dfrac{\cot\left(x/2^n\right)}{2^n}$

Now $\dfrac{\cot\left(x/2^n\right)}{2^n} = \dfrac{\cos\left(x/2^n\right)}{2^n\sin\left(x/2^n\right)} = \dfrac{\cos\left(x/2^n\right)}{x}\cdot\dfrac{\left(x/2^n\right)}{\sin\left(x/2^n\right)}$

Then $\quad \displaystyle\lim_{n\to\infty}\frac{\cot\left(x/2^n\right)}{2^n} = \lim_{n\to\infty}\frac{\cos\left(x/2^n\right)}{x}\cdot\frac{\left(x/2^n\right)}{\sin\left(x/2^n\right)}$

$$= \frac{1}{x}\cdot\lim_{n\to\infty}\frac{\left(x/2^n\right)}{\sin\left(x/2^n\right)}$$

Let $\quad x/2^n = \theta \qquad$ then $\theta \to 0 \qquad$ as $\qquad n \to \infty$

So $\quad \displaystyle\lim_{n\to\infty}\frac{\cot\left(x/2^n\right)}{2^n} = \frac{1}{x}\lim_{\theta\to 0}\frac{\theta}{\sin\theta}$

$$= \frac{1}{x}.1$$

Then $\quad \displaystyle\lim_{n\to\infty} s_n = -\cot x + \frac{1}{x}$

Sum $\quad \boxed{s = \dfrac{1}{x} - \cot x}$

Answer 3TFQ.

True
Since any subsequence of a convergent sequence is also converges to the same limit
$\{a_{2n+1}\}$ is a subsequence of $\{a_n\}$
Since if $n \to \infty$ then $2n+1 \to \infty$
So, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{2n+1} = L$

Answer 4CC.

$\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 3$

Answer 4E.

We have $a_n = \cos(n\pi/2)$

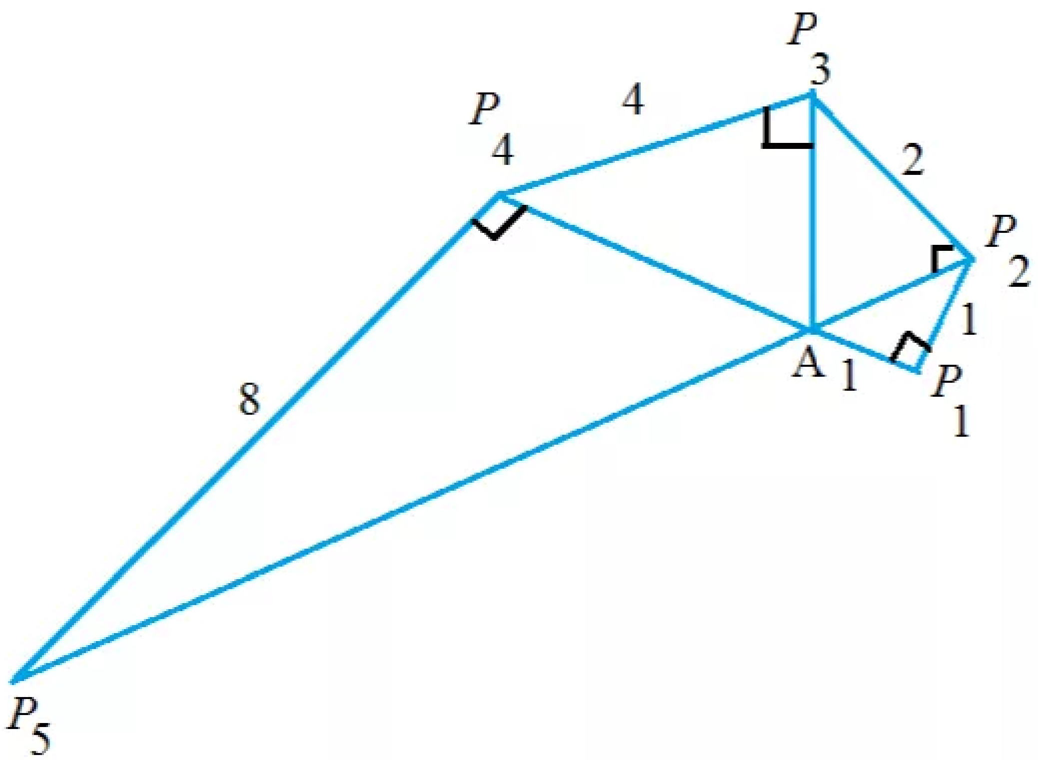
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n\pi/2)$$

When n is an odd number, $\cos(n\pi/2) = 0$
And when n is an even number,
The value of $\cos(n\pi/2)$ oscillates between -1 and 1

So $\cos(n\pi/2)$ does not tend to a fix number
Then $\lim_{n\to\infty} \cos(n\pi/2)$ does not exist
So the sequence a_n is $\boxed{\text{divergent}}$

Answer 4P.

Let $\{P_n\}$ be a sequence of points determined as shown in the below figure.



Let, $|AP_1| = 1$, $|P_nP_{n+1}| = 2^{n-1}$ **and** AP_nP_{n+1} **is a right angle.**

So, $\tan\left(\angle P_nAP_{n+1}\right) = \dfrac{|P_nP_{n+1}|}{|AP_n|}$ and $|AP_{n+1}|^2 - |AP_n|^2 = |P_nP_{n+1}|^2$

And we can write $|AP_n|^2 - |AP_1|^2$ in the following form.

$$\begin{aligned}|AP_n|^2 - |AP_1|^2 &= \left(|AP_n|^2 - |AP_{n-1}|^2\right) + \left(|AP_{n-1}|^2 - |AP_{n-2}|^2\right) + \ldots + \left(|AP_2|^2 - |AP_1|^2\right)\\ &= |P_{n-1}P_n|^2 + |P_{n-2}P_{n-1}|^2 + \ldots + |P_2P_3|^2 + |P_1P_2|^2\\ &= \left(2^{n-2}\right)^2 + \left(2^{n-3}\right)^2 + \ldots + (2)^2 + (1)^2\end{aligned}$$

$$= 2^{2(n-2)} + 2^{2(n-3)} + \ldots + 4 + 1$$

$$= 4^{n-2} + 4^{n-3} + \ldots + 4 + 1$$

$$= \frac{4^{n-1} - 1}{4 - 1} \qquad \text{Since by geometric series formula.}$$

$$|AP_n|^2 - |AP_1|^2 = \frac{4^{n-1} - 1}{4 - 1}$$

$$|AP_n|^2 = \frac{4^{n-1} - 1}{4 - 1} + |AP_1|^2$$

$$= \frac{4^{n-1} - 1}{3} + 1$$

$$= \frac{4^{n-1} + 2}{3}$$

$$|AP_n| = \sqrt{\frac{4^{n-1} + 2}{3}}$$

To find $\lim_{n\to\infty} \angle P_n A P_{n+1}$.

$$\lim_{n\to\infty} \angle P_n A P_{n+1} = \lim_{n\to\infty} \tan^{-1}\left(\frac{|P_n P_{n+1}|}{|AP_n|}\right)$$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{|P_n P_{n+1}|}{|AP_n|}\right)$$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{n-1}}{\sqrt{\frac{4^{n-1}+2}{3}}}\right)$$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{n-1}}{1} \times \frac{\sqrt{3}}{\sqrt{4^{n-1}+2}}\right)$$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{n-1}}{1} \times \frac{\sqrt{3}}{\sqrt{4^{n-1}\left(1+\frac{2}{4^{n-1}}\right)}}\right)$$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{n-1}}{1} \times \frac{\sqrt{3}}{\sqrt{2^{2(n-1)}\left(1+\frac{2}{4^{n-1}}\right)}}\right)$$

Solve the above equation.

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{n-1}}{1} \times \frac{\sqrt{3}}{2^{n-1}\sqrt{\left(1+\frac{2}{4^{n-1}}\right)}}\right)$$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{\left(1+\frac{2}{4^{n-1}}\right)}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{(1+0)}}\right)$$

$$= \tan^{-1}\left(\sqrt{3}\right)$$

$$= \boxed{\frac{\pi}{3}}$$

Answer 4TFQ.

True

Given

$$a_n = c_n(6)^n$$

Since the series is convergent by the Ratio Test we must have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{6c_{n+1}}{c_n}\right| \le 1$$

$$\Rightarrow \left|\frac{c_{n+1}}{c_n}\right| \le \frac{1}{6}$$

Let $b_n = c_n(-2)^n$

Then $\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{2c_{n+1}}{c_n}\right| \le \frac{1}{3} < 1$

So by the Ratio Test the series $\sum c_n(-2)^n$ is convergent.

Answer 5CC.

(a)

The Test for Divergence:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \ne 0$, then the series is divergent.

(b)

The Integral Test:

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$

And let $a_n = f(n)$. Then the series

$\sum a_n$ is convergent if and only if the improper Integral $\int_1^{\infty} f(x)dx$ is convergent

(c)

The Comparison Test:

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

(1) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(2) If $\sum a_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

(d)

The Limit Comparison Test:

Suppose that

$\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite and c>0, then either both series converge or both diverge.

(e)
Alternating Series Test:
If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ldots \quad b_n > 0$$

Satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

Then the series is convergent.

(f)
The Ratio Test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive: that is, no conclusion can be

Drawn about the convergence or divergence of $\sum a_n$

(g)
The Root Test:

(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(and therefore convergent).

(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ divergent.

(iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive: that is, no conclusion can be

Drawn about the convergence or divergence of $\sum a_n$

Answer 5E.

We have $a_n = \frac{n\sin n}{n^2+1}$

$$\Rightarrow a_n = \frac{n}{n^2+1}.\sin n$$

Taking limit *as* $n\to\infty$

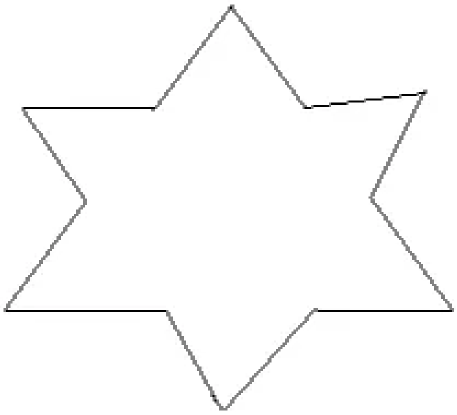
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n^2+1}\left(\lim_{n\to\infty}\sin n\right)$$

Since $-1 \le \sin n \le 1$ for all value of n.
So sin n tends to a finite value between -1 and 1, but $\sin n$ does not tend to a fixed value.

Answer 5P.

To construct a snowflake curve, start with an equilateral triangle with sides of length.

In the construction of it, divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part as shown.



Repeat the above process for each side of the resulting polygon.

(a)

Let s_n, l_n, and p_n represent the number of sides, the length of a side, and the total length of the nth approximating curve respectively.

At each stage each side is replaced by four shorter sides, each of the length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and l_0 for the number of sides and the length of the side of the initial triangle.

We generate the table as shown.

Sides	Length
$s_0 = 3$	$l_0 = 1$
$s_1 = 3 \cdot 4$	$l_1 = 1/3$
$s_2 = 3 \cdot 4^2$	$l_1 = 1/3^2$
$s_3 = 3 \cdot 4^3$	$l_1 = 1/3^3$
$\cdots$	$\cdots$

In general, we have

$s_n = 3 \cdot 4^n$ and $l_n = \left(\frac{1}{3}\right)^n$

So, the length of the perimeter at the n^{th} stage of construction is

$$\begin{aligned} p_n &= s_n l_n \\ &= 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n \\ &= 3 \cdot \left(\frac{4}{3}\right)^n \end{aligned}$$

Thus $p_n = 3 \cdot \left(\frac{4}{3}\right)^n$

(b)

$$\begin{aligned} \lim_{n\to\infty} p_n &= \lim_{n\to\infty} 3 \cdot \left(\frac{4}{3}\right)^n \\ &= \lim_{n\to\infty} 4 \cdot \left(\frac{4}{3}\right)^{n-1} \\ &= \infty \qquad \text{Since } \frac{4}{3} > 1, \left(\frac{4}{3}\right)^{n-1} \to \infty \text{ as } n \to \infty \end{aligned}$$

(c)

The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage.

Let a be the area of the original triangle.

Then the area a_n of each of the small triangles added at stage n is

$$\begin{aligned} a_n &= a \cdot \frac{1}{9^n} \\ &= \frac{a}{9^n} \end{aligned}$$

Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at n^{th} stage is

$$\begin{aligned} A_n &= s_{n-1} \cdot a_n \\ &= 3 \cdot 4^{n-1} \cdot \frac{a}{9^n} \\ &= a \cdot \frac{4^{n-1}}{3^{2n-1}} \end{aligned}$$

Then the total area enclose by the snowflake curve is

$$\begin{aligned} A &= a + A_1 + A_2 + A_3 + \cdots \\ &= a + a \cdot \frac{1}{3} + a \cdot \frac{4}{3^3} + a \cdot \frac{4^2}{3^5} + a \cdot \frac{4^3}{3^7} + \cdots \end{aligned}$$

After the first term, this is a geometric series with common ration $\frac{4}{9}$

So

$$A = a + \frac{a/3}{1-\frac{4}{9}}$$
$$= a + \frac{a}{3}\cdot\frac{9}{5}$$
$$= \frac{8a}{5}$$

But the area of original equilateral triangle with sides 1 is

$$a = \frac{1}{2}\cdot 1\cdot \sin\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{4}$$

So, the area enclosed by the snowflake curve is

$$\frac{8a}{5} = \frac{8}{5}\cdot\frac{\sqrt{3}}{4}$$
$$= \boxed{\frac{2\sqrt{3}}{5}}$$

Answer 5TFQ.

False

Example:

Let $c_n = \frac{(-1)^n}{n6^n}$

Then $\sum c_n = \sum \frac{(-1)^n}{n6^n}$ is convergent, but $\sum c_n(-6)^n = \sum \frac{1}{n}$ is divergent.

Answer 6CC.

(a)

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(b)

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

(c)

A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Answer 6E.

We have $a_n = \dfrac{\ln n}{\sqrt{n}}$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$$

This is the form of $\dfrac{\infty}{\infty}$ so we use L-hospital rule

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(\sqrt{x}\right)}$$

$$= \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}}$$

$$= 0$$

So $\boxed{\lim_{n\to\infty} a_n = 0}$

And then the sequence a_n *is* $\boxed{convergent}$

Answer 6P.

Find the sum of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\cdots$$

Here the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s

Suppose that $S = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\cdots$

Since the terms are reciprocals of the positive integers whose only prime factors are 2s and 3s, we write the series as

$$S = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{2^n \times 3^m}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^n \times 3^0}+\frac{1}{2^n \times 3^1}+\frac{1}{2^n \times 3^2}+\frac{1}{2^n \times 3^3}+\cdots\right)$$ Expand the summation over m

$$= \frac{1}{2^0 \times 3^0}+\frac{1}{2^0 \times 3^1}+\frac{1}{2^0 \times 3^2}+\frac{1}{2^0 \times 3^3}+\cdots$$
$$+\frac{1}{2^1 \times 3^0}+\frac{1}{2^1 \times 3^1}+\frac{1}{2^1 \times 3^2}+\frac{1}{2^1 \times 3^3}+\cdots$$
$$+\frac{1}{2^2 \times 3^0}+\frac{1}{2^2 \times 3^1}+\frac{1}{2^2 \times 3^2}+\frac{1}{2^2 \times 3^3}+\cdots$$
$$+\frac{1}{2^3 \times 3^0}+\frac{1}{2^3 \times 3^1}+\frac{1}{2^3 \times 3^2}+\frac{1}{2^3 \times 3^3}+\cdots$$

Expand the summation over n

Continuation to the above

$$S=1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots$$
$$+\frac{1}{2^1}+\frac{1}{2^1\times 3^1}+\frac{1}{2^1\times 3^2}+\frac{1}{2^1\times 3^3}+\cdots$$
$$+\frac{1}{2^2}+\frac{1}{2^2\times 3^1}+\frac{1}{2^2\times 3^2}+\frac{1}{2^2\times 3^3}+\cdots$$
$$+\frac{1}{2^3}+\frac{1}{2^3\times 3^1}+\frac{1}{2^3\times 3^2}+\frac{1}{2^3\times 3^3}+\cdots$$
$$=\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)+\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)+$$
$$+\frac{1}{2^1\times 3^1}+\frac{1}{2^1\times 3^2}+\frac{1}{2^2\times 3^1}+\frac{1}{2^2\times 3^2}+\frac{1}{2^1\times 3^3}+\frac{1}{2^2\times 3^1}+\cdots$$
$$=\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)+\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{2^p\times 3^q}$$

Continuation to the above

$$S=\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)+\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)+\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^{p+1}\times 3^{q+1}}$$
$$=\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)+\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)+\frac{1}{2\cdot 3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^p\times 3^q}$$
$$=\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)+\left(\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)-1\right)+\frac{1}{2\cdot 3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^p\times 3^q}$$
$$=\frac{1}{1-\frac{1}{3}}+\left(\frac{1}{1-\frac{1}{2}}-1\right)+\frac{1}{2\cdot 3}S$$
$$=\frac{1}{1-\frac{1}{3}}+(2-1)+\frac{1}{2\cdot 3}S$$
$$=\frac{3}{2}+1+\frac{1}{6}S$$

Thus

$$S = \frac{3}{2} + 1 + \frac{1}{6}S$$

As $S = \frac{3}{2} + 1 + \frac{1}{6}S$, we have that

$$\left(1 - \frac{1}{6}\right)S = \frac{3}{2} + 1$$

$$\Rightarrow \frac{5}{6}S = \frac{3+2}{2}$$

$$\Rightarrow \frac{5}{6}S = \frac{5}{2}$$

$$\Rightarrow S = 3$$

That is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots = \boxed{3}$$

Answer 6TFQ.

True

Given

$$a_n = c_n (6)^n$$

Since the series is divergent by the Ratio Test we must have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{6c_{n+1}}{c_n}\right| \geq 1$$

$$\Rightarrow \left|\frac{c_{n+1}}{c_n}\right| \geq \frac{1}{6}$$

Let $b_n = c_n (10)^n$

Then $\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{10c_{n+1}}{c_n}\right| \geq \frac{10}{6} > 1$

So by the Ratio Test the series $\sum c_n (10)^n$ is divergent.

Answer 7CC.

(a)

Remainder Estimate for the Integral Test:

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent.

If $R_n = s - s_n$

Then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$

(b)
Remainder Estimate for the Comparison Test:
If we have used the Comparison Test to show that a series

$\sum_{n=1}^{\infty} a_n$ is converges by comparison with a series $\sum_{n=1}^{\infty} b_n$, then we may be able to estimate

the sum $\sum_{n=1}^{\infty} a_n$ by comparing remainders.

We consider the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + \ldots$

For the comparison series $\sum_{n=1}^{\infty} b_n$ we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \ldots$$

Since $a_n \le b_n$ for all n, we have $R_n \le T_n$

(c)
Remainder Estimate for the Alternating Series Test:
If $s = \sum (-1)^{n-1} b_n$ is a sum of an alternating series that satisfies $b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$

Then

$$|R_n| = |s - s_n| \le b_{n+1}$$

Answer 7E.

We have $a_n = \left\{(1+3/n)^{4n}\right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1+3/n)^{4n}$$

Let $y = \left(1+\frac{3}{x}\right)^{4x}$

Taking logarithms of both sides

$$\ln y = 4x\ln(1+3/x)$$

Taking limit of both sides as $x \to \infty$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 4x\ln(1+3/x)$$

$$= \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/4x}$$

This is the form of $\frac{0}{0}$ so we use L-Hospital rule

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{1}{(1+3/x)}(-3/x^2)}{-(1/4x^2)}$$

$$= \lim_{x\to\infty} \frac{12}{(1+3/x)} = \frac{12}{1+0}$$

$$= 12$$

Then $\lim_{x\to\infty} \ln y = 12$ so $\lim_{x\to\infty} y = e^{12}$

Thus $\boxed{\lim_{n\to\infty} a_n = e^{12}}$ so the sequence a_n is $\boxed{\text{convergent}}$

Answer 7P.

(a) Let $a = \arctan x$ and $b = \arctan y$.

$$\text{Then } \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$$= \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x)\tan(\arctan y)}$$

$$= \frac{x-y}{1+xy}$$

Now, $\arctan x - \arctan y = a - b$

$$= \arctan(\tan(a-b))$$

$$= \arctan\frac{x-y}{1+xy} \qquad \text{Since } -\frac{\pi}{2} < a-b < \frac{\pi}{2}$$

(b) From part (a), we have

$$\arctan\frac{120}{119} - \arctan\frac{1}{239} = \arctan\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}}$$

$$= \arctan\frac{\frac{28,561}{28,441}}{\frac{28,561}{28,441}}$$

$$= \arctan 1$$

$$= \frac{\pi}{4}$$

(c) Replacing y by $-y$ in the formula of part (a), we get

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

So $$4\arctan\frac{1}{5} = 2\left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right)$$

$$= 2\arctan\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}}$$

$$= 2\arctan\frac{5}{12}$$

$$= \arctan\frac{5}{12} + \arctan\frac{5}{12}$$

$$= \arctan\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{5}{12}\cdot\frac{5}{12}}$$

$$= \arctan\frac{120}{119}$$

Therefore, from part (b), we have

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239}$$

$$= \frac{\pi}{4}$$

(d) We have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots\ldots$$

So $\arctan\frac{1}{5} = \frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} + \frac{1}{9.5^9} - \frac{1}{11.5^{11}} + \ldots.$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between

s_5 and s_6, that is $0.197395560 < \arctan\frac{1}{5} < 0.197395562$

(e) From the series in part (d) we get that

$$\arctan\frac{1}{239} = \frac{1}{239} - \frac{1}{3.239^3} + \frac{1}{5.239^5} - \ldots.$$

The third term is less than $2.6x10^{-3}$, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan\frac{1}{239} \approx s_2 \approx 0.004184076$.

Thus $0.004184075 < \arctan\frac{1}{239} < 0.004184077$

(f) From part (c), we have that $\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$.

So, from parts (d) and (e) we have that

$$16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075)$$

$$\Rightarrow 3.141592652 < \pi < 3.141592692$$

So, to 7 decimal places, $\pi \approx 3.1415927$

Answer 7TFQ.

False

Let $a_n = \frac{1}{n^3}$

$$\text{Then } \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}}\right| = \frac{1}{\left(1+\frac{1}{n}\right)^3} \to 1$$

So Ratio Test cannot be used

Ratio test:

(i) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive

Answer 8CC.

(a)

The general form of the power series is

$$\sum_{n=1}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

(b)

If there is positive number R such that the above power series converges if $|x-a| < R$ and diverges if $|x-a| > R$ then R is called the Radius of convergence.

(c)

The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

Answer 8E.

Let $a_n = \frac{(-10)^n}{n!}$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(10)^n}{n!}$

Both numerator and denominator tend to ∞ *as* $n \to \infty$, but we can not use L-Hospital rule with x!

We expand it

$$\lim_{n\to\infty} \frac{10^n}{n!} = \lim_{n\to\infty}\left[\left(\frac{10}{n}\right)\cdot\left(\frac{10}{n-1}\right)\cdot\left(\frac{10}{n-2}\right)\cdot\left(\frac{10}{n-3}\right)\cdots\cdots\frac{10}{1}\right]$$

$$= \lim_{n\to\infty}\left(\frac{10}{n}\right).\lim_{n\to\infty}\left(\frac{10}{n-1}\right).\lim_{n\to\infty}\left(\frac{10}{n-2}\right)\cdots\cdots\lim_{n\to\infty}(10)$$

$$= 0\left[\lim_{n\to\infty}\left(\frac{10}{n-1}\right).\lim_{n\to\infty}\left(\frac{10}{n-2}\right)---\lim_{n\to\infty}10\right]$$

$$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$$

So the sequence a_n *is* $\boxed{\textit{convergent}}$

Answer 8P.

(a)To show that for $xy \neq 1$,

$$arc\cot x - arc\cot y = arc\cot\left(\frac{xy+1}{x-y}\right)$$

That is, $\cot^{-1} x - \cot^{-1} y = \cot^{-1}\left(\frac{xy+1}{x-y}\right) \quad \cdots\cdots(1)$

Let $\cot^{-1} x = \alpha \Rightarrow x = \cot\alpha$

and $\cot^{-1} y = \beta \Rightarrow y = \cot\beta$

Now the formula of $\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\alpha-\cot\beta}$

The above formula can be written as

$$\alpha-\beta = \cot^{-1}\left(\frac{\cot\alpha\cot\beta+1}{\cot\alpha-\cot\beta}\right) \quad \cdots\cdots(2)$$

Substitute the above conditions in (2)

$$\cot^{-1} x - \cot^{-1} y = \cot^{-1}\left(\frac{xy+1}{x-y}\right)$$

Hence $\boxed{\cot^{-1} x - \cot^{-1} y = \cot^{-1}\left(\frac{xy+1}{x-y}\right)}$ for $xy \neq 1$.

(b)Consider the series

$$\sum_{n=0}^{\infty} arc\cot\left(n^2+n+1\right)$$

If $\alpha = arc\cot x$ and $\beta = arc\cot y$ for some $x, y > 0$, then the subtraction formula

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\alpha-\cot\beta}$$

Provided that $arc\cot x - arc\cot y \le \frac{\pi}{2}$ i.e., $arc\cot x \le arc\cot\left(\frac{1}{y}\right)$

Which is equivalent to $x \ge \frac{1}{y}$ and hence equivalent to $xy \ge 1$.

Verifying that $xy \ge 1$

Using the previous equation to compute the first few partial sums,

$$arc\cot 1 = arc\cot 1$$

$$arc\cot 3 - arc\cot 1 = arc\cot 2$$

$$\begin{aligned} arc\cot 7-\left(arc\cot 3-arc\cot 1\right) &= arc\cot 7 - arc\cot 2 \\ &= arc\cot 3 \end{aligned}$$

And guess that $\sum_{n=0}^{m-1} arc\cot\left(n^2+n+1\right)=arc\cot(m)$ for all $m \ge 1$.

This is easily proved by induction on *m*.

$$\begin{aligned} \sum_{n=0}^{m} arc\cot\left(n^2+n+1\right) &= arc\cot\left(m^2+m+1\right)+\sum_{n=0}^{m-1} arc\cot\left(n^2+n+1\right) \\ &= arc\cot\left(m^2+m+1\right)+arc\cot(m) \\ &= \frac{\left(m^2+m+1\right)(m)+1}{m^2+1} \\ &= \frac{(m+1)\left(m^2+1\right)}{m^2+1} \\ &= m+1 \end{aligned}$$

Therefore, $\boxed{\sum_{n=0}^{m} arc\cot\left(n^2+n+1\right)=m+1}$

Answer 8TFQ.

True

Let $a_n = \frac{1}{n!}$

Than $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \frac{1}{n+1} \to 0 < 1$

So by Ratio Test the given series is convergent.

Ratio test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L<1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive

Answer 9CC.

(a)

If the power series is $\sum_{n=1}^{\infty} c_n (x-a)^n$ has radius of convergence R>0, then the function f defined by

$$f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$$

The radius of the convergence of this power series is R.

(b)

If the power series is $\sum_{n=1}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

is integrable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$\int f(x) = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

$$= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots$$

The radius of the convergence of this power series is R

Answer 9E.

Assume that $a_{n-1} < a_n < 2 \qquad -(1)$

For n = 1 $\quad a_0 < a_1 < 2$

$$\Rightarrow a_0 + 4 < a_1 + 4 < 2+4$$

$$\Rightarrow \frac{1}{3}(a_0+4) < \frac{1}{3}(a_1+4) < \frac{1}{3}(2+4)$$

$$\Rightarrow a_1 < a_2 < 2 \qquad \text{Since } a_{n+1} = \frac{1}{3}(a_n+4)$$

$$\Rightarrow 1 < \frac{5}{3} < 2$$

So the statement (1) is true for n = 1

Assuming that (1) is true for n = k

So $\quad a_{k-1} < a_k < 2$

$$\Rightarrow a_{k-1} + 4 < a_k + 4 < 2+4$$

$$\Rightarrow \frac{1}{3}(a_{k-1}+4) < \frac{1}{3}(a_k+4) < \frac{1}{3}(2+4)$$

$\Rightarrow a_k < a_{k+1} < 2$ so it is true for n = k+1

So by mathematical induction

$a_n < a_{n+1} < 2 \quad$ *is true for all n*

Since $\{a_n\}$ is an increasing sequence then

$$a_n < a_{n+1}$$

Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = L \qquad (Let)$

Then $L = \frac{1}{3}(L+4)$

$$\Rightarrow L - \frac{L}{3} = \frac{4}{3} \quad \Rightarrow \quad \frac{2L}{3} = \frac{4}{3}$$

$$\Rightarrow 2L = 4$$

$$\Rightarrow L = 4/3 \qquad \Rightarrow L = 2$$

Then $\boxed{\lim\limits_{n\to\infty} a_n = 2}$ and so the sequence a_n is convergent

Answer 9P.

Consider the power series

$$\sum_{n=1}^{\infty} n^3 x^n$$

Its need to find the interval of convergence and its sum

Consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3 x^{n+1}}{n^3 x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^3\left(1+\frac{1}{n}\right)^3 x^{n+1}}{n^3 x^n}\right|$$

$$= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^3 x\right|$$

$$= |(1+0)\cdot x|$$

$$= |x|$$

Thus $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$

By the Ratio Test, the series $\sum a_n x^n$ converge if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

By the above fact, the given series converges if

$|x| < 1$

This means that given series converges if $|x| < 1$ and diverges if $|x| > 1$.

That is the radius of convergence is $R = 1$

As $|x| < 1 \Leftrightarrow -1 < x < 1$

So the series converges when $-1 < x < 1$ and diverges when $x < -1$ or $x > 1$

The Ratio test gives no information when $|x|=1$

$\Leftrightarrow \quad x=1$ and $x=-1$

So we must consider $x=-1$ and $x=1$ separately.

If we keep $x=-1$ in the series $\sum_{n=1}^{\infty} 3^n x^n$, it becomes

$$\sum_{n=1}^{\infty} 3^n(-1)^n=\sum_{n=1}^{\infty}(-3)^n$$

which is divergent series.

The Ratio test gives no information when $|x|=1$

$\Leftrightarrow \quad x=1$ and $x=-1$

So we must consider $x=-1$ and $x=1$ separately.

If we keep $x=-1$ in the series $\sum_{n=1}^{\infty} 3^n x^n$, it becomes

$$\sum_{n=1}^{\infty} 3^n(-1)^n=\sum_{n=1}^{\infty}(-3)^n$$

which is divergent series.

Answer 9TFQ.

False
Using comparison test, the given statement is false

The Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Answer 10CC.

(a)

Expression for the nth-degree Taylor polynomial of f centered at a:

$$f(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$=f(a)+\frac{f^{(1)}(a)}{1!}(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

(b)

Expression for the Taylor series of f centered at a:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$=f(a)+\frac{f^{(1)}(a)}{1!}(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\ldots$$

(c)

Expression for the Maclaurin series of f centered at a:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
$$=f(0)+\frac{f^{(1)}(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\ldots$$

(d)

If $f(x)=T_n(x)+R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and $\lim_{n\to\infty} R_n(x)=0$ for $|x-a|<R$, then f is equal to the sum of its Taylor series on the interval $|x-a|<R$

(e)

Taylor's Inequality:

If $\left|f^{n+1}(x)\right|\le M$ for $|x-a|\le d$ then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}\text{ for }|x-a|\le d$$

Answer 10E.

L'Hospital's rule:

Suppose f and g are differentiable and $g'(x)\ne 0$ on an open interval I that contains a.

Suppose that ,

$$\lim_{x\to a} f(x)=0 \quad\text{and}\quad \lim_{x\to a} g(x)=0$$

Or that

$$\lim_{x\to a} f(x)=\pm\infty \quad\text{and}\quad \lim_{x\to a} g(x)=\pm\infty$$

Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ if the limit on the right side exists.

Recollect that:

If $\lim_{x\to\infty} f(x) = L$ and $f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{x\to\infty} a_n = L$

Consider the limit, $\lim_{n\to\infty} n^4 e^{-n}$

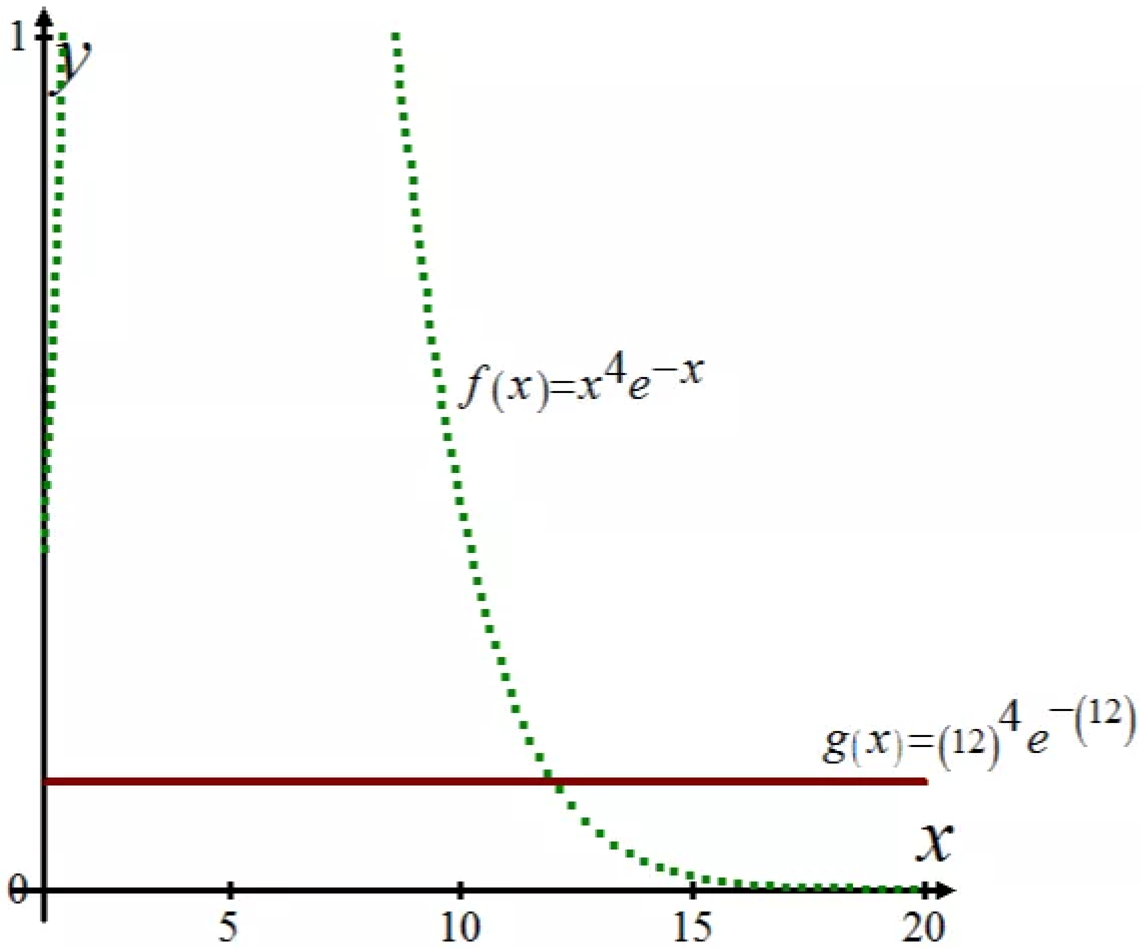
$$\begin{aligned}\lim_{n\to\infty} n^4 e^{-n} &= \lim_{n\to\infty} \frac{n^4}{e^n} \\ &= \lim_{n\to\infty} \frac{4n^3}{e^n} \quad \text{By L'Hospital's rule.} \\ &= \lim_{n\to\infty} \frac{12n^2}{e^n} \quad \text{Again use L'Hospital's rule.} \\ &= \lim_{n\to\infty} \frac{24n}{e^n} \quad \text{Again use L'Hospital's rule}\end{aligned}$$

$$= \lim_{n\to\infty} \frac{24}{e^n} \quad \text{Again use L'Hospital's rule}$$
$$= 0$$

The below figure it seem that $12^4 e^{-12} > 0.1$ but $n^4 e^{-n} < 0.1$ whenever $n > 12$

Therefore the smallest value of N corresponding to $\varepsilon = 0.1$

The definition of the limit is $N = 12$



Answer 10P.

Consider the following:

$a_0+a_1+a_2+\cdots+a_k=0$ ...... (1)

At $k=1$:

$$\begin{aligned} a_k\sqrt{n+k} &= a_1\sqrt{n+1} \\ &= a_1\sqrt{n+1} \\ &= 0 \qquad\qquad \text{From (1), } a_1=0 \end{aligned}$$

Hence, it is true.

At $k=2$:

$$\begin{aligned} a_k\sqrt{n+2} &= a_2\sqrt{n+2} \\ &= a_2\sqrt{n+2} \\ &= 0 \qquad\qquad \text{From (1), } a_2=0 \end{aligned}$$

Hence, it is true.

Now, to evaluate $\lim\limits_{n\to\infty}\left(a_0\sqrt{n}+a_1\sqrt{n+1}+a_2\sqrt{n+2}+\cdots+a_k\sqrt{n+k}\right)$, follow the steps:

$$\lim_{n\to\infty}\left(a_0\sqrt{n}+a_1\sqrt{n+1}+a_2\sqrt{n+2}+\cdots+a_k\sqrt{n+k}\right)$$

$$=\lim_{n\to\infty}\left(a_0\sqrt{n}+a_1\sqrt{n+1}+a_2\sqrt{n+2}+\cdots+\left(-a_1-a_2-\cdots-a_{k-1}\right)\sqrt{n+k}\right)$$

$$=\lim_{n\to\infty}\begin{pmatrix} a_0\left(\sqrt{n}-\sqrt{n+k}\right)+a_1\left(\sqrt{n+1}-\sqrt{n+k}\right)+\cdots \\ +a_{k-1}\left(\sqrt{n+(k-1)}-\sqrt{n+k}\right) \end{pmatrix}$$

$$=\lim_{n\to\infty}\frac{a_0\left(n-(n+k)\right)}{\sqrt{n}+\sqrt{n+k}}+\lim_{n\to\infty}\frac{a_1\left(n+1-(n+k)\right)}{\sqrt{n+1}+\sqrt{n+k}}+\cdots$$

$$+\lim_{n\to\infty}\frac{a_{k-1}\left(n+(k-1)-(n+k)\right)}{\sqrt{n+(k-1)}+\sqrt{n+k}}$$

$$=\lim_{n\to\infty}\frac{a_0k}{\sqrt{n}+\sqrt{n+k}}+\lim_{n\to\infty}\frac{a_1(1-k)}{\sqrt{n+1}+\sqrt{n+k}}+\cdots+\lim_{n\to\infty}\frac{a_{k-1}(1)}{\sqrt{n+(k-1)}+\sqrt{n+k}}$$

$$=\frac{a_0k}{\infty}+\frac{a_1(1-k)}{\infty}+\cdots+\frac{a_{k-1}}{\infty}$$

$$=0$$

Hence, it is proved.

Answer 10TFQ.

True

We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

Answer 11CC.

(a)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= 1 + x + x^2 + \ldots$$

Radius of Convergence R=1.

(b)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$$

Radius of Convergence R=∞.

(c)

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

Radius of Convergence R=∞.

(d)

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= x - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

Radius of Convergence R=∞.

(e)

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

Radius of Convergence R=1.

(f)

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Radius of Convergence R=1.

Answer 11E.

Series is $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

We use comparison test

Since $\frac{1}{n^3+1} \le \frac{1}{n^3}$

$$\Rightarrow \frac{n}{n^3+1} \le \frac{n}{n^3} = \frac{1}{n^2}$$

If $a_n = \frac{n}{n^3+1}$ and $b_n = 1/n^2$

Then $a_n \le b_n$ for all n.

Now $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$

So series $\sum_{n=0}^{\infty} b_n$ is convergent then

By comparison test series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ is also $\boxed{\text{convergent}}$

Answer 11P.

Consider the series

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$$

Its need to find the sum of the series

On rewriting the given series, we have that

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n=2}^{\infty} \ln\left(\frac{n^2-1}{n^2}\right)$$

$$= \sum_{n=2}^{\infty} \ln\left(\frac{n-1}{n}\frac{n+1}{n}\right)$$

$$= \sum_{n=2}^{\infty} \left[\ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n+1}{n}\right)\right]$$

$$= \sum_{n=2}^{\infty} \left[\ln\left(1-\frac{1}{n}\right) + \ln\left(1+\frac{1}{n}\right)\right]$$

$$= \ln\left(1-\frac{1}{2}\right) + \ln\left(1+\frac{1}{2}\right) + \ln\left(1-\frac{1}{3}\right) + \ln\left(1+\frac{1}{3}\right) + \cdots + \ln\left(1-\frac{1}{n-1}\right) +$$
$$\ln\left(1+\frac{1}{n-1}\right) + \ln\left(1-\frac{1}{n}\right) + \ln\left(1+\frac{1}{n}\right) + \cdots$$

Continuation to the above

$$\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{4}{3}\right) + \cdots + \ln\left(\frac{n-2}{n-1}\right) +$$
$$\ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n+1}{n}\right) + \cdots$$

$$= \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right) - \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \cdots + \ln\left(\frac{n-2}{n-1}\right) +$$
$$\ln\left(\frac{n}{n-1}\right) - \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n+1}{n}\right) + \cdots + 0 - 0$$

$$= \ln\left(\frac{1}{2}\right) + \cancel{\ln\left(\frac{3}{2}\right)} - \cancel{\ln\left(\frac{3}{2}\right)} + \cancel{\ln\left(\frac{4}{3}\right)} + \cdots + \cancel{\ln\left(\frac{n-2}{n-1}\right)} +$$
$$\cancel{\ln\left(\frac{n}{n-1}\right)} - \cancel{\ln\left(\frac{n}{n-1}\right)} + \cancel{\ln\left(\frac{n+1}{n}\right)} + \cdots + 0 - 0$$

$$= \ln\left(\frac{1}{2}\right)$$

Thus $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \boxed{\ln\left(\frac{1}{2}\right)}$

Answer 11TFQ.

True

As $n \to \infty$, the value of α^n are tends to 0.

The sequence $\{r^n\}$ is convergent if $-1 < r < 1$ and divergent for all other values of r

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Answer 12CC.

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$
$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \dots$$

Radius of Convergence R=1.

Answer 12E.

Given series is $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

Dominant part of the numerator is n^2

And dominant part of the denominator is n^3

So we take $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

Now we use limit comparison test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+1}{n^3+1}}{1/n}$$
$$= \lim_{n\to\infty} \frac{n^3+n}{n^3+1}$$
$$= \lim_{n\to\infty} \frac{1+1/n^2}{1+1/n^3} = \frac{1+0}{1+0} = 1$$

So $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$

And since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series with $p = 1$

Then the series $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$ is also $\boxed{\text{divergent}}$

Answer 12P.

To balance a book on the table, the center of gravity of the book must be somewhere over the table to achieve the maximum overhang, and the center of gravity should be just over the table's edge.

Assuming the book is one unit in length, the maximum overhang achieved is $\frac{1}{2}$ unit.

For two books, the center of gravity of the first should be directly over the edge of the second, while the center of gravity of the stack of two books should be directly above the edge of the table.

The center of gravity of the stack of two books is at the midpoint of the books overlap, that is,

$$\frac{\left(1+\frac{1}{2}\right)}{2}=\frac{3}{4}$$

The maximum overhang possible d_n for n books (without the stack falling over) is half the nth partial sum of the harmonic series,

$$\boxed{d_n=\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}}$$

The overhang in case of five books $=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)$

$$=\frac{1}{2}(2.28333)$$
$$=1.14166$$
$$>1$$

Therefore, no part of the top book is over the table.

Answer 12TFQ.

True

Since we have

$$|a_n|\geq a_n$$
$$\Rightarrow\sum|a_n|\geq\sum a_n$$

$\sum|a_n|$ is divergent by comparison test

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

(i) If $\sum b_n$ is convergent and $a_n\leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n\geq b_n$ for all n, then $\sum a_n$ is also divergent.

Answer 13E.

Series is $\sum_{n=1}^{\infty}\frac{n^3}{5^n}$

We use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left[\frac{(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{n^3}\right]$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\cdot\frac{1}{5}$$

$$=\frac{1}{5}\cdot\lim_{n\to\infty}\left(\frac{1+1/n}{1}\right)^3$$

$$=\frac{1}{5}\cdot\left(\frac{1+0}{1}\right)^3$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{5}<1$$

Then $\boxed{\sum_{n=1}^{\infty}\frac{n^3}{5^n}\text{ converges}}$ by ratio test.

Answer 13P.

(a)The graph of the function $f(x)=e^{\frac{-x}{10}}\sin x,\ x\geq 0$ intersects the x-axis an infinite number of times at $x=(k-1)\pi\ \ k=1,2,\ldots$ The volume of the n^{th} bead can then be found as:

$$V_n=\pi\int_{(n-1)\pi}^{n\pi}\left[f(x)\right]^2dx$$

Simplify:

$$= \pi \int_{(n-1)\pi}^{n\pi} \left[e^{\frac{-x}{10}} \sin x \right]^2 dx$$

$$= \pi \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} [\sin x]^2 dx$$

$$= \frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} [1 - \cos(2x)] dx$$

$$= \left(\frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} dx \right) - \left(\frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} [\cos(2x)] dx \right)$$

In first part plugin $u = \frac{-x}{5}$

$$= \left(\frac{-5\pi}{2} \int_{\frac{-(n-1)\pi}{5}}^{\frac{-n\pi}{5}} e^{u} du \right) - \left(\frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} \cos(2x) dx \right)$$

$$= \frac{-5\pi}{2} e^{u} \Big[_{\frac{-(n-1)\pi}{5}}^{\frac{-n\pi}{5}} - \frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} \cos(2x) dx$$

$$= \frac{-5\pi}{2} e^{\frac{-n\pi}{5}} \left(1 - e^{\frac{\pi}{5}} \right) - \frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} \cos(2x) dx$$

To calculate the second integral we carry out integration by parts:

$$\int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} \cos(2x) dx$$

$$= \int_{(n-1)\pi}^{n\pi} \left(\frac{e^{\frac{-x}{5}}}{-1/5} \right)' \cos(2x) dx$$

$$= \left(\frac{e^{\frac{-x}{5}}}{-1/5} \right) \cos(2x) \Big[_{(n-1)\pi}^{n\pi} - \int_{(n-1)\pi}^{n\pi} \left(\frac{e^{\frac{-x}{5}}}{-1/5} \right) (\cos(2x))' dx$$

Simplify:

$$=\left(\frac{e^{\frac{-n\pi}{5}}}{-1/5}\right)\cos(2n\pi)-\left(\frac{e^{\frac{-(n-1)\pi}{5}}}{-1/5}\right)\cos(2(n-1)\pi)+2\int_{(n-1)\pi}^{n\pi}\left(\frac{e^{\frac{-x}{5}}}{-1/5}\right)(\sin(2x))\,dx$$

$$=-5e^{\frac{-n\pi}{5}}+5e^{\frac{-(n-1)\pi}{5}}-10\int_{(n-1)\pi}^{n\pi}\left(\frac{e^{\frac{-x}{5}}}{-1/5}\right)'(\sin(2x))\,dx$$

$$=5e^{\frac{-n\pi}{5}}\left(e^{\frac{\pi}{5}}-1\right)-10\left[\left(\frac{e^{\frac{-x}{5}}}{-1/5}\right)\sin(2x)\Bigg|_{(n-1)\pi}^{n\pi}-\int_{(n-1)\pi}^{n\pi}\left(\frac{e^{\frac{-x}{5}}}{-1/5}\right)(\sin(2x))'\,dx\right]$$

$$=5e^{\frac{-n\pi}{5}}\left(e^{\frac{\pi}{5}}-1\right)-10\left(\frac{e^{\frac{-n\pi}{5}}}{-1/5}\right)\sin(2n\pi)+10\left(\frac{e^{\frac{-(n-1)\pi}{5}}}{-1/5}\right)\sin(2(n-1)\pi)-100\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)dx$$

$$=5e^{\frac{-n\pi}{5}}\left(e^{\frac{\pi}{5}}-1\right)-100\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)dx$$

Then

$$\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)dx=5e^{\frac{-n\pi}{5}}\left(5e^{\frac{\pi}{5}}-1\right)-100\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)\,dx$$

$$\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)dx+100\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)\,dx=5e^{\frac{-n\pi}{5}}\left(5e^{\frac{\pi}{5}}-1\right)$$

$$101\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)dx=5e^{\frac{-n\pi}{5}}\left(5e^{\frac{\pi}{5}}-1\right)$$

$$\Rightarrow\int_{(n-1)\pi}^{n\pi}e^{\frac{-x}{5}}\cos(2x)dx=\frac{5}{101}e^{\frac{-n\pi}{5}}\left(5e^{\frac{\pi}{5}}-1\right)$$

Using that expression in the equation for V_n gives:

$$V_n = \frac{-5\pi}{2} e^{\frac{-n\pi}{5}} \left(1 - e^{\frac{\pi}{5}}\right) - \frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{\frac{-x}{5}} \cos(2x)\, dx$$

$$= \frac{-5\pi}{2} e^{\frac{-n\pi}{5}} \left(1 - e^{\frac{\pi}{5}}\right) - \frac{5\pi}{2} \frac{1}{101} e^{\frac{-n\pi}{5}} \left(e^{\frac{\pi}{5}} - 1\right)$$

$$= \frac{5\pi}{2} e^{\frac{-n\pi}{5}} \left(e^{\frac{\pi}{5}} - 1\right)\left(1 - \frac{1}{101}\right)$$

$$= \frac{250\pi}{101} e^{\frac{-n\pi}{5}} \left(e^{\frac{\pi}{5}} - 1\right)$$

$$\Rightarrow \boxed{V_n = \frac{250\pi}{101} e^{\frac{-n\pi}{5}} \left(e^{\frac{\pi}{5}} - 1\right)}$$

(b)The volume of all the beads is the infinite sum of the all the volumes V_n.

Then

$$V = \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{250\pi}{101} e^{\frac{-n\pi}{5}} \left(e^{\frac{\pi}{5}} - 1\right)$$

$$= \frac{250\pi}{101} \left(e^{\frac{\pi}{5}} - 1\right) \sum_{n=1}^{\infty} e^{\frac{-n\pi}{5}}$$

$$= \frac{250\pi}{101} \left(e^{\frac{\pi}{5}} - 1\right) \lim_{N \to \infty} \sum_{n=1}^{N} \left(e^{\frac{-\pi}{5}}\right)^n$$

$$\Rightarrow V = \frac{250\pi}{101} \left(e^{\frac{\pi}{5}} - 1\right) \lim_{N \to \infty} \left[e^{\frac{-\pi}{5}} \frac{\left(e^{\frac{-\pi}{5}}\right)^N - 1}{\left(e^{\frac{-\pi}{5}}\right) - 1} \right]$$

$$= \frac{250\pi}{101} \left(1 - \lim_{N \to \infty} \left(e^{\frac{-\pi}{5}}\right)^N\right)$$

Since $e^{\frac{-\pi}{5}} < 1$, then $\lim_{N \to \infty} \left(e^{\frac{-\pi}{5}}\right)^N = 0$.

Thus $\boxed{V = \frac{250\pi}{101}}$.

Answer 13TFQ.

True

Since the Maclaurin series is

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$\Rightarrow 2x-x^2+\frac{1}{3}x^3-...=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$\frac{f^{(3)}(0)}{3!}x^3=\frac{1}{3}x^3$$

now comparing coefficients of x^3

$$\Rightarrow\frac{f^{(3)}(0)}{3!}=\frac{1}{3}$$

$$\Rightarrow f^{(3)}(0)=2$$

Answer 14E.

Series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$

This is an alternating series so we use alternating series test

here $\sum_{n=1}^{\infty}(-1)^n b_n=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$

$$\Rightarrow b_n=\frac{1}{\sqrt{n+1}}$$

Cleary it is an decreasing function so $b_{n+1}\le b_n$

Now $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}}$

$$=\lim_{n\to\infty}\frac{1/\sqrt{n}}{\sqrt{1+1/n}}=0$$

$$\Rightarrow\lim_{n\to\infty}b_n=0$$

Then by alternating series test the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ is $\boxed{convergent}$

Answer 14P.

Consider the expression

$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots}$$

Its need to evaluate the expression for $p>1$

On rewriting given expression, we have that

$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots}$$

$$=\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1+\frac{1}{2^p}-\frac{1}{2^p}-\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}-\frac{1}{4^p}-\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}-\frac{1}{6^p}-\frac{1}{6^p}+\cdots}$$

$$=\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{\left[1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots\right]-\frac{2}{2^p}-\frac{2}{4^p}-\frac{2}{6^p}+\cdots}$$

$$=\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{\left[1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots\right]-\frac{2}{2^p}\left[1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\right]}$$

Thus

$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots}=\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{\left[1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots\right]-\frac{2}{2^p}\left[1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\right]} \quad \text{...... (1)}$$

Suppose that $S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

Then from (1), we have that

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \frac{S}{[S] - \frac{2}{2^p}[S]}$$

$$= \frac{S}{S\left(1 - \frac{2}{2^p}\right)}$$

$$= \frac{1}{1 - \frac{2}{2^p}}$$

$$= \frac{2^p}{2^p - 2}$$

Hence

$$\frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \boxed{\frac{2^p}{2^p - 2}}$$

Answer 14TFQ.

False

Example : Let $a_n = n$ and $b_n = -n$

Then both $\{a_n\}$ and $\{b_n\}$ are divergent but $\{a_n + b_n\} = \{0\}$ is a convergent sequence.

Answer 15E.

Series is $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Since the integration of $\frac{1}{x\sqrt{\ln x}}$ is very simple so we use integral test

We find $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx$

Let $\ln x = t$ then $\frac{1}{x} dx = dt$

And $t \to \infty$ as $x \to \infty,$ $when$ $x = 2,$ $t = \ln 2$

$$\Rightarrow \int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \lim_{y \to \infty} \int_{\ln 2}^{y} \frac{1}{\sqrt{t}} dt$$

$$= \lim_{y \to \infty} \left[2\sqrt{t}\right]_{\ln 2}^{y}$$

$$= \lim_{y \to \infty} \left[2\sqrt{y} - 2\sqrt{\ln 2}\right]$$

$$= \infty$$

Since $\int_2^{\infty} \frac{2}{x\sqrt{\ln x}} dx$ is divergent

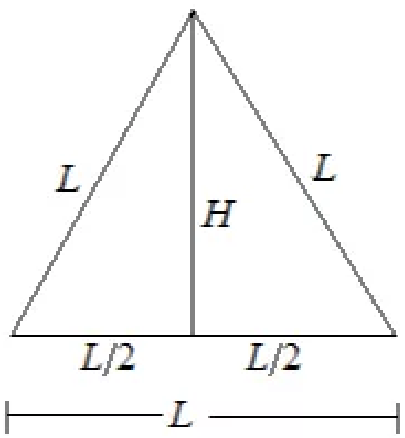
So the series $\sum_{n=2}^{\infty} \frac{1}{2\sqrt{\ln n}}$ is also $\boxed{divergent}$

Answer 15P.

If L is the length of a side of the equilateral triangle, then the area is given by the following relation:

$$A = \frac{1}{2} \times L \times H$$

Here, H is the height of the equilateral triangle.



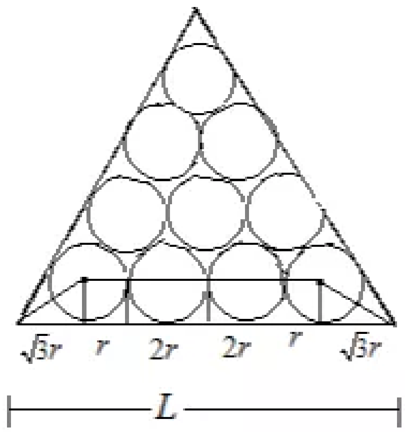
From the above triangle, Use the Pythagorean Theorem and solve as follows:

$$H^2 + (L/2)^2 = L^2$$
$$\Rightarrow H^2 = L^2 - \frac{L^2}{4}$$
$$\Rightarrow H = \frac{\sqrt{3}}{2}L$$

Thus, area of the equilateral triangle is calculated as follows:

$$A = \frac{1}{2} \times L \times \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$$
$$\Rightarrow L^2 = \frac{4}{\sqrt{3}}A$$

Let r be the radius of one of the circles. When there are n rows of circles, the radius in terms of length can be calculated as follows:



$$L = \sqrt{3}r + r + (n-2)(2r) + r + \sqrt{3}r$$
$$= r\left(2n - 2 + 2\sqrt{3}\right)$$

So, $r = \dfrac{L}{2\left(n + \sqrt{3} - 1\right)}$

The number of circles is calculated by using the following summation:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

So, the total area of the circles is calculated as follows:

$$\begin{aligned}
A_n &= \frac{n(n+1)}{2}\pi r^2 \\
&= \frac{n(n+1)}{2}\pi\frac{L^2}{4\left(n+\sqrt{3}-1\right)^2} \\
&= \frac{n(n+1)}{2}\pi\frac{4A/\sqrt{3}}{4\left(n+\sqrt{3}-1\right)^2} \\
&= \frac{n(n+1)}{\left(n+\sqrt{3}-1\right)^2}\frac{\pi A}{2\sqrt{3}}
\end{aligned}$$

$$\Rightarrow \frac{A_n}{A}=\frac{n(n+1)}{\left(n+\sqrt{3}-1\right)^2}\frac{\pi}{2\sqrt{3}}$$

Consider the limit of A_n/A as $n\to\infty$

$$\begin{aligned}
\lim_{n\to\infty}\frac{A_n}{A} &= \lim_{n\to\infty}\frac{n(n+1)}{\left(n+\sqrt{3}-1\right)^2}\frac{\pi}{2\sqrt{3}} \\
&= \lim_{n\to\infty}\frac{n\cdot n\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{\left(\sqrt{3}-1\right)}{n}\right)^2}\frac{\pi}{2\sqrt{3}} \\
&= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{\left(\sqrt{3}-1\right)}{n}\right)^2}\frac{\pi}{2\sqrt{3}} \\
&= \frac{(1+0)}{(1+0)^2}\frac{\pi}{2\sqrt{3}} \\
&= \frac{\pi}{2\sqrt{3}}
\end{aligned}$$

Thus, the final answer is as follows:

$$\lim_{n\to\infty}\frac{A_n}{A}=\boxed{\frac{\pi}{2\sqrt{3}}}$$

Answer 15TFQ.

False

Example : Let $a_n = (-1)^n, b_n = (-1)^n$

Then both $\{a_n\}$ and $\{b_n\}$ are divergent but $\{a_n b_n\} = \{1\}$ is a convergent sequence.

Answer 16E.

Series is $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

Here $\quad a_n = \ln\left(\frac{n}{3n+1}\right)$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n}{3n+1}\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{3+1/n}\right)$$
$$= \ln\left(\frac{1}{3+0}\right)$$
$$\Rightarrow \lim_{n\to\infty} a_n = \ln(1/3) \neq 0$$

Then by the test for divergence the series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$ is $\boxed{\text{divergent}}$

Answer 16P.

Suppose a sequence $\{a_n\}$ is defined recursively by the equations

$a_0 = 1, a_1 = 1, n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$

Find the sum of the series $\sum_{n=0}^{\infty} a_n$.

Put $n = 2$ into $n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$, then

$$2(2-1)a_2 = (2-1)(2-2)a_{2-1} - (2-3)a_{2-2}$$
$$2a_2 = a_0$$
$$2a_2 = 1 \qquad \text{Substitute } a_0 = 1.$$
$$a_2 = \frac{1}{2}$$

Put $n = 3$ into $n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$, then

$$3(3-1)a_3 = (3-1)(3-2)a_{3-1} - (3-3)a_{3-2}$$
$$6a_3 = 2a_2$$
$$6a_3 = 2\left(\frac{1}{2}\right) \qquad \text{Substitute } a_2 = \frac{1}{2}.$$
$$6a_3 = 1$$
$$a_3 = \frac{1}{6}$$

Put $n=4$ into $n(n-1)a_n=(n-1)(n-2)a_{n-1}-(n-3)a_{n-2}$, then

$$4(4-1)a_4=(4-1)(4-2)a_{4-1}-(4-3)a_{4-2}$$

$$12a_4=6a_3-a_2$$

$$12a_4=6\left(\frac{1}{6}\right)-\frac{1}{2} \qquad \text{Substitute } a_2=\frac{1}{2} \text{ and } a_3=\frac{1}{6}.$$

$$12a_4=\frac{1}{2}$$

$$a_4=\frac{1}{24}$$

Put $n=5$ into $n(n-1)a_n=(n-1)(n-2)a_{n-1}-(n-3)a_{n-2}$, then

$$5(5-1)a_5=(5-1)(5-2)a_{5-1}-(5-3)a_{5-2}$$

$$20a_5=12a_4-2a_3$$

$$20a_5=12\left(\frac{1}{24}\right)-2\left(\frac{1}{6}\right) \qquad \text{Substitute } a_3=\frac{1}{6} \text{ and } a_4=\frac{1}{24}.$$

$$20a_5=\frac{1}{6}$$

$$a_5=\frac{1}{120}$$

Recollect the series $\sum_{n=0}^{\infty}a_n=a_0+a_1+a_2+a_3+a_4+a_5+\ldots$

Substitute these values in the series $\sum_{n=0}^{\infty}a_n$, to get

$$\begin{aligned}\sum_{n=0}^{\infty}a_n&=a_0+a_1+a_2+a_3+a_4+a_5+\ldots\\&=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\ldots\\&=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots\\&=e \qquad \text{Since } e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots\end{aligned}$$

Therefore the sum of the series is $\boxed{\sum_{n=0}^{\infty}a_n=e}$.

Answer 16TFQ.

True

$\{a_n\}$ is decreasing and bounded below. So $\{a_n\}$ is convergent.

Answer 17E.

Series is $\sum_{n=1}^{\infty} \frac{\cos 3n}{1+(1.2)^n}$

Since $\cos 3n \le 1$ for all n.

Then $\frac{\cos 3n}{1+(1.2)^n} \le \frac{1}{1+(1.2)^n} < \frac{1}{(1.2)^n}$

Since $\sum_{n=1}^{\infty} \frac{1}{(1.2)^n}$ is a geometric series with $r = \frac{1}{1.2}$ and $|r| = 0.8\overline{3} < 1$

So $\sum_{n=1}^{\infty} \frac{1}{(1.2)^n}$ is convergent series

Then by comparison test $\sum_{n=1}^{\infty} \frac{1}{1+(1.2)^n}$ is also $\boxed{\text{convergent}}$

Answer 17P.

By taking x^x at 0 to be 1 and integrating a series term by term, its need to shows that

$$\int_0^1 x^x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$$

To do this, rewrite the function x^x as

$$x^x = \left(e^{\ln x}\right)^x$$

$$= e^{x \ln x}$$

Now, using the Maclaurin series for e^x, we have

$$e^{x \ln x} = \sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n (\ln x)^n}{n!}$$

As with power series, we can integrate this series term by term

$$\int_0^1 x^x dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n (\ln x)^n}{n!} dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\ln x)^n \, dx$$

Now integrate by parts with $u = (\ln x)^n$, $dv = x^n dx$

So,

$$du = \frac{n(\ln x)^{n-1}}{x} dx \text{ and } v = \frac{x^{n+1}}{n+1}$$

Thus

$$\int_0^1 x^n (\ln x)^n \, dx = \lim_{t \to 0^+} \int_t^1 x^n (\ln x)^n \, dx$$

$$= \lim_{t \to 0^+} \left[\frac{x^{n+1}}{n+1} (\ln x)^n \right]_t^1 - \lim_{t \to 0^+} \int_0^1 \frac{n}{n+1} x^n (\ln x)^{n-1} \, dx$$

Use L'Hospital rule to evaluate the first limit.

$$\int_0^1 x^n (\ln x)^n \, dx = 0 - \frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} \, dx$$

Further integration by parts gives

$$\int_0^1 x^n (\ln x)^k \, dx = -\frac{k}{n+1} \int_0^1 x^n (\ln x)^{k-1} \, dx$$

Combining the terms, we get

$$\int_0^1 x^n (\ln x)^n \, dx = \frac{(-1)^n n!}{(n+1)^n} \int_0^1 x^n dx$$

$$= \frac{(-1)^n n!}{(n+1)^{n+1}}$$

As $\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\ln x)^n \, dx$ and $\int_0^1 x^n (\ln x)^n \, dx = \frac{(-1)^n n!}{(n+1)^{n+1}}$, we have that

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$$

Therefore

$$\boxed{\int_0^1 x^x dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}}$$

Answer 17TFQ.

True

Since if a series $\{a_n\}$ is absolutely convergent, then it is convergent.

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

Answer 18E.

Series is $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n} = \sum_{n=1}^{\infty} \left(\frac{n^2}{(1+2n^2)} \right)^n$

Since this series is the form of $\sum b_n^n$

So we use root test

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \frac{n^2}{1+2n^2}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n^2}+2} = \frac{1}{0+2} = \frac{1}{2}$$

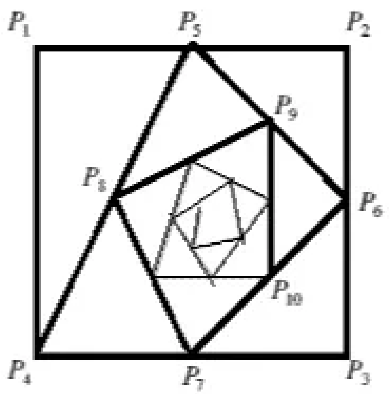
So $\lim_{n\to\infty} \sqrt[n]{|b_n|} = \frac{1}{2} < 1$

So by root test, the series $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$ *is* $\boxed{\text{convergent}}$

Answer 18P.

Consider a square with vertices $P_1(0,1), P_2(1,1), P_3(1,0), P_4(0,0)$

Now construct further points as shown in the figure below.



From the figure, P_5 is the midpoint of P_1P_2, P_6 is the midpoint of P_2P_3, P_7 is the midpoint of P_3P_4 and so on.

a)

Consider the coordinates of P_n are (x_n, y_n).

From the figure, P_n is the midpoint of $P_{n-4}P_{n-3}$

Now prove that the statement $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$ by using mathematical induction:

Let $S(n)$ be the given statement.

Let $n = 1$

$$\frac{1}{2}x_1 + x_{1+1} + x_{1+2} + x_{1+3} = 2$$

$$\frac{1}{2}x_1 + x_2 + x_3 + x_4 = 2$$

$$\frac{1}{2}(0) + 1 + 1 + 0 = 2$$

$$2 = 2 \quad \text{True.}$$

Thus, the statement $S(n)$ is true for $n = 1$

Assume that $S(n)$ is true for $n=k-1$

That implies that

$$\frac{1}{2}x_{k-1}+x_k+x_{k+1}+x_{k+2}=2 \text{ ...... (1)}$$

Now prove that $S(n)$ is true for $n=k$

$$\frac{1}{2}x_k+x_{k+1}+x_{k+2}+x_{k+3}=2$$

Since $P_{k+3}(x_{k+3},y_{k+3})$ is the midpoint of $P_{k+3-4}P_{k+3-3}$.

$$\frac{1}{2}x_k+x_{k+1}+x_{k+2}+x_{k+3}=2$$

$$\frac{1}{2}x_k+x_{k+1}+x_{k+2}+\frac{1}{2}(x_{k+3-4}+x_{k+3-3})=2$$

$$\frac{1}{2}x_k+x_{k+1}+x_{k+2}+\frac{1}{2}x_{k-1}+\frac{1}{2}x_k=2$$

$$\frac{1}{2}x_{k-1}+x_k+x_{k+1}+x_{k+2}=2$$

$2=2$ From (1)

Thus, the by induction hypothesis, $\frac{1}{2}x_n+x_{n+1}+x_{n+2}+x_{n+3}=2$ is true for all *n*

Now find an equation for the *y* coordinate:

Substitute $n=1$ in $\frac{1}{2}y_n+y_{n+1}+y_{n+2}+y_{n+3}$; the value obtained is as follows:

$$\frac{1}{2}y_1+y_2+y_3+y_4=\frac{1}{2}(1)+1+0+0$$
$$=\frac{3}{2}$$

Similarly, prove that $\frac{1}{2}y_n+y_{n+1}+y_{n+2}+y_{n+3}=\frac{3}{2}$ by using mathematical induction.

Thus, $\boxed{\frac{1}{2}y_n+y_{n+1}+y_{n+2}+y_{n+3}=\frac{3}{2}}$

b)

Find the coordinates of P:

$$\lim_{n\to\infty}\left(\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}\right) = \lim_{n\to\infty} 2$$

$$\frac{1}{2}\lim_{n\to\infty} x_n + \lim_{n\to\infty} x_{n+1} + \lim_{n\to\infty} x_{n+2} + \lim_{n\to\infty} x_{n+3} = 2$$

Since all limit values are same. So, equate and obtain the following value:

$$\frac{7}{2}\lim_{n\to\infty} x_n = 2$$

$$\lim_{n\to\infty} x_n = \frac{4}{7}$$

Now find the y coordinate:

$$\lim_{n\to\infty}\left(\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3}\right) = \lim_{n\to\infty} \frac{3}{2}$$

$$\frac{1}{2}\lim_{n\to\infty} y_n + \lim_{n\to\infty} y_{n+1} + \lim_{n\to\infty} y_{n+2} + \lim_{n\to\infty} y_{n+3} = \frac{3}{2}$$

Since all limit values are same. So, proceed as follows:

$$\frac{7}{2}\lim_{n\to\infty} y_n = \frac{3}{2}$$

$$\lim_{n\to\infty} y_n = \frac{3}{7}$$

Therefore, the coordinates of P are $\boxed{\left(\frac{4}{7}, \frac{3}{7}\right)}$

Answer 18TFQ.

True

By the ratio test, $\sum_{n=1}^{\infty}\{a_n\}$ is convergent.

So

$\lim_{n\to\infty}\{a_n\} = 0$

Ratio test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L<1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive

Answer 19E.

Series is $\sum_{n=1}^{\infty}\frac{1.3.5......(2n-1)}{5^n\ n!}$

Since this series involves factorial. So we use ratio test.

Let $\quad a_n=\frac{1.3.5.......(2n-1)}{5^n\ n!}$

And $\quad a_{n+1}=\frac{1.3.5.......(2n-1)(2n+1)}{5^{n+1}(n+1)!}$

Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left(\frac{1.3.5......(2n-1)(2n+1)}{5^{n+1}(n+1)!}\cdot\frac{5^n n!}{1.3.5......(2n-1)}\right)$

$$=\lim_{n\to\infty}\frac{2n+1}{5(n+1)}$$

$$=\lim_{n\to\infty}\frac{2+1/n}{5(1+1/n)}$$

$$=\frac{(2+0)}{5(1+0)}$$

$$\Rightarrow\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{2}{5}<1$$

Then by ratio test given series is $\boxed{\text{convergent}}$

Answer 19P.

Given series

$$\sum_{n=1}^{\infty}\frac{1.3.5...(2n-1)}{5^n n!}$$

We used the Ratio Test with $a_n=\frac{1.3.5...(2n-1)}{5^n n!}$

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{\frac{1.3.5...(2n-1)(2n+1)}{5^{n+1}(n+1)!}}{\frac{1.3.5...(2n-1)}{5^n n!}}\right|$$

$$=\frac{(2n+1)}{5(n+1)}$$

$$=\frac{\left(2+\frac{1}{n}\right)}{5\left(1+\frac{1}{n}\right)}$$

$$\therefore\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{5\left(1+\frac{1}{n}\right)}=\frac{2}{5}<1$$

Ratio test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L<1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive

Answer 19TFQ.

True

Since

$$\begin{aligned} 0.99999\ldots &= 0.\bar{9} \\ &= \frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots \\ &= \frac{\frac{9}{10}}{1-\frac{1}{10}} \\ &= 1 \end{aligned}$$

Since the series is a geometric series with $a=\frac{9}{10}$, and $r=\frac{1}{10}$

Sum of the series is $\frac{a}{1-r}$

Answer 20E.

Series is $\sum_{n=1}^{\infty}\frac{(-5)^{2n}}{n^2 9^n}$

Here we use ratio test.

$$\text{Let } a_n = \frac{(-5)^{2n}}{n^2 9^n}$$

$$\text{Then} \quad a_{n+1} = \frac{(-5)^{2(n+1)}}{(n+1)^2 9^{(n+1)}} = \frac{(-5)^{2n+2}}{(n+1)^2 9^{n+1}}$$

$$\text{Therefore } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-5)^{2n+2}}{(n+1)^2 9^{n+1}}\cdot\frac{n^2 9^n}{(-5)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-5)^2 n^2}{9(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\frac{25}{9}\frac{n^2}{(n+1)^2}$$

$$= \frac{25}{9}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2$$

$$= \frac{25}{9}\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^2$$

$$= \frac{25}{9}\left(\frac{1}{1+0}\right)^2 = \frac{25}{9}$$

$$\text{Since } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{25}{9} > 1$$

Then by ratio test the series $\sum_{n=1}^{\infty}\frac{(-5)^{2n}}{n^2 9^n}$ *is* $\boxed{\textit{divergent}}$

Answer 20P.

Given series $\sum_{n=1}^{\infty}\frac{(-5)^{2n}}{n^2 9^n}$

We use the Ratio Test with $a_n = \frac{(-5)^{2n}}{n^2 9^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-5)^{2n+2}}{(n+1)^2 9^{n+1}}}{\frac{(-5)^{2n}}{n^2 9^n}}\right|$$

$$= \frac{25}{9\left(1+\frac{1}{n^2}\right)}$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{25}{9\left(1+\frac{1}{n^2}\right)} = \frac{25}{9} > 1$$

Thus, by the Ratio Test, the given series is divergent.

Ratio test:

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L<1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive

Answer 20TFQ.

True

Since

$$\lim_{n\to\infty} a_n = 2$$

$$\Rightarrow \lim_{n\to\infty} a_{n+3} = 2$$

$$\lim_{n\to\infty}\left[a_{n+3} - a_n\right] = 0$$

Answer 21E.

Series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$

Since this is an alternating series so we use test for alternating series

Here $b_n = \frac{\sqrt{n}}{n+1}$

Let $f(x) = \frac{\sqrt{x}}{x+1}$

Then $f'(x) = \frac{(x+1)\left(1/2\sqrt{x}\right)-\sqrt{x}}{(x+1)^2}$

$$= \frac{\frac{1}{2\sqrt{x}}(x+1-2x)}{(x+1)^2}$$

$$\Rightarrow f'(x) = \frac{1-x}{2\sqrt{x}(x+1)^2} < 0 \quad \text{for } x>1$$

So $f(n) > f(n+1) \Rightarrow b_{n+1} \le b_n$

Now $$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$$
$$= \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/n} = 0$$
$$\Rightarrow \lim_{n\to\infty} b_n = 0$$
So by alternating series test given series is $\boxed{\text{convergent.}}$

Answer 21P.

Given series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{(n+1)}$$
Let $b_n = \frac{(-1)^{n-1}\sqrt{n}}{(n+1)}$

This series satisfies

(1)

$b_{n+1} < b_n$ because $\frac{\sqrt{n+1}}{n+2} < \frac{\sqrt{n}}{n+1}$

(2)

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\frac{1}{n}} = 0$$
Thus, by the Alternating series Test, the given series is convergent.

Alternating Series Test:
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 + b_2 + \ldots \qquad b_n > 0$$
satisfies

$(\mathrm{i})\, b_{n+1} < b_n$ for all n

$(\mathrm{i})\, \lim_{n\to\infty} b_n = 0$

then the series is convergent.

Answer 21TFQ.

True

Let $\sum_{n=1}^{\infty} a_n = a$

Suppose a finite number of terms $b_1, b_2, \ldots, b_k$ added to the above series.

Let $b_1 + b_2 + \ldots. + b_k = b$

Then $\sum_{n=1}^{\infty} a_n + b_1 + b_2 + \ldots. + b_k = a + b$

Hence the new series is also convergent.

Answer 22E.

The given series is $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

Here nth term of the given series

$$a_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$$

$$= \frac{\left(\sqrt{n+1}-\sqrt{n-1}\right)\left(\sqrt{n+1}+\sqrt{n-1}\right)}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$$

$$= \frac{(n+1)-(n-1)}{n\left[\sqrt{n}\sqrt{1+\frac{1}{n}}+\sqrt{n}\sqrt{1-\frac{1}{n}}\right]}$$

$$= \frac{2}{n^{3/2}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

Let us consider another series $\sum b_n$ where

$$b_n = \frac{1}{n^{3/2}}$$

Now $$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2}{n^{3/2}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)\cdot\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty}\frac{2}{\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{2} = 1 \ (\text{Finite}).$$

Therefore, By comparison test. Both the series $\sum a_n$ and $\sum b_n$ will converse or diverse simultaneously.

Now the series $\sum b_n = \sum \frac{1}{n^{3/2}}$ is convergent by p-series test).

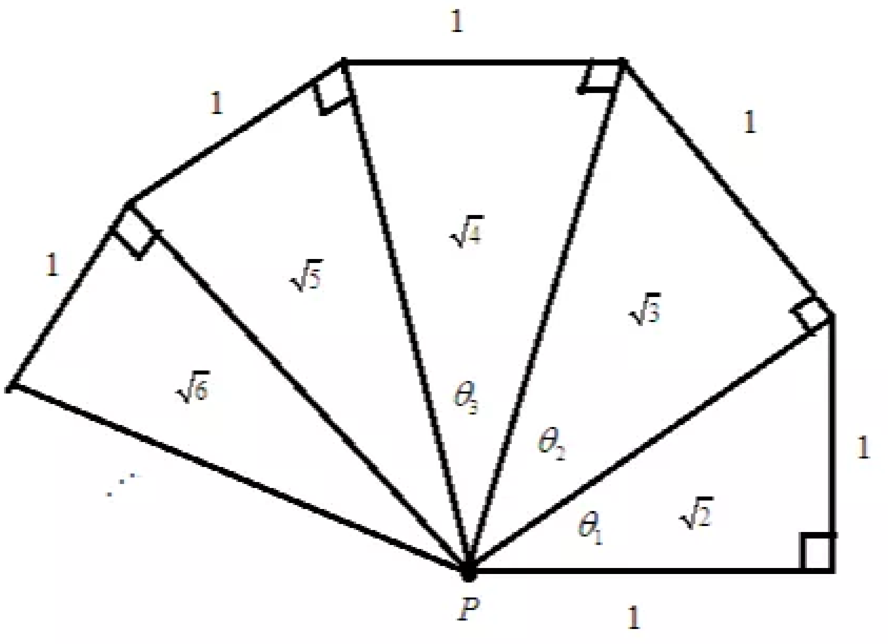
As here $p = \frac{3}{2} > 1$.

So the given series $\sum a_n$ i.e.

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$$ is convergent.

Answer 22P.

Consider the figure about Right-angled triangles.



From the figure,

$$\sin\theta_1 = \frac{1}{\sqrt{2}}$$

$$\sin\theta_2 = \frac{1}{\sqrt{3}}$$

$$\sin\theta_3 = \frac{1}{\sqrt{4}}$$

$$\sin\theta_n = \frac{1}{\sqrt{n+1}} \Rightarrow \theta_n = \sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$$

$$\sum\theta_n = \sum\sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)$$
$$= \sum u_n$$

Then $v_n = \frac{1}{\sqrt{n}} \Rightarrow v_n = \sum\frac{1}{\sqrt{n}}$ is divergent,

Since $\sum\frac{1}{n^p}$ is divergent if $p \le 1$

Now $l = \lim_{n\to\infty} \frac{u_n}{v_n}$

$$= \lim_{n\to\infty} \frac{\sin^{-1}\left(\frac{1}{\sqrt{n+1}}\right)}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{1-\frac{1}{n+1}}} \times \left(-\frac{1}{n+1}\right)\left(\frac{1}{2\sqrt{n+1}}\right)}{-\frac{1}{n} \times \frac{1}{2\sqrt{n}}} \quad \text{(By L'Hospital rule)}$$

$$= 2\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \times \left(\frac{1}{n+1}\right)\left(\frac{1}{2\sqrt{n+1}}\right) \times n\sqrt{n}$$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{\frac{1}{n}\to 0} \frac{1}{1+\frac{1}{n}}$$

$= 1 \neq 0$ and $\sum v_n$ is divergent.

So by limit comparison test, $\sum u_n$ is also divergent.

That is, $\sum \theta_n$ is divergent series.

Answer 22TFQ.

False

Example: let

$a_n = \frac{1}{2^n}, b_n = \frac{1}{3^n}$

Then $\sum_{n=1}^{\infty} a_n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and $\sum_{n=1}^{\infty} b_n = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

but

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{6^n}$$

$$= \frac{\frac{1}{6}}{1-\frac{1}{6}} = \frac{1}{5} \neq \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$$

Answer 23E.

Series is $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1/3}$

Comparing with alternating series form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$

$$\Rightarrow b_n = n^{-1/3} = 1/n^{1/3}$$

Clearly it is a deceasing function so

$$b_{n+1} \le b_n \qquad \textit{for all } n$$

And $\lim_{n\to\infty} b_n = \lim_{n\to\infty} n^{-1/3} = 0$

So given series is $\boxed{\textit{convergent}}$

Now we check for absolute convergence

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}n^{-1/3}\right| = \sum_{n=1}^{\infty}n^{-1/3}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{1/3}} \qquad \text{which is a } p\text{-series with } \left(p=\frac{1}{3}<1\right).$$

Therefore $\sum_{n=1}^{\infty}\left|(-1)^{n-1}n^{-1/3}\right|$ is a divergent series

Hence given series is $\boxed{\textit{conditionally convergent}}$

Answer 23P.

Consider a series whose terms are the reciprocals of the positive integers that can be written in base 10 notation. without using the digit 0.

Observe that, the terms in the series are as follows:

$$\underbrace{1+\frac{1}{2}+\cdots+\frac{1}{9}}_{\text{terms with single digit}}+\underbrace{\frac{1}{11}+\cdots\frac{1}{99}}_{\text{terms with two digits}}+\underbrace{\frac{1}{111}+\cdots+\frac{1}{999}}_{\text{terms with three digits}}+\cdots$$

Suppose that, the series is S, then the terms are as follows:

$$S=\underbrace{1+\frac{1}{2}+\cdots+\frac{1}{9}}_{\substack{\text{terms with single digit}\\ s_1}}+\underbrace{\frac{1}{11}+\cdots\frac{1}{99}}_{\substack{\text{terms with two digits}\\ s_2}}+\underbrace{\frac{1}{111}+\cdots+\frac{1}{999}}_{\substack{\text{terms with three digits}\\ s_3}}+\cdots$$

Now in group g_n we have 9^n terms because we have 9 choices for each of the n digits in the denominator.

Furthermore, each term in g_n is less than $\frac{1}{10^{n-1}}$ [except for the first term in g_1].

So, $g_n < 9^n \cdot \frac{1}{10^{n-1}} = 9\left(\frac{9}{10}\right)^{n-1}$

Now $\sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1}$ is a geometric series with $a = 9$ and $r = \frac{9}{10} < 1$.

Therefore, by comparison Test,

$$\begin{aligned} S &= \sum_{n=1}^{\infty} g_n \\ &< \sum_{n=1}^{\infty} 9\left(\frac{9}{10}\right)^{n-1} \\ &= \frac{9}{1-\frac{9}{10}} \\ &= 90 \end{aligned}$$

Answer 24E.

Given series is $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$

Here $b_n = n^{-3}$ this is a decreasing function

So $b_{n+1} \le b_n$ *for all n*

And $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^3} = 0$

So by alternating series test the series $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$ is convergent.

Now we check for absolutely convergence

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1} n^{-3}\right| = \sum_{n=1}^{\infty} n^{-3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Which is a p-series with p=3

Since p>1 , so series is convergent

Hence given series is $\boxed{\textit{absolutely convergent}}$

Answer 24P.

(a)

Consider the function

$$f(x)=\frac{x}{1-x-x^2}$$

Its need to shows that Maclaurin series of the given function is $\sum_{n=1}^{\infty} f_n x^n$

Where f_n is the n^{th} Fibonacci number, that is, $f_1=1$, $f_2=1$, and $f_n=f_{n-1}+f_{n-2}$ for $n\geq 3$

On rewriting the function, we have that

$$f(x)=\frac{x}{1-x-x^2}$$

$$=\frac{x}{1-(x+x^2)}$$

$$=x\left[1-(x+x^2)\right]^{-1}$$

$$=x\left(1+x+x^2+(x+x^2)^2+(x+x^2)^3+(x+x^2)^4+(x+x^2)^5+\cdots\right)$$

$$=x\left[1+x+x^2+(x^2+x^4+2x^3)+(x^3+x^6+3x^4+3x^5)+(x^4+4x^5+6x^6+4x^7)+\cdots\right]$$

$$=x\left[1+x+2x^2+3x^3+5x^4+8x^5+\cdots\right]$$

$$=x+x^2+2x^3+3x^4+5x^5+8x^6+\cdots$$

$$=\sum_{n=1}^{\infty} f_n x^n$$

From the above series, observe that

$f_1=1, f_2=1$ and $f_3=2=f_2+f_1$

$f_4=3=f_3+f_2$

$f_5=5=f_4+f_3$

$f_6=8=f_5+f_4$

And so on.

Thus successively we get $f_n=f_{n-1}+f_{n-2}$ for $n\geq 3$

(b)

Resolve the fraction $\frac{x}{1-x-x^2}$ in to partial fractions

Observe that

$$\frac{x}{1-x-x^2}=\frac{-x}{x^2+x-1}$$

And also that

$$x^2+x-1=0 \Rightarrow x=\frac{-1\pm\sqrt{1-4(1)(-1)}}{2}$$

$$=\frac{-1\pm\sqrt{1+4}}{2}$$

$$=\frac{-1\pm\sqrt{5}}{2}$$

Thus

$$x^2+x-1=\left(x-\left(\frac{-1+\sqrt{5}}{2}\right)\right)\left(x-\left(\frac{-1-\sqrt{5}}{2}\right)\right)$$

By using partial fractions, we get

$$\frac{-x}{x^2+x-1}=\frac{A}{x-\left(\frac{-1+\sqrt{5}}{2}\right)}+\frac{B}{x-\left(\frac{-1-\sqrt{5}}{2}\right)}$$

$$\Rightarrow \quad -x=A\left[x-\left(\frac{-1-\sqrt{5}}{2}\right)\right]+B\left[x-\left(\frac{-1+\sqrt{5}}{2}\right)\right]$$

$$-x=(A+B)x+\left[(A+B)+(A-B)\frac{\sqrt{5}}{2}\right]$$

By comparing the coefficients on each sides, get

$$A+B=-1$$

$$(A+B)+(A-B)\frac{\sqrt{5}}{2}=0 \;\Rightarrow\; A-B=\frac{2}{\sqrt{5}}$$

Solve the equations

$A+B=-1$ and $A-B=\dfrac{2}{\sqrt{5}}$

for unknowns.

By adding these two, get

$$A+\cancel{B}+A-\cancel{B}=-1+\frac{2}{\sqrt{5}}$$

$$2A=\frac{2-\sqrt{5}}{\sqrt{5}}$$

$$\Rightarrow A=\frac{2-\sqrt{5}}{2\sqrt{5}}$$

Use $A=\dfrac{2-\sqrt{5}}{2\sqrt{5}}$ in $A+B=-1$, to get

$$\begin{aligned}B&=-1-\left(\frac{2-\sqrt{5}}{2\sqrt{5}}\right)\\&=\frac{-2\sqrt{5}-2+\sqrt{5}}{2\sqrt{5}}\\&=\frac{-2-\sqrt{5}}{2\sqrt{5}}\end{aligned}$$

Thus

$$\frac{-x}{x^2+x-1}=\frac{2-\sqrt{5}}{2\sqrt{5}}\cdot\frac{1}{x-\left(\frac{-1+\sqrt{5}}{2}\right)}-\frac{2+\sqrt{5}}{2\sqrt{5}}\frac{1}{x-\left(\frac{-1-\sqrt{5}}{2}\right)}$$

$$=\frac{2-\sqrt{5}}{2\sqrt{5}}\cdot\frac{-2}{-1+\sqrt{5}}\left(1-\frac{x}{\left(\frac{-1+\sqrt{5}}{2}\right)}\right)^{-1}-\frac{2+\sqrt{5}}{2\sqrt{5}}\cdot\frac{2}{1+\sqrt{5}}\left(1-\frac{x}{\left(\frac{1+\sqrt{5}}{2}\right)}\right)^{-1}$$

$$=\frac{2-\sqrt{5}}{2\sqrt{5}}\cdot\frac{-2}{-1+\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{2}{\sqrt{5}-1}\right)^n x^n-\frac{2+\sqrt{5}}{2\sqrt{5}}\cdot\frac{2}{1+\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{2}{\sqrt{5}+1}\right)^n x^n$$

Answer 25E.

Given series is $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1) 3^n}{2^{2n+1}}$

We use ratio test for absolute convergence

$$a_n = \frac{(-1)^n (n+1) 3^n}{2^{2n+1}}, \quad a_{n+1} = \frac{(-1)^{n+1} (n+2) 3^{n+1}}{2^{2n+3}}$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (n+2) 3^{n+1}}{2^{2n+3}} \cdot \frac{2^{2n+1}}{(-1)^n (n+1) 3^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)3}{4} \frac{(n+2)}{(n+1)} \right| \\
&= \lim_{n\to\infty} \frac{3}{4} \cdot \frac{n+2}{n+1} \\
&= \frac{3}{4} \lim_{n\to\infty} \frac{1+2/n}{1+1/n} \\
&= \frac{3}{4} \cdot \frac{1+0}{1+0} = \frac{3}{4} < 1
\end{aligned}$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ *then given series is* $\boxed{\textit{absolutely convergent}}$

Answer 25P.

Let $u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$

$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$$

$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots$$

Its need to shows that $u^3 + v^3 + w^3 - 3uvw = 1$

From the data, we observe hat

$u' = w$, $w' = v$ and $v' = u$

Suppose that

$$u^3+v^3+w^3-3uvw=f(x) \quad \text{......} \ (1)$$

On differentiating both sides with respect to x, get

$$3u^2u'+3v^2v'+3w^2w'-3u'vw-3uv'w-3u'vw'=f'(x)$$

$$3u^2(w)+3v^2(u)+3w^2(v)-3(w)vw-3u(u)w-3uv(v)=f'(x)$$

$$3wu^2+3uv^2+3vw^2-3vw^2-3u^2w-3uv^2=f'(x)$$

$$\cancel{3wu^2}+\cancel{3uv^2}+\cancel{3vw^2}-\cancel{3vw^2}-\cancel{3u^2w}-\cancel{3uv^2}=f'(x)$$

$$\Rightarrow \quad f'(x)=0$$

$\Rightarrow \quad f(x)=C$ here C is a constant.

Replace x by 0 in (1), we get

$$[u(0)]^3+[v(0)]^3+[w(0)]^3-3[u(0)][v(0)][w(0)]=f(0)$$

$$[1]^3+[0]^3+[0]^3-3[1][0][0]=f(0)$$

$$\Rightarrow \quad f(0)=1$$

Since $f(x)=C$ and $f(0)=1$, we have that

$$C=1$$

Thus $f(x)=1$ for any value of x ...... (2)

From (1) and (2), we observe that

$$u^3+v^3+w^3-3uvw=1$$

Answer 26E.

L'Hospital's rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a.

Suppose that ,

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

Or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the limit on the right side exists.

The test for divergence:

If $\lim_{n \to \infty} a_n$ does not exists or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The alternating series test:

If the alternating series,

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \quad b_n > 0$ satisfies,

(1) $b_{n+1} \leq b_n$ for all n

(2) $\lim_{n \to \infty} b_n = 0$ then the series is convergent.

Consider the series, $\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{\ln n}$

Let $a_n=\frac{\sqrt{n}}{\ln n}$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{\sqrt{n}}{\ln n}$$

$$=\lim_{n\to\infty}\left(\frac{\frac{1}{2}n^{-\frac{1}{2}}}{\frac{1}{n}}\right) \quad \text{By L'Hospital's rule.}$$

$$=\lim_{n\to\infty}\left(\frac{n}{2n^{\frac{1}{2}}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{2n^{\frac{1}{2}}n^{-1}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{2n^{-\frac{1}{2}}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n^{\frac{1}{2}}}{2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt{n}}{2}\right)$$

$$=\infty$$

The above alternating series condition (2) is not satisfies

So, the limit of the nth term of the series,

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}}{\ln n}$ is does not exists.

Therefore, the above series is divergent by the test for divergence.

Answer 26P.

Let 2^k be the largest power of 2 that is less than or equal to n.

And let M be the product of all odd integers that are less than or equal to n.

Suppose that $s_n=m,$ an integer.

Then $M2^k s_n=M2^k m.$

The right side of this equation is even.

The harmonic series is the divergent infinite series:

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+---$$

Suppose the harmonic series converges with sum *S*.

Then $\frac{1}{2}+\frac{1}{4}+---+\frac{1}{2^n}+---=\frac{1}{2}S.$

Therefore, the sum of the odd-numbered terms must be other half of *S*.

That is, $1+\frac{1}{3}+---+\frac{1}{2n-1}+---.$

Therefore, the left side is odd by showing that each of its terms is an even integer, except for the last one.

However this is impossible since $\frac{1}{2n-1}>\frac{1}{2n}$ for each positive integer *n*.

Note: The difference between distinct harmonic numbers is never an integer.

No harmonic numbers are integers, except for $n=1$.

Answer 27E.

Consider the series, $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{2^{3n}}$

$$S_n=\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{2^{3n}}$$

$$=\sum_{n=1}^{\infty}\frac{(-3)^n(-3)^{-1}}{(2^3)^n}$$

$$=\sum_{n=1}^{\infty}\left(\frac{-3}{2^3}\right)^n\frac{1}{(-3)}$$

$$=\frac{1}{(-3)}\sum_{n=1}^{\infty}\left(\frac{-3}{2^3}\right)^n$$

$$=\frac{1}{(-3)}\left[\left(\frac{-3}{2^3}\right)^1+\left(\frac{-3}{2^3}\right)^2+\left(\frac{-3}{2^3}\right)^3+\ldots\right]$$

The above equation is in the geometric series, so that the sum of the above series,

$$=\frac{1}{(-3)}\left(\frac{-3}{2^3}\right)\left[\frac{\left(\frac{-3}{2^3}\right)^n-1}{\left(\frac{-3}{2^3}\right)-1}\right]$$

$$=\frac{1}{(-3)}\left(\frac{-3}{2^3}\right)\left[\frac{\left(\frac{-3}{2^3}\right)^n-1}{\frac{-3-2^3}{2^3}}\right]$$

$$=\frac{1}{(-3)}\left(\frac{-3}{2^3}\right)\left[\frac{\left(\frac{-3}{2^3}\right)^n-1}{\frac{-3-8}{8}}\right]$$

$$=\frac{1}{(-3)}\left(\frac{-3}{2^3}\right)\left[\frac{\left(\frac{-3}{2^3}\right)^n-1}{\frac{-11}{8}}\right]$$

$$=\frac{1}{(-3)}\times\frac{8}{(-11)}\left(\frac{-3}{2^3}\right)\left[\left(\frac{-3}{2^3}\right)^n-1\right]$$

$$=\frac{8}{33}\left(\frac{-3}{2^3}\right)\left[\left(\frac{-3}{2^3}\right)^n-1\right]$$

$$=\frac{-1}{11}\left[\left(\frac{-3}{2^3}\right)^n-1\right]$$

Applying limits on the both sides.

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{-1}{11}\left[\left(\frac{-3}{2^3}\right)^n-1\right]$$

$$=\frac{-1}{11}\lim_{n\to\infty}\left[\left(\frac{-3}{2^3}\right)^n-1\right]$$

$$=\frac{-1}{11}\lim_{n\to\infty}\left[\left(\frac{-3}{2^3}\right)^n-1\right]$$

$=\frac{-1}{11}[0-1]$ Since $n\to\infty$ then $\left(\frac{-3}{2^3}\right)^n\to 0$

$$=\frac{-1}{11}$$

Therefore the sum of the series of $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{2^{3n}}$ is $\boxed{\frac{-1}{11}}$

Answer 28E.

We have to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

We use partial fraction

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{(n+3)}$$

$$\Rightarrow A(n+3) + Bn = 1$$

Putting n = 0 $\quad A = 1/3 \qquad$ and putting $n = -3, \ \Rightarrow B = -1/3$

So $\quad \sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \left(\frac{1}{3n} - \frac{1}{3(n+3)} \right)$

$$s_n = \left(\frac{1}{3} - \frac{1}{3.4} \right) + \left(\frac{1}{6} - \frac{1}{3.5} \right) + \left(\frac{1}{9} - \frac{1}{3.6} \right) + \ldots\ldots + \left(\frac{1}{3n} - \frac{1}{3(n+3)} \right)$$

$$= \frac{1}{3}\left(1 - \frac{1}{4} \right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5} \right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{6} \right) + \ldots\ldots + \frac{1}{3}\left(\frac{1}{n} - \frac{1}{(n+3)} \right)$$

$$= \frac{1}{3}\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots + \frac{1}{n} \right) - \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots\ldots \frac{1}{(n+3)} \right) \right]$$

$$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right)$$

$$= \frac{1}{3}\left(\frac{6+3+2}{6} - \left\{ \frac{(n+2)(n+3) + (n+1)(n+3) + (n+1)(n+2)}{(n+3)(n+2)(n+1)} \right\} \right)$$

$$s_n = \frac{1}{3}\left(\frac{11}{6} - \frac{(n+2)(n+3) + (n+1)(n+3) + (n+1)(n+2)}{(n+3)(n+2)(n+1)} \right)$$

Therefore

$$s = \lim_{n\to\infty} s_n = \frac{1}{3}\left(\frac{11}{6} - \lim_{n\to\infty} \frac{(n+2)(n+3) + (n+1)(n+3) + (n+1)(n+2)}{(n+3)(n+2)(n+1)} \right)$$

$$= \frac{1}{3}\left(\frac{11}{6} - \lim_{n\to\infty} \frac{n^2\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right) + n^2\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right) + n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^3\left(1+\frac{3}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)} \right)$$

$$= \frac{1}{3}\left(\frac{11}{6} - \lim_{n\to\infty} \frac{\frac{1}{n}\left[\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right) + \left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right) + \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\right]}{\left(1+\frac{3}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)} \right)$$

$$= \frac{1}{3}\left(\frac{11}{6} - 0 \right)$$

$$\Rightarrow \boxed{s = \frac{11}{18}}$$

Answer 29E.

We have to find the sum of the series $\sum_{n=1}^{\infty}\left[\tan^{-1}(n+1)-\tan^{-1}(n)\right]$

$$s_n=\left[\left(\tan^{-1}2-\tan^{-1}1\right)+\left(\tan^{-1}3-\tan^{-1}2\right)+\left(\tan^{-1}4-\tan^{-1}3\right)+\dots\dots\right.$$

$$\left.\dots\dots+\left(\tan^{-1}(n+1)-\tan^{-1}n\right)\right]$$

$$\Rightarrow s_n=\left[\tan^{-1}(n+1)-\tan^{-1}1\right]$$

Then sum

$$s=\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left[\tan^{-1}(n+1)-\tan^{-1}1\right]$$

$$=\frac{\pi}{2}-\frac{\pi}{4} \qquad \begin{bmatrix}\text{since } n\to\infty \text{ so}\\ n+1\to\infty\\ \text{then } \tan^{-1}(n+1)\to\pi/2\\ \text{as } \quad n\to\infty\end{bmatrix}$$

$$\Rightarrow \boxed{s=\pi/4}$$

Answer 30E.

Given series is $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{3^{2n}(2n)!}$

This is an alternating series

$$s=\frac{1}{0!}-\frac{\pi}{3^2.(2)!}+\frac{\pi^2}{3^4.(4)!}-\frac{\pi^3}{3^6(6)!}+\frac{\pi^4}{3^8 8!}+\dots\dots$$

Or $\quad s=1-\frac{\pi}{18}+\frac{\pi^2}{1944}-\frac{\pi^3}{524880}+\frac{\pi^4}{3^8 8!}+\dots\dots$

Since $\quad b=\frac{\pi^4}{3^8.8!}\approx 3.7\times10^{-7}$

By Alternating series Estimation Theorem we know that

$$|s-s_3|\le 3.7\times10^{-7}<0.0000004$$

This error does not affect the affect the sixth decimal place

So we have

$$s\approx 1-\frac{\pi}{18}+\frac{\pi^2}{1944}-\frac{\pi^3}{524880}$$

$$\Rightarrow \boxed{s\approx 0.830485}$$

Alternate solution:

Since $\sum_{n=0}^{\infty}\frac{(-1)^n \pi^n}{3^{2n}(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3^2}\right)^n$

$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\sqrt{\pi}}{3}\right)^{2n}$

Comparing with $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x)^{2n}$ for all x

We have $\sum_{n=0}^{\infty}\frac{(-1)^n \pi^n}{3^{2n}(2n)!}=\boxed{\cos\left(\sqrt{\pi}/3\right)}$

Answer 31E.

Consider the series $1-e+\frac{e^2}{2!}-\frac{e^3}{3!}+\frac{e^4}{4!}-\cdots$.

Find the sum of the given series.

Use the Maclaurin Series for e^x.

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

Now substitute $x=-e$ in the Maclaurin Series for e^x.

$$e^{-e}=\sum_{n=0}^{\infty}\frac{(-e)^n}{n!}$$
$$=1+\frac{(-e)}{1!}+\frac{(-e)^2}{2!}+\frac{(-e)^3}{3!}+\frac{(-e)^4}{4!}+\cdots$$
$$=1-e+\frac{e^2}{2!}-\frac{e^3}{3!}+\frac{e^4}{4!}+\cdots$$

Thus,

$$1-e+\frac{e^2}{2!}-\frac{e^3}{3!}+\frac{e^4}{4!}+\cdots=e^{-e}$$
$$=\frac{1}{e^e}$$
$$=\frac{1}{(2.7183)^{2.7183}}$$
$$\approx\frac{1}{15.155}$$
$$\approx 0.065989$$

Therefore, the sum of the given series is $\boxed{e^{-e}}$ or approximately $\boxed{0.065989}$.

Answer 32E.

Since

$$
\begin{aligned}
4.17326326326\ldots\ldots &= 4.11+0.0632+0.0000632+0.0000000632+\ldots\ldots \\
&= 4.11+\frac{632}{10^4}+\frac{632}{10^7}+\frac{632}{10^{10}}+\ldots\ldots \\
&= 4.11+\left[\text{a geometric series with } a=\frac{632}{10^4}, r=\frac{1}{10^3}\right] \\
&= 4.11+[\text{sum of the series}] \\
&= 4.11+\left[\frac{a}{1-r}\right]=4.11+\frac{632/10^4}{1-1/10^3} \\
&= \frac{411}{100}+\frac{632}{9990}=\frac{410589+6320}{99900} \\
&= \frac{416909}{99900}
\end{aligned}
$$

$$\Rightarrow \boxed{4.17326326326\ldots\ldots=\frac{416909}{99900}}$$

Answer 33E.

We recall the following:

$$\cosh x=\frac{e^x+e^{-x}}{2}$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\ldots\ldots$$

$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\ldots\ldots$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{and} \quad e^{-x}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}$$

Using the above we have

$$\cosh x=\frac{1}{2}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)+\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\ldots\right)\right]$$

$$\Rightarrow \cosh x=\frac{1}{2}\left[2+\frac{2x^2}{2!}+\frac{2x^4}{4!}+\frac{2x^6}{6!}+\ldots\ldots\right]$$

$$\Rightarrow \cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}$$

$$\Rightarrow \cosh x=1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\ldots\geq 1+\frac{x^2}{2} \text{ for all } x$$

So $\boxed{\cosh x\geq 1+\frac{x^2}{2}}$ for all x

Answer 34E.

By the Root test, series converges when $|\ln x| < 1$

$$\Rightarrow -1 < \ln x < 1$$
$$\Rightarrow e^{-1} < x < e$$

Thus series $\sum_{n=1}^{\infty} (\ln x)^n$ converges when $\boxed{e^{-1} < x < e}$

Answer 35E.

Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$ is an alternating series

So $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16807} - \frac{1}{32768} + \ldots\ldots$$

Since $b_8 = \frac{1}{32768} \approx 0.0000305$

By Alternating series estimation method we have

$$|s - s_7| \le b_8 < 0.000031$$

This error does not affect 4th decimal place

So sum $s = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16807}$

$$\Rightarrow \boxed{s \approx 0.9721}$$

Answer 36E.

(A) Series is $\sum_{n=1}^{\infty} \frac{1}{n^6}$

$$s_5 = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6}$$
$$\Rightarrow \boxed{s_5 \approx 1.017305}$$

According to the remainder estimate, we have

$$R_5 \le \int_5^{\infty} \frac{1}{x^6}\,dx = \lim_{t\to\infty} \int_5^t \frac{1}{x^6}\,dx$$
$$\Rightarrow R_5 \le \lim_{t\to\infty} \left[-\frac{1}{5x^5}\right]_5^t$$
$$\Rightarrow R_5 \le \lim_{t\to\infty} \left[-\frac{1}{5t^5} + \frac{1}{5^6}\right]$$
$$\Rightarrow R_5 \le \left[0 + \frac{1}{5^6}\right] = 6.4 \times 10^{-5}$$

So the size of the error is at most $\boxed{6.4 \times 10^{-5}}$

(B) For getting the sum correct to five decimal places we must have the size of the error 10^{-6} $\Rightarrow R_n \le 10^{-6}$ first we find the number of terms by remainder estimate

$$R_n \le \int_n^{\infty} \frac{1}{x^6}\,dx = \frac{1}{5n^5}$$

We want $\dfrac{1}{5n^5} < 0.000001$

Solving this inequality, we get $5n^5 > \dfrac{1}{0.000001}$

$$\Rightarrow 5n^5 > 1000000$$
$$\Rightarrow n^5 > 200000$$
$$\Rightarrow n > 11.48$$

We have to add n = 12 terms for getting the sum correct up to 5 decimal places.

$$s \approx s_{12} = \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\frac{1}{8^6}+\frac{1}{9^6}+\frac{1}{10^6}+\frac{1}{11^6}+\frac{1}{12^6}$$

$$\boxed{s \approx 1.01734}$$

Answer 37E.

We have $\displaystyle\sum_{n=1}^{\infty}\left(2+5^n\right)^{-1} \approx \sum_{n=1}^{8}\left(2+5^n\right)^{-1}$

$$\Rightarrow s_8 = \frac{1}{(2+5)}+\frac{1}{(2+5^2)}+\frac{1}{(2+5^3)}+\frac{1}{(2+5^4)}+\ldots\ldots$$

$$\ldots\ldots+\frac{1}{(2+5^5)}+\frac{1}{(2+5^6)}+\frac{1}{(2+5^7)}+\frac{1}{(2+5^8)}$$

$$\boxed{\approx 0.18976224}$$

We see that $\dfrac{1}{2+5^n} < \dfrac{1}{5^n}$ *for all n*

So remainder term is $R_8 = \displaystyle\sum_{n=9}^{\infty}\frac{1}{2+5^n} < \sum_{n=9}^{\infty}\frac{1}{5^n}$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{5^n}$ is a geometric series with $a = \dfrac{1}{5^9}$ *and* $r = \dfrac{1}{n}$

Then $R_8 < \dfrac{1/5^9}{1-1/5} = 6.4\times10^{-7}$

So error $= \boxed{6.4\times10^{-7}}$

Answer 38E.

(A) Series is $\sum_{n=1}^{\infty}\frac{n^n}{(2n)!}$

We use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)(n+1)^n(2n)!}{(2n+2)(2n+1)(2n)!n^n}$$

$$= \lim_{n\to\infty}\frac{(n+1)(n+1)^n/n^n}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty}\frac{(1/n+1/n^2)(1+1/n)^n}{(2+2/n)(2+1/n)}$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right)\cdot\lim_{n\to\infty}\frac{(1+1/n)^n}{(2+2/n)(2+1/n)}$$

$$= 0\left(\lim_{n\to\infty}\frac{(1+1/n)^n}{(2+2/n)(2+1/n)}\right) = 0$$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ so by ratio test, series is $\boxed{\text{convergent}}$

(B) By the theorem, if the series $\sum_{n=1}^{\infty}a_n$ is convergent then $\lim_{n\to\infty}a_n = 0$

And so $\boxed{\lim_{n\to\infty}\frac{n^n}{(2n)!} = 0}$

Answer 39E.

We have $\sum_{n=0}^{\infty}a_n$ is absolutely convergent

We use limit comparison test

$$\lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{n}\right)a_n}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 > 0$$

And $\sum_{n=1}^{\infty}a_n$ is absolutely convergent then $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}a_n\right)$ is also absolutely convergent by limit comparison test.

Answer 40E.

We have to find radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}$$

Let $\quad a_n=(-1)^n \frac{x^n}{n^2 5^n} \quad$ then

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(-1)^{n+1} x^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{(-1)^n x^n}\right|$$

$$=\left|-x \frac{n^2}{5(n+1)^2}\right|$$

$$=\frac{1}{5} \frac{1}{(1+1/n)^2}|x|$$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{1}{5}\lim_{n\to\infty} \frac{1}{(1+1/n)^2}|x|$$

$$=\frac{1}{5}|x|$$

By the ratio test given series converges if $\frac{1}{5}|x|<1$

$\Rightarrow |x|<5$ and diverges when $\frac{1}{5}|x|>1$ or $|x|>5$

This means radius of convergence is $\boxed{R=5}$ and $-5 < x < 5$

If $x=5$

Then series becomes $\sum_{n=1}^{\infty}(-1)^n \frac{5^n}{n^2 5^n}=\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$

This is an alternating series with $b_n=\frac{1}{n^2}$

Clearly $\qquad b_{n+1} \le b_n$

And $\quad \lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{n^2}=0$, therefore series is convergent

If $x=-5$ then $\sum_{n=1}^{\infty}\frac{(-1)^n(-5)^n}{n^2 5^n}$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

This is a p-series with $p=2>1$ so it is convergent.

Then interval of convergence is $\boxed{[-5,5]}$

Answer 41E.

Series is $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n\,4^n}$

Let $\quad a_n=\frac{(x+2)^n}{n\,4^n}$

So $\quad \left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(x+2)^{n+1}(x4^n)}{(n+1)4^{n+1}(x+2)^n}\right|$

$$=\left|\frac{(x+2)}{4}\frac{n}{(n+1)}\right|$$

Then $\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x+2}{4}.\frac{n}{(n+1)}\right|$

$$=\frac{|x+2|}{4}\lim_{n\to\infty}\frac{1}{1+1/n}=\frac{|x+2|}{4}$$

The series converges when $\frac{|x+2|}{4}<1\Rightarrow|x+2|<4$

So radius of convergence is $\boxed{R=4}$

Since $\quad -4<x+2<4$

$\quad\quad \Rightarrow -6<x<2$

If $\quad$ x = -6, then series becomes $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n4^n}=\sum_{n=1}^{\infty}\frac{(-4)^n}{4^n\,n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

This is an alternating series with $b_{n+1}\le b_n$, and $\lim_{n\to\infty}\frac{1}{n}=0$ so by alternating series test this series converges.

If $\quad n=2 \quad$ then $\sum_{n=1}^{\infty}\frac{(x+2)^n}{4^n n}=\sum_{n=1}^{\infty}\frac{4^n}{4^n n}=\sum_{n=1}^{\infty}\frac{1}{n}$

Which is a p-series with p = 1, so it is divergent.

The interval of convergence is $\boxed{[-6,2)}$.

Answer 42E.

Series is $\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{(n+2)!}$

Let $\quad a_n=\frac{2^n(x-2)^n}{(n+2)!}$

Then $\quad \left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}.\frac{(n+2)!}{2^n(x-2)^n}\right|$

$$=\left|\frac{2(x-2)}{(n+3)}\right|$$

And so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-2|\lim_{n\to\infty}\frac{1}{n+3}$

$= 2|x-2|(0) = 0$ for all x

So radius of convergence is $\boxed{R=\infty}$ and interval of convergence is $\boxed{(-\infty,\infty)}$

Answer 43E.

Series is $\sum_{n=0}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n+3}}$

Let $a_n = \frac{2^n(x-3)^n}{\sqrt{n+3}}$

Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\frac{\sqrt{n+3}}{2^n(x-3)^n}\right|$

$= 2|x-3|.\sqrt{\frac{n+3}{n+4}}$

Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-3|\lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}}$

$= 2|x-3|.\lim_{n\to\infty}\sqrt{\frac{1+3/n}{1+4/n}}$

$= 2|x-3|$

Converges when $2|x-3|<1 \quad \Rightarrow |x-3|<1/2$

Radius of convergence is $\boxed{R=1/2}$

Since $|x-3|<1/2 \quad \Rightarrow -1/2<x-3<1/2$

$\Rightarrow 5/2<x<7/2$

If $x=5/2$, series becomes $\sum_{n=1}^{\infty}\frac{2^n(-1/2)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$

This is an alternating series with $b_n = \frac{1}{\sqrt{n+3}}$ clearly $b_{n+1}\le b_n$

And $\lim_{n\to\infty}\frac{1}{\sqrt{n+3}} = 0$ so this series is convergent.

If $n=7/2$ series becomes $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}} = \sum_{n=3}^{\infty}\frac{1}{n^{1/2}}$ it is a p-series with $p=\frac{1}{2}<1$

so it is divergent, then interval of convergence is $\boxed{[5/2,7/2)}$

Answer 44E.

Given series is $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)} x^n$

Let $a_n = \frac{(2n)!}{(n!)} x^n$ then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2n+2)!\,x^{n+1}}{\{(n+1)!\}^2} \frac{(n!)^2}{(2n)!\,x^n}\right|$$

$$= \left|\frac{(2n+2)(2n+1).x}{(n+1)(n+1)}\right|$$

Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x| \lim_{n\to\infty}\left|\frac{(2+2/n)(2+1/n)}{(1+1/n)^2}\right|$

$$= |x| . \frac{(2+0)(2+0)}{(1+0)^2}$$

$$= 4|x|$$

Series converges when $4|x| < 1 \Rightarrow |x| < 1/4$

So radius of convergence is $\boxed{R = 1/4}$

Answer 45E.

202-12.R-45E

We have $f(x) = \sin x, \quad a = \pi/6$

$$\Rightarrow f(x) = \sin x, \qquad f(\pi/6) = \frac{1}{2}$$

$$\Rightarrow f'(x) = \cos x \qquad f'(\pi/6) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow f''(x) = -\sin x \qquad f''(\pi/6) = -1/2$$

$$\Rightarrow f'''(x) = -\cos x \qquad f'''(\pi/6) = -\sqrt{3}/2$$

$$\Rightarrow f^{(iv)}(x) = \sin x \qquad f^{(iv)}(\pi/6) = 1/2$$

Seeing the trend we have $f^{(2n)}\left(\frac{\pi}{6}\right) = (-1)^n \frac{1}{2}$

And $\qquad f^{(2n+1)}(\pi/6) = (-1)^n \frac{\sqrt{3}}{2}$

Therefore Taylor series at $\pi/6$ is

$$\sin x = f\left(\frac{\pi}{6}\right) + \frac{f'(\pi/6)}{1!}\left(x - \frac{\pi}{6}\right) + \frac{f''(\pi/6)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f'''(\pi/6)}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{f^{(iv)}(\pi/6)}{4!}\left(x - \frac{\pi}{6}\right)^4 + \ldots..$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2.1!}\left(x - \frac{\pi}{6}\right) - \frac{1}{2.2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2.3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{2.4!}\left(x - \frac{\pi}{6}\right)^4 + \ldots\ldots$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2.(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1}}$$

Answer 46E.

We have $f(x) = \cos x \quad f(\pi/3) = 1/2$

So $\qquad f^{I}(x) = -\sin x \qquad f'(\pi/3) = -\sqrt{3}/2$

$$f''(x) = -\cos x \qquad f''(\pi/3) = -1/2$$

$$f^{(iv)}(x) = \cos x \qquad f^{(iv)}(\pi/3) = 1/2$$

Repeats this pattern indefinitely. Therefore the Taylor series at $a = \pi/3$ is

$$\cos x = f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{f^{(iv)}\left(\frac{\pi}{3}\right)}{4!}\left(x - \frac{\pi}{3}\right)^4 + .$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2.1!}\left(x - \frac{\pi}{3}\right) - \frac{1}{2.2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2.3!}\left(x - \frac{\pi}{3}\right) + \frac{1}{2.4!}\left(x - \frac{\pi}{3}\right)^4 + \ldots..$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x - \frac{\pi}{3}\right)^{2n}}{2.(2n)!} + \sum_{n=0}^{\infty} \frac{\sqrt{3}(-1)^{n+1}\left(x - \frac{\pi}{3}\right)^{2n+1}}{2.(2n+1)!}}$$

Answer 47E.

We have $f(x)=\dfrac{x^2}{1+x}$

Since $\dfrac{1}{1+x}=\dfrac{1}{1-(-x)}$

$$=\sum_{n=0}^{\infty}(-1)^n x^n \qquad \text{for} \qquad |x|<1$$

$\Rightarrow \dfrac{x^2}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^{n+2}$ With radius of convergence $R=1$

Maclaurin series for $f(x)$ is

$$\boxed{f(x)=\sum_{n=0}^{\infty}(-1)^n x^{n+2}}$$

And radius of convergence is $\boxed{R=1}$

Answer 48E.

We have $\qquad f(x)=\tan^{-1}(x^2)$

Since $\quad \tan^{-1}(x)=\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$

Replacing x by x^2, we have

$$\tan^{-1}(x^2)=\sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

Or $\qquad \boxed{f(x)=\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+2}}{(2n+1)}}$

This series converges when $\quad |x^2|<1 \Rightarrow |x|<1$

So radius of convergence is $\quad \boxed{R=1}$

Answer 49E.

We have $f(x)=\ln(1-x)$

$$\ln(1-x)=\int \frac{-1}{1-x}dx$$

$$=-\int \sum_{n=0}^{\infty} x^n dx \qquad \textit{with} \qquad |x|<1$$

$$=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}+c \qquad \textit{with} \qquad |x|<1$$

For $x=0$, $\qquad \ln(1-x)=0 \quad$ *and then* $\quad c=0$

So $\qquad \boxed{\ln(1-x)=-\sum_{n=0}^{\infty}\dfrac{x^{n+1}}{n+1}} \quad$ *or* $\quad \boxed{\ln(1-x)=\sum_{n=0}^{\infty}\dfrac{-x^{n+1}}{(n+1)}}$

Or $\qquad \boxed{\ln(1-x)=\sum_{n=1}^{\infty}\dfrac{-x^n}{n}}$

Radius of convergence is $\qquad \boxed{R=1}$

Answer 50E.

We have $f(x) = xe^{2x}$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

Replacing $2x$ in place of x

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \quad \text{for all } x$$

$$\Rightarrow \boxed{xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}} \quad \text{for all } x$$

Radius of convergence is $\boxed{R = \infty}$

Answer 51E.

We have $f(x) = \sin(x^4)$

We have Maclaurin series representation for sin x as

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \textit{for all } x$$

Replacing x^4 in place of x

$$\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} \qquad \textit{for all } x$$

$$\Rightarrow \boxed{\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!}} \qquad \textit{for all } x$$

Radius of convergence is $\boxed{R = \infty}$

Answer 52E.

We have $f(x) = 10^x$ $\qquad f(0) = 1$

$f'(x) = 10^x \ln 10$ $\qquad f'(0) = \ln 10$

$f''(x) = 10^x (\ln 10)^2$ $\qquad f''(0) = (\ln 10)^2$

$f'''(x) = 10^x (\ln 10)^3$ $\qquad f'''(0) = (\ln 10)^3$

$f^{(iv)}(x) = 10^x (\ln 10)^4$ $\qquad f^{(iv)}(0) = (\ln 10)^4$

The Maclaurin series for f(x) is

$$f(x)=f(0)+\frac{f'(0)}{1!}+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots\ldots$$

$$f(x)=1+\frac{\ln 10}{1!}x+\frac{(\ln 10)^2}{2!}x^2+\frac{(\ln 10)^3}{3!}x^3+\ldots..$$

$$f(x)=\sum_{n=0}^{\infty}\frac{(\ln 10)^n x^n}{n!} \qquad \text{for all } x$$

$$\Rightarrow \boxed{10^x=\sum_{n=0}^{\infty}\frac{(x\ln 10)^n}{n!}} \qquad \text{for all } x$$

For getting radius of convergence we take $a_n=\frac{(x\ln 10)^n}{n!}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(x\ln 10)^{n+1}}{(n+1)!}\frac{n!}{(x\ln 10)^n}\right|$$

$$=\left|\frac{x\ln 10}{n+1}\right|$$

$$=(\ln 10)|x|\frac{1}{n+1}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=(\ln 10)|x|.\lim_{n\to\infty}\frac{1}{n+1}=0<1$ so it converges for all x

Then radius of convergence is $\boxed{R=\infty}$

Answer 53E.

We have $f(x)=\frac{1}{\sqrt[4]{16-x}}=(16-x)^{-1/4}$

Then $f(x)=(16-x)^{-1/4}$ $\qquad f(0)=(16)^{-1/4}=\frac{1}{2}$

$f'(x)=\left(-\frac{1}{4}\right)(16-x)^{-5/4}$ $\qquad f'(0)=\left(-\frac{1}{4}\right)\left(\frac{1}{32}\right)$

$f''(x)=\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)(16-x)^{-9/4}$ $\qquad f''(0)=\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\frac{1}{512}$

$f'''(x)=\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)(16-x)^{-13/4}$ $\qquad f'''(0)=\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)(1/8152)$

Then Maclaurin series

$$f(x)=f(0)+\frac{f'(0)}{1!}(-x)+\frac{f''(0)}{2!}(-x)^2+\frac{f'''(0)}{3!}(-x)^3+\ldots\ldots$$

$$=\frac{1}{2}+\frac{(-1/4)}{1!}\left(\frac{-x}{32}\right)+\frac{(-1/4)(-5/4)}{2!}\frac{(-x)^2}{512}+\frac{(-1/4)(-5/4)(-9/4)}{3!}\frac{(-x)^3}{8192}+..$$

$$=\frac{1}{2}\left[1+\frac{(-1/4)}{1!}\left(\frac{-x}{16}\right)+\frac{(-1/4)(-5/4)}{2!}\frac{(-x)^2}{256}+\frac{(-1/4)(-5/4)(-9/4)}{3!}\frac{(-x^3)}{4096}+\ldots\right]$$

$$=\frac{1}{2}\left[1+\left(-\frac{1}{4}\right)\left(-\frac{x}{16}\right)+\frac{5}{4^2 2!}\left(-\frac{x}{16}\right)^2-\frac{5.9}{4^3 3!}\left(-\frac{x}{16}\right)^3+\ldots\right]$$

$$=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n\left(-\frac{x}{16}\right)^n.1.5.9\ldots\ldots(4n-3)}{2n!.4n}=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1.5.9\ldots\ldots(4n-3)}{2.4^n.16^n\ \ n!}x^n$$

$$\Rightarrow \boxed{f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1.5.9\ldots\ldots(4n-3)}{2^{6n+1}n!}x^n} \qquad for\ \left|-\frac{x}{16}\right|<1$$

So radius of convergence $\boxed{R=16}$

Answer 54E.

Given function is $f(x) = (1-3x)^{-5}$ $\quad \therefore f(0) = 1$

Successive derivatives of f(x) and their values for x = 0 are

$$f^{I}(x) = (-5)(1-3x)^{-6}(-3) \qquad \therefore f^{I}(0) = (-5)(-3)$$

$$f^{II}(x) = (-5)(-6)(1-3x)^{-7}(-3)^2 \qquad \therefore f^{II}(0) = (-5)(-6)(-3)^2$$

$$f(x) = (1-3x)^{-5} \text{ is given by}$$

$$f^{IV}(n) = (-5)(-6)(-7)(-8)(1-3x)^{-9}(-3)^4 \therefore f^{IV}(0) = (-5)(-6)(-7)(-8).(-3)^4$$

And so on

Therefore Maclaurins series for

$$\Rightarrow \boxed{\int \frac{e^x}{x}dx = \ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n\,.n!} + c} \qquad \text{when } c \text{ is a contant}$$

$$f(x) = 1 + \frac{(-5)(-3)}{1!}x + \frac{(-5)(-6)(-3)^2}{2!}x^2 + \frac{(-5)(-6)(-7)(-3)^3}{3!}x^3 + ---$$

$$= 1 - \frac{5}{1!}(-3x) + \frac{5.6}{2!}(-3x)^2 - \frac{5.6.7}{3!}(-3x)^3 + ---$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 5.6.7 --- (n+4)}{n!}(-3x)^n$$

$$= \sum_{n=0}^{\infty}\frac{5.6.7...(n+4)}{n!}.3^n x^n = \boxed{\sum_{n=0}^{\infty}\frac{(n+4)}{4!\ n!}3^n x^n}$$

The series is convergent for $|3x| < 1$ i.e. $|x| < \frac{1}{3}$.

So the radius of convergence is $\boxed{R = \frac{1}{3}}$

Answer 55E.

Consider the following integral:

$$\int \frac{e^x}{x}dx$$

To evaluate the above integral as an infinite series use the infinite series of e^x and evaluate the integral.

The infinite series of exponential is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Divide the above by x:

$$\frac{e^x}{x} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!}$$
$$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

Now, integrate:

$$\int \frac{e^x}{x} dx = \int \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} dx$$
$$= \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n.n!} + c$$

Therefore the required integral is:

$$\boxed{\int \frac{e^x}{x} dx = \ln|x| + \frac{1}{n} \sum_{n=1}^{\infty} \frac{x^n}{n!} + c}$$

Answer 56E.

Binomial series is

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \ldots\ldots$$

Putting $k = \frac{1}{2}$ and replace x^4 in place of x

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}(x^4) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 + \ldots\ldots$$
$$= 1 + \frac{x^4}{2} - \frac{1}{2^2 2!} x^8 + \frac{1.3}{2^3 3!} x^{12} - \frac{1.3.5}{2^4 4!} x^{16} + \ldots\ldots$$

Then $\int_0^1 (1+x^4)^{1/2} dx = \int_0^1 \left[1 + \frac{x^4}{2} - \frac{x^8}{2^2.2!} + \frac{1.3.x^{12}}{2^3.3!} - \frac{1.3.5}{2^4.4!} x^{16} + \ldots\ldots\right] dx$

$$= \left[x + \frac{x^5}{2.5} - \frac{x^9}{2^2.(2!)9} + \frac{1.3.x^{13}}{2^3.(3!)13} - \frac{1.3.5.x^{17}}{2^4.(4!)17} + \ldots\ldots\right]_0^1$$
$$= 1 + \frac{1}{10} - \frac{1}{72} + \frac{3}{624} - \frac{15}{6528} + \ldots\ldots$$
$$= 1 + \left(\frac{1}{10} - \frac{1}{72} + \frac{3}{624} - \frac{15}{6528} + \ldots\ldots\right)$$

We can approximate the sum using Alternating series estimation theorem

$$\text{since } \frac{15}{6528} \approx 2.3\times10^{-3}$$

This error will not affect second decimal place

$$\text{So } \int_0^1 \left(1+x^4\right)^{1/2} dx \approx 1+\left(\frac{1}{10}-\frac{1}{72}+\frac{3}{624}\right) \approx \boxed{1.09}$$

Answer 57E.

(A)

$$f(x)=\sqrt{x} \qquad a=1, \quad n=3, \quad 0.9 \le x \le 1.1$$

$$f(1)=1$$

$$f'(x)=\frac{1}{2}x^{-1/2} \qquad f'(1)=\frac{1}{2}$$

$$f''(x)=-\frac{1}{4}x^{-3/2} \qquad f''(1)=-\frac{1}{4}$$

$$f'''(x)=\frac{3}{8}x^{-5/2} \qquad f'''(1)=\frac{3}{8}$$

$$f^{(iv)}(x)=-\frac{15}{16}x^{-7/2} \qquad f^{(iv)}(1)=-\frac{15}{16}$$

Then

$$\sqrt{x} \approx T_3(x)=1+\frac{1/2}{1!}(x-1)+\frac{(-1/4)}{2!}(x-1)^2+\frac{3/8}{3!}(x-1)^3$$

$$\boxed{\sqrt{x} \approx T_3(x)=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)}$$

(B) Now we graph f and T_n on common series

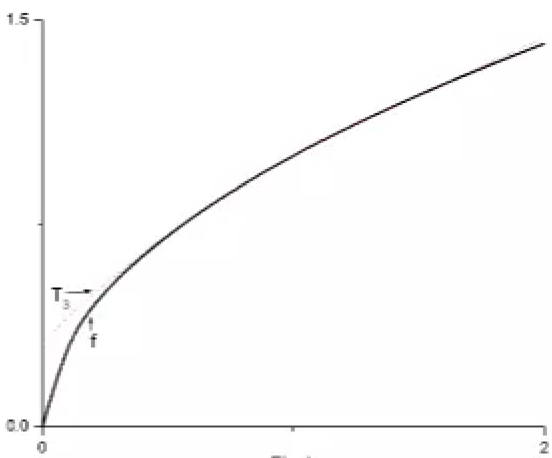


Fig.1

(C)

We have $0.9 \le x \le 1.1$

$$\Rightarrow \qquad (x-4)^4 \le (0.1)^4$$

By Taylor's inequality if $\left|f^{n+1}(x)\right| \le M$ for $|x-a| \le d$

Then $\quad \left|R_n(x)\right| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \le d$

$$\Rightarrow \qquad \left|R_3(x)\right| \le \frac{M}{4!}|x-1|^4 \quad , \quad \left|f^{(4)}(x)\right| \le M$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$$

Let $x = 0.9$

Then $\quad M = \dfrac{15}{16(0.9)^{7/2}}$

And so, $\left|R_3(x)\right| \le \dfrac{15}{16(0.9)^{7/2}\,4!}(0.1)^4 \approx \boxed{0.000005648}$

(D) Now we graph $\left|R_3(x)\right| = \left|\sqrt{x} - T_3(x)\right|$

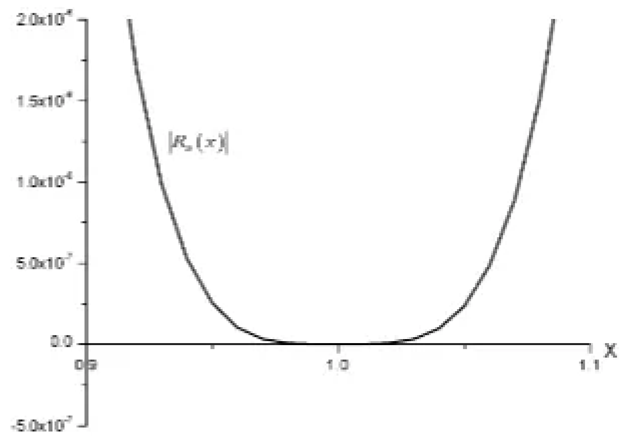


Fig.2

We see that error is less than 5×10^{-6} on [0.9, 1.1]

Answer 58E.

(A)

$$f(x) = \sec x, \quad a = 0, n = 2, 0 \le x \le \pi/6$$

$$f(0) = \sec 0 = 1$$

$$f'(x) = \sec x \tan x \qquad f'(0) = 0$$

$$f''(x) = \sec^3 x + \sec \tan^2 x \qquad f''(0) = 1$$

$$f'''(x) = 3\sec^3 x \tan x + 2\sec^3 x \tan x + \sec x \tan^2 x \qquad \Rightarrow f'''(0) = 0$$

Then $\quad \sec x \approx T_2(x) = 1 + \dfrac{0}{1!}(x-0) + \dfrac{1}{2!}(x-0)^2$

$$\Rightarrow \sec x \approx T_2(x) = 1 + \frac{1}{2!}x^2$$

$$\Rightarrow \boxed{\sec x \approx T_2(x) = 1 + \frac{x^2}{2}}$$

(B) Now we sketch the graphs of $T_2(x)$ and $f(x)$

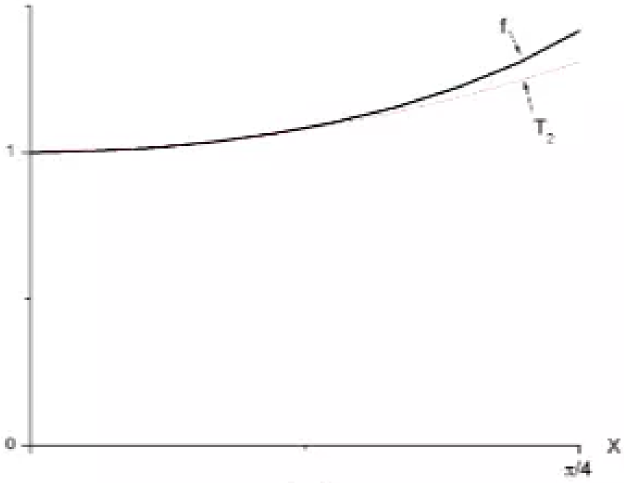


Fig.1

(C) We have $0 \le x \le \pi/6$

$$\Rightarrow \quad x^3 \le \left(\pi/6\right)^3$$

By Taylor's inequality If $\left|f^{n+1}(x)\right| \le M$ for $|x-a| \le d$

Then $\left|R_n(x)\right| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$

$$\Rightarrow \left|R_2(x)\right| \le \frac{M}{3!}(x)^3, \quad \left|f^3(x)\right| \le M$$

We have $f^{(3)}(x) = 5\sec^2 x \tan x + \sec x \tan^3 x$

Let $x = \pi/6$

Then $f^{(3)}(\pi/6) \approx 4.\overline{6}$

$\Rightarrow M = 4.\overline{6}$

So $\left|R_2(x)\right| \le \dfrac{4.\overline{6}}{3!}\left(\dfrac{\pi}{6}\right)^3 \approx \boxed{0.111648}$

(D)

Now we sketch graph of $\left|R_2(x)\right| = \left|\sec x - T_2(x)\right|$

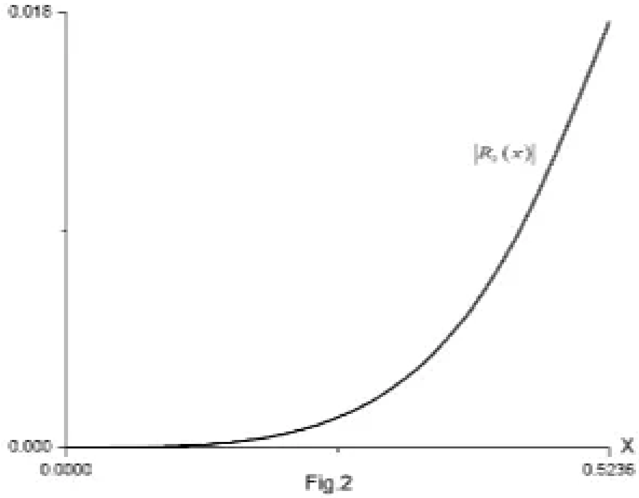


Fig.2

We see that error is ten then $\boxed{\approx 0.018}$ on $\left[0, \pi/6\right]$

Answer 59E.

We have to evaluate $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$

We have Sin $x = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$$

So $\quad \sin x - x = \dfrac{-x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots\ldots$

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6} + \lim_{x\to\infty}\left(\frac{x^2}{5!} - \frac{x^4}{7!} + \ldots\ldots\right)$$

$$= -\frac{1}{6} + 0$$

$$\Rightarrow \boxed{\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}}$$

Answer 60E.

(a)

Consider the force due to gravity on an object with mass m at a height h above the surface of the earth is,

$$F = \frac{mgR^2}{(R+h)^2}$$
$$= \frac{mg}{\left(1+\frac{h}{R}\right)^2}$$
$$= mg\left(1+\frac{h}{R}\right)^{-2}$$

$= mg\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{h}{R}\right)^n$ By use the Binomial series.

(b)

F as a series in power of $\frac{h}{R}$ then,

$$F = mg\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{h}{R}\right)^n$$
$$= mg\left[1-2\left(\frac{h}{R}\right)+3\left(\frac{h}{R}\right)^2-\cdots\right]$$

The above equation as an alternating series so by the Estimation theorem the error in the approximation $F = mg$ is less than $\frac{2mgh}{R}$ and the accuracy within 1% then,

$$\left|\frac{\left(\frac{2mgh}{R}\right)}{\left(\frac{mgR^2}{(R+h)^2}\right)}\right| < 0.01$$

$$\frac{2h(R+h)^2}{R^3} < 0.01$$

The above inequality should be difficult solve for h. So $R = 6400\ \text{km}$

$$\frac{2h(6400+h)^2}{(6400)^3} < 0.01$$
$$h < 31.68548279$$

Therefore, the approximation is accurate to within 1% for $\boxed{h < 31.68548279}$

Answer 61E.

(A) We have $f(x)=\sum_{n=0}^{\infty} c_n x^n \quad for\ all\ x$ ---(1)

Replacing – x in place of x

$$f(-x)=\sum_{n=0}^{\infty} c_n(-x)^n=\sum_{n=0}^{\infty}(-1)^n c_n x^n \qquad ---(2)$$

If f is odd function then $\sum_{n=0}^{\infty}(-1)^n c_n x^n=\sum_{n=0}^{\infty}-c_n x^n$

Coefficients are given by the formula $c_n=\dfrac{f^{(n)}(a)}{n!}$

And coefficients are uniquely determined. So $(-1)^n c_n=-c_n$

If n is even number then $(-1)^n=1$

So $c_n=-c_n$

$\Rightarrow c_n=0 \qquad \Rightarrow c_2=c_4=c_6=c_8=\ldots\ldots=0$

So all even coefficients are equal to 0

(B) If f is even then f(-x) = f (x)

$$\Rightarrow \sum_{n=0}^{\infty}(-1)^n c_n x^n=\sum_{n=0}^{\infty} c_n x^n$$

$$\Rightarrow (-1)^n c_n=c_n$$

$$\Rightarrow -c_n=c_n \qquad if\ n\ is\ odd\ then\ (-1)^n=-1$$

$$\Rightarrow 2c_n=0$$

$$\Rightarrow \boxed{c_n=0} \quad \Rightarrow \quad c_1=c_3=c_5=c_7=\ldots..=0$$

Thus all coefficients are equal to 0.

Answer 62E.

We have $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dfrac{x^5}{5!}+\ldots\ldots\ldots\ldots$

Then $\quad e^{x^2}=1+x^2+\dfrac{x^4}{2!}+\dfrac{x^6}{3!}+\dfrac{x^8}{4!}+\dfrac{x^{10}}{5!}+\ldots\ldots+\dfrac{x^{2n}}{n!}+\ldots. \qquad (1)$