Chapter 7

Rotations and Addition of Angular Momenta

In this chapter we deal with rotations, the properties of addition of angular momenta, and the properties of tensor operators.

7.1 Rotations in Classical Physics

A rotation is defined by an angle of rotation and an axis about which the rotation is performed. Knowing the rotation matrix R, we can determine how vectors transform under rotations; in a three-dimensional space, a vector $\vec{A}$ becomes $\vec{A}'$ when rotated: $\vec{A}' = R\vec{A}$. For instance, a rotation over an angle ϕ about the z-axis transforms the components A_x, A_y, A_z of the vector $\vec{A}$ into A'_x, A'_y, A'_z:

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{7.1}$$

or

$$\vec{A}' = R_z(\phi)\vec{A}, \tag{7.2}$$

where

$$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.3}$$

Similarly, the rotation matrices about the $x-$ and $y-$ axes are given by

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}, \qquad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}. \tag{7.4}$$

From classical physics we know that while rotations about the same axis commute, rotations about different axes do not. From (7.4) we can verify that $R_x(\phi)R_y(\phi) \neq R_y(\phi)R_x(\phi)$. In fact, using (7.4) we can have

$$R_x(\phi)R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ -\sin^2\phi & \cos\phi & \cos\phi\sin\phi \\ -\cos\phi\sin\phi & -\sin\phi & \cos^2\phi \end{pmatrix}, \tag{7.5}$$

$$R_y(\phi)R_x(\phi) = \begin{pmatrix} \cos\phi & -\sin^2\phi & \cos\phi\sin\phi \\ 0 & \cos\phi & \sin\phi \\ -\sin\phi & -\sin\phi\cos\phi & \cos^2\phi \end{pmatrix}; \tag{7.6}$$

hence $R_x(\phi)R_y(\phi) - R_y(\phi)R_x(\phi)$ is given by

$$\begin{pmatrix} 0 & \sin^2\phi & \sin\phi - \cos\phi\sin\phi \\ -\sin^2\phi & 0 & \cos\phi\sin\phi - \sin\phi \\ \sin\phi - \cos\phi\sin\phi & \cos\phi\sin\phi - \sin\phi & 0 \end{pmatrix}. \tag{7.7}$$

In the case of infinitesimal rotations of angle δ about the $x-$ $y-$, $z-$ axes, and using $\cos\delta \simeq 1 - \delta^2/2$ and $\sin\delta \simeq \delta$, we can reduce (7.7) to

$$R_x(\delta)R_y(\delta) - R_y(\delta)R_x(\delta) = \begin{pmatrix} 0 & \delta^2 & 0 \\ -\delta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.8}$$

which, when combined with $R_z(\delta^2)$ of (7.3),

$$R_z(\delta) = \begin{pmatrix} 1 - \frac{\delta^2}{2} & \delta & 0 \\ -\delta & 1 - \frac{\delta^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies R_z(\delta^2) = \begin{pmatrix} 1 & \delta^2 & 0 \\ -\delta^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.9}$$

leads to

$$R_x(\delta)R_y(\delta) - R_y(\delta)R_x(\delta) = R_z(\delta^2) - 1 = \begin{pmatrix} 1 & \delta^2 & 0 \\ -\delta^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.10}$$

We will show later that this relation can be used to derive the commutation relations between the components of the angular momentum (7.26).

The rotation matrices R are orthogonal, i.e.,

$$RR^T = R^T R = 1, \tag{7.11}$$

where R^T is the transpose of the matrix R. In addition, the orthogonal matrices conserve the magnitude of vectors:

$$|\vec{A}'| = |\vec{A}|, \tag{7.12}$$

since $\vec{A}' = \hat{R}\vec{A}$ yields $\vec{A}'^2 = \vec{A}^2$ or $A'^2_x + A'^2_y + A'^2_z = A^2_x + A^2_y + A^2_z$.

It is easy to show that the matrices of orthogonal rotations form a (nonabelian) group and that they satisfy this relation

$$\det(R) = 1. \tag{7.13}$$

This group is called the *special* three-dimensional orthogonal group, $SO(3)$, because the rotation group is a special case of a more general group, the group of three-dimensional orthogonal transformations, $O(3)$, which consist of both rotations and reflections and for which

$$\det(R) = \pm 1. \tag{7.14}$$

The group $SO(3)$ transforms a vector $\vec{A}$ into another vector $\vec{A}'$ while conserving the size of its length.

7.2 Rotations in Quantum Mechanics

In this section we study the relationship between the angular momentum and the rotation operator and then study the properties as well as the representation of the rotation operator. The connection is analogous to that between the linear momentum operator and translations. We will see that the angular momentum operator acts as a generator for rotations.

A rotation is specified by an angle and by a unit vector $\vec{n}$ about which the rotation is performed. Knowing the rotation operator $\hat{R}$, we can determine how state vectors and operators transform under rotations; as shown in Chapter 2, a state $\mid \psi\rangle$ and an operator $\hat{A}$ transform according to

$$\mid \psi'\rangle = \hat{R} \mid \psi\rangle, \qquad \hat{A}' = \hat{R}\hat{A}\hat{R}^{\dagger}. \tag{7.15}$$

The problem reduces then to finding $\hat{R}$. We may now consider infinitesimal as well as finite rotations.

7.2.1 Infinitesimal Rotations

Consider a rotation of the coordinates of a *spinless* particle over an *infinitesimal* angle $\delta\phi$ about the z-axis. Denoting this rotation by the operator $\hat{R}_z(\delta\phi)$, we have

$$\hat{R}_z(\delta\phi)\psi(r,\theta,\phi) = \psi(r,\theta,\phi-\delta\phi). \tag{7.16}$$

Taylor expanding the wave function to the first order in $\delta\phi$, we obtain

$$\psi(r,\theta,\phi-\delta\phi) \simeq \psi(r,\theta,\phi) - \delta\phi\frac{\delta\psi}{\delta\phi} = \left(1 - \delta\phi\frac{\delta}{\delta\phi}\right)\psi(r,\theta,\phi). \tag{7.17}$$

Comparing (7.16) and (7.17) we see that $\hat{R}_z(\delta\phi)$ is given by

$$\hat{R}_z(\delta\phi) = 1 - \delta\phi\frac{\delta}{\delta\phi}. \tag{7.18}$$

Since the z-component of the orbital angular momentum is

$$\hat{L}_z = -i\hbar\frac{\delta}{\delta\phi}, \tag{7.19}$$

we can rewrite (7.18) as

$$\boxed{\hat{R}_z(\delta\phi) = 1 - \frac{i}{\hbar}\delta\phi\hat{L}_z.} \tag{7.20}$$

We may generalize this relation to a rotation of angle $\delta\phi$ about an arbitrary axis whose direction is given by the unit vector $\vec{n}$:

$$\boxed{\hat{R}(\delta\phi) = 1 - \frac{i}{\hbar}\delta\phi\,\vec{n}\cdot\hat{\vec{L}}.} \tag{7.21}$$

This is the operator corresponding to an *infinitesimal* rotation of angle $\delta\phi$ about $\vec{n}$ for a *spinless* system. The orbital angular momentum is thus the *generator* of infinitesimal spatial rotations.

Rotations and the commutation relations

We can show that the relation (7.10) leads to the commutation relations of angular momentum $[\hat{L}_x,\ \hat{L}_y] = i\hbar\hat{L}_z$. The operators corresponding to infinitesimal rotations of angle δ about the x and y axes can be inferred from (7.20):

$$\hat{R}_x(\delta) = 1 - \frac{i\delta}{\hbar}\hat{L}_x - \frac{\delta^2}{2\hbar^2}\hat{L}_x^2, \qquad \hat{R}_y(\delta) = 1 - \frac{i\delta}{\hbar}\hat{L}_y - \frac{\delta^2}{2\hbar^2}\hat{L}_y^2, \tag{7.22}$$

where we have extended the expansions to the second power in δ. On the one hand, the following useful relation can be obtained from (7.22):

$$\begin{aligned}
\hat{R}_x(\delta)\hat{R}_y(\delta) - \hat{R}_y(\delta)\hat{R}_x(\delta) &= \left(1 - \frac{i\delta}{\hbar}\hat{L}_x - \frac{\delta^2}{2\hbar^2}\hat{L}_x^2\right)\left(1 - \frac{i\delta}{\hbar}\hat{L}_y - \frac{\delta^2}{2\hbar^2}\hat{L}_y^2\right) \\
&\quad - \left(1 - \frac{i\delta}{\hbar}\hat{L}_y - \frac{\delta^2}{2\hbar^2}\hat{L}_y^2\right)\left(1 - \frac{i\delta}{\hbar}\hat{L}_x - \frac{\delta^2}{2\hbar^2}\hat{L}_x^2\right) \\
&= -\frac{\delta^2}{\hbar^2}\left(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x\right) \\
&= -\frac{\delta^2}{\hbar^2}[\hat{L}_x,\ \hat{L}_y],
\end{aligned} \tag{7.23}$$

where we have kept only terms up to the second power in δ; the terms in δ cancel out automatically.

On the other hand, according to (7.10), we have

$$R_x(\delta)R_y(\delta) - R_y(\delta)R_x(\delta) = R_z(\delta^2) - 1. \tag{7.24}$$

Since $\hat{R}_z(\delta^2) = 1 - (i\delta^2/\hbar)\hat{L}_z$ this relations leads to

$$R_x(\delta)R_y(\delta) - R_y(\delta)R_x(\delta) = R_z(\delta^2) - 1 = -\frac{i\delta^2}{\hbar}\hat{L}_z. \tag{7.25}$$

Finally, equating (7.23) and (7.25), we end up with

$$[\hat{L}_x,\ \hat{L}_y] = i\hbar\hat{L}_z. \tag{7.26}$$

Similar calculations for $R_y(\delta)R_z(\delta) - R_z(\delta)R_y(\delta)$ and $R_z(\delta)R_x(\delta) - R_x(\delta)R_z(\delta)$ lead to the other two commutation relations $[\hat{L}_y,\ \hat{L}_z] = i\hbar\hat{L}_x$ and $[\hat{L}_z,\ \hat{L}_x] = i\hbar\hat{L}_y$.

7.2.2 Finite Rotations

The operator $\hat{R}_z(\phi)$ corresponding to a rotation (of the coordinates of a spinless particle) over a *finite* angle ϕ about the z-axis can be constructed in terms of the infinitesimal rotation operator (7.20) as follows. We divide the angle ϕ into N infinitesimal angles $\delta\phi$: $\phi = N\,\delta\phi$. The rotation over the finite angle ϕ can thus be viewed as a series of N consecutive infinitesimal rotations, each over the angle $\delta\phi$, about the z-axis, applied consecutively one after the other:

$$\hat{R}_z(\phi) = \hat{R}_z(N\delta\phi) = (R_z(\delta\phi))^N = \left(1 - i\frac{\delta\phi}{\hbar}\hat{L}_z\right)^N. \tag{7.27}$$

Since $\delta\phi = \phi/N$, and if $\delta\phi$ is infinitesimally small, we have

$$\hat{R}_z(\phi) = \lim_{N\to\infty}\prod_{k=1}^{N}\left(1 - \frac{i}{\hbar}\frac{\phi}{N}\hat{n}\cdot\vec{L}\right) = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\frac{\phi}{N}\hat{L}_z\right)^N, \tag{7.28}$$

or

$$\boxed{\hat{R}_z(\phi) = e^{-i\phi\hat{L}_z/\hbar}.} \tag{7.29}$$

We can generalize this result to infer the rotation operator $\hat{R}_n(\phi)$ corresponding to a rotation over a finite angle ϕ around an axis $\vec{n}$:

$$\boxed{\hat{R}_n(\phi) = e^{-i\phi\vec{n}\cdot\hat{\vec{L}}/\hbar},} \tag{7.30}$$

where $\vec{L}$ is the orbital angular momentum. This operator represents the rotation of the coordinates of a spinless particle over an angle ϕ about an axis $\vec{n}$.

The discussion that led to (7.30) was carried out for a spinless system. A more general study for a system with spin would lead to a relation similar to (7.30):

$$\boxed{\hat{R}_n(\phi) = e^{-\frac{i}{\hbar}\phi\vec{n}\cdot\hat{\vec{J}}},} \tag{7.31}$$

where $\hat{\vec{J}}$ is the total angular momentum operator; this is known as the *rotation operator*. For instance, the rotation operator $\vec{R}_x(\phi)$ of a rotation through an angle ϕ about the x-axis is given by

$$\hat{R}_x(\phi) = e^{-i\phi\hat{J}_x/\hbar}. \tag{7.32}$$

The properties of $\hat{R}_n(\phi)$ are determined by those of the operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$.

Remark

The Hamiltonian of a particle in a *central potential*, $\hat{H} = \hat{P}^2/(2m) + \hat{V}(r)$, is *invariant under spatial rotations* since, as shown in Chapter 6, it commutes with the orbital angular momentum:

$$[\hat{H},\ \hat{\vec{L}}] = 0 \qquad \Longrightarrow \qquad \left[\hat{H},\ e^{-i\phi\vec{n}\cdot\hat{\vec{L}}/\hbar}\right] = 0. \tag{7.33}$$

Due to this symmetry of space isotropy or rotational invariance, the *orbital angular momentum is conserved*[1]. So, in the case of particles moving in central potentials, the orbital angular momentum is a constant of the motion.

[1] In classical physics when a system is invariant under rotations, its total angular momentum is also conserved.

7.2.3 Properties of the Rotation Operator

The rotation operators constitute a representation of the rotation group and satisfy the following properties:

- The product of any two rotation operators is another rotation operator:
$$\hat{R}_{n_1}\hat{R}_{n_2} = \hat{R}_{n_3}. \tag{7.34}$$
- The associative law holds for rotation operators:
$$\left(\hat{R}_{n_1}\hat{R}_{n_2}\right)\hat{R}_{n_3} = \hat{R}_{n_1}\left(\hat{R}_{n_2}\hat{R}_{n_3}\right). \tag{7.35}$$
- The identity operator (corresponding to no rotation) satisfies the relation
$$\hat{I}\hat{R}_n = \hat{R}_n\hat{I} = \hat{R}_n. \tag{7.36}$$
From (7.31) we see that for each rotation operator $\hat{R}_n$, there exists an inverse operator $\hat{R}_n^{-1}$ so that
$$\hat{R}_n\hat{R}_n^{-1} = \hat{R}_n^{-1}\hat{R}_n = \hat{I}. \tag{7.37}$$
The operator $\hat{R}_{-n}$, which is equal to $\hat{R}_n^{-1}$, corresponds to a rotation in the opposite sense to $\hat{R}_n$.

In sharp contrast to the translation group[2] in three dimensions, the rotation group is not commutative (nonabelian). *The product of two rotation operators depends on the order in which they are performed*:
$$\hat{R}_{n_1}(\phi)\hat{R}_{n_2}(\theta) \neq \hat{R}_{n_2}(\theta)\hat{R}_{n_1}(\phi); \tag{7.38}$$
this is due to the fact that the commutator $[\vec{n}_1 \cdot \hat{\vec{J}},\ \vec{n}_2 \cdot \hat{\vec{J}}]$ is not zero. In this way, the rotation group is in general nonabelian.

But if the two rotations were performed about the *same* axis, the corresponding operators would *commute*:
$$\hat{R}_n(\phi)\hat{R}_n(\theta) = \hat{R}_n(\theta)\hat{R}_n(\phi) = \hat{R}_n(\phi+\theta). \tag{7.39}$$

Note that, since the angular momentum operator $\hat{J}$ is Hermitian, equation (7.31) yields
$$\boxed{\hat{R}_n^\dagger(\phi) = \hat{R}_n^{-1}(\phi) = \hat{R}_n(-\phi) = e^{i\phi\vec{n}\cdot\hat{\vec{J}}/\hbar};} \tag{7.40}$$
hence the rotation operator (7.31) is unitary:
$$\boxed{\hat{R}_n^\dagger(\phi) = \hat{R}_n^{-1}(\phi) \qquad \Longrightarrow \qquad \hat{R}_n^\dagger(\phi)\hat{R}_n(\phi) = \hat{I}.} \tag{7.41}$$
The operator $\hat{R}_n(\phi)$ therefore conserves the scalar product of kets, notably the norm of vectors. For instance, using
$$\mid \psi' \rangle = \hat{R}_n(\phi) \mid \psi \rangle, \qquad \mid \chi' \rangle = \hat{R}_n(\phi) \mid \chi \rangle, \tag{7.42}$$
along with (7.41), we can show that $\langle \chi' \mid \psi' \rangle = \langle \chi \mid \psi \rangle$, since
$$\langle \chi' \mid \psi' \rangle = \langle \chi \mid \hat{R}_n^\dagger(\phi)\hat{R}_n(\phi) \mid \psi \rangle = \langle \chi \mid \psi \rangle. \tag{7.43}$$

[2]The linear momenta $\hat{P}_i$ and $\hat{P}_j$—which are the generators of translation—commute even when $i \neq j$; hence the translation group is said to be abelian.

7.2.4 Euler Rotations

It is known from classical mechanics that an arbitrary rotation of a rigid body can be expressed in terms of three consecutive rotations, called the Euler rotations. In quantum mechanics, instead of expressing the rotation operator $\hat{R}_n(\phi) = e^{-i\phi\vec{n}\cdot\hat{\vec{J}}/\hbar}$ in terms of a rotation through an angle ϕ about an arbitrary axis $\vec{n}$, it is more convenient to parameterize it, as in classical mechanics, in terms of the three *Euler angles* (α, β, γ) where $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \pi$, and $0 \leq \gamma \leq 2\pi$. The Euler rotations transform the space-fixed set of axes xyz into a new set $x'y'z'$, having the same origin O, by means of three consecutive counterclockwise rotations:

- First, rotate the space-fixed $Oxyz$ system through an angle α about the z-axis; this rotation transforms the $Oxyz$ system into $Ouvz$: $Oxyz \longrightarrow Ouvz$.
- Second, rotate the uvz system through an angle β about the v-axis; this rotation transforms the $Ouvz$ system into $Owvz'$: $Ouvz \longrightarrow Owvz'$.
- Third, rotate the wvz' system through an angle γ about the z'-axis; this rotation transforms the $Owvz'$ system into $Ox'y'z'$: $Owvz' \longrightarrow Ox'y'z'$.

The operators representing these three rotations are given by $\hat{R}_z(\alpha)$, $\hat{R}_v(\beta)$, and $\hat{R}_{z'}(\gamma)$, respectively. Using (7.31) we can represent these three rotations by

$$\hat{R}(\alpha, \beta, \gamma) = \hat{R}_{z'}(\gamma)\hat{R}_v(\beta)\hat{R}_z(\alpha) = \exp\left[-i\gamma J_{z'}/\hbar\right]\exp\left[-i\beta J_v/\hbar\right]\exp\left[-i\alpha J_z/\hbar\right]. \quad (7.44)$$

The form of this operator is rather inconvenient, for it includes rotations about axes belonging to different systems (i.e., z', v, and z); this form would be most convenient were we to express (7.44) as a product of three rotations about the space-fixed axes x, y, z. So let us express $\hat{R}_{z'}(\gamma)$ and $\hat{R}_v(\beta)$ in terms of rotations about the x, y, z axes. Since the first Euler rotation described above, $\hat{R}_z(\alpha)$, transforms the operator $\hat{J}_y$ into $\hat{J}_v$, i.e., $\hat{J}_v = \hat{R}_z(\alpha)\hat{J}_y\hat{R}_z(-\alpha)$ by (7.15), we have

$$\hat{R}_v(\beta) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(-\alpha) = e^{-i\alpha J_z/\hbar}e^{-i\beta J_y/\hbar}e^{i\alpha J_z/\hbar}. \quad (7.45)$$

Here $\hat{J}_{z'}$ is obtained from $\hat{J}_z$ by the consecutive application of the second and third Euler rotations, $\hat{J}_{z'} = \hat{R}_v(\beta)\hat{R}_z(\alpha)\hat{J}_z\hat{R}_z(-\alpha)\hat{R}_v(-\beta)$; hence

$$\hat{R}_{z'}(\gamma) = \hat{R}_v(\beta)\hat{R}_z(\alpha)\hat{R}_z(\gamma)\hat{R}_z(-\alpha)\hat{R}_v(-\beta). \quad (7.46)$$

Since $\hat{R}_v(-\beta) = \hat{R}_z(\alpha)\hat{R}_y(-\beta)\hat{R}_z(-\alpha)$, substituting (7.45) into (7.46) we obtain

$$\begin{aligned}\hat{R}_{z'}(\gamma) &= \left[\hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(-\alpha)\right]\hat{R}_z(\alpha)\hat{R}_z(\gamma)\hat{R}_z(-\alpha)\left[\hat{R}_z(\alpha)\hat{R}_y(-\beta)\hat{R}_z(-\alpha)\right] \\ &= \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma)\hat{R}_y(-\beta)\hat{R}_z(-\alpha) \\ &= e^{-i\alpha J_z/\hbar}e^{-i\beta J_y/\hbar}e^{-i\gamma J_z/\hbar}e^{i\beta J_y/\hbar}e^{i\alpha J_z/\hbar}, \quad (7.47)\end{aligned}$$

where we used the fact that $\hat{R}_z(-\alpha)\hat{R}_z(\alpha) = e^{-i\alpha J_z/\hbar}e^{i\alpha J_z/\hbar} = 1$.

Finally, inserting (7.45) and (7.47) into (7.44) and simplifying (i.e., using $\hat{R}_z(-\alpha)\hat{R}_z(\alpha) = 1$ and $\hat{R}_y(-\beta)\hat{R}_y(\beta) = 1$), we end up with a product of three rotations about the space-fixed axes y and z:

$$\boxed{\hat{R}(\alpha, \beta, \gamma) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma) = e^{-i\alpha J_z/\hbar}e^{-i\beta J_y/\hbar}e^{-i\gamma J_z/\hbar}.} \quad (7.48)$$

The inverse transformation of (7.48) is obtained by taking three rotations in reverse order over the angles $(-\gamma, -\beta, -\alpha)$:

$$\boxed{\hat{R}^{-1}(\alpha, \beta, \gamma) = \hat{R}_z(-\gamma)\hat{R}_y(-\beta)\hat{R}_z(-\alpha) = \hat{R}^{\dagger}(\alpha, \beta, \gamma) = e^{i\gamma J_z/\hbar}e^{i\beta J_y/\hbar}e^{i\alpha J_z/\hbar}.} \quad (7.49)$$

7.2.5 Representation of the Rotation Operator

The rotation operator $\hat{R}(\alpha, \beta, \gamma)$ as given by (7.48) implies that its properties are determined by the algebraic properties of the angular momentum operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$. Since $\hat{R}(\alpha, \beta, \gamma)$ commutes with $\hat{\vec{J}}^2$, we may look for a representation of $\hat{R}(\alpha, \beta, \gamma)$ in the basis spanned by the eigenvectors of $\hat{\vec{J}}^2$ and J_z, i.e., the $\mid j,\ m\rangle$ states.

From (7.48), we see that$\hat{\vec{J}}^2$ commutes with the rotation operator, $[\hat{\vec{J}}^2,\ \hat{R}(\alpha, \beta, \gamma)] = 0$; thus, the total angular momentum is conserved under rotations

$$\hat{\vec{J}}^2\hat{R}(\alpha, \beta, \gamma) \mid j,\ m\rangle = \hat{R}(\alpha, \beta, \gamma)\hat{\vec{J}}^2 \mid j,\ m\rangle = j(j+1)\hat{R}(\alpha, \beta, \gamma) \mid j,\ m\rangle. \quad (7.50)$$

However, the z-component of the angular momentum changes under rotations, unless the axis of rotation is along the z-axis. That is, when $\hat{R}(\alpha, \beta, \gamma)$ acts on the state $\mid j,\ m\rangle$, we end up with a new state having the same j but with a different value of m:

$$\begin{aligned} \hat{R}(\alpha, \beta, \gamma) \mid j,\ m\rangle &= \sum_{m'=-j}^{j} \mid j,\ m'\rangle\langle j,\ m' \mid \hat{R}(\alpha, \beta, \gamma) \mid j,\ m\rangle \\ &= \sum_{m'=-j}^{j} D^{(j)}_{m'm}(\alpha, \beta, \gamma) \mid j,\ m'\rangle, \end{aligned} \quad (7.51)$$

where

$$\boxed{D^{(j)}_{m'm}(\alpha, \beta, \gamma) = \langle j,\ m' \mid \hat{R}(\alpha, \beta, \gamma) \mid j,\ m\rangle.} \quad (7.52)$$

These are the matrix elements of $\hat{R}(\alpha, \beta, \gamma)$ for the $\mid j,\ m\rangle$ states; $D^{(j)}_{m'm}(\alpha, \beta, \gamma)$ is the amplitude of $\mid j,\ m'\rangle$ when $\mid j,\ m\rangle$ is rotated. The rotation operator is thus represented by a $(2j+1) \times (2j+1)$ square matrix in the $\{\mid j,\ m\rangle\}$ basis. The matrix of $D^{(j)}(\alpha, \beta, \gamma)$ is known as the *Wigner D-matrix* and its elements $D^{(j)}_{m'm}(\alpha, \beta, \gamma)$ as the *Wigner functions*. This matrix representation is often referred to as the $(2j+1)$-dimensional *irreducible representation* of the rotation operator $\hat{R}(\alpha, \beta, \gamma)$.

Since $\mid j,\ m\rangle$ is an eigenstate of J_z, it is also an eigenstate of the rotation operator $e^{i\alpha J_z/\hbar}$, because

$$e^{i\alpha J_z/\hbar} \mid j,\ m\rangle = e^{i\alpha m} \mid j,\ m\rangle. \quad (7.53)$$

We may thus rewrite (7.52) as

$$\boxed{D^{(j)}_{m'm}(\alpha, \beta, \gamma) = e^{-i(m'\alpha+m\gamma)}d^{(j)}_{m'm}(\beta),} \quad (7.54)$$

where

$$\boxed{d^{(j)}_{m'm}(\beta) = \langle j,\ m'|e^{-i\beta\hat{J}_y/\hbar} \mid j,\ m\rangle.} \quad (7.55)$$

This shows that only the middle rotation operator, $e^{-i\beta\hat{J}_y/\hbar}$, mixes states with different values of m. Determining the matrix elements $D^{(j)}_{m'm}(\alpha, \beta, \gamma)$ therefore reduces to evaluation of the quantities $d^{(j)}_{m'm}(\beta)$.

A general expression of $d^{(j)}_{m'm}(\beta)$, called the *Wigner formula*, is given by the following explicit expression:

$$d^{(j)}_{m'm}(\beta) = \sum_k (-1)^{k+m'-m} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j-m'-k)!(j+m-k)!(k+m'-m)!k!} \times \left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2k} \left(\sin\frac{\beta}{2}\right)^{m'-m+2k}. \tag{7.56}$$

The summation over k is taken such that none of the arguments of factorials in the denominator are negative.

We should note that, since the D-function $D^{(j)}_{m'm}(\alpha, \beta, \gamma)$ is a joint eigenfunction of $\vec{J}^2$ and J_z, we have

$$\boxed{\hat{\vec{J}}^2 D^{(j)}_{m'm}(\alpha, \beta, \gamma) = j(j+1)\hbar^2 D^{(j)}_{m'm}(\alpha, \beta, \gamma),} \tag{7.57}$$

$$\boxed{\hat{J}_z D^{(j)}_{m'm}(\alpha, \beta, \gamma) = \hbar m D^{(j)}_{m'm}(\alpha, \beta, \gamma),} \tag{7.58}$$

$$\boxed{\hat{J}_\pm D^{(j)}_{m'm}(\alpha, \beta, \gamma) = \hbar\sqrt{(j \pm m)(j \mp m + 1)} D^{(j)}_{m'm\pm 1}(\alpha, \beta, \gamma).} \tag{7.59}$$

Properties of the D-functions

We now list some of the most useful properties of the rotation matrices. The complex conjugate of the D-functions can be expressed as

$$\begin{aligned}\left[D^{(j)}_{m'm}(\alpha, \beta, \gamma)\right]^* &= \langle j,\ m' \mid \hat{R}(\alpha, \beta, \gamma) \mid j,\ m\rangle^* = \langle j\ m \mid \hat{R}^\dagger(\alpha, \beta, \gamma) \mid j,\ m'\rangle \\ &= \langle j\ m \mid \hat{R}^{-1}(\alpha, \beta, \gamma) \mid j,\ m'\rangle \\ &= D^{(j)}_{mm'}(-\gamma, -\beta, -\alpha).\end{aligned} \tag{7.60}$$

We can easily show that

$$\boxed{\left[D^{(j)}_{m'm}(\alpha, \beta, \gamma)\right]^* = (-1)^{m'-m} D^{(j)}_{-m'-m}(\alpha, \beta, \gamma) = D^{(j)}_{mm'}(-\gamma, -\beta, -\alpha).} \tag{7.61}$$

The D-functions satisfy the following unitary relations:

$$\boxed{\sum_m \left[D^{(j)}_{km}(\alpha, \beta, \gamma)\right]^* D^{(j)}_{k'm}(\alpha, \beta, \gamma) = \delta_{k,k'},} \tag{7.62}$$

$$\boxed{\sum_m \left[D^{(j)}_{mk}(\alpha, \beta, \gamma)\right]^* D^{(j)}_{mk'}(\alpha, \beta, \gamma) = \delta_{k,k'},} \tag{7.63}$$

since

$$\begin{aligned}\sum_m \left[D^{(j)}_{mk}(\alpha,\beta,\gamma)\right]^* D^{(j)}_{mk'}(\alpha,\beta,\gamma) &= \sum_m \langle j\ k|\hat{R}^{-1}(\alpha,\beta,\gamma)\mid j,\ m\rangle\langle j\ m\mid R(\alpha,\beta,\gamma)|j\ k'\rangle \\ &= \langle j\ k|\hat{R}^{-1}(\alpha,\beta,\gamma)\hat{R}(\alpha,\beta,\gamma)(|j\ k'\rangle \\ &= \langle j\ k|j\ k'\rangle \\ &= \delta_{k,\,k'}. \end{aligned} \tag{7.64}$$

From (7.55) we can show that the d-functions satisfy the following relations:

$$d^{(j)}_{m'm}(\pi) = (-1)^{j-m}\delta_{m',\,-m}, \qquad d^{(j)}_{m'm}(0) = \delta_{m',\,m}. \tag{7.65}$$

Since $d^j_{m'm}$ are elements of a unitary real matrix, the matrix $d^{(j)}(\beta)$ must be orthogonal. We may thus write

$$\boxed{d^{(j)}_{m'm}(\beta) = \left(d^{(j)}_{m'm}(\beta)\right)^{-1} = d^{(j)}_{mm'}(-\beta)} \tag{7.66}$$

and

$$\boxed{d^{(j)}_{m'm}(\beta) = (-1)^{m'-m}d^{(j)}_{mm'}(\beta) = (-1)^{m'-m}d^{(j)}_{-m'-m}(\beta).} \tag{7.67}$$

The unitary matrices $D^{(j)}$ form a $(2j+1)$ dimensional irreducible representation of the $SO(3)$ group.

7.2.6 Rotation Matrices and the Spherical Harmonics

In the case where the angular momentum operator $\hat{\vec{J}}$ is purely orbital (i.e., the values of j are integer, $j = l$), there exists a connection between the D-functions and the spherical harmonics $Y_{lm}(\theta,\varphi)$. The operator $\hat{R}(\alpha,\beta,\gamma)$ when applied to a vector $|\vec{r}\rangle$ pointing in the direction (θ,φ) would generate a vector $|\vec{r}\,'\rangle$ along a new direction (θ',φ'):

$$|\vec{r}\,'\rangle = \hat{R}(\alpha,\beta,\gamma)\mid\vec{r}\rangle. \tag{7.68}$$

An expansion in terms of $|\ l,\ m'\rangle$ and a multiplication by $\langle l,\ m\mid$ leads to

$$\langle l,\ m\mid\vec{r}\,'\rangle = \sum_{m'}\langle l,\ m|\hat{R}(\alpha,\beta,\gamma)\mid l,\ m'\rangle\langle l,\ m'\mid\vec{r}\rangle, \tag{7.69}$$

or to

$$Y^*_{lm}(\theta',\varphi') = \sum_{m'} D^{(l)}_{m\ m'}(\alpha,\beta,\gamma)Y^*_{lm'}(\theta,\varphi), \tag{7.70}$$

since $\langle l,\ m\mid\vec{r}\,'\rangle = Y^*_{lm}(\theta',\varphi')$ and $\langle l,\ m'\mid\vec{r}\rangle = Y^*_{lm'}(\theta,\varphi)$.

In the case where the vector $\vec{r}$ is along the z-axis, we have $\theta = 0$; hence $m' = 0$. From Chapter 5, $Y^*_{l0}(0,\varphi)$ is given by

$$Y^*_{lm'}(0,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\delta_{m',\,0}. \tag{7.71}$$

We can thus reduce (7.70) to

$$Y^*_{lm}(\beta,\ \alpha) = D^{(l)}_{m\ 0}(\alpha,\beta,\gamma)Y^*_{l0}(0,\varphi) = \sqrt{\frac{2l+1}{4\pi}}D^{(l)}_{m\ 0}(\alpha,\beta,\gamma), \tag{7.72}$$

or to

$$\boxed{D^{(l)}_{m0}(\alpha, \beta, \gamma) = \sqrt{\frac{4\pi}{2l+1}}\, Y^*_{lm}(\beta,\ \alpha).} \tag{7.73}$$

This means that a rotation through the Euler angles (α, β, γ) of the vector $\vec{r}$, when it is along the z-axis, produces a vector $\vec{r}\ '$ whose azimuthal and polar angles are given by β and α, respectively. Similarly, we can show that

$$\boxed{D^{(l)}_{0m}(\gamma, \beta, \alpha) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\beta,\ \alpha)} \tag{7.74}$$

and

$$D^{(l)}_{00}(0, \theta, 0) = P_l(\cos\theta), \tag{7.75}$$

where $P_l(\cos\theta)$ is the Legendre polynomial.

We are now well equipped to derive the theorem for the addition of spherical harmonics. Let (θ, φ) be the polar coordinates of the vector $\vec{r}$ with respect to the space-fixed x, y, z system and let (θ', φ') be its polar coordinates with respect to the rotated system x', y', z'; taking the complex conjugate of (7.70) we obtain

$$Y_{lm}(\theta', \varphi') = \sum_{m'} \left[D^{(l)}_{m\ m'}(\alpha, \beta, \gamma)\right]^* Y_{lm'}(\theta, \varphi). \tag{7.76}$$

For the case $m = 0$, since (from Chapter 5)

$$Y_{l0}(\theta', \varphi') = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta') \tag{7.77}$$

and since from (7.74)

$$\left[D^{(l)}_{0m'}(\alpha, \beta, \gamma)\right]^* = \sqrt{\frac{4\pi}{2l+1}}\, Y^*_{lm'}(\beta, \gamma), \tag{7.78}$$

we can reduce (7.76) to

$$\sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta') = \sum_{m'} \sqrt{\frac{4\pi}{2l+1}}\, Y^*_{lm'}(\beta, \gamma) Y_{lm'}(\theta, \varphi), \tag{7.79}$$

or to

$$P_l(\cos\theta') = \frac{4\pi}{2l+1} \sum_{m'} Y^*_{lm'}(\beta, \gamma) Y_{lm'}(\theta, \varphi). \tag{7.80}$$

Integrals involving the D-functions

Let ω denote the Euler angles; hence

$$\int d\omega = \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma. \tag{7.81}$$

Using the relation

$$\begin{aligned} \int D^{(j)}_{m'm}(\omega)\, d\omega &= \int_0^{\pi} d^{(j)}_{m'm}(\beta) \sin\beta d\beta \int_0^{2\pi} e^{-im'\alpha} d\alpha \int_0^{2\pi} e^{-im\gamma}\, d\gamma \\ &= 8\pi^2 \delta_{j,\,0}\delta_{m',\,0}\delta_{m,\,0}, \end{aligned} \tag{7.82}$$

we may write

$$\begin{aligned}
\int D_{mk}^{(j)*}(\omega)D_{m'k'}^{(j')}(\omega)\,d\omega &= (-1)^{m-k}\int D_{-m-k}^{(j)}(\omega)D_{m'k'}^{(j')}(\omega)\,d\omega \\
&= (-1)^{m-k}\int_0^{\pi} d_{-m-k}^{(j)}(\beta)d_{m'k'}^{(j')}(\beta)\sin\beta d\beta \\
&\quad\times\int_0^{2\pi} e^{-i(m'-m)\alpha}d\alpha\int_0^{2\pi} e^{-i(k'-k)\gamma}\,d\gamma \\
&= \frac{8\pi^2}{2j+1}\delta_{j,\,j'}\delta_{m,\,m'}\delta_{k,\,k'}.
\end{aligned} \tag{7.83}$$

Example 7.1

Find the rotation matrices $d^{(1/2)}$ and $D^{(1/2)}$ corresponding to $j=\frac{1}{2}$.

Solution

On the one hand, since the matrix of $\hat{J}_y$ for $j=\frac{1}{2}$ (Chapter 5) is given by

$$\hat{J}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y, \tag{7.84}$$

and since the square of the Pauli matrix σ_y is equal to the unit matrix, $\sigma_y^2=1$, the even and odd powers of σ_y are given by

$$\sigma_y^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_y^{2n+1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y. \tag{7.85}$$

On the other hand, since the rotation operator

$$\hat{R}_y(\beta) = e^{-i\beta\hat{J}_y/\hbar} = e^{-i\beta\sigma_y/2} \tag{7.86}$$

can be written as

$$e^{-i\beta\sigma_y/2} = \sum_{n=0}^{\infty}\frac{(-i)^{2n}}{(2n)!}\left(\frac{\beta}{2}\right)^{2n}\sigma^{2n} + \sum_{n=0}^{\infty}\frac{(-i)^{2n+1}}{(2n+1)!}\left(\frac{\beta}{2}\right)^{2n+1}\sigma_y^{2n+1}, \tag{7.87}$$

a substitution of (7.85) into (7.87) yields

$$\begin{aligned}
e^{-i\beta\sigma_y/2} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\beta}{2}\right)^{2n} - i\sigma_y\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\beta}{2}\right)^{2n+1} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\left(\frac{\beta}{2}\right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin\left(\frac{\beta}{2}\right);
\end{aligned} \tag{7.88}$$

hence

$$d^{(1/2)}(\beta) = e^{-i\beta J_y/\hbar} = \begin{pmatrix} d_{\frac{1}{2}\frac{1}{2}}^{(1/2)} & d_{\frac{1}{2}-\frac{1}{2}}^{(1/2)} \\ d_{-\frac{1}{2}\frac{1}{2}}^{(1/2)} & d_{-\frac{1}{2}-\frac{1}{2}}^{(1/2)} \end{pmatrix} = \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}. \tag{7.89}$$

Since as shown in (7.54) $D^{(j)}_{m'm}(\alpha, \beta, \gamma) = e^{-i(m'\alpha + m\gamma)} d^{(j)}_{m'm}(\beta)$, we have

$$D^{(1/2)}(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}. \tag{7.90}$$

7.3 Addition of Angular Momenta

The addition of angular momenta is encountered in all areas of modern physics. Mastering its techniques is essential for an understanding of the various subatomic phenomena. For instance, the total angular momentum of the electron in a hydrogen atom consists of two parts, an orbital part $\hat{\vec{L}}$, which is due to the orbiting motion of the electron around the proton, and a spin part $\hat{\vec{S}}$, which is due to the spinning motion of the electron about itself. The properties of the hydrogen atom cannot be properly discussed without knowing how to add the orbital and spin parts of the electron's total angular momentum.

In what follows we are going to present the formalism of angular momentum addition and then consider some of its most essential applications.

7.3.1 Addition of Two Angular Momenta: General Formalism

In this section we present the general formalism corresponding to the problem of adding two *commuting* angular momenta.

Consider two angular momenta $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ which belong to different subspaces 1 and 2; $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ may refer to two distinct particles or to two different properties of the same particle[3]. The latter case may refer to the orbital and spin angular momenta of the same particle. Assuming that the spin–orbit coupling is sufficiently weak, then the space and spin degrees of freedom of the electron evolve *independently* of each other.

The components of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ satisfy the usual commutation relations of angular momentum:

$$\left[\hat{J}_{1_x},\ \hat{J}_{1_y}\right] = i\hbar\hat{J}_{1_z}, \qquad \left[\hat{J}_{1_y},\ \hat{J}_{1_z}\right] = i\hbar\hat{J}_{1_x}, \qquad \left[\hat{J}_{1_z},\ \hat{J}_{1_x}\right] = i\hbar\hat{J}_{1_y}, \tag{7.91}$$

$$\left[\hat{J}_{2_x},\ \hat{J}_{2_y}\right] = i\hbar\hat{J}_{2_z}, \qquad \left[\hat{J}_{2_y},\ \hat{J}_{2_z}\right] = i\hbar\hat{J}_{2_x}, \qquad \left[\hat{J}_{2_z},\ \hat{J}_{2_x}\right] = i\hbar\hat{J}_{2_y}. \tag{7.92}$$

Since $\hat{\vec{J}}_1$, and $\hat{\vec{J}}_2$ belong to different spaces, their components commute:

$$\left[\hat{J}_{1_j},\ \hat{J}_{2_k}\right] = 0, \qquad (j,\ k = x, y, z). \tag{7.93}$$

[3]Throughout this section we shall use the labels 1 and 2 to refer to quantities relevant to the two particles or the two subspaces.

Now, denoting the joint eigenstates of $\hat{\vec{J}}_1^2$ and $\hat{J}_{1_z}$ by $\mid j_1,\ m_1\rangle$ and those of $\hat{\vec{J}}_2^2$ and $\hat{J}_{2_z}$ by $\mid j_2,\ m_2\rangle$, we have

$$\begin{aligned}
\hat{\vec{J}}_1^2 \mid j_1,\ m_1\rangle &= j_1(j_1+1)\hbar^2 \mid j_1,\ m_1\rangle, && (7.94)\\
\hat{J}_{1_z} \mid j_1,\ m_1\rangle &= m_1\hbar \mid j_1,\ m_1\rangle, && (7.95)\\
\hat{\vec{J}}_2^2 \mid j_2,\ m_2\rangle &= j_2(j_2+1)\hbar^2 \mid j_2,\ m_2\rangle, && (7.96)\\
\hat{J}_{2_z} \mid j_2,\ m_2\rangle &= m_2\hbar \mid j_2,\ m_2\rangle. && (7.97)
\end{aligned}$$

The dimensions of the spaces to which $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ belong are given by $(2j_1+1)$ and $(2j_2+1)$, respectively[4]. The operators $\hat{\vec{J}}_1^2$ and $\hat{J}_{1_z}$ are represented within the $\{\mid j_1,\ m_1\rangle\}$ basis by square matrices of dimension $(2j_1+1)\times(2j_1+1)$, while $\hat{\vec{J}}_2^2$ and $\hat{J}_{2_z}$ are representation by square matrices of dimension $(2j_2+1)\times(2j_2+1)$ within the $\{\mid j_2,\ m_2\rangle\}$ basis.

Consider now the two particles (or two subspaces) 1 and 2 together. The four operators $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{J}_{1_z}$, $\hat{J}_{2_z}$ form a complete set of commuting operators; they can thus be jointly diagonalized by the same states. Denoting their joint eigenstates by $\mid j_1, j_2;\ m_1, m_2\rangle$, we can write them as *direct products* of $\mid j_1,\ m_1\rangle$, and $\mid j_2,\ m_2\rangle$

$$\mid j_1, j_2;\ m_1, m_2\rangle = \mid j_1,\ m_1\rangle \mid j_2,\ m_2\rangle, \tag{7.98}$$

because the coordinates of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ are *independent*. We can thus rewrite (7.94)–(7.97) as

$$\begin{aligned}
\hat{\vec{J}}_1^2 \mid j_1, j_2;\ m_1, m_2\rangle &= j_1(j_1+1)\hbar^2 \mid j_1, j_2;\ m_1, m_2\rangle, && (7.99)\\
\hat{J}_{1_z} \mid j_1, j_2;\ m_1, m_2\rangle &= m_1\hbar \mid j_1, j_2;\ m_1, m_2\rangle, && (7.100)\\
\hat{\vec{J}}_2^2 \mid j_1, j_2;\ m_1, m_2\rangle &= j_2(j_2+1)\hbar^2 \mid j_1, j_2;\ m_1, m_2\rangle, && (7.101)\\
\hat{J}_{2_z} \mid j_1, j_2;\ m_1, m_2\rangle &= m_2\hbar \mid j_1, j_2;\ m_1, m_2\rangle. && (7.102)
\end{aligned}$$

The kets $\mid j_1, j_2;\ m_1, m_2\rangle$ form a complete and orthonormal basis. Using

$$\sum_{m_1 m_2} \mid j_1, j_2;\ m_1, m_2\rangle\langle j_1, j_2;\ m_1, m_2 \mid = \left(\sum_{m_1} \mid j_1,\ m_1\rangle\langle j_1,\ m_1 \mid\right)\left(\sum_{m_2} \mid j_2,\ m_2\rangle\langle j_2,\ m_2 \mid\right), \tag{7.103}$$

and since $\{\mid j_1,\ m_1\rangle\}$ and $\{\mid j_2,\ m_2\rangle\}$ are complete (i.e., $\sum_{m_1} \mid j_1,\ m_1\rangle\langle j_1,\ m_1 \mid = 1$) and orthonormal (i.e., $\langle j_1',\ m_1' \mid j_1,\ m_1\rangle = \delta_{j_1', j_1}\delta_{m_1', m_1}$ and similarly for $\mid j_2,\ m_2\rangle$), we see that the basis $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ is complete,

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \mid j_1, j_2;\ m_1, m_2\rangle\langle j_1, j_2;\ m_1, m_2 \mid = 1, \tag{7.104}$$

and orthonormal,

$$\begin{aligned}
\langle j_1', j_2';\ m_1', m_2' \mid j_1, j_2;\ m_1, m_2\rangle &= \langle j_1',\ m_1' \mid j_1,\ m_1\rangle\langle j_2',\ m_2' \mid j_2,\ m_2\rangle \\
&= \delta_{j_1', j_1}\delta_{j_2', j_2}\delta_{m_1', m_1}\delta_{m_2', m_2}. && (7.105)
\end{aligned}$$

[4]This is due to the fact that the number of basis vectors spanning the spaces to which $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ belong are equal to $(2j_1+1)$ and $(2j_2+1)$, respectively; these vectors are $|j_1,\ -j_1\rangle$, $|j_1,\ -j_1+1\rangle, \ldots, |j_1,\ j_1-1\rangle$, $|j_1,\ j_1\rangle$ and $|j_2,\ -j_2\rangle$, $|j_2,\ -j_2+1\rangle, \ldots, |j_2,\ j_2-1\rangle$, $|j_2,\ j_2\rangle$.

The basis $\{|\ j_1, j_2;\ m_1, m_2\rangle\}$ clearly spans the total space which is made of subspaces 1 and 2. From (7.98) we see that the dimension N of this space is equal to the product of the dimensions of the two subspaces spanned by $\{|\ j_1,\ m_1\rangle\}$ and $\{|\ j_2,\ m_2\rangle\}$:

$$N = (2j_1 + 1) \times (2j_2 + 1). \tag{7.106}$$

We can now introduce the *step* operators $\hat{J}_{1\pm} = \hat{J}_{1_x} \pm i\hat{J}_{1_y}$ and $\hat{J}_{2\pm} = \hat{J}_{2_x} \pm i\hat{J}_{2_y}$; their actions on $|j_1 j_2; m_1 m_2\rangle$ are given by

$$\hat{J}_{1\pm}\ |\ j_1, j_2;\ m_1, m_2\rangle \ = \ \hbar\sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\,|j_1, j_2;\ m_1 \pm 1, m_2\rangle, \tag{7.107}$$

$$\hat{J}_{2\pm}\ |\ j_1, j_2;\ m_1, m_2\rangle \ = \ \hbar\sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\,|j_1, j_2;\ m_1, m_2 \pm 1\rangle. \tag{7.108}$$

The problem of adding two angular momenta, $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$,

$$\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2, \tag{7.109}$$

consists of finding the eigenvalues and eigenvectors of $\hat{\vec{J}}^2$ and $\hat{J}_z$ in terms of the eigenvalues and eigenvectors of $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{J}_{1_z}$, and $\hat{J}_{2_z}$. Since the matrices of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ have in general different dimensions, the addition specified by (7.109) is not an addition of matrices; it is a symbolic addition.

By adding (7.91) and (7.92), we can easily ascertain that the components of $\hat{\vec{J}}$ satisfy the commutation relations of angular momentum:

$$\left[\hat{J}_x,\ \hat{J}_y\right] = i\hbar\hat{J}_z, \qquad \left[\hat{J}_y,\ \hat{J}_z\right] = i\hbar\hat{J}_x, \qquad \left[\hat{J}_z,\ \hat{J}_x\right] = i\hbar\hat{J}_y. \tag{7.110}$$

Note that $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{\vec{J}}^2$, $\hat{J}_z$ jointly commute; this can be ascertained from the relation:

$$\hat{\vec{J}}^2 = \hat{\vec{J}}_1^2 + \hat{\vec{J}}_2^2 + 2\hat{J}_{1_z}\hat{J}_{2_z} + J_{1+}J_{2+} + J_{1-}J_{2-}, \tag{7.111}$$

which leads to

$$\left[\hat{\vec{J}}^2,\ \hat{\vec{J}}_1^2\right] = \left[\hat{\vec{J}}^2,\ \hat{\vec{J}}_2^2\right] = 0, \tag{7.112}$$

and to

$$\left[\hat{\vec{J}}^2,\ \hat{J}_z\right] = \left[\hat{\vec{J}}_1^2,\ \hat{J}_z\right] = \left[\hat{\vec{J}}_2^2,\ \hat{J}_z\right] = 0. \tag{7.113}$$

But in spite of the fact that $\left[\hat{\vec{J}}^2,\ \hat{J}_z\right] = 0$, the operators $\hat{J}_{1_z}$ and $\hat{J}_{2_z}$ do not commute separately with $\hat{\vec{J}}^2$:

$$\left[\hat{\vec{J}}^2,\ \hat{J}_{1_z}\right] \neq 0, \qquad \left[\hat{\vec{J}}^2,\ \hat{J}_{2_z}\right] \neq 0. \tag{7.114}$$

Now, since $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{\vec{J}}^2$, $\hat{J}_z$ form a complete set of commuting operators, they can be diagonalized simultaneously by the same states; designating these joint eigenstates by $|j_1, j_2;\ j,\ m\rangle$, we have

$$\hat{\vec{J}}_1^2\ |\ j_1, j_2;\ j,\ m\rangle \ = \ j_1(j_1 + 1)\hbar^2\ |\ j_1, j_2;\ j,\ m\rangle, \tag{7.115}$$

$$\hat{\vec{J}}_2^2\ |\ j_1, j_2;\ j,\ m\rangle \ = \ j_2(j_2 + 1)\hbar^2\ |\ j_1, j_2;\ j,\ m\rangle, \tag{7.116}$$

$$\hat{\vec{J}}^2\ |\ j_1, j_2;\ j,\ m\rangle \ = \ j(j + 1)\hbar^2\ |\ j_1, j_2;\ j,\ m\rangle, \tag{7.117}$$

$$\hat{J}_z\ |\ j_1, j_2;\ j,\ m\rangle \ = \ m\hbar\ |\ j_1, j_2;\ j,\ m\rangle. \tag{7.118}$$

For every j, the number m has $(2j+1)$ allowed values: $m = -j, -j+1, \ldots, j-1, j$.

Since j_1 and j_2 are usually fixed, *we will be using, throughout the rest of this chapter, the shorthand notation* $\mid j,\ m\rangle$ *to abbreviate* $\mid j_1,\ j_2;\ j,\ m\rangle$. The set of vectors $\{\mid j,\ m\rangle\}$ form a complete and orthonormal basis:

$$\sum_j \sum_{m=-j}^{j} \mid j,\ m\rangle\langle j,\ m \mid = 1, \tag{7.119}$$

$$\langle j',\ m' \mid j,\ m\rangle = \delta_{j,\,j'}\delta_{m',\,m}. \tag{7.120}$$

The space where the total angular momentum $\hat{\vec{J}}$ operates is spanned by the basis $\{\mid j,\ m\rangle\}$; this space is known as a *product space*. It is important to know that this space is the same as the one spanned by $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$; that is, the space which includes both subspaces 1 and 2. So the dimension of the space which is spanned by the basis $\{\mid j,\ m\rangle\}$ is also equal to $N = (2j_1+1)\times(2j_2+1)$ as specified by (7.106).

The issue now is to find the transformation that connects the bases $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ and $\{\mid j,\ m\rangle\}$.

7.3.1.1 Transformation between Bases: Clebsch–Gordan Coefficients

Let us now return to the addition of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$. This problem consists in essence of obtaining the eigenvalues of $\hat{\vec{J}}^2$ and $\hat{J}_z$ and of expressing the states $\mid j,\ m\rangle$ in terms of $\mid j_1, j_2;\ m_1, m_2\rangle$. We should mention that $\mid j,\ m\rangle$ is the state in which $\hat{\vec{J}}^2$ and $\hat{\vec{J}}_z$ have fixed values, $j(j+1)$ and m, but in general not a state in which the values of $\hat{\vec{J}}_{1z}$ and $\hat{\vec{J}}_{2z}$ are fixed; as for $\mid j_1, j_2;\ m_1, m_2\rangle$, it is the state in which $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{\vec{J}}_{1z}$, and $\hat{\vec{J}}_{2z}$ have fixed values.

The $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ and $\{\mid j,\ m\rangle\}$ bases can be connected by means of a transformation as follows. Inserting the identity operator as a sum over the complete basis $\mid j_1, j_2;\ m_1, m_2\rangle$, we can write

$$\begin{aligned}\mid j,\ m\rangle &= \left(\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} \mid j_1, j_2;\ m_1, m_2\rangle\langle j_1, j_2;\ m_1, m_2 \mid\right)\mid j,\ m\rangle \\ &= \sum_{m_1 m_2}\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \mid j_1, j_2;\ m_1, m_2\rangle, \end{aligned}\tag{7.121}$$

where we have used the normalization condition (7.104); since the bases $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ and $\{\mid j,\ m\rangle\}$ are both *normalized*, this transformation must be *unitary*. The coefficients $\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle$, which depend only on the quantities j_1, j_2, j, m_1, m_2, and m, *are the matrix elements of the unitary transformation which connects the* $\{\mid j,\ m\rangle\}$ and $\{\mid j_1, j_2;\ m_1, m_2\rangle\}$ bases. These coefficients are called the *Clebsch–Gordan coefficients*.

The problem of angular momentum addition reduces then to finding the Clebsch–Gordan coefficients $\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle$. These coefficients are taken to be *real by convention*; hence

$$\boxed{\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle = \langle j,\ m \mid j_1, j_2;\ m_1, m_2\rangle.} \tag{7.122}$$

Using (7.104) and (7.120) we can infer the orthonormalization relation for the Clebsch–Gordan coefficients:

$$\sum_{m_1 m_2}\langle j',\ m' \mid j_1, j_2;\ m_1, m_2\rangle\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle = \delta_{j',\,j}\delta_{m',\,m}, \tag{7.123}$$

and since the Clebsch–Gordan coefficients are real, this relation can be rewritten as

$$\boxed{\sum_{m_1 m_2} \langle j_1, j_2;\ m_1, m_2 \mid j',\ m' \rangle \langle j_1, j_2;\ m_1, m_2 \mid j,\ m \rangle = \delta_{j', j} \delta_{m', m},} \tag{7.124}$$

which leads to

$$\boxed{\sum_{m_1 m_2} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m \rangle^2 = 1.} \tag{7.125}$$

Likewise, we have

$$\boxed{\sum_{j} \sum_{m=-j}^{j} \langle j_1, j_2;\ m_1', m_2' \mid j,\ m \rangle \langle j_1, j_2;\ m_1, m_2 \mid j,\ m \rangle = \delta_{m_1', m_1} \delta_{m_2', m_2}} \tag{7.126}$$

and, in particular,

$$\boxed{\sum_{j} \sum_{m} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m \rangle^2 = 1.} \tag{7.127}$$

7.3.1.2 Eigenvalues of $\hat{\vec{J}}^2$ and $\hat{J}_z$

Let us study how to find the eigenvalues of $\hat{\vec{J}}^2$ and $\hat{J}_z$ in terms of those of $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{J}_{1_z}$, and $\hat{J}_{2_z}$; that is, obtain j and m in terms of j_1, j_2, m_1 and m_2. First, since $\hat{J}_z = \hat{J}_{1_z} + \hat{J}_{2_z}$, we have $m = m_1 + m_2$. Now, to find j in terms of j_1 and j_2, we proceed as follows. Since the maximum values of m_1 and m_2 are $m_{1max} = j_1$ and $m_{2max} = j_2$, we have $m_{max} = m_{1max} + m_{2max} = j_1 + j_2$; but since $|m| \leq j$, then $j_{max} = j_1 + j_2$.

Next, to find the minimum value j_{min} of j, we need to use the fact that there are a total of $(2j_1+1) \times (2j_2+1)$ eigenkets $\mid j,\ m\rangle$. To each value of j there correspond $(2j+1)$ eigenstates $\mid j,\ m\rangle$, so we have

$$\sum_{j=j_{min}}^{j_{max}} (2j+1) = (2j_1+1)(2j_2+1), \tag{7.128}$$

which leads to (see Example 7.2, page 408, for the proof)

$$j_{min}^2 = (j_1 - j_2)^2 \qquad \Longrightarrow \qquad j_{min} = |j_1 - j_2|. \tag{7.129}$$

Hence the allowed values of j are located within the range

$$\boxed{|j_1 - j_2| \leq j \leq j_1 + j_2.} \tag{7.130}$$

This expression can also be inferred from the well-known triangle relation[5]. So the allowed values of j proceed in integer steps according to

$$\boxed{j = |j_1 - j_2|,\ |j_1 - j_2| + 1,\ \ldots, j_1 + j_2 - 1,\ j_1 + j_2.} \tag{7.131}$$

[5] The length of the sum of two classical vectors, $\vec{A} + \vec{B}$, must be located between the sum and the difference of the lengths of the two vectors, $A + B$ and $|A - B|$, i.e., $|A - B| \leq |\vec{A} + \vec{B}| \leq A + B$.

Thus, for every j the allowed values of m are located within the range $-j \leq m \leq j$.

Note that the coefficient $\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle$ vanishes unless $m_1 + m_2 = m$. This can be seen as follows: since $\hat{J}_z = \hat{J}_{1_z} + \hat{J}_{2_z}$, we have

$$\langle j_1, j_2;\ m_1, m_2 \mid \hat{J}_z - \hat{J}_{1_z} - \hat{J}_{2_z} \mid j,\ m\rangle = 0, \tag{7.132}$$

and since $\hat{J}_z \mid j,\ m\rangle = m\hbar \mid j,\ m\rangle$, $\langle j_1, j_2;\ m_1, m_2 \mid \hat{J}_{1_z} = m_1\hbar\langle j_1, j_2;\ m_1, m_2 \mid$, and $\langle j_1, j_2;\ m_1, m_2 \mid \hat{J}_2 z = m_2\hbar\langle j_1, j_2;\ m_1, m_2 \mid$, we can write

$$(m - m_1 - m_2)\,\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle = 0, \tag{7.133}$$

which shows that $\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle$ is not zero only when $m - m_1 - m_2 = 0$.

$$\boxed{\text{If} \quad m_1 + m_2 \neq m \quad \Longrightarrow \quad \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle = 0.} \tag{7.134}$$

So, for the Clebsch–Gordan coefficient $\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle$ not to be zero, we must simultaneously have

$$\boxed{m_1 + m_2 = m \qquad \text{and} \qquad \mid j_1 - j_2 \mid \leq j \leq j_1 + j_2.} \tag{7.135}$$

These are known as the *selection rules* for the Clebsch–Gordan coefficients.

Example 7.2

Starting from $\sum_{j=j_{min}}^{j_{max}} (2j+1) = (2j_1+1)(2j_2+1)$, prove (7.129).

Solution

Let us first work on the left-hand side of

$$\sum_{j=j_{min}}^{j_{max}} (2j+1) = (2j_1+1)\,(2j_2+1)\,. \tag{7.136}$$

Since $j_{max} = j_1 + j_2$ we can write the left-hand side of this equation as an arithmetic sum which has $(j_{max} - j_{min} + 1) = [(j_1 + j_2 + 1) - j_{min}]$ terms:

$$\sum_{j=j_{min}}^{j_{max}} (2j+1) = (2j_{min}+1) + (2j_{min}+3) + (2j_{min}+5) + \cdots + \big[2(j_1+j_2)+1\big]\,. \tag{7.137}$$

To calculate this sum, we simply write it in the following two equivalent ways:

$$S = (2j_{min}+1) + (2j_{min}+3) + (2j_{min}+5) + \cdots + [2(j_1+j_2)+1], \tag{7.138}$$

$$S = [2(j_1+j_2)+1] + [2(j_1+j_2)-1] + [2(j_1+j_2)-3] + \cdots + (2j_{min}+1). \tag{7.139}$$

Adding these two series term by term, we obtain

$$2S = 2[(j_1+j_2+1)+j_{min}] + 2[(j_1+j_2+1)+j_{min}] + \cdots + 2[(j_1+j_2+1)+j_{min}]. \tag{7.140}$$

Since this expression has $(j_{max} - j_{min} + 1) = [(j_1 + j_2 + 1) - j_{min}]$ terms, we have

$$2S = 2[(j_1+j_2+1)+j_{min}][(j_1+j_2+1)-j_{min}]; \tag{7.141}$$

hence

$$S = [(j_1 + j_2 + 1) + j_{min}][(j_1 + j_2 + 1) - j_{min}] = (j_1 + j_2 + 1)^2 - j_{min}^2. \tag{7.142}$$

Now, equating this expression with the right-hand side of (7.136), we obtain

$$(j_1 + j_2 + 1)^2 - j_{min}^2 = (2j_1 + 1)\,(2j_2 + 1)\,, \tag{7.143}$$

which in turn leads to

$$j_{min}^2 = (j_1 - j_2)^2\,. \tag{7.144}$$

7.3.2 Calculation of the Clebsch–Gordan Coefficients

First, we should point out that the Clebsch–Gordan coefficients corresponding to the two limiting cases where $m_1 = j_1, m_2 = j_2, j = j_1 + j_2, m = j_1 + j_2$ and $m_1 = -j_1, m_2 = -j_2$, $j = j_1 + j_2, m = -(j_1 + j_2)$ are equal to one:

$$\boxed{\langle j_1, j_2;\ j_1, j_2|(j_1 + j_2),\ (j_1 + j_2)\rangle = 1, \quad \langle j_1, j_2;\ -j_1, -j_2|(j_1 + j_2),\ -(j_1 + j_2)\rangle = 1.} \tag{7.145}$$

These results can be inferred from (7.121), since $|(j_1+j_2),\ (j_1+j_2)\rangle$, and $|(j_1+j_2),\ -(j_1+j_2)\rangle$ have one element each:

$$|(j_1 + j_2),\ (j_1 + j_2)\rangle = \langle j_1, j_2;\ j_1,\ j_2|(j_1 + j_2),\ (j_1 + j_2)\rangle |j_1, j_2;\ j_1,\ j_2\rangle, \tag{7.146}$$

$$|(j_1+j_2),\ -(j_1+j_2)\rangle = \langle j_1, j_2;\ -j_1,\ -j_2|(j_1+j_2),\ -(j_1+j_2)\rangle |j_1, j_2;\ -j_1,\ -j_2\rangle, \tag{7.147}$$

where $|(j_1 + j_2),\ (j_1 + j_2)\rangle$, $|(j_1 + j_2),\ -(j_1 + j_2)\rangle$, $|j_1, j_2;\ j_1,\ j_2\rangle$, and $|j_1, j_2;\ -j_1,\ -j_2\rangle$ are all normalized.

The calculations of the other coefficients are generally more involved than the two limiting cases mentioned above. For this, we need to derive the recursion relations between the matrix elements of the unitary transformation between the $\{|\ j,\ m\rangle\}$ and $\{|j_1, j_2;\ m_1, m_2\rangle\}$ bases, since, when $j_1,\ j_2$ and j are fixed, the various Clebsch–Gordan coefficients are related to one another by means of recursion relations. To find the recursion relations, we need to evaluate the matrix elements $\langle j_1, j_2;\ m_1, m_2 \mid \hat{J}_\pm \mid j,\ m\rangle$ in two different ways. First, allow $\hat{J}_\pm$ to act to the right, i.e., on $|\ j,\ m\rangle$:

$$\langle j_1, j_2;\ m_1, m_2 \mid \hat{J}_\pm \mid j,\ m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\langle j_1, j_2;\ m_1, m_2 \mid j,\ m \pm 1\rangle. \tag{7.148}$$

Second, make $\hat{J}_\pm = \hat{J}_{1\pm} + \hat{J}_{2\pm}$ act to the left[6], i.e., on $\langle j_1, j_2;\ m_1, m_2 \mid$:

$$\begin{aligned}\langle j_1, j_2;\ m_1, m_2 \mid \hat{J}_\pm \mid j,\ m\rangle &= \hbar\sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\langle j_1, j_2;\ m_1 \mp 1, m_2 \mid j,\ m\rangle \\ &\quad + \hbar\sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}\langle j_1, j_2;\ m_1, m_2 \mp 1 \mid j,\ m\rangle. \end{aligned} \tag{7.149}$$

[6]Recall that $\langle j_1, j_2;\ m_1, m_2|\hat{J}_{1\pm} = \hbar\sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\langle j_1, j_2;\ m_1 \mp 1, m_2|$.

Equating (7.148) and (7.149) we obtain the desired recursion relations for the Clebsch–Gordan coefficients:

$$\boxed{\begin{aligned} &\sqrt{(j \mp m)(j \pm m + 1)}\ \langle j_1, j_2;\ m_1, m_2 \mid j,\ m \pm 1\rangle \\ &\qquad = \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\langle j_1, j_2; m_1 \mp 1, m_2 \mid j,\ m\rangle \\ &\qquad\quad + \sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}\langle j_1, j_2;\ m_1, m_2 \mp 1 \mid j,\ m\rangle. \end{aligned}} \tag{7.150}$$

These relations, together with the orthonormalization relation (7.125), determine all Clebsch–Gordan coefficients for any given values of j_1, j_2, and j. To see this, let us substitute $m_1 = j_1$ and $m = j$ into the lower part of (7.150). Since m_2 can be equal only to $m_2 = j - j_1 - 1$, we obtain

$$\begin{aligned} \sqrt{2j}\langle j_1, j_2;\ j_1, (j - j_1 - 1)|j,\ j - 1\rangle\ =\ &\sqrt{(j_2 - j + j_1 + 1)(j_2 + j - j_1)} \\ &\times\langle j_1, j_2;\ j_1, (j - j_1)|j,\ j\rangle. \end{aligned} \tag{7.151}$$

Thus, knowing $\langle j_1, j_2;\ j_1, (j - j_1)|j,\ j\rangle$, we can determine $\langle j_1, j_2;\ j_1, (j - j_1 - 1)|j,\ j - 1\rangle$. In addition, substituting $m_1 = j_1$, $m = j - 1$ and $m_2 = j - j_1$ into the upper part of (7.150), we end up with

$$\begin{aligned} \sqrt{2j}\ \langle j_1, j_2;\ j_1, (j - j_1) \mid j,\ j\rangle &= \sqrt{2j_1}\langle j_1, j_2;\ (j_1 - 1), (j - j_1) \mid j,\ j - 1\rangle \\ &+ \sqrt{(j_2 + j - j_1)(j_2 - j + j_1 + 1)}\langle j_1, j_2;\ j_1, (j - j_1 - 1) \mid j,\ j - 1\rangle. \end{aligned} \tag{7.152}$$

Thus knowing $\langle j_1, j_2;\ j_1, (j - j_1) \mid j,\ j\rangle$ and $\langle j_1, j_2;\ j_1, (j - j_1 - 1) \mid j,\ j - 1\rangle$, we can determine $\langle j_1, j_2;\ (j_1 - 1), (j - j_1) \mid j,\ j - 1\rangle$. Repeated application of the recursion relation (7.150) will determine all the other Clebsch–Gordan coefficients, provided we know only one of them: $\langle j_1, j_2;\ j_1, (j - j_1) \mid j,\ j\rangle$. As for the absolute value of this coefficient, it can be determined from the normalization condition (7.124). Thus, the recursion relation (7.150), in conjunction with the normalization condition (7.124), determines all the Clebsch–Gordan coefficients except for a sign. But how does one determine this sign?

The convention, known as the phase convention, is to consider $\langle j_1, j_2;\ j_1, (j - j_1)|j,\ j\rangle$ to be *real* and *positive*. This phase convention implies that

$$\boxed{\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle = (-1)^{j-j_1-j_2}\langle j_2, j_1;\ m_2, m_1 \mid j,\ m\rangle;} \tag{7.153}$$

hence

$$\boxed{\begin{aligned} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle &= (-1)^{j-j_1-j_2}\langle j_1, j_2;\ -m_1,\ -m_2|j,\ -m\rangle \\ &= \langle j_2, j_1;\ -m_2,\ -m_1|j,\ -m\rangle. \end{aligned}} \tag{7.154}$$

Note that, since all the Clebsch–Gordan coefficients are obtained from a single coefficient $\langle j_1, j_2;\ j_1, (j - j_1)|j,\ j\rangle$, and since this coefficient is real, all other Clebsch–Gordan coefficients must also be real numbers.

Following the same method that led to (7.150) from $\langle j_1, j_2;\ m_1, m_2 \mid J_\pm \mid j,\ m\rangle$, we can show that a calculation of $\langle j_1, j_2;\ m_1, m_2 \mid J_\pm | j\ m \mp 1\rangle$ leads to the following recursion relation:

$$\boxed{\begin{aligned}&\sqrt{(j \mp m + 1)(j \pm m)}\ \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \\ &\qquad = \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\langle j_1, j_2;\ m_1 \mp 1,\ m_2 | j,\ m \mp 1\rangle \\ &\qquad\quad + \sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}\langle j_1, j_2;\ m_1, m_2 \mp 1 | j,\ m \mp 1\rangle.\end{aligned}} \tag{7.155}$$

We can use the recursion relations (7.150) and (7.155) to obtain the values of the various Clebsch–Gordan coefficients. For instance, if we insert $m_1 = j_1$, $m_2 = j_2 - 1$, $j = j_1 + j_2$, and $m = j_1 + j_2$ into the lower sign of (7.150), we obtain

$$\langle j_1, j_2;\ j_1, (j_2 - 1) | (j_1 + j_2),\ (j_1 + j_2 - 1)\rangle = \sqrt{\frac{j_2}{j_1 + j_2}}. \tag{7.156}$$

Similarly, a substitution of $m_1 = j_1 - 1$, $m_2 = j_2$, $j = j_1 + j_2$, and $m = j_1 + j_2$ into the lower sign of (7.150) leads to

$$\langle j_1, j_2;\ (j_1 - 1), j_2 | (j_1 + j_2),\ (j_1 + j_2 - 1)\rangle = \sqrt{\frac{j_1}{j_1 + j_2}}. \tag{7.157}$$

We can also show that

$$\langle j, 1;\ m, 0 \mid j,\ m\rangle = \frac{m}{\sqrt{j(j+1)}}, \qquad \langle j, 0;\ m, 0 \mid j,\ m\rangle = 1. \tag{7.158}$$

Example 7.3

(a) Find the Clebsch–Gordan coefficients associated with the coupling of the spins of the electron and the proton of a hydrogen atom in its ground state.

(b) Find the transformation matrix which is formed by the Clebsch–Gordan coefficients. Verify that this matrix is unitary.

Solution

In their ground states the proton and electron have no orbital angular momenta. Thus, the total angular momentum of the atom is obtained by simply adding the spins of the proton and electron.

This is a simple example to illustrate the general formalism outlined in this section. Since $j_1 = \frac{1}{2}$ and $j_2 = \frac{1}{2}$, j has two possible values $j = 0,\ 1$. When $j = 0$, there is only a single state $| j,\ m\rangle = | 0,\ 0\rangle$; this is called the *spin singlet*. On the other hand, there are three possible values of $m = -1,\ 0,\ 1$ for the case $j = 1$; this corresponds to a *spin triplet state* $| 1,\ -1\rangle$, $| 1,\ 0\rangle$, $| 1,\ 1\rangle$.

From (7.121), we can express the states $| j,\ m\rangle$ in terms of $| \frac{1}{2}, \frac{1}{2};\ m_1, m_2\rangle$ as follows:

$$| j,\ m\rangle = \sum_{m_1=-1/2}^{1/2} \sum_{m_2=-1/2}^{1/2} \langle \frac{1}{2}, \frac{1}{2};\ m_1, m_2 \mid j,\ m\rangle \mid \frac{1}{2}, \frac{1}{2};\ m_1, m_2\rangle, \tag{7.159}$$

which, when applied to the two cases $j = 0$ and $j = 1$, leads to

$$| \, 0, \; 0\rangle \; = \; \langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle \left|\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2}\right\rangle + \; \langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 0, \; 0\rangle \left|\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2}\right\rangle, \tag{7.160}$$

$$| \, 1, \; 1\rangle \; = \; \langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, \frac{1}{2} \mid 1, \; 1\rangle \left|\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, \frac{1}{2}\right\rangle, \tag{7.161}$$

$$| \, 1, \; 0\rangle \; = \; \langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 1, \; 0\rangle \left|\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2}\right\rangle + \; \langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 1, \; 0\rangle \left|\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2}\right\rangle, \tag{7.162}$$

$$| \, 1, \; -1\rangle \; = \; \langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, -\frac{1}{2} \mid 1, \; -1\rangle \left|\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, -\frac{1}{2}\right\rangle. \tag{7.163}$$

To calculate the Clebsch–Gordan coefficients involved in (7.160)–(7.163), we are going to adopt two separate approaches: the first approach uses the recursion relations (7.150) and (7.155), while the second uses the algebra of angular momentum.

First approach: using the recursion relations

First, to calculate the two coefficients $\langle\frac{1}{2}, \frac{1}{2}; \; \pm\frac{1}{2}, \mp\frac{1}{2} \mid 0, \; 0\rangle$ involved in (7.160), we need, on the one hand, to substitute $j = 0, m = 0, m_1 = m_2 = \frac{1}{2}$ into the upper sign relation of (7.150):

$$\langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 0, \; 0\rangle = -\langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle. \tag{7.164}$$

On the other hand, the substitution of $j = 0$ and $m = 0$ into (7.125) yields

$$\langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 0, \; 0\rangle^2 + \langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle^2 = 1 \tag{7.165}$$

Combining (7.164) and (7.165) we end up with

$$\langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle = \pm\frac{1}{\sqrt{2}}. \tag{7.166}$$

The sign of $\langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2}|0, \; 0\rangle$ has to be positive because, according to the phase convention, the coefficient $\langle j_1, j_2; \; j_1, (j - j_1) \mid j, \; j\rangle$ is positive; hence

$$\langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle = \frac{1}{\sqrt{2}}. \tag{7.167}$$

As for $\langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 0, \; 0\rangle$, its value can be inferred from (7.164) and (7.167):

$$\langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 0, \; 0\rangle = -\frac{1}{\sqrt{2}}. \tag{7.168}$$

Second, the calculation of the coefficients involved in (7.161) to (7.163) goes as follows. The orthonormalization relation (7.125) leads to

$$\langle\frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, \frac{1}{2} \mid 1, \; 1\rangle^2 = 1, \qquad \langle\frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, -\frac{1}{2} \mid 1, \; -1\rangle^2 = 1, \tag{7.169}$$

and since $\langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2} \mid 1,\ 1\rangle$ and $\langle \frac{1}{2}, \frac{1}{2};\ -\frac{1}{2}, -\frac{1}{2} \mid 1,\ -1\rangle$ are both real and positive, we have

$$\langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2} \mid 1,\ 1\rangle = 1, \qquad \langle \frac{1}{2}, \frac{1}{2}; -\frac{1}{2} - \frac{1}{2} \mid 1,\ -1\rangle = 1. \tag{7.170}$$

As for the coefficients $\langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, -\frac{1}{2} \mid 1,\ 0\rangle$ and $\langle \frac{1}{2}, \frac{1}{2};\ -\frac{1}{2}, \frac{1}{2} \mid 1,\ 0\rangle$, they can be extracted by setting $j = 1, m = 0, m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}$ and $j = 1, m = 0, m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}$, respectively, into the lower sign case of (7.155):

$$\sqrt{2}\,\langle \frac{1}{2}, \frac{1}{2};\ -\frac{1}{2}, \frac{1}{2} \mid 1,\ 0\rangle = \langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2} \mid 1,\ 1\rangle, \tag{7.171}$$

$$\sqrt{2}\,\langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, -\frac{1}{2} \mid 1,\ 0\rangle = \langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2} \mid 1,\ 1\rangle. \tag{7.172}$$

Combining (7.170) with (7.171) and (7.172), we find

$$\langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, -\frac{1}{2} \mid 1,\ 0\rangle = \langle \frac{1}{2}, \frac{1}{2};\ -\frac{1}{2}, \frac{1}{2} \mid 1,\ 0\rangle = \frac{1}{\sqrt{2}}. \tag{7.173}$$

Finally, substituting the Clebsch–Gordan coefficients (7.167), (7.168) into (7.160) and (7.170), and substituting (7.173) into (7.161) to (7.163), we end up with

$$\mid 0,\ 0\rangle \;=\; -\frac{1}{\sqrt{2}}\left|\frac{1}{2}, \frac{1}{2};\ -\frac{1}{2}, \frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, -\frac{1}{2}\right\rangle, \tag{7.174}$$

$$\mid 1,\ 1\rangle \;=\; \left|\frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2}\right\rangle, \tag{7.175}$$

$$\mid 1,\ 0\rangle \;=\; \frac{1}{\sqrt{2}}\left|\frac{1}{2}, \frac{1}{2};\ -\frac{1}{2}, \frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, -\frac{1}{2}\right\rangle, \tag{7.176}$$

$$\mid 1,\ -1\rangle \;=\; \left|\frac{1}{2}, \frac{1}{2};\ -\frac{1}{2}, -\frac{1}{2}\right\rangle. \tag{7.177}$$

Note that the singlet state $|0,\ 0\rangle$ is antisymmetric, whereas the triplet states $|1,\ -1\rangle$, $|1,\ 0\rangle$, and $|1,\ 1\rangle$ are symmetric.

Second approach: using angular momentum algebra

Beginning with $j = 1$, and since $\mid 1,\ 1\rangle$ and $\mid \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2}\rangle$ are both normalized, equation (7.161) leads to

$$\langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2} \mid 1,\ 1\rangle^2 = 1. \tag{7.178}$$

From the phase convention, which states that $\langle j_1, j_2;\ j, (j - j_1)|j,\ j\rangle$ must be positive, we see that $\langle \frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2} \mid 1,\ 1\rangle = 1$, and hence

$$\mid 1,\ 1\rangle = \left|\frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2}\right\rangle. \tag{7.179}$$

Now, to find the Clebsch–Gordan coefficients in $\mid 1,\ 0\rangle$, we simply apply $\hat{J}_-$ on $\mid 1,\ 1\rangle$:

$$\hat{J}_- \mid 1,\ 1\rangle = (\hat{J}_{1-} + \hat{J}_{2-})\left|\frac{1}{2}, \frac{1}{2};\ \frac{1}{2}, \frac{1}{2}\right\rangle, \tag{7.180}$$

which leads to

$$| \, 1, \; 0\rangle = \frac{1}{\sqrt{2}} \left| \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \right\rangle + \frac{1}{\sqrt{2}} \left| \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \right\rangle, \tag{7.181}$$

hence $\langle \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 1, \; 0\rangle = 1/\sqrt{2}$ and $\langle \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 1, \; 0\rangle = 1/\sqrt{2}$. Next, applying $\hat{J}_-$ on (7.181), we get

$$| \, 1, \; -1\rangle = \left| \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, -\frac{1}{2} \right\rangle. \tag{7.182}$$

Finally, to find $| \, 0, \; 0\rangle$, we proceed in two steps: first, since

$$| \, 0, \; 0\rangle = a \left| \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \right\rangle + b \left| \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \right\rangle, \tag{7.183}$$

where $a = \langle \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 0, \; 0\rangle$ and $b = \langle \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle$, a combination of (7.181) with (7.183) leads to

$$\langle 0, \; 0 \mid 1, \; 0\rangle = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 0; \tag{7.184}$$

second, since $| \, 0, \; 0\rangle$ is normalized, we have

$$\langle 0, \; 0 \mid 0, \; 0\rangle = a^2 + b^2 = 1. \tag{7.185}$$

Combining (7.184) and (7.185), and since $\langle \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle$ must be positive, we obtain $a = \langle \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \mid 0, \; 0\rangle = -1/\sqrt{2}$ and $b = \langle \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \mid 0, \; 0\rangle = 1/\sqrt{2}$. Inserting these values into (7.183) we obtain

$$| \, 0, \; 0\rangle = -\frac{1}{\sqrt{2}} \left| \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \frac{1}{2} \right\rangle + \frac{1}{\sqrt{2}} \left| \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2} \right\rangle. \tag{7.186}$$

(b) Writing (7.174) to (7.177) in a matrix form:

$$\begin{pmatrix} |0, \; 0\rangle \\ |1, \; 1\rangle \\ |1, \; 0\rangle \\ |1, -1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} | \, \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, \; \frac{1}{2}\rangle \\ | \, \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2}\rangle \\ | \, \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \; \frac{1}{2}\rangle \\ | \, \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, -\frac{1}{2}\rangle \end{pmatrix}, \tag{7.187}$$

we see that the elements of the transformation matrix

$$U = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.188}$$

which connects the $\{| \, j, \; m\rangle\}$ vectors to their $\{| \, j_1, j_2; \; m_1, m_2\rangle\}$ counterparts, are given by the Clebsch–Gordan coefficients derived above. Inverting (7.187) we obtain

$$\begin{pmatrix} | \, \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, \; \frac{1}{2}\rangle \\ | \, \frac{1}{2}, \frac{1}{2}; \; \frac{1}{2}, -\frac{1}{2}\rangle \\ | \, \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, \; \frac{1}{2}\rangle \\ | \, \frac{1}{2}, \frac{1}{2}; \; -\frac{1}{2}, -\frac{1}{2}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |0, \; 0\rangle \\ |1, \; 1\rangle \\ |1, \; 0\rangle \\ |1, -1\rangle \end{pmatrix}. \tag{7.189}$$

From (7.187) and (7.189) we see that the transformation matrix U is unitary; this is expected since $U^{-1} = U^{\dagger}$.

7.3.3 Coupling of Orbital and Spin Angular Momenta

We consider here an important application of the formalism of angular momenta addition to the coupling of an orbital and a spin angular momentum: $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$. In particular, we want to find Clebsch–Gordan coefficients associated with this coupling for a spin $s = \frac{1}{2}$ particle. In this case we have: $j_1 = l$ (integer), $m_1 = m_l$, $j_2 = s = \frac{1}{2}$, and $m_2 = m_s = \pm\frac{1}{2}$. The allowed values of j as given by (7.130) are located within the interval $|l - \frac{1}{2}| \le j \le |l + \frac{1}{2}|$. If $l = 0$ the problem would be obvious: the particle would have only spin and no orbital angular momentum. But if $l > 0$ then j can take only two possible values $j = l \pm \frac{1}{2}$. There are $2(l+1)$ states $\{| \, l + \frac{1}{2}, m\rangle\}$ corresponding to the case $j = l + 1/2$ and $2l$ states $\{| \, l - \frac{1}{2}, m\rangle\}$ corresponding to $j = l - \frac{1}{2}$. Let us study in detail each one of these two cases.

Case $j = l + 1/2$

Applying the relation (7.121) to the case where $j = l + \frac{1}{2}$, we have

$$\begin{aligned}\left|l+\frac{1}{2}, m\right\rangle &= \sum_{m_l=-l}^{l} \sum_{m_2=-1/2}^{1/2} \left\langle l, \frac{1}{2};\ m_l, m_2 \left| l + \frac{1}{2},\ m \right\rangle \right| l, \frac{1}{2};\ m_l, m_2 \right\rangle \\ &= \sum_{m_l} \left\langle l, \frac{1}{2};\ m_l, -\frac{1}{2} \left| l + \frac{1}{2},\ m \right\rangle \right| l, \frac{1}{2};\ m_l, -\frac{1}{2} \right\rangle \\ &\quad + \sum_{m_l} \left\langle l, \frac{1}{2};\ m_l, \frac{1}{2} \left| l + \frac{1}{2},\ m \right\rangle \right| l, \frac{1}{2};\ m_l, \frac{1}{2} \right\rangle. \end{aligned} \tag{7.190}$$

Using the selection rule $m_l + m_2 = m$ or $m_l = m - m_2$, we can rewrite (7.190) as follows:

$$\begin{aligned}\left|l+\frac{1}{2}, m\right\rangle &= \left\langle l, \frac{1}{2};\ m + \frac{1}{2}, -\frac{1}{2} \left| l + \frac{1}{2}, m \right\rangle \right| l, \frac{1}{2};\ m + \frac{1}{2}, -\frac{1}{2} \right\rangle \\ &\quad + \left\langle l, \frac{1}{2};\ m - \frac{1}{2}, \frac{1}{2} \left| l + \frac{1}{2}, m \right\rangle \right| l, \frac{1}{2};\ m - \frac{1}{2}, \frac{1}{2} \right\rangle. \end{aligned} \tag{7.191}$$

We need now to calculate $\langle l, \frac{1}{2};\ m + \frac{1}{2}, -\frac{1}{2} | l + \frac{1}{2},\ m\rangle$ and $\langle l, \frac{1}{2};\ m - \frac{1}{2}, \frac{1}{2} | l + \frac{1}{2},\ m\rangle$. We begin with the calculation of $\langle l, \frac{1}{2};\ m + \frac{1}{2}, -\frac{1}{2} | l + \frac{1}{2},\ m\rangle$. Substituting $j = l + \frac{1}{2}$, $j_1 = l$, $j_2 = \frac{1}{2}$, $m_1 + m = \frac{1}{2}$, $m_2 = -\frac{1}{2}$ into the upper sign case of (7.155), we obtain

$$\begin{aligned}&\sqrt{\left(l - m + \frac{3}{2}\right)\left(l + m + \frac{1}{2}\right)} \left\langle l, \frac{1}{2};\ m + \frac{1}{2}, -\frac{1}{2} \middle| l + \frac{1}{2}, m \right\rangle \\ &\quad = \sqrt{\left(l + m + \frac{1}{2}\right)\left(l - m + \frac{1}{2}\right)} \left\langle l, \frac{1}{2};\ m - \frac{1}{2}, -\frac{1}{2} \middle| l + \frac{1}{2}, m - 1 \right\rangle \end{aligned} \tag{7.192}$$

or

$$\left\langle l, \frac{1}{2};\ m+\frac{1}{2}, -\frac{1}{2} \middle| l + \frac{1}{2},\ m \right\rangle = \sqrt{\frac{l - m + 1/2}{l - m + 3/2}} \left\langle l, \frac{1}{2};\ m-\frac{1}{2}, -\frac{1}{2} \middle| l + \frac{1}{2},\ m - 1 \right\rangle. \tag{7.193}$$

By analogy with $\langle l, \frac{1}{2};\ m+\frac{1}{2}, -\frac{1}{2}|l+\frac{1}{2},\ m\rangle$ we can express the Clebsch–Gordan coefficient $\langle l, \frac{1}{2};\ m+\frac{1}{2}, -\frac{1}{2}|l+\frac{1}{2},\ m-1\rangle$ in terms of $\langle l, \frac{1}{2};\ m-\frac{3}{2}, -\frac{1}{2}|l+\frac{1}{2},\ m-2\rangle$:

$$\left\langle l, \frac{1}{2};\ \ m+\frac{1}{2}, -\frac{1}{2}\right|l+\frac{1}{2},\ m\Big\rangle = \sqrt{\frac{l-m+1/2}{l-m+3/2}}\sqrt{\frac{l-m+3/2}{l-m+5/2}} \times \left\langle l, \frac{1}{2};\ m-\frac{3}{2}, -\frac{1}{2}\right|l+\frac{1}{2}, m-2\Big\rangle. \tag{7.194}$$

We can continue this procedure until m reaches its lowest values, $-l-\frac{1}{2}$:

$$\left\langle l, \frac{1}{2};\ \ m+\frac{1}{2}, -\frac{1}{2}\right|l+\frac{1}{2}, m\Big\rangle = \sqrt{\frac{l-m+1/2}{l-m+3/2}}\sqrt{\frac{l-m+3/2}{l-m+5/2}} \times\cdots\times\sqrt{\frac{2l}{2l+1}}\left\langle l, \frac{1}{2};\ \ -l, -\frac{1}{2}\right|+\frac{1}{2}, -l-\frac{1}{2}\Big\rangle, \tag{7.195}$$

or

$$\left\langle l, \frac{1}{2};\ \ m+\frac{1}{2}, -\frac{1}{2}\right|l+\frac{1}{2},\ m\Big\rangle = \sqrt{\frac{l-m+1/2}{2l+1}}\left\langle l, \frac{1}{2};\ \ -l, -\frac{1}{2}\right|l+\frac{1}{2},\ -l-\frac{1}{2}\Big\rangle. \tag{7.196}$$

From (7.125) we can easily obtain $\left\langle l, -\frac{1}{2};\ \ -l, -\frac{1}{2}\right|l+\frac{1}{2}, -l-\frac{1}{2}\Big\rangle^2 = 1$, and since this coefficient is real we have $\left\langle l, -\frac{1}{2};\ \ -l, -\frac{1}{2}\right|l+\frac{1}{2}, -l-\frac{1}{2}\Big\rangle = 1$. Inserting this value into (7.196) we end up with

$$\boxed{\left\langle l, \frac{1}{2};\ \ m+\frac{1}{2}, -\frac{1}{2}\right|l+\frac{1}{2},\ m\Big\rangle = \sqrt{\frac{l-m+1/2}{2l+1}}.} \tag{7.197}$$

Now we turn to the calculation of the second coefficient, $\langle l, \frac{1}{2};\ m-\frac{1}{2}, \frac{1}{2}|l+\frac{1}{2}\ m\rangle$, involved in (7.191). We can perform this calculation in two different ways. The first method consists of following the same procedure adopted above to find $\langle l, \frac{1}{2};\ m+\frac{1}{2}, -\frac{1}{2}|l+\frac{1}{2}, m\rangle$. For this, we need only to substitute $j=l+\frac{1}{2}, j_1=l, j_2=\frac{1}{2}, m_1=m-\frac{1}{2}, m_2=\frac{1}{2}$ in the lower sign case of (7.155) and work our way through. A second, simpler method consists of substituting (7.197) into (7.191) and then calculating the norm of the resulting equation:

$$1 = \frac{l-m+1/2}{2l+1} + \left\langle l, \frac{1}{2};\ \ m-\frac{1}{2}, \frac{1}{2}\right|l+\frac{1}{2},\ m\Big\rangle^2, \tag{7.198}$$

where we have used the facts that the three kets $|l+\frac{1}{2}, m\rangle$ and $\left|l, \frac{1}{2};\ m\pm\frac{1}{2}, \mp\frac{1}{2}\right\rangle$ are normalized. Again, since $\langle l, \frac{1}{2};\ m-\frac{1}{2}, \frac{1}{2}|l+\frac{1}{2}, m\rangle$ is real, (7.198) leads to

$$\boxed{\left\langle l, \frac{1}{2};\ \ m-\frac{1}{2}, \frac{1}{2}\right|l+\frac{1}{2},\ m\Big\rangle = \sqrt{\frac{l+m+1/2}{2l+1}}.} \tag{7.199}$$

A combination of (7.191), (7.197), and (7.199) yields

$$\left|l+\frac{1}{2},\ m\right\rangle = \sqrt{\frac{l-m+1/2}{2l+1}}\left|l,\frac{1}{2};\ m+\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{l+m+1/2}{2l+1}}\left|l,\frac{1}{2};\ m-\frac{1}{2},\frac{1}{2}\right\rangle, \tag{7.200}$$

where the possible values of m are given by

$$m = -l-\frac{1}{2}, -l+\frac{1}{2}, -l+\frac{3}{2}, \ldots, l-\frac{3}{2}, l-\frac{1}{2}, l+\frac{1}{2}. \tag{7.201}$$

Case $j = l - 1/2$

There are $2l$ states, $\{|l-\frac{1}{2},\ m\rangle\}$, corresponding to $j = l-\frac{1}{2}$; these are $\left|l-\frac{1}{2},\ -l+\frac{1}{2}\right\rangle$, $\left|l-\frac{1}{2},\ -l+\frac{3}{2}\right\rangle, \ldots, \left|l-\frac{1}{2},\ l-\frac{1}{2}\right\rangle$. Using (7.121) we write any state $\left|l-\frac{1}{2}, m\right\rangle$ as

$$\begin{aligned}\left|l-\frac{1}{2}, m\right\rangle = &\left\langle l,\frac{1}{2};\ m+\frac{1}{2},-\frac{1}{2}\right|\left. l-\frac{1}{2},\ m\right\rangle\left|l,\frac{1}{2};\ m+\frac{1}{2};-\frac{1}{2}\right\rangle \\ &+\left\langle l,\frac{1}{2};\ m-\frac{1}{2},\frac{1}{2}\right|\left. l-\frac{1}{2},\ m\right\rangle\left|l,\frac{1}{2};\ m-\frac{1}{2},\frac{1}{2}\right\rangle.\end{aligned} \tag{7.202}$$

The two Clebsch–Gordan coefficients involved in this equation can be calculated by following the same method that we adopted above for the case $j = l+\frac{1}{2}$. Thus, we can ascertain that $|l-\frac{1}{2}, m\rangle$ is given by

$$\left|l-\frac{1}{2},\ m\right\rangle = \sqrt{\frac{l+m+1/2}{2l+1}}\left|l,\frac{1}{2};\ m+\frac{1}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{l-m+1/2}{2l+1}}\left|l,\frac{1}{2};\ m-\frac{1}{2},\frac{1}{2}\right\rangle, \tag{7.203}$$

where

$$m = -l+\frac{1}{2}, -l+\frac{3}{2},\ \ldots, l-\frac{3}{2},\ l-\frac{1}{2}. \tag{7.204}$$

We can combine (7.200) and (7.203) into

$$\boxed{\left|l\pm\frac{1}{2},\ m\right\rangle = \sqrt{\frac{l\mp m+\frac{1}{2}}{2l+1}}\left|l,\frac{1}{2};\ m+\frac{1}{2},-\frac{1}{2}\right\rangle \pm \sqrt{\frac{l\pm m+\frac{1}{2}}{2l+1}}\left|l,\frac{1}{2};\ m-\frac{1}{2},\frac{1}{2}\right\rangle.} \tag{7.205}$$

Illustration on a particle with $l = 1$

As an illustration of the formalism worked out above, we consider the particular case of $l = 1$. Inserting $l = 1$ and $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ into the upper sign of (7.205), we obtain

$$\left|\frac{3}{2},\ \frac{3}{2}\right\rangle = \left|1,\frac{1}{2};\ 1,\frac{1}{2}\right\rangle, \tag{7.206}$$

$$\left|\frac{3}{2},\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1,\frac{1}{2};\ 0,\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|1,\frac{1}{2};\ 1,-\frac{1}{2}\right\rangle, \tag{7.207}$$

$$\left|\frac{3}{2},\ -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1,\frac{1}{2};\ -1,\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|1,\frac{1}{2};\ 0,-\frac{1}{2}\right\rangle, \tag{7.208}$$

$$\left|\frac{3}{2},\ -\frac{3}{2}\right\rangle = \left|1,\frac{1}{2};\ -1,-\frac{1}{2}\right\rangle. \tag{7.209}$$

Similarly, an insertion of $l = 1$ and $m = \frac{1}{2}, -\frac{1}{2}$ into the lower sign of (7.205) yields

$$\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|1, \frac{1}{2};\ 1, -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|1, \frac{1}{2};\ 0, \frac{1}{2}\right\rangle, \tag{7.210}$$

$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|1, \frac{1}{2};\ 0, -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|1, \frac{1}{2};\ -1, \frac{1}{2}\right\rangle. \tag{7.211}$$

Spin–orbit functions

The eigenfunctions of the particle's total angular momentum $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ may be represented by the direct product of the eigenstates of the orbital and spin angular momenta, $\left|l, m - \frac{1}{2}\right\rangle$ and $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$. From (7.205) we have

$$\left|l \pm \frac{1}{2}, m\right\rangle = \sqrt{\frac{l \mp m + \frac{1}{2}}{2l+1}}\left|l, m + \frac{1}{2}\right\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \pm \sqrt{\frac{l \pm m + \frac{1}{2}}{2l+1}}\left|l, m - \frac{1}{2}\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle. \tag{7.212}$$

If this particle moves in a central potential, its complete wave function consists of a space part, $\langle r\theta\varphi | n, l, m \pm \frac{1}{2}\rangle = R_{nl}(r) Y_{l,m\pm\frac{1}{2}}$, and a spin part, $\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$:

$$\Psi_{n,l,j=l\pm\frac{1}{2},m} = R_{nl}(r)\left[\sqrt{\frac{l \mp m + \frac{1}{2}}{2l+1}}Y_{l,m+\frac{1}{2}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \pm \sqrt{\frac{l \pm m + \frac{1}{2}}{2l+1}}Y_{l,m-\frac{1}{2}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle\right], \tag{7.213}$$

Using the spinor representation for the spin part, $\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can write (7.213) as follows:

$$\Psi_{n,l,j=l\pm\frac{1}{2},m}(r, \theta, \varphi) = \frac{R_{nl}(r)}{\sqrt{2l+1}}\begin{pmatrix} \pm\sqrt{l \pm m + \frac{1}{2}}\,Y_{l,m-\frac{1}{2}}(\theta, \varphi) \\ \sqrt{l \mp m + \frac{1}{2}}\,Y_{l,m+\frac{1}{2}}(\theta, \varphi) \end{pmatrix}, \tag{7.214}$$

where m is half-integer. The states (7.213) and (7.214) are simultaneous eigenfunctions of $\hat{\vec{J}}^2$, $\hat{\vec{L}}^2$, $\hat{\vec{S}}^2$, and $\hat{J}_z$ with eigenvalues $\hbar^2 j(j+1)$, $\hbar^2 l(l+1)$, $\hbar^2 s(s+1) = 3\hbar^2/4$, and $\hbar m$, respectively. The wave functions $\Psi_{n,l,j=l\pm\frac{1}{2},m}(r, \theta, \varphi)$ are eigenstates of $\hat{\vec{L}} \cdot \hat{\vec{S}}$ as well, since

$$\begin{aligned}\hat{\vec{L}} \cdot \hat{\vec{S}}|nljm\rangle &= \frac{1}{2}\left(\hat{\vec{J}}^2 - \hat{\vec{L}}^2 - \hat{\vec{S}}^2\right)|nljm\rangle \\ &= \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]|nljm\rangle.\end{aligned} \tag{7.215}$$

Here j takes only two values, $j = l \pm \frac{1}{2}$, so we have

$$\langle nljm|\hat{\vec{L}} \cdot \hat{\vec{S}}|nljm\rangle = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - \frac{3}{4}\right] = \begin{cases} \frac{1}{2}l\hbar^2, & j = l + \frac{1}{2}, \\ -\frac{1}{2}(l+1)\hbar^2, & j = l - \frac{1}{2}. \end{cases} \tag{7.216}$$

7.3.4 Addition of More Than Two Angular Momenta

The formalism for adding two angular momenta may be generalized to those cases where we add three or more angular momenta. For instance, to add three mutually commuting angular momenta $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2 + \hat{\vec{J}}_3$, we may follow any of these three methods. (a) Add $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ to obtain $\hat{\vec{J}}_{12} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$, and then add $\hat{\vec{J}}_{12}$ to $\hat{\vec{J}}_3$: $\hat{\vec{J}} = \hat{\vec{J}}_{12} + \hat{\vec{J}}_3$. (b) Add $\hat{\vec{J}}_2$ and $\hat{\vec{J}}_3$ to form $\hat{\vec{J}}_{23} = \hat{\vec{J}}_2 + \hat{\vec{J}}_3$, and then add $\hat{\vec{J}}_{23}$ to $\hat{\vec{J}}_1$: $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_{23}$. (c) Add $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_3$ to form $\hat{\vec{J}}_{13} = \hat{\vec{J}}_1 + \hat{\vec{J}}_3$, and then add $\hat{\vec{J}}_{13}$ to $\hat{\vec{J}}_2$: $\hat{\vec{J}} = \hat{\vec{J}}_2 + \hat{\vec{J}}_{13}$.

Considering the first method and denoting the eigenstates of $\hat{\vec{J}}_1^2$ and $\hat{J}_{1_z}$ by $\mid j_1,\ m_1\rangle$, those of $\hat{\vec{J}}_2^2$, and $\hat{J}_{2_z}$ by $\mid j_2,\ m_2\rangle$, and those of $\hat{\vec{J}}_3^2$ and $\hat{J}_{3_z}$ by $\mid j_3,\ m_3\rangle$, we may express the joint eigenstates $|j_{12}, j, m\rangle$ of $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{\vec{J}}_3^2$, $\hat{\vec{J}}_{12}^2$, $\hat{\vec{J}}^2$ and J_z in terms of the states

$$|j_1, j_2, j_3;\ m_1, m_2, m_3\rangle = \mid j_1,\ m_1\rangle \mid j_2,\ m_2\rangle |j_3,\ m_3\rangle \tag{7.217}$$

as follows. First, the coupling of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ leads to

$$|j_{12},\ m_{12}\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1, j_2;\ m_1, m_2 \mid j_{12},\ m_{12}\rangle |j_1, j_2;\ m_1, m_2\rangle, \tag{7.218}$$

where $m_{12} = m_1 + m_2$ and $|j_1 - j_2| \le j_{12} \le |j_1 + j_2|$. Then, adding $\hat{\vec{J}}_{12}$ and $\hat{\vec{J}}_3$, the state $|j_{12}, j, m\rangle$ is given by

$$\sum_{m_{12}=-j_{12}}^{j_{12}} \sum_{m_3=-j_3}^{j_3} \langle j_1, j_2; m_1, m_2 \mid j_{12}, m_{12}\rangle \langle j_{12}, j_3; m_{12}, m_3 | j_{12}, j, m\rangle |j_1, j_2, j_3; m_1, m_2, m_3\rangle, \tag{7.219}$$

with $m = m_{12} + m_3$ and $|j_{12} - j_3| \le j \le |j_{12} + j_3|$; the Clebsch–Gordan coefficients $\langle j_1, j_2;\ m_1, m_2 | j_{12},\ m_{12}\rangle$ and $\langle j_{12}, j_3;\ m_{12}, m_3 | j_{12}, j, m\rangle$ correspond to the coupling of $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ and of $\hat{\vec{J}}_{12}$ and $\hat{\vec{J}}_3$, respectively. The calculation of these coefficients is similar to that of two angular momenta. For instance, in Problem 7.4, page 438, we will see how to add three spins and how to calculate the corresponding Clebsch–Gordan coefficients.

We should note that the addition of $\hat{\vec{J}}_1$, $\hat{\vec{J}}_2$, and $\hat{\vec{J}}_3$ in essence consists of constructing the eigenvectors $|j_{12}, j, m\rangle$ in terms of the $(2j_1+1)(2j_2+1)(2j_3+1)$ states $\mid j_1, j_2, j_3;\ m_1, m_2, m_3\rangle$. We may then write

$$\hat{J}_{\pm} |j_{12}, j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,|j_{12}, j, m \pm 1\rangle, \tag{7.220}$$

$$\hat{J}_{1_\pm} \mid j_1, j_2, j_3;\ m_1, m_2, m_3\rangle = \hbar\sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}\,|j_1, j_2, j_3;\ (m_1 \pm 1), m_2, m_3\rangle, \tag{7.221}$$

$$\hat{J}_{2_\pm} \mid j_1, j_2, j_3;\ m_1, m_2, m_3\rangle = \hbar\sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}\,|j_1, j_2, j_3;\ m_1, (m_2 \pm 1), m_3\rangle, \tag{7.222}$$

$$\hat{J}_{3_\pm} \mid j_1, j_2, j_3;\ m_1, m_2, m_3\rangle = \hbar\sqrt{j_3(j_3+1) - m_3(m_3 \pm 1)}\,|j_1, j_2, j_3;\ m_1, m_2, (m_3 \pm 1)\rangle. \tag{7.223}$$

The foregoing method can be generalized to the coupling of more than three angular momenta: $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2 + \hat{\vec{J}}_3 + \cdots + \hat{\vec{J}}_N$. Each time we couple two angular momenta, we reduce the problem to the coupling of $(N-1)$ angular momenta. For instance, we may start by adding $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ to generate $\hat{\vec{J}}_{12}$; we are then left with $(N-1)$ angular momenta. Second, by adding $\hat{\vec{J}}_{12}$ and $\hat{\vec{J}}_3$ to form $\hat{\vec{J}}_{123}$, we are left with $(N-2)$ angular momenta. Third, an addition of $\hat{\vec{J}}_{123}$ and $\hat{\vec{J}}_4$ leaves us with $(N-3)$ angular momenta, and so on. We may continue in this way till we add all given angular momenta.

7.3.5 Rotation Matrices for Coupling Two Angular Momenta

We want to find out how to express the rotation matrix associated with an angular momentum $\hat{\vec{J}}$ in terms of the rotation matrices corresponding to $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$ such that $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$. That is, knowing the rotation matrices $d^{(j_1)}(\beta)$ and $d^{(j_2)}(\beta)$, how does one calculate $d^{(j)}_{mm'}(\beta)$?

Since

$$d^{(j)}_{m'm}(\beta) = \langle j,\ m' \mid \hat{R}_y(\beta) \mid j,\ m\rangle, \tag{7.224}$$

where

$$\mid j,\ m\rangle \ = \ \sum_{m_1 m_2} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \mid j_1, j_2;\ m_1, m_2\rangle, \tag{7.225}$$

$$\mid j,\ m'\rangle \ = \ \sum_{m'_1 m'_2} \langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle |j_1, j_2;\ m'_1, m'_2\rangle, \tag{7.226}$$

and since the Clebsch–Gordan coefficients are real,

$$\langle j,\ m' \mid = \sum_{m'_1 m'_2} \langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle\langle j_1, j_2;\ m'_1, m'_2|, \tag{7.227}$$

we can rewrite (7.224) as

$$\begin{aligned} d^{(j)}_{m'm}(\beta) \ = \ & \sum_{m_1 m_2}\sum_{m'_1 m'_2} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle\langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle \\ & \times\langle j_1, j_2;\ m'_1, m'_2|\hat{R}_y(\beta) \mid j_1, j_2;\ m_1, m_2\rangle. \end{aligned} \tag{7.228}$$

Since $\hat{R}_y(\beta) = \exp[-\beta\hat{J}_y/\hbar] = \exp[-\beta\hat{J}_{1_y}/\hbar]\exp[-\beta\hat{J}_{2_y}/\hbar]$, because $\hat{J}_y = \hat{J}_{1_y} + \hat{J}_{2_y}$, and since $\langle j_1, j_2;\ m'_1, m'_2| = \langle j_1,\ m'_1 \mid \langle j_2,\ m'_2 \mid$ and $\mid j_1, j_2;\ m_1, m_2\rangle = \mid j_1,\ m_1\rangle \mid j_2,\ m_2\rangle$, we have

$$\begin{aligned} d^{(j)}_{m'm}(\beta) \ = \ & \sum_{m_1 m_2}\sum_{m'_1 m'_2} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle\langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle \\ & \times\langle j_1,\ m'_1 \mid \exp\left[-\frac{i}{\hbar}\beta\hat{J}_{1_y}\right] \mid j_1,\ m_1\rangle\langle j_2,\ m'_2 \mid \exp\left[-\frac{i}{\hbar}\beta\hat{J}_{2_y}\right] \mid j_2,\ m_2\rangle, \end{aligned} \tag{7.229}$$

or

$$\boxed{d^{(j)}_{m'm}(\beta) = \sum_{m_1 m_2}\sum_{m'_1 m'_2} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle\langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle d^{(j_1)}_{m'_1 m_1}(\beta) d^{(j_2)}_{m'_2 m_2}(\beta),} \tag{7.230}$$

with

$$d^{(j_1)}_{m'_1 m_1}(\beta) = \langle j_1,\ m'_1 \mid \exp\left[-\frac{i}{\hbar}\beta \hat{J}_{1_y}\right] \mid j_1,\ m_1\rangle, \tag{7.231}$$

$$d^{(j_2)}_{m'_2 m_2}(\beta) = \langle j_2,\ m'_2 \mid \exp\left[-\frac{i}{\hbar}\beta \hat{J}_{2_y}\right] \mid j_2,\ m_2\rangle. \tag{7.232}$$

From (7.54) we have

$$d^{(j)}_{m'm}(\beta) = e^{i(m'\alpha + m\gamma)} D^{(j)}_{m'm}(\alpha, \beta, \gamma); \tag{7.233}$$

hence can rewrite (7.230) as

$$\boxed{D^{(j)}_{m'm}(\alpha, \beta, \gamma) = \sum_{m_1 m_2}\sum_{m'_1 m'_2} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle D^{(j_1)}_{m'_1 m_1}(\alpha, \beta, \gamma) D^{(j_2)}_{m'_2 m_2}(\alpha, \beta, \gamma),} \tag{7.234}$$

since $m = m_1 + m'_1$ and $m' = m_2 + m'_2$.

Now, let us see how to express the product of the rotation matrices $d^{(j_1)}(\beta)$ and $d^{(j_2)}(\beta)$ in terms of $d^{(j)}_{mm'}(\beta)$. Sandwiching both sides of

$$\exp\left[-\frac{i}{\hbar}\beta \hat{J}_{1_y}\right] \exp\left[-\frac{i}{\hbar}\beta \hat{J}_{2_y}\right] = \exp\left[-\frac{i}{\hbar}\beta \hat{J}_y\right] \tag{7.235}$$

between

$$\mid j_1, j_2;\ m_1, m_2\rangle = \sum_{jm} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \mid j,\ m\rangle \tag{7.236}$$

and

$$\langle j_1, j_2;\ m'_1, m'_2 \mid = \sum_{jm'} \langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle \langle j,\ m' \mid, \tag{7.237}$$

and since $\langle j_1, j_2;\ m'_1, m'_2 \mid = \langle j_1,\ m'_1 \mid \langle j_2,\ m'_2 \mid$ and $\mid j_1, j_2;\ m_1, m_2\rangle = \mid j_1,\ m_1\rangle \mid j_2,\ m_2\rangle$, we have

$$\begin{aligned}\langle j_1,\ m'_1 \mid \exp\left[-\frac{i}{\hbar}\beta \hat{J}_{1_y}\right] \mid j_1,\ m_1\rangle \langle j_2,\ m'_2 \mid \exp\left[-\frac{i}{\hbar}\beta \hat{J}_{2_y}\right] \mid j_2,\ m_2\rangle \\ = \sum_{jmm'} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle \langle j,\ m' \mid \hat{R}_y(\beta) \mid j,\ m\rangle\end{aligned} \tag{7.238}$$

or

$$\boxed{d^{(j_1)}_{m'_1 m_1}(\beta) d^{(j_2)}_{m'_2 m_2}(\beta) = \sum_{|j_1 - j_2|}^{|j_1 + j_2|}\sum_{mm'} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle d^{(j)}_{m'm}(\beta).} \tag{7.239}$$

Following the same procedure that led to (7.234), we can rewrite (7.239) as

$$\boxed{D^{(j_1)}_{m'_1 m_1}(\alpha, \beta, \gamma) D^{(j_2)}_{m'_2 m_2}(\alpha, \beta, \gamma) = \sum_{jmm'} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle \langle j_1, j_2;\ m'_1, m'_2 \mid j,\ m'\rangle D^{(j)}_{m'm}(\alpha, \beta, \gamma).} \tag{7.240}$$

This relation is known as the *Clebsch–Gordan series*.

The relation (7.240) has an important application: the derivation of an integral involving three spherical harmonics. When j_1 and j_2 are both integers (i.e., $j_1 = l_1$ and $j_2 = l_2$) and m_1 and m_2 are both zero (hence $m = 0$), equation (7.240) finds a useful application:

$$D^{(l_1)}_{m'_1 0}(\alpha, \beta, \gamma) D^{(l_2)}_{m'_2 0}(\alpha, \beta, \gamma) = \sum_{lm'} \langle l_1, l_2;\ 0, 0 \mid l,\ 0\rangle \langle l_1, l_2;\ m'_1, m'_2 \mid l,\ m'\rangle D^{(l)}_{m'0}(\alpha, \beta, \gamma). \tag{7.241}$$

Since the expressions of $D^{(l_1)}_{m'_1 0}$, $D^{(l_2)}_{m'_2 0}$, and $D^{(l)}_{m'0}$ can be inferred from (7.73), notably

$$D^{(l)}_{m'0}(\alpha, \beta, 0) = \sqrt{\frac{4\pi}{2l+1}}\, Y^*_{lm'}(\beta, \alpha), \tag{7.242}$$

we can reduce (7.241) to

$$\boxed{Y_{l_1 m_1}(\beta, \alpha) Y_{l_2 m_2}(\beta, \alpha) = \sum_{lm} \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \langle l_1, l_2;\ 0, 0 \mid l,\ 0\rangle \langle l_1, l_2;\ m_1, m_2 \mid l,\ m\rangle Y_{lm}(\beta, \alpha),} \tag{7.243}$$

where we have removed the primes and taken the complex conjugate. Multiplying both sides by $Y^*_{lm}(\beta, \alpha)$ and integrating over α and β, we obtain the following frequently used integral:

$$\boxed{\begin{aligned} \int_0^{2\pi} d\alpha \int_0^{\pi} Y^*_{lm}(\beta, \alpha) Y_{l_1 m_1}(\beta, \alpha) Y_{l_2 m_2}(\beta, \alpha) \sin\beta \, d\beta = \sqrt{\tfrac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \langle l_1, l_2; 0, 0 \mid l,\ 0\rangle \\ \times \langle l_1, l_2;\ m_1, m_2 \mid l,\ m\rangle. \end{aligned}} \tag{7.244}$$

7.3.6 Isospin

The ideas presented above—spin and the addition of angular momenta—find some interesting applications to other physical quantities. For instance, in the field of nuclear physics, the quantity known as *isotopic spin* can be represented by a set of operators which not only obey the same algebra as the components of angular momentum, but also couple in the same way as ordinary angular momenta.

Since the nuclear force does not depend on the electric charge, we can consider the proton and the neutron to be separate manifestations (states) of the same particle, the *nucleon*. The nucleon may thus be found in two different states: a proton and a neutron. In this way, as the protons and neutrons are identical particles with respect to the nuclear force, we will need an additional quantum number (or label) to indicate whether the nucleon is a proton or a neutron. Due to its formal analogy with ordinary spin, this label is called the *isotopic spin* or, in short, the *isospin*. If we take the isospin quantum number to be $\frac{1}{2}$, its z-component will then be represented by a quantum number having the values $\frac{1}{2}$ and $-\frac{1}{2}$. The difference between a proton and a neutron then becomes analogous to the difference between spin-up and spin-down particles.

The fundamental difference between ordinary spin and the isospin is that, unlike the spin, the isospin has nothing to do with rotations or spinning in the coordinate space, it hence cannot be coupled with the angular momenta of the nucleons. Nucleons can thus be distinguished by $\langle \hat{t}_3 \rangle = \pm\frac{1}{2}$, where $\hat{t}_3$ is the third or z-component of the isospin vector operator $\hat{\vec{t}}$.

7.3.6.1 Isospin Algebra

Due to the formal analogy between the isospin and the spin, their formalisms have similar structures from a mathematical viewpoint. The algebra obeyed by the components $\hat{t}_1$, $\hat{t}_2$, $\hat{t}_3$ of the isospin operator $\hat{\vec{t}}$ can thus be inferred from the properties and commutation relations of the spin operator. For instance, the components of the isospin operator can be constructed from the Pauli matrices $\vec{\tau}$ in the same way as we did for the angular momentum operators of spin $\frac{1}{2}$ particles:

$$\hat{\vec{t}} = \frac{1}{2}\vec{\tau}, \tag{7.245}$$

with

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.246}$$

The components $\hat{t}_1$, $\hat{t}_2$, $\hat{t}_3$ obey the same commutation relations as those of angular momentum:

$$\boxed{\left[\hat{t}_1,\ \hat{t}_2\right] = i\hat{t}_3, \qquad \left[\hat{t}_2,\ \hat{t}_3\right] = i\hat{t}_1, \qquad \left[\hat{t}_3,\ \hat{t}_1\right] = i\hat{t}_2.} \tag{7.247}$$

So the nucleon can be found in two different states: when $\hat{t}_3$ acts on a nucleon state, it gives the eignvalues $\pm\frac{1}{2}$. By convention the $\hat{t}_3$ of a proton is taken to be $\hat{t}_3 = +\frac{1}{2}$ and that of a neutron is $\hat{t}_3 = -\frac{1}{2}$. Denoting the proton and neutron states, respectively, by $\mid p\rangle$ and $\mid n\rangle$,

$$\mid p\rangle = \left| t = \frac{1}{2}, t_3 = \frac{1}{2} \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mid n\rangle = \left| t = \frac{1}{2}, t_3 = -\frac{1}{2} \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{7.248}$$

we have

$$\hat{t}_3 \mid p\rangle = \hat{t}_3 \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \tag{7.249}$$

$$\hat{t}_3 \mid n\rangle = \hat{t}_3 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = -\frac{1}{2} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle. \tag{7.250}$$

We can write (7.249) and (7.250), respectively, as

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7.251}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7.252}$$

By analogy with angular momentum, denoting the joint eignstates of $\hat{\vec{t}}^2$ and $\hat{t}_3$ by $\mid t,\ t_3\rangle$, we have

$$\boxed{\hat{\vec{t}}^2 \mid t,\ t_3\rangle = t(t+1) \mid t,\ t_3\rangle, \qquad \hat{t}_3 \mid t,\ t_3\rangle = t_3 \mid t,\ t_3\rangle.} \tag{7.253}$$

We can also introduce the *raising and lowering* isospin operators:

$$\hat{t}_+ = \hat{t}_1 + i\hat{t}_2 = \frac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{7.254}$$

$$\hat{t}_- = \hat{t}_1 - i\hat{t}_2 = \frac{1}{2}(\tau_1 - i\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{7.255}$$

hence

$$\boxed{\hat{t}_\pm \mid t,\ t_3\rangle = \sqrt{t(t+1) - t_3(t_3 \pm 1)}\ \mid t,\ t_3 \pm 1\rangle.} \tag{7.256}$$

Note that $\hat{t}_+$ and $\hat{t}_-$ are operators which, when acting on a nucleon state, convert neutron states into proton states and proton states into neutron states, respectively:

$$\hat{t}_+ \mid n\rangle = \mid p\rangle, \qquad \hat{t}_- \mid p\rangle = \mid n\rangle. \tag{7.257}$$

We can also define a *charge operator*

$$\hat{Q} = e\left(\hat{t}_3 + \frac{1}{2}\right), \tag{7.258}$$

where e is the charge of the proton, with

$$\hat{Q} \mid p\rangle = e \mid p\rangle, \qquad \hat{Q} \mid n\rangle = 0. \tag{7.259}$$

We should mention that strong interactions conserve isospin. For instance, a reaction like

$$d + d \to \alpha + \pi^0 \tag{7.260}$$

is forbidden since the isospin is not conserved, because the isospins of d and α are both zero and the isospin of the pion is equal to one (i.e., $T(d) = T(\alpha) = 0$, but $T(\pi) = 1$); this leads to isospin zero for $(d + d)$ and isospin one for $(\alpha + \pi^0)$. The reaction was confirmed experimentally to be forbidden, since its cross-section is negligibly small. However, reactions such as

$$p + p \to d + \pi^+, \quad p + n \to d + \pi^0 \tag{7.261}$$

are allowed, since they conserve isospin.

7.3.6.2 Addition of Two Isospins

We should note that the isospins of different nucleons can be added in the same way as adding angular momenta. For a nucleus consisting of several nucleons, the total isospin is given by the vector sum of the isospins of all individual nucleons: $\hat{\vec{T}} = \sum_i^A \hat{\vec{t}}_i$. For instance, the total isospin of a system of two nucleons can be obtained by coupling their isospins $\hat{\vec{t}}_1$ and $\hat{\vec{t}}_2$:

$$\hat{\vec{T}} = \hat{\vec{t}}_1 + \hat{\vec{t}}_2. \tag{7.262}$$

Denoting the joint eigenstates of $\hat{\vec{t}}_1^{\,2}$, $\hat{\vec{t}}_2^{\,2}$, $\hat{\vec{T}}^2$, and $\hat{T}_3$ by $\mid T,\ N\rangle$, we have:

$$\hat{\vec{T}}^{\,2} \mid T,\ N\rangle = T(T+1) \mid T,\ N\rangle, \qquad \hat{T}_3 \mid T,\ N\rangle = N \mid T,\ N\rangle. \tag{7.263}$$

Similarly, if we denote the joint eigenstates of $\hat{\vec{t}}_1^{\,2}$, $\hat{\vec{t}}_2^{\,2}$, $\hat{t}_{1_3}$, and $\hat{t}_{2_3}$ by $|t_1, t_2;\ n_1, n_2\rangle$, we have

$$\hat{\vec{t}}_1^{\,2}|t_1, t_2;\ n_1, n_2\rangle = t_1(t_1+1)|t_1, t_2;\ n_1, n_2\rangle, \tag{7.264}$$

$$\hat{\vec{t}}_2^{\,2}|t_1, t_2;\ n_1, n_2\rangle = t_2(t_2+1)|t_1, t_2;\ n_1, n_2\rangle, \tag{7.265}$$

$$\hat{t}_{1_3}|t_1, t_2;\ n_1, n_2\rangle = n_1|t_1, t_2;\ n_1, n_2\rangle, \tag{7.266}$$

$$\hat{t}_{2_3}|t_1, t_2;\ n_1, n_2\rangle = n_2|t_1, t_2;\ n_1, n_2\rangle, \tag{7.267}$$

The matrix elements of the unitary transformation connecting the $\{|T,\ N\rangle\}$ and $\{|t_1, t_2;\ n_1, n_2\rangle\}$ bases,

$$|T,\ N\rangle = \sum_{n_1, n_2} \langle t_1, t_2;\ n_1 n_2 | T,\ N\rangle | t_1, t_2;\ n_1, n_2\rangle, \tag{7.268}$$

are given by the coefficients $\langle t_1, t_2;\ n_1 n_2 | T,\ N\rangle$; these coefficients can be calculated in the same way as the Clebsch–Gordan coefficients; see the next example.

Example 7.4
Find the various states corresponding to a two-nucleon system.

Solution
Let $\vec{\hat{T}}$ be the total isospin vector operator of the two-nucleon system:

$$\vec{\hat{T}} = \vec{\hat{t}}_1 + \vec{\hat{t}}_2. \tag{7.269}$$

This example is similar to adding two spin $\frac{1}{2}$ angular momenta. Thus, the values of T are 0 and 1. The case $T = 0$ corresponds to a singlet state:

$$|\ 0,\ 0\rangle = \frac{1}{\sqrt{2}}\left[|\ p\rangle_1\ |\ n\rangle_2 - |\ n\rangle_1\ |\ p\rangle_2\right], \tag{7.270}$$

where $|\ p\rangle_1$ means that nucleon 1 is a proton, $|\ n\rangle_2$ means that nucleon 2 is a neutron, and so on. This state, which is an antisymmetric isospsin state, describes a bound (p-n) system such as the ground state of deuterium ($T = 0$).

The case $T = 1$ corresponds to the triplet states $|1,\ N\rangle$ with $N = 1, 0, -1$:

$$|\ 1,\ 1\rangle = |\ p\rangle_1\ |\ p\rangle_2, \tag{7.271}$$

$$|\ 1,\ 0\rangle = \frac{1}{\sqrt{2}}\left[|\ p\rangle_1\ |\ n\rangle_2 + |\ n\rangle_1 | p\rangle_2\right], \tag{7.272}$$

$$|\ 1,\ -1\rangle = |\ n\rangle_1\ |\ n\rangle_2. \tag{7.273}$$

The state $|\ 1,\ 1\rangle$ corresponds to the case where both nucleons are protons (p-p) and $|\ 1,\ -1\rangle$ corresponds to the case where both nucleons are neutrons (n-n).

7.4 Scalar, Vector, and Tensor Operators

In this section we study how operators transform under rotations. Operators corresponding to various physical quantities can be classified as scalars, vectors, and tensors as a result of their behavior under rotations.

Consider an operator $\hat{A}$, which can be a scalar, a vector, or a tensor. The transformation of $\hat{A}$ under a rotation of infinitesimal angle $\delta\theta$ about an axis $\vec{n}$ is[7]

$$\hat{A}' = \hat{R}_n^\dagger(\delta\theta)\hat{A}\hat{R}_n(\delta\theta), \tag{7.274}$$

[7]The expectation value of an operator $\hat{A}$ with respect to the rotated state $|\ \psi'\rangle = \hat{R}_n(\delta\theta)\ |\ \psi\rangle$ is given by $\langle\psi'\ |\ \hat{A}\ |\ \psi'\rangle = \langle\psi|\hat{R}_n^\dagger(\delta\theta)\hat{A}\hat{R}_n(\delta\theta)\ |\ \psi\rangle = \langle\psi|\hat{A}'\ |\ \psi\rangle$.

where $\hat{R}_n(\delta\theta)$ can be inferred from (7.20)

$$\hat{R}_n(\delta\theta) = 1 - \frac{i}{\hbar}\delta\theta\vec{n}\cdot\hat{\vec{J}}. \tag{7.275}$$

Substituting (7.275) into (7.274) and keeping terms up to first order in $\delta\theta$, we obtain

$$\boxed{\hat{A}' = \hat{A} - \frac{i}{\hbar}\,\delta\theta\,[\hat{A},\ \vec{n}\cdot\hat{\vec{J}}\,].} \tag{7.276}$$

In the rest of this section we focus on the application of this relation to scalar, vector, and tensor operators.

7.4.1 Scalar Operators

Since scalar operators are invariant under rotations (i.e., $\hat{A}' = \hat{A}$), equation (7.276) implies that they commute with the angular momentum

$$\boxed{[\hat{A},\ \hat{J}_k\,] = 0 \qquad\qquad (k = x, y, z).} \tag{7.277}$$

This is also true for pseudo-scalars. A pseudo-scalar is defined by the product of a vector $\vec{A}$ and a pseudo-vector or axial vector $\vec{B}\times\vec{C}$: $\vec{A}\cdot(\vec{B}\times\vec{C})$.

7.4.2 Vector Operators

On the one hand, a vector operator $\hat{\vec{A}}$ transforms according to (7.276):

$$\boxed{\hat{\vec{A}}\,' = \hat{\vec{A}} - \frac{i}{\hbar}\,\delta\theta\,[\hat{\vec{A}},\ \vec{n}\cdot\hat{\vec{J}}\ ].} \tag{7.278}$$

On the other hand, from the classical theory of rotations, when a vector $\hat{\vec{A}}$ is rotated through an angle $\delta\theta$ around an axis $\vec{n}$, it is given by

$$\hat{\vec{A}}\,' = \hat{\vec{A}} + \delta\theta\,\vec{n}\times\hat{\vec{A}}. \tag{7.279}$$

Comparing (7.278) and (7.279), we obtain

$$\boxed{[\hat{\vec{A}},\ \vec{n}\cdot\hat{\vec{J}}\,] = i\hbar\vec{n}\times\hat{\vec{A}}.} \tag{7.280}$$

The jth component of this equation is given by

$$[\hat{\vec{A}},\ \vec{n}\cdot\hat{\vec{J}}\,]_j = i\hbar(\vec{n}\times\hat{\vec{A}})_j \qquad (j = x, y, z), \tag{7.281}$$

which in the case of $j = x, y, z$ leads to

$$\left[\hat{A}_x, \hat{J}_x\right] = \left[\hat{A}_y, \hat{J}_y\right] = \left[\hat{A}_z, \hat{J}_z\right] = 0, \tag{7.282}$$

$$\left[\hat{A}_x,\ \hat{J}_y\right] = i\hbar\hat{A}_z, \qquad \left[\hat{A}_y,\ \hat{J}_z\right] = i\hbar\hat{A}_x, \quad \left[\hat{A}_z,\ \hat{J}_x\right] = i\hbar\hat{A}_y, \tag{7.283}$$

$$\left[\hat{A}_x,\ \hat{J}_z\right] = -i\hbar\hat{A}_y, \quad \left[\hat{A}_y,\ \hat{J}_x\right] = -i\hbar\hat{A}_z, \quad \left[\hat{A}_z,\ \hat{J}_y\right] = -i\hbar\hat{A}_x. \tag{7.284}$$

Some interesting applications of (7.280) correspond to the cases where the vector operator $\hat{\vec{A}}$ is either the angular momentum, the position, or the linear momentum operator. Let us consider these three cases separately. First, substituting $\hat{\vec{A}} = \hat{\vec{J}}$ into (7.280), we recover the usual angular momentum commutation relations:

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z, \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y. \tag{7.285}$$

Second, in the case of a spinless particle (i.e., $\hat{\vec{J}} = \hat{\vec{L}}$), and if $\hat{\vec{A}}$ is equal to the position operator, $\hat{\vec{A}} = \vec{R}$, then (7.280) will yield the following relations:

$$\left[\hat{x}, \hat{L}_x\right] = 0, \qquad \left[\hat{x}, \hat{L}_y\right] = i\hbar\hat{z}, \qquad \left[\hat{x}, \hat{L}_z\right] = -i\hbar\hat{y}, \tag{7.286}$$

$$\left[\hat{y}, \hat{L}_y\right] = 0, \qquad \left[\hat{y}, \hat{L}_z\right] = i\hbar\hat{x}, \qquad \left[\hat{y}, \hat{L}_x\right] = -i\hbar\hat{z}, \tag{7.287}$$

$$\left[\hat{z}, \hat{L}_z\right] = 0, \qquad \left[\hat{z}, \hat{L}_x\right] = i\hbar\hat{y}, \qquad \left[\hat{z}, \hat{L}_y\right] = -i\hbar\hat{x}. \tag{7.288}$$

Third, if $\hat{\vec{J}} = \hat{\vec{L}}$ and if $\hat{\vec{A}}$ is equal to the momentum operator, $\hat{\vec{A}} = \hat{\vec{P}}$, then (7.280) will lead to

$$\left[\hat{P}_x, \hat{L}_x\right] = 0, \qquad \left[\hat{P}_x, \hat{L}_y\right] = i\hbar\hat{P}_z, \qquad \left[\hat{P}_x, \hat{L}_z\right] = -i\hbar\hat{P}_y, \tag{7.289}$$

$$\left[\hat{P}_y, \hat{L}_y\right] = 0, \qquad \left[\hat{P}_y, \hat{L}_z\right] = i\hbar\hat{P}_x, \qquad \left[\hat{P}_y, \hat{L}_x\right] = -i\hbar\hat{P}_z, \tag{7.290}$$

$$\left[\hat{P}_z, \hat{L}_z\right] = 0, \qquad \left[\hat{P}_z, \hat{L}_x\right] = i\hbar\hat{P}_y, \qquad \left[\hat{P}_z, \hat{L}_y\right] = -i\hbar\hat{P}_x. \tag{7.291}$$

Now, introducing the operators

$$\hat{A}_\pm = \hat{A}_x \pm i\hat{A}_y, \tag{7.292}$$

and using the relations (7.282) to (7.284), we can show that

$$\left[\hat{J}_x,\ \hat{A}_\pm\right] = \mp\hbar\hat{A}_z, \qquad \left[J_y,\ \hat{A}_\pm\right] = -i\hbar\hat{A}_z, \qquad \left[J_z,\ \hat{A}_\pm\right] = \pm\hbar\hat{A}_\pm. \tag{7.293}$$

These relations in turn can be shown to lead to

$$\left[\hat{J}_\pm,\ \hat{A}_\pm\right] = 0, \qquad \left[\hat{J}_\pm,\ \hat{A}_\mp\right] = \pm 2\hbar\hat{A}_z. \tag{7.294}$$

Let us introduce the *spherical components* $\hat{A}_{-1}$, $\hat{A}_0$, $\hat{A}_1$ of the vector operator $\hat{A}$; they are defined in terms of the Cartesian coordinates $\hat{A}_x$, $\hat{A}_y$, $\hat{A}_z$ as follows:

$$\hat{A}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\hat{A}_x \pm \hat{A}_y), \qquad \hat{A}_0 = \hat{A}_z. \tag{7.295}$$

For the particular case where $\hat{A}$ is equal to the position vector $\hat{R}$, we can express the components $\hat{R}_q$ (where $q = -1,\ 0,\ 1$),

$$\hat{R}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\hat{x} \pm \hat{y}), \qquad \hat{R}_0 = \hat{z}, \tag{7.296}$$

in terms of the spherical coordinates (recall that $\hat{R}_1 = \hat{x} = r\sin\theta\cos\phi$, $\hat{R}_2 = \hat{y} = r\sin\theta\sin\phi$, and $\hat{R}_3 = \hat{z} = r\cos\theta$) as follows:

$$\hat{R}_{\pm 1} = \mp\frac{1}{\sqrt{2}} r e^{\pm i\phi}\sin\theta, \qquad \hat{R}_0 = r\cos\theta. \tag{7.297}$$

Using the relations (7.282) to (7.284) and (7.292) to (7.294), we can ascertain that

$$\boxed{\left[\hat{J}_z, \hat{A}_q\right] = \hbar q\hat{A}_q \qquad (q = -1, 0, 1),} \tag{7.298}$$

$$\boxed{\left[\hat{J}_\pm,\ \hat{A}_q\right] = \hbar\sqrt{2 - q(q\pm 1)}\hat{A}_{q\pm 1} \qquad (q = -1, 0, 1).} \tag{7.299}$$

7.4.3 Tensor Operators: Reducible and Irreducible Tensors

In general, a tensor of rank k has 3^k components, where 3 denotes the dimension of the space. For instance, a tensor such as

$$\hat{T}_{ij} = A_i B_j \qquad (i, j = x, y, z), \tag{7.300}$$

which is equal to the product of the components of two vectors $\vec{\hat{A}}$ and $\vec{B}$, is a *second-rank* tensor; this tensor has 3^2 components.

7.4.3.1 Reducible Tensors

A Cartesian tensor $\hat{T}_{ij}$ can be decomposed into three parts:

$$\hat{T}_{ij} = \hat{T}_{ij}^{(0)} + \hat{T}_{ij}^{(1)} + \hat{T}_{ij}^{(2)}, \tag{7.301}$$

with

$$\hat{T}_{ij}^{(0)} = \frac{1}{3}\delta_{i,j}\sum_{i=1}^{3}\hat{T}_{ii}, \tag{7.302}$$

$$\hat{T}_{ij}^{(1)} = \frac{1}{2}(\hat{T}_{ij} - \hat{T}_{ji}) \qquad (i \neq j), \tag{7.303}$$

$$\hat{T}_{ij}^{(2)} = \frac{1}{2}(\hat{T}_{ij} + \hat{T}_{ji}) - \hat{T}_{ij}^{(0)}. \tag{7.304}$$

Notice that if we add equations (7.302), (7.303), and (7.304), we end up with an identity relation: $\hat{T}_{ij} = \hat{T}_{ij}$.

The term $\hat{T}_{ij}^{(0)}$ has only one component and transforms like a scalar under rotations. The second term $\hat{T}_{ij}^{(1)}$ is an antisymmetric tensor of rank 1 which has three independent components; it transforms like a vector. The third term $\hat{T}_{ij}^{(2)}$ is a symmetric second-rank tensor with zero trace, and hence has five independent components; $\hat{T}_{ij}^{(2)}$ cannot be reduced further to tensors of lower rank. These five components define an irreducible second-rank tensor.

In general, any tensor of rank k can be decomposed into tensors of lower rank that are expressed in terms of linear combinations of its 3^k components. However, there always remain

$(2k+1)$ components that behave as a tensor of rank k which cannot be reduced further. These $(2k+1)$ components are symmetric and traceless with respect to any two indices; they form the components of an *irreducible tensor* of rank k.

Equations (7.301) to (7.304) show how to decompose a Cartesian tensor operator, $\hat{T}_{ij}$, into a sum of irreducible *spherical* tensor operators $\hat{T}_{ij}^{(0)}$, $T_{ij}^{(1)}$, $T_{ij}^{(2)}$. Cartesian tensors are not very suitable for studying transformations under rotations, because they are reducible whenever their rank exceeds 1. In problems that display spherical symmetry, such as those encountered in subatomic physics, spherical tensors are very useful simplifying tools. It is therefore interesting to consider irreducible spherical tensor operators.

7.4.3.2 Irreducible Spherical Tensors

Let us now focus only on the representation of irreducible tensor operators in spherical coordinates. An irreducible spherical tensor operator of rank k (k is integer) is a set of $(2k+1)$ operators $T_q^{(k)}$, with $q=-k,\ldots,k$, which transform in the same way as angular momentum under a rotation of axes. For example, the case $k=1$ corresponds to a vector. The quantities $T_q^{(1)}$ are related to the components of the vector $\vec{\hat{A}}$ as follows (see (7.295)):

$$\hat{T}_{\pm 1}^{(1)} = \mp\frac{1}{\sqrt{2}}(\hat{A}_x \pm \hat{A}_y), \qquad\qquad \hat{T}_0^{(1)} = \hat{A}_z. \tag{7.305}$$

In what follows we are going to study some properties of spherical tensor operators and then determine how they transform under rotations.

First, let us look at the various commutation relations of spherical tensors with the angular momentum operator. Since a vector operator is a tensor of rank 1, we can rewrite equations (7.298) to (7.299), respectively, as follows:

$$\left[\hat{J}_z,\ \hat{T}_q^{(1)}\right] = \hbar q \hat{T}_q^{(1)} \qquad\qquad (q=-1,0,1), \tag{7.306}$$

$$\left[\hat{J}_\pm,\ \hat{T}_q^{(1)}\right] = \hbar\sqrt{1(1+1)-q(q\pm 1)}\,\hat{T}_{q\pm 1}^{(1)}, \tag{7.307}$$

where we have adopted the notation $\hat{A}_q = \hat{T}_q^{(1)}$. We can easily generalize these two relations to any spherical tensor of rank k, $\hat{T}_q^{(k)}$, and obtain these commutators:

$$\boxed{\left[\hat{J}_z,\ \hat{T}_q^{(k)}\right] = \hbar q \hat{T}_q^{(k)} \qquad (q=-k,-k+1,\ldots,k-1,k),} \tag{7.308}$$

$$\boxed{\left[\hat{J}_\pm,\ \hat{T}_q^{(k)}\right] = \hbar\sqrt{k(k+1)-q(q\pm 1)}\,\hat{T}_{q\pm 1}^{(k)}.} \tag{7.309}$$

Using the relations

$$\langle k,\ q' \mid \hat{J}_z \mid k,\ q\rangle = \hbar q \langle k,\ q' \mid k,\ q\rangle = \hbar q \delta_{q',\,q}, \tag{7.310}$$

$$\langle k,\ q' \mid \hat{J}_\pm \mid k,\ q\rangle = \hbar\sqrt{k(k+1)-q(q\pm 1)}\,\delta_{q',\,q\pm 1}, \tag{7.311}$$

along with (7.308) and (7.309), we can write

$$\sum_{q'=-k}^{k} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid \hat{J}_z \mid k,\ q\rangle = \hbar q \hat{T}_q^{(k)} = \left[\hat{J}_z,\ \hat{T}_q^{(k)}\right], \tag{7.312}$$

$$\sum_{q'=-k}^{k} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid \hat{J}_{\pm} \mid k,\ q\rangle \ = \ \hbar\sqrt{k(k+1) - q(q \pm 1)}\hat{T}_{q\pm 1}^{(k)} = \left[\hat{J}_{\pm},\ \hat{T}_q^{(k)}\right]. \quad (7.313)$$

The previous two relations can be combined into

$$\boxed{\left[\hat{\vec{J}},\ \hat{T}_q^{(k)}\right] = \sum_{q'=-k}^{k} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid \hat{\vec{J}} \mid k,\ q\rangle} \quad (7.314)$$

or

$$\boxed{\left[\vec{n} \cdot \hat{\vec{J}},\ \hat{T}_q^{(k)}\right] = \sum_{q'=-k}^{k} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid \vec{n} \cdot \hat{\vec{J}} \mid k,\ q\rangle.} \quad (7.315)$$

Having determined the commutation relations of the tensor operators with the angular momentum (7.315), we are now well equipped to study how irreducible spherical tensor operators transform under rotations. Using (7.276) we can write the transformation relation of a spherical tensor $T_q^{(k)}$ under an infinitesimal rotation as follows:

$$\hat{R}_n^{\dagger}(\delta\theta)\hat{T}_q^{(k)}\hat{R}_n(\delta\theta) = T_q^{(k)} + \frac{i}{\hbar}\,\delta\theta\left[\vec{n} \cdot \hat{\vec{J}},\ \hat{T}_q^{(k)}\right]. \quad (7.316)$$

Inserting (7.315) into (7.316), we obtain

$$\hat{R}^{\dagger}(\delta\theta)\hat{T}_q^{(k)}\hat{R}(\delta\theta) = \sum_{q'=-k}^{k} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid 1 + \frac{i}{\hbar}\,\delta\theta\,\vec{n}\cdot\hat{\vec{J}} \mid k,\ q\rangle = \sum_{q'} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid e^{i\delta\theta\,\vec{n}\cdot\hat{\vec{J}}/\hbar} \mid k,\ q\rangle. \quad (7.317)$$

This result also holds for finite rotations

$$\boxed{\hat{R}^{\dagger}(\alpha,\beta,\gamma)\hat{T}_q^{(k)}\hat{R}(\alpha,\beta,\gamma) = \sum_{q'=-k}^{k} \hat{T}_{q'}^{(k)} \langle k,\ q' \mid \hat{R}^{\dagger}(\alpha,\beta,\gamma) \mid k,\ q\rangle = \sum_{q'} \hat{T}_{q'}^{(k)} D_{q'q}^{(k)}(\alpha,\beta,\gamma).} \quad (7.318)$$

7.4.4 Wigner–Eckart Theorem for Spherical Tensor Operators

Taking the matrix elements of (7.308) between eigenstates of $\hat{\vec{J}}^2$ and $\hat{J}_z$, we find

$$\langle j',\ m' \mid \left[\hat{J}_z,\ \hat{T}_q^{(k)}\right] - \hbar q \hat{T}_q^{(k)} \mid j,\ m\rangle = 0 \quad (7.319)$$

or

$$(m' - m - q)\langle j',\ m' \mid \hat{T}_q^{(k)} \mid j,\ m\rangle = 0. \quad (7.320)$$

This implies that $\langle j',\ m' \mid \hat{T}_q^{(k)} \mid j,\ m\rangle$ vanishes unless $m' = m + q$. This property suggests that the quantity $\langle j',\ m' \mid \hat{T}_q^{(k)} \mid j,\ m\rangle$ must be proportional to the Clebsch–Gordan coefficient $\langle j',\ m' \mid j, k;\ m, q\rangle$; hence (7.320) leads to

$$(m' - m - q)\langle j',\ m' \mid j, k;\ m, q\rangle = 0. \quad (7.321)$$

Now, taking the matrix elements of (7.309) between $|\ j,\ m\rangle$ and $|j',\ m'\rangle$, we obtain

$$\begin{aligned}
&\sqrt{(j' \pm m')(j' \mp m' + 1)}\,\langle j', m' \mp 1|\hat{T}_q^{(k)}\ |\ j,\ m\rangle \\
&\quad = \sqrt{(j \mp m)(j \pm m + 1)}\,\langle j', m'|\hat{T}_q^{(k)}\ |\ j,\ m \pm 1\rangle \\
&\quad + \sqrt{(k \mp q)(k \pm q + 1)}\,\langle j', m'|\hat{T}_{q\pm 1}^{(k)}\ |\ j,\ m\rangle.
\end{aligned} \tag{7.322}$$

This equation has a structure which is identical to the recursion relation (7.150). For instance, substituting $j = j', m = m', j_1 = j, m_1 = m, j_2 = k, m_2 = q$ into (7.150), we end up with

$$\begin{aligned}
&\sqrt{(j' \pm m')(j' \mp m' + 1)}\,\langle j', m' \mp 1\ |\ j, k;\ m, q\rangle \\
&\quad = \sqrt{(j \mp m)(j \pm m + 1)}\,\langle j', m'\ |\ j, k;\ m \pm 1, q\rangle \\
&\quad + \sqrt{(k \mp q)(k \pm q + 1)}\,\langle j', m'\ |\ j, k;\ m, q \pm 1\rangle.
\end{aligned} \tag{7.323}$$

A comparison of (7.320) with (7.321) and (7.322) with (7.323) suggests that the dependence of $\langle j', m'|T_q^{(k)}\ |\ j,\ m\rangle$ on m', m, q is through a Clebsch–Gordan coefficient. The dependence, however, of $\langle j', m'|T_q^{(k)}\ |\ j,\ m\rangle$ on j', j, k has yet to be determined.

We can now state the **Wigner–Eckart theorem:** *The matrix elements of spherical tensor operators $\hat{T}_q^{(k)}$ with respect to angular momentum eigenstates* $|\ j,\ m\rangle$ are given by

$$\boxed{\langle j', m'|\hat{T}_q^{(k)}\ |\ j,\ m\rangle = \langle j, k;\ m, q|j',\ m'\rangle\langle j'\ \|\ \hat{T}^{(k)}\ \|\ j\rangle.} \tag{7.324}$$

The factor $\langle j'\ \|\ \hat{T}^{(k)}\ \|\ j\rangle$, which depends only on j', j, k, is called the *reduced matrix element* of the tensor $\hat{T}_q^{(k)}$ (note that the double bars notation is used to distinguish the reduced matrix elements, $\langle j'\ \|\ \hat{T}^{(k)}\ \|\ j\rangle$, from the matrix elements, $\langle j',\ m'|\hat{T}_q^{(k)}\ |\ j,\ m\rangle$). The theorem implies that the matrix elements $\langle j',\ m'|\hat{T}_q^{(k)}\ |\ j,\ m\rangle$ are written as the product of two terms: a Clebsch–Gordan coefficient $\langle j, k;\ m, q|j',\ q'\rangle$—which depends on the geometry of the system (i.e., the orientation of the system with respect to the z-axis), but not on its dynamics (i.e., j', j, k)—and a dynamical factor, the reduced matrix element, which does not depend on the orientation of the system in space (m', q, m). The quantum numbers m', m, q—which specify the projections of the angular momenta $\hat{\vec{J}}\,'$, $\hat{\vec{J}}$, and $\vec{k}$ onto the z-axis—give the orientation of the system in space, for they specify its orientation with respect to the z-axis. As for j', j, k, they are related to the dynamics of the system, not to its orientation in space.

Wigner–Eckart theorem for a scalar operator

The simplest application of the Wigner–Eckart theorem is when dealing with a scalar operator $\hat{B}$. As seen above, a scalar is a tensor of rank $k = 0$; hence $q = 0$ as well; thus, equation (7.324) yields

$$\langle j', m'|\hat{B}\ |\ j,\ m\rangle = \langle j, 0;\ m, 0|j',\ m'\rangle\langle j'\ \|\ \hat{B}\ \|\ j\rangle = \langle j'\ \|\ \hat{B}\ \|\ j\rangle\delta j' j\ \delta m' m, \tag{7.325}$$

since $\langle j, 0;\ m, 0|j',\ m'\rangle = \delta j' j\ \delta m' m$.

Wigner–Eckart theorem for a vector operator

As shown in (7.305), a vector is a tensor of rank 1: $T^{(1)} = A^{(1)} = \hat{\vec{A}}$, with $A_0^{(1)} = A_0 = A_z$ and $A_{\pm 1}^{(1)} = A_{\pm 1} = \mp(\hat{A}_x \pm \hat{A}_y)/\sqrt{2}$. An application of (7.324) to the q-component of a vector operator $\hat{\vec{A}}$ leads to

$$\langle j', m'|\hat{A}_q \mid j,\ m\rangle = \langle j, 1;\ m, q|j',\ m'\rangle\langle j' \parallel \hat{\vec{A}} \parallel j\rangle. \tag{7.326}$$

For instance, in the case of the angular momentum $\hat{\vec{J}}$, we have

$$\langle j', m'|\hat{J}_q \mid j,\ m\rangle = \langle j, 1;\ m, q|j',\ m'\rangle\langle j' \parallel \hat{\vec{J}} \parallel j\rangle. \tag{7.327}$$

Applying this relation to the component $\hat{J}_0$,

$$\langle j', m'|\hat{J}_0 \mid j,\ m\rangle = \langle j, 1;\ m, 0|j',\ m'\rangle\langle j' \parallel \hat{\vec{J}} \parallel j\rangle. \tag{7.328}$$

Since $\langle j', m'|\hat{J}_0 \mid j,\ m\rangle = \hbar m\, \delta j' j\, \delta m' m$ and the coefficient $\langle j, 1;\ m, 0 \mid j,\ m\rangle$ is equal to $\langle j, 1;\ m, 0 \mid j,\ m\rangle = m/\sqrt{j(j+1)}$, we have

$$\hbar m\, \delta j' j\, \delta m' m = \frac{m}{\sqrt{j(j+1)}}\langle j' \parallel \hat{\vec{J}} \parallel j\rangle \quad \Longrightarrow \quad \langle j' \parallel \hat{\vec{J}} \parallel j\rangle = \hbar\sqrt{j(j+1)}\delta_{j',\,j}. \tag{7.329}$$

Due to the selection rules imposed by the Clebsch–Gordan coefficients, we see from (7.326) that *a spin zero particle cannot have a dipole moment*. Since $\langle 0, 1;\ 0, q|0,\ 0\rangle = 0$, we have $\langle 0,\ 0|\hat{L}_q \mid 0,\ 0\rangle = \langle 0, 1;\ 0, q|0,\ 0\rangle\langle 0 \parallel \hat{\vec{L}} \parallel 0\rangle = 0$; the dipole moment is $\hat{\vec{\mu}} = -q\hat{\vec{L}}/(2mc)$. Similarly, *a spin $\frac{1}{2}$ particle cannot have a quadrupole moment*, because as $\langle \frac{1}{2}, 2;\ m, q|\frac{1}{2},\ m'\rangle = 0$, we have $\langle \frac{1}{2},\ m'|\hat{T}_q^{(2)}|\frac{1}{2},\ m\rangle = \langle \frac{1}{2}, 2;\ m, q|\frac{1}{2}m'\rangle\langle \frac{1}{2} \parallel \hat{T}^{(2)} \parallel \frac{1}{2}\rangle = 0$.

Wigner–Eckart theorem for a scalar product $\hat{\vec{J}} \cdot \hat{\vec{A}}$

On the one hand, since $\hat{\vec{J}} \cdot \hat{\vec{A}} = \hat{J}_0\hat{A}_0 - \hat{J}_{+1}\hat{A}_{-1} - \hat{J}_{-1}\hat{A}_{+1}$ and since $\hat{J}_0 \mid j,\ m\rangle = \hbar m \mid j,\ m\rangle$ and $\hat{J}_{\pm 1} \mid j,\ m\rangle = (\hbar/2)\sqrt{j(j+1) - m(m\pm 1)}|jm \pm 1\rangle$, we have

$$\begin{aligned}\langle j, m|\hat{\vec{J}} \cdot \hat{\vec{A}} \mid j, m\rangle = {}&\hbar m\langle j, m \mid \hat{A}_0 \mid j, m\rangle - \frac{\hbar}{2}\sqrt{j(j+1) - m(m+1)}\langle j, m+1|\hat{A}_{+1} \mid j, m\rangle \\ &+ \frac{\hbar}{2}\sqrt{j(j+1) - m(m-1)}\langle j, m-1|\hat{A}_{+1} \mid j, m\rangle.\end{aligned} \tag{7.330}$$

On the other hand, from the Wigner–Eckart theorem (7.324) we have $\langle j,\ m \mid \hat{A}_0 \mid j,\ m\rangle = \langle j, 1;\ m, 0 \mid j,\ m\rangle\langle j \parallel \hat{\vec{A}} \parallel j\rangle$, $\langle j,\ m+1|\hat{A}_{+1} \mid j,\ m\rangle = \langle j, 1;\ m, 1|j,\ m+1\rangle\langle j \parallel \hat{\vec{A}} \parallel j\rangle$ and $\langle j, m-1|\hat{A}_{-1} \mid j,\ m\rangle = \langle j, 1;\ m, -1|j,\ m-1\rangle\langle j \parallel \hat{\vec{A}} \parallel j\rangle$; substituting these terms into (7.330) we obtain

$$\begin{aligned}\langle j,\ m|\hat{\vec{J}} \cdot \hat{\vec{A}} \mid j,\ m\rangle = \Big[&\hbar m\langle j, 1;\ m, 0 \mid j,\ m\rangle \\ &- \frac{\hbar}{2}\langle j, 1;\ m, 1|j,\ m+1\rangle\sqrt{j(j+1) - m(m+1)} \\ &+ \frac{\hbar}{2}\langle j, 1;\ m, -1|j,\ m-1\rangle\sqrt{j(j+1) - m(m-1)}\Big]\langle j \parallel \hat{\vec{A}} \parallel j\rangle.\end{aligned} \tag{7.331}$$

When $\hat{\vec{A}} = \hat{\vec{J}}$ this relation leads to

$$\begin{aligned}\langle j,\ m|\hat{\vec{J}}^{\,2} \mid j,\ m\rangle \ =\ \Big[\hbar m\langle j, 1;\ m, 0 \mid j,\ m\rangle \\ -\frac{\hbar}{2}\langle j, 1;\ m, 1|j,\ m+1\rangle\sqrt{j(j+1)-m(m+1)} \\ +\ \frac{\hbar}{2}\langle j, 1;\ m, -1|j,\ m-1\rangle\sqrt{j(j+1)-m(m-1)}\Big]\langle j \parallel \hat{\vec{J}} \parallel j\rangle.\end{aligned} \tag{7.332}$$

We are now equipped to obtain a relation between the matrix elements of a vector operator $\hat{\vec{A}}$ and the matrix elements of the scalar operator $\hat{\vec{J}} \cdot \hat{\vec{A}}$; this relation is useful in the calculation of the hydrogen's energy corrections due to the Zeeman effect (see Chapter 9). For this, we need to calculate two ratios: the first is between (7.326) and (7.327)

$$\frac{\langle j,\ m'|\hat{A}_q \mid j,\ m\rangle}{\langle j,\ m'|\hat{J}_q \mid j,\ m\rangle} = \frac{\langle j \parallel \hat{\vec{A}} \parallel j\rangle}{\langle j \parallel \hat{\vec{J}} \parallel j\rangle} \tag{7.333}$$

and the second is between (7.331) and (7.332)

$$\frac{\langle j,\ m|\hat{\vec{J}} \cdot \hat{\vec{A}} \mid j,\ m\rangle}{\langle j,\ m|\hat{\vec{J}}^{\,2} \mid j,\ m\rangle} = \frac{\langle j \parallel \hat{\vec{A}} \parallel j\rangle}{\langle j \parallel \hat{\vec{J}} \parallel j\rangle} \quad\Longrightarrow\quad \frac{\langle j\ m \mid \hat{\vec{J}} \cdot \hat{\vec{A}} \mid j,\ m\rangle}{\hbar^2 j(j+1)^2} = \frac{\langle j \parallel \hat{\vec{A}} \parallel j\rangle}{\langle j \parallel \hat{\vec{J}} \parallel j\rangle}, \tag{7.334}$$

since $\langle j\ m \mid \hat{\vec{J}}^{\,2} \mid j,\ m\rangle = \hbar^2 j(j+1)$. Equating (7.333) and (7.334) we obtain

$$\boxed{\langle j,\ m'|\hat{A}_q \mid j,\ m\rangle = \frac{\langle j,\ m|\hat{\vec{J}} \cdot \hat{\vec{A}} \mid j,\ m\rangle}{\hbar^2 j(j+1)}\langle j,\ m'|\hat{J}_q \mid j,\ m\rangle.} \tag{7.335}$$

An important application of this relation pertains to the case where the vector operator $\hat{\vec{A}}$ is a spin angular momentum $\hat{\vec{S}}$. Since

$$\begin{aligned}\hat{\vec{J}} \cdot \hat{\vec{S}} \ &=\ (\hat{\vec{L}} + \hat{\vec{S}}) \cdot \hat{\vec{S}} = \hat{\vec{L}} \cdot \hat{\vec{S}} + \hat{S}^2 = \frac{(\hat{\vec{L}} + \hat{\vec{S}})^2 - \hat{L}^2 - \hat{S}^2}{2} + \hat{S}^2 = \frac{\hat{J}^2 - \hat{L}^2 - \hat{S}^2}{2} + \hat{S}^2 \\ &=\ \frac{\hat{J}^2 - \hat{L}^2 + \hat{S}^2}{2},\end{aligned} \tag{7.336}$$

and since $\mid j,\ m\rangle$ is a joint eigenstate of $\hat{\vec{J}}^2$, $\hat{\vec{L}}^2$, $\hat{\vec{S}}^2$ and $\hat{J}_z$ with eigenvalues $\hbar^2 j(j+1)$, $\hbar^2 l(l+1)$, $\hbar^2 s(s+1)$, and $\hbar m$, respectively, the matrix element of $\hat{S}_z$ then becomes easy to calculate from (7.335):

$$\langle j, m|\hat{S}_z \mid j, m\rangle = \frac{\langle j, m \mid \hat{\vec{J}} \cdot \hat{\vec{S}} \mid j, m\rangle}{\hbar^2 j(j+1)}\langle j, m|\hat{J}_z \mid j, m\rangle = \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\hbar m. \tag{7.337}$$

7.5 Solved Problems

Problem 7.1

(a) Show how $\hat{J}_x$ and $\hat{J}_y$ transform under a rotation of (finite) angle α about the z-axis. Using these results, determine how the angular momentum operator $\hat{\vec{J}}$ transform under the rotation.

(b) Show how a vector operator $\hat{\vec{A}}$ transforms under a rotation of angle α about the y-axis.

(c) Show that $e^{i\pi\hat{J}_z/\hbar}e^{i\alpha\hat{J}_y/\hbar}e^{-i\pi\hat{J}_z/\hbar} = e^{-i\alpha\hat{J}_y/\hbar}$.

Solution

(a) The operator corresponding to a rotation of angle α about the z-axis is given by $\hat{R}_z(\alpha) = e^{-i\alpha\hat{J}_z/\hbar}$. Under this rotation, an operator $\hat{B}$ transforms like $\hat{B}' = \hat{R}_z^\dagger\hat{B}\hat{R}_z = e^{i\alpha\hat{J}_z/\hbar}\hat{B}e^{-i\alpha\hat{J}_z/\hbar}$. Using the relation

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A},\,\hat{B}\right] + \frac{1}{2!}\left[\hat{A},\,\left[\hat{A},\,\hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A},\,\left[\hat{A},\,\left[\hat{A},\,\hat{B}\right]\right]\right] + \cdots, \tag{7.338}$$

along with the commutation relations $\left[\hat{J}_z,\ \hat{J}_y\right] = -i\hbar\hat{J}_x$ and $\left[\hat{J}_z,\ \hat{J}_x\right] = i\hbar\hat{J}_y$, we have

$$\begin{aligned}
e^{i\alpha\hat{J}_z/\hbar}\hat{J}_x e^{-i\alpha\hat{J}_z/\hbar} &= \hat{J}_x + \frac{i\alpha}{\hbar}\left[\hat{J}_z,\ \hat{J}_x\right] - \frac{\alpha^2}{2!\hbar^2}\left[\hat{J}_z,\ \left[\hat{J}_z,\ \hat{J}_x\right]\right] \\
&\quad - \frac{i\alpha^3}{3!\hbar^3}\left[\hat{J}_z,\ \left[\hat{J}_z,\ \left[\hat{J}_z, \hat{J}_x\right]\right]\right] + \cdots \\
&= \hat{J}_x - \alpha\hat{J}_y - \frac{\alpha^2}{2!}\hat{J}_x + \frac{\alpha^3}{3!}\hat{J}_y + \frac{\alpha^4}{4!}\hat{J}_x - \frac{\alpha^5}{5!}\hat{J}_y + \cdots \\
&= \hat{J}_x\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots\right) - \hat{J}_y\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right) \\
&= \hat{J}_x\cos\alpha - \hat{J}_y\sin\alpha.
\end{aligned} \tag{7.339}$$

Similarly, we can show that

$$e^{i\alpha\hat{J}_z/\hbar}\hat{J}_y e^{-i\alpha\hat{J}_z/\hbar} = \hat{J}_y\cos\alpha + \hat{J}_x\sin\alpha. \tag{7.340}$$

As $\hat{J}_z$ is invariant under an arbitrary rotation about the z-axis ($e^{i\alpha\hat{J}_z/\hbar}\hat{J}_z e^{-i\alpha\hat{J}_z/\hbar} = \hat{J}_z$), we can condense equations (7.339) and (7.340) into a single matrix relation:

$$e^{i\alpha\hat{J}_z/\hbar}\hat{\vec{J}}e^{-i\alpha\hat{J}_z/\hbar} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{pmatrix}. \tag{7.341}$$

(b) Using the commutation relations $\left[\hat{J}_y,\ \hat{A}_x\right] = -i\hbar\hat{A}_z$ and $\left[\hat{J}_y,\ \hat{A}_z\right] = i\hbar\hat{A}_x$ (see (7.282) to (7.284)) along with (7.338), we have

$$e^{i\alpha\hat{J}_y/\hbar}\hat{A}_x e^{-i\alpha\hat{J}_y/\hbar} = \hat{A}_x + \frac{i\alpha}{\hbar}\left[\hat{J}_y,\ \hat{A}_x\right] - \frac{\alpha^2}{2!\hbar^2}\left[\hat{J}_y,\ \left[\hat{J}_y,\ \hat{A}_x\right]\right]$$

$$
\begin{aligned}
& - \frac{i\alpha^3}{3!\hbar^3}\left[\hat{J}_y,\left[\hat{J}_y,\left[\hat{J}_y,\hat{A}_x\right]\right]\right]+\cdots \\
&= \hat{A}_x + \alpha\hat{A}_z - \frac{\alpha^2}{2!}\hat{A}_x - \frac{\alpha^3}{3!}\hat{A}_z + \frac{\alpha^4}{4!}\hat{A}_x + \frac{\alpha^5}{5!}\hat{J}_z + \cdots \\
&= \hat{A}_x\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots\right) + \hat{A}_z\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \cdots\right) \\
&= \hat{A}_x \cos\alpha + \hat{A}_z \sin\alpha. \qquad (7.342)
\end{aligned}
$$

Similarly, we can show that

$$
\hat{A}'_z = e^{i\alpha\hat{J}_y/\hbar}\hat{A}_z e^{-i\alpha\hat{J}_y/\hbar} = -\hat{A}_x \sin\alpha + \hat{A}_z\cos\alpha. \qquad (7.343)
$$

Also, since $\hat{A}_y$ is invariant under an arbitrary rotation about the y-axis, we may combine equations (7.342) and (7.343) to find the vector operator $\hat{\vec{A}}\,'$ obtained by rotating $\hat{\vec{A}}$ through an angle α about the y-axis:

$$
\hat{\vec{A}}\,' = e^{i\alpha\hat{J}_y/\hbar}\hat{\vec{A}}e^{-i\alpha\hat{J}_y/\hbar} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}\begin{pmatrix} \hat{A}_x \\ \hat{A}_y \\ \hat{A}_z \end{pmatrix}. \qquad (7.344)
$$

(c) Expanding $e^{i\alpha\hat{J}_y/\hbar}$ and then using (7.340), we obtain

$$
\begin{aligned}
e^{i\pi\hat{J}_z/\hbar}e^{i\alpha\hat{J}_y/\hbar}e^{-i\pi\hat{J}_z/\hbar} &= \sum_{n=0}^{\infty}\frac{(i\alpha/\hbar)^n}{n!}e^{i\pi\hat{J}_z/\hbar}\left(\hat{J}_y\right)^n e^{-i\pi\hat{J}_z/\hbar} \\
&= \sum_{n=0}^{\infty}\frac{(i\alpha/\hbar)^n}{n!}\left(\hat{J}_y\cos\pi + \hat{J}_x\sin\pi\right)^n = \sum_{n=0}^{\infty}\frac{(-i\alpha/\hbar)^n}{n!}\left(\hat{J}_y\right)^n \\
&= e^{-i\alpha\hat{J}_y/\hbar}. \qquad (7.345)
\end{aligned}
$$

Problem 7.2

Use the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, to show that

(a) $e^{-i\alpha\sigma_x} = I\cos\alpha - i\sigma_x\sin\alpha$, where I is the unit matrix,
(b) $e^{i\alpha\sigma_x}\sigma_z e^{-i\alpha\sigma_x} = \sigma_z\cos(2\alpha) + \sigma_y\sin(2\alpha)$.

Solution

(a) Using the expansion

$$
e^{-i\alpha\sigma_x} = \sum_{n=0}^{\infty}\frac{(-i)^{2n}}{(2n)!}(\alpha)^{2n}\sigma_x^{2n} + \sum_{n=0}^{\infty}\frac{(-i)^{2n+1}}{(2n+1)!}(\alpha)^{2n+1}\sigma_x^{2n+1}, \qquad (7.346)
$$

and since $\sigma_x^2 = 1$, $\sigma_x^{2n} = I$, and $\sigma_x^{2n+1} = \sigma_x$, where I is the unit matrix, we have

$$
\begin{aligned}
e^{-i\alpha\sigma_x} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\alpha)^{2n} - i\sigma_x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\alpha)^{2n+1} \\
&= I\cos\alpha - i\sigma_x\sin\alpha. \qquad (7.347)
\end{aligned}
$$

(b) From (7.347) we can write

$$\begin{aligned} e^{i\alpha\sigma_x}\sigma_z e^{-i\alpha\sigma_x} &= (\cos\alpha + i\sigma_x\sin\alpha)\sigma_z(\cos\alpha - i\sigma_x\sin\alpha) \\ &= \sigma_z\cos^2\alpha + \sigma_x\sigma_z\sigma_x\sin^2\alpha + i[\sigma_x,\ \sigma_z]\sin\alpha\cos\alpha, \end{aligned} \tag{7.348}$$

which, when using the facts that $\sigma_x\sigma_z = -\sigma_z\sigma_x$, $\sigma_x^2 = I$, and $[\sigma_x,\ \sigma_z] = -2i\sigma_y$, reduces to

$$\begin{aligned} e^{i\alpha\sigma_x}\sigma_z e^{-i\alpha\sigma_x} &= \sigma_z\cos^2\alpha - \sigma_z\sigma_x^2\sin^2\alpha + 2\sigma_y\sin\alpha\cos\alpha \\ &= \sigma_z(\cos^2\alpha - \sin^2\alpha) + \sigma_y\sin(2\alpha) \\ &= \sigma_z\cos(2\alpha) + \sigma_y\sin(2\alpha). \end{aligned} \tag{7.349}$$

Problem 7.3

Find the Clebsch–Gordan coefficients associated with the addition of two angular momenta $j_1 = 1$ and $j_2 = 1$.

Solution

The addition of $j_1 = 1$ and $j_2 = 1$ is encountered, for example, in a two-particle system where the angular momenta of both particles are orbital.

The allowed values of the total angular momentum are between $|j_1 - j_2| \le j \le j_1 + j_2$; hence $j = 0, 1, 2$. To calculate the relevant Clebsch–Gordan coefficients, we need to find the basis vectors $\{|j,\ m\rangle\}$, which are common eigenvectors of $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\vec{J}^2$ and $\hat{J}_z$, in terms of $\{|1, 1;\ m_1, m_2\rangle\}$.

Eigenvectors $|j,\ m\rangle$ associated with $j = 2$

The state $|\,2,\ 2\rangle$ is simply given by

$$\boxed{|\,2,\ 2\rangle = |1, 1;\ 1, 1\rangle;} \tag{7.350}$$

the corresponding Clebsch–Gordan coefficient is thus given by $\langle 1, 1;\ 1, 1\,|\,2,\ 2\rangle = 1$.

As for $|\,2,\ 1\rangle$, it can be found by applying J_- to $|\,2,\ 2\rangle$ and $(J_{1-} + J_{2-})$ to $|1, 1;\ 1, 1\rangle$, and then equating the two results

$$J_-\,|\,2,\ 2\rangle = \left(J_{1-} + J_{2-}\right)|1, 1;\ 1, 1\rangle. \tag{7.351}$$

This leads to

$$2\hbar\,|\,2,\ 1\rangle = \sqrt{2}\hbar\left(|1, 1;\ 1, 0\rangle + |1, 1;\ 0, 1\rangle\right) \tag{7.352}$$

or to

$$\boxed{|\,2,\ 1\rangle = \frac{1}{\sqrt{2}}\left(|1, 1;\ 1, 0\rangle + |1, 1;\ 0, 1\rangle\right);} \tag{7.353}$$

hence $\langle 1, 1;\ 1, 0\,|\,2,\ 1\rangle = \langle 1, 1;\ 0, 1\,|\,2,\ 1\rangle = 1/\sqrt{2}$. Using (7.353), we can find $|2,\ 0\rangle$ by applying J_- to $|\,2,\ 1\rangle$ and $(J_{1-} + J_{2-})$ to $[|1, 1;\ 1, 0\rangle + |1, 1;\ 0, 1\rangle]$:

$$J_-\,|\,2,\ 1\rangle = \frac{1}{\sqrt{2}}\hbar\left(J_{1-} + J_{2-}\right)[|1, 1;\ 1, 0\rangle + |1, 1;\ 0, 1\rangle], \tag{7.354}$$

which leads to

$$\boxed{|\,2,\ 0\rangle \ = \frac{1}{\sqrt{6}}\Big(\,|1,1;\ 1,-1\rangle + 2|1,1;\ 0,0\rangle + |1,1;\ -1,1\rangle\Big)\,;} \tag{7.355}$$

hence $\langle 1,1;\ 1,-1 \mid 2,\ 0\rangle = \langle 1,1;\ -1,1 \mid 2,\ 0\rangle = 1/\sqrt{6}$ and $\langle 1,1;\ 0,0 \mid 2,\ 0\rangle = 2/\sqrt{6}$.

Similarly, by repeated applications of J_- and $(J_{1_-} + J_{2_-})$, we can show that

$$\boxed{|\,2,\ -1\rangle = \frac{1}{\sqrt{2}}\Big(\,|1,1;\ 0,-1\rangle + |1,1;\ -1,0\rangle\Big)\,,} \tag{7.356}$$

$$\boxed{|\,2,\ -2\rangle = |1,1;\ -1,-1\rangle,} \tag{7.357}$$

with $\langle 1,1;\ 0,-1|2,\ -1\rangle = \langle 1,1;\ -1,0|2,\ -1\rangle = 1/\sqrt{2}$ and $\langle 1,1;\ -1,-1|2,\ -2\rangle = 1$.

Eigenvectors $|\,j,\ m\rangle$ associated with $j = 1$
The relation

$$|1,\ m\rangle = \sum_{m_1=-1}^{1}\ \sum_{m_2=-1}^{1} \langle 1,1;\ m_1, m_2|1,\ m\rangle |1,1;\ m_1, m_2\rangle \tag{7.358}$$

leads to

$$|\,1,\ 1\rangle = a|1,1;\ 1,0\rangle + b|1,1;\ 0,1\rangle, \tag{7.359}$$

where $a = \langle 1,1;\ 1,0 \mid 1,\ 1\rangle$ and $b = \langle 1,1;\ 0,1|1,\ 1\rangle$. Since $|\ 1,\ 1\rangle$, $|1,1;\ 1,0\rangle$ and $|1,1;\ 0,1\rangle$ are all normalized, and since $|1,1;\ 1,0\rangle$ is orthogonal to $|1,1;\ 0,1\rangle$ and a and b are real, we have

$$\langle 1,1 \mid 1,\ 1\rangle = a^2 + b^2 = 1. \tag{7.360}$$

Now, since $\langle 2,1 \mid 1,\ 1\rangle = 0$, equations (7.353) and (7.359) yield

$$\langle 2,1 \mid 1,\ 1\rangle = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = 0. \tag{7.361}$$

A combination of (7.360) and (7.361) leads to $a = -b = \pm 1/\sqrt{2}$. The signs of a and b have yet to be found. The phase convention mandates that coefficients like $\langle j_1, j_2;\ j_1, (j - j_1)|j,\ j\rangle$ must be positive. Thus, we have $a = 1/\sqrt{2}$ and $b = -1/\sqrt{2}$, which when inserted into (7.359) give

$$\boxed{|\,1,\ 1\rangle = \frac{1}{\sqrt{2}}\Big(\,|1,1;\ 1,0\rangle - |1,1;\ 0,1\rangle\Big)\,.} \tag{7.362}$$

This yields $\langle 1,1;\ 1,0 \mid 1,\ 1\rangle = \frac{1}{2}$ and $\langle 1,1;\ 0,1 \mid 1,\ 1\rangle = -\frac{1}{2}$.

To find $|1,\ 0\rangle$ we proceed as we did above when we obtained the states $|\,2,\ 1\rangle, |\,2,\ 0\rangle, \ldots, |2,\ -2\rangle$ by repeatedly applying J_- on $|\,2,\ 2\rangle$. In this way, the application of J_- on $|\,1,\ 1\rangle$ and $\big(J_{1_-} + J_{2_-}\big)$ on $[|1,1;\ 1,0\rangle - |1,1;\ 0,1\rangle]$,

$$J_- \mid 1,\ 1\rangle = \frac{1}{2}\big(J_{1_-} + J_{2_-}\big)\,[|1,1;\ 1,0\rangle - |1,1;\ 0,1\rangle] \tag{7.363}$$

gives

$$\sqrt{2}\hbar \mid 1,\ 0\rangle = \frac{\sqrt{2}\hbar}{2}\,[|1,1;\ 1,-1\rangle - |1,1;\ -1,1\rangle]\,, \tag{7.364}$$

or

$$\boxed{|\,1,\ 0\rangle = \frac{1}{\sqrt{2}}\Big(|1,1;\ 1,-1\rangle - |1,1;\ -1,1\rangle\Big),} \tag{7.365}$$

with $\langle 1,1;\ 1,-1 \mid 1,\ 0\rangle = \frac{1}{\sqrt{2}}$ and $\langle 1,1;\ -1,1 \mid 1,\ 0\rangle = -1/\sqrt{2}$.

Similarly, we can show that

$$\boxed{|\,1,\ -1\rangle = \frac{1}{\sqrt{2}}\Big(|1,1;\ 0,-1\rangle - |1,1;\ -1,0\rangle\Big);} \tag{7.366}$$

hence $\langle 1,1;\ 0,-1 \mid 1,\ -1\rangle = 1/\sqrt{2}$ and $\langle 1,1;\ -1,0 \mid 1,\ -1\rangle = -1/\sqrt{2}$.

Eigenvector $|0,\ 0\rangle$ associated with $j = 0$

Since

$$|0,\ 0\rangle = a|1,1;\ 1,-1\rangle + b|1,1;\ 0,0\rangle + c|1,1;\ -1,1\rangle, \tag{7.367}$$

where $a = \langle 1,1;\ 1,-1 \mid 0,\ 0\rangle$, $b = \langle 1,1;\ 0,0 \mid 0,\ 0\rangle$, and $c = \langle 1,1;\ -1,1 \mid 0,\ 0\rangle$ are real, and since the states $|0,\ 0\rangle$, $|1,1;\ 1,-1\rangle$, $|1,1;\ 0,0\rangle$, and $|1,1;\ -1,1\rangle$ are normal, we have

$$\langle 0,\ 0 \mid 0,\ 0\rangle = a^2 + b^2 + c^2 = 1. \tag{7.368}$$

Now, combining (7.355), (7.365), and (7.367), we obtain

$$\langle 2,\ 0 \mid 0,\ 0\rangle = \frac{a}{\sqrt{6}} + \frac{2b}{\sqrt{6}} + \frac{c}{\sqrt{6}} = 0, \tag{7.369}$$

$$\langle 1,\ 0 \mid 0,\ 0\rangle = \frac{a}{\sqrt{2}} - \frac{c}{\sqrt{2}} = 0. \tag{7.370}$$

Since a is by convention positive, we can show that the solutions of (7.368), (7.369), and (7.370) are given by $a = 1/\sqrt{3}$, $b = -1/\sqrt{3}$, $c = 1/\sqrt{3}$, and consequently

$$\boxed{|0,\ 0\rangle = \frac{1}{\sqrt{3}}\Big(|1,1;\ 1,-1\rangle - |1,1;\ 0,0\rangle + |1,1;\ -1,1\rangle\Big),} \tag{7.371}$$

with $\langle 1,1;\ 1,-1 \mid 0,\ 0\rangle = \langle 1,1;\ -1,1 \mid 0,\ 0\rangle = 1/\sqrt{3}$ and $\langle 1,1;\ 0,0 \mid 0,\ 0\rangle = -1/\sqrt{3}$.

Note that while the quintuplet states $|2,\ m\rangle$ (with $m = \pm 2, \pm 1, 0$) and the singlet state $|\,0,\ 0\rangle$ are symmetric, the triplet states $|1,\ m\rangle$ (with $m = \pm 1, 0$) are antisymmetric under space inversion.

Problem 7.4

(a) Find the total spin of a system of three spin $\frac{1}{2}$ particles and derive the corresponding Clebsch–Gordan coefficients.

(b) Consider a system of three nonidentical spin $\frac{1}{2}$ particles whose Hamiltonian is given by $\hat{H} = -\epsilon_0(\vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3)/\hbar^2$. Find the system's energy levels and their degeneracies.

Solution

(a) To add $j_1 = \frac{1}{2}$, $j_2 = \frac{1}{2}$, and $j_3 = \frac{1}{2}$, we begin by coupling j_1 and j_2 to form $j_{12} = j_1 + j_2$, where $|j_1 - j_2| \le j_{12} \le |j_1 + j_2|$; hence $j_{12} = 0,\ 1$. Then we add j_{12} and j_3; this leads to $|j_{12} - j_3| \le j \le |j_{12} + j_3|$ or $j = \frac{1}{2},\ \frac{3}{2}$.

We are going to denote the joint eigenstates of $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{\vec{J}}_3^2$, $\hat{\vec{J}}_{12}^2$, $\hat{\vec{J}}^2$, and J_z by $|j_{12}, j, m\rangle$ and the joint eigenstates of $\hat{\vec{J}}_1^2$, $\hat{\vec{J}}_2^2$, $\hat{\vec{J}}_3^2$, $\hat{J}_{1_z}$, $\hat{J}_{2_z}$, and $\hat{J}_{3_z}$ by $|\ j_1, j_2, j_3;\ m_1, m_2, m_3\rangle$; since $j_1 = j_2 = j_3 = \frac{1}{2}$ and $m_1 = \pm\frac{1}{2}$, $m_2 = \pm\frac{1}{2}$, $m_3 = \pm\frac{1}{2}$, we will be using throughout this problem the lighter notation $|j_1, j_2, j_3; \pm, \pm, \pm\rangle$ to abbreviate $|\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle$.

In total there are eight states $|j_{12}, j, m\rangle$ since $(2j_1+1)(2j_2+1)(2j_3+1) = 8$. Four of these correspond to the subspace $j = \frac{3}{2}$: $|\ 1, \frac{3}{2}, \frac{3}{2}\rangle$, $|\ 1, \frac{3}{2}, \frac{1}{2}\rangle$, $|\ 1, \frac{3}{2}, -\frac{1}{2}\rangle$, and $|\ 1, \frac{3}{2}, -\frac{3}{2}\rangle$. The remaining four belong to the subspace $j = \frac{1}{2}$: $|\ 0, \frac{1}{2}, \frac{1}{2}\rangle$, $|\ 0, \frac{1}{2}, -\frac{1}{2}\rangle$, $|\ 1, \frac{1}{2}, \frac{1}{2}\rangle$, and $|\ 1, \frac{1}{2}, -\frac{1}{2}\rangle$. To construct the states $|j_{12}, j, m\rangle$ in terms of $|j_1, j_2, j_3; \pm, \pm, \pm\rangle$, we are going to consider the two subspaces $j = \frac{3}{2}$ and $j = \frac{1}{2}$ separately.

Subspace $j = \frac{3}{2}$
First, the states $|\ 1, \frac{3}{2}, \frac{3}{2}\rangle$ and $|\ 1, \frac{3}{2}, -\frac{3}{2}\rangle$ are clearly given by

$$\left|1, \frac{3}{2}, \frac{3}{2}\right\rangle = |j_1, j_2, j_3; +, +, +\rangle, \qquad \left|1, \frac{3}{2}, -\frac{3}{2}\right\rangle = |j_1, j_2, j_3; -, -, -\rangle. \tag{7.372}$$

To obtain $|\ 1, \frac{3}{2}, \frac{1}{2}\rangle$, we need to apply, on the one hand, $\hat{J}_-$ on $|\ 1, \frac{3}{2}, \frac{3}{2}\rangle$ (see (7.220)),

$$\hat{J}_- \left|1, \frac{3}{2}, \frac{3}{2}\right\rangle = \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}-1\right)}\left|1, \frac{3}{2}, \frac{1}{2}\right\rangle = \hbar\sqrt{3}\left|1, \frac{3}{2}, \frac{1}{2}\right\rangle, \tag{7.373}$$

and, on the other hand, apply $(\hat{J}_{1-} + \hat{J}_{2-} + \hat{J}_{3-})$ on $|j_1, j_2, j_3; +, +, +\rangle$ (see (7.221) to (7.223)). This yields

$$\begin{aligned}(\hat{J}_{1-} + \hat{J}_{2-} + \hat{J}_{3-})|j_1, j_2, j_3; +, +, +\rangle \ &= \ \hbar\Big(|j_1, j_2, j_3; -, +, +\rangle + |j_1, j_2, j_3; +, -, +\rangle \\ &\quad + |j_1, j_2, j_3; +, +, -\rangle\Big),\end{aligned} \tag{7.374}$$

since $\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)} = 1$. Equating (7.373) and (7.374) we infer

$$\left|1, \frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\Big(|j_1, j_2, j_3; -, +, +\rangle + |j_1, j_2, j_3; +, -, +\rangle + |j_1, j_2, j_3; +, +, -\rangle\Big). \tag{7.375}$$

Following the same method—applying $\hat{J}_-$ on $|\ 1, \frac{3}{2}, \frac{1}{2}\rangle$ and $(\hat{J}_{1-} + \hat{J}_{2-} + \hat{J}_{3-})$ on the right-hand side of (7.375) and then equating the two results—we find

$$\left|1, \frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\Big(|j_1, j_2, j_3; +, -, -\rangle + |j_1, j_2, j_3; -, +, -\rangle + |j_1, j_2, j_3; -, -, +\rangle\Big). \tag{7.376}$$

Subspace $j = \frac{1}{2}$
We can write $|\ 0, \frac{1}{2}, \frac{1}{2}\rangle$ as a linear combination of $|j_1, j_2, j_3; +, +, -\rangle$ and $|j_1, j_2, j_3; -, +, +\rangle$:

$$\left|0, \frac{1}{2}, \frac{1}{2}\right\rangle = \alpha|j_1, j_2, j_3; +, +, -\rangle + \beta|j_1, j_2, j_3; -, +, +\rangle. \tag{7.377}$$

Since $\mid 0, \frac{1}{2}, \frac{1}{2}\rangle$ is normalized, while $|j_1, j_2, j_3; +, +, -\rangle$ and $|j_1, j_2, j_3; -, +, +\rangle$ are orthonormal, and since the Clebsch–Gordan coefficients, such as α and β, are real numbers, equation (7.377) yields

$$\alpha^2 + \beta^2 = 1. \tag{7.378}$$

On the other hand, since $\langle 1, \frac{3}{2}, \frac{1}{2} \mid 0, \frac{1}{2}, \frac{1}{2}\rangle = 0$, a combination of (7.375) and (7.377) leads to

$$\frac{1}{\sqrt{3}}(\alpha + \beta) = 0 \quad \Longrightarrow \quad \alpha = -\beta. \tag{7.379}$$

A substitution of $\alpha = -\beta$ into (7.378) yields $\alpha = -\beta = 1/\sqrt{2}$, and substituting this into (7.377) we obtain

$$\left|0, \frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\Big(|j_1, j_2, j_3; +, +, -\rangle - |j_1, j_2, j_3; -, +, +\rangle \Big). \tag{7.380}$$

Following the same procedure that led to (7.375)—applying $\hat{J}_-$ on the left-hand side of (7.380) and $(\hat{J}_{1-} + \hat{J}_{2-} + \hat{J}_{3-})$ on the right-hand side and then equating the two results—we find

$$\left|0, \frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\Big(-|j_1, j_2, j_3; +, -, -\rangle + |j_1, j_2, j_3; -, -, +\rangle \Big). \tag{7.381}$$

Now, to find $\mid 1, \frac{1}{2}, \frac{1}{2}\rangle$, we may write it as a linear combination of $|j_1, j_2, j_3; +, +, -\rangle$, $|j_1, j_2, j_3; +, -, +\rangle$, and $|j_1, j_2, j_3; -, +, +\rangle$:

$$\left|1, \frac{1}{2}, \frac{1}{2}\right\rangle = \alpha|j_1, j_2, j_3; +, +, -\rangle + \beta|j_1, j_2, j_3; +, -, +\rangle + \gamma\,|j_1, j_2, j_3; -, +, +\rangle. \tag{7.382}$$

This state is orthogonal to $\mid 0, \frac{1}{2}, \frac{1}{2}\rangle$, and hence $\alpha = \gamma$; similarly, since this state is also orthogonal to $\mid 1, \frac{3}{2}, \frac{1}{2}\rangle$, we have $\alpha + \beta + \gamma = 0$, and hence $2\alpha + \beta = 0$ or $\beta = -2\alpha = -2\gamma$. Now, since all the states of (7.382) are orthonormal, we have $\alpha^2 + \beta^2 + \gamma^2 = 1$, which when combined with $\beta = -2\alpha = -2\gamma$ leads to $\alpha = \gamma = -1/\sqrt{6}$ and $\beta = 2/\sqrt{6}$. We may thus write (7.382) as

$$\left|1, \frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}\Big(-|j_1, j_2, j_3; +, +, -\rangle + 2|j_1, j_2, j_3; +, -, +\rangle - |j_1, j_2, j_3; -, +, +\rangle \Big). \tag{7.383}$$

Finally, applying $\hat{J}_-$ on the left-hand side of (7.383) and $(\hat{J}_{1-}+\hat{J}_{2-}+\hat{J}_{3-})$ on the right-hand side and equating the two results, we find

$$\left|1, \frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}\Big(|j_1, j_2, j_3; +, -, -\rangle - 2|j_1, j_2, j_3; -, +, -\rangle + |j_1, j_2, j_3; -, -, +\rangle \Big). \tag{7.384}$$

(b) Since we have three different (nonidentical) particles, their spin angular momenta mutually commute. We may thus write their Hamiltonian as $\hat{H} = -(\epsilon_0/\hbar^2)(\hat{\vec{S}}_1 + \hat{\vec{S}}_2) \cdot \hat{\vec{S}}_3$. Due to this suggestive form of $\hat{H}$, it is appropriate, as shown in (a), to start by coupling $\hat{\vec{S}}_1$ with $\hat{\vec{S}}_2$ to obtain $\hat{\vec{S}}_{12} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$, and then add $\hat{\vec{S}}_{12}$ to $\hat{\vec{S}}_3$ to generate the total spin: $\hat{\vec{S}} = \hat{\vec{S}}_{12} + \hat{\vec{S}}_3$. We may thus write $\hat{H}$ as

$$\hat{H} = -\frac{\epsilon_0}{\hbar^2}\left(\hat{\vec{S}}_1 + \hat{\vec{S}}_2\right) \cdot \hat{\vec{S}}_3 = -\frac{\epsilon_0}{\hbar^2}\hat{\vec{S}}_{12} \cdot \hat{\vec{S}}_3 = -\frac{\epsilon_0}{2\hbar^2}\left(\hat{S}^2 - \hat{S}_{12}^2 - \hat{S}_3^2\right), \tag{7.385}$$

since $\hat{\vec{S}}_{12} \cdot \hat{\vec{S}}_3 = \frac{1}{2}[(\hat{\vec{S}}_{12} + \hat{\vec{S}}_3)^2 - \hat{S}_{12}^2 - \hat{S}_3^2]$. Since the operators $\hat{H}$, $\hat{S}^2$, $\hat{S}_{12}^2$, and $\hat{S}_3^2$ mutually commute, we may select as their joint eigenstates the kets $|s_{12},\ s,\ m\rangle$; we have seen in (a) how to construct these states. The eigenvalues of $\hat{H}$ are thus given by

$$\begin{aligned} \hat{H}|s_{12}, s, m\rangle &= -\frac{\epsilon_0}{2\hbar^2}\left(\hat{S}^2 - \hat{S}_{12}^2 - \hat{S}_3^2\right)|s_{12}, s, m\rangle \\ &= -\frac{\epsilon_0}{2}\left[s(s+1) - s_{12}(s_{12}+1) - \frac{3}{4}\right]|s_{12}, s, m\rangle, \end{aligned} \tag{7.386}$$

since $s_3 = \frac{1}{2}$ and $\hat{S}_3^2|s_{12}, s, m\rangle = \hbar^2 s_3(s_3+1)|s_{12}, s, m\rangle = (3\hbar^2/4)|s_{12},\ s,\ m\rangle$.

As shown in (7.386), the energy levels of this system are degenerate with respect to m, since they depend on the quantum numbers s and s_{12} but not on m:

$$E_{s_{12},s} = -\frac{\epsilon_0}{2}\left[s(s+1) - s_{12}(s_{12}+1) - \frac{3}{4}\right]. \tag{7.387}$$

For instance, the energy $E_{s_{12},s} = E_{1,3/2} = -\epsilon_0/2$ is fourfold degenerate, since it corresponds to four different states: $|s_{12},\ s,\ m\rangle = |\ 1, \frac{3}{2}, \pm\frac{3}{2}\rangle$ and $|\ 1, \frac{3}{2}, \pm\frac{1}{2}\rangle$. Similarly, the energy $E_{0,1/2} = 0$ is twofold degenerate; the corresponding states are $|\ 0, \frac{1}{2}, \pm\frac{1}{2}\rangle$. Finally, the energy $E_{1,1/2} = \epsilon_0$ is also twofold degenerate since it corresponds to $|\ 1, \frac{1}{2}, \pm\frac{1}{2}\rangle$.

Problem 7.5

Consider a system of four nonidentical spin $\frac{1}{2}$ particles. Find the possible values of the total spin S of this system and specify the number of angular momentum eigenstates, corresponding to each value of S.

Solution

First, we need to couple two spins at a time: $\hat{\vec{S}}_{12} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$ and $\hat{\vec{S}}_{34} = \hat{\vec{S}}_3 + \hat{\vec{S}}_4$. Then we couple $\hat{\vec{S}}_{12}$ and $\hat{\vec{S}}_{34}$: $\hat{\vec{S}} = \hat{\vec{S}}_{12} + \hat{\vec{S}}_{34}$. From Problem 7.4, page 438, we have $s_{12} = 0, 1$ and $s_{34} = 0, 1$. In total there are 16 states $|sm\rangle$ since $(2s_1+1)(2s_2+1)(2s_3+1)(2s_4+1) = 2^4 = 16$.

Since $s_{12} = 0, 1$ and $s_{34} = 0, 1$, the coupling of $\hat{\vec{S}}_{12}$ and $\hat{\vec{S}}_{34}$ yields the following values for the total spin s:

- When $s_{12} = 0$ and $s_{34} = 0$ we have only one possible value, $s = 0$, and hence only one eigenstate, $|sm\rangle = |\ 0,\ 0\rangle$.
- When $s_{12} = 1$ and $s_{34} = 0$, we have $s = 1$; there are three eigenstates: $|s\ m\rangle = |\ 1,\ \pm 1\rangle$, and $|\ 1,\ 0\rangle$.
- When $s_{12} = 0$ and $s_{34} = 1$, we have $s = 1$; there are three eigenstates: $|sm\rangle = |\ 1,\ \pm 1\rangle$, and $|\ 1,\ 0\rangle$.
- When $s_{12} = 1$ and $s_{34} = 1$ we have $s = 0, 1, 2$; we have here nine eigenstates (see Problem 7.3, page 436): $|\ 0,\ 0\rangle$, $|\ 1,\ \pm 1\rangle$, $|\ 1,\ 0\rangle$, $|\ 2,\ \pm 2\rangle$, $|\ 2,\ \pm 1\rangle$, and $|\ 2,\ 0\rangle$.

In conclusion, the possible values of the total spin when coupling four $\frac{1}{2}$ spins are $s = 0,\ 1,\ 2$; the value $s = 0$ occurs twice, $s = 1$ three times, and $s = 2$ only once.

Problem 7.6
Work out the coupling of the isospins of a pion–nucleon system and infer the various states of this system.

Solution
Since the isospin of a pion meson is 1 and that of a nucleon is $\frac{1}{2}$, the total isospin of a pion–nucleon system can be obtained by coupling the isospins $t_1 = 1$ and $t_2 = \frac{1}{2}$. The various values of the total isospin lie in the range $|t_1 - t_2| < T < t_1 + t_2$; hence they are given by $T = \frac{3}{2}, \frac{1}{2}$.

The coupling of the isospins $t_1 = 1$ and $t_2 = \frac{1}{2}$ is analogous to the addition of an orbital angular momentum $l = 1$ and a spin $\frac{1}{2}$; the expressions pertaining to this coupling are listed in (7.206) to (7.211). Note that there are three different π-mesons:

$$| 1,\ 1\rangle = | \pi^+\rangle, \qquad | 1,\ 0\rangle = | \pi^0\rangle, \qquad | 1,\ -1\rangle = | \pi^-\rangle, \tag{7.388}$$

and two nucleons, a proton and a neutron:

$$\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = | p\rangle, \qquad \left|\frac{1}{2},\ -\frac{1}{2}\right\rangle = | n\rangle. \tag{7.389}$$

By analogy with (7.206) to (7.211) we can write the states corresponding to $T = \frac{3}{2}$ as

$$\left|\frac{3}{2},\ \frac{3}{2}\right\rangle = | 1,\ 1\rangle \left|\frac{1}{2},\ \frac{1}{2}\right\rangle = | \pi^+\rangle\ | p\rangle, \tag{7.390}$$

$$\left|\frac{3}{2},\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\ | 1,\ 0\rangle \left|\frac{1}{2},\ \frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\ | 1,\ 1\rangle \left|\frac{1}{2},\ -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\ | \pi^0\rangle\ | p\rangle + \frac{1}{\sqrt{3}}\ | \pi^+\rangle\ | n\rangle, \tag{7.391}$$

$$\left|\frac{3}{2},\ -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\ | 1, -1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\ | 1, 0\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\ | \pi^-\rangle\ | p\rangle + \sqrt{\frac{2}{3}}\ | \pi^0\rangle\ | n\rangle, \tag{7.392}$$

$$\left|\frac{3}{2},\ -\frac{3}{2}\right\rangle = | 1,\ -1\rangle \left|\frac{1}{2},\ -\frac{1}{2}\right\rangle = | \pi^-\rangle\ | n\rangle, \tag{7.393}$$

and those corresponding to $T = \frac{1}{2}$ as

$$\left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\ | 1,\ 1\rangle \left|\frac{1}{2},\ -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\ | 1,\ 0\rangle \left|\frac{1}{2},\ \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\ | \pi^+\rangle\ | n\rangle - \frac{1}{\sqrt{3}}\ | \pi^0\rangle\ | p\rangle, \tag{7.394}$$

$$\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\ | 1, 0\rangle \left|\frac{1}{2},\ -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\ | 1,\ -1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\ | \pi^0\rangle\ | n\rangle - \sqrt{\frac{2}{3}}\ | \pi^-\rangle\ | p\rangle. \tag{7.395}$$

Problem 7.7
(a) Calculate the expression of $\langle 2,\ 0|Y_{10}\ |\ 1,\ 0\rangle$.

(b) Use the result of (a) along with the Wigner–Eckart theorem to calculate the reduced matrix element $\langle 2\ \|\ Y_1\ \|\ 1\rangle$.

Solution

(a) Since

$$\langle 2,\ 0 \mid Y_{10} \mid 1,\ 0\rangle = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} Y_{20}^*(\theta,\varphi) Y_{10}(\theta,\varphi) Y_{10}(\theta,\varphi)\, d\varphi, \tag{7.396}$$

and using the relations $Y_{20}(\theta,\varphi) = \sqrt{5/(16\pi)}(3\cos^2\theta - 1)$ and $Y_{10}(\theta,\varphi) = \sqrt{3/(4\pi)}\cos\theta$, we have

$$\begin{aligned}\langle 2,\ 0 \mid Y_{10} \mid 1,\ 0\rangle &= \frac{3}{4\pi}\sqrt{\frac{5}{16\pi}} \int_0^{\pi} \cos^2\theta(3\cos^2\theta - 1)\sin\theta\, d\theta \int_0^{2\pi} d\varphi \\ &= \frac{3}{2}\sqrt{\frac{5}{16\pi}} \int_0^{\pi} \cos^2\theta(3\cos^2\theta - 1)\sin\theta\, d\theta.\end{aligned} \tag{7.397}$$

The change of variables $x = \cos\theta$ leads to

$$\begin{aligned}\langle 2,\ 0 \mid Y_{10} \mid 1,\ 0\rangle &= \frac{3}{2}\sqrt{\frac{5}{16\pi}} \int_0^{\pi} \cos^2\theta(3\cos^3\theta - 1)\sin\theta\, d\theta \\ &= \frac{3}{2}\sqrt{\frac{5}{16\pi}} \int_{-1}^{1} x^2(3x^2 - 1)\, dx = \frac{1}{\sqrt{5\pi}}.\end{aligned} \tag{7.398}$$

(b) Applying the Wigner–Eckart theorem to $\langle 2,\ 0 \mid Y_{10} \mid 1,\ 0\rangle$ and using the Clebsch–Gordan coefficient $\langle 1, 1;\ 0, 0 \mid 2,\ 0\rangle = 2/\sqrt{6}$, we have

$$\langle 2,\ 0 \mid Y_{10} \mid 1,\ 0\rangle = \langle 1, 1;\ 0, 0 \mid 2,\ 0\rangle\langle 2 \parallel Y_1 \parallel 1\rangle = \frac{2}{\sqrt{6}}\langle 2 \parallel Y_1 \parallel 1\rangle. \tag{7.399}$$

Finally, we may obtain $\langle 2 \parallel Y_1 \parallel 1\rangle$ from (7.398) and (7.399):

$$\langle 2 \parallel Y_1 \parallel 1\rangle = \sqrt{\frac{3}{10\pi}}. \tag{7.400}$$

Problem 7.8

(a) Find the reduced matrix elements associated with the spherical harmonic $Y_{kq}(\theta,\varphi)$.

(b) Calculate the dipole transitions $\langle n'l'm' \mid \vec{r} \mid nlm\rangle$.

Solution

On the one hand, an application of the Wigner–Eckart theorem to Y_{kq} yields

$$\langle l',\ m' \mid Y_{kq} \mid l,\ m\rangle = \langle l, k;\ m, q \mid l',\ m'\rangle\langle l' \parallel Y^{(k)} \parallel l\rangle \tag{7.401}$$

and, on the other hand, a straightforward evaluation of

$$\begin{aligned}\langle l',\ m' \mid Y_{kq} \mid l,\ m\rangle &= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \langle l',\ m \mid \theta\varphi\rangle Y_{kq}(\theta,\varphi)\langle\theta\varphi \mid l,\ m\rangle \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta Y_{l'm'}^*(\theta,\varphi) Y_{kq}(\theta,\varphi) Y_{lm}(\theta,\varphi)\end{aligned} \tag{7.402}$$

can be inferred from the triple integral relation (7.244):

$$\boxed{\langle l',\ m' \mid Y_{kq} \mid l,\ m\rangle = \sqrt{\frac{(2l+1)(2k+1)}{4\pi(2l'+1)}}\langle l, k;\ 0, 0|l',\ 0\rangle\langle l, k;\ m, q \mid l',\ m'\rangle.} \tag{7.403}$$

We can then combine (7.401) and (7.403) to obtain the reduced matrix element

$$\boxed{\langle l' \parallel Y^{(k)} \parallel l\rangle = \sqrt{\frac{(2l+1)(2k+1)}{4\pi(2l'+1)}}\langle l, k;\ 0, 0|l',\ 0\rangle.} \tag{7.404}$$

(b) To calculate $\langle n'l'm' \mid \vec{r} \mid nlm\rangle$ it is more convenient to express the vector $\vec{r}$ in terms of the spherical components $\vec{r} = (r_{-1}, r_0, r_1)$, which are given in terms of the Cartesian coordinates x, y, z as follows:

$$r_1 = -\frac{x+iy}{\sqrt{2}} = \frac{r}{\sqrt{2}}e^{i\varphi}\sin\theta, \qquad r_0 = z = r\cos\theta, \qquad r_{-1} = \frac{x-iy}{\sqrt{2}} = \frac{r}{\sqrt{2}}e^{-i\varphi}\sin\theta, \tag{7.405}$$

which in turn may be condensed into a single relation

$$r_q = \sqrt{\frac{4\pi}{3}}rY_{1q}(\theta, \varphi), \qquad q = 1, 0, -1. \tag{7.406}$$

Next we may write $\langle n'l'm' \mid r_q \mid nlm\rangle$ in terms of a radial part and an angular part:

$$\langle n'l'm' \mid r_q \mid nlm\rangle = \sqrt{\frac{4\pi}{3}}\langle n'l' \mid r_q|nl\rangle\langle l',\ m'|Y_{1q}(\theta, \varphi) \mid l,\ m\rangle. \tag{7.407}$$

The calculation of the radial part, $\langle n'l' \mid r_q \mid nl\rangle = \int_0^\infty r^3 R^*_{n'l'}(r)R^*_{nl}(r)\,dr$, is straightforward and is of no concern to us here; see Chapter 6 for its calculation. As for the angular part $\langle l',\ m' \mid Y_{1q}(\theta, \varphi) \mid l,\ m\rangle$, we can infer its expression from (7.403)

$$\langle l',\ m' \mid Y_{1q} \mid l,\ m\rangle = \sqrt{\frac{3(2l+1)}{4\pi(2l'+1)}}\langle l, 1;\ 0, 0|l',\ 0\rangle\langle l, 1;\ m, q \mid l',\ m'\rangle. \tag{7.408}$$

The Clebsch–Gordan coefficients $\langle l, 1;\ m, q \mid l',\ m'\rangle$ vanish unless $m' = m + q$ and $l - 1 \leq l' \leq l + 1$ or $\Delta m = m' - m = q = 1, 0, -1$ and $\Delta l = l' - l = 1, 0, -1$. Notice that the case $\Delta l = 0$ is ruled out from the parity selection rule; so, the only permissible values of l' and l are those for which $\Delta l = l' - l = \pm 1$. Obtaining the various relevant Clebsch–Gordan coefficients from standard tables, we can ascertain that the only terms of (7.408) that survive are

$$\langle l+1,\ m+1|Y_{11} \mid l,\ m\rangle = \sqrt{\frac{3(l+m+1)(l+m+2)}{8\pi(2l+1)(2l+3)}}, \tag{7.409}$$

$$\langle l-1,\ m+1|Y_{11} \mid l,\ m\rangle = \sqrt{\frac{3(l-m-1)(l-m)}{8\pi(2l+1)(2l+3)}}, \tag{7.410}$$

$$\langle l+1,\ m|Y_{10} \mid l,\ m\rangle = \sqrt{\frac{3[(l+1)^2-m^2]}{4\pi(2l+1)(2l+3)}}, \tag{7.411}$$

$$\langle l-1,\ m|Y_{10}\mid l,\ m\rangle \ = \ \sqrt{\frac{3(l^2-m^2)}{4\pi(2l+1)(2l-1)}}, \tag{7.412}$$

$$\langle l+1,\ m-1|Y_{1-1}\mid l,\ m\rangle \ = \ \sqrt{\frac{3(l-m+1)(l-m+2)}{8\pi(2l+1)(2l+3)}}, \tag{7.413}$$

$$\langle l-1,\ m-1|Y_{1-1}\mid l,\ m\rangle \ = \ \sqrt{\frac{3(l+m)(l+m-1)}{8\pi(2l+1)(2l-1)}}. \tag{7.414}$$

Problem 7.9
Find the rotation matrix $d^{(1)}$ corresponding to $j=1$.

Solution
To find the matrix of $d^{(1)}(\beta)=e^{-i\beta\hat{J}_y/\hbar}$ for $j=1$, we need first to find the matrix representation of $\hat{J}_y$ within the joint eigenstates $\{\mid j,\ m\rangle\}$ of $\hat{\vec{J}}^2$ and $\hat{J}_z$. Since the basis of $j=1$ consists of three states $\mid 1,\ -1\rangle$, $\mid 1,\ 0\rangle$, $\mid 1,\ 1\rangle$, the matrix representing $\hat{J}_y$ within this basis is given by

$$\begin{aligned} J_y \ &= \ \frac{\hbar}{2}\begin{pmatrix} \langle 1,\ 1\mid \hat{J}_y\mid 1,\ 1\rangle & \langle 1,\ 1\mid \hat{J}_y\mid 1,\ 0\rangle & \langle 1,\ 1\mid \hat{J}_y\mid 1,\ -1\rangle \\ \langle 1,\ 0\mid \hat{J}_y\mid 1,\ 1\rangle & \langle 1,\ 0\mid \hat{J}_y\mid 1,\ 0\rangle & \langle 1,\ 0\mid \hat{J}_y\mid 1,\ -1\rangle \\ \langle 1,\ -1\mid \hat{J}_y\mid 1,\ 1\rangle & \langle 1,\ -1\mid \hat{J}_y\mid 1,\ 0\rangle & \langle 1,\ -1\mid \hat{J}_y\mid 1,\ -1\rangle \end{pmatrix} \\ &= \ \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{7.415}$$

We can easily verify that $J_y^3=J_y$:

$$J_y^2=\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad J_y^3=\frac{i\hbar^3}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}=\hbar^2 J_y. \tag{7.416}$$

We can thus infer

$$J_y^{2n}=\hbar^{2n-2}J_y^2 \qquad (n>0), \qquad\qquad J_y^{2n+1}=\hbar^{2n}J_y. \tag{7.417}$$

Combining these two relations with

$$\begin{aligned} e^{-i\beta\hat{J}_y/\hbar} \ &= \ \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\beta}{\hbar}\right)^n J_y^n \\ &= \ \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(-\frac{i\beta}{\hbar}\right)^{2n} J_y^{2n}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\frac{i\beta}{\hbar}\right)^{2n+1} J_y^{2n+1}, \end{aligned} \tag{7.418}$$

we obtain

$$e^{-i\beta\hat{J}_y/\hbar} \ = \ \hat{I}+\left(\frac{\hat{J}_y}{\hbar}\right)^2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(\beta)^{2n}-i\frac{\hat{J}_y}{\hbar}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\beta^{2n+1}$$

$$= \hat{I} + \left(\frac{\hat{J}_y}{\hbar}\right)^2 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\beta)^{2n} - 1\right] - i\frac{\hat{J}_y}{\hbar}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\beta^{2n+1}, \tag{7.419}$$

where $\hat{I}$ is the 3×3 unit matrix. Using the relations $\sum_{n=0}^{\infty}[(-1)^n/(2n)!](\beta)^{2n} = \cos\beta$ and $\sum_{n=0}^{\infty}[(-1)^n/(2n+1)!]\beta^{2n+1} = \sin\beta$, we may write

$$e^{-i\beta\hat{J}_y/\hbar} = \hat{I} + \left(\frac{\hat{J}_y}{\hbar}\right)^2 [\cos\beta - 1] - i\frac{\hat{J}_y}{\hbar}\sin\beta. \tag{7.420}$$

Inserting now the matrix expressions for J_y and J_y^2 as listed in (7.415) and (7.416), we obtain

$$e^{-i\beta\hat{J}_y/\hbar} = \hat{I} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}(\cos\beta - 1) - i\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\sin\beta \tag{7.421}$$

or

$$d^{(1)}(\beta) = \begin{pmatrix} d_{11}^{(1)} & d_{1,0}^{(1)} & d_{1-1}^{(1)} \\ d_{01}^{(1)} & d_{00}^{(1)} & d_{0-1}^{(1)} \\ d_{-11}^{(1)} & d_{-10}^{(1)} & d_{-1-1}^{(1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}. \tag{7.422}$$

Since $\frac{1}{2}(1+\cos\beta) = \cos^2(\beta/2)$ and $\frac{1}{2}(1-\cos\beta) = \sin^2(\beta/2)$, we have

$$d^{(1)}(\beta) = e^{-i\beta J_y/\hbar} = \begin{pmatrix} \cos^2(\beta/2) & -\frac{1}{\sqrt{2}}\sin(\beta) & \sin^2(\beta/2) \\ \frac{1}{\sqrt{2}}\sin(\beta) & \cos(\beta) & -\frac{1}{\sqrt{2}}\sin(\beta) \\ \sin^2(\beta/2) & \frac{1}{\sqrt{2}}\sin(\beta) & \cos^2(\beta/2) \end{pmatrix}. \tag{7.423}$$

This method becomes quite intractable when attempting to derive the matrix of $d^{(j)}(\beta)$ for large values of j. In Problem 7.10 we are going to present a simpler method for deriving $d^{(j)}(\beta)$ for larger values of j; this method is based on the addition of angular momenta.

Problem 7.10

(a) Use the relation

$$d_{mm'}^{(j)}(\beta) = \sum_{m_1 m_2}\sum_{m_1' m_2'} \langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle\langle j_1, j_2;\ m_1', m_2' \mid j,\ m'\rangle\ d_{m_1 m_1'}^{(j_1)}(\beta) d_{m_2 m_2'}^{(j_2)}(\beta),$$

for the case where $j_1 = 1$ and $j_2 = \frac{1}{2}$ along with the Clebsch–Gordan coefficients derived in (7.206) to (7.209), and the matrix elements of $d^{(1/2)}(\beta)$ and $d^{(1)}(\beta)$, which are given by (7.89) and (7.423), respectively, to find the expressions of the matrix elements of $d_{\frac{3}{2}\frac{3}{2}}^{(3/2)}(\beta)$, $d_{\frac{3}{2}\frac{1}{2}}^{(3/2)}(\beta)$, $d_{\frac{3}{2}-\frac{1}{2}}^{(3/2)}(\beta)$, $d_{\frac{3}{2}-\frac{3}{2}}^{(3/2)}(\beta)$, $d_{\frac{1}{2}\frac{1}{2}}^{(3/2)}(\beta)$, and $d_{\frac{1}{2}-\frac{1}{2}}^{(3/2)}(\beta)$.

(b) Use the six expressions derived in (a) to infer the matrix of $d^{(3/2)}(\beta)$.

Solution

(a) Using $\langle 1, \frac{1}{2};\ 1, \frac{1}{2} \mid \frac{3}{2},\ \frac{3}{2}\rangle = 1$, $d_{11}^{(1)}(\beta) = \cos^2(\beta/2)$ and $d_{\frac{1}{2}\frac{1}{2}}^{(1/2)}(\beta) = \cos(\beta/2)$, we have

$$d_{\frac{3}{2}\frac{3}{2}}^{(3/2)}(\beta) = \left\langle 1, \frac{1}{2};\ 1, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{3}{2}\right\rangle\left\langle 1, \frac{1}{2};\ 1, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{3}{2}\right\rangle d_{11}^{(1)}(\beta) d_{\frac{1}{2}\frac{1}{2}}^{(1/2)}(\beta) = \cos^3\left(\frac{\beta}{2}\right). \quad (7.424)$$

Similarly, since $\langle 1, \frac{1}{2};\ 0, \frac{1}{2} \mid \frac{3}{2},\ \frac{1}{2}\rangle = \sqrt{2/3}$, $\langle 1, \frac{1}{2};\ 1, -\frac{1}{2} \mid \frac{3}{2},\ \frac{1}{2}\rangle = 1/\sqrt{3}$, and since $d_{10}^{(1)}(\beta) = -(1/\sqrt{2})\sin(\beta)$ and $d_{\frac{1}{2}-\frac{1}{2}}^{(1/2)}(\beta) = -\sin(\beta/2)$, we have

$$\begin{aligned} d_{\frac{3}{2}\frac{1}{2}}^{(3/2)}(\beta) &= \left\langle 1, \frac{1}{2};\ 1, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{3}{2}\right\rangle\left\langle 1, \frac{1}{2};\ 0, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{1}{2}\right\rangle d_{10}^{(1)}(\beta) d_{\frac{1}{2}\frac{1}{2}}^{(1/2)}(\beta) \\ &\quad + \left\langle 1, \frac{1}{2};\ 1, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{3}{2}\right\rangle\left\langle 1, \frac{1}{2};\ 1, -\frac{1}{2}\ \middle|\frac{3}{2},\ \frac{1}{2}\right\rangle d_{11}^{(1)}(\beta) d_{\frac{1}{2}-\frac{1}{2}}^{(1/2)}(\beta) \\ &= -\frac{1}{\sqrt{3}}\sin\beta\cos\left(\frac{\beta}{2}\right) - \frac{1}{\sqrt{3}}\cos^2\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) \\ &= -\sqrt{3}\sin\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right). \end{aligned} \quad (7.425)$$

To calculate $d_{\frac{3}{2}-\frac{1}{2}}^{(3/2)}(\beta)$, we need to use the coefficients $\langle 1, \frac{1}{2};\ 0, -\frac{1}{2} \mid \frac{3}{2},\ -\frac{1}{2}\rangle = \sqrt{2/3}$ and $\langle 1, \frac{1}{2};\ -1, \frac{1}{2} \mid \frac{3}{2},\ -\frac{1}{2}\rangle = 1/\sqrt{3}$ along with $d_{1,-1}^{(1)}(\beta) = \sin^2(\beta/2)$:

$$\begin{aligned} d_{\frac{3}{2}-\frac{1}{2}}^{(3/2)}(\beta) &= \left\langle 1, \frac{1}{2};\ 1, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{3}{2}\right\rangle\left\langle 1, \frac{1}{2};\ -1, \frac{1}{2}\ \middle|\frac{3}{2},\ -\frac{1}{2}\right\rangle d_{1-1}^{(1)}(\beta) d_{\frac{1}{2}\frac{1}{2}}^{(1/2)}(\beta) \\ &\quad + \left\langle 1, \frac{1}{2};\ 1, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{3}{2}\right\rangle\left\langle 1, \frac{1}{2};\ 0, -\frac{1}{2}\ \middle|\frac{3}{2},\ -\frac{1}{2}\right\rangle d_{10}^{(1)}(\beta) d_{\frac{1}{2}-\frac{1}{2}}^{(1/2)}(\beta) \\ &= \frac{1}{\sqrt{3}}\sin^2\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) + \frac{1}{\sqrt{3}}\sin\beta\sin\left(\frac{\beta}{2}\right) \\ &= \sqrt{3}\sin^2\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right). \end{aligned} \quad (7.426)$$

For $d_{\frac{3}{2}-\frac{3}{2}}^{(3/2)}(\beta)$ we have

$$d_{\frac{3}{2}-\frac{3}{2}}^{(3/2)}(\beta) = \left\langle 1, \frac{1}{2}; 1, \frac{1}{2}\ \middle|\frac{3}{2}, \frac{3}{2}\right\rangle\left\langle 1, \frac{1}{2};\ -1, -\frac{1}{2}\ \middle|\frac{3}{2},\ -\frac{3}{2}\right\rangle d_{1-1}^{(1)}(\beta) d_{\frac{1}{2},-\frac{1}{2}}^{(1/2)}(\beta) = -\sin^3\left(\frac{\beta}{2}\right), \quad (7.427)$$

because $\langle 1, \frac{1}{2};\ -1, -\frac{1}{2} \mid \frac{3}{2},\ -\frac{3}{2}\rangle = 1$, $d_{1-1}^{(1)}(\beta) = \sin^2(\beta/2)$, and $d_{\frac{1}{2}-\frac{1}{2}}^{(1/2)}(\beta) = -\sin(\beta/2)$.

To calculate $d_{\frac{1}{2}\frac{1}{2}}^{(3/2)}(\beta)$, we need to use the coefficients $\langle 1, \frac{1}{2};\ 0, \frac{1}{2} \mid \frac{3}{2},\ \frac{1}{2}\rangle = \sqrt{2/3}$ and $\langle 1, \frac{1}{2};\ 1, -\frac{1}{2} \mid \frac{3}{2},\ \frac{1}{2}\rangle = 1/\sqrt{3}$ along with $d_{1-1}^{(1)}(\beta) = \sin^2(\beta/2)$:

$$\begin{aligned} d_{\frac{1}{2}\frac{1}{2}}^{(3/2)}(\beta) &= \left\langle 1, \frac{1}{2};\ 1, -\frac{1}{2}\ \middle|\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1, \frac{1}{2};\ 1, -\frac{1}{2}\ \middle|\frac{3}{2},\ \frac{1}{2}\right\rangle d_{11}^{(1)}(\beta) d_{-\frac{1}{2}-\frac{1}{2}}^{(1/2)}(\beta) \\ &\quad + \left\langle 1, \frac{1}{2};\ 0, \frac{1}{2}\ \middle|\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1, \frac{1}{2};\ 1, -\frac{1}{2}\ \middle|\frac{3}{2},\ \frac{1}{2}\right\rangle d_{01}^{(1)}(\beta) d_{\frac{1}{2}-\frac{1}{2}}^{(1/2)}(\beta) \end{aligned}$$

$$
\begin{aligned}
&+\left\langle 1,\frac{1}{2};\ 0,\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1,\frac{1}{2};\ 0,\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle d^{(1)}_{00}(\beta)d^{(1/2)}_{\frac{1}{2}\frac{1}{2}}(\beta)\\
&+\left\langle 1,\frac{1}{2};\ 1,-\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1,\frac{1}{2};\ 0,\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle d^{(1)}_{10}(\beta)d^{(1/2)}_{-\frac{1}{2}\frac{1}{2}}(\beta)\\
=\ &\frac{1}{3}\cos^3\left(\frac{\beta}{2}\right)-\frac{1}{3}\sin(\beta)\sin\left(\frac{\beta}{2}\right)+\frac{2}{3}\cos(\beta)\cos\left(\frac{\beta}{2}\right)-\frac{1}{3}\sin(\beta)\sin\left(\frac{\beta}{2}\right)\\
=\ &\left[3\cos^2\left(\frac{\beta}{2}\right)-2\right]\cos\left(\frac{\beta}{2}\right)\\
=\ &\frac{1}{2}\left(3\cos\beta-1\right)\cos\left(\frac{\beta}{2}\right).
\end{aligned}
\tag{7.428}
$$

Similarly, we have

$$
\begin{aligned}
d^{(3/2)}_{\frac{1}{2}-\frac{1}{2}}(\beta)\ =\ &\left\langle 1,\frac{1}{2};\ 1,-\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1,\frac{1}{2};\ -1,\frac{1}{2}\,\middle|\,\frac{3}{2},\ -\frac{1}{2}\right\rangle d^{(1)}_{1-1}(\beta)d^{(1/2)}_{-\frac{1}{2}\frac{1}{2}}(\beta)\\
&+\left\langle 1,\frac{1}{2};\ 1,-\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1,\frac{1}{2};\ 0,-\frac{1}{2}\,\middle|\,\frac{3}{2},\ -\frac{1}{2}\right\rangle d^{(1)}_{10}(\beta)d^{(1/2)}_{-\frac{1}{2}-\frac{1}{2}}(\beta)\\
&+\left\langle 1,\frac{1}{2};\ 0,\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1,\frac{1}{2};\ -1,\frac{1}{2}\,\middle|\,\frac{3}{2},\ -\frac{1}{2}\right\rangle d^{(1)}_{0-1}(\beta)d^{(1/2)}_{\frac{1}{2}\frac{1}{2}}(\beta)\\
&+\left\langle 1,\frac{1}{2};\ 0,\frac{1}{2}\,\middle|\,\frac{3}{2},\ \frac{1}{2}\right\rangle\left\langle 1,\frac{1}{2};\ 0,-\frac{1}{2}\,\middle|\,\frac{3}{2},\ -\frac{1}{2}\right\rangle d^{(1)}_{00}(\beta)d^{(1/2)}_{\frac{1}{2}-\frac{1}{2}}(\beta)\\
=\ &\frac{1}{3}\sin^3\left(\frac{\beta}{2}\right)-\frac{1}{3}\sin(\beta)\cos\left(\frac{\beta}{2}\right)-\frac{1}{3}\sin(\beta)\cos\left(\frac{\beta}{2}\right)-\frac{2}{3}\cos(\beta)\sin\left(\frac{\beta}{2}\right)\\
=\ &-\left[3\cos^2\left(\frac{\beta}{2}\right)-1\right]\sin\left(\frac{\beta}{2}\right)\\
=\ &-\frac{1}{2}\left(3\cos\beta+1\right)\sin\left(\frac{\beta}{2}\right).
\end{aligned}
\tag{7.429}
$$

(b) The remaining ten matrix elements of $d^{(3/2)}(\beta)$ can be inferred from the six elements derived above by making use of the properties of the d-function listed in (7.67). For instance, using $d^{(j)}_{m'm}(\beta)=(-1)^{m'-m}d^{(j)}_{-m'-m}(\beta)$, we can verify that

$$
d^{(3/2)}_{-\frac{3}{2}-\frac{3}{2}}(\beta)\ =\ d^{(3/2)}_{\frac{3}{2}\frac{3}{2}}(\beta),\qquad d^{(3/2)}_{-\frac{1}{2}-\frac{1}{2}}(\beta)=d^{(3/2)}_{\frac{1}{2}\frac{1}{2}}(\beta),\qquad d^{(3/2)}_{-\frac{3}{2}-\frac{1}{2}}(\beta)=-d^{(3/2)}_{\frac{3}{2}\frac{1}{2}}(\beta),
\tag{7.430}
$$

$$
d^{(3/2)}_{-\frac{3}{2}\frac{1}{2}}(\beta)\ =\ d^{(3/2)}_{\frac{3}{2}-\frac{1}{2}}(\beta),\qquad d^{(3/2)}_{-\frac{3}{2}\frac{3}{2}}(\beta)=-d^{(3/2)}_{\frac{3}{2}-\frac{1}{2}}(\beta),\qquad d^{(3/2)}_{-\frac{1}{2}\frac{1}{2}}(\beta)=-d^{(3/2)}_{\frac{3}{2}-\frac{1}{2}}(\beta).
\tag{7.431}
$$

Similarly, using $d^{(j)}_{m'm}(\beta)=(-1)^{m'-m}d^{(j)}_{mm'}(\beta)$ we can obtain the remaining four elements:

$$
d^{(3/2)}_{\frac{1}{2}\frac{3}{2}}(\beta)\ =\ -d^{(3/2)}_{\frac{3}{2}\frac{1}{2}}(\beta),\qquad d^{(3/2)}_{-\frac{1}{2}\frac{3}{2}}(\beta)=d^{(3/2)}_{\frac{3}{2}-\frac{1}{2}}(\beta),
\tag{7.432}
$$

$$
d^{(3/2)}_{\frac{1}{2}-\frac{3}{2}}(\beta)\ =\ d^{(3/2)}_{-\frac{3}{2}\frac{1}{2}}(\beta),\qquad d^{(3/2)}_{-\frac{1}{2}-\frac{3}{2}}(\beta)=-d^{(3/2)}_{-\frac{3}{2}-\frac{1}{2}}(\beta).
\tag{7.433}
$$

Collecting the six matrix elements calculated in (a) along with the ten elements inferred above, we obtain the matrix of $d^{(3/2)}(\beta)$:

$$\begin{pmatrix} \cos^3\left(\frac{\beta}{2}\right) & -\sqrt{3}\sin\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right) & \sqrt{3}\sin^2\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) & -\sin^3\left(\frac{\beta}{2}\right) \\ \sqrt{3}\sin\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right) & \frac{1}{2}\left(3\cos\beta-1\right)\cos\left(\frac{\beta}{2}\right) & -\frac{1}{2}\left(3\cos\beta+1\right)\sin\left(\frac{\beta}{2}\right) & \sqrt{3}\sin^2\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) \\ \sqrt{3}\sin^2\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) & \frac{1}{2}\left(3\cos\beta+1\right)\sin\left(\frac{\beta}{2}\right) & \frac{1}{2}\left(3\cos\beta-1\right)\cos\left(\frac{\beta}{2}\right) & -\sqrt{3}\sin\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right) \\ \sin^3\left(\frac{\beta}{2}\right) & \sqrt{3}\sin^2\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) & \sqrt{3}\sin\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right) & \cos^3\left(\frac{\beta}{2}\right) \end{pmatrix}, \tag{7.434}$$

which can be reduced to

$$d^{3/2}(\beta) = \frac{\sin\beta}{2}\begin{pmatrix} \frac{\cos^2(\beta/2)}{\sin(\beta/2)} & -\sqrt{3}\cos\left(\frac{\beta}{2}\right) & \sqrt{3}\sin\left(\frac{\beta}{2}\right) & -\frac{\sin^2(\beta/2)}{\cos(\beta/2)} \\ \sqrt{3}\cos\left(\frac{\beta}{2}\right) & \frac{3\cos\beta-1}{2\sin(\beta/2)} & -\frac{3\cos\beta+1}{2\cos(\beta/2)} & \sqrt{3}\sin\left(\frac{\beta}{2}\right) \\ \sqrt{3}\sin\left(\frac{\beta}{2}\right) & \frac{3\cos\beta+1}{\cos(\beta/2)} & \frac{3\cos\beta-1}{2\sin(\beta/2)} & -\sqrt{3}\cos\left(\frac{\beta}{2}\right) \\ \frac{\sin^2(\beta/2)}{\cos(\beta/2)} & \sqrt{3}\sin\left(\frac{\beta}{2}\right) & \sqrt{3}\cos\left(\frac{\beta}{2}\right) & \frac{\cos^2(\beta/2)}{\sin(\beta/2)} \end{pmatrix}. \tag{7.435}$$

Following the method outlined in this problem, we can in principle find the matrix of any d-function. For instance, using the matrices of $d^{(1)}$ and $d^{(1/2)}$ along with the Clebsch–Gordan coefficients resulting from the addition of $j_1 = 1$ and $j_2 = 1$, we can find the matrix of $d^{(2)}(\beta)$.

Problem 7.11

Consider two nonidentical particles each with angular momenta 1 and whose Hamiltonian is given by

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}(\hat{\vec{L}}_1 + \hat{\vec{L}}_2)\cdot\hat{\vec{L}}_2 + \frac{\varepsilon_2}{\hbar^2}(\hat{L}_{1_z} + \hat{L}_{2_z})^2,$$

where ε_1 and ε_2 are constants having the dimensions of energy. Find the energy levels and their degeneracies for those states of the system whose total angular momentum is equal to $2\hbar$.

Solution

The total angular momentum of the system is obtained by coupling $l_1 = 1$ and $l_2 = 1$: $\hat{\vec{L}} = \hat{\vec{L}}_1 + \hat{\vec{L}}_2$. This leads to $\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2 = \frac{1}{2}(\hat{L}^2 - \hat{L}_1^2 - \hat{L}_2^2)$, and when this is inserted into the system's Hamiltonian it yields

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}(\hat{\vec{L}}_1\cdot\hat{\vec{L}}_2 + \hat{L}_2^2) + \frac{\varepsilon_2}{\hbar^2}\hat{L}_z^2 = \frac{\varepsilon_1}{2\hbar^2}(\hat{L}^2 - \hat{L}_1^2 + \hat{L}_2^2) + \frac{\varepsilon_2}{\hbar^2}\hat{L}_z^2. \tag{7.436}$$

Notice that the operators $\hat{H}$, $\hat{L}_1^2$, $\hat{L}_2^2$, $\hat{L}^2$, and $\hat{L}_z$ mutually commute; we denote their joint eigenstates by $\mid l,\ m\rangle$. The energy levels of (7.436) are thus given by

$$E_{lm} = \frac{\varepsilon_1}{2}\Big[\, l(l+1) - l_1(l_1+1) + l_2(l_2+1)\Big] + \varepsilon_2 m^2 = \frac{\varepsilon_1}{2}l(l+1) + \varepsilon_2 m^2, \tag{7.437}$$

since $l_1 = l_2 = 1$.

The calculation of $\mid l,\ m\rangle$ in terms of the states $|l_1,\ m_1\rangle|l_2,\ m_2\rangle = |l_1, l_2;\ m_1, m_2\rangle$ was carried out in Problem 7.3, page 436; the states corresponding to a total angular momentum of $l = 2$ are given by

$$\mid 2,\ \pm 2\rangle = |1, 1;\ \pm 1, \pm 1\rangle, \qquad |2,\ \pm 1\rangle = \frac{1}{\sqrt{2}}\Big(\,|1, 1;\ \pm 1, 0\rangle + |1, 1;\ 0, \pm 1\rangle\Big), \tag{7.438}$$

$$| 2, \ 0\rangle = \frac{1}{\sqrt{6}}\Big(|1, 1; \ 1, -1\rangle + 2|1, 1; \ 0, 0\rangle + |1, 1; \ -1, 1\rangle\Big). \tag{7.439}$$

From (7.437) we see that the energy corresponding to $l = 2$ and $m = \pm 2$ is doubly degenerate, because the states $| 2, \ \pm 2\rangle$ have the same energy $E_{2,\pm 2} = 3\varepsilon_1 + 4\varepsilon_2$. The two states $| 2, \ \pm 1\rangle$ are also degenerate, for they correspond to the same energy $E_{2,\pm 1} = 3\varepsilon_1 + \varepsilon_2$. The energy corresponding to $| 2, \ 0\rangle$ is not degenerate: $E_{20} = 3\varepsilon_1$.

7.6 Exercises

Exercise 7.1
Show that the linear transformation $y = Rx$ where

$$R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is a counterclockwise rotation of the Cartesian x_1x_2 coordinate system in the plane about the origin with an angle ϕ.

Exercise 7.2
Show that the nth power of the rotation matrix

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

is equal to

$$R^n(\phi) = \begin{pmatrix} \cos(n\phi) & -\sin(n\phi) \\ \sin(n\phi) & \cos(n\phi) \end{pmatrix}.$$

What is the geometrical meaning of this result?

Exercise 7.3
Using the space displacement operator $U(\hat{\vec{A}}) = e^{-i\hat{\vec{A}}\cdot\hat{\vec{P}}/\hbar}$, where $\hat{\vec{P}}$ is the linear momentum operator, show that $e^{i\hat{\vec{A}}\cdot\hat{\vec{P}}/\hbar}\ \hat{\vec{R}}\ e^{-i\hat{\vec{A}}\cdot\hat{\vec{P}}/\hbar} = \hat{\vec{R}} + \hat{\vec{A}}$.

Exercise 7.4
The components A_j (with $j = x, y, z$) of a vector $\hat{\vec{A}}$ transform under space rotations as $A'_i = R_{ij}A_j$, where R is the rotation matrix.

(a) Using the invariance of the scalar product of any two vectors (e.g., $\hat{\vec{A}}\cdot\hat{\vec{B}}$) under rotations, show that the rows and columns of the rotation matrix R are orthonormal to each other (i.e., show that $R_{lj}R_{lk} = \delta_{j,k}$).

(b) Show that the transpose of R is equal to the inverse of R and that the determinant of R is equal to ± 1.

Exercise 7.5
The operator corresponding to a rotation of angle θ about an axis $\vec{n}$ is given by

$$U_{\vec{n}}(\theta) = e^{-i\theta\vec{n}\cdot\hat{\vec{J}}/\hbar}.$$

Show that the matrix elements of the position operator $\hat{\vec{R}}$ are rotated through an infinitesimal rotation like $\vec{R}' = \hat{\vec{R}} + \theta\vec{n} \times \hat{\vec{R}}$. (i.e., in the case where θ is infinitesimal, show that $U_n^+(\theta)\hat{R}_j U_n(\theta) = \hat{R}_j + \theta(\vec{n} \times \hat{\vec{R}})_j$).

Exercise 7.6
Consider the wave function of a particle $\psi(\vec{r}) = (\sqrt{2}x + \sqrt{2}y + z) f(r)$, where $f(r)$ is a spherically symmetric function.

(a) Is $\psi(\vec{r})$ an eigenfunction of $\vec{L}^2$? If so, what is the eigenvalue?

(b) What are the probabilities for the particle to be found in the state $m_l = -1$, $m_l = 0$, and $m_l = 1$?

(c) If $\psi(\vec{r})$ is an energy eigenfunction with eigenvalues E and if $f(r) = 3r^2$, find the expression of the potential $V(r)$ to which this particle is subjected.

Exercise 7.7
Consider a particle whose wave function is given by

$$\psi(\vec{r}) = \left(\frac{1}{\sqrt{5}} Y_{11}(\theta, \varphi) - \frac{1}{5} Y_{1-1}(\theta, \varphi) + \frac{1}{\sqrt{2}} Y_{10}(\theta, \varphi)\right) f(r),$$

where $f(r)$ is a normalized radial function, i.e., $\int_0^\infty r^2 f^2(r)\, dr = 1$.

(a) Calculate the expectation values of $\hat{\vec{L}}^2$, $\hat{L}_z$, and $\hat{L}_x$ in this state.

(b) Calculate the expectation value of $V(\theta) = 2\cos^2\theta$ in this state.

(c) Find the probability that the particle will be found in the state $m_l = 0$.

Exercise 7.8
A particle of spin $\frac{1}{2}$ is in a d state of orbital angular momentum (i.e., $l = 2$). Work out the coupling of the spin and orbital angular momenta of this particle, and find all the states and the corresponding Clebsch–Gordan coefficients.

Exercise 7.9
The spin-dependent Hamiltonian of an electron–positron system in the presence of a uniform magnetic field in the z-direction ($\vec{B} = B\vec{k}$) can be written as

$$\hat{H} = \lambda \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 + \left(\frac{eB}{mc}\right)\left(\hat{S}_{1_z} - \hat{S}_{2_z}\right),$$

where λ is a real number and $\vec{S}_1$ and $\vec{S}_2$ are the spin operators for the electron and the positron, respectively.

(a) If the spin function of the system is given by $|\frac{1}{2},\ -\frac{1}{2}\rangle$, find the energy eigenvalues and their corresponding eigenvectors.

(b) Repeat (a) in the case where $\lambda = 0$, but $B \neq 0$.

(c) Repeat (a) in the case where $B = 0$, but $\lambda \neq 0$.

Exercise 7.10

(a) Show that $e^{-i\pi \hat{J}_z/2}\, e^{-i\pi \hat{J}_x}\, e^{i\pi \hat{J}_z/2} = e^{-i\pi \hat{J}_y}$.

(b) Prove $\hat{J}_- e^{-i\pi \hat{J}_x} = e^{-i\pi \hat{J}_x} \hat{J}_+$ and then show that $e^{-i\pi \hat{J}_x}\, |\, j,\ m\rangle = e^{-i\pi j} |j,\ -m\rangle$.

(c) Using (a) and (b), show that $e^{-i\pi \hat{J}_y}|j,\ m\rangle = (-1)^{j-m}|j,\ -m\rangle$.

Exercise 7.11
Using the commutation relations between the Pauli matrices, show that:
(a) $e^{i\alpha\sigma_y}\sigma_x e^{-i\alpha\sigma_y} = \sigma_x\cos(2\alpha) + \sigma_z\sin(2\alpha)$,
(b) $e^{i\alpha\sigma_z}\sigma_x e^{-i\alpha\sigma_z} = \sigma_x\cos(2\alpha) - \sigma_y\sin(2\alpha)$,
(c) $e^{i\alpha\sigma_x}\sigma_y e^{-i\alpha\sigma_x} = \sigma_y\cos(2\alpha) - \sigma_z\sin(2\alpha)$.

Exercise 7.12
(a) Show how $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ transform under a rotation of (finite) angle α about the x-axis.

(b) Using the results of part (a), determine how the angular momentum operator $\hat{\vec{J}}$ transforms under the rotation.

Exercise 7.13
(a) Show how the operator $\hat{J}_\pm$ transforms under a rotation of angle π about the x-axis.
(b) Use the result of part (a) to show that $\hat{J}_\pm e^{-i\pi\hat{J}_x/\hbar} = e^{-i\pi\hat{J}_x/\hbar}\hat{J}_\mp$.

Exercise 7.14
Consider a rotation of finite angle α about an axis $\vec{n}$ which transforms unit vector $\vec{a}$ into another unit vector $\vec{b}$. Show that $e^{-i\beta\hat{J}_b/\hbar} = e^{i\alpha\hat{J}_n/\hbar}e^{-i\beta\hat{J}_a/\hbar}e^{-i\alpha\hat{J}_n/\hbar}$.

Exercise 7.15
(a) Show that $e^{i\pi\hat{J}_y/2\hbar}\hat{J}_x e^{-i\pi\hat{J}_y/2\hbar} = \hat{J}_z$.
(b) Show also that $e^{i\pi\hat{J}_y/2\hbar}e^{i\alpha\hat{J}_x/\hbar}e^{-i\pi\hat{J}_y/2\hbar} = e^{i\alpha\hat{J}_z/\hbar}$.
(c) For any vector operator $\hat{\vec{A}}$, show that $e^{i\alpha\hat{J}_z/\hbar}\hat{A}_x e^{-i\alpha\hat{J}_z/\hbar} = \hat{A}_x\cos\alpha + \hat{A}_y\sin\alpha$.

Exercise 7.16
Using $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$ show that

$$d^{(j)}_{mm'}(\beta) = \sum_{m_1 m_2}\sum_{m'_1 m'_2}\langle j_1, j_2;\ m_1, m_2 \mid j,\ m\rangle\langle j_1, j_2;\ m'_1 m'_2 | j,\ m'\rangle d^{(j_1)}_{m_1 m'_1}(\beta) d^{(j_2)}_{m_2 m'_2}(\beta).$$

Exercise 7.17
Consider the tensor $A(\theta, \varphi) = \cos\theta\sin\theta\cos\varphi$.
(a) Calculate all the matrix elements $A_{m'm} = \langle l,\ m' \mid A \mid l,\ m\rangle$ for $l = 1$.
(b) Express $A(\theta, \varphi)$ in terms of the components of a spherical tensor of rank 2 (i.e., in terms of $Y_{2m}(\theta, \varphi)$).
(c) Calculate again all the matrix elements $A_{m'm}$, but this time using the Wigner–Eckart theorem. Compare these results with those obtained in (a). (The Clebsch–Gordan coefficients may be obtained from tables.)

Exercise 7.18
(a) Express xz/r^2 and $(x^2 - y^2)/r^2$ in terms of the components of a spherical tensor of rank 2.
(b) Using the Wigner–Eckart theorem, calculate the values of $\langle 1,\ 0 \mid xz/r^2 \mid 1,\ 1\rangle$ and $\langle 1,\ 1 \mid (x^2 - y^2)/r^2 \mid 1,\ -1\rangle$.

Exercise 7.19
Show that $\langle j,\ m' \mid e^{-i\beta\hat{J}_y/\hbar}\hat{J}_z^2 e^{i\beta\hat{J}_y/\hbar} \mid j,\ m'\rangle = \sum_{m=-j}^{m=j} m^2 \mid d^{(j)}_{m'm}(\beta)\mid^2$.

Exercise 7.20
Calculate the trace of the rotation matrix $D^{(1/2)}(\alpha, \beta, \gamma)$ for (a) $\beta = \pi$ and (b) $\alpha = \gamma = \pi$ and $\beta = 2\pi$.

Exercise 7.21
The quadrupole moment operator of a charge q is given by $\hat{Q}_{20} = q(3\hat{z}^2 - r^2)$. Write $\hat{Q}_{20}$ in terms of an irreducible spherical tensor of rank 2 and then express $\langle j, \ j|\hat{Q}_{20}|j, \ j\rangle$ in terms of j and the reduced matrix element $\langle j \ \| \ r^2 Y^{(2)} \ \| \ j\rangle$. *Hint:* You may use the coefficient $\langle j, 2; \ m, 0 \mid j, \ m\rangle = (-1)^{j-m}[3m^2 - j(j+1)]/\sqrt{(2j-1)j(j+1)(2j+3)}$.

Exercise 7.22
Prove the following commutation relations:

(a) $\left[J_x, [J_x, T_q^{(k)}]\right] = \sum_{q'} T_{q'}^{(k)} \langle k, \ q' \mid J_x^2 \mid k, \ q\rangle$,

(b) $\left[J_x, [J_x, T_q^{(k)}]\right] + \left[J_y, [J_y, T_q^{(k)}]\right] + \left[J_z, [J_z, T_q^{(k)}]\right] = k(k+1)\hbar^2 T_q^{(k)}$.

Exercise 7.23
Consider a spin $\frac{1}{2}$ particle which has an orbital angular momentum $l = 1$. Find all the Clebsch–Gordan coefficients involved in the addition of the orbital and spin angular momenta of this particle. *Hint:* The Clebsch–Gordan coefficient $\langle j_1, j_2; \ j_1, (j_2 - j_1) \mid j_2, \ j_2\rangle$ is real and positive.

Exercise 7.24
This problem deals with another derivation of the matrix elements of $d^{(1)}(\beta)$. Use the relation

$$d_{mm'}^{(j)}(\beta) = \sum_{m_1 m_2} \sum_{m_1' m_2'} \langle j_1, j_2; \ m_1, m_2 \mid j, \ m\rangle \langle j_1, j_2; \ m_1', m_2' \mid j, \ m'\rangle \, d_{m_1 m_1'}^{(j_1)}(\beta) d_{m_2 m_2'}^{(j_2)}(\beta)$$

for the case where $j_1 = j_2 = \frac{1}{2}$ along with the matrix elements of $d^{(1/2)}(\beta)$, which are given by (7.89), to derive all the matrix elements of $d^{(1)}(\beta)$.

Exercise 7.25
Consider the tensor $A(\theta, \varphi) = \sin^2\theta \cos(2\varphi)$.

(a) Calculate the reduced matrix element $\langle 2 \ \| \ Y_2 \ \| \ 2\rangle$. *Hint:* You may calculate explicitly $\langle 2, \ 1|Y_{20} \mid 2, \ 1\rangle$ and then use the Wigner–Eckart theorem to calculate it again.

(b) Express $A(\theta, \varphi)$ in terms of the components of a spherical tensor of rank 2 (i.e., in terms of $Y_{2m}(\theta, \varphi)$).

(c) Calculate $A_{m'\pm 1} = \langle 2, \ m'|A|2, \ \pm 1\rangle$ for $m' = \pm 2, \pm 1, 0$. You may need this Clebsch–Gordan coefficient: $\langle j, 2; \ m, 0 \mid j, \ m\rangle = [3m^2 - j(j+1)]/\sqrt{(2j-1)j(j+1)(2j+3)}$.

Exercise 7.26

(a) Calculate the reduced matrix element $\langle 1 \ \| \ Y_1 \ \| \ 2\rangle$. *Hint:* For this, you may need to calculate $\langle 1, \ 0 \mid Y_{10} \mid 2, \ 0\rangle$ directly and then from the Wigner–Eckart theorem.

(b) Using the Wigner–Eckart theorem and the relevant Clebsch–Gordan coefficients from the table, calculate $\langle 1, \ m|Y_{1m'}|2, \ m''\rangle$ for all possible values of m, m', and m''. *Hint:* You may find the integral $\int_0^\infty r^3 R_{21}^*(r) R_{32}(r)\, dr = \frac{64a_0}{15\sqrt{5}}\left(\frac{6}{5}\right)^5$ and the following coefficients useful:
$\langle j, 1; \ m, 0|(j-1), \ m\rangle = -\sqrt{(j-m)(j+m)/[j(2j+1)]}$,
$\langle j, 1; \ (m-1), 1|(j-1), \ m\rangle = \sqrt{(j-m)(j-m+1)/[2j(2j+1)]}$, and
$\langle j, 1; \ (m+1), -1|(j-1), \ m\rangle = \sqrt{(j+m)(j+m+1)/[2j(2j+1)]}$.

Exercise 7.27
A particle of spin $\frac{1}{2}$ is in a d state of orbital angular momentum (i.e., $l = 2$).

(a) What are its possible states of total angular momentum.

(b) If its Hamiltonian is given by $H = a + b\hat{\vec{L}} \cdot \hat{\vec{S}} + c\hat{\vec{L}}^2$, where a, b, and c are numbers, find the values of the energy for each of the different states of total angular momentum. Express your answer in terms of a, b, c.

Exercise 7.28
Consider an h-state electron. Calculate the Clebsch–Gordan coefficients involved in the following $\{|\ j,\ m\rangle\}$ states of the electron: $|\ \frac{11}{2},\ \frac{9}{2}\rangle$, $|\ \frac{11}{2},\ \frac{7}{2}\rangle$, $|\ \frac{9}{2},\ \frac{9}{2}\rangle$, $|\ \frac{9}{2},\ \frac{7}{2}\rangle$.

Exercise 7.29
Let the Hamiltonian of two nonidentical spin $\frac{1}{2}$ particles be

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}(\hat{\vec{S}}_1 + \hat{\vec{S}}_2) \cdot \hat{\vec{S}}_1 - \frac{\varepsilon_2}{\hbar}(\hat{S}_{1_z} + \hat{S}_{2_z}),$$

where ε_1 and ε_2 are constants having the dimensions of energy. Find the energy levels and their degeneracies.

Exercise 7.30
Find the energy levels and their degeneracies for a system of two nonidentical spin $\frac{1}{2}$ particles with Hamiltonian

$$\hat{H} = \frac{\varepsilon_0}{\hbar^2}(\hat{S}_1^2 + \hat{S}_2^2) - \frac{\varepsilon_0}{\hbar}(\hat{S}_{1_z} + \hat{S}_{2_z}),$$

where ε_0 is a constant having the dimensions of energy.

Exercise 7.31
Consider two nonidentical spin $s = \frac{1}{2}$ particles with Hamiltonian

$$\hat{H} = \frac{\varepsilon_0}{\hbar^2}(\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2 - \frac{\varepsilon_0}{\hbar^2}(\hat{S}_{1_z} + \hat{S}_{2_z})^2,$$

where ε_0 is a constant having the dimensions of energy. Find the energy levels and their degeneracies.

Exercise 7.32
Consider a system of three nonidentical particles, each of spin $s = \frac{1}{2}$, whose Hamiltonian is given by

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}(\hat{\vec{S}}_1 + \hat{\vec{S}}_3) \cdot \hat{\vec{S}}_2 + \frac{\varepsilon_2}{\hbar^2}(\hat{S}_{1_z} + \hat{S}_{2_z} + \hat{S}_{3_z})^2,$$

where ε_1 and ε_2 are constants having the dimensions of energy. Find the system's energy levels and their degeneracies.

Exercise 7.33
Consider a system of three nonidentical particles, each with angular momentum $\frac{3}{2}$. Find the possible values of the total spin S of this system and specify the number of angular momentum eigenstates corresponding to each value of S.