4.2 ELECTRIC OSCILLATIONS

4.94 If the electron (charge of each electron = $-e$) are shifted by a small distance x, a net $+ve$ charge density (per unit area) is induced on the surface. This will result in an electric field $E = n e x/\varepsilon_0$ in the direction of x and a restoring force on an electron of

$$-\frac{n e^2 x}{\varepsilon_0},$$

Thus
$$m \ddot{x} = -\frac{n e^2 x}{\varepsilon_0}$$

or
$$\ddot{x} + \frac{n e^2}{m \varepsilon_0} x = 0$$

This gives
$$\omega_p = \sqrt{\frac{n e^2}{m \varepsilon_0}} = 1.645 \times 10^{16} \text{ s}^{-1}.$$

as the plasma frequency for the problem.

4.95 Since there are no sources of emf in the circuit, Ohm's 1 law reads

$$\frac{q}{c} = -L \frac{dI}{dt}$$

where q = change on the capacitor, $I = \frac{dq}{dt}$ = current through the coil. Then

$$\frac{d^2 q}{dt^2} + \omega_0^2 q = 0, \quad \omega_0^2 = \frac{1}{LC}.$$

The solution fo this equation is

$$q = q_m \cos(\omega_0 t + \alpha)$$

From the problem $V_m = \frac{q_m}{C}$. Then

$$I = -\omega_0 C V_m \sin(\omega_0 t + \alpha)$$

and
$$V = V_m \cos(\omega_0 t - \alpha)$$

$$V^2 + \frac{I^2}{\omega_0^2 C^2} = V_m^2$$

or
$$V^2 + \frac{L I^2}{C} = V_m^2.$$

By energy conservation
$$\frac{1}{2} L I^2 + \frac{q^2}{2C} = \text{constant}$$

When the P.D. across the capacitor takes its maximum value V_m, the current I must be zero.

Thus "constant" $= \frac{1}{2} C V_m^2$

Hence
$$\frac{L I^2}{C} + V^2 = V_m^2 \text{ once again.}$$

4.96 After the switch was closed, the circuit satisfies

$$-L\frac{dI}{dt} = \frac{q}{C}$$

or $$\frac{d^2q}{dt^2} + \omega_0^2 q = 0 \Rightarrow q = C V_m \cos\omega_0 t$$

where we have used the fact that when the switch is closed we must have

$$V = \frac{q}{C} = V_m,\ I = \frac{dq}{dt} = 0 \text{ at } t = 0.$$

Thus (a) $$I = \frac{dq}{dt} = -C V_m \omega_0 \sin\omega_0 t$$

$$= -V_m\sqrt{\frac{C}{L}}\sin\omega_0 t$$

(b) The electrical energy of the capacitor is $\frac{q^2}{2C} \propto \cos^2\omega_0 t$ and of the inductor is

$$\frac{1}{2}L I^2 \propto \sin^2\omega_0 t.$$

The two are equal when

$$\omega_0 t = \frac{\pi}{4}$$

At that instant the emf of the self-inductance is

$$-L\frac{di}{dt} = V_m\cos\omega_0 t = V_m/\sqrt{2}$$

4.97 In the oscillating circuit, let

$$q = q_m\cos\omega t$$

be the change on the condenser where

$\omega^2 = \frac{1}{LC}$ and C is the instantaneous capacity of the condenser (S = area of plates)

$$C = \frac{\varepsilon_0 S}{y}$$

y = distance between the plates. Since the oscillation frequency increases η fold, the quantity

$$\omega^2 = \frac{y}{\varepsilon_0 S L}$$

changes η^2 fold and so does y i.e. changes from y_0 initially to $\eta^2 y_0$ finally. Now the P.D. across the condenser is

$$V = \frac{q_m}{C}\cos\omega t = \frac{y\,q_m}{\varepsilon_0 S}\cos\omega t$$

and hence the electric field between the plates is

$$E = \frac{q_m}{\varepsilon_0 S}\cos\omega t$$

Thus, the charge on the plate being $q_m \cos \omega t$, the force on the plate is

$$F = \frac{q_m^2}{\varepsilon_0 S} \cos^2 \omega t$$

Since this force is always positive and the plate is pulled slowly we can use the average force

$$\overline{F} = \frac{q_m^2}{2\,\varepsilon_0 S}$$

and work done is $\quad A = \overline{F}(\eta^2 y_0 - y_0) = (\eta^2 - 1)\dfrac{q_m^2 y_0}{2\,\varepsilon_0 S}$

But $\dfrac{q_m^2 y_0}{2\,\varepsilon_0 S} = \dfrac{q_m^2}{2\,C_0} = W$ the initial stored energy. Thus.

$$A = (\eta^2 - 1)\,W.$$

4.98 The equations of the $L-C$ circuit are

$$L\frac{d}{dt}(I_1 + I_) = \frac{C_1 V - \int I_1\,dt}{C_1} = \frac{C_2 V - \int I_2\,dt}{C_2}$$

Differentiating again $\quad L\,(I_1 + I_2) = -\dfrac{1}{C_1} I_1 = -\dfrac{1}{C_2} I_2$

Then $\quad I_1 = \dfrac{C_1}{C_1 - C_2} I,\; I_2 = \dfrac{C_2}{C_1 + C_2} I,$

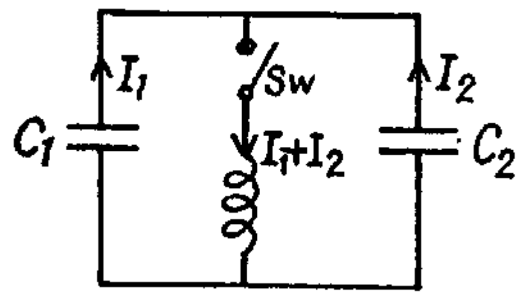
$$I = I_1 + I_2$$

so $\quad L\,(C_1 + C_2)\,I + I = 0$

or $\quad I = I_0 \sin(\omega_0 t + \alpha)$

where $\quad \omega_0^2 = \dfrac{1}{L\,(C_1 + C_2)}$ (Part a)

(Hence $T = \dfrac{2\pi}{\omega_0} = 0.7$ ms)



At $\quad t = 0,\ I = 0$ so $\alpha = 0$

$$I = I_0 \sin \omega_0 t$$

The peak value of the current is I_0 and it is related to the voltage V by the first equation

$$L\,I = V - \int I\,dt/(C_1 + C_2)$$

or $\quad +L\,\omega_0 I_0 \cos \omega_0 t = V - \dfrac{1}{C_1 + C_2}\displaystyle\int_0^t I_0 \sin \omega_0 t\,dt$

(The P.D. across the inductance is V at $t = 0$)

$$= V + \frac{1}{C_1 + C_2} \cdot \frac{I_0}{\omega_0}(\cos \omega_0 t - 1)$$

Hence $\quad I_0 = (C_1 + C_2)\,\omega_0 V = V\sqrt{\dfrac{C_1 + C_2}{L}} = 8.05$ A.

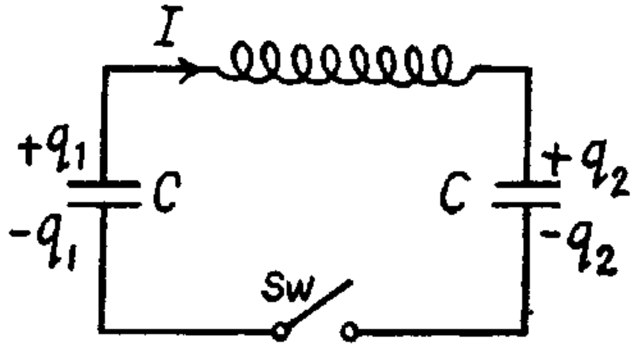
4.99 Initially $q_1 = C V_0$ and $q_2 = 0$. After the switch is closed change flows and we get

$$q_1 + q_2 = C V_0$$

$$\frac{q_1}{C} + L\frac{dI}{dt} - \frac{q_2}{C} = 0 \qquad (1)$$

Also $I = \dot{q}_1 = -\dot{q}_2$. Thus

$$L\,\ddot{I} + \frac{2I}{C} = 0$$



Hence $\ddot{I} + \omega_0^2 I = 0 \quad \omega_0^2 = \frac{2}{LC}$,

The soloution of this equation subject to

$I = 0$ at $t = 0$

is $I = I_0 \sin \omega_0 t$.

Integrating
$$q_1 = A - \frac{I_0}{\omega_0} \cos \omega_0 t$$

$$q_2 = B + \frac{I_0}{\omega_0} \cos \omega_0 t$$

Finally substituting in (1)

$$\frac{A-B}{C} - \frac{2I_0}{\omega_0 C}\cos \omega_0 t + L I_0 \omega_0 \cos \omega_0 t = 0$$

Thus
$$A = B = \frac{C V_0}{2} \text{ and}$$

$$\frac{C V_0}{2} + \frac{I_0}{\omega_0} = 0$$

so
$$q_1 = \frac{C V_0}{2}(1 + \cos \omega_0 t)$$

$$q_2 = \frac{C V_0}{2}(1 - \cos \omega_0 t)$$

4.100 The flux in the coil is

$$\Phi(t) = \begin{cases} \Phi & t < 0 \\ 0 & t > 0 \end{cases}$$

The equation of the current is
$$-L\frac{dI}{dt} = \frac{\int_0^t I\,dt}{C} \qquad (1)$$

This mean that
$$LC\frac{d^2 I}{dt^2} + I = 0$$

or with $\omega_0^2 = \frac{1}{LC}$
$$I = I_0 \sin(\omega_0 t + \alpha)$$

Putting in (1) $-L I_0 \omega_0 \cos(\omega_0 t + \alpha) = -\frac{I_0}{\omega_0 C}[\cos(\omega_0 t + \alpha) - \cos\alpha]$

This implies $\cos\alpha = 0 \quad \therefore \quad I = \pm I_0 \cos\omega_0 t$. From Faraday's law

$$\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

or integrating from $t = -\varepsilon$ to $-\varepsilon$ where $\varepsilon \to 0$

$$\Phi = L I_0 \text{ with } + \text{ sign in } I$$

so, $$I = \frac{\Phi}{L}\cos\omega_0 t\,.$$

4.101 Given $V = V_m\, e^{-\beta t}\cos\omega t$

(a) The phrase 'peak values' is not clear. The answer is obtained on taking $|\cos\omega t| = 1$

i.e $$t = \frac{\pi n}{\omega}.$$

(b) For extrema $\frac{dV}{dt} = 0$

$$-\beta\cos\omega t - \omega\sin\omega t = 0$$

or $$\tan\omega t = -\beta/\omega$$

i.e. $$\omega t = n\pi + \tan^{-1}\left(\frac{-\beta}{\omega}\right).$$

4.102 The equation of the circuit is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

where Q = charge on the capacitor,

This has the solution $$Q = Q_m\, e^{-\beta t}\sin(\omega t + \alpha)$$

where $$\beta = \frac{R}{2L},\ \omega = \sqrt{\omega_0^2 - \beta^2},\ \omega_0^2 = \frac{1}{LC}.$$

Now $$I = \frac{dQ}{dt} = 0 \quad \text{at } t = 0$$

so, $$Q_m\, e^{-\beta t}(-\beta\sin(\omega t + \alpha) + \omega\cos(\omega t + \alpha)) = 0 \quad \text{at } t = 0$$

Thus $$\omega\cos\alpha = \beta\sin\alpha \quad \text{or} \quad \alpha = \tan^{-1}\frac{\omega}{\beta}$$

Now $$V_m = \frac{Q_m}{C} \quad \text{and} \quad V_0 = \text{P.D. at } t = 0 = \frac{Q_m}{C}\sin\alpha$$

$$\therefore\quad \frac{V_0}{V_m} = \sin\alpha = \frac{\omega}{\sqrt{\omega^2 + \beta^2}} = \frac{\omega}{\omega_0} = \sqrt{1 - \beta^2/\omega_0^2} = \sqrt{1 - \frac{R^2 C}{4L^2}}$$

4.103 We write

$$-\frac{dQ}{dt} = I = I_m\, e^{-\beta t} \sin \omega t$$

$$= gm\ I_m\, e^{-\beta t + i\omega t} \qquad (gm \text{ means imaginary part})$$

Then

$$Q = gm\ I_m \frac{e^{-\beta t + i\omega t}}{-\beta + i\omega}$$

$$Q = gm\ I_m \frac{e^{-\beta t + i\omega t}}{\beta - i\omega}$$

$$= gm\ I_m \frac{(\beta + i\omega)\, e^{-\beta t + i\omega t}}{\beta^2 + \omega^2}$$

$$= I_m\, e^{-\beta t}\, \frac{\beta \sin \omega t + \omega \cos \omega t}{\beta^2 + \omega^2}$$

$$= I_m\, e^{-\beta t}\, \frac{\sin(\omega t + \delta)}{\sqrt{\beta^2 + \omega^2}}, \quad \tan \delta = \frac{\omega}{\beta}.$$

(An arbitrary constant of integration has been put equal to zero.)

Thus

$$V = \frac{Q}{C} = I_m \sqrt{\frac{L}{C}}\, e^{-\beta t} \sin(\omega t + \delta)$$

$$V(0) = I_m \sqrt{\frac{L}{C}} \sin \delta = I_m \sqrt{\frac{L}{C}} \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$$

$$= I_m \sqrt{\frac{L}{C(1 + \beta^2/\omega^2)}}$$

4.104 $I = I_m\, e^{-\beta t} \sin \omega t$

$$\beta = \frac{R}{2L}, \quad \omega_0 = \sqrt{\frac{1}{LC}}, \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

$I = -\dot{q}$, $q =$ charge on the capacitor

Then
$$q = I_m\, e^{-\beta t}\, \frac{\sin(\omega t + \delta)}{\sqrt{\omega^2 + \beta^2}}, \quad \tan \delta = \frac{\omega}{\beta}.$$

Thus
$$W_M = \frac{1}{2} L\, I_m^2\, e^{-2\beta t} \sin^2 \omega t$$

$$W_E = \frac{I_m^2}{2C} \frac{e^{-2\beta t} \sin^2(\omega t + \delta)}{\omega^2 + \beta^2} = \frac{L I_m^2}{2} e^{-2\beta t} \sin^2(\omega t + \delta)$$

Current is maximum when $\frac{d}{dt}\, e^{-\beta t} \sin \omega t = 0$

Thus $- \beta \sin \omega t + \omega \cos \omega t = 0$

or $\tan \omega t = \frac{\omega}{\beta} = \tan \delta$

i.e. $\omega t = n \pi + \delta$

and hence $$\frac{W_M}{W_E} = \frac{\sin^2 (\omega t)}{\sin^2 (\omega t + \delta)} = \frac{\sin^2 \delta}{\sin^2 2 \delta} = \frac{1}{4 \cos^2 \delta}$$

$$= \frac{1}{4 \beta^2 / \omega_0^2} = \frac{\omega_0^2}{4 \beta^2} = \frac{1}{LC} \times \frac{L^2}{R^2} = \frac{L}{CR^2} = 5.$$

(W_M is the magnetic energy of the inductance coil and W_E is the electric energy of the capacitor.)

4.105 Clearly

$$L = L_1 + L_2 ,\ R = R_1 + R_2$$

4.106 $Q = \frac{\pi}{\beta T}$ or $\beta = \frac{\pi}{Q T}$

Now $\beta t = \ln \eta$ so $t = \frac{\ln \eta}{\pi} Q T$

$$= \frac{Q \ln \eta}{\pi \nu} = 0.5 \text{ ms}$$

4.107 Current decreases e fold in time

$$t = \frac{1}{\beta} = \frac{2L}{R} \text{ sec} = \frac{2L}{RT} \text{ oscillations}$$

$$= \frac{2L}{R} \frac{\omega}{2 \pi}$$

$$= \frac{L}{\pi R} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2 \pi} \sqrt{\frac{4L}{R^2 C} - 1} = 15.9 \text{ oscillations}$$

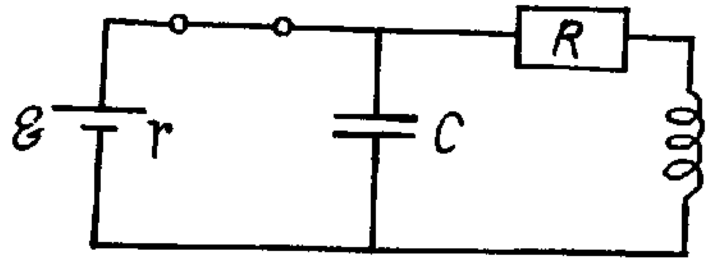
4.108 $Q = \frac{\pi}{\beta T} = \frac{\omega}{2 \beta}$

$$\therefore\ \omega = 2 \beta Q ,\ \beta = \frac{\omega}{2Q} .$$

Now $\omega_0 = \omega \sqrt{1 + \frac{1}{4Q^2}}$ or $\omega = \frac{\omega_0}{\sqrt{1 + \frac{1}{4Q^2}}}$

so $$\left| \frac{\omega_0 - \omega}{\omega_0} \right| \times 100\ \% = \frac{1}{8Q^2} \times 100\ \% = 0.5\%$$

4.109



At $t = 0$ current through the coil $= \dfrac{\varepsilon}{R + r}$

P.D. across the condenser $= \dfrac{\varepsilon}{R + r}$

(a) At $t = 0$, energy stored $= W_0$

$$= \frac{1}{2} L \left(\frac{\varepsilon}{R + r}\right)^2 + \frac{1}{2} C \left(\frac{\varepsilon R}{R + r}\right)^2 = \frac{1}{2} \varepsilon^2 \frac{(L + C R^2)}{(R + r)^2} = 2.0 \text{ mJ.}$$

(b) The current and the change stored decrease as $e^{-tR/2L}$ so energy decreases as $e^{-tR/L}$

$$\therefore \quad W = W_0 \, e^{-tR/L} = 0.10 \text{ mJ.}$$

4.110 $Q = \dfrac{\pi}{\beta T} = \dfrac{\pi \nu}{\beta} = \dfrac{\omega}{2\beta} = \dfrac{\sqrt{\omega_0^2 - \beta^2}}{2\beta}$

or $$\frac{\omega_0}{\beta} = \sqrt{1 + 4Q^2} \quad \text{or} \quad \beta = \frac{\omega_0}{\sqrt{1 + Q^2}}$$

Now $$W = W_0 \, e^{-2\beta t}$$

Thus energy decreases η times in $\dfrac{\ln \eta}{2\beta}$ sec.

$$= \ln \eta \, \frac{\sqrt{1 + 4Q^2}}{2\omega_0} = \frac{Q \ln \eta}{2\pi \nu_0} \text{ sec.} = 1.033 \text{ ms.}$$

4.111 In a leaky condenser

$$\frac{dq}{dt} = I - I' \quad \text{where} \quad I' = \frac{V}{R} = \text{leak current}$$

Now $$V = \frac{q}{C} = -L\frac{dI}{dt} = -L\frac{d}{dt}\left(\frac{dq}{dt} + \frac{V}{R}\right)$$

$$= -L\frac{d^2 q}{dt^2} - \frac{L}{RC}\frac{dq}{dt}$$

or $$\ddot{q} + \frac{1}{RC}\frac{dQ}{dt} + \frac{1}{LC} q = 0$$

Then $$q = q_m e^{-\beta t} \sin(\omega t + \alpha)$$

(a) $\beta = \frac{1}{2RC}$, $\omega_0^2 = \frac{1}{LC}$, $\omega = \sqrt{\omega_0^2 - \beta^2}$

$$= \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

(b) $Q = \frac{\omega}{2\beta} = RC\sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$

$$= \frac{1}{2}\sqrt{\frac{4CR^2}{L} - 1}$$

4.112 Given $V = V_m e^{-\beta t}\sin\omega t$, $\omega = \omega_0$ $\beta T << 1$

$$\text{Power loss} = \frac{\text{Energy loss per cycle}}{T}$$

$$= \frac{1}{2} C V_m^2 \times 2\beta$$

(energy decreases as $W_0 e^{-2\beta t}$ so loss per cycle is $W_0 \times 2\beta T$)

Thus $$\langle P \rangle = \frac{1}{2} C V_m^2 \times \frac{R}{L}$$

or $$R = \frac{2\langle P \rangle}{V_m^2}\frac{L}{C}$$

Hence $Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \sqrt{\frac{C}{L}}\frac{V_m^2}{2\langle P \rangle} = 100$ on putting the vales.

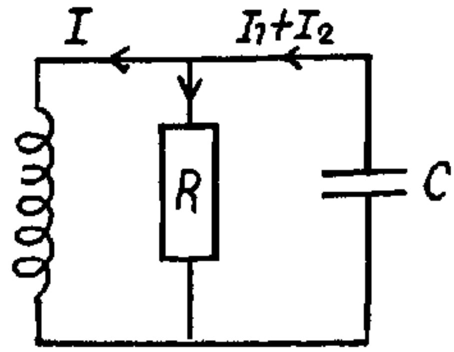
4.113 Energy is lost across the resistance and the mean power lass is

$$\langle P \rangle = R\langle I^2 \rangle = \frac{1}{2} R I_m^2 = 0.2 \text{ mW}.$$

This power should be fed to the circuit to maintain undamped oscillations.

4.114 $\langle P \rangle = \frac{RCV_m^2}{2L}$ as in (4.112). We get $\langle P \rangle = 5$ mW.

4.115 Given $q = q_1 + q_2$



$I_1 = -\dot{q}_1$, $I_2 = -\dot{q}_2$

$L\dot{I}_1 = RI_2 = \frac{q}{C}$.

Thus $CL\ddot{q}_1 + (q_1 + q_2) = 0$

$RC\dot{q}_2 + q_1 + q_2 = 0$

Putting $q_1 = Ae^{i\omega t}$ $q_2 = Be^{+i\omega t}$

$$(1 - \omega^2 LC)A + B = 0$$

$$A+(1+i\omega RC)B = 0$$

A solution exists only if

$$(1-\omega^2 LC)(1+i\omega RC) = 1$$

or
$$i\omega RC - \omega^2 LC - i\omega^3 LRC^2 = 0$$

or
$$LRC^2\omega^2 - i\omega LC - RC = 0$$

$$\omega^2 - i\omega\frac{1}{RC} - \frac{1}{LC} = 0$$

$$\omega = \frac{i}{2RC} \pm \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} = i\beta \pm \omega_0$$

Thus
$$q_1 = (A_1 \cos\omega_0 t + A_2 \sin\omega_0 t)\, e^{-\beta t} \text{ etc.}$$

ω_0 is the oscillation frequency. Oscillations are possible only if $\omega_0^2 > 0$

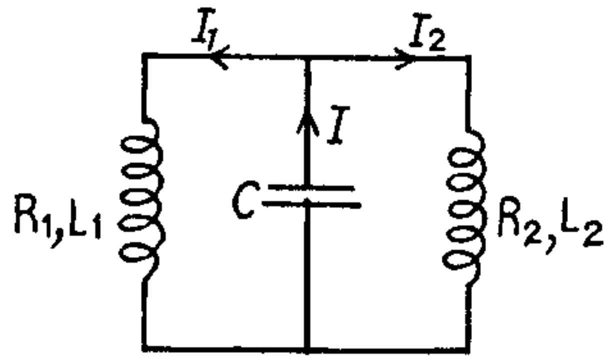
i.e.
$$\frac{1}{4R^2} < \frac{C}{L}.$$

4.116 We have

$$L_1 \dot{I}_1 + R_1 I_1 = L_2 \dot{I}_2 + R_2 I_2$$

$$= -\frac{\int I\,dt}{C}$$

$$I = I_1 + I_2$$



Then differentiating we have the equations

$$L_1 C \ddot{I}_1 + R_1 C \dot{I}_1 + (I_1 + I_2) = 0$$

$$L_2 C \ddot{I}_2 + R_2 C \dot{I}_2 + (I_1 + I_2) = 0$$

Look for a solution

$$I_1 = A_1 e^{\alpha t},\ I_2 = A_2 e^{\alpha t}$$

Then
$$(1+\alpha^2 L_1 C + \alpha R_1 C)A_1 + A_2 = 0$$

$$A_1 + (1+\alpha^2 L_2 C + \alpha R_2 C)A_2 = 0$$

This set of simultaneous equations has a nontrivial solution only if

$$(1+\alpha^2 L_1 C + \alpha R_1 C)(1+\alpha^2 L_2 C + \alpha R_2 C) = 1$$

or
$$\alpha^3 + \alpha^2\frac{L_1R_2 + L_2R_1}{L_1L_2} + \alpha\frac{L_1 + L_2 + R_1R_2C}{L_1L_2C} + \frac{R_1+R_2}{L_1L_2C} = 0$$

This cubic equation has one real root which we ignore and two complex conjugate roots. We require the condition that this pair of complex conjugate roots is identical with the roots of the equation

$$\alpha^2 LC + \alpha RC + 1 = 0$$

The general solution of this problem is not easy. We look for special cases. If $R_1 = R_2 = 0$, then $R = 0$ and $L = \frac{L_1 L_2}{L_1 + L_2}$. If $L_1 = L_2 = 0$, then

$L = 0$ and $R = R_1 R_2 / (R_1 + R_2)$. These are the quoted solution but they are misleading.

We shall give the solution for small R_1, R_2. Then we put $\alpha = -\beta + i\omega$ when β is small

We get $(1 - \omega^2 L_1 C - 2i\beta\omega L_1 C - \beta \not{R}_1 C + i\omega R_1 C)$

$$(1 - \omega^2 L_2 C - 2i\beta\omega L_2 C - \beta \not{R}_2 C + i\omega R_2 C) = 1$$

(we neglect β^2 & βR_1, βR_2). Then

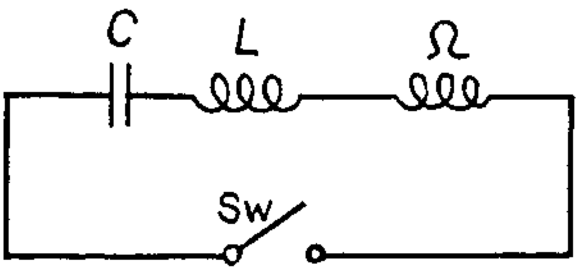
$$(1 - \omega^2 L_1 C)(1 - \omega^2 L_2 C) = 1 \Rightarrow \omega^2 = \frac{L_1 + L_2}{L_1 L_2 C}$$

This is identical with $\omega^2 = \frac{1}{LC}$ if $L = \frac{L_1 L_2}{L_1 + L_2}$.

also $\quad (2\beta L_1 - R_1)(1 - \omega^2 L_2 C) + (2\beta L_2 - R_2)(1 - \omega^2 L_2 C) = 0$

This gives $\beta = \frac{R}{2L} = \frac{R_1 L_2^2 + R_2 L_1^2}{2 L_1 L_2 (L_1 + L_2)} \Rightarrow R = \frac{R_1 L_2^2 + R_2 L_1^2}{(L_1 + L_2)^2}$.

4.117 $o = \frac{q}{C} + L\frac{dI}{dt} + RI$, $I = +\frac{dq}{dt}$



For the critical case $R = 2\sqrt{\frac{L}{C}}$

Thus $LC\ddot{q} + 2\sqrt{LC}\,\dot{q} + q = 0$

Look for a solution with $q \propto e^{\alpha t}$

$$\alpha = -\frac{1}{\sqrt{LC}}.$$

An independent solution is $t e^{\alpha t}$. Thus

$$q = (A + Bt)\, e^{-t/\sqrt{LC}},$$

At $\quad t = 0 \quad q = C V_0$ thus $A = C V_0$

Also at $\quad t = 0 \quad \dot{q} = I = 0$

$$0 = B - A\frac{1}{\sqrt{lC}} \Rightarrow B = V_0\sqrt{\frac{C}{L}}$$

Thus finally $$I = \frac{dq}{dt} = V_0 \sqrt{\frac{C}{L}}\; e^{-t/\sqrt{LC}}$$

$$- \frac{1}{\sqrt{LC}} \left(C V_0 + V_0 \sqrt{\frac{C}{L}}\; t \right) e^{-t/\sqrt{LC}}$$

$$= - \frac{V_0}{L} t\; e^{-t/\sqrt{LC}}$$

The current has been defined to increase the charge. Hence the minus sign.

The current is maximum when

$$\frac{dI}{dt} = - \frac{V_0}{L}\, e^{-t/\sqrt{LC}} \left(1 - \frac{t}{\sqrt{LC}} \right) = 0$$

This gives $t = \sqrt{LC}$ and the magnitude of the maximum current is

$$|I_{max}| = \frac{V_0}{e} \sqrt{\frac{C}{L}}.$$

4.118 The equation of the circuit is (I is the current)

$$L \frac{dI}{dt} + RI = V_m \cos \omega t$$

From the theory of differential equations

$$I = I_P + I_C$$

where I_P is a particular integral and I_C is the complementary function (Solution of the differential equation with the RHS = 0). Now

$$I_C = I_{CO}\; e^{-tR/L}$$

and for I_P we write $I_P = I_m \cos (\omega t - \varphi)$

Substituting we get

$$I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}, \quad \varphi = \tan^{-1} \frac{\omega L}{R}$$

Thus $$I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos (\omega t - \varphi) + I_{CO}\, e^{-tR/L}$$

Now in an inductive circuit $I = 0$ at $t = 0$

because a current cannot change suddenly.

Thus $$I_{CO} = - \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos \varphi$$

and so

$$I = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \left[\cos (\omega t - \varphi) - \cos \varphi\, e^{-tR/L} \right]$$

4.119 Here the equation is (Q is charge on the capacitor)

$$\frac{Q}{C} + R\frac{dQ}{dt} = V_m \cos\omega t$$

A solution subject to $Q = 0$ at $t = 0$ is of the form (as in the previous problem)

$$Q = Q_m\left[\cos(\omega t - \overline{\varphi}) - \cos\overline{\varphi}\, e^{-t/RC}\right]$$

Substituting back

$$\frac{Q_m}{C}\cos(\omega t - \overline{\varphi}) - \omega R Q_m \sin(\omega t - \overline{\varphi})$$

$$= V_m \cos\omega t$$

$$= V_m\left\{\cos\overline{\varphi}\cos(\omega t - \overline{\varphi}) - \sin\overline{\varphi}\sin(\omega t - \overline{\varphi})\right\}$$

so

$$Q_m = C V_m \cos\overline{\varphi}$$

$$\omega R Q_m = V_m \sin\overline{\varphi}$$

This leads to

$$Q_m = \frac{C V_m}{\sqrt{1 + (\omega R C)^2}}, \quad \tan\overline{\varphi} = \omega R C$$

Hence

$$I = \frac{dQ}{dt} = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\left[-\sin(\omega t - \overline{\varphi}) + \frac{\cos^2\overline{\varphi}}{\sin\overline{\varphi}}\, e^{-t/RC}\right]$$

The solution given in the book satisfies $I = 0$ at $t = 0$. Then $Q = 0$ at $t = 0$ but this will not satisfy the equation at $t = 0$. Thus $I \neq 0$, (Equation will be satisfied with $I = 0$ only if $Q \neq 0$ at $t = 0$)

With our I, $$I(t = 0) = \frac{V_m}{R}$$

4.120 The current lags behind the voltage by the phase angle

$$\varphi = \tan^{-1}\frac{\omega L}{R}$$

Now $L = \mu_0 n^2 \pi a^2 l$, l = length of the solenoid

$R = \dfrac{\rho \cdot 2\pi a n \cdot l}{\pi b^2}$, $2b$ = diameter of the wire

But $$2bn = 1 \qquad \therefore \quad b = \frac{1}{2n}$$

Then $$\varphi = \tan^{-1}\frac{\mu_0 n^2 l \pi a^2 \cdot 2\pi\nu}{\rho \cdot 2\pi a n l} \times \pi\frac{1}{4n^2}$$

$$= \tan^{-1}\frac{\mu_0 \pi^2 a \nu}{4\rho n}.$$

4.121 Here $V = V_m \cos \omega t$

$$I = I_m \cos(\omega t + \varphi)$$

where $$I_m = \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}, \quad \tan \varphi = \frac{1}{\omega R C}$$

Now $$R^2 + \frac{1}{(\omega C)^2} = \left(\frac{V_m}{I_m}\right)^2$$

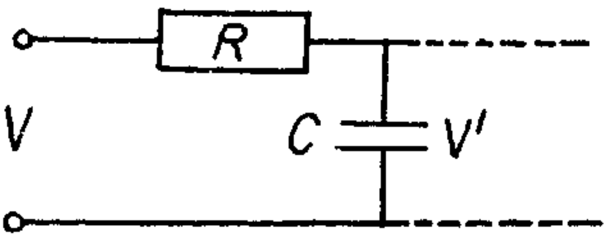
$$\frac{1}{\omega R C} = \sqrt{\left(\frac{V_m}{R I_m}\right)^2 - 1}$$

Thus the current is ahead of the voltage by

$$\varphi = \tan^{-1} \frac{1}{\omega R C} = \tan^{-1} \sqrt{\left(\frac{V_m}{R I_m}\right)^2 - 1} = 60^\circ$$

4.122 Here $$V = I R + \frac{\int_0^t I\, dt}{C}$$

or $$R \dot{I} + \frac{1}{C} I = \dot{V} = -\omega V_0 \sin \omega t$$



Ignoring transients, a solution has the form

$$I = I_0 \sin(\omega t - \alpha)$$

$$\omega R I_0 \cos(\omega t - \alpha) + \frac{I_0}{C} \sin(\omega t - \alpha) = -\omega V_0 \sin \omega t$$

$$= -\omega V_0 \{\sin(\omega t - \alpha)\cos\alpha + \cos(\omega t - \alpha)\sin\alpha\}$$

so $$R I_0 = -V_0 \sin\alpha$$

$$\frac{I_0}{\omega C} = -V_0 \cos\alpha \qquad \alpha = \pi + \tan^{-1}(\omega R C)$$

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

$$I = I_0 \sin(\omega t - \tan^{-1} \omega R C - \pi) = -I_0 \sin(\omega t - \tan^{-1} \omega R C)$$

Then $$Q = \int_0^t I\, dt = Q_0 + \frac{I_o}{\omega} \cos(\omega t - \tan^{-1} \omega R C)$$

It satisfies $$V_0(1 + \cos \omega t) = R \frac{dQ}{dt} + \frac{Q}{C}$$

if $$V_0(1+\cos\omega t) = -RI_0\sin(\omega t-\tan^{-1}\omega RC) + \frac{Q_0}{C} + \frac{I_0}{\omega C}\cos(\omega t - \tan^{-1}\omega RC)$$

Thus $$Q_0 = CV_0$$

and $$\left.\begin{aligned}\frac{I_0}{\omega C} &= V_0\Big/\sqrt{1+(\omega RC)^2}\\ RI_0 &= \frac{V_0\,\omega RC}{\sqrt{1+(\omega RC)^2}}\end{aligned}\right\}\text{ checks}$$

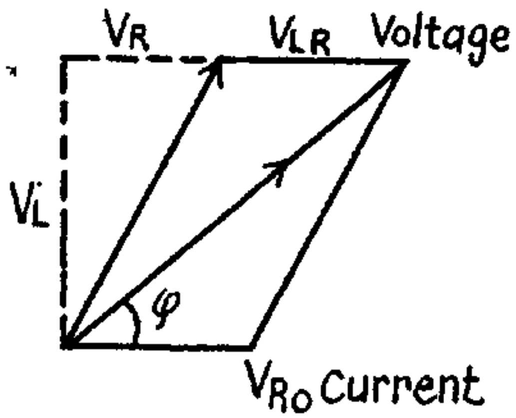
Hence $$V' = \frac{Q}{C} = V_0 + \frac{V_0}{\sqrt{1+(\omega RC)^2}}\cos(\omega t-\alpha)$$

(b) $$\frac{V_0}{\eta} = \frac{V_0}{\sqrt{1+(\omega RC)^2}}$$

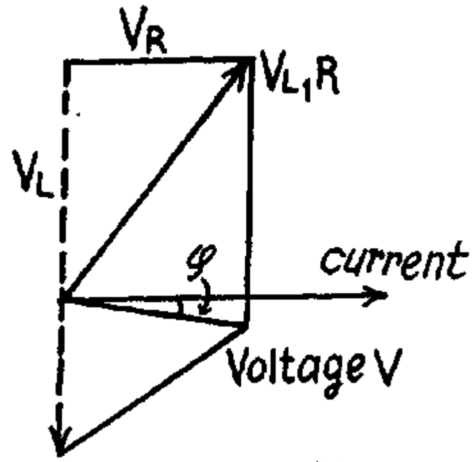
or $$\eta^2 - 1 = \omega^2(RC)^2$$

or $$RC = \sqrt{\eta^2-1}\Big/\omega = 22\text{ ms.}$$

4.123



(a)



(b)

(b) $$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R} = -ve$$

as $$\omega^2 < \frac{1}{LC}$$

4.124 (a) $$I_m = \frac{V_m}{\sqrt{R^2+\left(\omega L - \frac{1}{\omega C}\right)^2}} = 4{\cdot}48\text{ A}$$

(b) $$\tan\varphi = \frac{\omega L - \frac{1}{\omega C}}{R},\ \varphi = -60°$$

Current lags behind the voltage V by φ

(c) $V_C = \frac{I_m}{\omega C} = 0{\cdot}65\ \text{kV}$

$$V_{L/R} = I_m \sqrt{R^2 + \omega^2 L^2} = 0{\cdot}5\ \text{kV}$$

4.125 (a) $V_C = \frac{1}{\omega C} \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

$$= \frac{V_m}{\sqrt{(\omega R C)^2 + (\omega^2 L C - 1)^2}} = \frac{V_m}{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1\right)^2 + 4\beta^2 \omega^2/\omega_0^4}}$$

$$= \frac{V_m}{\sqrt{\left(\frac{\omega^2}{\omega_0^2} - 1 + \frac{2\beta^2}{\omega_0^2}\right)^2 + \frac{4\beta^2}{\omega_0^2} - \frac{4\beta^4}{\omega_0^4}}}$$

This is maximum when $\omega^2 = \omega_0^2 - 2\beta^2 = \frac{1}{LC} - \frac{R^2}{2L^2}$

(b) $V_L = I_m \omega L = V_m \frac{\omega L}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

$$= \frac{V_m L}{\sqrt{\frac{R^2}{\omega^2} + \left(L - \frac{1}{\omega^2 C}\right)^2}} = \frac{V_m L}{\sqrt{L^2 - \frac{1}{\omega^2}\left(\frac{2L}{C} - R^2\right) + \frac{1}{\omega^4 C^2}}}$$

$$= \frac{V_m L}{\sqrt{\left(\frac{1}{\omega^2 C} - \left(L - \frac{CR^2}{2}\right)\right)^2 + L^2 - \left(L - \frac{1}{2} C R^2\right)^2}}$$

This is maximum when

$$\frac{1}{\omega^2 C} = L - \frac{1}{2} C R^2$$

or $$\omega^2 = \frac{1}{LC - \frac{1}{2} C^2 R^2} = \frac{1}{\frac{1}{\omega_0^2} - \frac{2\beta^2}{\omega_0^4}}$$

$$= \frac{\omega_0^4}{\omega_0^2 - 2\beta^2} \quad \text{or} \quad \omega = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2\beta^2}}.$$

4.126. $V_L = I_m \sqrt{R^2 + \omega^2 L^2}$

$$= \frac{V_m \sqrt{R^2 + \omega^2 L^2}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

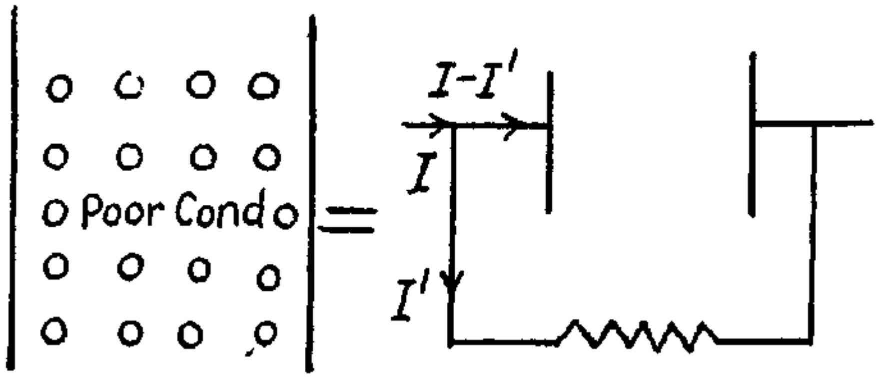
for a given ω, L, R, this is maximum when

$$\frac{1}{\omega C} = \omega L \quad \text{or} \quad C = \frac{1}{\omega^2 L} = 28.2\ \mu\text{F}.$$

For that C, $\quad V_L = \dfrac{V\sqrt{R^2 + \omega^2 L^2}}{R} = V\sqrt{1 + (\omega L/R)^2} = 0.540$ kV

At this C $\qquad V_C = \dfrac{1}{\omega C}\dfrac{V_m}{R} = \dfrac{V_m \omega L}{R} = .509$ kV

4.127



We use the complex voltage $V = V_m\, e^{i\omega t}$. Then the voltage across the capacitor is

$$(I - I')\frac{1}{i\omega C}$$

and that across the resistance $R I'$ and both equal V. Thus

$$I' = \frac{V_m}{R}\, e^{i\omega t},\ I - I' = i\omega C V_m\, e^{i\omega t}$$

Hence

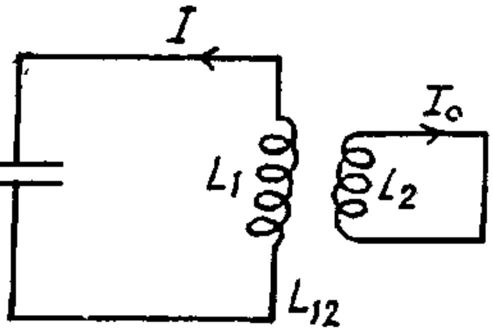
$$I = \frac{V_m}{R}\,(1 + i\omega R C)\, e^{i\omega t}$$

The actual voltage is obtained by taking the real part. Then

$$I = \frac{V_m}{R}\sqrt{1 + (\omega R C)^2}\,\cos(\omega t + \varphi)$$

Where $\qquad \tan\varphi = \omega R C$

Note → A condenser with poorly conducting material (dielectric of high resistance) be the plates is equvalent to an an ideal condenser with a high resistance joined in p between its plates.



4.128 $$L_1 \frac{dI_1}{dt} + \frac{\int I_1\, dt}{C} = -L_{12}\frac{dI_2}{dt}$$

$$L_2 \frac{dI_2}{dt} = -L_{12}\frac{dI_1}{dt}$$

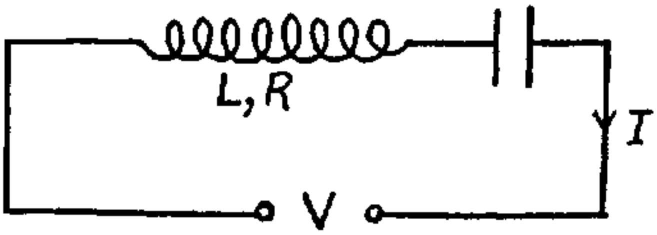
from the second equation

$$L_2 I_2 = -L_{12} I_1$$

Then
$$\left(L_1 - \frac{L_{12}^2}{L_2}\right)\ddot{I}_1 + \frac{I_1}{C} = 0$$

Thus the current oscillates with frequency

$$\omega = \frac{1}{\sqrt{C\left(L_1 - \frac{L_{12}^2}{L_2}\right)}}$$



4.129 Given $V = V_m \cos \omega t$

$$I = I_m \cos(\omega t - \varphi)$$

where

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Then,
$$V_C = \frac{\int I\, dt}{C} = \frac{I_m \sin(\omega t - \varphi)}{\omega C}$$

$$= \frac{V_m}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}} \sin(\omega t - \varphi)$$

As resonance the voltage amplitude across the capacitor

$$= \frac{V_m}{RC\frac{1}{\sqrt{LC}}} = \sqrt{\frac{L}{CR^2}}\, V_m = n V_m$$

So
$$\frac{L}{CR^2} = n^2$$

Now
$$Q = \sqrt{\frac{L}{CR^2} - \frac{1}{4}} = \sqrt{n^2 - \frac{1}{4}}$$

4.130 For maximum current amplitude

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$L = \frac{1}{\omega^2 C} \text{ and then } I_{m0} = \frac{V_m}{R}$$

Now
$$\frac{I_{m0}}{\eta} = \frac{V_m}{\sqrt{R^2 + \frac{(n-1)^2}{\omega^2 C^2}}}$$

So
$$\eta = \sqrt{1 + \frac{(n-1)^2}{(\omega R C)^2}}$$

$$\omega R C = \frac{n-1}{\sqrt{\eta^2 - 1}}$$

Now
$$Q = \sqrt{\left(\frac{L}{CR^2}\right)^2 - \frac{1}{4}} = \sqrt{\left(\frac{1}{\omega R C}\right)^2 - \frac{1}{4}} = \sqrt{\frac{\eta^2 - 1}{(n-1)^2} - \frac{1}{4}}$$

4.131 At resonance

$$\omega_0 L = (\omega_0 C)^{-1} \text{ or } \omega_0 = \frac{1}{\sqrt{LC}},$$

and
$$(I_m)_{res} = \frac{V_m}{R}.$$

Now
$$\frac{V_m}{nR} = \frac{V_m}{\sqrt{R^2 + \left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2}} = \frac{V_m}{\sqrt{R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2}}$$

Then
$$\omega_1 L - \frac{1}{\omega_1 C} = \sqrt{n^2 - 1}\, R$$

$$\omega_2 L - \frac{1}{\omega_2 C} = + \sqrt{n^2 - 1}\, R \qquad \text{(assuming } \omega_2 > \omega_1\text{)}$$

or
$$\omega_1 - \frac{\omega_0^2}{\omega_1} = -\omega_2 + \frac{\omega_0^2}{\omega_2} = -\sqrt{n^2 - 1}\,\frac{R}{L}$$

or
$$\omega_1 + \omega_2 = \frac{\omega_0^2}{\omega_1 \omega_2}(\omega_1 + \omega_2) \Rightarrow \omega_0 = \sqrt{\omega_1 \omega_2}$$

and
$$\omega_2 - \omega_1 = \sqrt{n^2 - 1}\,\frac{R}{L}$$

$$\beta = \frac{R}{2L} = \frac{\omega_2 - \omega_1}{2\sqrt{n^2 - 1}}$$

and
$$Q = \sqrt{\frac{\omega_0^2}{4\beta^2} - \frac{1}{4}} = \sqrt{\frac{(n^2 - 1)\,\omega_1 \omega_2}{(\omega_2 - \omega_1)^2} - \frac{1}{4}}$$

4.132 $Q = \frac{\omega}{2\beta} \approx \frac{\omega_0}{2\beta}$ for low damping.

Now $\frac{I_m}{\sqrt{2}} = \frac{R I_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$, I_m= current amplitude at resouance

or $$\omega - \frac{\omega_0^2}{\omega} = \pm \frac{R}{L} = \pm 2\beta$$

Thus $$\omega \approx \omega_0 \pm \beta$$

So $$\Delta\omega = 2\beta \text{ and } Q = \frac{\omega_0}{\Delta\omega}.$$

4.133 At resonance $\omega = \omega_0$

$$I_m(\omega_0) = \frac{V_m}{R}$$

Then $$I_m(\eta\,\omega_0) = \frac{V_m}{\sqrt{R^2 + \left(\eta\,\omega_0 L - \frac{1}{\eta\,\omega_0 C}\right)^2}}$$

$$= \frac{V_m}{\sqrt{R^2 + \left(\eta - \frac{l}{\eta}\right)^2 \frac{L}{C}}} = \frac{V_m}{\sqrt{1 + \left(Q^2 + \frac{l}{4}\right)\left(\eta - \frac{1}{\eta}\right)^2 \frac{L}{C}}}$$

4.134 The a.c. current must be

$$I = I_0 \sqrt{2}\, \sin \omega t$$

Then D.C. component of the rectified current is

$$\langle I' \rangle = \frac{1}{T} \int_0^{T/2} I_0 \sqrt{2}\, \sin \omega t\, dt$$

$$= I_0 \sqrt{2}\, \frac{1}{2\pi} \int_0^{\pi} \sin\theta\, d\theta$$

$$= \frac{I_0 \sqrt{2}}{\pi}$$

Since the charge deposited must be the same

$$I_0 t_0 = \frac{I_0 \sqrt{2}}{\pi} t \quad \text{or} \quad t = \frac{\pi t_0}{\sqrt{2}}$$

The answer is incorrect.

4.135 (a) $I(t) = I_1 \frac{t}{T} \quad 0 \le t < T$

$$I(t \pm T) = I(t)$$

Now mean current

$$<I> = \frac{1}{T} \int_0^T I_1 \frac{t}{T}\, dt = I_1 \frac{T^2/2}{T^2} = I_1/2$$

Then $\qquad I_1 = 2 I_0$ since $<I> = I_0$.

Now mean square current $<I^2>$

$$= 4 I_0^2 \frac{1}{T} \int_0^T \frac{t^2}{T^2}\, dt = \frac{4 I_0^2}{3}$$

so effective current $= \dfrac{2 I_0}{\sqrt{3}}$.

(b) In this case $I = I_1 |\sin \omega t|$

and $\qquad I_0 = \dfrac{1}{T} \displaystyle\int_0^T I_1 |\sin \omega t|\, dt$

$$= \frac{1}{2\pi} I_1 \int_0^{2\pi} |\sin \theta|\, d\theta = \frac{I_1}{\pi} \int_0^{\pi} \sin \theta\, d\theta = \frac{2 I_1}{\pi}$$

So $\qquad I_1 = \dfrac{\pi I_0}{2}$

Then, mean square current $= <I^2> = \dfrac{\pi^2 I_0^2}{4T} \displaystyle\int_0^T \sin^2 \omega t\, dt$

$$= \frac{\pi^2 I_0^2}{4} \times \frac{1}{2\pi} \int_0^{2\pi} \sin^2 \theta\, d\theta = \frac{\pi^2 I_0^2}{8}$$

so effective current $= \dfrac{\pi I_0}{\sqrt{8}}$.

4.136 $P_{d.c.} = \dfrac{V_0^2}{R}$

$$P_{a.c.} = \frac{V_0^2}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{R}{\sqrt{R^2 + \omega^2 L^2}} = \frac{v_0^2/R}{1 + \left(\frac{\omega L}{R}\right)^2} = \frac{P_{d.c.}}{\eta}$$

Thus $$\frac{\omega L}{R} = \sqrt{\eta - 1}$$

or $$\omega = \frac{R}{L}\sqrt{\eta - 1}$$

$$\nu = \frac{R}{2\pi L}\sqrt{\eta - 1} = 2 \text{ kH of on putting the values.}$$

4.137 $Z = \sqrt{R^2 + X_L^2}$ or $R_0 = \sqrt{Z^2 - X_L^2}$

The $$\tan\theta = \frac{X_L}{\sqrt{Z^2 - X_L^2}}$$

So $$\cos\varphi = \frac{\sqrt{Z^2 - X_L^2}}{Z} = \sqrt{1 - \left(\frac{X_L}{Z}\right)^2}$$

$$\varphi = \cos^{-1}\sqrt{1 - \left(\frac{X_L}{Z}\right)^2} = 37°.$$

The current lags by φ behind the voltage.

also $$P = VI\cos\varphi = \frac{V^2}{Z^2}\sqrt{Z^2 - X_L^2} = .160 \text{ kW.}$$

4.138 $$P = \frac{V^2(R + r)}{(R + r)^2 + \omega^2 L^2}$$

This is maximum when $R + r = \omega L$ for

$$P = \frac{V^2}{R + r + \frac{(\omega L)^2}{R + r}} = \frac{V^2}{\left[\sqrt{R + r} - \frac{\omega L}{\sqrt{R + r}}\right]^2 + 2\omega L}$$

Thus $R = \omega L - r$ for maximum power and $P_{max} = \frac{V^2}{2\omega L}$.

Substituting the values, we get $R = 200\ \Omega$ and $P_{max} = .114$ kW.

4.139 $$P = \frac{V^2 R}{R^2 + (X_L - X_C)^2}$$

Varying the capacitor does not change R so if P increases n times

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \text{ must decreases } \sqrt{n} \text{ times}$$

Thus $$\cos\varphi = \frac{R}{Z} \text{ increases } \sqrt{n} \text{ times}$$

$\therefore$ % increase in $\cos\varphi = \left(\sqrt{n} - 1\right) \times 100\,\% = 30.4\%$.

4.140 $P = \dfrac{V^2 R}{R^2 + (X_L - X_C)^2}$

At resonance $X_L = X_C \Rightarrow \omega_0 = \dfrac{1}{\sqrt{LC}}$.

Power generated will decrease n times when

$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 = (n-1)R^2$$

or $$\omega - \frac{\omega_0^2}{\omega} = \pm\sqrt{n-1}\,\frac{R}{L} = \pm\sqrt{n-1}\,2\beta.$$

Thus $$\omega^2 \mp 2\sqrt{n-1}\,\beta\,\omega - \omega_0^2 = 0$$

$$\left(\omega \mp \sqrt{n-1}\,\beta\right)^2 = \omega_0^2 + (n-1)\beta^2$$

or $$\frac{\omega}{\omega_0} = \sqrt{1 + (n-1)\beta^2/\omega_0^2} \pm \sqrt{n-1}\,\beta/\omega_0$$

(taking only the positive sign in the first term to ensure positive value for $\dfrac{\omega}{\omega_0}$.)

Now $$Q = \frac{\omega}{2\beta} = \frac{1}{2}\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}$$

$$\frac{\omega_0}{\beta} = \sqrt{1 + 4Q^2}$$

Thus $$\frac{\omega}{\omega_0} = \sqrt{1 + \frac{n-1}{(1+4Q^2)}} \pm \sqrt{n-1}\Big/\sqrt{1+4Q^2}$$

For large Q

$$\left|\frac{\omega - \omega_0}{\omega_0}\right| = \frac{\sqrt{en-1}}{2Q} = \frac{\sqrt{en-1}}{2Q} \times 100\,\% = 0.5\%$$

4.141 We have

$$V_1 = \frac{VR}{\sqrt{(R+R_1)^2 + X_L^2}}, \quad V_2 = \frac{V\sqrt{R_1^2 + X_L^2}}{\sqrt{(R+R_1)^2 + X_L^2}}$$

so $$(R+R_1)^2 + X_L^2 = \left(\frac{VR}{V_1}\right)^2, \quad R_1^2 + X_L^2 = \left(\frac{V_2 R}{V_1}\right)^2$$

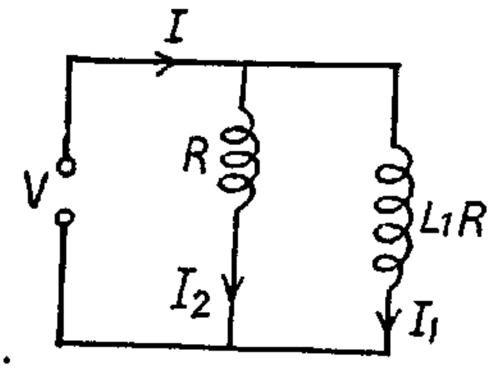
Hence $$R^2 + 2RR_1 = \frac{R^2}{V_1^2}(V^2 - V_2^2)$$

or $$R_1 = \frac{R}{2V_1^2}(V^2 - V_2^2 - V_1^2)$$

Heat generated in the coil $= \dfrac{V^2 R_1}{(R_1 + R_2)^2 + X_L^2} = \dfrac{V_1^2}{R^2} \times R_1 = \dfrac{V_1^2}{R^2} \times \dfrac{R^2}{2V_1^2}(V^2 - V_1^2 - V_2^2)$

$$= \frac{V^2 - V_1^2 - V_2^2}{2R} = 30 \text{ W}$$

4.142 Here $I_2 = \dfrac{V}{R}$, $V =$ effective voltage



$$I_1 = \frac{V}{\sqrt{R^2 + X_L^2}}$$

and $I = \dfrac{V\sqrt{(R + R_1)^2 + X_L^2}}{R\sqrt{R_1^2 + X_L^2}} = \dfrac{V}{R_{eff}}$

R_{eff} is the impedance of the coil & the resistance in parallel.

Now $$\frac{I^2 - I_2^2}{I_2^2} = \frac{R^2 + 2RR_1}{R_1^2 + X_L^2} = \left(\frac{I_1}{I_2}\right)^2 + \frac{2RR_1}{R^2 + X_L^2}$$

$$\frac{I^2 - I_2^2 - I_1^2}{I_2^2} = \frac{2RR_1}{R^2 + X_L^2}$$

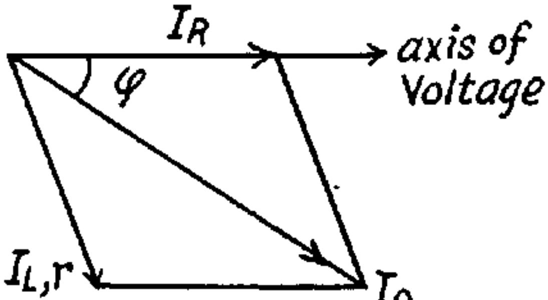
Now mean power consumed in the coil

$$= I_1^2 R_1 = \frac{V^2 R_1}{R_2 + X_L^2} = I_2^2 R \cdot \frac{I^2 - I_1^2 - I_2^2}{2I_2^2} = \frac{1}{2} R (I^2 - I_1^2 - I_2^2) = 2.5 \text{ W.}$$

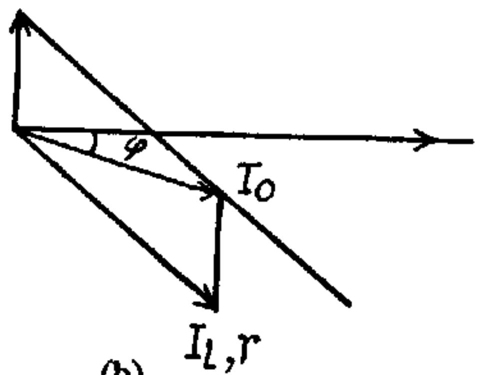
4.143 $\dfrac{1}{Z} = \dfrac{1}{R} + \dfrac{1}{\dfrac{1}{i\omega C}} = \dfrac{1}{R} + i\omega C = \dfrac{1 + i\omega RC}{R}$

$$|Z| = \frac{R}{\sqrt{1 + (\omega RC)^2}} = 40\ \Omega$$

4.144 (a) For the resistance, the voltage and the current are in phase. For the coil the voltage is ahead of the current by less than 90°. The current is obtained by addition because the elements are in parallel.



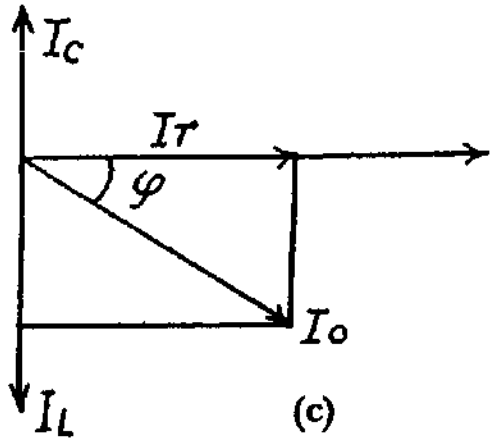
(a)



(b)

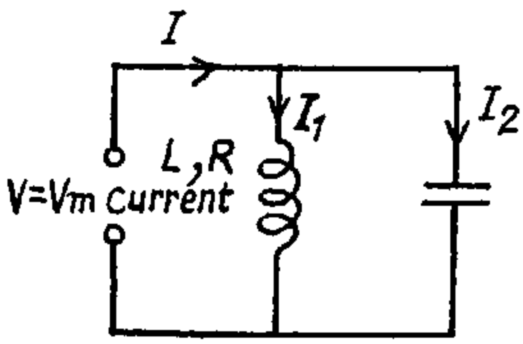
(b) I_C is ahead of the voltage by 90°.

(c) The coil has no resistance so I_L is 90° behind the voltage.



(c)

4.145 When the coil and the condenser are in parallel, the equation is



$$L\frac{dI_1}{dt} + RI_1 = \frac{\int I_2\,dt}{C} = V_m \cos\omega t$$

$$I = I_1 + I_2$$

Using complex voltages

$$I_1 = \frac{V_m\, e^{i\omega t}}{R + i\omega L}, \quad I_2 = i\omega C V_m\, e^{i\omega t}$$

and

$$I = \left(\frac{1}{R + i\omega L} + i\omega C\right) V_m\, e^{i\omega t} = \left[\frac{R - i\omega L + i\omega C(R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2}\right] V_m\, e^{i\omega t}$$

Thus, taking real parts $\qquad I = \dfrac{V_m}{|Z(\omega)|}\cos(\omega t - \varphi)$

where $\qquad \dfrac{1}{|Z(\omega)|} = \dfrac{[R^2 + \{\omega C(R^2 + \omega^2 L^2) - \omega L\}^2]}{(R^2 + \omega^2 L^2)^{1/2}}$

and $\qquad \tan\varphi = \dfrac{\omega L - \omega C(R^2 + \omega^2 L^2)}{R}$

(a) To get the frequency of resonance we must define what we mean by resouance. One definition requires the extremum (maximum or minimum) of current amplitude. The other definition requires rapid change of phase with φ passing through zero at resonance. For the series circuit.

$$I_m = \frac{V_m}{\left\{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2\right\}^{1/2}} \quad \text{and} \quad \tan\varphi = \frac{\omega L - \dfrac{1}{\omega C}}{R}$$

both definitions give $\omega^2 = \frac{1}{LC}$ at resonance. In the present case the two definitions do not agree (except when $R = 0$). The definition that has been adopted in the answer given in the book is the vanishing of phase. This requires

$$C(R^2 + \omega^2 L^2) = L$$

or
$$\omega^2 = \frac{1}{LC} - \frac{R^2}{L^2} = \omega_{res}^2, \quad \omega_{res} = 31.6 \times 10^3 \text{ rad/s.}$$

Note that for small R, φ rapidly changes from $= -\frac{\pi}{2}$ to $+\frac{\pi}{2}$ as ω passes through ω_{res} from $< \omega_{res}$ to $> \omega_{res}$.

(b) At resonance
$$I_m = \frac{V_m R}{L/C} = V_m \frac{CR}{L}$$

so I = effective value of total current = $V \frac{CR}{L} = 3.1$ mA.

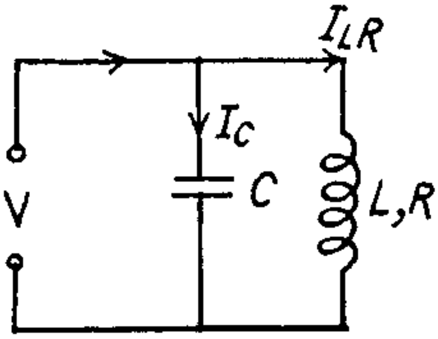
similarly
$$I_L = \frac{V}{\sqrt{L/C}} = V\sqrt{\frac{C}{L}} = 0.98 \text{ A.}$$

$$I_C = \omega C V = V\sqrt{\frac{C}{L} - \frac{R^2 C^2}{L^2}} = 0.98 \text{ A.}$$

Note :- The vanishing of phase (its passing through zero) is considered a more basic definition of resonance.

4.146 We use the method of complex voltage

$V = V_0 e^{i\omega t}$



Then $I_C = \dfrac{V_0 e^{i\omega t}}{\dfrac{1}{i\omega C}} = i\omega C V_0 e^{i\omega t}$

$$I_{L,R} = \frac{V_0 e^{i\omega t}}{R + i\omega L}$$

$$I = I_C + I_{L,R} = V_0 \frac{R - i\omega L + i\omega C(R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2} e^{i\omega t}$$

Then taking the real part

$$I = \frac{V_0\sqrt{R^2 + \{\omega C(R^2 + \omega^2 L^2) - \omega L\}^2}}{R^2 + \omega^2 L^2} \cos(\omega t - \varphi)$$

where
$$\tan\varphi = \frac{\omega L - \omega C(R^2 + \omega^2 L^2)}{R}$$

4.147 From the previous problem

$$Z = \frac{R^2+\omega^2 L^2}{\sqrt{R^2+\left\{\omega C(R^2+\omega^2L^2)-\omega L\right\}^2}}$$

$$= \frac{R^2+\omega^2 L^2}{\sqrt{(R^2+\omega^2L^2)(1-2\omega^2LC)+\omega^2C^2(R^2+\omega^2L^2)^2}}$$

$$= \frac{\sqrt{R^2+\omega^2L^2}}{\sqrt{(1-2\omega^2LC)+\omega^2C^2(R^2+\omega^2L^2)}} = \frac{\sqrt{R^2+\omega^2L^2}}{\sqrt{(1-\omega^2LC)^2+(\omega RC)^2}}$$

4.148 (a) We have

$$\varepsilon = -\frac{d\Phi}{dt} = \omega\Phi_0 \sin\omega t = L\dot{I}+RI$$

Put $\quad I = I_m \sin(\omega t-\varphi)$. Then

$$\omega\Phi_0\sin\omega t = \omega\Phi_0\left\{\sin(\omega t-\varphi)\cos\varphi+\cos(\omega t-\varphi)\sin\varphi\right\}$$

$$= LI_m\omega\cos(\omega t-\varphi)+RI_m\sin(\omega t-\varphi)$$

so $\quad RI_m = \omega\Phi_0\cos\varphi \quad \text{and} \quad LI_m = \Phi_0\sin\varphi$

or $\quad I_m = \frac{\omega\Phi_0}{\sqrt{R^2+\omega^2L^2}} \quad \text{and} \quad \tan\varphi = \frac{\omega L}{R}.$

(b) Mean mechanical power required to maintain rotation = energy loss per unit time

$$= \frac{1}{T}\int_0^T RI^2\,dt = \frac{1}{2}RI_m^2 = \frac{1}{2}\frac{\omega^2\Phi_0^2R}{R^2+\omega^2L^2}$$

4.149 We consider the force $\vec{F}_{12}$ that a circuit 1 exerts on another closed circuit 2 :-

$$\vec{F}_{12} = \oint I_2\, d\vec{l}_2\times\vec{B}_{12}$$

Here $\vec{B}_{12}$ = magnetic field at the site of the current element $d\vec{l}_2$ due to the current I_1 flowing in 1.

$$= \frac{\mu_0}{4\pi}\int\frac{I_1\,d\vec{l}_1\times\vec{r}_{12}}{r_{12}^3}$$

where $\vec{r}_{12} = \vec{r}_2-\vec{r}_1$ = vector from current element $d\vec{l}_1$ to the current element $d\vec{l}_2$

Now

$$\vec{F}_{12} = \frac{\mu_0}{4\pi}\iint I_1I_2\frac{d\vec{l}_2\times(d\vec{l}_1\times\vec{r}_{12})}{r_{12}^3} = \frac{\mu_0}{4\pi}\iint I_1I_2\frac{d\vec{l}_1(d\vec{l}_2\cdot\vec{r}_{12})-(d\vec{l}_1\cdot d\vec{l}_2)\vec{r}_{12}}{r_{12}^3}$$

In the first term, we carry out the integration over $d\vec{l}_2$ first. Then

$$\iint \frac{d\vec{l_1}\,(d\vec{l_2}\cdot\vec{r_{12}})}{r_{12}^3} = \int d\vec{l_1}\oint \frac{d\vec{l_2}\cdot\vec{r_{12}}}{r_{12}^3} = -\int d\vec{l_1}\oint d\vec{l_2}\cdot\nabla_2\frac{1}{r_{12}} = 0$$

because $$\oint d\,l_2\cdot\nabla_2\frac{1}{r_{12}} = \int d\vec{S_2}\ \text{curl}\left(\nabla\frac{1}{r_{12}}\right) = 0$$

Thus $$F_{12} = -\frac{\mu_0}{4\pi}\iint I_1 I_2\ d\vec{l_1}\cdot d\vec{l_2}\,\frac{\vec{r_{12}}}{r_{12}^3}$$

The integral involved will depend on the vector $\vec{a}$ that defines the separation of the (suitably chosen)centre of the coils. Let C_1 and C_2 be the centres of the two coil suitably defined. Write

$$\vec{r_{12}} = \vec{r_2} - \vec{r_1} = \vec{\rho_2} - \vec{\rho_1} + \vec{a}$$

where $\vec{\rho_1}$ ($\vec{\rho_2}$) is the distance of $d\vec{l_1}(d\vec{l_2})$ from $C_1\,(C_2)$ and $\vec{a}$ stands for the vector $\overrightarrow{C_1C_2}$.

Then $$\frac{\vec{r_{12}}}{r_{12}^3} = -\vec{\nabla}_{\vec{a}}\frac{1}{r_{12}}$$

and $$\vec{F_{12}} = \vec{\nabla}_a\left[I_1 I_2\,\frac{\mu_0}{4\pi}\iint\frac{d\vec{l_1}\cdot d\vec{l_2}}{r_{12}}\right]$$

The bracket defines the mutual inductance L_{12}. Thus noting the definition of x

$$\langle F_x\rangle = \frac{\partial L_{12}}{\partial x}\langle I_1 I_2\rangle$$

where < > denotes time average. Now

$$I_1 = I_0\cos\omega t = \text{Real part of } I_0 e^{i\omega t}$$

The current in the coil 2 satisfies $R I_2 + L_2\dfrac{dI_2}{dt} = -L_{12}\dfrac{dI_1}{dt}$

or $$I_2 = \frac{-i\omega L_{12}}{R + i\omega L_2} I_0 e^{i\omega t}\ \text{(in the complex case)}$$

taking the real part

$$I_2 = -\frac{\omega L_{12} I_0}{R^2 + \omega^2 L_2^2}(\omega L_2\cos\omega t - R\sin\omega t) = -\frac{\omega L_{12}}{\sqrt{R^2 + \omega^2 L_\tau^2}} I_0\cos(\omega t + \varphi)$$

Where $\tan\varphi = \dfrac{R}{\omega L_2}$. Taking time average, we get

$$\langle F_x\rangle = \frac{\partial L_{12}}{\partial x} I_0\,\frac{\omega L_{12} I_0}{\sqrt{R^2 + \omega^2 L_2^2}}\cdot\frac{1}{2}\cos\varphi = \frac{\omega^2 L_2 L_{12} I_0^2}{2\left(R^2 + \omega^2 L_2^2\right)}\frac{\partial L_{12}}{\partial x}$$

The repulsive nature of the force is also consistent with Lenz's law, assuming, of comse, that L_{12} decreases with x.