3. Algebraic Expressions

- A **variable** is something that does not have a fixed value. The value of a variable varies.
- Variables are represented by English letters such as *x*, *y*, *z*, *a*, *b*, *c* etc.
- A combination of variables, numbers and operators (+, –, × and ÷) is known as **expression**.
- Using different operations on variables and numbers, expressions such as $\frac{1}{7}-4y, 9x-5$, can be formed.

Example:

Meena's age is 4 years less than 7 times the age of Ravi. Express it using variables.

Solution:

Let the age of Ravi be *x* years.

7 times the age of Ravi can be expressed as 7*x*.

4 years less than 7 times the age of Ravi can be written as 7*x* – 4.

∴ Age of Meena = (7*x* – 4)

- Algebraic expressions are formed by combining variables with constants using operations of addition, subtraction, multiplication and division.

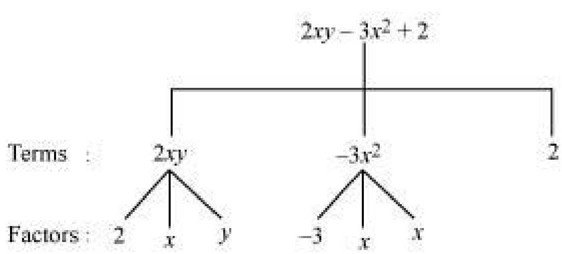
For example: 4xy, 2x2 – 3, 7xy + 2x, etc.

In an algebraic expression, say 2xy – 3x2 + 2; 2xy, (–3x2), 2 are known as the terms of the expression.

The expression 2xy – 3x2 +2 is formed by adding the terms 2xy, (–3x2) and 2 where 2, x, y are factors of the term 2xy; (–3), x, x are factors of the term (–3x2); 2 is the factor of the term 2.

For an expression, the terms and its factors can be represented easily and elegantly by a tree diagram.

Tree diagram for the expression 2xy – 3x2 +2:



Note: In an expression, 1 is not taken as separate factor.

- The numerical factor of a term is known as its coefficient. For example, for the term –3x2y, the coefficient is (–3).
- The terms having the same algebraic factors are called like terms, while the terms having different algebraic factors are called unlike terms.

For example: 13x2y, – 23x2y are like terms; 12xy, 3x2 are unlike terms

- Expressions can be classified on the basis of the number of terms present in them.
    - Expression containing only one term is called a **monomial**.

      For example, $2x$, $-3x^2$, $2xy$ etc.

    - Expression containing only two unlike terms is called a **binomial**.

      For example, $2x + 3$, $3x^2 - 2$, $-2xy + 3y^2$ etc.

    - Expression with three terms, where the terms are unlike is called a **trinomial**.

      For example, $2x^2 - 3x + 1$, $-2xy + 5y + 6x$ etc.

  In general, the expression with one or more terms is called a polynomial.

- **Degree** of the polynomial is the highest exponent of the variable in the polynomial.

  For example, polynomial $2x^2 - 3x + 1$ has degree 2.

- **Factorization** is the decomposition of an algebraic expression into product of factors. Factors of an algebraic term can be numbers or algebraic variables or algebraic expressions.

  For example, the factors of $2a^2b$ are 2, a, a, b, since $2a^2b = 2 \times a \times a \times b$

  The factors, 2, a, a, b, are said to be irreducible factors of $2a^2b$ since they cannot be expressed further as a product of factors.

  Also, $2a^2b = 1 \times 2 \times a \times a \times b$

  Therefore, 1 is also a factor of $2a^2b$. In fact, 1 is a factor of every term. However, we do not represent 1 as a separate factor of any term unless it is specially required.

  For example, the expression, $2x^2(x + 1)$, can be factorized as $2 \times x \times x \times (x + 1)$.

  Here, the algebraic expression $(x + 1)$ is a factor of $2x^2(x + 1)$.

- **Factorization of expressions by the method of common factors**

  This method involves the following steps.

  Step 1: Write each term of the expression as a product of irreducible factors.

  Step 2: Observe the factors, which are common to the terms and separate them.

  Step 3: Combine the remaining factors of each term by making use of distributive law.

  Example: Factorize $12p^2q + 8pq^2 + 18pq$.

  Solution: We have,

  $12p^2q = 2 \times 2 \times 3 \times p \times p \times q$

$8pq^2 = 2 \times 2 \times 2 \times p \times q \times q$

$18pq = 2 \times 3 \times 3 \times p \times q$

The common factors are 2, p, and q.

$\therefore 12p^2q + 8pq^2 + 18pq$

$= 2 \times p \times q\ [(2 \times 3 \times p) + (2 \times 2 \times q) + (3 \times 3)]$

$= 2pq\ (6p + 4q + 9)$

- **Factorization by regrouping terms**

  Sometimes, all terms in a given expression do not have a common factor. However, the terms can be grouped by trial and error method in such a way that all the terms in each group have a common factor. Then, there happens to occur a common factor amongst each group, which leads to the required factorization.

  Example:

  Factorize $2a^2 - b + 2a - ab$.

  Solution:

  $2a^2 - b + 2a - ab = 2a^2 + 2a - b - ab$

  The terms,$2a^2$ and $2a$, have common factors, 2 and a.

  The terms, $-b$ and $-ab$ have common factors, -1 and b.

  Therefore,

  $2a^2 - b + 2a - ab = 2a^2 + 2a - b - ab$

  $= 2a\ (a + 1) - b\ (1 + a)$

  $= (a + 1)\ (2a - b)$ (As the factor, $(1 + a)$, is common to both the terms)

  Thus, the factors of the given expression are $(a + 1)$ and $(2a - b)$.

- **Some of the expressions can also be factorized by making use of the following identities.**
  1. $a^2 + 2ab + b^2 = (a + b)^2$
  2. $a^2 - 2ab + b^2 = (a - b)^2$
  3. $a^2 - b^2 = (a + b)\ (a - b)$

  For example, the expression $4x^2 + 12xy + 9y^2 - 4$ can be factorized as follows:
  $4x^2 + 12xy + 9y^2 - 4$
  $= (2x^2) + 2(2x)(3y) + (3y)^2 - 4$
  $= (2x + 3y)^2 - 4$ [Using the identity, $a^2 + 2ab + b^2 = (a + b)^2$]
  $= (2x + 3y)^2 - (2)^2$
  $= (2x + 3y + 2)\ (2x + 3y - 2)$ [Using the identity, $a^2 - b^2 = (a + b)\ (a - b)$]

- **Factorization by using the identity, $x^2 + (a + b)x + ab = (x + a)(x + b)$.**

  To apply this identity in an expression of the type $x^2 + px + q$, we observe the coefficient of x and the constant term.

  Two numbers, a and b, are chosen such that their product is q and their sum is p.

  i.e., $a + b = p$ and $ab = q$

  Then, the expression, $x^2 + px + q$, becomes $(x + a)(x + b)$.

  Example:

  Factorize $a^2 - 2a - 8$.

  Solution:

  Observe that, $-8 = (-4) \times 2$ and $(-4) + 2 = -2$

  Therefore, $a^2 - 2a - 8 = a^2 - 4a + 2a - 8$

  $= a(a - 4) + 2(a - 4)$

  $= (a - 4)(a + 2)$

- **Polynomial**

An algebraic expression in which the exponents of the variables are non-negative integers are called polynomials. For example, $3x^4 + 2x^3 + x + 9$, $3x^4$ etc are polynomials.

- **Constant polynomial:** A constant polynomial is of the form $p(x) = k$, where k is a real number. For example, –9, 10, 0 are constant polynomials.
- **Zero polynomial:** A constant polynomial '0' is called zero polynomial.

General form of a polynomial:

A polynomial of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1 \ldots a_r$ are constants and $a_n \neq 0$.
Here, $a_0, a_1 \ldots a_n$ are the respective coefficients of $x^0, x^1, x^2, \ldots x^n$ and n is the power of the variable x.
$a_n x^n, a_{n-1} x^{n-1} - a_0$ and $a_0 \neq 0$ are called the terms of $p(x)$.

- **Classification of polynomials on the basis of number of terms**
  - A polynomial having one term is called a monomial e.g. $3x$, $25t^3$ etc.
  - A polynomial having two terms is called a binomial e.g. $2t - 6$, $3x^4 + 2x$ etc.
  - A polynomial having three terms is called a trinomial. e.g. $3x^4 + 8x + 7$ etc.

- **Degree**

The degree of a polynomial is the highest exponent of the variable of the polynomial. For example, the degree of polynomial $3x^4 + 2x^3 + x + 9$ is 4.

The degree of a term of a polynomial is the value of the exponent of the term.

- **Classification of polynomial according to their degrees**

- A polynomial of degree one is called a linear polynomial e.g. $3x + 2$, $4x$, $x + 9$.
- A polynomial of degree two is called a quadratic polynomial. e.g. $x^2 + 9$, $3x^2 + 4x + 6$.
- A polynomial of degree three is called a cubic polynomial e.g. $10x^3 + 3$, $9x^3$.

Note: The degree of a non-zero constant polynomial is zero and the degree of a zero polynomial is not defined.

- Addition and subtraction of algebraic expressions:
  - The sum or difference of two like terms is a like term, with its numerical coefficient equal to the sum or difference of the numerical coefficients of the two like terms.
  - When algebraic expressions are added, the like terms are added and unlike terms are left as they were.

Example : Subtract $(x^2 – 2y^2 + y)$ from the sum of $(–2x^2 + 3x + 2)$ and $\left(-2y+3x^2+5x\right)$

Solution:

$$\left(-2x^2+3x+2\right)+\left(-2y+3x^2+5x\right)$$

$$=\left(-2x^2+3x^2\right)+\left(3x+5x\right)-2y+2 \qquad \text{[Rearranging terms]}$$

$$=x^2+8x-2y+2$$

$$\therefore \left(x^2+8x-2y+2\right)-\left(x^2-2y^2+y\right)$$

ultiplying a polynomial by a binomial (or trinomial) in horizontal a
erm. That is, every term of the polynomial is multiplied by every t

$$=x^2+8x-2y+2-x^2+2y^2-y$$

$$=\left(x^2-x^2\right)+2y^2+8x+\left(-2y-y\right)+2 \qquad \text{[Rearranging terms]}$$

$$=2y^2+8x-3y+2$$

- While multiplying a polynomial by a binomial (or trinomial) in horizontal arrangement, we multiply it term by term. That is, every term of the polynomial is multiplied by every term of the binomial (or trinomial).

Example:

Simplify $(x + 2y)(x + 3) – (2x + 1)(y + x + 1)$.

Solution:

$(x + 2y)(x + 3) = x(x + 3) + 2y(x + 3)$

$= x^2 + 3x + 2xy + 6y$

$(2x + 1)(y + x + 1) = 2x(y + x + 1) + 1(y + x + 1)$

$= 2xy + 2x^2 + 2x + y + x + 1$

$= 2xy + 2x^2 + 3x + y + 1$

$\therefore (x + 2y)(x + 3) - (2x + 1)(y + x + 1) = x^2 + 3x + 2xy + 6y - 2xy - 2x^2 - 3x - y - 1$

$= -x^2 + 5y - 1$

- We can also perform multiplication of two polynomials using vertical arrangement.

  For example,

$$\begin{array}{r} l + 6m + 7n \\ \times \qquad\qquad l + 3m \\ \hline 3lm + 18m^2 + 21mn \\ +l^2 + 6lm + 7nl \qquad\qquad \\ \hline l^2 + 9lm + 18m^2 + 21mn + 7nl \end{array}$$

- **Values of polynomials at different points**

A polynomial is made up of constants and variables. Hence, the value of the polynomial changes as the value of the variable in the polynomial changes. Thus, for the different values of the variable *x*, we get different values of the polynomial.

Example:
Find the value of polynomial $p(x) = 3x^2 + 2x + 9$ at $x = -2$.

Solution:

The variable in the given polynomial is *x*. Hence, replacing *x* by –2.

$p(x) = 3x^3 + 2x + 9$

$\therefore p(-2) = 3(-2)^3 + 2(-2) + 9$

$= -24 - 4 + 9$

$= -19$

- **Remainder Theorem**

  If $p(x)$ is a polynomial of degree greater than or equal to one and a is any real number then if $p(x)$ is divided by the linear polynomial $x - a$, the remainder is $p(a)$.

  Example:
  Find the remainder when $x^5 - x^2 + 5$ is divided by $x - 2$.
  Solution:
  $p(x) = x^5 - x^2 + 5$
  The zero of $x - 2$ is 2.
  $p(2) = 2^5 - 2^2 + 5 = 32 - 4 + 5 = 33$
  Therefore, by remainder theorem, the remainder is 33.

- **Factor Theorem**

If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then

  - $x - a$ is a factor of $p(x)$, if $p(a) = 0$.
  - $p(a) = 0$, if $(x - a)$ is a factor of $p(x)$.

Example:
Determine whether $x + 3$ is a factor of $x^3 + 5x^2 + 5x - 3$.
Solution:
The zero of $x + 3$ is -3.
Let $p(x) = x^3 + 5x^2 + 5x - 3$
$p(-3) = (-3)^3 + 5(-3)^2 + 5(-3) - 3$
$= -27 + 45 - 15 - 3$
$= -45 + 45$
$= 0$
Therefore, by factor theorem, $x + 3$ is the factor of $p(x)$.

- Factorisation of quadratic polynomials of the form $ax^2 + bx + c$ can be done using Factor theorem and splitting the middle term.

Example 1:
Factorize $x^2 - 7x + 10$ using the factor theorem.

Solution:
Let $p(x) = x^2 - 7x + 10$
The constant term is 10 and its factors are ±1, ±2, $\pm5$and ±10.
Let us check the value of the polynomial for each of these factors of 10.
$p(1) = 1^2 - 7 \cdot 1 + 10 = 1 - 7 + 10 = 4 \neq 0$
Hence, $x - 1$ is not a factor of $p(x)$.
$p(2) = 2^2 - 7 \cdot 2 + 10 = 4 - 14 + 10 = 0$
Hence, $x - 2$ is a factor of $p(x)$.
$p(5) = 5^2 - 7 \cdot 5 + 10 = 25 - 35 + 10 = 0$
Hence, $x - 5$ is a factor of $p(x)$.
We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $x - 2$ and $x - 5$.
Thus, we can write the given polynomial as:
$p(x) = x^2 - 7x + 10 = (x - 2)(x - 5)$

Example 2:
Factorize $2x^2 - 11x + 15$ by splitting the middle term.

Solution:
The given polynomial is $2x^2 - 11x + 15$.
Here, $a\,c = 2 \times 15 = 30$. The middle term is -11. Therefore, we have to split -11 into two numbers such that their product is 30 and their sum is -11. These numbers are -5 and -6 [As $(-5) + (-6) = -11$ and $(-5) \times (-6) = 30$].
Thus, we have:
$2x^2 - 11x + 15 = 2x^2 - 5x - 6x + 15$
$= x(2x - 5) - 3(2x - 5)$
$= (2x - 5)(x - 3)$

Note: A quadratic polynomial can have a maximum of two factors.