ADJOINT AND INVERSE OF A MATRIX (XII, R. S. AGGARWAL)

EXERCISE 7 (Pg.No.: 293)

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A). $A = |A|. I$.

1. $\begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$

Sol. $\text{adj}\, A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(9) = 9$; $C_{12} = (-1)^{1+2}(5) = -5$

$C_{21} = (-1)^{2+1}(3) = -3$; $C_{22} = (-1)^{2+2}(2) = 2$

$\therefore\ \text{adj}\, A = \begin{bmatrix} 9 & -5 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 \\ 5 & 9 \end{vmatrix} \Rightarrow |A| = [2(9) - 5(3)] \Rightarrow |A| = (18-15) \Rightarrow |A| = 3$

$\text{M.H.S} = (\text{adj}\, A).\, A = \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 18-15 & 27-27 \\ -10+10 & -15+18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$\text{L.H.S.} = A(\text{adj}\, A) = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 18-15 & -6+6 \\ 45-45 & -15+18 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

and $\text{R.H.S.} = |A|.I = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Hence $A.(adj\ A) = (adj\ A).A = |A|.I$

2. $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Sol. $\text{adj}\, A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(2) = 2$; $C_{12} = (-1)^{1+2}(-1) = 1$

$C_{21} = (-1)^{2+1}(-5) = 5$; $C_{22} = (-1)^{2+2}(3) = 3$

$\therefore\ \text{adj}\, A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}' = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} \Rightarrow |A| = [3(2) - (-1)(-5)] \Rightarrow |A| = (6-5) \Rightarrow |A| = 1$

Now, $\text{M.H.S.} = (\text{adj}\, A).\, A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, L.H.S. $= A.(\text{adj } A) = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, R.H.S. $= |A|.I = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence, $A(\text{adj } A) = (\text{adj } A).A = |A|.I$

3. $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Sol. $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(\cos\alpha) = \cos\alpha$; $C_{12} = (-1)^{1+2}(\sin\alpha) = -\sin\alpha$

$C_{21} = (-1)^{2+1}(\sin\alpha) = -\sin\alpha$; $C_{22} = (-1)^{2+2}(\cos\alpha) = \cos\alpha$

$\therefore$ $\text{adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ $\Rightarrow |A| = \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$

$\Rightarrow |A| = [\cos\alpha.\cos\alpha - \sin\alpha.\sin\alpha]$ $\Rightarrow |A| = (\cos^2\alpha - \sin^2\alpha)$ $\Rightarrow |A| = \cos 2\alpha$

Now, L.H.S. $= A.(\text{adj } A) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & -\sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$

Now, M.H.S. $= (\text{adj } A).A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \sin\alpha\cos\alpha - \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$

Now, R.H.S $= |A|.I = \cos 2\alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$. Hence, $A.(\text{adj } A) = (\text{adj } A).A = |A|.I$

4. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Sol. $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = [1(3) - 0(-2)] = (3-0) = 3$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -[3(3) - (-2)1] = -(9+2) = -11$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = [3(0) - 1(1)] = (0-1) = -1$

$C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -[(-1)3 - 0(2)] = -(-3-0) = 3$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = [1(3)-1(2)] = (3-2) = 1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -[1(0)-1(-1)] = -(0+1) = -1$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} = [(-1)(-2)-1(2)] = (2-2) = 0$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -[1(-2)-3(2)] = -(-2-6) = 8$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = [1(1)-3(-1)] = (1+3) = 4$$

$$\therefore\ \operatorname{adj} A = \begin{bmatrix} 3 & -11 & -1 \\ 3 & 1 & -1 \\ 0 & 8 & 4 \end{bmatrix}' = \begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix} \quad \Rightarrow |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix}$$

$$\Rightarrow |A| = 1\begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \ \Rightarrow\ |A| = 1(3-0)+1(9+2)+2(0-1)$$

$$\Rightarrow |A| = 3+11-2 \quad \Rightarrow |A| = 12$$

$$\text{Now, L.H.S.} = A(\operatorname{adj} A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3+11-2 & 3-1-2 & 0-8+8 \\ 9-11-2 & 9+1+2 & 0+8-8 \\ 3-0-3 & 3+0-3 & 0+0+12 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$\text{Now, M.H.S.} = (\operatorname{adj} A).A = \begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+9+0 & -3+3+0 & 6-6+0 \\ -11+3+8 & 11+1+0 & -22-2+24 \\ -1+(-3)+4 & 1-1+0 & -2+2+12 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

Now, R.H.S. $= |A|.I$.

$$12.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}. \quad \text{Hence, } A.(\operatorname{adj} A) = (\operatorname{adj} A).A = |A|.I$$

5. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

Sol. $\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}6 & -5\\ -2 & 2\end{vmatrix}=(12-10)=2$; $C_{12}=(-1)^{1+2}\begin{vmatrix}-15 & -5\\ 5 & 2\end{vmatrix}=-(-30+25)=5$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}-15 & 6\\ 5 & -2\end{vmatrix}=(30-30)=0;\quad C_{21}=(-1)^{2+1}\begin{vmatrix}-1 & 1\\ -2 & 2\end{vmatrix}=-(-2+2)=0$$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}3 & 1\\ 5 & 2\end{vmatrix}=(6-5)=1;\quad C_{23}=(-1)^{2+3}\begin{vmatrix}3 & -1\\ 5 & -2\end{vmatrix}=-1(-6+5)=1$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}-1 & 1\\ 6 & -5\end{vmatrix}=(5-6)=-1;\quad C_{32}=(-1)^{3+2}\begin{vmatrix}3 & 1\\ -15 & -5\end{vmatrix}=-(-15+15)=0$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}3 & -1\\ -15 & 6\end{vmatrix}=(18-15)=3$$

$$\operatorname{adj} A=\begin{bmatrix}2 & 5 & 0\\ 0 & 1 & 1\\ -1 & 0 & 3\end{bmatrix}'=\begin{bmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}$$

$$|A|=\begin{vmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{vmatrix}\Rightarrow |A|=3\begin{vmatrix}6 & -5\\ -2 & 2\end{vmatrix}+1\begin{vmatrix}-15 & -5\\ 5 & 2\end{vmatrix}+1\begin{vmatrix}-15 & 6\\ 5 & -2\end{vmatrix}$$

$$\Rightarrow |A|=3(12-10)+(-30+25)+(30-30)\quad \Rightarrow |A|=3(2)+(-5)$$

$$\Rightarrow |A|=6-5\quad \Rightarrow |A|=1$$

Now, L.H.S. $=A(\operatorname{adj} A)=\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}$

$$=\begin{bmatrix}6-5+0 & 0-1+1 & -3-0+3\\ -30+30+0 & 0+6-5 & 15+0-15\\ 10-10+0 & 0-2+2 & -5+0+6\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Now, M.H.S $=(\operatorname{adj} A).A=\begin{bmatrix}2 & 0 & -1\\ 5 & 1 & 0\\ 0 & 1 & 3\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$

$$=\begin{bmatrix}6-0-5 & -2+0+2 & 2-0-2\\ 15-15+0 & -5+6-0 & 5-5+0\\ 0-15+15 & 0+6-6 & 0-5+6\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Now, R.H.S. $=|A|.I=1.\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$. Hence, $A(\operatorname{adj} A)=(\operatorname{adj} A).A=|A|.I$

6. $\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}$

Sol. $\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = (2-3) = -1$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -(1-9) = 8$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = (1-6) = -5$; $C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(-1-2) = 1$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = (0-6) = -6$; $C_{23} = (-1)^{2+3}\begin{vmatrix} 0 & 1 \\ 3 & \end{vmatrix} = -(0-3) = 3$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = (3-4) = -1$; $C_{32} = (-1)^{3+2}\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(-0-2) = 2$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = (0-1) = -1$

$$\operatorname{adj} A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}' \Rightarrow \operatorname{adj} A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} \Rightarrow |A| = 0\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \Rightarrow |A| = 0 - 1(1-9) + 2(1-6)$$

$\Rightarrow |A| = (8-10) \Rightarrow |A| = -2$

Now, L.H.S. $= A(\operatorname{adj} A) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 0+8-10 & 0-6+6 & 0+2-2 \\ -1+16-15 & 1-12+9 & -1+4-3 \\ -3+8-5 & 3-6+3 & -3+2-1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Now, M.H.S $= (\operatorname{adj} A).A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0+1-3 & -1+2-1 & -2+3-1 \\ 0-6+6 & 8-12+2 & 16-18+2 \\ 0+3-3 & -5+6-1 & -10+9-1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Now, R.H.S. $= |A|.I = (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Hence, $A(\operatorname{adj} A) = (\operatorname{adj} A).A = |A|.I$

7. $\begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$

Sol. $\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 4 \\ 8 & 2 \end{vmatrix} = (-2-32) = -34$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 4 \\ 6 & 2 \end{vmatrix} = -(10-24) = 14$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 5 & -1 \\ 6 & 8 \end{vmatrix} = (40+6) = 46$; $C_{21} = (-1)^{2+1}\begin{vmatrix} 7 & 3 \\ 8 & 2 \end{vmatrix} = -(14-24) = 10$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 9 & 3 \\ 6 & 2 \end{vmatrix} = (18-18) = 0$; $C_{23} = (-1)^{2+3}\begin{vmatrix} 9 & 7 \\ 6 & 8 \end{vmatrix} = -(72-42) = -30$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 7 & 3 \\ -1 & 4 \end{vmatrix} = (28+3) = 31$; $C_{32} = \begin{vmatrix} 9 & 3 \\ 5 & 4 \end{vmatrix} = -(36-15) = -21$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 9 & 7 \\ 5 & -1 \end{vmatrix} = (-9-35) = -44$

$$\operatorname{adj} A = \begin{bmatrix} -34 & 14 & 46 \\ 10 & 0 & -30 \\ 31 & -21 & -44 \end{bmatrix}' = \begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & 21 \\ 46 & -30 & -44 \end{bmatrix}$$

Now, L.H.S. $= A.(\operatorname{adj} A) = \begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}\begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix}$

$$= \begin{bmatrix} -306+98+138 & 90+0-90 & 279-147-132 \\ -70-14+184 & 50-0-120 & 155+21-176 \\ -204+112+92 & 60+0-60 & 186-168-88 \end{bmatrix} = \begin{bmatrix} -70 & 0 & 0 \\ 0 & -70 & 0 \\ 0 & 0 & -70 \end{bmatrix}$$

Now, M.H.S. $= (\operatorname{adj} A).A = \begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & 21 \\ 46 & -30 & -44 \end{bmatrix}\begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -306+50+186 & 238-10+248 & -132-40+62 \\ 126+0-126 & 98-0-168 & 42+0-142 \\ 414-150-264 & 322+30-352 & 138-120-88 \end{bmatrix} = \begin{bmatrix} -70 & 0 & 0 \\ 0 & -70 & 0 \\ 0 & 0 & -70 \end{bmatrix}$$

$A = \begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix} \Rightarrow |A| = 9\begin{vmatrix} -1 & 4 \\ 8 & 2 \end{vmatrix} - 7\begin{vmatrix} 5 & 4 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & -1 \\ 6 & 8 \end{vmatrix}$

$\Rightarrow |A| = 9(-2-32) - 7(10-24) + 3(40+6) \Rightarrow |A| = -306+98+138 \Rightarrow |A| = -70$

Now, R.H.S. $= |A|.I = (-70)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -70 & 0 & 0 \\ 0 & -70 & 0 \\ 0 & 0 & -70 \end{bmatrix}$

Hence, $A(\operatorname{adj} A)=(\operatorname{adj} A).A=|A|.I$

8. $\begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}$

Sol. $\operatorname{adj} A=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix} 0 & 6 \\ 7 & 9 \end{vmatrix}=(0-42)=-42$; $C_{12}=(-1)^{1+2}\begin{vmatrix} 1 & 6 \\ 2 & 9 \end{vmatrix}=-(9-12)=3$

$C_{13}=(-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix}=(7-0)=7$; $C_{21}=(-1)^{2+1}\begin{vmatrix} 5 & 3 \\ 7 & 9 \end{vmatrix}=-(45-21)=-24$

$C_{22}=(-1)^{2+2}\begin{vmatrix} 4 & 3 \\ 2 & 9 \end{vmatrix}=(36-6)=30$; $C_{23}=(-1)^{2+3}\begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix}=-(28-10)=-18$

$C_{31}=(-1)^{3+1}\begin{vmatrix} 5 & 3 \\ 0 & 6 \end{vmatrix}=(30-0)=30$; $C_{32}=\begin{vmatrix} 4 & 3 \\ 1 & 6 \end{vmatrix}=-(24-3)=-21$

$C_{33}=(-1)^{3+3}\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix}=(0-5)=-5$

$$\operatorname{adj} A=\begin{bmatrix} -42 & 3 & 7 \\ -24 & 30 & -18 \\ 30 & -21 & -5 \end{bmatrix}' \Rightarrow \operatorname{adj} A=\begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{vmatrix} \Rightarrow |A|=4\begin{vmatrix} 0 & 6 \\ 7 & 9 \end{vmatrix}-5\begin{vmatrix} 1 & 6 \\ 2 & 9 \end{vmatrix}+3\begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} \Rightarrow |A|=4(0-42)-5(9-12)+3(7-0)$$

$$\Rightarrow |A|=4(-42)-5(-3)+3(7) \Rightarrow |A|=-168+15+21 \Rightarrow |A|=-132$$

Now, L.H.S. $=A.(\operatorname{adj} A)=\begin{bmatrix} 4 & 5 & 9 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}\begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$

$$=\begin{bmatrix} -168+15+21 & -96+150-54 & 120-105-15 \\ -42+0+42 & -24+0-108 & 30-0-30 \\ -84+21+63 & -48+210-162 & 60-147-45 \end{bmatrix}=\begin{bmatrix} -132 & 0 & 0 \\ 0 & -132 & 0 \\ 0 & 0 & -132 \end{bmatrix}$$

Now, M.H.S. $=(\operatorname{adj} A)..A=\begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}\begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}$

$$=\begin{bmatrix} -168-24+60 & -210-0+21 & -126-144+270 \\ 12+30-42 & 15+0-147 & 9+180-189 \\ 28-18-10 & 35-0-35 & 21-108-45 \end{bmatrix}=\begin{bmatrix} -132 & 0 & 0 \\ 0 & -132 & 0 \\ 0 & 0 & -132 \end{bmatrix}$$

Now, R.H.S. $=|A|.I=(-132)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-132&0&0\\0&-132&0\\0&0&-132\end{bmatrix}$

Hence, $A(\text{adj}\,A)=(\text{adj}\,A).A=|A|.I$

9. $\begin{bmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}$

Sol. $\text{adj}\,A=\begin{bmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}\cos\alpha&0\\0&1\end{vmatrix}=\cos\alpha$; $C_{12}=(-1)^{1+2}\begin{vmatrix}\sin\alpha&0\\0&1\end{vmatrix}=-\sin\alpha$

$C_{13}=(-1)^{1+3}\begin{vmatrix}\sin\alpha&\cos\alpha\\0&0\end{vmatrix}=-(0-0)=0$; $C_{21}=(-1)^{2+1}\begin{vmatrix}-\sin\alpha&0\\0&1\end{vmatrix}=-(-\sin\alpha)=\sin\alpha$

$C_{22}=(-1)^{2+2}\begin{vmatrix}\cos\alpha&0\\0&1\end{vmatrix}=\cos\alpha$; $C_{23}=(-1)^{2+3}\begin{vmatrix}\cos\alpha&-\sin\alpha\\0&0\end{vmatrix}=0$

$C_{31}=(-1)^{3+1}\begin{vmatrix}-\sin\alpha&0\\\cos\alpha&0\end{vmatrix}=(0-0)=0$; $C_{32}=(-1)^{3+2}\begin{vmatrix}\cos\alpha&0\\\sin\alpha&0\end{vmatrix}=(0-0)=0$

$C_{33}=(-1)^{3+3}\begin{vmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{vmatrix}=(\cos^2\alpha+\sin^2\alpha)=1$

$\therefore\ \text{adj}\,A=\begin{bmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}'\ \Rightarrow \text{adj}\,A=\begin{bmatrix}\cos\alpha&\sin\alpha&0\\-\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}$

$|A|=\begin{vmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{vmatrix}\Rightarrow |A|=\cos\alpha\begin{vmatrix}\cos\alpha&0\\0&1\end{vmatrix}+\sin\alpha\begin{vmatrix}\sin\alpha&0\\0&1\end{vmatrix}+0\begin{vmatrix}\sin\alpha&\cos\alpha\\0&0\end{vmatrix}$

$\Rightarrow |A|=\cos^2\alpha+\sin^2\alpha+0\ \ \Rightarrow |A|=1$

Now, L.H.S. $=A(\text{adj}\,A)=\begin{bmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}\begin{bmatrix}\cos\alpha&\sin\alpha&0\\-\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}$

$=\begin{bmatrix}\cos^2\alpha+\sin^2\alpha&\sin\alpha\cos\alpha-\sin\alpha\cos\alpha&0\\\sin\alpha\cos\alpha-\sin\alpha\cos\alpha&\sin^2\alpha+\cos^2\alpha&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Now, M.H.S. $=(\text{adj}\,A).A=\begin{bmatrix}\cos\alpha&\sin\alpha&0\\-\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}\begin{bmatrix}\cos\alpha&-\sin\alpha&0\\\sin\alpha&\cos\alpha&0\\0&0&1\end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & -\sin\alpha\cos\alpha+\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha+\sin\alpha\cos\alpha & \sin^2\alpha+\cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, R.H.S. $= |A|.I = 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence, $A.(\text{adj } A) = (\text{adj } A).A = |A|.I$

10. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that adj $A = A$.

Sol. $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = (0-4) = -4$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(3-4) = 1$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = (4-0) = 4$; $C_{21} = (-1)^{2+1}\begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -(-9+12) = -3$

$C_{22} = (-1)^{2+2}\begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = (-12+12) = 0$; $C_{23} = (-1)^{2+3}\begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = -(-16+12) = 4$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = (-3-0) = -3$; $C_{32} = (-1)^{3+2}\begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = -(-4+3) = 1$

$C_{33} = (-1)^{3+3}\begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = (0+3) = 3$;

$$\text{adj } A = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}' \Rightarrow \text{adj } A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = A. \text{ Hence, adj } A = A.$$

11. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that adj $A = 3A'$.

Sol. $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = (1-4) = -3$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2+4) = -6$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = (-4-2) = -6$; $C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} = -(-2-4) = 6$

$C_{22} = (-1)^{2+2}\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = (-1+4) = 3$; $C_{23} = (-1)^{2+3}\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2+4) = -6$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}-2 & -2\\ 1 & -2\end{vmatrix}=(4+2)=6;\ C_{32}=(-1)^{3+2}\begin{vmatrix}-1 & -2\\ 2 & -2\end{vmatrix}=-(2+4)=-6$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}-1 & -2\\ 2 & 1\end{vmatrix}=(-1+4)=3$$

$$\therefore\ \text{adj}\,A=\begin{bmatrix}-3 & -6 & -6\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix}' \Rightarrow \text{adj}\,A=\begin{bmatrix}-3 & 6 & 6\\ -6 & 3 & -6\\ -6 & -6 & 3\end{bmatrix}$$

$$3A'=3\begin{bmatrix}-1 & 2 & 2\\ -2 & 1 & -2\\ -2 & -2 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}-3 & 6 & 6\\ -6 & 3 & -6\\ -6 & -6 & 3\end{bmatrix}.\quad \text{Hence, adj}\,A=3A'$$

Find the inverse of each of the matrices given below :

12. $\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}$

Sol. $|A|=\begin{vmatrix}3 & -5\\ -1 & 2\end{vmatrix} \Rightarrow |A|=(6-5) \Rightarrow |A|=1,\ |A|\neq 0$

$\therefore\ A^{-1}$ exist. $\quad$ $\text{adj}\,A=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}(2)=2;\ C_{12}=(-1)^{1+2}(-1)=1;\ C_{21}=(-1)^{2+1}(-5)=5;\ C_{22}=(-1)^{2+2}(3)=3$

$$\therefore\ \text{adj}\,A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}' \Rightarrow \text{adj}\,A=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(\text{adj}\,A)=\frac{1}{1}.\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$$

13. $\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}$

Sol. $|A|=\begin{vmatrix}4 & 1\\ 2 & 3\end{vmatrix} \Rightarrow |A|=(12-2)=10$

$\therefore\ |A|\neq 0,\ A^{-1}$ exist. $\quad$ $\text{adj}\,A=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}(3)=3;\ C_{12}=(-1)^{1+2}(2)=-2;\ C_{21}=(-1)^{2+1}(1)=-1;\ C_{22}=(-1)^{2+2}(4)=4$

$$\therefore\ \text{adj}\,A=\begin{bmatrix}3 & -2\\ -1 & 4\end{bmatrix}' \Rightarrow \text{adj}\,A=\begin{bmatrix}3 & -1\\ -2 & 4\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}.(\text{adj}\,A)=\frac{1}{10}\begin{bmatrix}3 & -1\\ -2 & 4\end{bmatrix}=\begin{bmatrix}3/10 & -1/10\\ -1/5 & 2/5\end{bmatrix}$$

14. $\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$

Sol. $|A| = \begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix} \Rightarrow |A| = (12+12) \Rightarrow |A| = 24$

$\therefore\ |A| \neq 0,\ A^{-1}$ exist. $\quad \operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(6) = 6$; $C_{12} = (-1)^{1+2}(4) = -4$; $C_{21} = (-1)^{2+1}(-3) = 3$; $C_{22} = (-1)^{2+2}(2) = 2$

$\operatorname{adj}(A) = \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}' \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$;

$\therefore\ A^{-1} = \frac{1}{|A|}(\operatorname{adj} A) = \frac{1}{24}\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/8 \\ -1/6 & 1/12 \end{bmatrix}$

15. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $(ad - bc) \neq 0$

Sol. $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow |A| = (ad - bc)$

$\therefore\ |A| \neq 0,\ A^{-1}$ exist. $\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(d) = d$; $C_{12} = (-1)^{1+2}(c) = -c$; $C_{21} = (-1)^{2+1}(b) = -b$; $C_{22} = (-1)^{2+2}(a) = a$

$\operatorname{adj}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}' \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\therefore\ A^{-1} = \frac{1}{|A|}(\operatorname{adj} A) = \frac{1}{(ad - bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

Sol. $|A| = \begin{vmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{vmatrix} \Rightarrow |A| = 1\begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$

$\Rightarrow |A| = 1(1+3) - 2(-1+2) + 5(3+2) \Rightarrow |A| = 1(4) - 2(1) + 5(5) = 4 - 2 + 25 \Rightarrow |A| = 27$

$\therefore\ |A| \neq 0,\ A^{-1}$ exist. $\quad \operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} = (1+3) = 4$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -(-1+2) = -1$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = (3+2) = 5$; $C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = -(-2-15) = 17$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}1 & 5\\ 2 & -1\end{vmatrix}=(-1-10)=-11;\ C_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}=-(3-4)=1$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}2 & 5\\ -1 & -1\end{vmatrix}=(-2+5)=3;\ C_{32}=(-1)^{3+2}\begin{vmatrix}1 & 5\\ 1 & -1\end{vmatrix}=-(-1-5)=6$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ 1 & -1\end{vmatrix}=(-1-2)=-3$$

$$\operatorname{adj} A=\begin{bmatrix}4 & -1 & 5\\ 17 & -11 & 1\\ 3 & 6 & -3\end{bmatrix}' \Rightarrow \operatorname{adj} A=\begin{bmatrix}4 & 17 & 3\\ -1 & -11 & 6\\ 5 & 1 & -3\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(\operatorname{adj} A)=\frac{1}{27}\begin{bmatrix}4 & 17 & 3\\ -1 & -11 & 6\\ 5 & 1 & 3\end{bmatrix}$$

17. $\begin{bmatrix}2 & -1 & 1\\ 3 & 0 & -1\\ 2 & 6 & 0\end{bmatrix}$

Sol. $|A|=\begin{vmatrix}2 & -1 & 1\\ 3 & 0 & -1\\ 2 & 6 & 0\end{vmatrix} \quad \Rightarrow |A|=2\begin{vmatrix}0 & -1\\ 6 & 0\end{vmatrix}-(-1)\begin{vmatrix}3 & -1\\ 2 & 0\end{vmatrix}+1\begin{vmatrix}3 & 0\\ 2 & 6\end{vmatrix}$

$\Rightarrow |A|=2(0+6)+(0+2)+(18-0) \quad \Rightarrow |A|=2(0+6)+(0+2)+(18-0)$

$\Rightarrow |A|=12+2+18 \quad \Rightarrow |A|=32$

$\therefore\ |A|\neq 0$, A^{-1} exist. $\quad \operatorname{adj} A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}0 & -1\\ 6 & 0\end{vmatrix}=(0+6)=6;\ C_{12}=(-1)^{1+2}\begin{vmatrix}3 & -1\\ 2 & 0\end{vmatrix}=-(0+2)=-2$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}3 & 0\\ 2 & 6\end{vmatrix}=(18-0)=18;\ C_{21}=(-1)^{2+1}\begin{vmatrix}-1 & 1\\ 6 & 0\end{vmatrix}=-(0-6)=6$$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}2 & 1\\ 2 & 0\end{vmatrix}=(0-2)=-2;\ C_{23}=(-1)^{2+3}\begin{vmatrix}2 & -1\\ 2 & 6\end{vmatrix}=-(12+2)=-14$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}-1 & 1\\ 0 & -1\end{vmatrix}=(1-0)=1;\ C_{32}=(-1)^{3+2}\begin{vmatrix}2 & 1\\ 3 & -1\end{vmatrix}=-(-2-3)=5$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}2 & -1\\ 3 & 0\end{vmatrix}=(0+3)=3$$

$$\operatorname{adj} A=\begin{bmatrix}6 & -2 & 18\\ 6 & -2 & -14\\ 1 & 5 & 3\end{bmatrix}' \Rightarrow \operatorname{adj} A=\begin{bmatrix}6 & 6 & 1\\ -2 & -2 & 5\\ 18 & -14 & 3\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(\operatorname{adj}A)=\frac{1}{32}\begin{bmatrix}6 & 6 & 1\\ -2 & -2 & 5\\ 18 & -14 & 3\end{bmatrix}$$

18. $\begin{bmatrix}2 & -3 & 3\\ 2 & 2 & 3\\ 3 & -2 & 2\end{bmatrix}$

Sol. $|A|=\begin{vmatrix}2 & -3 & 3\\ 2 & 2 & 3\\ 3 & -2 & 2\end{vmatrix}\quad \Rightarrow |A|=2\begin{vmatrix}2 & 3\\ -2 & 2\end{vmatrix}+3\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}+3\begin{vmatrix}2 & 2\\ 3 & -2\end{vmatrix}$

$\Rightarrow |A|=2(4+6)+3(4-9)+3(-4-6)\quad \Rightarrow |A|=20-15-30\quad \Rightarrow |A|=-25$

$\therefore\ |A|\neq 0,\ A^{-1}$ exist. $\qquad \operatorname{adj}A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}2 & 3\\ -2 & 2\end{vmatrix}=(4+6)=10$; $C_{12}=(-1)^{1+2}\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}=-(4-9)=5$

$C_{13}=(-1)^{1+3}\begin{vmatrix}2 & 2\\ 3 & -2\end{vmatrix}=(-4-6)=-10$; $C_{21}=(-1)^{2+1}\begin{vmatrix}-3 & 3\\ -2 & 2\end{vmatrix}=-(-6+6)=0$

$C_{22}=(-1)^{2+2}\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix}=(4-9)=-5$; $C_{23}=(-1)^{2+3}\begin{vmatrix}2 & -3\\ 3 & -2\end{vmatrix}=-(-4+9)=-5$

$C_{31}=(-1)^{3+1}\begin{vmatrix}-3 & 3\\ 2 & 3\end{vmatrix}=(-9-6)=-15$; $C_{32}=(-1)^{3+2}\begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix}=-(6-6)=0$

$C_{33}=(-1)^{3+3}\begin{vmatrix}2 & -3\\ 2 & 2\end{vmatrix}=(4+6)=10$

$$\operatorname{adj}A=\begin{bmatrix}10 & 5 & -10\\ 0 & -5 & -5\\ -15 & 0 & 10\end{bmatrix}'\ \Rightarrow\ \operatorname{adj}A=\begin{bmatrix}10 & 0 & -15\\ 5 & -5 & 0\\ -10 & -5 & 10\end{bmatrix}$$

Now, $A^{-1}=\frac{1}{|A|}(\operatorname{adj}A)=\frac{1}{-25}\begin{bmatrix}10 & 0 & -15\\ 5 & -5 & 0\\ -10 & -5 & 10\end{bmatrix}=\frac{1}{25}\begin{bmatrix}-10 & 0 & 15\\ -5 & 5 & 0\\ 10 & 5 & -10\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-2 & 0 & 3\\ -1 & 1 & 0\\ 2 & 1 & -2\end{bmatrix}$

19. $\begin{bmatrix}0 & 0 & -1\\ 3 & 4 & 5\\ -2 & -4 & -7\end{bmatrix}$

Sol. $|A|=\begin{vmatrix}0 & 0 & -1\\ 3 & 4 & 5\\ -2 & -4 & -7\end{vmatrix}\quad \Rightarrow |A|=0\begin{vmatrix}4 & 5\\ -4 & -7\end{vmatrix}-0\begin{vmatrix}3 & 5\\ -2 & -7\end{vmatrix}-1\begin{vmatrix}3 & 4\\ -2 & -4\end{vmatrix}$

$\Rightarrow |A|=0-1(-12+8)=4$

$\therefore\ |A| \neq 0,\ A^{-1}$ exist. $\quad \operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 5 \\ -4 & -7 \end{vmatrix} = (-28+20) = -8$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} = -(-21+10) = 11$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 4 \\ -2 & -4 \end{vmatrix} = (-12+8) = -4;\ C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & -1 \\ -4 & -7 \end{vmatrix} = -(0-4) = 4$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 0 & -1 \\ -2 & -7 \end{vmatrix} = (0-2) = -2;\ C_{23} = (-1)^{2+3}\begin{vmatrix} 0 & 0 \\ -2 & -2 \end{vmatrix} = -(0+0) = 0$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = (0+4) = 4;\ C_{32} = (-1)^{3+2}\begin{vmatrix} 0 & -1 \\ 3 & 5 \end{vmatrix} = -(0+3) = -3$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 0 & 0 \\ 3 & 4 \end{vmatrix} = (0-0) = 0$$

$$\therefore\ \operatorname{adj} A = \begin{bmatrix} -8 & 11 & -4 \\ 4 & -2 & 0 \\ 4 & -3 & 0 \end{bmatrix}' \Rightarrow \operatorname{adj} A = \begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}(\operatorname{adj} A) = \frac{1}{4}\begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

20. $\begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

Sol. $|A| = \begin{vmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} \quad \Rightarrow |A| = 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix}$

$$\Rightarrow |A| = 2(0-1) + 1(-6+1) + 4(-3+0) \quad \Rightarrow |A| = -2-5-12 \quad \Rightarrow |A| = -19$$

$\therefore\ |A| \neq 0,\ A^{-1}$ exist. $\quad \operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = (0-1) = -1$; $C_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} = -(-6+1) = 5$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix} = (-3+0) = -3;\ C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} = -(-2-4) = 6$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = (4+4) = 8;\ C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -(2-1) = -1$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} = (-1-0) = -1;\ C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = -(2+12) = -14$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}2 & -1\\ -3 & 0\end{vmatrix}=(0-3)=-3$$

$$\text{adj}\, A=\begin{bmatrix}-1 & 5 & -3\\ 6 & 8 & -1\\ -1 & -14 & -3\end{bmatrix}' \Rightarrow \text{adj}\, A=\begin{bmatrix}-1 & 6 & -1\\ 5 & 8 & -14\\ -3 & -1 & -3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(\text{adj}\, A)=\frac{1}{-19}\begin{bmatrix}-1 & 6 & -1\\ 5 & 8 & -14\\ -3 & -1 & -3\end{bmatrix}=\frac{1}{19}\begin{bmatrix}1 & -6 & 1\\ -5 & -8 & 14\\ 3 & 1 & 3\end{bmatrix}$$

21. $\begin{bmatrix}8 & -4 & 1\\ 10 & 0 & 6\\ 8 & 1 & 6\end{bmatrix}$

Sol. $|A|=\begin{vmatrix}8 & -4 & 1\\ 10 & 0 & 6\\ 8 & 1 & 6\end{vmatrix} \Rightarrow |A|=8\begin{vmatrix}0 & 6\\ 1 & 6\end{vmatrix}+4\begin{vmatrix}10 & 6\\ 8 & 6\end{vmatrix}+1\begin{vmatrix}10 & 0\\ 8 & 1\end{vmatrix}$

$\Rightarrow |A|=8(0-6)+4(60-48)+1(10-0) \quad \Rightarrow |A|=-48+48+10 \quad \Rightarrow |A|=10$

$\therefore \ |A|\neq 0,\ A^{-1}$ exist. $\qquad \text{adj}\, A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}0 & 6\\ 1 & 6\end{vmatrix}=(0-6)=-6$; $C_{12}=(-1)^{1+2}\begin{vmatrix}10 & 6\\ 8 & 6\end{vmatrix}=-(60-48)=-12$

$C_{13}=(-1)^{1+3}\begin{vmatrix}10 & 0\\ 8 & 1\end{vmatrix}=(10-0)=10$; $C_{21}=(-1)^{2+1}\begin{vmatrix}-4 & 1\\ 1 & 6\end{vmatrix}=-(-24-1)=25$

$C_{22}=(-1)^{2+2}\begin{vmatrix}8 & 1\\ 8 & 6\end{vmatrix}=(48-8)=40$; $C_{23}=(-1)^{2+3}\begin{vmatrix}8 & -4\\ 8 & 1\end{vmatrix}=-(8+32)=-40$

$C_{31}=(-1)^{3+1}\begin{vmatrix}-4 & 1\\ 0 & 6\end{vmatrix}=(-24+0)=-24$; $C_{32}=(-1)^{3+2}\begin{vmatrix}8 & 1\\ 10 & 6\end{vmatrix}=-(48-10)=-38$

$C_{33}=(-1)^{3+3}\begin{vmatrix}8 & -4\\ 10 & 0\end{vmatrix}=(0+40)=40$

$$\text{adj}\, A=\begin{bmatrix}-6 & -12 & 10\\ 25 & 40 & -40\\ -24 & -38 & 40\end{bmatrix}' \Rightarrow \text{adj}\, A=\begin{bmatrix}-6 & 25 & -24\\ -12 & 40 & -38\\ 10 & -40 & 40\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(\text{adj}\, A)=\frac{1}{10}\begin{bmatrix}-6 & 25 & -24\\ -12 & 40 & -38\\ 10 & -40 & 40\end{bmatrix}$$

22. If $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$, show that $A^{-1}=\frac{1}{19}A$.

Sol. $\text{adj}\, A=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}(-2)=-2$; $C_{12}=(-1)^{1+2}(5)=-5$;

$C_{21}=(-1)^{2+1}(3)=-3$; $C_{22}=(-1)^{2+2}(2)=2$

$\therefore\ \text{adj}\, A=\begin{bmatrix}-2 & -5\\ -3 & 2\end{bmatrix}' \Rightarrow \text{adj}\, A=\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}$

$\therefore\ A^{-1}=\frac{1}{|A|}(\text{adj}\, A)=\frac{1}{-19}\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}=\frac{1}{19}\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$. Hence, $A^{-1}=\frac{1}{19}.A$

23. If $A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$, show that $A^{-1}=A^2$.

Sol. $\text{adj}\, A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}-1 & 0\\ 0 & 0\end{vmatrix}=(0-0)=0$; $C_{12}=(-1)^{1+2}\begin{vmatrix}2 & 0\\ 1 & 0\end{vmatrix}=-(0-0)=0$

$C_{13}=(-1)^{1+3}\begin{vmatrix}2 & -1\\ 1 & 0\end{vmatrix}=(0+1)=1$; $\quad C_{21}=(-1)^{2+1}\begin{vmatrix}-1 & 1\\ 0 & 0\end{vmatrix}=-(0-0)=0$

$C_{22}=(-1)^{2+2}\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}=(0-1)=-1$; $\quad C_{23}=(-1)^{2+3}\begin{vmatrix}1 & -1\\ 1 & 0\end{vmatrix}=-(1-0)=-1$

$C_{31}=(-1)^{3+1}\begin{vmatrix}-1 & 1\\ -1 & 0\end{vmatrix}=(0+1)=1$; $\quad C_{32}=(-1)^{3+2}\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix}=-(0-2)=2$

$C_{33}=(-1)^{3+3}\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix}=(-1+2)=1$

$\therefore\ \text{adj}\, A=\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & -1\\ 1 & 2 & 1\end{bmatrix}' \Rightarrow \text{adj}\, A=\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}$. Also, $A^{-1}=\frac{1}{|A|}\text{adj}\, A=\begin{vmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{vmatrix}$

$|A|=\begin{vmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{vmatrix} \Rightarrow |A|=1\begin{vmatrix}-1 & 0\\ 0 & 0\end{vmatrix}+1\begin{vmatrix}2 & 0\\ 1 & 0\end{vmatrix}+1\begin{vmatrix}2 & -1\\ 1 & 0\end{vmatrix}$

$\Rightarrow |A|=1(0-0)+1(-0-0)+1(0+1) \quad \Rightarrow |A|=0+0+1 \quad \Rightarrow |A|=1$

$A^2=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix}1-2+1 & -1+1+0 & 1-0+0\\ 2-2+0 & -2+1+0 & 2-0+0\\ 1+0+0 & -1-0+0 & 1+0+0\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 2\\ 1 & -1 & 1\end{bmatrix}=A^{-1}$.

Hence, $A^{-1}=A^2$.

24. If $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$, prove that $A^{-1}=A^3$.

Sol. $A^3=A^2.A \Rightarrow A^3=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}.A$

$$\Rightarrow A^3=\begin{bmatrix}9-6+0 & -9+9-4 & 12-12+4\\ 6-6+0 & -6+9-4 & 8-12+4\\ 0-2+0 & 0+3-1 & 0-4+1\end{bmatrix}.A \Rightarrow A^3=\begin{bmatrix}3 & -4 & 4\\ 0 & -1 & 0\\ -2 & 2 & -3\end{bmatrix}\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$$

$$\Rightarrow A^3=\begin{bmatrix}9-8+0 & -9+12-4 & 12-16+4\\ 0-2+0 & 0+3-0 & 0-4+0\\ -6+4-0 & 6-6+3 & -8+8-3\end{bmatrix} \Rightarrow A^3=\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{bmatrix}$$

$$A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix} \quad \text{adj}\,A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}'$$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}-3 & 4\\ -1 & 1\end{vmatrix}=(-3+4)=1$; $C_{12}=(-1)^{1+2}\begin{vmatrix}2 & 4\\ 0 & 1\end{vmatrix}=-(2-0)=-2$

$C_{13}=(-1)^{1+3}\begin{vmatrix}2 & -3\\ 0 & -1\end{vmatrix}=(-2+0)=-2$; $C_{21}=(-1)^{2+1}\begin{vmatrix}-3 & 4\\ -1 & 1\end{vmatrix}=-(-3+4)=-1$

$C_{22}=(-1)^{2+2}\begin{vmatrix}3 & 4\\ 0 & 1\end{vmatrix}=(3-0)=3$; $C_{23}=(-1)^{2+3}\begin{vmatrix}3 & -3\\ 0 & -1\end{vmatrix}=-(-3+0)=3$

$C_{31}=(-1)^{3+1}\begin{vmatrix}-3 & 4\\ -3 & 4\end{vmatrix}=(-12+12)=0$; $C_{32}=(-1)^{3+2}\begin{vmatrix}3 & 4\\ 2 & 4\end{vmatrix}=-(12-8)=-4$

$C_{33}=(-1)^{3+3}\begin{vmatrix}3 & -3\\ 2 & -3\end{vmatrix}=(-9+6)=-3$

$$\therefore \text{adj}\,A=\begin{bmatrix}1 & -2 & -2\\ -1 & 3 & 3\\ 0 & -4 & -3\end{bmatrix}' \Rightarrow \text{adj}\,A=\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{bmatrix}$$

$$|A|=\begin{vmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{vmatrix} \Rightarrow |A|=3\begin{vmatrix}-3 & 4\\ -1 & 1\end{vmatrix}+3\begin{vmatrix}2 & 4\\ 0 & 1\end{vmatrix}+4\begin{vmatrix}2 & -3\\ 0 & -1\end{vmatrix}$$

$$\Rightarrow |A|=3(-3+4)+3(2-0)+4(-2+0) \Rightarrow |A|=3+6-8 \Rightarrow |A|=1$$

$$\therefore A^{-1}=\frac{1}{|A|}(\text{adj}\,A) \Rightarrow A^{-1}=\frac{1}{1}\begin{bmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{bmatrix}=A^3.$$ Hence, $A^{-1}=A^3$.

25. If $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, show that $A^{-1}=A'$.

Sol. $A=\begin{bmatrix}-\frac{8}{9} & \frac{1}{9} & \frac{4}{9}\\ \frac{4}{9} & \frac{4}{9} & \frac{7}{9}\\ \frac{1}{9} & \frac{-8}{9} & \frac{4}{9}\end{bmatrix}$. Now, $\operatorname{adj} A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}\frac{4}{9} & \frac{7}{9}\\ \frac{-8}{9} & \frac{4}{9}\end{vmatrix}=\left(\frac{16}{81}+\frac{56}{81}\right)=\frac{8}{9}$; $C_{12}=(-1)^{1+2}\begin{vmatrix}\frac{4}{9} & \frac{7}{9}\\ \frac{1}{9} & \frac{4}{9}\end{vmatrix}=-\left(\frac{16}{81}-\frac{7}{81}\right)=\frac{-1}{9}$

$C_{13}=(-1)^{1+3}\begin{vmatrix}\frac{4}{9} & \frac{4}{9}\\ \frac{1}{9} & \frac{-8}{9}\end{vmatrix}=\left(\frac{-32}{81}-\frac{4}{81}\right)=\frac{-4}{9}$; $C_{21}=(-1)^{2+1}\begin{vmatrix}\frac{1}{9} & \frac{4}{9}\\ \frac{-8}{9} & \frac{4}{9}\end{vmatrix}=-\left(\frac{4}{81}+\frac{32}{81}\right)=-\frac{4}{9}$

$C_{22}=(-1)^{2+2}\begin{vmatrix}\frac{-8}{9} & \frac{4}{9}\\ \frac{1}{9} & \frac{4}{9}\end{vmatrix}=\left(\frac{-32}{81}-\frac{4}{9}\right)=\frac{-4}{9}$; $C_{23}=(-1)^{2+3}\begin{vmatrix}\frac{-8}{9} & \frac{1}{9}\\ \frac{1}{9} & \frac{-8}{9}\end{vmatrix}=-\left(\frac{64}{81}-\frac{1}{81}\right)=\frac{-7}{9}$

$C_{31}=(-1)^{3+1}\begin{vmatrix}\frac{1}{9} & \frac{4}{9}\\ \frac{4}{9} & \frac{7}{9}\end{vmatrix}=\left(\frac{7}{81}-\frac{16}{81}\right)=\frac{-1}{9}$; $C_{32}=(-1)^{3+2}\begin{vmatrix}\frac{-8}{9} & \frac{4}{9}\\ \frac{4}{9} & \frac{7}{9}\end{vmatrix}=-\left(\frac{-56}{81}-\frac{16}{81}\right)=\frac{8}{9}$

$C_{33}=(-1)^{3+3}\begin{vmatrix}\frac{-8}{9} & \frac{1}{9}\\ \frac{4}{9} & \frac{4}{9}\end{vmatrix}=\left(\frac{-32}{81}-\frac{4}{81}\right)=\frac{-4}{9}$

$$\operatorname{adj} A=\begin{bmatrix}\frac{8}{9} & \frac{-1}{9} & \frac{-4}{9}\\ \frac{-4}{9} & \frac{-4}{9} & \frac{-7}{9}\\ \frac{-1}{9} & \frac{8}{9} & \frac{-4}{9}\end{bmatrix}' \Rightarrow \operatorname{adj} A=\begin{bmatrix}\frac{8}{9} & \frac{-4}{9} & \frac{-1}{9}\\ \frac{-1}{9} & \frac{-4}{9} & \frac{8}{9}\\ \frac{-4}{9} & \frac{-7}{9} & \frac{-4}{9}\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\operatorname{adj} A=\frac{-1}{81}\begin{bmatrix}\frac{8}{9} & \frac{-4}{9} & \frac{-1}{9}\\ \frac{-1}{9} & \frac{-4}{9} & \frac{8}{9}\\ \frac{-4}{9} & \frac{-7}{9} & \frac{-4}{9}\end{bmatrix}=\frac{1}{-9}\begin{bmatrix}8 & -4 & -1\\ -1 & -4 & 8\\ -4 & -7 & -4\end{bmatrix}$$

$$\Rightarrow A^{-1}=\frac{1}{9}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix}=A' \text{ proved}$$

26. Let $D=\operatorname{diag}[d_1,\ d_2\ \ d_3]$, where none of d_1, d_2, d_3 is 0; prove that $D^{-1}=diag\left[d_1^{-1}, d_2^{-1}, d_3^{-1}\right]$.

Sol. $\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} d_2 & 0 \\ 0 & d_3 \end{vmatrix} = d_2 d_3$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ 0 & d_3 \end{vmatrix} = 0$; $C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & d_2 \\ 0 & 0 \end{vmatrix} = 0$;

$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ 0 & d_3 \end{vmatrix} = 0$; $C_{22} = (-1)^{2+2}\begin{vmatrix} d_1 & 0 \\ 0 & d_3 \end{vmatrix} = d_1 d_3$; $C_{23} = (-1)^{2+3}\begin{vmatrix} d_1 & 0 \\ 0 & 0 \end{vmatrix} = 0$;

$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ d_2 & 0 \end{vmatrix} = 0$; $C_{32} = (-1)^{3+2}\begin{vmatrix} d_1 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $C_{33} = (-1)^{3+3}\begin{vmatrix} d_1 & 0 \\ 0 & d_2 \end{vmatrix} = d_1 d_2$

$\therefore\ \operatorname{adj} D = \begin{bmatrix} d_2 d_3 & 0 & 0 \\ 0 & d_2 d_3 & 0 \\ 0 & 0 & d_1 d_2 \end{bmatrix}' \Rightarrow \operatorname{adj} D = \begin{bmatrix} d_2 d_3 & 0 & 0 \\ 0 & d_1 d_3 & 0 \\ 0 & 0 & d_1 d_2 \end{bmatrix}$

$|D| = \begin{vmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_1 d_2 \end{vmatrix} \Rightarrow |D| = d_1\begin{vmatrix} d_2 & 0 \\ 0 & d_3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & d_3 \end{vmatrix} + 0\begin{vmatrix} 0 & d_2 \\ 0 & 0 \end{vmatrix}$

$\Rightarrow |A| = d_1(d_2 d_3) - 0 + 0 \Rightarrow |A| = (d_1 d_2 d_3)$

Now, $D^{-1} = \frac{1}{|D|}(\operatorname{adj} D) = \frac{1}{(d_1 d_2 d_3)}\begin{bmatrix} d_2 d_3 & 0 & 0 \\ 0 & d_1 d_3 & 0 \\ 0 & 0 & d_1 d_2 \end{bmatrix} = \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \end{bmatrix}$

$\therefore\ D^{-1} = diag\begin{bmatrix} d_1^{-1} & d_2^{-1} & d_3^{-1} \end{bmatrix}$

27. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. $AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 18+16 & 21+18 \\ 42+40 & 49+45 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 34 & 39 \\ 82 & 94 \end{bmatrix}$

$\operatorname{adj} AB = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(94) = 94$; $C_{12} = (-1)^{1+2}(82) = -82$;

$C_{21} = (-1)^{2+1}(39) = -39$, $C_{22} = (-1)^{2+2}(34) = 34$

$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = (15-14) = 1$; $|B| = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix} = (54-56) = -2$

$\therefore\ |AB| = \begin{vmatrix} 34 & 39 \\ 82 & 94 \end{vmatrix} = (3196-3198) = -2$

$\therefore\ (AB)^{-1} = \frac{1}{|AB|}(\operatorname{adj} AB) = \frac{1}{-2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -94 & 39 \\ 82 & -34 \end{bmatrix}$

R.H.S. $= B^{-1}A^{-1} \Rightarrow B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}(9)=9$; $C_{12}=(-1)^{1+2}(8)=-8$; $C_{21}=(-1)^{2+1}(7)=-7$, $C_{22}=(-1)^{2+2}(6)=6$

$\therefore\ \operatorname{adj} B=\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}' \Rightarrow \operatorname{adj} B=\begin{bmatrix}9 & -7\\ -8 & 6\end{bmatrix}$. Also, $A=\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix} \Rightarrow \operatorname{adj} A=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}(5)=5$; $C_{12}=(-1)^{1+2}(7)=-7$; $C_{21}=(-1)^{2+1}(2)=-2$, $C_{22}=(-1)^{2+2}(3)=3$

$\therefore\ \operatorname{adj} A=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}' \Rightarrow \operatorname{adj} A=\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}$

$|A|=\begin{vmatrix}3 & 2\\ 7 & 5\end{vmatrix}=(15-14)=1;\quad |B|=\begin{vmatrix}6 & 7\\ 8 & 9\end{vmatrix}=(54-56)=-2$

Now, $A^{-1}=\frac{1}{|A|}(\operatorname{adj} A)=\frac{1}{1}\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}$ and $B^{-1}=\frac{1}{|B|}(\operatorname{adj} B)=\frac{1}{(-2)}\begin{bmatrix}9 & -7\\ -8 & 6\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-9 & 7\\ 8 & -6\end{bmatrix}$

Now, $B^{-1}A^{-1}=\frac{1}{2}\begin{bmatrix}-9 & 7\\ 8 & -6\end{bmatrix}\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-45-49 & 18+21\\ 40+42 & -16-18\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-94 & 39\\ 82 & -34\end{bmatrix}$

Hence, $(AB)^{-1}=B^{-1}A^{-1}$.

28. If $A=\begin{bmatrix}9 & -1\\ 6 & -2\end{bmatrix}$ and $B=\begin{bmatrix}-4 & 3\\ 5 & -4\end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

Sol. L.H.S. $=(AB)^{-1} \Rightarrow AB=\begin{bmatrix}9 & -1\\ 6 & -2\end{bmatrix}\begin{bmatrix}-4 & 3\\ 5 & -4\end{bmatrix}=\begin{bmatrix}-36-5 & 27+4\\ -24-10 & 18+8\end{bmatrix}=\begin{bmatrix}-41 & 31\\ -34 & 26\end{bmatrix}$

$\operatorname{adj} AB=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}(26)=26$; $C_{12}=(-1)^{1+2}(-34)=34$

$C_{21}=(-1)^{2+1}(31)=-31$, $C_{22}=(-1)^{2+2}(-41)=-41$

$\therefore\ \operatorname{adj} AB=\begin{bmatrix}26 & 34\\ -31 & -41\end{bmatrix}' \Rightarrow \operatorname{adj} AB=\begin{bmatrix}26 & -31\\ 34 & -41\end{bmatrix}$

$|AB|=\begin{vmatrix}-41 & 31\\ -34 & 26\end{vmatrix}=(1066-1054)=12$. $\quad\therefore\ (AB)^{-1}=\frac{1}{|AB|}(\operatorname{adj} AB)=\frac{1}{12}\begin{bmatrix}26 & -31\\ 34 & -41\end{bmatrix}$

R.H.S. $=B^{-1}A^{-1} \Rightarrow B=\begin{bmatrix}-4 & 3\\ 5 & -4\end{bmatrix} \Rightarrow \operatorname{adj} B=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}(-4)=-4$; $C_{12}=(-1)^{1+2}(5)=-5$

$C_{21}=(-1)^{2+1}(3)=-3$, $C_{22}=(-1)^{2+2}(-4)=-4$

$\operatorname{adj} B=\begin{bmatrix}-4 & -5\\ -3 & -4\end{bmatrix}' \Rightarrow \operatorname{adj} B=\begin{bmatrix}-4 & -3\\ -5 & -4\end{bmatrix} \Rightarrow |B|=\begin{vmatrix}-4 & 3\\ 5 & -4\end{vmatrix}=(16-15)=1$

$\therefore B^{-1}=\frac{1}{|B|}(\operatorname{adj} B)=\frac{1}{1}\begin{bmatrix}-4 & -3\\ -5 & -4\end{bmatrix}$

$$A=\begin{bmatrix}9 & -1\\ 6 & -2\end{bmatrix} \Rightarrow \text{adj}\, A=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}'$$

Cofactor of $C_{11}=(-1)^{1+1}(-1)=-2$; $C_{12}=(-1)^{1+2}(6)=-6$; $C_{21}=(-1)^{2+1}(-1)=1$, $C_{22}=(-1)^{2+2}(9)=9$

$$|A|=\begin{vmatrix}9 & -1\\ 6 & -2\end{vmatrix}=(-18+6)=-12 \quad \therefore\ A^{-1}=\frac{1}{|A|}(\text{adj}\, A)=\frac{1}{12}\begin{bmatrix}-2 & 1\\ -6 & 9\end{bmatrix}$$

Now, $B^{-1}A^{-1}=\frac{1}{12}\begin{bmatrix}-4 & -3\\ -5 & -4\end{bmatrix}\begin{bmatrix}-2 & 1\\ -6 & 9\end{bmatrix}=\frac{1}{12}\begin{bmatrix}8+18 & -4-27\\ 10+24 & -5-36\end{bmatrix}=\frac{1}{12}\begin{bmatrix}26 & -31\\ 34 & -41\end{bmatrix}$

Hence, $(AB)^{-1}=B^{-1}.A^{-1}$

29. Compute $(AB)^{-1}$ when $A=\begin{bmatrix}1 & 1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$ and $B^{-1}=\begin{bmatrix}1 & 2 & 0\\ 0 & 3 & -1\\ 1 & 0 & 2\end{bmatrix}$

Sol. $\text{adj}\, A=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}'$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}2 & -3\\ -2 & 4\end{vmatrix}=(8-6)=2$; $C_{12}=(-1)^{1+2}\begin{vmatrix}0 & -3\\ 3 & 4\end{vmatrix}=-(0+9)=-9$

$C_{13}=(-1)^{1+3}\begin{vmatrix}0 & 2\\ 3 & -2\end{vmatrix}=(0-6)=-6$; $C_{21}=(-1)^{2+1}\begin{vmatrix}1 & 2\\ -2 & 4\end{vmatrix}=-(4+4)=-8$

$C_{22}=(-1)^{2+2}\begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix}=(4-6)=-2$; $C_{23}=(-1)^{2+3}\begin{vmatrix}1 & 1\\ 3 & -2\end{vmatrix}=-(-2-3)=5$

$C_{31}=(-1)^{3+1}\begin{vmatrix}1 & 2\\ 2 & -3\end{vmatrix}=(-3-4)=-7$; $C_{32}=(-1)^{3+2}\begin{vmatrix}1 & 2\\ 0 & -3\end{vmatrix}=-(-3-0)=3$

$C_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}=(2-0)=2$.

$$\therefore\ \text{adj}\, A=\begin{bmatrix}2 & -9 & -6\\ -8 & -2 & 5\\ -7 & 3 & 2\end{bmatrix}' \Rightarrow \text{adj}\, A=\begin{bmatrix}2 & -8 & -7\\ -9 & -2 & 3\\ -6 & 5 & 2\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{vmatrix} \Rightarrow |A|=1\begin{vmatrix}2 & -3\\ -2 & 4\end{vmatrix}-1\begin{vmatrix}0 & -3\\ 3 & 4\end{vmatrix}+2\begin{vmatrix}0 & 2\\ 3 & -2\end{vmatrix}$$

$$=1(8-6)-1(0+9)+2(0-6)=2-9-12=(-19)$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(\text{adj}\, A)=\frac{1}{-19}\begin{bmatrix}2 & -8 & -7\\ -9 & -2 & 3\\ -6 & 5 & 2\end{bmatrix}=\frac{1}{19}\begin{bmatrix}-2 & 8 & 7\\ 9 & 2 & -3\\ 6 & -5 & -2\end{bmatrix}$$

$$\Rightarrow (AB)^{-1}=B^{-1}.A^{-1}=\frac{1}{19}\begin{bmatrix}1 & 2 & 0\\ 0 & 3 & -1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}-2 & 0 & 7\\ 9 & 2 & -3\\ 6 & -5 & -2\end{bmatrix}$$

$$= \frac{1}{19}\begin{bmatrix} -2+18+0 & 0+4-0 & 7-6+0 \\ 0+27-6 & 0+6+5 & 0-9+2 \\ -2+0+12 & 0+0-10 & 7-0-4 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 16 & 4 & 1 \\ 21 & 11 & -7 \\ 10 & -10 & 3 \end{bmatrix}$$

30. Obtain the inverses of the matrices $\begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$ And, hence find the inverse of the matrix $\begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$ $R = \begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} = (1-0) = 1$; $C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & p \\ 0 & 1 \end{vmatrix} = 0$; $C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$;

$C_{21} = (-1)^{2+1}\begin{vmatrix} p & 0 \\ 0 & 1 \end{vmatrix} = -p$; $C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & p \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^{3+1}\begin{vmatrix} p & 0 \\ 1 & p \end{vmatrix} = p^2$; $C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 0 & p \end{vmatrix} = -p$; $C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} = 1$

$$\therefore \quad \operatorname{adj} A = \begin{bmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ p^2 & -p & 1 \end{bmatrix}' \Rightarrow \operatorname{adj} A = \begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix} \Rightarrow \operatorname{adj} B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}'$$

Cofactor of $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 0 \\ q & 1 \end{vmatrix} = (1-0) = 1$; $C_{12} = (-1)^{1+2}\begin{vmatrix} q & 0 \\ 0 & 1 \end{vmatrix} = -q$; $C_{13} = (-1)^{1+3}\begin{vmatrix} q & 1 \\ 0 & q \end{vmatrix} = q^2$;

$C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 0 \\ q & 1 \end{vmatrix} = 0$; $C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 0 & q \end{vmatrix} = -q$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$; $C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 0 \\ q & 0 \end{vmatrix} = 0$; $C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ q & 1 \end{vmatrix} = 1$

$$\therefore \quad \operatorname{adj} B = \begin{bmatrix} 1 & -q & q^2 \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{bmatrix}' \Rightarrow \operatorname{adj} B = \begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 0 \end{bmatrix}$$

$$|A|=\begin{vmatrix}1&p&0\\0&1&p\\0&0&1\end{vmatrix}\Rightarrow |A|=1\begin{vmatrix}1&p\\0&1\end{vmatrix}-p\begin{vmatrix}0&p\\0&1\end{vmatrix}+0\begin{vmatrix}0&1\\0&0\end{vmatrix}=1(1-0)-p(0-0)+0(0-0)=1$$

$$|B|=\begin{vmatrix}1&0&0\\q&1&0\\0&q&1\end{vmatrix}\Rightarrow |B|=1\begin{vmatrix}1&0\\q&1\end{vmatrix}-0\begin{vmatrix}q&0\\0&1\end{vmatrix}+0\begin{vmatrix}q&1\\0&q\end{vmatrix}=1(1-0)-0(q-0)-0(q-0)=1$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(\operatorname{adj}A)=\frac{1}{1}\begin{bmatrix}1&-p&p^2\\0&1&-p\\0&0&1\end{bmatrix}\text{ and } B^{-1}=\frac{1}{|B|}(\operatorname{adj}B)=\frac{1}{1}\begin{bmatrix}1&0&0\\-q&1&0\\q^2&-q&1\end{bmatrix}$$

$$R=\begin{bmatrix}1+pq&p&0\\q&1+pq&p\\0&q&1\end{bmatrix}\Rightarrow |R|=(1+pq)\begin{vmatrix}1+pq&p\\q&1\end{vmatrix}-p\begin{vmatrix}q&p\\0&1\end{vmatrix}+0\begin{vmatrix}q&1+pq\\0&q\end{vmatrix}$$

$$\Rightarrow |R|=(1+pq)(1+pq-pq)-p(q-0)+0\quad\Rightarrow |R|=(1+pq)(1)-pq$$

$$\Rightarrow |R|=1+pq-pq\quad\Rightarrow |R|=1$$

$$\operatorname{adj}R=\begin{bmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{bmatrix}'$$

Cofactor of $C_{11}=(-1)^{1+1}\begin{vmatrix}1+pq&p\\q&1\end{vmatrix}=(1+pq-pq)=1;\ C_{12}=(-1)^{1+2}\begin{vmatrix}q&0\\0&1\end{vmatrix}=-(q-0)=-q$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}q&1+pq\\0&q\end{vmatrix}=(q^2-0)=q^2;\ C_{21}=(-1)^{2+1}\begin{vmatrix}p&0\\q&1\end{vmatrix}=-(p-0)=-p$$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}1+pq&0\\0&1\end{vmatrix}=(1+pq-0)=1+pq;\ C_{23}=(-1)^{2+3}\begin{vmatrix}1+pq&p\\0&q\end{vmatrix}=-(q+p^2q)$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}p&0\\1+pq&p\end{vmatrix}=(p^2-0)=p^2;$$

$$C_{32}=(-1)^{3+2}\begin{vmatrix}1+pq&0\\q&p\end{vmatrix}=-(p+p^2q-0)=-(p+p^2q)$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}1+pq&p\\q&1+pq\end{vmatrix}=(1+2pq+p^2q^2-pq)=(1+pq+p^2q^2)$$

$$\operatorname{adj}R=\begin{bmatrix}1&-q&q^2\\-p&1+pq&-(q+p^2q)\\p^2&-(p+q^2p)&(1+pq+p^2q^2)\end{bmatrix}'\Rightarrow \operatorname{adj}R=\begin{bmatrix}1&-p&p^2\\-q&1+pq&-(p+q^2p)\\q^2&-(q+p^2q)&(1+pq+p^2q^2)\end{bmatrix}$$

$$\therefore R^{-1}=\begin{bmatrix}1&-p&p^2\\-q&(1+pq)&-(p+q^2p)\\q^2&-(q+p^2q)&(1+pq+p^2q^2)\end{bmatrix}$$

31. If $A=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$ verify that $A^2-4A-I=0$ and hence find A^{-1}

Sol. $A^2-4A-I=0$, multiplying both side A^{-1}

$\Rightarrow A^{-1}\left(A^2-4A-I\right)=0.A^{-1} \Rightarrow A^{-1}A^2-4A.A^{-1}-I.A^{-1}=0$

$\Rightarrow A^{-1}.A.A-4A.A^{-1}-I.A^{-1}=0 \Rightarrow AI-4I-IA^{-1}=0$

$\Rightarrow A-4I-IA^{-1}=0 \Rightarrow IA^{-1}=A-4I \Rightarrow IA^{-1}=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}-4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow A^{-1}=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}-\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} \Rightarrow A^{-1}=\begin{bmatrix}3-4 & 2-0\\ 2-0 & 1-4\end{bmatrix} \Rightarrow A^{-1}=\begin{bmatrix}-1 & 2\\ 2 & -3\end{bmatrix}$

Hence, $A^{-1}=\begin{bmatrix}-1 & 2\\ 2 & -3\end{bmatrix}$.

32. Show that the matrix $A=\begin{bmatrix}-8 & 5\\ 2 & 4\end{bmatrix}$ satisfies the equation $x^2+4x-42=0$ and hence find A^{-1}

Sol. $A^2+4A-42=0$, multiplying both side by A^{-1}.

$\Rightarrow A^{-1}\left(A^2+4A-42\right)=0.A^{-1} \Rightarrow A^{-1}A^2+4AA^{-1}-42A^{-1}=0$

$\Rightarrow IA+4I-42A^{-1}=0 \Rightarrow A+4I-42A^{-1}=0 \Rightarrow 42A^{-1}=A+4I$

$\Rightarrow 42A^{-1}=\begin{bmatrix}-8 & 5\\ 2 & 4\end{bmatrix}+4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow 42A^{-1}=\begin{bmatrix}-8 & 5\\ 2 & 4\end{bmatrix}+\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}$

$\Rightarrow 42A^{-1}=\begin{bmatrix}-4 & 5\\ 2 & 8\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{42}\begin{bmatrix}-4 & 5\\ 2 & 8\end{bmatrix}$. Hence, $A^{-1}=\frac{1}{42}\begin{bmatrix}-4 & 5\\ 2 & 8\end{bmatrix}$.

33. If $A=\begin{bmatrix}-1 & -1\\ 2 & -2\end{bmatrix}$ show that $A^2+3A+4I_2=0$ and hence find A^{-1}

Sol. $A^2+3A+4I_2=0$, Multiplying both side by A^{-1}.

$\Rightarrow A^{-1}\left(A^2+3A+4I_2\right)=0.A^{-1} \Rightarrow A^{-1}A^2+3AA^{-1}+4I_2A^{-1}=0$

$\Rightarrow A^{-1}A.A+3A.A^{-1}+4I_2A^{-1}=0 \Rightarrow IA+3I+4I_2A^{-1}=0 \Rightarrow 4A^{-1}=-A-3I$

$\Rightarrow 4A^{-1}=-\begin{bmatrix}-1 & -1\\ 2 & -2\end{bmatrix}-3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow 4A^{-1}=\begin{bmatrix}1 & 1\\ -2 & 2\end{bmatrix}-\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

$\Rightarrow 4A^{-1}=\begin{bmatrix}-2 & 1\\ -2 & -1\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{4}\begin{bmatrix}-2 & 1\\ -2 & -1\end{bmatrix} \therefore A^{-1}=\begin{bmatrix}-1/2 & 1/4\\ -1/2 & -1/4\end{bmatrix}$

34. If $A=\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$ find x and y such that $A^2+xI=yA$ hence find A^{-1}

Sol. $A=\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}9+7 & 3+5\\ 21+35 & 7+25\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix}$

$\therefore A^2+xI=yA \Rightarrow \begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix}+x\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=y\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$

$\Rightarrow \begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix}+\begin{bmatrix}x & 0\\ 0 & x\end{bmatrix}=\begin{bmatrix}3y & y\\ 7y & 5y\end{bmatrix}$

$\Rightarrow 16+x=3y \Rightarrow x-3y=-16 \quad ...(1) \qquad \Rightarrow y=8 \quad ...(2)$

Putting the value of y in equation (1), then, $x-3y=-16 \Rightarrow x-3(8)=-16$

$\Rightarrow x=-16+24 \Rightarrow x=8$. Hence, $x=8$ and $y=8$.

Now, $A^2+8I=8A \Rightarrow A^2+8I-8A=0$

Multiplying both side by A^{-1}, we get, $A^{-1}\left(A^2+8I-8A\right)=0.A^{-1} \Rightarrow A^{-1}A^2+8IA^{-1}-8AA^{-1}=0$

$\Rightarrow A^{-1}A.A+8IA^{-1}-8I=0 \Rightarrow 1A+8IA^{-1}-8I=0 \Rightarrow A+8A^{-1}-8I=0$

$\Rightarrow 8A^{-1}=8I-A \Rightarrow 8A^{-1}=8\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix} \Rightarrow 8A^{-1}=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}-\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$

$\Rightarrow 8A^{-1}=\begin{bmatrix}5 & -1\\ -7 & 3\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{8}\begin{bmatrix}5 & -1\\ -7 & 3\end{bmatrix}$

35. If $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}$, find the value of λ so that $A^2=\lambda A-2I$. Hence, find A^{-1}

Sol. $A=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}\begin{bmatrix}3 & 2\\ 4 & -2\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}9-8 & -6+4\\ 12-8 & -8+4\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}1 & -2\\ 4 & -2\end{bmatrix}$

$\therefore A^2=\lambda A-2I \Rightarrow \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}=\lambda\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}=\begin{bmatrix}3\lambda & -2\lambda\\ 4\lambda & -2\lambda\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

$\Rightarrow 3\lambda-2=1 \Rightarrow 3\lambda=1+2 \Rightarrow 3\lambda=3 \therefore \lambda=1$

$\Rightarrow A^2=A-2I \Rightarrow A^2-A+2I=0$

Multiplying both side by A^{-1}, we get, $A^{-1}\left(A^2-A+2I\right)=0.A^{-1} \Rightarrow A^{-1}A^2-A.A^{-1}+2I\,A^{-1}=0$

$\Rightarrow A^{-1}.A.A-A.A^{-1}+2A^{-1}=0 \Rightarrow IA-I+2A^{-1}=0 \Rightarrow 2A^{-1}=I-A$

$\Rightarrow 2A^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}3 & -2\\ 4 & -2\end{bmatrix} \Rightarrow 2A^{-1}=\begin{bmatrix}-2 & 2\\ -4 & 3\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{2}\begin{bmatrix}-2 & 2\\ -4 & 3\end{bmatrix}$

36. Show that the matrix $A=\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix}$ satisfies the equation $A^3-A^2-3A-I=0$, and hence find A^{-1}.

Sol. $A=\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}1-0-6 & 0-0-8 & -2+0-2\\ -2+2+6 & 0+1+8 & 4-2+2\\ 3-8+3 & 0-4+4 & -6+8+1\end{bmatrix}=\begin{bmatrix}-5 & -8 & -4\\ 6 & 9 & 4\\ -2 & 0 & 3\end{bmatrix}$

$\Rightarrow A^3=\begin{bmatrix}-5 & -8 & -4\\ 6 & 9 & 4\\ -2 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{bmatrix} \Rightarrow A^3=\begin{bmatrix}-5+16-12 & 0+8-16 & 10-16-4\\ 6-18+12 & 0-9+16 & -12+18+4\\ -2-0+9 & 0-0+12 & 4+0+3\end{bmatrix}$

$\Rightarrow A^3=\begin{bmatrix}-1 & -8 & -20\\ 0 & 7 & 2\\ 7 & 12 & 7\end{bmatrix}$. $\therefore A^3-A^2-3A-I=0$, multiplying both side by A^{-1}.

We get, $A^{-1}\left(A^3-A^2-3A-I\right)=0.A^{-1} \Rightarrow A^{-1}.A.A.A-A^{-1}A.A-3A.A^{-1}-IA^{-1}=0$

$\Rightarrow IAA - IA - 3I - A^{-1} = 0 \Rightarrow A^2 - A - 3I - A^{-1} = 0$

$\Rightarrow A^{-1} = A^2 - A - 3I \quad \Rightarrow A^{-1} = \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} -6 & -8 & -2 \\ 8 & 10 & 2 \\ -5 & -4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \Rightarrow A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

37. Prove that : (i) adj $I = I$ (ii) adj $O = O$ (iii) $I^{-1} = I$.

Sol. **(i)** R.H.S. Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, adj $I = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(1) = 1$; $C_{12} = (-1)^{1+2}(0) = 0$; $C_{21} = (-1)^{2+1}(0) = 0$; $C_{22}(-1)^{2+2}(1) = 1$

adj $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}' \Rightarrow$ adj $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \therefore$ adj $I = I$

(ii) Let the zero matrix $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, adj $0 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(0) = 0$; $C_{12} = (-1)^{1+2}(0) = 0$; $C_{21} = (-1)^{2+1}(0) = 0$; $C_{22} = (-1)^{2+2}(0) = 0$

adj $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}' \Rightarrow$ adj $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore$ adj $A = 0$

(iii) Let I be the identify matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |I| = (1-0) = 1$ adj $I = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}'$

Cofactor of $C_{11} = (-1)^{1+1}(1) = 1$; $C_{12} = (-1)^{1+2}(0) = 0$; $C_{21} = (-1)^{2+1}(0) = 0$; $C_{22} = (-1)^{2+2}(1) = 1$

adj $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}' \Rightarrow$ adj $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

$I^{-1} = \frac{1}{|I|}$ adj $I \quad \Rightarrow I^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \therefore I^{-1} = I$