CHAPTER 01

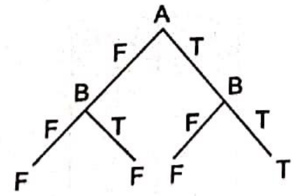
Propositional Logic

1.1 Propositional Logic; First Order Logic

Logic: In general logic is about reasoning. It is about the validity of arguments. Consistency among statements and matters of truth and falsehood. In a formal sense logic is concerned only with the form of arguments and the principle of valid inferencing. It deals with the notion of truth in an abstract sense.

Truth Tables: Logic is mainly concerned with valid deductions. The basic ingredients of logic are logical connectives, and, or, not , if.... then ...., if and only if etc. We are concerned with expressions involving these connectives. We want to know how the truth of a compound sentence like, "$x = 1$ and $y = 2$" is affected by, or determined by, the truth of the separate simple sentences "$x = 1$", "$y = 2$".

Truth tables present an exhaustive enumeration of the truth values of the component propositions of a logical expression, as a function of the truth values of the simple propositions contained in them. An example of a truth table is shown in table 1 below. The information embodied in them can also be usefully presented in tree form.



The branches descending from the node A are labelled with the two possible truth values for A. The branches emerging from the nodes marked B give the two possible values for B for each value of A. The leaf nodes at the bottom of the tree are marked with the values of A ∧ B for each truth combination of A and B.

1.2 Logical Connectives or Operators

The following symbols are used to represent the logical connectives or operators.

And	∧(Conjuction)
or	∨(Disjunction)
not	¬(Not)
Ex - or	⊕
Nand	↑
Nor	↓
if....then	→(Implication)
ifandonlyif	↔(Biconditional)

1. **∧ (And / Conjuction):** We use the letters F and T to stand for false and true respectively.

Table-1

A	B	A ∧ B
F	F	F
F	T	F
T	F	F
T	T	T

It tells us that the conjuctive operation ∧ is being treated as a **binary** logical connective–it operates on two logical statements. The letters A and B are **"Propositional Variables"**.

The table tells us that the compound proposition A ∧ B is true only when both A and B are true separately. The truth table tells us how to do this for the operator. A ∧ B is called a truth function of A and B as its value is dependent on and determined by the truth values of A and B.

A and B can be made to stand for the truth values of propositions as follows:

A : The cat sat on the mat

B : The dog barked

Each of which may be true or false. Then A ∧ B would represent the compound proposition "The cat sat on the mat and the dog barked"

A ∧ B is written as A.B in Boolean Algebra.

2. **∨ (Disjunction):** The truth table for the disjunctive binary operation ∨ tells us that the compound proposition A ∨ B is false only if A and B are both false, otherwise it is true.

A	B	A ∨ B
F	F	F
F	T	T
T	F	T
T	T	T

This is inclusive use of the operator 'or'.

In Boolean Algebra A ∨ B is written as A + B.

3. **¬ (Not):**

The negation operator is a **"unary operator"** rather than a binary operator like ∧ and its truth table is

A	¬ A
F	T
T	F

The table presents $\neg$ in its role i.e the negation of true is false, and the negation of false is true. Notice that $\neg A$ is sometimes written as $\sim A$ or A'.

4. **$\oplus$ (Exclusive OR or Ex - OR):**
$A \oplus B$ is true only when either A or B is true but not when both are true or when both are false. $A \oplus B$ is also denoted by $A \vee - B$.

A	B	$A \oplus B$
F	F	F
F	T	T
T	F	T
T	T	F

5. **$\uparrow$ (NAND):**
$P \uparrow Q \equiv \neg(P \wedge Q)$

6. **$\downarrow$ (NOR):**
$P \downarrow Q \equiv \neg(P \vee Q)$

Note:
$$P \uparrow P \equiv \neg P$$
$$P \downarrow P \equiv \neg P$$
$$(P \downarrow Q) \downarrow (P \downarrow Q) \equiv P \vee Q$$
$$(P \uparrow Q) \uparrow (P \uparrow Q) \equiv P \wedge Q$$
$$(P \uparrow P) \uparrow (Q \uparrow Q) \equiv P \vee Q$$

7. **$\rightarrow$ (Implication):**

A	B	$A \rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

Note that $A \rightarrow B$ is false only when A is true and B is false. Also, note that $A \rightarrow B$ is true, whenever A is false, irrespective of the truth value of B.

8. **$\leftrightarrow$ (if and only if):** The truth table is

A	B	$A \leftrightarrow B$
F	F	T
F	T	F
T	F	F
T	T	T

Note that Bi-conditional (if and only if) is true only when both A & B have the same truth values. ($A \leftrightarrow B$ may be written as $A \rightleftarrows B$)

Equivalences: $B \wedge A$ always takes on the same truth value is as $A \wedge B$.
We say that $B \wedge A$ is logically equivalent to $A \wedge B$ and we can write this as follows $B \wedge A \equiv A \wedge B$

Definition: Two expression are logically equivalent if each one always has the same truth value as the other.

Also,

$$B \wedge A \equiv A \wedge B$$
$$B \vee A \equiv A \vee B$$
$$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$$
$$A \vee (B \vee C) \equiv (A \vee B) \vee C$$

These equivalence reveal $\wedge$ and $\vee$ to be commutative and associative operations. But these are not the only important equivalences that hold between logical forms.

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

A	B	$\neg A$	$\neg A \vee B$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

In the last columns in ta bles (Shown Bracketed) we have exactly same sequences of truth values. So, $A \to B \equiv \neg A \vee B$. Thus, we could do without the operation $\to$.
Now, consider the following two truth tables.

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\neg(\neg A \vee \neg B)$
F	F	T	T	T	F
F	T	T	F	T	F
T	F	F	T	T	F
T	T	F	F	F	T

We see from the bracketed truth values that $A \wedge B$ is logically equivalent to $\neg(\neg A \vee \neg B)$. Thus $\wedge$ could be replaced by a combination of $\neg$ and $\vee$.
Similarly we could show that $\leftrightarrow$ can be replaced by a combination of $\neg$ and $\vee$.
We say therefore that $(\neg, \vee)$ forms a functionally complete set of connectives.
We can also show that $(\neg, \wedge)$ also form a functionally complete set of connectives.
The NAND operator $(\uparrow)$ by itself is also a functionally complete set. So is the Nor operator $(\downarrow)$. These both are minimal functionally complete set.
Notice that $(\vee, \wedge)$ is not a functionally complete set. Neither is $(\neg)$, $(\vee)$ or $(\wedge)$ by themselves functionally complete.

Example - 1.1 **Obtain the truth table for $\alpha = (P \vee Q) \wedge (P \to Q) \wedge (Q \to P)$**

Solution:

P	Q	$P \vee Q$	$P \to Q$	$(P \vee Q) \wedge (P \to Q)$	$Q \to P$	α
T	T	T	T	T	T	T
T	F	T	F	F	T	F
F	T	T	T	T	F	F
F	F	F	T	F	T	F

1.2.1 List of Important Equivalences

$A \wedge 0 \equiv 0$	Domination law	$A \cdot 0 \equiv 0$
$A \vee 1 \equiv 1$	Domination law	$A + 1 \equiv 1$
$A \wedge 1 \equiv A$	Identity of $\vee$	$A \cdot 1 \equiv 1$
$A \vee 0 \equiv A$	Identity of $\wedge$	$A + 0 \equiv A$
$A \wedge A \equiv A$	Idempotence	$A \cdot A = A$
$A \vee A \equiv A$	Idempotence	$A + A = A$
$A \wedge \neg A \equiv 0$	Complement law	$A \cdot A = 0$
$A \wedge \neg A \equiv 1$	Complement law	$A + A' = 1$
$\neg\neg A \equiv A$	Law of double negation	$(A')' = A$
$A \wedge B \equiv B \wedge A$	Commutativity	$A \cdot B = B \cdot A$
$A \vee B \equiv B \vee A$	Commutativity	$A + B = B + A$
$A \vee (B \vee C) \equiv (A \vee B) \vee C$	Associativity	$A + (B + C) = (A + B) + C$
$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$	Associativity	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$
$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$	Distributivity	$A \cdot (B + C) = A \cdot B + A \cdot C$
$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	Distributivity	$A + (B \cdot C) \equiv (A + B) \cdot (A + C)$
$A \wedge (A \vee B) \equiv A$	Absorption law	$A \cdot (A + B) \equiv A$
$A \vee (A \wedge B) \equiv A$	Absorption law	$A + (A \cdot B) \equiv A$
$\neg(A \wedge B) \equiv \neg A \vee \neg B$	De Morgan's law	$(A \cdot B)' \equiv A' + B'$
$\neg(A \vee B) \equiv \neg A \vee \neg B$	De Morgan's law	$(A + B)' \equiv A' \cdot B'$
$A \rightarrow B \equiv \neg A \vee B$		$A \rightarrow B \equiv A' + B$
$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$		$A \leftrightarrow B \equiv (A' \rightarrow B) \cdot (B' \rightarrow A)$

NOTE: $\oplus$ (EX – OR) is commutative and associative, (NAND) and (NOR) are both commutative but not associative. $P \wedge (Q \oplus R) \equiv (P \wedge Q) \oplus (P \wedge R)$

Simplication

The various equivalence between logical forms provide us with a means of simplifying logical expressions. For example, we can simply the logical forms.

$(A \vee 0) \wedge (A \vee \neg A)$ as follows:

$$(A \vee 0) \wedge (A \vee \neg A) = A \wedge (A \vee \neg A)$$
$$= A \wedge 1 \text{ (since } A \vee \neg A \equiv 1)$$
$$= A$$

Similarly, $(A \wedge \neg B) \wedge (A \wedge B \wedge C) \equiv A \wedge (\neg B \vee (B \wedge C))$

$$\equiv A \wedge ((\neg B \vee B) \wedge (\neg B \vee C))$$
$$\equiv A \wedge (1 \wedge (\neg B \vee C))$$
$$\equiv A \wedge (\neg B \vee C)$$

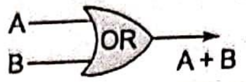
Since logic is a boolean algebra, it is always much easier to do simplification of logical expressions by converting them first into its boolean algebra equivalents.

Example: $(A \vee 0) \wedge (A \vee \neg A) \equiv (A + 0) \cdot (A + A') = (A) \cdot (1) = A$

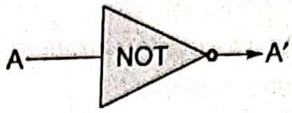
Application to Circuit Design

One of most important application of Boolean algebra is to the design of electronic circuits and especially to the design of computer logic. The basic logic functions 'and', 'or' and 'not' can be realised by electronic devices called gates and these can be combined together to form complicated circuits. The 'and' connective is realised by an 'and-gate' which is symbolised as follows:

The idea is that if the AND-gate receives input signals on both the A and B input lines, there will be an output signal on the line marked A.B. If there is no input on either A or B or both there will be no output signal. Similarly, an 'OR-gate' is symbolised as follows:

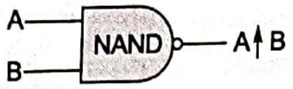


If there is an input signal on either A or B or both, there will be an output signal on A + B.
Finally a NOT gate looks alike



In this case, if there is an input A, there will be no output and If there is no input A, there will be an output. The input is said to be inverted or negated.

Similarly we have the NAND gate and NOR gate represented *y* represented by.



1.3 Well-Formed Formulas (WFFs)

Consider $P \wedge Q$ and $Q \wedge P$, where P and Q are any two propositions (logical statements). The truth table of these two propositions are identical. This happens when we have any proposition in place of P and any propositions in place of Q.

So we can develop the concept of propositional variable (corresponding to propositions) and well formed formulas (corresponding to propositions involving connectives).

Definition: A propositional variable is a symbol representing any proposition. We note that in algebra, a real variable is represented by the symbol x. This means that x is not a real number but can take a real value. Similarly, a propositional variable is not a proposition but can be replaced by a proposition.

Definition: A **well formed formula is** defined as follows:

(*i*) If P is a propositional variable then it is wff.

(*ii*) If a is a wff, then $\neg$ a is a wff.

(*iii*) If α and β are well formed formula, then $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ are well formed formula.

NOTE: A wff is not a proposition, but if we substitute the proposition in place of propositional variable, we get a proposition e.g., $(\neg P \wedge Q) \leftrightarrow Q$ is a wff.

Also note that "$P \neg \wedge Q \leftrightarrow Q$" is not a wff. Similarly "$P \wedge Q \vee$" is not a wff. This is because the above two cannot be derived using rules *i*, *ii* and *iii* given above for wffs.

Duality Law

Two formulas A and A' are said to be duals of each other, if either can be obtained from the other by replacing $\wedge$ by $\vee$, $\vee$ by $\vee\wedge$, 0 by 1 (F by T) and 1 by 0 (T by F). e.g., Dual of $(P \wedge Q) \vee T$ is $(P \vee Q) \wedge F$, Dual of $A + 1 = 1$ is $A . 0 = 0$

Let A and A' be duals consisting of $P_1, P_2, \ldots P_n$ Propositional variables. By repeated application of De Morgan's Law, it can be shown that $\neg A(P_1, P_2, \ldots, P_n) = A'(\neg P_1, \neg P_2, \ldots, \neg P_n)$.

1.3.1 Truth Table for a Well-Formed Formula

If we replace the propositional variables in a formula a by propositions, we get a proposition involving connectives. The table giving the truth value of such proposition obtained by replacing the propositional variables by arbitrary proposition is called the truth table of a.

If a involves n propositional variables, we have 2^n possible combinations of truth values of propositions replacing the variables.

Tautology, Contradiction and Contingency

Definition: A **tautology** is a well formed formula whose truth value is T for all possible assignments of truth values to the propositional variables. Such a wff is also called valid (always true).

NOTE: When it is not clear whether a given formula is tautology, we can construct a truth table and verify that the truth value is T for all possible combinations of truth value of the propositional variables appearing in given formula.

Example - 1.2 Show that $\alpha = (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$ is a tautology.

Solution:

Truth table for α:

P	Q	R	Q→R	P→(Q→R)	P→Q	P→R	(P→Q)→(P→R)	α
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Since the truth value of α is T for every possible Combination of truth values of P, Q and R, we can say that α is a tautology.

Definition: A **contradiction** (or absurdity) is a wff whose truth value is F for all possible assignments of truth values to the propositional variables.

e.g., $P \wedge \neg P$ and $(P \wedge Q) \wedge \neg Q$ ← are examples of contradictions

Truth table for $P \wedge \neg P$

P	¬P	P∧¬P
T	F	F
F	T	F

Truth table for $(P \wedge Q \wedge \neg Q)$

P	Q	P∧Q	¬Q	P∧Q∧¬Q
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	F	T	F

NOTE: α is contradiction if and only if $\neg a$ is tautology.

Definition: A **contingency** is a wff which is neither a tautology nor a contradiction. In otherwords, a contingency is a wff which is sometimes true of sometimes false.

Examples of contigency are $P \wedge Q, \neg P \vee Q, P \wedge (P \vee Q)$

Truth table for $P \wedge (P \vee Q)$

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Notice that $P \wedge (P \vee Q)$ is sometimes true and sometimes false.

Satisfiable and Unsatisfiable wffs

- A wff which is either a tautology or a contingency is called **satisfiable.**
- A wff which is a contradiction is called **unsatisfiable.**

Equivalence of Well-Formed Formulas

Definition: Two wff a and b in propositional variables $P_1, P_2, \ldots P_n$ are **equivalent** if the formula $a \leftrightarrow b$ is a tautology.

When α and β are equivalent we write $\alpha \equiv \beta$.

NOTE: α and β are equivalent if and only if truth tables of *a* and *b* are the same.

1.4 Normal forms of Well-Formed Formulas

We know that two formulas are equivalent if and only if they have the same truth table. The number of distinct truth tables for formulas in P and Q is 2^4 (As the possible combination of truth values of P and Q are TT, TF, FT, FF, the truth table of any formula in P and Q has 4 rows. So the number of distinct truth tables is 2^4). Each row may be associated with either T or F value for the function involving P and Q. Thus there are only 16 distinct formulas and any formula in P and Q is equivalent to one of these 16 formulas.

Here there is a method of reducing a given formula to an equivalent form called a 'normal form'. We use 'sum' for disjunctions, 'product' for conjuction, and 'literal' either for P or for $\neg$ P, where P is any propositional variable.

Definition:

1. An elementary product is a product of literals. An elementary sum is a sum of literals. e.g. $P \wedge \neg Q, \neg P \wedge Q, P \wedge Q, P$ are elementary products $PV \vee \neg Q, P \vee \neg R, P$ are elementary sums.
2. A formula is in disjunctive normal form (DNF) if it is a sum of elementary products. e.g. $(P) \vee (Q \wedge R)$ AND $(P) \vee (\neg Q \wedge R)$ are in disjunctive normal form.

Construction to obtain a disjunctive normal form of a given formula:

Step-1: Eliminate $\rightarrow$ and $\leftrightarrow$ using logical identities.

Step-2: Use De Morgan's law to eliminate $\neg$ before sums or products. The resulting formula has $\neg$ only before propositional variables. i.e. it involve sum, product and literals.

Step-3: Apply distributive laws repeatedly to eliminate product of sums. The resulting formula will be a sum of products of literals i.e. sum of elementary products.

Definition: A min term in n propositional variables $P_1 \ldots P_n$ is $Q_1 \wedge Q_2 \ldots \wedge Q_n$ where each Q_i is either P_i or $\neg P_i$ e.g. The min term in P_1 and P_2 are $P_1 \wedge P_2$. The number of min items in n variables is 2^n.

Definition: A formula α is in principal disjunctive normal form (PDNF), if α is a sum of min terms.

Construction to obtain the Principal Disjunctive Normal form of a given formula:

Step-1: Obtain a disjunctive normal form

Step-2: Drop elementary products which are contradictions (such as $P \wedge \neg P$)

Step-3: P_i and $\neg P_i$ are missing in an elementary product a replace a by $(\alpha \wedge P_i) \vee (\alpha \wedge \neg P_i)$

Step-4: Repeat step 3 until all elementary products are reduced to sum of min terms. Use idempotent laws to avoid repetition of min terms.

Definition:

1. A max term in n propositional variables $P_1, P_2, \ldots P_n$ is $Q_1 \vee Q_2 \ldots \vee Q_n$, where each Q_i is either P_i or $\neg P_i$.
2. A formula α is in **principal conjunctive normal form** if α (PCNF) is a product of max terms.

For obtaining the principal conjunctive normal form of α, we can constant the principal disjunctive normal form of $\neg \alpha$ and apply negation ($\neg$).

NOTE: For a given wff the PDNF form is unique the PCNF form is unique, if PDNF form or PCNF of 2 wffs are same, they are equivalent.

Example-1.3 Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

$$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \quad \text{(By using } p \rightarrow q \equiv \neg p \vee q)$$
$$\equiv \neg(\neg p) \wedge \neg q \quad \text{(By using Demorgans law)}$$
$$= p \wedge \neg q \quad \text{(By using double negation law)}$$

Example-1.4 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution:

$$\text{LHS} \equiv \neg(p \vee (\neg p \wedge q))$$
$$\equiv (p + (p'q))'$$
$$\equiv (p + q)'$$
$$\equiv p'q'$$
$$\text{RHS} \equiv \neg p \wedge \neg q$$
$$\equiv p'q'$$

Therefore, LHS $\equiv$ RHS

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Example-1.5 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$(p \wedge q) \rightarrow (p \vee q) \equiv pq \rightarrow (p + q)$$
$$\equiv (pq)' + (p + q)$$
$$\equiv p' + q' + p + q$$
$$\equiv p' + p + q' + q$$
$$\equiv 1 + q' + q \equiv 1$$

Therefore, $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

1.5 Rules of Inferences for Propositional Calculus

In logical reasoning (an argument or proof), a certain number of propositions are assumed to be true and based on that assumption some other propositions are derived. There are some important reasoning or rules of inferences.

The propositions that are assumed to be true are called **hypotheses or premises.** The proposition derived by using the rules of inference is called a **conclusion.**

The process of deriving conclusions based on assumption of premises is called **argument.** An argument is valid iff the conclusion is true whenever the premises are all true.

The rules of inference are commonly known tautologies in the form of implication (i.e. $\alpha \to \beta$).

e.g. $P \to (P \vee Q)$ is such a tautology and it is a rule of inference.

We write this in the form of $\dfrac{P}{P \vee Q}$. Here P denotes a premise. The proposition below the line i.e. $P \vee Q$, is the conclusion.

Rules of inference specify which conclusion may be inferred legitimately from known, assumed or previously established premises.

Therefore these are commonly used in mathematical proofs and logical arguments. Infact, most math proofs uses only one or more of the rules of inferences.

1.5.1 Rules of Inference

Rules of Inference		Implication form
1. Addition	$\dfrac{P}{\therefore P \vee Q}$	$P \to (P \vee Q)$
2. Conjunction	$\dfrac{P \quad Q}{\therefore P \wedge Q}$	$P \wedge Q \to P \wedge Q$
3. Simplification	$\dfrac{P \wedge Q}{\therefore Q}$	$(P \wedge Q) \to Q$
4. Modus ponens	$\dfrac{P \quad P \to Q}{\therefore Q}$	$(P \wedge (P \to Q)) \to Q$
5. Modus tollens	$\dfrac{\neg Q \quad P \to Q}{\therefore \neg P}$	$(\neg Q \wedge (P \to Q) \to \neg P)$
6. Disjunctive syllogism	$\dfrac{\neg P \quad P \vee Q}{\therefore Q}$	$(\neg P \wedge (P \vee Q)) \to Q$
7. Hypothetical syllogism	$\dfrac{P \to Q \quad Q \to R}{\therefore P \Rightarrow R}$	$((P \to Q) \wedge (Q \to R)) \to (P \to R)$

8. Constructive Dilemma $(P \rightarrow Q) \wedge (R \rightarrow S)$ $\quad (P \rightarrow Q) \wedge (R \rightarrow S) \wedge (P \vee R) \rightarrow (Q \vee S)$

$$\frac{P \vee R}{\therefore Q \vee S}$$

9. Destructive Dilemma $(P \rightarrow Q) \wedge (R \rightarrow S)$ $\quad (P \rightarrow Q) \wedge (R \rightarrow S) \wedge (\neg Q \vee \neg S) (\neg P \vee \neg R)$

$$\frac{\neg Q \vee \neg S}{\therefore \neg P \vee \neg R}$$

Arguments: An **argument** is a set of premises followed by a conclusion.

An argument is **valid** if the conjunction of premises $\rightarrow$ conclusion is a tautology.

An **invalid** argument is also called as **fallacy**.

Example: "If you study hard you will pass the exam. I studied hard. Therefore I will pass the exam", can be translated as

P : I study hard

Q : I will pass the exam $(P \rightarrow Q, P) \dashv Q$.

This argument is valid if the wff $(P \rightarrow Q) \wedge (P) \rightarrow Q$ is a tautology.

It can be verified that, it is indeed a tautology & therefore, the given argument is valid.

Inconsistency and Consistency

A set of wff's $H_1, H_2, H_3 \ldots H_n$ are **inconsistent** if $H_1 \wedge H_2 \wedge H_3 \ldots\ldots \wedge H_n$ is a contradiction (unsatisfiable). The set is **consistent** if $H_1 \wedge H_2 \ldots\ldots \wedge H_n$ is satisfiable (i.e. either a tautology or a contingency).

1.6 Predicate Calculus

Let us consider two propositions **"Rita is a student"** and **"Sita is a student"**.

As propositions, there is no relation between them but there is something common between the two statements. Both Rita and Sita share a property of being a student.

We can replace the two proposition by a single statement **"x is a student"**. By replacing x by Rita or Sita (or any other name), we get many propositions. The common feature expressed by **"is a student "** is called a predicate. In predicate calculus we deal with sentences involving **predicates.**

A **predicate** $P(x)$ is a propositional function such as $P(x) : x$ is a student.

Now, P(Rita) has the truth value of the Statement, "Rita is a student".

Another Example: $P(x, y) : x + y = 4$. Here, $P(3, 1)$ is true but $P(3, 2)$ is false.

Statements involving predicates occur in Mathematics and programming languages e.g. **"$2x + 3y = 4z$"**. **"IF (D.GE.O.O)GO TO 20"** are statements in Mathematics and FORTRAN, respectively involving predicates.

Predicates

A part of a declarative sentence describing the properties of an object or relation among objects is in English called a **predicate.** e.g. **"is a student"** is a predicate.

The sentence **"*x* is the father of *y*"** also involves a predicate **"is the father of"**. Here the predicate the describes relation between two persons.

We can write this sentence as $P(x, y)$.

Similarly, $2x + 3y = 4z$ can be described by $P(x, y, z)$.

NOTE: Although, $P(x)$, involving a predicate looks like a proposition, it is not a proposition.

As $P(x)$ involves a variable x, we cannot assign a truth value to $P(x)$.

However, if we replace x by a specific object, then we get a proposition.

1.7 Universal and Existential Quantifiers

The phrase 'for all' (denoted by $\forall$) is called the **Universal Quantifier.**
Using this symbol, we can write "for all x, $x^2 = (-x)^2$" as $\forall x\, Q(x)$, where $Q(x)$ is "$x^2 = (-x)^2$".
The phrase 'there exists' (denoted by $\exists$) is called the **Existential Quantifier.**
The sentence, "there exists x such that $x^2 = 5$" can be written as $\exists x\, R(x)$, where $R(x)$ is $x^2 = 5$.
$P(x)$ in $\forall x P(x)$ or in $\exists x\, P(x)$ is called the **'scope of quantifier'** $\forall$ or $\exists$.

NOTE: $\forall$ also be written as $(x)\, P(x)$.

Consider $P(x, y): x + y = y + x$
Now $\forall x \forall y\, P(x, y)$ is true.
Whereas $\forall x \forall y\, Q(x, y)$ is false, where, $Q(x, y): x = y^2$
Similarly if $P(x): x^2 \equiv 4$ and $Q(x): x^2 = -1$, then
$\exists x P(x)$ is true while $\exists x\, Q(x)$ is false.

NOTE: Default domain for numbers is R. Domain may also be specified with quanitifier as follows:
$\exists x \in Z, P(x)$ (here x takes only integer values)

1.7.1 Well-Formed Formulas of Predicate Calculus

A well formed formula of predicate calculus is a string of variables such as $x_1, x_2 .. x_n$, connectives, parenthesis, and quantifiers defined recursively by the following rules:

1. $P(x_1 .. x_n)$ is a wff, when P is a predicate involving n variables $x_1, x_2, ... x_n$.
2. If α is a wff, then $\neg\alpha$ is a wff.
3. If α and β are wffs then $\alpha \vee \beta$, $\alpha \wedge \beta$, $\alpha \rightarrow \beta$, are also wff.
4. If α is a wff and x is any variable, then $\forall x\, (\alpha)$, $\exists x\, (\alpha)$ are wff.
5. A string is wff if and only if it is obtained by finitely applications rules (1) — (4).

A proposition can be viewed as sentences involving a predicate with O variables. So propositions are wff of predicate calculus by rule (1).

Definition: Let α and β be two predicate formulas in variables $x_1, .. x_n$ and let U be a universe of discourse of α and β. Then, α and β are equivalent to each other over U if, for every possible assignment of values to each variable in α and β, the resulting statements have the same truth values.

We can write $a \equiv b$ over U.

We say that a and b are equivalent to each other $(a \equiv \beta)$ if $a \equiv b$ over U for every universe of discourse U.

Remark: In predicate formulas, the predicate variables may or may not be quantified. We can classify, the predicate variables in a predicate formula, depending on whether they are quantified or not. This gives to the following definitions.

Definition: If a formula of the form $\exists x\, (Px)$ or $\forall x\, P(x)$ occurs as part of a predicate formula α, then such part is called an **x-bound** part of α, and the occurrence of x is called a bound occurrence of x.

An occurence of x is **free** if it is not a bound occurrence.

A predicate variable in α is free if its occurrence is free in any part of α.

In a = $(\exists x_1 P(x_1, x_2)) \wedge (\forall x_2 Q(x_2, x_3))$; for example the occurence of x_1 in $\exists x_1\, P(x_1, x_2)$ is a bound occurence and that of x_2 is free. In $\forall x_2\, Q(x_2, x_3)$, the occurence of x_2 is a bound occurrence the occurence of x_3 in a is free.

NOTE: Quantified parts of predicate formula such as $\forall x\, P(x)$ or $\exists x\, P(x)$ are propositions. We can assign values from the universe of discourse only to free variables in a predicate formula α.

Definition:

1. A predicate formula is **valid** if for all possible assignments of values from any universe of discourse to free variables, the resulting propositions have truth value T.
2. A predicate formula is **satisfiable** if, for some assignments of values to predicate variables the resulting proposition has truth value T.
3. A predicate formula is **unsatisfiable** if, for all possible assignments of values from any universe of discourse to predicate variables, the resulting propositions have truth value F.

Rules of Inference for Predicate Calculus

(*i*) Proposition formula are also predicate formulas.

(*ii*) Predicate formulas where all the variables are quantified are proposition formulas. There fore, all the rules of inference for proposition formulas are also applicable for predicate calculus where ever necessary.

In addition, we have the following 4 laws applicable for predicate calculus.

(US) **Universal Instantiation** (Specification) $\forall x\, A(x) \Rightarrow A(y)$

(ES) **Existential Instantiation** $\exists x\, A(x) \Rightarrow A(y)$

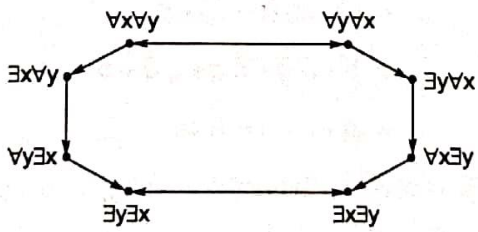
(UG) **Universal Generalisation** $A(x) \Rightarrow \forall y\, A(y)$

(ES) **Existential Generalisation** $A(x) \Rightarrow \exists y\, A(y)$

Equivalence Involving the Two Quantifiers and Valid Implications

1. $\neg(\forall x\, P(x)) \equiv \exists x \neg P(x)$
2. $\neg(\exists x\, Px)) \equiv \forall x \neg P(x)$
3. $\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$
4. $\forall x\, P(x) \wedge \forall Q(x) \Rightarrow \forall x(P(x) \vee Q(x))$
5. $\exists x\, (P(x) \vee Q(x)) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$
6. $\exists x\, (P(x) \wedge Q(x)) \Rightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$
7. $\forall x\, P(x) \Rightarrow \exists x\, P(x)$
8. $\forall x\, P \wedge Q(x)) \equiv P \wedge (\forall x\, Q(x))$
9. $\forall x\, (P \vee Q(x)) \equiv P \vee (\forall x\, Q(x))$
10. $\exists x\, (P \wedge Q(x)) \equiv P \wedge (\exists x\, Q(x)$
11. $\exists x\, (P \vee Q(x)) \equiv P \vee (\exists x\, Q(x)$

Graphical Representation of Relation between Sentences Involving Two Quantifiers



***Example*-1.6** Let, *p*: "Maria learns discrete mathematics". *q*: "Maria will find a good job". Express the statement $p \rightarrow q$ as a statement in English.

Solution:

"If Maria learns discrete mathematics, then she will find a good job" $\{p \rightarrow q\}$.

" Maria will find a good job when she learns discrete mathematics" {*q* when *p*}.

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics" {*p* is sufficient for *q*}.

"Maria will find a good job unless she does not learn discrete mathematics"

$\{q$ unless $\neg p \equiv q \vee \neg p \equiv p \rightarrow q\}$.

NOTE: p unless q is same as $p \vee q$.
p nevertheless q is same as $p \wedge q$.

Example - 1.7 **Express statement using predicates and quantifiers. "For every person x, if person x is a student in this class then x has studied calculus".**

Solution:

We take $C(x)$: "x has studied calculus" consequently if the domain for x consists of the students in the class. We can translate our statement as $\forall x\, C(x)$.

If $S(x)$ represents the statement that person x is in this class, the see that our statement can be expressed as $\forall x\,(S(x) \rightarrow C(x))$.

Note that the statement cannot be expressed as $\forall x\,(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus.

Example - 1.8 Consider the following formulas:

(*i*) $((p \rightarrow q) \Rightarrow (p \wedge q)) \rightarrow p$

(*ii*) $\neg(\forall x(Q(x) \wedge p(x)) \wedge \exists y \neg p(y))$

Which of the above are tautologies?

(a) Only (*i*)
(b) Only (*ii*)
(c) Both (*i*) and (*ii*)
(d) Neither (*i*) nor (*ii*)

Solution:

(*i*)
$$\begin{aligned}((p \rightarrow q) \rightarrow (p \wedge q)) \rightarrow p &= ((\neg p \vee q) \rightarrow (p \wedge q)) \rightarrow p \\ &= ((p \wedge \neg q) \vee (p \wedge q)) \rightarrow p \\ &\equiv \neg[(p \wedge \neg q) \vee (p \wedge q)] \vee p \\ &= ((\neg p \vee q) \wedge (\neg p \vee \neg q)) \vee p \\ &= (\bar{p}+q)(\bar{p}+\bar{q})+p \\ &= \bar{p}+\bar{p}\bar{q}+\bar{p}q+q\bar{q}+p \\ &= \bar{p}+p = 1 = \text{True}\end{aligned}$$

(*ii*)
$$\begin{aligned}\neg(\forall x(q(x) \wedge p(x)) \wedge \exists x \neg p(x)) &\equiv \exists x(\neg q(x) \vee \neg p(x)) \vee \forall x\, p(x) \\ &\equiv \exists x(\neg q(x)) \vee \exists x(\neg p(x)) \vee \forall x\, p(x) \\ &\equiv \exists x(\neg q(x)) \vee \neg(\forall x p(x)) \vee \forall x\, p(x) \\ &\equiv \exists x(\neg q(x)) \vee 1 \\ &\equiv 1 \\ &\equiv \text{True}\end{aligned}$$

$\therefore$ Both (*i*) and (*ii*) are tautologies.

Summary

- Two expression are logically equivalent if each one always has the same truth value as the other.
- ⊕ (EX–OR) is commutative and associative, (NAND) and (NOR) are both commutative but not associative. $P \wedge (Q \oplus R) \equiv (P \wedge Q) \oplus (P \wedge R)$
- A wff is not a proposition, but if we substitute the proposition in place of propositional variable, we get a proposition e.g., $(\neg P \wedge Q) \leftrightarrow Q$ is a wff.
- When it is not clear whether a given formula is tautology, we can construct a truth table and verify that the truth value is T for all possible combinations of truth value of the propositional variables appearing in given formula.
- A **contradiction** (or absurdity) is a wff whose truth value is F for all possible assignments of truth values to the propositional variables.
- A **contingency** is a wff which is neither a tautology nor a contradiction. In otherwords, a contingency is a wff which is sometimes true of sometimes false.
- Two wff a and b in propositional variables $P_1, P_2, \ldots P_n$ are equivalent if the formula $a \leftrightarrow b$ is a tautology.
- For a given wff the PDNF form is unique the PCNF form is unique, if PDNF form or PCNF of 2 wffs are same, they are equivalent.
- Quantified parts of predicate formula such as $\forall x\, P(x)$ or $\exists x\, P(x)$ are propositions. We can assign values from the universe of discourse only to free variables in a predicate formula α.

Student's Assignments

Q.1 The logical expression $((P \wedge Q) \Rightarrow (R' \wedge P)) \Rightarrow P$
(a) a tautology
(b) a contradiction
(c) a contingency
(d) All the above

Q.2 The principal conjunctive normal form is
(a) sum of products
(b) product of sums
(c) sum of max-terms
(d) product of max-terms

Q.3 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I
A. Associative law
B. Absorption law
C. Demorgans law
D. Commutative

List-II
1. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
2. $P \vee Q \equiv Q \vee P$
3. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
4. $P \vee (P \wedge Q) \equiv P$

Codes:

	A	B	C	D
(a)	1	2	3	4
(b)	4	3	1	2
(c)	1	4	3	2
(d)	2	1	4	2

Q.4 Consider the following statements:
S_1: $R \vee (P \vee Q)$
is a valid conclusion from the premises
$P \vee Q,\ Q \rightarrow R,\ P \rightarrow M$ and $\neg M$
S_2: $a \rightarrow b,\ \neg(f \vee c) \Rightarrow \neg b$
then
(a) S_1 is true and S_2 is invalid
(b) S_1 is false and S_2 is invalid
(c) Both are true
(d) Both are false

Q.5 The following propositional statement is

$[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

(a) tautology
(b) contradiction
(c) neither tautology nor contradiction
(d) not decidable

Q.6 Identify the correct translation into logical notation of the following assertion

"All connected bipartite graphs are nonplanar"

(a) $\forall x \left[\begin{matrix}\sim\text{connected}(x) \vee \sim \text{biparite} \\ (x) \wedge \sim \text{planar}(x)\end{matrix}\right]$

(b) $\forall x \left[\begin{matrix}\sim\text{connected}(x) \vee \sim \text{biparite} \\ (x) \vee \sim \text{planar}(x)\end{matrix}\right]$

(c) $\forall x \left[\begin{matrix}\sim\text{connected}(x) \wedge \sim \text{biparite} \\ (x) \wedge \sim \text{planar}(x)\end{matrix}\right]$

(d) $\forall x \left[\begin{matrix}\sim\text{connected}(x) \wedge \sim \text{biparite} \\ (x) \vee \sim \text{planar}(x)\end{matrix}\right]$

Q.7 Which of the following statements are true?

(*I*) It is not possible for the propositions $P \vee Q$ and $\neg P \vee \neg Q$ to be both false. to be both false.

(*II*) It is possible for the proposition $P \to (\neg P \to Q)$ to be false.

(a) Only (*I*) is true
(b) Only (*II*) is true
(c) Both (*I*) and (*II*) are true
(d) Both (*I*) and (*II*) are false

Q.8 Which of the following statements are true?

(*I*) $((P \to Q) \to R) \to ((R \to Q) \to P)$ is a tautology

(*II*) Let A, B be finite sets, with $|A| = m$ and $|B| = n$. The number of distinct functions $f : A \to B$ is there from A to B is m^n.

(a) Only (*I*) is true
(b) Only (*II*) is true
(c) Both (*I*) and (*II*) are true
(d) Both (*I*) and (*II*) are false

Q.9 State whether the following statements are true or false?

(*I*) $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ always holds, for all proposition P, Q.

(*II*) $((P \vee Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow (P \vee Q))$ always holds, for all propositions P.

(a) (*I*) is true, (*II*) is false
(b) Both (*I*) and (*II*) are true
(c) (*I*) is false, (*II*) is true
(d) Both (*I*) and (*II*) are false

Q.10 Which of the following is tautology?

(a) $x \vee y \to y \wedge z$
(b) $x \wedge y \to y \vee z$
(c) $x \vee y \to y \to z$
(d) $x \to y \to (y \to z)$

Q.11 Suppose

$P(x)$: x is a person.

$F(x, y)$: x is the father of y.

$M(x, y)$: x is mother of y.

What does the following indicates

$(\exists z)\ (P(z) \wedge F(x, z) \wedge M(z, y))$

(a) x is father of mother of y
(b) y is father of mother of x
(c) x is father of y
(d) None of the above

Q.12 Give the converse of "If it is raining then I get wet".

(a) If it is not raining then I get wet
(b) If it is not raining then I do not get wet
(c) If it get wet then it is raining
(d) If I do not get wet then it is not raining

Q.13 Which of the following is true?

(a) $\neg(p \Rightarrow q) \equiv p \wedge \neg q$
(b) $\neg(p \leftrightarrow q) \equiv ((p \vee \neg q) \vee (q \wedge \neg p))$
(c) $\neg(\exists x\,(p(x) \Rightarrow q(x))) \equiv \forall x\,(p(x) \Rightarrow q(x))$
(d) $\exists x\, p(x) \equiv \forall x\, p(x)$

Answer Key:

1. (c)	2. (d)	3. (c)	4. (a)	5. (a)
6. (b)	7. (a)	8. (d)	9. (c)	10. (b)
11. (a)	12. (c)	13. (a)		

Student's Assignments — Explanations

1. (c)

The logical expression

$((P \wedge Q) \Rightarrow (R' \wedge P)) \Rightarrow P$

can be converted in Boolean Algebra notation as,

$(pq \Rightarrow r'p) \Rightarrow p$

$\equiv (pq)' + r'p \Rightarrow p$

$\equiv (p' + q' + r'p) \Rightarrow p$

$\equiv ((p' + r'p) + q') \Rightarrow p$

$\equiv ((p' + p) \cdot (p' + r') + q') \Rightarrow p$

$\equiv (p' + r' + q') \Rightarrow p$

$\equiv (p' + r' + q')' + p \equiv prq + p$

$\equiv p$

$\therefore$ The given expression is a contingency.

4. (a)

S_1 : $P \vee Q,\ Q \Rightarrow R,\ P \rightarrow M,\ \sim M \Rightarrow R \vee (P \vee Q)$

In boolean algebra notation the above expression is written as

$(p + q)\cdot(q + r')\cdot(p + m')\cdot m' \Rightarrow r + p + q$

$\equiv (q + pr')(m') \Rightarrow r + p + q$

$\equiv qm' + pr'm' \Rightarrow r + p + q$

$\equiv (qm' + pr'm') + r + p + q$

$\equiv (q' + m)(p' + r + m)\, r + p + q$

$\equiv q'p' + q'r + q'm + mp' + mr + m + r + p + q$

(by absorption law) $\equiv q'p' + r + m + p + q$

$\equiv (p + p')\cdot(p + q') + r + m + q$

$\equiv p + q' + r + m + q$

$\equiv p + r + m + 1 \equiv 1$

$\therefore S_1$ is true

S_2 : $a \rightarrow b, \neg(f \vee c) \Rightarrow \neg b$

In boolean Algebra notation

$S_2 \equiv (a \rightarrow b)\cdot(f \vee c)' \Rightarrow b'$

$\equiv (a' + b)\cdot(f'c') \Rightarrow b'$

$\equiv [(a' + b)\cdot(f'c')] + b'$

$\equiv (a' + b)' + (f'c')^1 + b'$

$\equiv ab' + f + c + b'$

$\equiv f + c + b'$

which is a contingency

$\therefore S_2$ is invalid.

5. (a)

$[(p \rightarrow r) \vee (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$

$\equiv (p' + r)(q' + r) \rightarrow (p + q)' + r$

$\equiv (r + p'q')' \rightarrow (p + q)' + r$

$\equiv (r + p'q')' + (p + q)' + r$

$\equiv r'(p'q')' + p'q' + r$

$\equiv r'(p + q)' + p'q' + r$

$\equiv r'p + r'q + p'q' + r$

$\equiv (r + r')\cdot(r + p) + r'q + p'q'$

$\equiv r + p + r'q + p'q'$

$\equiv (r + r')(r + q) + (p + p')(p + q')$

$\equiv r + q + p + q'$

$\equiv r + p + 1 \equiv 1$

$\therefore$ tautology

6. (b)

The correct translation is

$\forall x[(\text{connected}(x) \wedge \text{bipartite}(x)) \rightarrow \sim \text{planar}(x)]$

however, since $p \rightarrow q \equiv \sim p \vee q$, we can write the above expression also as,

$\forall x[\sim\text{connected}(x) \vee \sim\text{bipartite}(x) \vee \sim \text{planar}(x)]$

7. (a)

If $P \vee Q$ is false, then both P and Q are false.

So, $\neg P \vee \neg Q \equiv \neg F \vee \neg F \equiv T \vee T \equiv T$

$\therefore$ (*i*) is true

Consider (*ii*)

$P \rightarrow (\neg P \rightarrow Q) \equiv P \rightarrow (P^1 \rightarrow Q)$

$\equiv P \rightarrow P + Q$

$\equiv P^1 + P + Q \equiv 1 + Q \equiv 1$

It is a tautology, So (*ii*) is false.

8. (d)

(*i*) Using boolean algebra, we can shown that the given expression reduces to $P + R' + Q'$ which is not a tautology.

(*ii*) For each element $a \in A$, we have n possible choices for value of $f(a)$. Thus there are n^m possible functions.

9. (c)

(*i*) $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$

$\equiv (P' + Q) \rightarrow (Q' + P)$

$\equiv (P' + Q)' + Q' + P$

$\equiv PQ' + Q' + P \equiv P + Q'$

Since $P + Q'$ is a contingency and not a tautology (i) is false

(*ii*) $((P \vee Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow (P \vee Q))$

$\equiv ((P + Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow P + Q)$

$\equiv (P + Q)' + \Rightarrow Q \Rightarrow Q' + P + Q$

$\equiv P'Q' + Q \Rightarrow 1 + P$

$\equiv (Q + P')\cdot(Q + P') \Rightarrow 1$

$\equiv (Q + P') \Rightarrow 1$

$\equiv (Q + P')' + 1 \equiv 1$

$\therefore$ (*ii*) always holds.

10. (b)

(*i*) $x \vee y \rightarrow y \wedge z \equiv x + y \rightarrow yz$

$\equiv (x + y)' + yz$

$\equiv x'y' + yz$

$\not\equiv 1$

$\therefore$ not a tautology

(*ii*) $x \wedge y \rightarrow y \vee z \equiv xy \rightarrow y + z$

$\equiv (xy)' + y + z$

$\equiv x' + y' + y + z$

$\equiv 1 + x' + z$

$\equiv 1$

$\therefore$ It is a tautology

(*iii*) $x \vee y \rightarrow (y \rightarrow z) \equiv x + y \rightarrow (y \rightarrow z)$

$\equiv (x + y)' + (y \rightarrow z)$

$\equiv x'y' + y' + z$

$\equiv y' + z$

$\not\equiv 1$

$\therefore$ not a tautology

(*iv*) $x \rightarrow y \rightarrow (y \rightarrow z) \equiv x' + y \rightarrow (y' + z)$

$\equiv (x' + y)' + y' + z$

$\equiv xy' + y' + z$

$\equiv y' + z$

$\not\equiv 1$

$\therefore$ not a tautology

13. (a)

Option (a) $\neg(p \Rightarrow q) \equiv p \wedge \neg q$ is true, since $\neg(p \Rightarrow q) \equiv \neg(p' + q) \equiv pq' \equiv p \wedge \neg q$