CHAPTER

3

Progression and Series

- Introduction
- Arithmetic Progression (A.P.)
- Geometric Progression (G.P.)
- Harmonic Progression (H.P.)
- Miscellaneous Series

INTRODUCTION

In mathematics, a **sequence** is an ordered list of objects (or events). Like a set, it contains members (also called *elements* or *terms*), and the number of terms (possibly infinite) is called the *length* of the sequence. Unlike a set, order matters, and exactly the same elements can appear multiple times at different positions in the sequence. A sequence is a discrete function.

The '*n*th' term is a formula with 'n' in it which enables you to find **any** term of a sequence without having to go up from one term to the next.

'***n***' stands for the ***term number*** so to find the 50th term we would just substitute 50 in the formula in place of '***n***'.

Thus *n*th term of A.P. is linear in n. Infact $I_n = a + bn$, where $n \in N$,

Real Sequence

A sequence whose range is a subset of R is called a real sequence.

Finite and Infinite Sequences

On the basis of the number of terms, there are two types of sequences.

(i) Finite sequences: A sequence is said to be finite if it has finite number of terms.

(ii) Infinite sequences: A sequence is said to be infinite if it has infinite number of terms i.e. sequence of all even natural numbers (2, 4, 6, 8, ...).

Example 3.1 **Write down the sequence whose n^{th} term is a. $2^n/n$ and b. $[3 + (-1)^n]/3^n$.**

Sol.

a. Let $t_n = 2^n/n$ and put $n = 1, 2, 3, 4, \ldots$. We get

$t_1 = 2, t_2 = 2, t_3 = 8/3, t_4 = 4$

So, the sequence is 2, 2, 8/3, 4, ...

b. Let $t_n = [3 + (-1)^n]/3^n$ and put $n = 1, 2, 3, 4, \ldots$

So, the sequence is 2/3, 4/9, 2/27, 4/81, ...

Example 3.2 **Find the sequence of the numbers defined by** $a_n = \begin{cases} \frac{1}{n}, & \text{when } n \text{ is odd} \\ -\frac{1}{n}, & \text{when } n \text{ is even} \end{cases}$

Sol. We have $a_1 = 1, a_3 = \frac{1}{3}, a_5 = \frac{1}{5}, \ldots$

and $a_2 = -\frac{1}{2}, a_4 = -\frac{1}{4}, a_6 = -\frac{1}{6}, \ldots$

Hence the sequence is $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots\ldots\ldots$

Example 3.3 **Write the first three terms of the sequence defined by $a_1 = 2, a_{n+1} = \frac{2a_n + 3}{a_n + 2}$.**

Sol. Put $n = 1$ in $a_{n+1} = \frac{2a_n + 3}{a_n + 2}$, we have

$$a_{1+1} = a_2 = \frac{2a_1 + 3}{a_1 + 2} = \frac{2(2)+3}{(2)+2} = \frac{7}{4}$$

Put $n = 2$, then we have $a_{2+1} = a_3 = \frac{2a_2 + 3}{a_2 + 2}$

$$= \frac{2\left(\frac{7}{4}\right)+3}{\left(\frac{7}{4}\right)+2} = \frac{26}{15}$$

Example 3.4 **Consider the sequence defined by $a_n = an^2 + bn + c$. If $a_1 = 1$, $a_2 = 5$ and $a_3 = 11$ then find the value of a_{10}.**

Sol. $a_1 = 1 \Rightarrow a + b + c = 1$ (i)

$a_2 = 5 \Rightarrow 4a + 2b + c = 5$ (ii)

$a_3 = 11, \Rightarrow 9a + 3b + c = 11$ (iii)

Now from (ii) – (i), we have $3a + b = 4$ (iv)

From (iii) – (ii), we have $5a + b = 6$ (v)

From (v) – (iv), we have $2a = 2$ or $a = 1$

$\Rightarrow b = 1$ (from (iv)), and $c = -1$ (from (i))

Hence $a_n = n^2 + n - 1$

Hence $a_{10} = 100 + 10 - 1 = 109$

Series

By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, \ldots, a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \cdots + a_n$ is a series. For example,

(i) $1 + 2 + 3 + 4 + \cdots + n$

(ii) $2 + 4 + 8 + 16 + \cdots$

Progression

It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the *n*th term. Those sequences whose terms follow certain patterns are called progressions.

ARITHMETIC PROGRESSION (A.P.)

A.P. is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference.

If a is the first term and d is the common difference, then A.P. can be written as

$$a, a + d, a + 2d, \ldots, a + (n - 1) d, \ldots$$

n^{th} term: $T_n = a + (n - 1) d = l$ (last term), where $d = T_n - T_{n-1}$.

n^{th} term from end: $T_n' = l - (n - 1)d$.

The n^{th} term of A.P. is linear in n.

Example 3.5 **Show that the sequence 9, 12, 15, 18, ... is an A.P. Find its 16^{th} term and the general term.**

Sol. Since (12 – 9) = (15 – 12) = (18 – 15) = 3, therefore the given sequence is an A.P. with common difference 3. First term is 9. Therefore, the 16th term is

$$a_{16} = a + (16 - 1)d \quad [\because a_n = a + (n-1)d]$$
$$= a + 15d$$
$$\Rightarrow a_{16} = 9 + 15 \times 3 = 54$$

The general term (n^{th} term) is given by

$$a_n = a + (n-1)d$$
$$= 9 + (n-1) \times 3 = 3n + 6$$

Example 3.6 **Show that the sequence log *a*, log (*ab*), log (ab^2), log(ab^3), ... is an A.P. Find its n^{th} term.**

Sol. We have,

$$\log(ab) - \log a = \log\left(\frac{ab}{a}\right) = \log b$$
$$\log(ab^2) - \log(ab) = \log\left(\frac{ab^2}{ab}\right) = \log b$$
$$\log(ab^3) - \log(ab^2) = \log\left(\frac{ab^3}{ab^2}\right) = \log b$$

It follows from the above results that the difference of a term and the preceding term is always same. So, the given sequence is an A.P. with common difference log *b*. Now,

$$a_n = a + (n-1)d$$
$$= \log a + (n-1)\log b$$
$$= \log a + \log b^{n-1}$$
$$= \log(ab^{n-1})$$

Example 3.7 **Find the sum to *n* terms of the sequence $\langle a_n \rangle$, where $a_n = 5 - 6n$, $n \in N$.**

Sol. We have,

$$a_n = 5 - 6n \Rightarrow a_{n+1} = 5 - 6(n+1) = -1 - 6n$$
$$\therefore \quad a_{n+1} - a_n = (-1 - 6n) - (5 - 6n) = -6, \text{ for all } n \in N$$

Since $a_{n+1} - a_n$ is constant for all $n \in N$. So, the given sequence is an A.P. with common difference –6. Putting $n = 1$ in $a_n = 5 - 6n$, we get $a_1 = -1$. So, the sum S_n to n terms is given by

$$S_n = (n/2)(a_1 + a_n) = (n/2)(-1 + 5 - 6n) = n(2 - 3n)$$

Example 3.8 **How many terms are there in the A.P. 3, 7, 11, ..., 407?**

Sol. We know that last term $a_n = a + (n-1)d$

Where d = common difference = 4

and a = first term = 3

$$407 = 3 + (n-1)4$$
$$\Rightarrow n = 102$$

Hence there are 102 terms in A.P.

Example 3.9 **If *a*, *b*, *c*, *d*, *e* are in A.P., then find the the value of $a - 4b + 6c - 4d + e$.**

Sol. $E = (a + e) - 4(b + d) + 6c$.

Now b, c, d in A.P. $\Rightarrow b + d = 2c$

Again a, c, e are also in A.P.

$$\therefore \quad a + e = 2c$$
$$\therefore \quad E = 2c - 4(2c) + 6c = 0$$

Example 3.10 **In a certain A.P., 5 times the 5th term is equal to 8 times the 8th term, then prove that its 13th term is 0.**

Sol. $5T_5 = 8T_8$

$$\Rightarrow 5(a + 4d) = 8(a + 7d)$$
$$\Rightarrow 3a + 36d = 0$$
$$\Rightarrow a + 12d = 0$$
$$\Rightarrow T_{13} = 0$$

Example 3.11 **Find the term of the series 25, $22\frac{3}{4}, 20\frac{1}{2}, 18\frac{1}{4}$, ... which is numerically the smallest.**

Sol. The given series is an A.P. $a = 25$, $d = -9/4$.

$$T_n = a + (n-1)d = \left(25 + \frac{9}{4}\right) - \frac{9}{4}n$$

or $$T_n = \frac{109}{4} - \frac{9}{4}n$$

Now T_n will be –ive if $\frac{109}{4} - \frac{9}{4}n < 0$ or $n > 12\frac{1}{9}$

Above shows that T_{13} will be first –ive terms and hence T_{12} will be smallest +ive terms. $T_{13} = -2$, $T_{12} = \frac{1}{4}$ is numerically smallest

Example 3.12 **Given two A.P.'s 2, 5, 8, 11, ..., T_{60} and 3, 5, 7, 9, ... T_{50}. Then find the number of terms which are identical.**

Sol. 2, 5, 8, 11, ..., $T_{60} \Rightarrow T_{60} = 2 + (60 - 1)3 = 179$

3, 5, 7, 9, ... $T_{50} \Rightarrow T_{50} = 3 + (50 - 1)2 = 101$

Hence, last common term ≤ 101.

Now common difference of first A.P. is 3 and common difference of second A.P. is 2.

Hence common difference of A.P. formed by common terms is L.C.M. of 3 and 2 which 6. Also common terms are are 5, 11, ...

For last term let $101 = 5 + (n-1)6$

$$\Rightarrow n = 17$$

Hence 101 is the actual last common term.

Example 3.13 **Consider two A.P.s:**

S_1: 2, 7, 12, 17, ... 500 terms

and S_2: 1, 8, 15, 22, ... 300 terms

Find the number of common terms. Also find the last common term.

Sol. S_1 :2, 7, 12, 17,... 500 terms

$$\Rightarrow T_{500} = 2 + (500 - 1)5 = 2497$$

S_2 :1,8, 15, 22, ... 300 terms

$$\Rightarrow T_{300} = 1 + (300 - 1)7 = 2094$$

Common differences of S_1 and S_2 are 5 and 7 respectively.

Hence common difference of common term series is 35

A.P. of common terms is 22, 57, 92, ...

Let last term is 2094 ⇒ 22 + (n – 1)35 = 2094 ⇒ n = 60.2
But n is natural number ⇒ n = 60
Then actual last common term = 22 + (60 – 1)35 = 2062

Example 3.14 **If p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c, respectively, then show that**

a. $a(q-r)+b(r-p)+c(p-q)=0$
b. $(a-b)r+(b-c)p+(c-a)q=0$

Sol. Let A be the first term and D be the common difference of the given A.P. Then,

$a = p^{th}$ term $\Rightarrow a = A + (p-1)D$ (i)
$b = q^{th}$ term $\Rightarrow b = A + (q-1)D$ (ii)
$c = r^{th}$ term $\Rightarrow c = A + (r-1)D$ (iii)

a. $a(q-r)+b(r-p)+c(p-q) = \{A+(p-1)D\}(q-r) + \{A+(q-1)D\}(r-p) + \{A+(r-1)D\}(p-q)$ [Using (i), (ii) and (iii)]
$= A\{(q-r)+(r-p)+(p-q)\} + D\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$
$= A\times 0 + D\{p(q-r)+q(r-p)+r(p-q) - (q-r)-(r-p)-(p-q)\}$
$= A\times 0 + D\times 0 = 0$

b. On subtracting (ii) from (i), (iii) from (ii) and (i) from (iii), we get
$a-b=(p-q)D$ (iv)
$b-c=(q-r)D$ (v)
$c-a=(r-p)D$ (vi)

Now,
$(a-b)r+(b-c)p+(c-a)q$
$=(p-q)Dr+(q-r)Dp+(r-p)Dq$
$=D[(p-q)r+(q-r)p+(r-p)q]$
$=D.0=0$

Some Important Facts about A.P.

1. If a fixed number is added or subtracted to each term of a given A.P., then the resulting series is also an A.P., and its common difference remains the same.
2. If each term of an A.P. is multiplied by a fixed constant or divided by a fixed non-zero constant, then the resulting series in also an A.P.
3. If $x_1+x_2+x_3+\cdots$ and $y_1+y_2+y_3+\cdots$ are two A.P.'s, then $x_1\pm y_1, x_2\pm y_2, x_3\pm y_3, \ldots$ are also A.P.'s.
4. Three terms in an A.P. should preferably be taken as $a-d, a, a+d$ and four terms as $a-3d, a-d, a+d, a+3d$.
5. In A.P., $a_n = \dfrac{a_{n-k}+a_{n+k}}{2}$, for $k \le n$.
6. It $a, a_2, a_3, \ldots a_n$ are in A.P. Then $a_1+a_n = a_2+a_{n-1} = a_3 + a_{n-2} = \ldots = a_r + a_{n-r+1}$.

Example 3.15 **If $(b+c-a)/a$, $(c+a-b)/b$, $(a+b-c)/c$ are in A.P., then prove that $1/a$, $1/b$, $1/c$ are also in A.P.**

Sol. $\dfrac{b+c-a}{a}, \dfrac{c+a-b}{b}, \dfrac{a+b-c}{c}$ are in A.P.

$\Rightarrow \left\{\dfrac{b+c-a}{a}+2\right\}, \left\{\dfrac{c+a-b}{b}+2\right\}, \left\{\dfrac{a+b-c}{c}+2\right\}$ are in A.P. [Adding 2 to each term]

$\Rightarrow \dfrac{a+b+c}{a}, \dfrac{a+b+c}{b}, \dfrac{a+b+c}{c}$ are in A.P.

$\Rightarrow \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P. [Dividing each term by $a+b+c$]

Example 3.16 **If $a, b, c \in R+$ form an A.P., then prove that $a + 1/(bc)$, $b + 1/(ac)$, $c + 1/(ab)$ are also in A.P.**

Sol. a, b, c are in A.P.

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are also in A.P. [Dividing by abc]

$\Rightarrow a+\dfrac{1}{bc}, b+\dfrac{1}{ca}, c+\dfrac{1}{ab}$ will also be in A.P.

[∵ Sum of two A.P.'s is also an A.P.]

Example 3.17 **If a, b, c are in A.P., then prove that the following are also in A.P.**

a. $a^2(b+c), b^2(c+a), c^2(a+b)$

b. $\dfrac{1}{\sqrt{b}+\sqrt{c}}, \dfrac{1}{\sqrt{c}+\sqrt{a}}, \dfrac{1}{\sqrt{a}+\sqrt{b}}$

c. $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$

Sol.

a. Let $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.
$\Rightarrow b^2(c+a)-a^2(b+c) = c^2(a+b)-b^2(c+a)$
$\Rightarrow c(b^2-a^2)+ab(b-a) = a(c^2-b^2)+bc(c-b)$
$\Rightarrow (b-a)(ab+bc+ca) = (c-b)(ab+bc+ca)$
$\Rightarrow b-a=c-b$
$\Rightarrow 2b=a+c$
$\Rightarrow a, b, c$ are in A.P.
$\Rightarrow a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

b. Let $\dfrac{1}{\sqrt{b}+\sqrt{c}}, \dfrac{1}{\sqrt{c}+\sqrt{a}}, \dfrac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

$\Rightarrow \dfrac{1}{\sqrt{c}+\sqrt{a}}-\dfrac{1}{\sqrt{b}+\sqrt{c}} = \dfrac{1}{\sqrt{a}+\sqrt{b}}-\dfrac{1}{\sqrt{c}+\sqrt{a}}$

$\Rightarrow \dfrac{\sqrt{b}-\sqrt{a}}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})} = \dfrac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

$\Rightarrow \dfrac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \dfrac{\sqrt{c}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

$\Rightarrow b-a=c-b$
$\Rightarrow 2b=a+c$
$\Rightarrow a, b, c$ are in A.P.

$\Rightarrow \dfrac{1}{\sqrt{b}+\sqrt{c}}, \dfrac{1}{\sqrt{c}+\sqrt{a}}, \dfrac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

c. a, b, c are in A.P.

$\Rightarrow \dfrac{a}{abc}, \dfrac{b}{abc}, \dfrac{c}{abc}$ are in A.P.

[On dividing each term by abc]

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in A.P.

$\Rightarrow \dfrac{ab+bc+ca}{bc}, \dfrac{ab+bc+ca}{ca}, \dfrac{ab+bc+ca}{ab}$ are in A.P.

[On multiplying each term by $ab + bc + ca$]

$\Rightarrow \dfrac{ab+bc+ca}{bc}-1, \dfrac{ab+bc+ca}{ca}-1, \dfrac{ab+bc+ca}{ab}-1$ are in A.P. [On adding -1 to each term]

$\Rightarrow \dfrac{ab+ac}{bc}, \dfrac{ab+bc}{ca}, \dfrac{bc+ca}{ab}$ are in A.P.

$\Rightarrow a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in A.P.

Example 3.18 **The sum of three numbers in A.P. is –3 and their product is 8. Find the numbers.**

Sol. Let the numbers be $(a - d), a, (a + d)$. Therefore,

$(a - d) + a + (a + d) = -3$

$\Rightarrow 3a = -3$

$\Rightarrow a = -1$

And

$(a - d)(a)(a + d) = 8$

$\Rightarrow a(a^2 - d^2) = 8$

$\Rightarrow (-1)(1 - d^2) = 8 \quad [\because a = -1]$

$\Rightarrow d^2 = 9$

$\Rightarrow d = \pm 3$

If $d = 3$, the numbers are $-4, -1, 2$. If $d = -3$, the numbers are $2, -1, -4$. So, the numbers are $-4, -1, 2$ or $2, -1, -4$.

Example 3.19 **Divide 32 into four parts which are in A.P. such that the ratio of the product of extremes to the product of means is 7:15.**

Sol. Let the four parts be $(a - 3d), (a - d), (a + d)$ and $(a + 3d)$. Then,

$(a - 3d) + (a - d) + (a + d) + (a + 3d) = 32$

$\Rightarrow 4a = 32$

$\Rightarrow a = 8$

Also,

$$\frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{7}{15}$$

$$\Rightarrow \frac{a^2-9d^2}{a^2-d^2} = \frac{7}{15}$$

$$\Rightarrow \frac{64-9d^2}{64-d^2} = \frac{7}{15}$$

$\Rightarrow 128d^2 = 512$

$\Rightarrow d^2 = 4$

$\Rightarrow d = \pm 2$

Thus, the four parts are 2, 6, 10, 14.

Example 3.20 **The digits of a positive integer, having three digits, are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.**

Sol. Let the digits at ones, tens and hundreds place be $(a - d)$, a and $(a + d)$, respectively. Then the number is

$(a + d) \times 100 + a \times 10 + (a - d) = 111a + 99d$

The number obtained by reversing the digits is,

$(a - d) \times 100 + a \times 10 + (a + d) = 111a - 99d$

It is given that

$(a - d) + a + (a + d) = 15 \quad$ (i)

and

$111a - 99d = 111a + 99d - 594 \quad$ (ii)

$\therefore 3a = 15$ and $198d = 594$

$\Rightarrow a = 5$ and $d = 3$

So, the number is $111 \times 5 + 99 \times 3 = 852$.

Sum of n terms of an A.P.

The sum S_n of n terms of an A.P. with the first term 'a' and the common difference 'd' is

$$S_n = \frac{n}{2}[2a + (n - 1)d] \text{ or } S_n = \frac{n}{2}[a + l]$$

where l = last term $= a + (n - 1)d$.

Proof: $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n \quad$ (i)

$= a_n + a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1 \quad$ (ii)

Adding corresponding terms in (i) and (ii), we get

$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots + (a_{n-1} + a_2) + (a_n + a_1)$

$= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n)$

$= n(a_1 + a_n)$

$[\because a_1 + a_n = a_k + a_{n-k+1}$ for $k = 2, 3, \ldots, n]$

$\Rightarrow S_n = \dfrac{n}{2}(a_1 + a_n)$

$= \dfrac{n}{2}\{a_1 + a_1 + (n - 1)d\} \quad [\because a_n = a_1 + (n - 1)d]$

$= \dfrac{n}{2}[2a_1 + (n - 1)d]$

Example 3.21 **If the sum of the series 2, 5, 8, 11, ... is 60100, then find the value of n.**

Sol. Here first term is $a = 2$ and common difference $d = 3$

Hence sum of n terms of A.P. is, $(n/2)(4 + 3(n - 1)) = 60100$

$\Rightarrow 3n^2 + n - 120200 = 0$

$\Rightarrow (n - 200)(3n + 601) = 0$

$\Rightarrow n = 200$

Example 3.22 **In an A.P. if $S_1 = T_1 + T_2 + T_3 + \cdots + T_n$ (n odd), $S_2 = T_2 + T_4 + T_6 + \cdots + T_{n-1}$, then find the value of S_1/S_2 in terms of n.**

Sol. S_1 is an A.P. of n terms, but S_2 is an A.P. of $\frac{n-1}{2}$ terms with common difference $2d$

$$S_1 = \frac{n}{2}\left[T_1 + T_n\right] \quad (1)$$

$$S_1 = \frac{1}{2}\left(\frac{n-1}{2}\right)\left[T_2 + T_{n-1}\right] = \frac{1}{2}\left(\frac{n-1}{2}\right)\left[T_1 + T_n\right]$$

$$\therefore \frac{S_1}{S_2} = \frac{2n}{n-1}$$

Example 3.23 **Prove that sum of n number of terms of two different A.P.s can be same for only one value of n.**

Sol. According to the given condition

$$\frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[2A + (n-1)D]$$

$\Rightarrow$ $[2a + (n-1)d] = [2A + (n-1)D]$, from this we get one integral value of n or no value of n.

Example 3.24 **In an A.P. of 99 terms, the sum of all the odd numbered terms is 2550. Then find the sum of all the 99 terms of the A.P.**

Sol. Given $\frac{50}{2}\left[a_1 + a_{99}\right] = 2550$

$\Rightarrow$ $a_1 + a_{99} = 102$

Now sum of all the terms is

$$\frac{99}{2}\left[a_1 + a_{99}\right] = \frac{99}{2} \times 102 = 5049$$

Example 3.25 **Find the degree of the expression $(1 + x)(1 + x^6)(1 + x^{11}) \ldots (1 + x^{101})$.**

Sol. The degree of the expression is $1 + 6 + 11 + \cdots + 101$ which is an A.P.

Now $101 = 1 + 5(n - 1) \Rightarrow n = 21$

$\Rightarrow$ $1 + 6 + 11 + \ldots + 101$

$$= \frac{21}{2}[1 + 101] = 21 \times 51 = 1071$$

Example 3.26 **Find the number of terms in the series 20, $19\frac{1}{3}$, $18\frac{2}{3}$, ... the sum of which is 300. Explain the answer.**

Sol. The given sequence is an A.P. with first term $a = 20$ and the common difference $d = -2/3$. Let the sum of n terms be 300. Then,

$$S_n = 300 \Rightarrow \frac{n}{2}[2a + (n-1)d] = 300$$

$\Rightarrow$ $\frac{n}{2}[2 \times 20 + (n-1)(-2/3)] = 300$

$\Rightarrow$ $n^2 - 61n + 900 = 0$

$\Rightarrow$ $(n - 25)(n - 36) = 0$

$\Rightarrow$ $n = 25$ or 36

So, sum of 25 terms is equal to sum of 36 terms, which is equal to 300.

Here the common difference is negative, therefore terms go on diminishing and the 31st term becomes zero. All terms after the 31st term are negative. These negative terms when added to positive terms from 26th term to 30th term, they cancel out each other and the sum remains same. Hence, the sum of 25 terms as well as that of 36 terms is 300.

Example 3.27 **Find the sum of all three-digit natural numbers, which are divisible by 7.**

Sol. The smallest and the largest numbers of three digits, which are divisible by 7 are 105 and 994, respectively. So, the sequence of three digit numbers which are divisible by 7 is 105, 112, 119, ..., 994. Clearly, it is an A.P. with first term $a = 105$ and common difference $d = 7$. Let there be n terms in this sequence. Then,

$a_n = 994$

$\Rightarrow$ $a + (n - 1)d = 994$

$\Rightarrow$ $105 + (n - 1) \times 7 = 994$

$\Rightarrow$ $n = 128$

Now, required sum is

$$\frac{n}{2}[2a + (n-1)d]$$

$$= \frac{128}{2}[2 \times 105 + (128 - 1) \times 7]$$

$$= 70336$$

Example 3.28 **Prove that a sequence is an A.P. if the sum of its n terms is of the form $An^2 + Bn$, where A, B are constants.**

Sol. Let S_n be the sum of n terms of an A.P. with first term a and common difference d. Then,

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= an + \frac{n^2}{2}d - \frac{n}{2}d$$

$$= \left(\frac{d}{2}\right)n^2 + \left(a - \frac{d}{2}\right)n$$

$$= An^2 + Bn$$

where $A = d/2$ and $B = a - d/2$.

Thus, the sum of n terms of an A.P. is of the form $An^2 + Bn$. Conversely, let the sum S_n of n terms of a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ be of the form $An^2 + Bn$.

Then, we have to show that the sequence is an A.P. We have,

$S_n = An^2 + Bn$

$\Rightarrow$ $S_{n-1} = A(n - 1)^2 + B(n - 1)$ [On replacing n by $(n - 1)$]

Now,

$a_n = S_n - S_{n-1}$

$\Rightarrow$ $a_n = \{An^2 + Bn\} - \{A(n - 1)^2 + B(n - 1)\}$

$= 2An + (B - A)$

$\Rightarrow a_{n+1} = 2A(n+1) + (B - A)$ [On replacing n by $(n+1)$]

$\Rightarrow a_{n+1} - a_n = \{2A(n+1) + B - A\} - \{2An + (B - A)\} = 2A$

Since $a_{n+1} - a_n = 2A$ for all $n \in N$, so the sequence is an A.P. with common difference $2A$.

Example 3.29 **If the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ forms an A.P., then prove that**

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2 = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$$

Sol. Let d be the common difference of the A.P. Then,

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = a_{2n} - a_{2n-1}$$

Now,

$$a_1^2 - a_2^2 + a_3^2 - a_4^2 + \cdots + a_{2n-1}^2 - a_{2n}^2$$

$$= (a_1 + a_2)(a_1 - a_2) + (a_3 + a_4)(a_3 - a_4) + \cdots + (a_{2n-1} + a_{2n}) \times (a_{2n-1} - a_{2n})$$

$$= -d(a_1 + a_2 + a_3 + \cdots + a_{2n})$$

$$= -d\frac{2n}{2}(a_1 + a_{2n})$$

$$= -dn\frac{(a_1^2 - a_{2n}^2)}{a_1 - a_{2n}}$$

$$= \frac{dn(a_1^2 - a_{2n}^2)}{a_{2n} - a_1}$$

$$= \frac{n}{2n-1}(a_1^2 - a_{2n}^2) \quad [\text{Using } a_{2n} = a_1 + (2n-1)d]$$

Example 3.30 **Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.**

Sol. We know that in an A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term, i.e., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \cdots$.

So, if an A.P. consists of 24 terms, then

$$a_1 + a_{24} = a_5 + a_{20}$$
$$= a_{10} + a_{15}$$
$$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$$

$\Rightarrow (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow 3(a_1 + a_{24}) = 225$

$\Rightarrow a_1 + a_{24} = \frac{225}{3} = 75$ (i)

$\therefore S_{24} = \frac{24}{2}(a_1 + a_{24}) \quad \left[\text{Using } S_n = \frac{n}{2}(a_1 + a_n)\right]$

$= 12\,(75)$

$= 900$ [Using (i)]

Example 3.31 **If the arithmetic progression whose common difference is non-zero, the sum of first $3n$ terms is equal to the sum of next n terms. Then, find the ratio of the sum of the first $2n$ terms to the sum of next $2n$ terms.**

Sol.

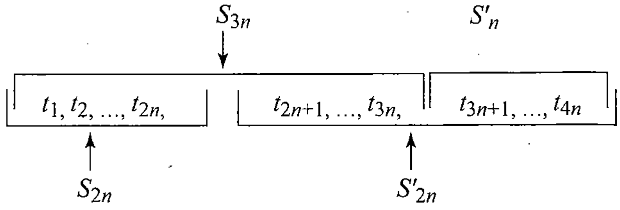

Fig. 3.1

Given,

$$S_{3n} = S_n' = S_{4n} - S_{3n}$$

$\Rightarrow 2S_{3n} = S_{4n}$

$\Rightarrow 2\frac{3n}{2}(2a + (3n-1)d) = \frac{4n}{2}(2a + (4n-1)d)$

$\Rightarrow 12a + (18n - 6)d = 8a + (16n - 4)d$

$\Rightarrow 4a = (-2n + 2)d$

$\Rightarrow 2a = (1 - n)d$ (1)

Now we have to find $\frac{S_{2n}}{S_{2n}'}$.

$$\frac{S_{2n}}{S_{2n}'} = \frac{S_{2n}}{S_{4n} - S_{2n}}$$

$$= \frac{\frac{2n}{2}(2a + (2n-1)d)}{\frac{4n}{2}[2a + (4n-1)d] - \frac{2n}{2}[2a + (2n-1)d]}$$

$$= \frac{2[(1-n)d + (2n-1)d]}{4[(1-n)d + (4n-1)d] - 2[(1-n)d + (2n-1)d]}$$

$$= \frac{2nd}{10nd} = \frac{1}{5}$$

Arithmetic Means

If between a and b, two given quantities, we have to insert n quantities $A_1, A_2, \ldots, A_n$ such that $a, A_1, A_2, \ldots, A_n, b$ forms an A.P., then we say that $A_1, A_2, \ldots, A_n$ are arithmetic means between a and b.

For example, 15, 11, 7, 3, –1, –5 are in A.P. It follows that 11, 7, 3, –1 are four arithmetic means between 15 and –5.

If a, A, b are in A.P., we say that A is the arithmetic mean between a and b.

Insertion of n Arithmetic Means Between a and b

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. Let d be the common difference of this A.P. Clearly, it contains $n + 2$ terms.

$\therefore b = (n+2)^{\text{th}}$ term

$\Rightarrow b = a + (n+1)d$

$\Rightarrow d = \frac{b - a}{n + 1}$

Now,

$$A_1 = a + d \Rightarrow A_1 = \left(a + \frac{b-a}{n+1}\right)$$

$$A_2 = a + 2d \Rightarrow A_2 = \left(a + \frac{2(b-a)}{n+1}\right)$$

$$A_n = a + nd \Rightarrow A_n = \left(a + n\frac{(b-a)}{n+1}\right)$$

These are the required arithmetic means between a and b.

An Important Property of A.M.'s

The sum of n arithmetic means between two numbers is n times the single A.M. between them.

Proof:

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. with common difference $d = \frac{b-a}{n+1}$.

Now,

$$A_1 + A_2 + \cdots + A_n$$

$$= \frac{n}{2}[A_1 + A_n]$$

$$= \frac{n}{2}[a+b]$$

$[\because a, A_1, A_2, \ldots, A_n, b$ is an A.P., $\therefore a + b = A_1 + A_n]$

$$= n\left(\frac{a+b}{2}\right)$$

$= n \times$ (A.M. between a and b)

Example 3.32 **Insert three arithmetic means between 3 and 19.**

Sol. Let A_1, A_2, and A_3 be three A.M.'s between 3 and 19. Then 3, A_1, A_2, A_3, 19 are in A.P. whose common difference is

$$d = \frac{19-3}{3+1} = 4$$

$\therefore \; A_1 = 3 + d = 7$

$\Rightarrow \; A_2 = 3 + 2d = 11$

$A_3 = 3 + 3d = 15$

Hence, the required A.M.'s are 7, 11, 15.

Example 3.33 **If eleven A.M.'s are inserted between 28 and 10, then find the number of integral A.M.'s.**

Sol. Assume $A_1, A_2, A_3, \ldots, A_{11}$ be the eleven A.M.'s between 28 and 10, so $28, A_1, A_2, \ldots, A_{11}$, 10 are in A.P. Let d be the common difference of the A.P. The number of terms is 13. Now,

$$10 = T_{13} = T_1 + 12d = 28 + 12d$$

$$\Rightarrow d = \frac{10-28}{12} = -\frac{18}{12} = -\frac{3}{2}$$

Here integral A.M.'s are

$$28 - 2\left(\frac{3}{2}\right), 28 - 4\left(\frac{3}{2}\right), 28 - 6\left(\frac{3}{2}\right),$$

$$28 - 8\left(\frac{3}{2}\right), 28 - 10\left(\frac{3}{2}\right).$$

Thus, the number of integral A.M.'s is 5.

Example 3.34 **Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7th and $(m-1)$th means is 5:9. Find the value of m.**

Sol. Let $A_1, A_2, \ldots, A_m$ be m arithmetic means between 1 and 31. Then, $1, A_1, A_2, \ldots, A_m$, 31 is an A.P. with common difference

$$d = \frac{31-1}{m+1} = \frac{30}{m+1} \qquad \left[\text{Using } d = \frac{b-a}{n+1}\right]$$

Now,

$$A_7 = 1 + 7d$$

$$\Rightarrow \quad A_7 = 1 + \frac{7 \times 30}{m+1} = \frac{m+211}{m+1}$$

$$A_{m-1} = 1 + (m-1)d$$

$$= 1 + \frac{30}{m+1}(m-1)$$

$$= \frac{31m-29}{m+1}$$

It is given that

$$\frac{A_7}{A_{m-1}} = \frac{5}{9}$$

$$\Rightarrow \quad \frac{m+211}{31m-29} = \frac{5}{9}$$

$\Rightarrow \; 9m + 1899 = 155m - 145$

$\Rightarrow \; 146m = 2044$

$\Rightarrow \; m = 14$

Example 3.35 **For what value of n, $(a^{n+1} + b^{n+1})/(a^n + b^n)$ is the arithmetic mean of a and b?**

Sol. Since A.M. of a and b is $(a + b)/2$, we have

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$

$\Rightarrow \; 2(a^{n+1} + b^{n+1}) = (a^n + b^n)(a + b)$

$\Rightarrow \; 2a^{n+1} + 2b^{n+1} = a^{n+1} + a^n b + b^n a + b^{n+1}$

$\Rightarrow \; a^{n+1} + b^{n+1} = a^n b + b^n a$

$\Rightarrow \; a^n(a - b) = b^n(a - b)$

$\Rightarrow \; a^n = b^n$

$\Rightarrow \; \frac{a^n}{b^n} = 1$

$\Rightarrow \; \left(\frac{a}{b}\right)^n = 1$

$\Rightarrow \; \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0$

$\Rightarrow \; n = 0$

Concept Application Exercise 3.1

1. If the p^{th} term of an A.P. is q and the q^{th} term is p, then find its r^{th} term.
2. If x is a positive real number different from 1, then prove that the numbers $\frac{1}{1+\sqrt{x}}, \frac{1}{1-x}, \frac{1}{1-\sqrt{x}}, \ldots$ are in A.P. Also find their common difference.
3. In a certain A.P., 5 times the 5^{th} term is equal to 8 times the 8^{th} term, then find its 13^{th} term.
4. If $S_n = nP + \frac{n(n-1)}{2}Q$, where S_n denotes the sum of the first n terms of an A.P., then find the common difference.
5. Find the first negative term of the sequence 20, $19\frac{1}{4}$, $18\frac{1}{2}$, $17\frac{3}{4}$, …
6. Solve the equation $(x + 1) + (x + 4) + (x + 7) + \cdots + (x + 28) = 155$.
7. The p^{th} term of an A.P. is a and q^{th} term is b. Then find the sum of its $(p + q)$ terms.
8. The sum of n, $2n$, $3n$ terms of an A.P. are S_1, S_2, S_3, respectively. Prove that $S_3 = 3(S_2 - S_1)$.
9. The ratio of the sums of m and n terms of an A.P. is $m^2{:}n^2$. Show that the ratio of the m^{th} and n^{th} terms is $(2m - 1)$: $(2n - 1)$.
10. Find the number of common terms to the two sequences 17, 21, 25, …, 417 and 16, 21, 26, …, 466.
11. If a, b, c, d are distinct integers in an A.P. such that $d = a^2 + b^2 + c^2$, then find the value of $a + b + c + d$.
12. Let S_n denote the sum of first n terms of an A.P. If $S_{2n} = 3S_n$, then find the ratio S_{3n}/S_n.
13. Find four numbers in an A.P. whose sum is 20 and sum of their squares is 120.
14. Divide 28 into four parts in an A.P. so that the ratio of the product of first and third with the product of second and fourth is 8:15.
15. If $(b - c)^2$, $(c - a)^2$, $(a - b)^2$ are in A.P. then prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are also in A.P.
16. If n arithmetic means are inserted between 2 and 38, then the sum of the resulting series is obtained as 200. Then find the value of n.
17. If a, b, c, d, e, f are A.M.'s between 2 and 12, then find the sum: $a + b + c + d + e + f$.
18. n arithmetic means are inserted between x and $2y$ and then between $2x$ and y. If the r^{th} means in each case be equal, then find the ratio x/y.

GEOMETRIC PROGRESSION (G.P.)

G.P. is a sequence of numbers whose first term is non-zero and each of the succeeding terms is equal to the proceeding terms multiplied by a constant. Thus in a G.P., the ratio of successive terms is constant. This constant factor is called the common ratio of the series and is obtained by dividing any term by that which immediately proceeds it.

If a is the first term and r is the common ratio, then G.P. can be written as $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}$.

n^{th} term: $T_n = ar^{n-1} = l$ (last term), where $r = \frac{T_n}{T_{n-1}}$.

n^{th} term from end: $T_n' = \frac{l}{r^{n-1}}$.

Increasing and Decreasing G.P.

For a G.P. to be increasing or decreasing, $r > 0$. Since if $r < 0$, terms of G.P. are alternately positive and negative and so neither increasing nor decreasing.

If $a > 0$, then G.P. is increasing if $r > 1$ and decreasing if $0 < r < 1$. If $a < 0$, then G.P. is decreasing if $r > 1$ and increasing if $0 < r < 1$. The above discussion can be exhibited as follows:

a	$a > 0$	$a > 0$	$a < 0$	$a < 0$
r	$r > 1$	$0 < r < 1$	$r > 1$	$0 < r < 1$
Result	Increasing	Decreasing	Decreasing	Increasing

Example 3.36 **Which term of the G.P. 2, 1, 1/2, 1/4, … is 1/128?**

Sol. Clearly, the given progression is a G.P. with first term $a = 2$ and common ratio $r = 1/2$. Let the n^{th} term be 1/128. Then,

$$a_n = \frac{1}{128}$$

$$\Rightarrow ar^{n-1} = \frac{1}{128}$$

$$\Rightarrow 2\left(\frac{1}{2}\right)^{n-1} = \frac{1}{128}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^{7}$$

$$\Rightarrow n - 2 = 7$$

$$\Rightarrow n = 9$$

Thus, 9^{th} term of the given G.P. is 1/128.

Example 3.37 **The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.**

Sol. Let r be the common ratio of the G.P. It is given that the first term is $a = 1$. Now,

$$a_3 + a_5 = 90$$

$$\Rightarrow ar^2 + ar^4 = 90$$

$$\Rightarrow r^2 + r^4 = 90$$

$$\Rightarrow r^4 + r^2 - 90 = 0$$

$\Rightarrow r^4 + 10r^2 - 9r^2 - 90 = 0$

$\Rightarrow (r^2 + 10)(r^2 - 9) = 0$

$\Rightarrow r^2 - 9 = 0$

$\Rightarrow r = \pm 3$

Example 3.38 **Fifth term of a G.P. is 2. Find the product of its first nine terms.**

Sol. $t_5 = ar^4 = 2$

Product of its first 9 terms is

$$a(ar)(ar^2)\cdots(ar^8) = a^9 r^{1+2+\cdots+8}$$
$$= a^9 r^{\frac{8}{2}(1+8)}$$
$$= a^9 r^{36}$$
$$= (ar^4)^9 = 2^9 = 512$$

Example 3.39 **The fourth, seventh and the last term of a G.P. are 10, 80 and 2560, respectively. Find the first term and the number of terms in the G.P.**

Sol. Let a be the first term and r be the common ratio of the given G.P. Then,

$$a_4 = 10,\ a_7 = 80 \Rightarrow ar^3 = 10 \text{ and } ar^6 = 80$$

$$\Rightarrow \frac{ar^6}{ar^3} = \frac{80}{10} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Putting $r = 2$ in $ar^3 = 10$, we get $a = 10/8$.

Let there be n terms in the given G.P. Then,

$$a_n = 2560 \Rightarrow ar^{n-1} = 2560$$

$$\Rightarrow \frac{10}{8}(2^{n-1}) = 2560$$

$$\Rightarrow 2^{n-4} = 256 \Rightarrow 2^{n-4} = 2^8$$

$$\Rightarrow n - 4 = 8 \Rightarrow n = 12$$

Example 3.40 **Three numbers are in G.P. If we double the middle term, we get an A.P. Then find the common ratio of the G.P.**

Sol. Let the three numbers in G.P. be a, ar, and ar^2. By the given condition, a, $2ar$, and ar^2 are in A.P. Hence,

$$4ar = a + ar^2$$
$$\Rightarrow 4r = 1 + r^2$$
$$\Rightarrow r^2 - 4r + 1 = 0$$
$$\Rightarrow r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Example 3.41 **If p, q, and r are in A.P., show that the p^{th}, q^{th} and r^{th} terms of any G.P. are in G.P.**

Sol. Let A be the first term and R the common ratio of a G.P. Then, $a_p = AR^{p-1}$, $a_q = AR^{q-1}$ and $a_r = AR^{r-1}$. We have to prove that a_p, a_q, and a_r are in G.P. For this, it is sufficient to show that $(a_q)^2 = a_p a_r$. We have,

$$(a_q)^2 = (AR^{q-1})^2$$
$$= A^2 R^{2q-2}$$
$$= A^2 R^{p+r-2} \quad [\because p, q, r \text{ are in A.P. } \therefore 2q = p + r]$$
$$= (AR^{p-1})(AR^{r-1}) = a_p a_r$$

Hence, a_p, a_q and a_r are in G.P.

Example 3.42 **If a, b, c, and d are in G.P., show that $(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$.**

Sol. Let r be the common ratio of the G.P., a, b, c, d. Then $b = ar$, $c = ar^2$ and $d = ar^3$.

$$\text{L.H.S.} = (ab + bc + cd)^2$$
$$= (aar + arar^2 + ar^2ar^3)^2$$
$$= a^4r^2(1 + r^2 + r^4)^2$$

$$\text{R.H.S.} = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$$
$$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$$
$$= a^2(1 + r^2 + r^4)a^2r^2(1 + r^2 + r^4)$$
$$= a^4r^2(1 + r^2 + r^4)^2$$

$\therefore$ L.H.S. = R.H.S.

Example 3.43 **Three non-zero numbers a, b and c are in A.P. Increasing a by 1 or increasing c by 2, the numbers are a in G.P. Then find b.**

Sol. a, b, and c are in A.P. Hence,

$$2b = a + c \qquad (1)$$

Again by the given condition, $a + 1$, b, and c are in G.P. and a, b, c, and $+ 2$ are in G.P. Hence,

$$b^2 = (a + 1)c \qquad (2)$$

and

$$b^2 = a(c + 2) \qquad (3)$$

By (2) and (3),

$$(a + 1)c = a(c + 2)$$
$$\Rightarrow ac + c = ac + 2a$$
$$\Rightarrow c = 2a$$

Equation (2) gives $b^2 = (a + 1)2a$

Also, Eq. (1) gives

$$2b = a + 2a = 3a$$
$$\Rightarrow b = \frac{3a}{2}$$
$$\Rightarrow \frac{9a^2}{4} = (a + 1)2a$$
$$\Rightarrow \frac{9a}{8} = a + 1$$
$$\Rightarrow a = 8$$
$$\Rightarrow c = 2(8) = 16$$
$$\Rightarrow 2b = 8 + 16 = 24$$
$$\Rightarrow b = 12$$

Example 3.44 **Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.**

Sol. Let the numbers be a, ar, and ar^2. Then,

$$a(1 + r + r^2) = 70 \quad (1)$$

It is given that $4a$, $5ar$, and $4ar^2$ are in A.P. Therefore,

$$2(5ar) = 4a + 4ar^2$$
$$\Rightarrow 5r = 2 + 2r^2$$
$$\Rightarrow 2r^2 - 5r + 2 = 0$$
$$\Rightarrow (2r - 1)(r - 2) = 0$$
$$\Rightarrow r = 2 \text{ or } r = 1/2$$

Putting $r = 2$ in (1), we obtain $a = 10$. Putting $r = 1/2$ in (i), we get $a = 40$. Hence, the numbers are 10, 20, 40 or 40, 20, 10.

Some Important Facts about G.P.

1. If each term of a G.P. is multiplied or divided by some fixed non-zero number, then the resulting sequence is also a G.P.
2. If $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are two G.P.'s, then $x_1y_1, x_2y_2, x_3y_3, \ldots$ and $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}, \cdots$ are also G.P.'s.
3. If $x_1, x_2, x_3, \ldots$ is a G.P. of positive terms, then $\log x_1, \log x_2, \log x_3, \ldots$ is an A.P. and vice versa.
4. Three terms of a G.P. can be taken as a/r, a, ar and four terms in G.P. as a/r^3 a/r, ar, ar^3. This presentation is useful if the product of terms is involved in the problem. In other problems, terms should be taken as $a, ar, ar^2, \ldots$.

Example 3.45 **If the continued product of three numbers in a G.P. is 216 and the sum of their products in pairs is 156, find the numbers.**

Sol. Let the three numbers be a/r, a, and ar. Then, product = 216. Hence, $a/r \times a \times ar = 216 \Rightarrow a^3 = 216 \Rightarrow a = 6$ Sum of the products in pairs is 156. Hence,

$$\frac{a}{r}a + aar + \frac{a}{r}ar = 156$$
$$\Rightarrow a^2\left(\frac{1}{r} + r + 1\right) = 156$$
$$\Rightarrow 36\left(\frac{1 + r^2 + r}{r}\right) = 156$$
$$\Rightarrow 3(r^2 + r + 1) = 13r$$
$$\Rightarrow 3r^2 - 10r + 3 = 0$$
$$\Rightarrow (3r - 1)(r - 3) = 0$$
$$\Rightarrow r = \frac{1}{3} \text{ or } r = 3$$

Hence, putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18.

Sum of *n* Terms of a G.P.

The sum of n terms of a G.P. with first term 'a' and common ratio 'r' is given by

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \text{ or } S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \neq 1$$

Proof:

Let S_n denote the sum of n terms of the G.P. with first term 'a' and common ration 'r'. Then,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \quad (1)$$

Multiplying both sides by r, we get

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \quad (2)$$

On subtracting (2) from (1), we get

$$S_n - rS_n = a - ar^n$$
$$\Rightarrow S_n(1 - r) = a(1 - r^n)$$
$$\Rightarrow S_n = a\left(\frac{1 - r^n}{1 - r}\right)$$
$$\Rightarrow S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

The above formulas do not hold for $r = 1$. For $r = 1$, the sum of n terms of the G.P. is $S_n = na$.

Example 3.46 **Determine the number of terms in a G.P., if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.**

Sol. Let r be the common ratio of the given G.P. Then,

$$a_n = 96$$
$$\Rightarrow a_1 r^{n-1} = 96$$
$$\Rightarrow 3r^{n-1} = 96$$
$$\Rightarrow r^{n-1} = 32 \quad (1)$$

Now,

$$S_n = 189$$
$$\Rightarrow a_1\left(\frac{r^n - 1}{r - 1}\right) = 189$$
$$\Rightarrow 3\left(\frac{(r^{n-1})r - 1}{r - 1}\right) = 189$$
$$\Rightarrow 3\left(\frac{32r - 1}{r - 1}\right) = 189 \quad \text{[Using (1)]}$$
$$\Rightarrow 32r - 1 = 63r - 63$$
$$\Rightarrow 31r = 62$$
$$\Rightarrow r = 2$$

Putting $r = 2$ in (1), we get

$$2^{n-1} = 32$$
$$\Rightarrow 2^{n-1} = 2^5$$
$$\Rightarrow n - 1 = 5$$
$$\Rightarrow n = 6$$

Example 3.47 **Find the sum to n terms of the sequence $(x + 1/x)^2, (x^2 + 1/x^2)^2, (x^3 + 1/x^3)^2, \ldots$.**

Sol. Let S_n denote the sum to n terms of the given sequence. Then,

$$S_n = \left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \cdots + \left(x^n + \frac{1}{x^n}\right)^2$$
$$= \left(x^2 + \frac{1}{x^2} + 2\right) + \left(x^4 + \frac{1}{x^4} + 2\right)$$
$$+ \left(x^6 + \frac{1}{x^6} + 2\right) + \cdots + \left(x^{2n} + \frac{1}{x^{2n}} + 2\right)$$

$$= (x^2 + x^4 + x^6 + \cdots + x^{2n}) + \left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\cdots+\frac{1}{x^{2n}}\right)+\underbrace{(2+2+\cdots)}_{n \text{ times}}$$

$$= x^2\left(\frac{(x^2)^n-1}{x^2-1}\right)+\frac{1}{x^2}\left(\frac{(1/x^2)^n-1}{(1/x^2)-1}\right)+2n$$

$$= x^2\left(\frac{x^{2n}-1}{x^2-1}\right)+\frac{1}{x^{2n}}\left(\frac{1-x^{2n}}{1-x^2}\right)+2n$$

$$= x^2\left(\frac{x^{2n}-1}{x^2-1}\right)+\frac{1}{x^{2n}}\left(\frac{x^{2n}-1}{x^2-1}\right)+2n$$

$$= \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right)+2n$$

Example 3.48 **Prove that the sum to n terms of the series $11 + 103 + 1005 + \cdots$ is $(10/9)(10^n - 1) + n^2$.**

Sol. Let S_n denote the sum to n terms of the given series. Then,

$$S_n = 11 + 103 + 1005 + \cdots \text{ to } n \text{ terms}$$

$$= (10 + 1) + (10^2 + 3) + (10^3 + 5) + \cdots + \{10^n + (2n - 1)\}$$

$$= (10 + 10^2 + \cdots + 10^n) + \{1 + 3 + 5 + \cdots + (2n - 1)\}$$

$$= \frac{10(10^n-1)}{(10-1)} + \frac{n}{2}(1+2n-1)$$

$$= \frac{10}{9}(10^n-1)+n^2$$

Example 3.49 **Find the sum of the following series: $5 + 55 + 555 + \cdots$ to n terms.**

Sol. We have,

$$5 + 55 + 555 + \cdots \text{ to } n \text{ terms}$$

$$= 5[1 + 11 + 111 + \cdots \text{ to } n \text{ terms}]$$

$$= \frac{5}{9}[9+99+999+\cdots \text{ to } n \text{ terms}]$$

$$= \frac{5}{9}[(10-1)+(10^2-1)+(10^3-1)+\cdots+(10^n-1)]$$

$$= \frac{5}{9}[(10+10^2+10^3+\cdots+10^n)-(1+1+1+\cdots n \text{ times})]$$

$$= \frac{5}{9}\left[10\times\frac{(10^n-1)}{10-1}-n\right]$$

$$= \frac{5}{9}\left[\frac{10}{9}(10^n-1)-n\right]$$

$$= \frac{5}{81}[10^{n+1}-10-9n]$$

Important Result

- $a^n - b^n$ is divisible by $a - b$ for any $n \in N$.

$$\frac{a^n-b^n}{a-b} = \frac{a^n\left(1-\left(\frac{b}{a}\right)^n\right)}{a\left(1-\frac{b}{a}\right)}$$

$$= a^{n-1}\left(\frac{1-\left(\frac{b}{a}\right)^n}{1-\frac{b}{a}}\right)$$

$$= a^{n-1}\left(1+\frac{b}{a}+\left(\frac{b}{a}\right)^2+\left(\frac{b}{a}\right)^3+\cdots+\left(\frac{b}{a}\right)^{n-1}\right)$$

$$= a^{n-1} + ba^{n-2} + b^2a^{n-3} + \cdots + b^{n-1}$$

- $a^n + b^n$ is divisible by $a + b$ for odd positive natural numbers.

$$\frac{a^n+b^n}{a+b} = \frac{a^n\left(1-\left(-\frac{b}{a}\right)^n\right)}{a\left(1-\left(-\frac{b}{a}\right)\right)}$$

$$= a^{n-1}\left(\frac{1-\left(-\frac{b}{a}\right)^n}{1-\left(-\frac{b}{a}\right)}\right)$$

$$= a^{n-1}\left(1-\frac{b}{a}+\left(\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)^3+\cdots+(-1)^{n-1}\left(\frac{b}{a}\right)^{n-1}\right)$$

$$= a^{n-1} - ba^{n-2} + b^2a^{n-3} - \cdots + (-1)^{n-1}b^{n-1}$$

Example 3.50 **If $p(x) = (1 + x^2 + x^4 + \cdots + x^{2n-2})(1 + x + x^2 + \cdots + x^{n-1})$ is a polynomial in x, then find possible values of n.**

Sol. $$p(x)=\left(\frac{1-x^{2n}}{1-x^2}\right)\left(\frac{1-x}{1-x^n}\right)=\frac{1+x^n}{1+x}$$

As $p(x)$ is a polynomial, $x = -1$ must be a zero of $1 + x^n = 0$, i.e., $1 + (-1)^n = 0$. Hence, n is odd.

Sum of an Infinite G.P.

The sum of an infinite G.P. with first term a and common ratio $r(-1 < r < 1, r \neq 0$ or $0 < |r| < 1)$ is $S = a/(1 - r)$

Proof:

Consider an infinite G.P. with first term a and common ratio r, where $0 < |r| < 1$. The sum of n terms of the G.P. is given by

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \frac{ar^n}{1-r} \quad (1)$$

Since $0 < |r| < 1$, therefore r^n decreases as n increases and r^n tends to zero as n tends to infinity, i.e., $r^n \to 0$ as $n \to \infty$.

$\therefore \quad \dfrac{ar^n}{1-r} \to 0$ as $n \to \infty$

Hence, from (1), the sum of an infinite G.P. is given by

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right) = \frac{a}{1-r} \text{ if } 0 < |r| < 1$$

Note: *If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.*

Example 3.51 **Find the sum of the following series:**

a. $(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \cdots \infty$

b. $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \cdots \infty$

Sol. a. The given series is a geometric series with first term

$a = \sqrt{2}+1$ and the common ratio

$$r = \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2}-1$$

Hence, the sum to infinity is given by

$$S = \frac{a}{1-r}$$

$$= \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)}$$

$$= \frac{\sqrt{2}+1}{2-\sqrt{2}}$$

$$= \frac{\sqrt{2}+1}{\sqrt{2}(\sqrt{2}-1)}$$

$$= \frac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}+1)}$$

$$= \frac{3+2\sqrt{2}}{\sqrt{2}}$$

$$= \frac{4+3\sqrt{2}}{2}$$

b. We have,

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \cdots \text{ to } \infty$$

$$= \left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right)$$

$$= \left(\frac{(1/2)}{1-(1/2^2)}\right) + \left(\frac{(1/3^2)}{1-(1/3^2)}\right)$$

$$= \frac{2}{3} + \frac{1}{8}$$

$$= \frac{19}{24}$$

Example 3.52 **If each term of an infinite G.P. is twice the sum of the terms following it, then find the common ratio of the G.P.**

Sol. Let a be the first term and r the common ratio of the G.P. Then,

$$a_n = 2[a_{n+1} + a_{n+2} + a_{n+3} + \cdots \infty], \text{ for all } n \in N \quad \text{[Given]}$$

$$\Rightarrow ar^{n-1} = 2[ar^n + ar^{n+1} + \cdots \infty]$$

$$\Rightarrow ar^{n-1} = \frac{2ar^n}{1-r}$$

$$\Rightarrow 1 = \frac{2r}{1-r}$$

$$\Rightarrow r = \frac{1}{3}$$

Example 3.53 **If $x = a + a/r + a/r^2 + \cdots \infty$, $y = b - b/r + b/r^2 - \cdots \infty$ and $z = c + c/r^2 + c/r^4 + \cdots \infty$, then prove that $xy/z = ab/c$.**

Sol. We have,

$$x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}$$

$$y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r}$$

$$z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

$$\therefore \quad xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$$

$$\therefore \quad \frac{xy}{z} = \left[\frac{\frac{abr^2}{r^2-1}}{\frac{cr^2}{r^2-1}}\right] = \frac{ab}{c}$$

Example 3.54 **Prove that $6^{1/2} \times 6^{1/4} \times 6^{1/8} \cdots \infty = 6$.**

Sol. We have,

$$6^{1/2} \times 6^{1/4} \times 6^{1/8} \cdots \infty$$

$$= 6^{[1/2 + 1/4 + 1/8 + \cdots \text{ to } \infty]}$$

$$= 6^{[(1/2)/(1-1/2)]} = 6^1 = 6$$

$$\left[\because \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ to } \infty = \frac{1/2}{1-1/2} = 1\right]$$

Example 3.55 **Sum of infinite number of terms in G.P. is 20 and sum of their squares is 100. Then find the common ratio of G.P.**

Sol. $a + ar + ar^2 + \cdots \text{ to } \infty = 20$

$$\Rightarrow \frac{a}{1-r} = 20 \quad (1)$$

$$a^2 + a^2r^2 + a^2r^4 + \cdots \text{ to } \infty = 100$$

$$\Rightarrow \frac{a^2}{1-r^2} = 100 \qquad (2)$$

Squaring (1), we have

$$\frac{a^2}{(1-r)^2} = 400 \qquad (3)$$

Dividing (3) by (2), we get

$$\frac{\dfrac{a^2}{(1-r)^2}}{\dfrac{a^2}{1-r^2}} = \frac{400}{100}$$

$$\Rightarrow \frac{1-r^2}{(1-r)^2} = 4$$

$$\Rightarrow \frac{1+r}{1-r} = 4$$

$$\Rightarrow 1 + r = 4 - 4r$$

$$\Rightarrow 5r = 3$$

$$\Rightarrow r = \frac{3}{5}$$

Geometric Means (G.M.'s)

Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then the numbers $G_1, G_2, \ldots, G_n$ are known as n G.M.'s between a and b.

If a single G.M. G is inserted between two given numbers a and b, then G is known as the G.M. between a and b. Thus, G is the G.M. between a and b.

$\Leftrightarrow$ a, G, b are in G.P.

$\Leftrightarrow$ $G^2 = ab \Leftrightarrow G = \sqrt{ab}$

For example, the G.M. between 4 and 9 is given by

$$G = \sqrt{4 \times 9} = 6$$

The G.M. between –9 and –4 is given by

$$G = \sqrt{-9 \times -4} = -6$$

Note: *If a and b are two numbers of opposite signs, then geometric mean between them does not exist.*

Insertion of n G.M.'s Between Two Given Numbers a and b

Let $G_1, G_2, \ldots, G_n$ be n G.M. between two given numbers a and b. Then, $a, G_1, G_2, \ldots, G_n, b$ is a G.P. consisting of $n + 2$ terms. Let r be the common ratio of this G.P. Then,

$$b = (n+2)^{\text{th}} \text{ term} = ar^{n+1}$$

$$\Rightarrow r^{n+1} = \frac{b}{a}$$

$$\Rightarrow r = (b/a)^{\frac{1}{n+1}}$$

$$\Rightarrow G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)}$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{n/(n+1)}$$

An Important Property of G.M.'s

If n G.M.'s are inserted between two quantities, then the product of n G.M.'s is the n^{th} power of the single G.M. between the two quantities.

Proof:

Let $G_1, G_2, G_3, \ldots, G_n$ be n G.M.'s between two quantities a and b. Then, $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Let r be the common ratio of this G.P. Then, $r = (b/a)^{1/(n+1)}$ and $G_1 = ar$, $G_2 = ar^2, G_3 = ar^3, \ldots, G_n = ar^n$. Now,

$$G_1 G_2 G_3 \cdots G_n = (ar)(ar^2)(ar^3)\cdots(ar^n)$$

$$= a^n r^{\frac{n(n+1)}{2}}$$

$$= a^n \left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}}$$

$$= a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2} b^{n/2}$$

$$= (\sqrt{ab})^n$$

$$= G^n$$

where $G = \sqrt{ab}$ is the single G.M. between a and b.

Example 3.56 **If G be the geometric mean of x and y, then prove that $1/(G^2 - x^2) + 1/(G^2 - y^2) = 1/G^2$.**

Sol. Given, $G = \sqrt{xy}$

$$\therefore \frac{1}{G^2 - x^2} + \frac{1}{G^2 - y^2}$$

$$= \frac{1}{xy - x^2} + \frac{1}{xy - y^2}$$

$$= \frac{1}{x-y}\left\{-\frac{1}{x} + \frac{1}{y}\right\}$$

$$= \frac{1}{xy} = \frac{1}{G^2}$$

Example 3.57 **Insert four G.M.'s between 2 and 486.**

Sol. Common ratio of the series is given by

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$= \left(\frac{486}{2}\right)^{\frac{1}{4+1}}$$
$$= (243)^{1/5}$$
$$r = 3$$

Hence, the four G.M.'s are 6, 18, 54, 162.

Example 3.58 **Find the product of three geometric means between 4 and 1/4.**

Sol. Product of n G.M.'s is $(\sqrt{ab})^n = \left(\sqrt{4\frac{1}{4}}\right)^3 = 1$

Example 3.59 **Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.**

Sol. Let the two numbers be a and b. Then,

$$\text{A.M.} = 34 \Rightarrow \frac{a+b}{2} = 34 \Rightarrow a + b = 68,$$
$$\text{G.M.} = 16$$
$$\Rightarrow \sqrt{ab} = 16 \Rightarrow ab = 256 \quad (1)$$
$$\therefore (a-b)^2 = (a+b)^2 - 4ab$$
$$\Rightarrow (a-b)^2 = (68)^2 - 4 \times 256 = 3600$$
$$\Rightarrow a - b = 60 \quad (2)$$

On solving (1) and (2), we get $a = 64$ and $b = 4$. Hence, the required numbers are 64 and 4.

Example 3.60 **If the A.M. and G.M. between two numbers is in the ratio $m:n$, then prove that the numbers are in the ratio $(m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2})$.**

Sol. Let the two numbers be a and b. Let A and G, be, respectively, the arithmetic and geometric means between a and b. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$
$$\therefore a + b = 2A \text{ and } G^2 = ab \quad (1)$$

The equation having a, and b as its roots is

$$x^2 - (a+b)x + ab = 0$$

or

$$x^2 - 2Ax + G^2 = 0 \quad \text{[Using (1)]}$$
$$\Rightarrow x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2}$$
$$\Rightarrow x = A \pm \sqrt{A^2 - G^2}$$

So, the two numbers are $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$. It is given that $A:G = m:n$. Therefore, $A = \lambda m$ and $G = \lambda n$ for some λ Substituting the values of A and G in $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$, we get

$$\frac{a}{b} = \frac{\lambda m + \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}{\lambda m - \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}$$
$$\Rightarrow \frac{a}{b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}}$$
$$\Rightarrow a:b = (m+\sqrt{m^2-n^2}):(m-\sqrt{m^2-n^2})$$

Example 3.61 **If a is the A.M. of b and c and the two geometric means are G_1 and G_2, then prove that $G_1^3 + G_2^3 = 2abc$.**

Sol. It is given that a is the A.M. of b and c. So,

$$a = \frac{b+c}{2} \Rightarrow b + c = 2a \quad (1)$$

Since G_1 and G_2 are two geometric means between b and c, so b, G_1, G_2, c is a G.P. with common ratio $r = (c/b)^{1/3}$.

$$\therefore G_1^3 = b^2c \text{ and } G_2^3 = bc^2$$
$$\Rightarrow G_1^3 + G_2^3 = b^2c + bc^2$$
$$\Rightarrow G_1^3 + G_2^3 = bc(b+c)$$
$$\Rightarrow G_1^3 + G_2^3 = 2abc \quad \text{[Using (1)]}$$

Concept Application Exercise 3.2

1. The first and second terms of a G.P. are x^{-4} and x^n, respectively. If x^{52} is the 8th term, then find the value of n.
2. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying odd places, then find the common ratio.
3. If the sum of n terms of a G.P. is $3 - \frac{3^{n+1}}{4^{2n}}$, then find the common ratio.
4. Prove that $(\underbrace{666\cdots6}_{n\text{ digits}})^2 + (\underbrace{888\cdots8}_{n\text{ digits}}) = \underbrace{4444\cdots4}_{2n\text{ digits}}$.
5. Find the sum $(x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \cdots n$ terms.
6. Find the sum of n terms of the series $4/3 + 10/9 + 28/27 + \cdots$.
7. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then find the common ratio.
8. If a, b, c, d are in G.P., then prove that $(a^3+b^3)^{-1}, (b^3+c^3)^{-1}, (c^3+d^3)^{-1}$ are also in G.P.
9. If the sum of the series $\sum_{n=0}^{\infty} r^n$, $|r| < 1$, is s, then find the sum of the series $\sum_{n=0}^{\infty} r^{2n}$.
10. Let T_r denote the r^{th} term of a G.P. for $r = 1, 2, 3, \ldots$. If for some postivie integers m and n, we have $T_m = 1/n^2$ and $T_n = 1/m^2$, then find the value of $T_{(m+n)/2}$.
11. If a, b, and c are in G.P., then prove that $\frac{1}{a^2-b^2} + \frac{1}{b^2} = \frac{1}{b^2-c^2}$.
12. Find the value of $(32)(32)^{1/6}(32)^{1/36}\cdots\infty$.
13. If a, b, and c are, respectively, the p^{th}, q^{th} and r^{th} terms of a G.P., show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.
14. If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \leq 0$, then prove that a, b, c, d are in G.P.
15. The product of three numbers in G.P. is 125 and sum of their products taken in pairs is 87/2. Find them.
16. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the geometric mean of a and b?

If a, b, c, are in H.P., then $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$ or $b=\frac{2ac}{a+c}$, here b is called harmonic mean of a and c.

HARMONIC PROGRESSION (H.P.)

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of non-zero numbers is called a harmonic progression or a harmonic sequence, if the sequence $1/a_1, 1/a_2, 1/a_3, \ldots, 1/a_n, \ldots$ is an arithmetic progression.
If a, b, c are in H.P., then

$$\frac{2}{b}=\frac{1}{a}+\frac{1}{c} \Rightarrow b=\frac{2ac}{a+c}$$

Hence b is called harmonic mean of a and c.

n^{th} Term of a H.P.

The n^{th} term of a H.P. is the reciprocal of the nth term of the corresponding A.P. Thus, if $a_1, a_2, a_3, \ldots, a_n$ is a H.P. and the common difference of the corresponding A.P. is d, i.e., $d = 1/a_{n+1} - 1/a_n$, then the n^{th} term of the H.P. is given by

$$a_n=\frac{1}{\frac{1}{a_1}+(n-1)d}$$

In other words, the n^{th} term of a H.P. is the reciprocal of the n^{th} term of the corresponding A.P.

Example 3.62 **The 8th and 14th term of a H.P. are 1/2 and 1/3, respectively. Find its 20th term. Also, find its general term.**

Sol. Let the H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a+(n-1)d}, \ldots$

Then,

$$a_8=\frac{1}{2} \text{ and } a_{14}=\frac{1}{3}.$$

$$\Rightarrow \frac{1}{a+7d}=\frac{1}{2} \text{ and } \frac{1}{a+13d}=\frac{1}{3} \quad \left[\because a_n=\frac{1}{a+(n-1)d}\right]$$

$$\Rightarrow a+7d=2 \text{ and } a+13d=3$$

$$\Rightarrow a=\frac{5}{6}, d=\frac{1}{6}$$

Now,

$$a_{20}=\frac{1}{a+19d}=\frac{1}{\frac{5}{6}+\frac{19}{6}}=\frac{1}{4}$$

and

$$a_n=\frac{1}{a+(n-1)d}=\frac{1}{\frac{5}{6}+(n-1)\times\frac{1}{6}}=\frac{6}{n+4}$$

Example 3.63 **If the 20th term of a H.P. is 1 and the 30th term is –1/17, then find its largest term.**

Sol. Let the H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$. We have,

$$a_{20}=1 \text{ and } a_{30}=-\frac{1}{17}$$

$$\Rightarrow \frac{1}{a+19d}=1 \text{ and } \frac{1}{a+29d}=\frac{-1}{17}.$$

$$\Rightarrow a+19d=1 \text{ and } a+29d=-17$$

$$\Rightarrow a=\frac{176}{5}, d=-\frac{9}{5}$$

Let n^{th} term be the largest term. We have,

$$a_n=\frac{1}{a+(n-1)d}=\frac{1}{\frac{176}{5}+(n-1)\left(\frac{-9}{5}\right)}=\frac{5}{176-9(n-1)}=\frac{5}{185-9n}$$

Now, a_n is the largest term if $185 - 9n$ is the least positive integer.

Clearly, $185 - 9n$ attains the least positive integral value, if $n = 20$. Thus, 20th term of the given H.P. is the largest term which is equal to 1.

Example 3.64 **If a, b, c, and d are in H.P., then prove that $(b + c + d)/a$, $(c + d + a)/b$, $(d + a + b)/c$, and $(a + b + c)/d$, are in A.P.**

Sol. a, b, c, and d are in H.P.

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \text{ are in A.P.}$$

$$\Rightarrow \frac{a+b+c+d}{a}, \frac{a+b+c+d}{b}, \frac{a+b+c+d}{c}, \frac{a+b+c+d}{d} \text{ are in A.P.}$$

$$\Rightarrow 1+\frac{b+c+d}{a}, \frac{a+c+d}{b}+1, \frac{a+b+d}{c}+1, \frac{a+b+c}{d}+1 \text{ are in A.P.}$$

$$\Rightarrow \frac{b+c+d}{a}, \frac{a+c+d}{b}, \frac{a+b+d}{c}, \frac{a+b+c}{d} \text{ are in A.P.}$$

Example 3.65 **The m^{th} term of a H.P. is n and the n^{th} term is m. Prove that its r^{th} term is mn/r.**

Sol. Let the H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$

Then,

$a_m = n$ and $a_n = m$

$\Rightarrow \dfrac{1}{a+(m-1)d} = n$ and $\dfrac{1}{a+(n-1)d} = m$

$\Rightarrow a+(m-1)d = \dfrac{1}{n}$ and $a+(n-1)d = \dfrac{1}{m}$

$\Rightarrow \{a+(m-1)d\} - \{a+(n-1)d\} = \dfrac{1}{n} - \dfrac{1}{m}$ [On subtracting]

$\Rightarrow (m-n)d = \dfrac{m-n}{mn}$

$\Rightarrow d = \dfrac{1}{mn}$

Putting $d = \dfrac{1}{mn}$ in $a+(m-1)d = \dfrac{1}{n}$, we get

$a + \dfrac{(m-1)}{mn} = \dfrac{1}{n}$

$\Rightarrow a + \dfrac{1}{n} - \dfrac{1}{mn} = \dfrac{1}{n}$

$\Rightarrow a = \dfrac{1}{mn}$

Now,

$$a_r = \frac{1}{a+(r-1)d} = \frac{1}{\frac{1}{mn}+\frac{(r-1)}{mn}} = \frac{mn}{1+r-1} = \frac{mn}{r}$$

Example 3.66 **If $a > 1$, $b > 1$, and $c > 1$ are in G.P., then show that $\dfrac{1}{1+\log_e a}$, $\dfrac{1}{1+\log_e b}$, and $\dfrac{1}{1+\log_e c}$ are in H.P.**

Sol. It is given that a, b, c are in G.P. Hence,

$b^2 = ac$

$\Rightarrow 2\log_e b = \log_e a + \log_e c$

$\Rightarrow \log_e a, \log_e b$, and $\log_e c$ are in A.P.

$\Rightarrow 1+\log_e a$, $1+\log_e b$, and $1+\log_e c$ are in A.P.

$\Rightarrow \dfrac{1}{1+\log_e a}, \dfrac{1}{1+\log_e b}$, and $\dfrac{1}{1+\log_e c}$ are in H.P.

Example 3.67 **If a, b, and c be in G.P. and $a+x, b+x$, and $c+x$ in H.P. then find the value of x (a, b, and c are distinct numbers).**

Sol. $a+x$, $b+x$, and $c+x$ are in H.P.

$\Rightarrow b+x = \dfrac{2(a+x)(c+x)}{(a+x)+(c+x)}$

$\Rightarrow (b+x)(a+c+2x) = 2(a+x)(c+x)$

$\Rightarrow (a+c+2b)x + 2x^2 + ab + bc = 2ac + 2x(a+c) + 2x^2$

$\Rightarrow x(c+a-2b) = bc + ab - 2ac$

$\Rightarrow x(c+a-2b) = bc + ab - 2b^2$ ($\because a, b, c$ are in G.P.)

$\Rightarrow x(c+a-2b) = b(c+a-2b)$

$\Rightarrow x = b$, (as a, b, c, are G.P. and distinct hence a, b, c, cannot be in A.P.)

Example 3.68 **If first three terms of the sequence 1/16, a, b, 1/6 are in geometric series and last three terms are in harmonic series, then find the values of a and b.**

Sol. 1/16, a, b are in G.P. Hence,

$$a^2 = \frac{b}{16} \text{ or } 16a^2 = b \qquad (1)$$

$a, b, \dfrac{1}{6}$ are in H.P. Hence,

$$b = \frac{2a\frac{1}{6}}{a+\frac{1}{6}} = \frac{2a}{6a+1} \qquad (2)$$

From (1) and (2),

$16a^2 = \dfrac{2a}{6a+1}$

$\Rightarrow 2a\left(8a - \dfrac{1}{6a+1}\right) = 0$

$\Rightarrow 8a(6a+1) - 1 = 0$

$\Rightarrow 48a^2 + 8a - 1 = 0$ ($\because a \neq 0$)

$\Rightarrow (4a+1)(12a-1) = 0$

$\therefore a = -\dfrac{1}{4}, \dfrac{1}{12}$

When $a = -1/4$, then from (1),

$b = 16\left(-\dfrac{1}{4}\right)^2 = 1$

When $a = 1/12$, then from (1),

$b = 16\left(\dfrac{1}{12}\right)^2 = \dfrac{1}{9}$.

Therefore, $a = -1/4$, $b = 1$ or $a = 1/12$, $b = 1/9$.

Harmonic Means

Let a and b be two given numbers. If n numbers $H_1, H_2, \ldots, H_n$ are inserted between a and b such that the sequence a, $H_1, H_2, H_3, \ldots, H_n, b$ is a H.P., then $H_1, H_2, \ldots, H_n$ are called n harmonic means between a and b. Now, $a, H_1, H_2, \ldots, H_n, b$ are in H.P. Hence,

$\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}; \ldots, \dfrac{1}{H_n}, \dfrac{1}{b}$ are in A.P.

Let D be the common difference of this A.P. Then,

$\dfrac{1}{b} = (n+2)^{\text{th}}$ term

$\Rightarrow \quad \frac{1}{b} = \frac{1}{a} + (n+1)D$

$\Rightarrow \quad D = \frac{a-b}{(n+1)ab}$

Thus, if n harmonic means are inserted between two given numbers a and b, then the common diference of the corresponding A.P. is given by

$$D = \frac{a-b}{(n+1)ab}$$

Also,

$$\frac{1}{H_1} = \frac{1}{a} + D, \frac{1}{H_2} = \frac{1}{a} + 2D, \ldots, \frac{1}{H_n} = \frac{1}{a} + nD$$

On putting the value of D, we can obtain the values of H_1, H_2, ..., H_n.

Harmonic Means of Two Given Numbers

If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the sequence a, H, and b is a H.P. Now, a, H, and b is a H.P. Hence,

$\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ is an A.P.

$\Rightarrow \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b}$

$\Rightarrow \quad H = \frac{2ab}{a+b}$

Thus, the harmonic mean H between two numbers a and b is given by $H = (2ab)/(a+b)$.

Example 3.69 **Insert four H.M.'s between 2/3 and 2/13.**

Sol. Let d be the common difference of corresponding A.P. So,

$$d = \frac{\frac{13}{2} - \frac{3}{2}}{5} = 1$$

$\therefore \quad \frac{1}{H_1} = \frac{3}{2} + 1 = \frac{5}{2}$ or $H_1 = \frac{2}{5}$

$\frac{1}{H_2} = \frac{3}{2} + 2 = \frac{7}{2}$ or $H_2 = \frac{2}{7}$

$\frac{1}{H_3} = \frac{3}{2} + 3 = \frac{9}{2}$ or $H_3 = \frac{2}{9}$

$\frac{1}{H_4} = \frac{3}{2} + 4 = \frac{11}{2}$ or $H_4 = \frac{2}{11}$

Example 3.70 **If H is the harmonic mean between P and Q, then find the value of $H/P + H/Q$.**

Sol. $\frac{H}{P} + \frac{H}{Q} = H\left(\frac{1}{P} + \frac{1}{Q}\right)$

$$= \frac{2PQ}{P+Q}\frac{P+Q}{PQ} = 2$$

Example 3.71 **If nine arithmetic means and 9 harmonic means are inserted between 2 and 3 alternatively, then prove that $A + 6/H = 5$ (where A is any of the A.M.'s and H the corresponding H.M.).**

Sol. Let $H_1, H_2, \ldots, H_9$ be nine harmonic means between 2 and 3. Then 2, $H_1, H_2, \ldots, H_9$, 3 are in H.P. Therefore, 1/2, $1/H_1$, $1/H_2$, $\cdots$, $1/H_9$, 1/3 are in A.P. with common difference

$$D = \frac{2-3}{(9+1) \times 2 \times 3} = -\frac{1}{60} \quad \left[\because D = \frac{a-b}{(n+1)ab}\right]$$

$\therefore \quad \frac{1}{H_i} = \frac{1}{2} + iD; \; i = 1, 2, 3, \ldots, 9$

$\Rightarrow \quad \frac{1}{H_i} = \frac{1}{2} - \frac{i}{60}$

$\Rightarrow \quad \frac{6}{H_i} = 3 - \frac{i}{10} \qquad (1)$

Let $A_1, A_2, \ldots, A_9$ be 9 A.M.'s between 2 and 3. Then, 2, $A_1, A_2, \ldots, A_9$, 3 are in A.P. with common difference

$$d = \frac{3-2}{9+1} = \frac{1}{10} \quad \left[\because d = \frac{b-a}{n+1}\right]$$

$\therefore \quad A_i = 2 + id; \; i = 1, 2, \ldots, 9$

$\Rightarrow \quad A_i = 2 + \frac{i}{10} \qquad (2)$

From (1) and (2), we have

$$A_i + \frac{6}{H_i} = 3 - \frac{i}{10} + 2 + \frac{i}{10} = 5 \text{ for } i = 1, 2, \ldots, 9$$

Concept Application Exercise 3.3

1. If the first two terms of a H.P. are 2/5 and 12/13, respectively. Then find the largest term.
2. If a, b, c are in G.P. and $a-b$, $c-a$ and $b-c$ are in H.P., then prove that $a + 4b + c$ is equal to 0.
3. If x, y, and z are in A.P., ax, by, and cz in G.P. and a, b, c in H.P., then prove that $\frac{x}{z} + \frac{z}{x} = \frac{a}{c} + \frac{c}{a}$.
4. If a, b, c, and d are in H.P., then find the value of $\frac{a^{-2} - d^{-2}}{b^{-2} - c^{-2}}$.
5. If $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ and p, q, and r be in A.P., then prove that x, y, z are in H.P.
6. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$, where a, b, and c are in A.P. and $|a| < 1$, $|b| < 1$, and $|c| < 1$, then prove that x, y, and z are in H.P.
7. If x, 1, and z are in A.P. and x, 2, and z are in G.P., then prove that x, and 4, z are in H.P.

8. If $a, a_1, a_2, a_3, \ldots, a_{2n}, b$ are in A.P. and $a, g_1, g_2, g_3, \ldots, g_{2n}, b$ are in G.P. and h is the H.M. of a and b, then prove that $\frac{a_1+a_{2n}}{g_1g_{2n}}+\frac{a_2+a_{2n-1}}{g_2g_{2n-1}}+\cdots+\frac{a_n+a_{n+1}}{g_ng_{n+1}}=\frac{2n}{h}$
9. If the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ terms of an A.P. are in G.P. and m, n, r are in H.P., then find the value of the ratio of the common difference to the first term of the A.P.
10. If the sum of the roots of the quadratic equation $ax^2+bx+c=0$ is equal to the sum of the squares of their reciprocals, then prove that $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in H.P.

Properties of A.M., G.M. and H.M. of Two Positive Real Numbers

Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b. Then,

$$A=\frac{a+b}{2},\ G=\sqrt{ab} \text{ and } H=\frac{2ab}{a+b}$$

These three means possess the following properties.

1. $A \geq G \geq H$

Proof: We have,

$$A=\frac{a+b}{2},\ G=\sqrt{ab} \text{ and } H=\frac{2ab}{a+b}$$

$$\Rightarrow A-G=\frac{a+b}{2}-\sqrt{ab}=\frac{(\sqrt{a}-\sqrt{b})^2}{2}\geq 0$$

$$\Rightarrow A\geq G \quad \text{(i)}$$

$$G-H=\sqrt{ab}-\frac{2ab}{a+b}$$

$$=\sqrt{ab}\left\{\frac{a+b-2\sqrt{ab}}{a+b}\right\}$$

$$=\frac{\sqrt{ab}}{a+b}(\sqrt{a}-\sqrt{b})^2\geq 0$$

$$\Rightarrow G\geq H \quad \text{(ii)}$$

From (i) and (ii), we get $A \geq G \geq H$.

Note: *The equality holds in the above result only when a = b.*

A.M., G.M. and H.M. of n positive quantities, $a_1, a_2, a_3, \ldots, a_n$ is given by

$$A=\frac{a_1+a_2+a_3+\cdots+a_n}{n}$$

$$G=(a_1a_2a_3\cdots a_n)^{1/n}$$

$$\text{and } H=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}$$

$A \geq G \geq H$ also holds here.

2. A, G, and H form a G.P., i.e., $G^2=AH$.

Proof: We have,

$$\therefore AH=\frac{a+b}{2}\times\frac{2ab}{a+b}$$

$$=ab=(\sqrt{ab})^2=G^2$$

Hence, $G^2=AH$..

3. The equation having a and b as its roots is $x^2-2Ax+G^2=0$.

Proof: The equation having a and b as its roots is

$$x^2-(a+b)x+ab=0$$

$$\Rightarrow x^2-2Ax+G^2=0 \quad \left[\because A=\frac{a+b}{2} \text{ and } G=\sqrt{ab}\right]$$

4. If A and G be the A.M. and G.M. between two positive numbers, then the numbers are $A\pm\sqrt{A^2-G^2}$.

Proof: The equation having its roots as the given numbers is

$$x^2-2Ax+G^2=0$$

$$\Rightarrow x=\frac{2A\pm\sqrt{4A^2-4G^2}}{2}$$

$$=A\pm\sqrt{A^2+G^2}$$

5. If A, G, and H are arithmetic, geometric and harmonic means between three given numbers a, b and c, then the equation having a, b, and c as its roots is

$$x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$$

Proof: We have,

$$A=\frac{a+b+c}{3},\ G=(abc)^{1/3},$$

$$\frac{1}{H}=\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

$$\therefore a+b+c=3A,\ abc=G^3$$

and

$$\frac{3G^3}{H}=ab+bc+ca$$

The equation having a, b, and c as its roots is

$$x^3-(a+b+c)x^2+(ab+bc+ca)x-abc=0$$

$$\Rightarrow x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$$

Example 3.72 **The A.M. and H.M. between two numbers are 27 and 12, respectively, then find their G.M.**

Sol. Let A, G and H denote, respectively, the A.M., G.M. and H.M. between the two numbers. Then, $A=27$ and $H=12$. Since A, G, and H are in G.P. Therefore,

$$G^2=AH$$

$= 27 \times 12$

$\Rightarrow \quad G = 18$

Example 3.73 **If the A.M. between two numbers exceeds their G.M. by 2 and the G.M. exceeds their H.M. by 8/5, find the numbers.**

Sol. $A - G = 2$ (1)

$G - H = 8/5$ (2)

$G^2 = AH$

$= (G + 2)(G - 8/5)$

$\Rightarrow \quad G = 8$

$\Rightarrow \quad ab = 64$ (3)

From (1),

$A = 10$

$\Rightarrow \quad a + b = 20$ (4)

Solving (3) and (4), we get $a = 4$ and $b = 16$ or $a = 16$ and $b = 4$.

Concept Application Exercise 3.4

1. If the arithmetic mean of two positive numbers a and b ($a > b$) is twice their geometric mean, then find the ratio $a:b$.
2. The A.M. of two given positive numbers is 2. If the larger number is increased by 1, the G.M. of the numbers becomes equal to the A.M. of the given numbers. Then find the H.M.
3. The harmonic mean between two numbers is 21/5, their A.M. 'A' and G.M. 'G' satisfy the relation $3A + G^2 = 36$. Then find the sum of square of numbers.

MISCELLANEOUS SERIES

Arithmetico-Geometric Sequence

Let a, $(a + d)r$, $(a + 2d)r^2$, $(a + 3d)r^3$, ... be an arithmetico-geometric sequence. Then, $a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + \cdots$ is an arithmetico-geometric series.

Sum of n Terms of an Arithmetico-geometric Sequence

Sum of n terms of an arithmetico-geometric sequence a, $(a + d)r$, $(a + 2d)r^2$, $(a + 3d)r^3$, ... is given by

$$S_n = \begin{cases} \dfrac{a}{1-r} + dr\dfrac{(1-r^{n-1})}{(1-r)^2} - \dfrac{\{a+(n-1)d\}r^n}{1-r}, & \text{when } r \neq 1 \\ \dfrac{n}{2}[2a+(n-1)d], & \text{when } r = 1 \end{cases}$$

Proof:

Let S_n denote the sum of n terms of the given sequence. Then,

$$S_n = a + (a + d)r + (a + 2d)r^2 + \cdots + \{a + (n-1)d\}r^{n-1} \quad (1)$$

$$\Rightarrow rS_n = ar + (a + d)r^2 + \cdots + \{a + (n-2)d\}r^{n-1} + \{a + (n-1)d\}r^n \quad (2)$$

Subtracting (2) from (1), we get

$$S_n - rS_n = a + [dr + dr^2 + \cdots + dr^{n-1}] - \{a + (n-1)d\}\, r^n$$

$$\Rightarrow S_n(1 - r) = a + dr\left(\frac{1-r^{n-1}}{1-r}\right) - [a+(n-1)d]r^n$$

$$\Rightarrow S_n = \frac{a}{1-r} + dr\frac{1-r^{n-1}}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r} \quad (3)$$

Note: *Generally we dont use this formula to find sum of n terms infact we use the mechanism by which we derived this formula.*

Example 3.74 **Find the sum of the series $1 + 3x + 5x^2 + 7x^3 + \cdots$ to n terms.**

Sol. The given series is an arithmetico-geometric series whose corresponding A.P. and G.P. are 1, 3, 5, 7, ... and 1, x, x^2, ..., respectively. The n^{th} term of A.P. is $[1 + (n - 1) \times 2] = (2n - 1)$. The n^{th} term of G.P. is $[1 \times x^{n-1}] = x^{n-1}$. So, the n^{th} term of the given series is $(2n - 1)x^{n-1}$. Let S_n denote the sum of n terms of the given series. Then,

$$S_n = 1 + 3x + 5x^2 + 7x^3 + \cdots + (2n-3)\, x^{n-2} + (2n-1)x^{n-1} \quad (1)$$

$$\therefore \quad xS_n = x + 3x^2 + 5x^3 + \cdots + (2n-3)\, x^{n-1} + (2n-1)\, x^n \quad (2)$$

Subtracting (2) from (1), we have

$$S_n - xS_n = 1 + [2x + 2x^2 + 2x^3 + \cdots + 2x^{n-1}] - (2n-1)x^n$$

$$\Rightarrow S_n(1 - x) = 1 + 2x\left(\frac{1-x^{n-1}}{1-x}\right) - (2n-1)x^n$$

$$\Rightarrow S_n = \frac{1}{1-x} + 2x\frac{(1-x^{n-1})}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$$

Example 3.75 **Find the sum of n terms of the series $1 + 4/5 + 7/5^2 + 10/5^3 + \cdots$.**

Sol. Clearly, the given series is an arithmetico-geometric series whose corresponding A.P. and G.P. are, respectively, 1, 4, 7, 10, ... and 1, 1/5, $1/5^2$, $1/5^3$, ...

The n^{th} term of A.P. is $[1 + (n - 1) \times 3] = 3n - 2$.

The n^{th} term of G.P. is $[1 \times (1/5)^{n-1}] = (1/5)^{n-1}$.

So, the n^{th} term of the given series is $(3n - 2) \times (1/5^{\,n-1}) = (3n - 2)/5^{\,n-1}$. Let,

$$S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots + \frac{3n-5}{5^{n-2}} + \frac{3n-2}{5^{n-1}} \quad (1)$$

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots + \frac{(3n-5)}{5^{n-1}} + \frac{3n-2}{5^n} \quad (2)$$

Subtracting (2) from (1), we get

$$S_n - \frac{1}{5}S_n = 1 + \left[\frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^2} + \cdots + \frac{3}{5^{n-1}}\right] - \frac{(3n-2)}{5^n}$$

$$\Rightarrow \quad \frac{4}{5}S_n = 1+\frac{3}{5}\frac{\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{\left(1-\frac{1}{5}\right)} - \frac{(3n-2)}{5^n}$$

$$= 1+\frac{3}{5}\frac{\left[1-\frac{1}{5^{n-1}}\right]}{\left(\frac{4}{5}\right)}-\frac{(3n-2)}{5^n}$$

$$= 1+\frac{3}{4}\left(1-\frac{1}{5^{n-1}}\right)-\frac{(3n-2)}{5^n}$$

$$\Rightarrow \quad S_n = \frac{5}{4}+\frac{15}{16}\left(1-\frac{1}{5^{n-1}}\right)-\frac{(3n-2)}{4(5^{n-1})}$$

Sum of an Infinite Arithmetico-geometric Sequence

If $|r| < 1$, then $r^n, r^{n-1} \to 0$ as $n \to \infty$ and it can also be shown that $nr^n \to 0$ as $n \to \infty$. So we have

$$S_n \to \frac{a}{1-r}+\frac{dr}{(1-r)^2}, \text{ as } n \to \infty$$

In other words, when $|r| < 1$, the sum to infinity of an arithmetico-geometric series is

$$S_\infty = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

Example 3.76 **Find the sum to infinity of the series $1 - 3x + 5x^2 + 7x^3 + \cdots \infty$ when $|x| < 1$.**

Sol. Let S_∞ denote the sum of the given infinite series. Now,

$$S_\infty = 1 - 3x + 5x^2 - 7x^3 + \cdots \infty$$
$$= 1 + 3(-x) + 5(-x)^2 + 7(-x)^3 + \cdots \infty$$

Here, $a = 1$, $r = -x$ and $d = 2$. Hence,

$$S_\infty = \frac{1}{1-(-x)}+\frac{2(-x)}{[1-(-x)]^2}$$

$$= \frac{1}{1+x}-\frac{2}{(1+x)^2}$$

$$= \frac{1-x}{(1+x)^2}$$

Sum to Infinity of the Series Reducible to Arithmetico-geometric Series

Example 3.77 **Find the sum to infinity of the series $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots \infty$.**

Sol. The given series is not an arithmetico-geometric series, because $1^2, 2^2, 3^2, 4^2, \ldots$ are not in A.P. However, their successive differences $(2^2 - 1^2), (3^2 - 2^2), (4^2 - 3^2), \ldots$, i.e., $3, 5, 7, \ldots$ form an A.P. So, the process of finding the sum to infinity of an arithmetico-geometric series will be repeated twice as given below. Let,

$$S_\infty = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots \infty \quad (1)$$

$$\Rightarrow \quad xS_\infty = 1^2x + 2^2x^2 + 3^2x^3 + \cdots \infty \quad (2)$$

Subtracting (2) from (1), we get

$$(1-x)S_\infty = 1^2 + (2^2 - 1^2)x + (3^2 - 2^2)x^2 + (4^2 - 3^2)x^3 + \cdots$$

$$\Rightarrow \quad (1-x)S_\infty = 1 + 3x + 5x^2 + 7x^3 + \cdots \quad (3)$$

This is an arithmetico-geometric series in which $a = 1$, $d = 2$, $r = x$.

$$\therefore \quad (1-x)S_\infty = \frac{1}{1-x}+\frac{2x}{(1-x)^2} = \frac{1+x}{(1-x)^2}$$

$$\Rightarrow \quad S_\infty = \frac{1+x}{(1-x)^3}.$$

Summation by Sigma (Σ) Operator

Properties of Sigma Operator

1. $\sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \cdots + T_n$, where T_r is the general term of the series.
2. $\sum_{r=1}^{n}(T_r \pm T_r') = \sum_{r=1}^{n} T_r \pm \sum_{r=1}^{n} T_r'$ (sigma operator is distributive over addition and subtraction)
3. $\sum_{r=1}^{n} T_r T_r' \neq \left(\sum_{r=1}^{n} T_r\right)\left(\sum_{r=1}^{n} T_r'\right)$ (sigma operator is not distributive over multiplication)
4. $\sum_{r=1}^{n} \frac{T_r}{T_r'} \neq \frac{\sum_{r=1}^{n} T_r}{\sum_{r=1}^{n} T_r'}$ (sigma operator is not distributive over division)
5. $\sum_{r=1}^{n} 1 = 1 + 1 + 1 + \cdots n \text{ times} = n$
6. $\sum_{r=1}^{n} aT_r = a\sum_{r=1}^{n} T_r$ (where a is constant)
7. $\sum_{j=1}^{n}\sum_{i=1}^{n} T_i T_j = \left(\sum_{i=1}^{n} T_i\right)\left(\sum_{j=1}^{n} T_j\right)$ (here i and j are independent)

Sum of the Squares of the First n Natural Numbers

Let the sum be denoted by S; then $S = 1^2 + 2^2 + 3^2 + \cdots + n^2$.

We have, $n^3 - (n-1)^3 = 3n^2 - 3n + 1$; and by changing n to $n - 1$, we get

$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$
$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$$
$$\vdots$$
$$3^3 - 2^3 = 3 \times 3^2 - 3 \times 3 + 1$$
$$2^3 - 1^2 = 3 \times 2^2 - 3 \times 2 + 1$$
$$1^2 - 0^2 = 3 \times 1^2 - 3 \times 1 + 1$$

Hence, by addition,

$$n^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) - 3(1 + 2 + 3 + \cdots + n) + n$$
$$= 3S - \frac{3n(n+1)}{2} + n$$

$$\Rightarrow \quad 3S = n^3 - n + \frac{3n(n+1)}{2}$$

$$= n(n+1)\left(n-1+\frac{3}{2}\right)$$

$$\Rightarrow \quad S = \frac{n(n+1)(2n+1)}{6}$$

Sum of the Cubes of the First n Natural Numbers

Let the sum be denoted by S; then $S = 1^3 + 2^3 + 3^3 + \cdots + n^3$.

We have,

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$$
$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$
$$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1$$
$$\vdots$$
$$3^4 - 2^4 = 4 \times 3^3 - 6 \times 3^2 + 4 \times 3 - 1$$
$$2^4 - 1^4 = 4 \times 2^3 - 6 \times 2^2 + 4 \times 2 - 1$$
$$1^4 - 0^4 = 4 \times 1^3 - 6 \times 1^2 + 4 \times 1 - 1$$

Hence, by addition,

$$n^4 = 4S - 6(1^2 + 2^2 + \cdots + n^2) + 4(1 + 2 + \cdots + n) - n$$

$$\therefore \quad 4S = n^4 + n + 6(1^2 + 2^2 + \cdots + n^2) - 4(1 + 2 + \cdots + n)$$
$$= n^4 + n + n(n+1)(2n+1) - 2n(n+1)$$
$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2)$$
$$= n(n+1)(n^2 + n)$$

$$\therefore \quad S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2$$

Thus, the sum of the cubes of the first n natural numbers is equal to the square of the sum of these numbers.

Example 3.78 **Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots$.**

Sol. $T_n = n(n+1)(n+2)$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{4}\{n^2 + n + 4n + 2 + 4\}$$

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Example 3.79 **Find the sum of the series $1 \times n + 2(n-1) + 3 \times (n-2) + \cdots + (n-1) \times 2 + n \times 1$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = r \times \{n - (r-1)\}$$
$$= r(n - r + 1)$$
$$= r\{(n+1) - r\}$$
$$= (n+1)r - r^2$$

$$\therefore \quad \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left[(n+1)r - r^2\right]$$

$$= (n+1)\left(\sum_{r=1}^{n} r\right) - \left(\sum_{r=1}^{n} r^2\right)$$

$$= (n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)(n+2)}{6}$$

Example 3.80 **Find the sum of the series:**

$$\frac{1^2}{1} + \frac{1^2+2^2}{1+2} + \frac{1^2+2^2+3^2}{1+2+3} + \cdots n \text{ terms.}$$

Sol. $T_r = \dfrac{1^2 + 2^2 + \cdots + r^2}{1 + 2 + \cdots + r}$

$$= \frac{r(r+1)(2r+1)2}{6r(r+1)}$$

$$= \frac{1}{3}(2r+1)$$

$$\Rightarrow \quad \sum_{r=1}^{n} T_r = \frac{2}{3}\left(\sum_{r=1}^{n} r\right) + \frac{n}{3}$$
$$= \frac{1}{3}n(n+1) + \frac{n}{3}$$

$$= \frac{n(n+2)}{3}$$

Example 3.81 **Find the sum of the series $31^3 + 32^3 + \cdots + 50^3$.**

Sol. $S = (1^3 + 2^3 + \cdots + 50^3) - (1^3 + 2^3 + \cdots + 30^3)$

$$= \left(\frac{50 \times 51}{2}\right)^2 - \left(\frac{30 \times 31}{2}\right)^2$$

$$\left[\text{Using } \Sigma n^3 = \left(\frac{n(n+1)}{2}\right)^2\right]$$

$$= \frac{1}{4}(50 \times 51 - 30 \times 31)(50 \times 51 + 30 \times 31)$$

$$= 1409400$$

Example 3.82 **Find the sum of first n terms of the series $1^3 + 3 \times 2^2 + 3^3 + 3 \times 4^2 + 5^3 + 3 \times 6^2 + \cdots$ when**

a. n is even

b. n is odd

Sol. a. n is even. Let $n = 2m$.

$$S_n = S_{2m} = \sum_{r=1}^{m}(2r-1)^3 + 3\sum_{r=1}^{m}(2r)^2$$

$$= \sum_{r=1}^{m}[8r^3 - 3(2r)^2 + 3(2r) - 1)] + 12\sum_{r=1}^{m} r^2$$

$$= 8\sum_{r=1}^{m} r^3 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= 2m^2 (m + 1)^2 + 3m(m + 1) - m$$

$$= m[2m^3 + 4m^2 + 5m + 2]$$

Put $2m = n$ or $m = n/2$.

$$\therefore\ S_n = \frac{n}{8}[n^3 + 4n^2 + 10n + 8] \quad \text{(i)}$$

b. If n is odd, then $n + 1$ is even. Now,

$$S_n = S_{n+1} - T_{n+1} \quad \text{(ii)}$$

S_{n+1} is obtained from (1) by replacing n by $n + 1$ and $T_{n+1} = (n + 1)^{\text{th}}$ even term $= 3(n + 1)^2$. Hence from (ii),

$$S_n\,(n \text{ odd}) = \frac{n+1}{8}[(n+1)^3 + 4(n+1)^2 + 10(n+1) + 8] - 3(n+1)^2$$

$$S_n = \frac{n+1}{8}[n^3 + 7n^2 - 3n - 1] \quad \text{(iii)}$$

Equations (i) and (iii) give the required results.

Example 3.83 **Find the sum to n terms of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots$.**

Sol. Clearly, n^{th} term of the given series is negative or positive accordingly as n is even or odd, respectively.

a. n is even:

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots + (n - 1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \cdots + ((n - 1)^2 - n^2)$$

$$= (1 - 2)(1 + 2) + (3 - 4)(3 + 4) + (5 - 6)(5 + 6) + \cdots + ((n - 1) - (n))(n - 1 + n)$$

$$= -(1 + 2 + 3 + 4 + \cdots + (n - 1) + n)$$

$$= -\frac{n(n + 1)}{2}$$

b. n is odd:

$$(1^2 - 2^2) + (3^2 - 4^2) + \cdots + \{(n - 2)^2 - (n - 1)^2\} + n^2$$

$$= (1 - 2)(1 + 2) + (3 - 4)(3 + 4) + \cdots + [(n - 2) - (n - 1)][(n - 2) + (n - 1)] + n^2$$

$$= -(1 + 2 + 3 + 4 + \cdots + (n - 2) + (n - 1)) + n^2$$

$$= -\frac{(n - 1)(n - 1 + 1)}{2} + n^2$$

$$= \frac{n(n + 1)}{2}$$

Getting n^{th} Term T_n from Sum of n Terms

Example 3.84 **If $\sum_{r=1}^{n} T_r = n(2n^2 + 9n + 13)$, then find the sum $\sum_{r=1}^{n} \sqrt{T_r}$.**

Sol. $S_n = \sum_{r=1}^{n} T_r = n(2n^2 + 9n + 13)$

$$\Rightarrow\ T_r = S_r - S_{r-1}$$

$$= r(2r^2 + 9r + 13) - (r - 1)(2(r - 1)^2 + 9(r - 1) + 13)$$

$$= 6r^2 + 12r + 6 = 6(r + 1)^2$$

$$\Rightarrow\ \sqrt{T_r} = \sqrt{6}(r + 1)$$

$$\Rightarrow\ \sum_{r=1}^{n} \sqrt{T_r} = \sqrt{6}\sum_{r=1}^{n}(r + 1)$$

$$= \sqrt{6}\left(\frac{n^2 + 3n}{2}\right)$$

$$= \sqrt{\frac{3}{2}}(n^2 + 3n)$$

Example 3.85 **If $\sum_{r=1}^{n} T_r = (3^n - 1)$, then find the sum of $\sum_{r=1}^{n} \frac{1}{T_r}$.**

Sol. $\sum_{r=1}^{n} T_r = 3^n - 1$

$$\Rightarrow\ T_r = (3^r - 1) - (3^{r-1} - 1) = 3^{r-1}(3 - 1) = 2(3^{r-1})$$

$$\Rightarrow\ \frac{1}{T_r} = \frac{1}{2} \times \left(\frac{1}{3}\right)^{r-1}$$

$$\Rightarrow\ \sum_{r=1}^{n} \frac{1}{T_r} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{r-1}$$

$$= \frac{1}{2}\,\frac{\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}}$$

$$= \frac{3}{4}\left(1 - \left(\frac{1}{3}\right)^n\right)$$

Sum of Series by Method of Difference

Sometimes, the n^{th} term of a series cannot be determined by the methods discussed so far. If a series is such that the difference between successive terms are either in A.P. or in G.P., then we determine its n^{th} term by the method of difference and then find the sum of the series by using the formulas for Σn, Σn^2 and Σn^3. The method of difference is illustrated in the following examples.

Example 3.86 **Find the sum to n terms of the series $3 + 15 + 35 + 63 + \cdots$.**

Sol. The differences between the successive terms are $15 - 3 = 12$, $35 - 15 = 20$, $63 - 35 = 28$, ... Clearly, these differences are in A.P. Let T_n be the n^{th} term and S_n denote the sum to n terms of the given series. Then,

$$S_n = 3 + 15 + 35 + 63 + \cdots + T_{n-1} + T_n \quad (1)$$

$$S_n = \quad 3 + 15 + 35 + 63 \cdots + T_{n-1} + T_n \quad (2)$$

$$0 = 3 + [12 + 20 + 28 + \cdots + (n-1) \text{ terms}] - T_n$$

[Subtracting (2) from (1)]

$$\Rightarrow T_n = 3 + \frac{(n-1)}{2}[2\times 12 + (n-1-1)\times 8]$$

$$= 3 + (n-1)(12 + 4n - 8)$$

$$= 3 + (n-1)(4n+4)$$

$$= 4n^2 - 1$$

$$\Rightarrow S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(4k^2 - 1)$$

$$= 4\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} 1$$

$$= 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} - n$$

$$= \frac{n}{3}(4n^2 + 6n - 1)$$

Example 3.87 **Find the sum of the following series to n terms $5 + 7 + 13 + 31 + 85 + \cdots$.**

Sol. The sequence of differences between successive terms is 2, 6, 18, 54, ... Clearly, it is a G.P. Let T_n be the n^{th} term of the given series and S_n be the sum of its n terms. Then,

$$S_n = 5 + 7 + 13 + 31 + \cdots + T_{n-1} + T_n \quad (1)$$

$$S_n = \quad 5 + 7 + 13 + 31 + \cdots + T_{n-1} + T_n \quad (2)$$

$$0 = 5 + [2 + 6 + 18 + \cdots + (n-1) \text{ terms}] - T_n$$

[Subtracting (2) from (1)]

$$\Rightarrow 0 = 5 + 2\,\frac{(3^{n-1} - 1)}{(3-1)} - T_n$$

$$\Rightarrow T_n = 5 + (3^{n-1} - 1) = 4 + 3^{n-1}$$

$$\therefore S_n = \sum_{k=1}^{n} T_k$$

$$= \sum_{k=1}^{n}(4 + 3^{k-1})$$

$$= \sum_{k=1}^{n} 4 + \sum_{k=1}^{n} 3^{k-1}$$

$$= 4n + (1 + 3 + 3^2 + \cdots + 3^{n-1})$$

$$= 4n + 1 \times \left(\frac{3^n - 1}{3-1}\right)$$

$$= 4n + \left(\frac{3^n - 1}{2}\right)$$

$$= \frac{1}{2}[3^n + 8n - 1]$$

Sum of Some Special Series

Consider the series

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \cdots + \frac{1}{(a+(n-2)d)(a+(n-1)d)}$$

In order to find the sum of a finite number of terms of such series, we write its each term as the difference of two terms as given below:

$$\frac{1}{a(a+d)} = \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+d}\right)$$

$$\frac{1}{(a+d)(a+2d)} = \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+2d}\right)$$

$$\frac{1}{(a+2d)(a+3d)} = \frac{1}{d}\left(\frac{1}{a+2d} - \frac{1}{a+3d}\right)$$

and so on. Therefore,

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \cdots + \frac{1}{(a+(n-2)d)(a+(n-1)d)}$$

$$= \frac{1}{d}\left[\left(\frac{1}{a} - \frac{1}{a+d}\right) + \left(\frac{1}{a+d} - \frac{1}{a+2d}\right) + \cdots + \left(\frac{1}{a+(n-2)d} - \frac{1}{a+(n-1)d}\right)\right]$$

$$= \frac{1}{d}\left[\frac{1}{a} - \frac{1}{a+(n-1)d}\right]$$

$$= \frac{n-1}{a(a+(n-1)d)}$$

Example 3.88 **Find the sum to n terms of the series $1/(1 \times 2) + 1/(2 \times 3) + 1/(3 \times 4) + \cdots + 1/n(n+1)$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{1}{r(r+1)}, \quad r = 1, 2, \ldots, n$$

$$= \frac{1}{r} - \frac{1}{r+1}$$

Hence, the required sum is

$$\sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Example 3.89 **Find the sum to n terms of the series $1/(1 \times 3) + 1/(3 \times 5) + 1/(5 \times 7) + \cdots$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{1}{(2r-1)(2r+1)}, \quad r = 1, 2, 3, \ldots, n$$

$$= \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

Hence, the required sum is

$$\sum_{r=1}^{n} T_r = \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]$$

$$= \frac{n}{2n+1}$$

Example 3.90 **Find the sum $\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$.**

Sol. $$\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)} = \sum_{r=1}^{n} \frac{1}{a}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right)$$

$$= \frac{1}{a}\sum_{r=1}^{n}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right)$$

$$= \frac{1}{b}\left[\left(\frac{1}{a+b}-\frac{1}{2a+b}\right)+\left(\frac{1}{2a+b}-\frac{1}{3a+b}\right)+\cdots+\left(\frac{1}{na+b}-\frac{1}{(n+1)a+b}\right)\right]$$

$$= \frac{1}{a}\left(\frac{1}{a+b} - \frac{1}{(n+1)a+b}\right)$$

$$= \frac{n}{(a+b)((n+1)a+b)}$$

Example 3.91 **Find the sum to n terms of the series $3/(1^2 \times 2^2) + 5/(2^2 \times 3^2) + 7/(3^2 \times 4^2) + \cdots$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{(2r+1)}{r^2(r+1)^2}, \quad r = 1, 2, 3, \ldots$$

$$= \left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right], \quad r = 1, 2, 3, \ldots$$

Hence, the required sum is

$$\sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$= \left(\frac{1}{1^2}-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\cdots+\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$

$$= 1 - \frac{1}{(n+1)^2}.$$

$$= \frac{2n+n^2}{(n+1)^2}$$

Example 3.92 **Find the sum to n terms of the series $\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \cdots$.**

Sol. Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{r}{1+r^2+r^4}, \quad r = 1, 2, 3, \ldots, n$$

$$= \frac{r}{(r^2+r+1)(r^2-r+1)}$$

$$= \frac{1}{2}\left[\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right]$$

Therefore sum of the series is

$$\sum_{r=1}^{n} T_r = \frac{1}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right)\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{n^2+n+1}\right]$$

$$= \frac{n^2+n}{2(n^2+n+1)}$$

Example 3.93 **Find the sum**

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+n}$$

Sol. $T_r = \frac{1}{1+2+3+\cdots+r} = \frac{2}{r(r+1)} = 2\left[\frac{1}{r}-\frac{1}{r+1}\right]$

$$\Rightarrow \sum_{r=1}^{n} T_r = \frac{2n}{n+1}$$

Example 3.94 **Find the sum**

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$$

Sol. $T_r = \frac{1}{r(r+1)(r+2)(r+3)}$

Here factors in determinator are in A.P. In such cases we multiply T_r by difference of last factor and first factor

$$\Rightarrow T_r = \frac{r+3-r}{3[r(r+1)(r+2)(r+3)]}$$

$$= -\frac{1}{3}\left[\frac{1}{(r+1)(r+2)(r+3)}-\frac{1}{r(r+1)(r+2)}\right]$$

$$= -\frac{1}{3}[V(r)-V(r-1)]$$

$$\Rightarrow \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)}$$

$$= -\frac{1}{3}[V(n)-V(0)]$$

$$= \frac{1}{3}\left[\frac{1}{6}-\frac{1}{(n+1)(n+2)(n+3)}\right]$$

Example 3.95 **Find the sum** $\sum_{r=1}^{n} \frac{r}{(r+1)!}$ **where** $n!$ **= 1.2.3 ... n.**

Sol. $T_r = \frac{r}{(r+1)!}$

$$= \frac{r+1-1}{(r+1)!}$$

$$= \frac{1}{r!}-\frac{1}{(r+1)!}$$

$$\Rightarrow \sum_{r=1}^{n}\left(\frac{1}{r!}-\frac{1}{(r+1)!}\right)$$

$$= \left(\frac{1}{1!}-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\cdots+\left(\frac{1}{ni}-\frac{1}{(n+1)!}\right)$$

$$= 1-\frac{1}{(n+1)!}$$

Example 3.96 **Find the sum** $\sum_{r=1}^{n} r(r+1)(r+2)(r+3)$.

Sol. $T_r = r(r+1)(r+2)(r+3)$

Here factors are in A.P., we multiply T_r by difference of factor after $r+3$ and factor before r.

$$\Rightarrow T_r = \frac{1}{5}r(r+1)(r+2)(r+3)[r+4-(r-1)]$$

$$= \frac{1}{5}[r(r+1)(r+2)(r+3)(r+4) \cdot$$

$$-(r-1)r(r+1)(r+2)(r+3)]$$

$$= \frac{1}{5}[V(r)-V(r-1)]$$

$$\Rightarrow \sum_{r=1}^{n} r(r+1)(r+2)(r+3)$$

$$= \frac{1}{5}[V(n)-V(0)]$$

$$= \frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$$

Note: *Also we have*

$$\sum_{r=1}^{n} r(r+1)(r+2)(r+3)(r+4)$$

$$= \frac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5)$$

$$\sum_{r=1}^{n} r(r+1)(r+2)$$

$$= \frac{1}{4}n(n+1)(n+2)(n+3) \text{ etc.}$$

Example 3.97 **Find the sum**

$$\frac{1^4}{1\times 3}+\frac{2^4}{3\times 5}+\frac{3^4}{5\times 7}+\cdots+\frac{n^4}{(2n-1)(2n+1)}$$

Sol. $T_n = \frac{1}{16}\frac{16n^4-1+1}{(4n^2-1)}$

$$= \frac{1}{16}\left[4n^2+1+\frac{1}{(2n-1)(2n+1)}\right]$$

$$\therefore T_n = \frac{1}{16}[4n^2+1]+\frac{1}{32}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$

Now putting $n = 1, 2, 3, \ldots, n$ and adding, we have

$$S_n = \frac{1}{16}[4\Sigma n^2 + n] + \frac{1}{32}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

$$= \frac{1}{16}\times 4\frac{n(n+1)(2n+1)}{6} + \frac{1}{16}n + \frac{1}{32}\left(1-\frac{1}{2n+1}\right)$$

$$= \frac{1}{16}\left[\frac{2}{3}\times n(n+1)(2n+1)+n+\frac{n}{2n+1}\right]$$

$$= \frac{1}{16}\left[\frac{2}{3}n(n+1)(2n+1)+\frac{n\times 2(n+1)}{2n+1}\right]$$

$$= \frac{1}{16}2n(n+1)\left[\frac{(2n+1)^2+3}{3(2n+1)}\right]$$

$$= \frac{n(n+1)}{8}\times\frac{4n^2+4n+4}{3(2n+1)}$$

$$= \frac{n(n+1)(n^2+n+1)}{6(2n+1)}$$

Example 3.98 **Find the sum of the series**

$$\sum_{k=1}^{360}\left(\frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\right)$$

Sol. $T_k = \dfrac{1}{\sqrt{k}\sqrt{k+1}\,[\sqrt{k}+\sqrt{k+1}]}$

$$= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}}$$

$$= \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

$$\therefore\ S = \sum_{k=1}^{360}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$$

$$= 1 - \frac{1}{\sqrt{361}}$$

$$= 1 - \frac{1}{19} = \frac{18}{19}$$

Example 3.99 **Find the sum of the series**

$$\frac{1}{3^2+1}+\frac{1}{4^2+2}+\frac{1}{5^2+3}+\frac{1}{6^2+4}+\cdots\infty$$

Sol. $T_n = \dfrac{1}{n^2+(n-2)} = \dfrac{1}{(n+2)(n-1)}$, where $n = 3, 4, 5\ldots$

$$= \frac{1}{3}\left[\frac{1}{n-1}-\frac{1}{n+2}\right]$$

$$\therefore\ S = \sum_{n=3}^{\infty}T_n = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$+\frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right)$$

$$+\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right)$$

$$+\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)$$

$$\vdots\ \vdots$$

$$S = \frac{1}{3}\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right] = \frac{1}{3}\left[\frac{6+4+3}{12}\right] = \frac{13}{36}$$

Example 3.100 **Find the sum of firs 100 terms of the series whose general term is given by $a_k = (k^2+1)\,k\,!$**

Sol. $a_k = (k^2+1)\,k\,!$

$= (k(k+1)-(k-1))\,k\,!$

$= k\,(k+1)\,! - (k-1)\,k\,!$

so $\quad k\,(k+1)\,! - (k-1)\,k\,!$

$a_1 = 1\,.2\,! - 0$

$a_2 = 2\,.3\,! - 1\,.2\,!$

$a_3 = 3\,.4\,! - 2\,.3\,!$

....

....

$a_{100} = 100\,.101\,! - 99.100\,!$

$a_1 + a_2 + \cdots + a_{100} = 100.\ 101\,!$

Sum of the Series when *i* and *j* are Dependent

Consider sum of the series $\sum\sum_{0\le i<j\le n} ij$. In the given summation, i and j are not independent. In the sum of series $\sum_{i=1}^{n}\sum_{j=1}^{n} ij = \sum_{i=1}^{n}\left(i\left(\sum_{j=1}^{n} j\right)\right)$. Here i and j are independent. In this summation, there are three types of terms, those when $i<j$ (upper triangle), $i>j$ (lower triangle) and $i=j$ (diagonal) as shown in the diagram below.

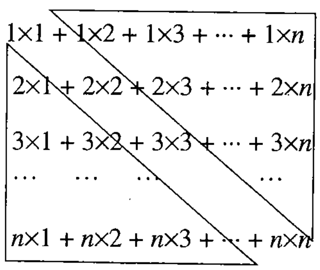

Fig. 3.2

Also, the sum of terms when $i < j$ is equal to the sum of the terms when $i > j$ as terms in both the triangles are symmetrical. So, in that case

$\sum_{i=1}^{n}\sum_{j=1}^{n} ij$ = Sum of terms in upper triangle + sum of terms in lower triangle + sum of terms in diagonal

$= 2\sum_{0\le i<j\le n}\sum ij + \sum_{i=j}\sum ij$ ($\because$ sums of terms in upper and lower triangles are same)

$$\Rightarrow \sum_{0\le i<j\le n}\sum ij = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} ij - \sum_{i=j}\sum ij}{2}$$

$$= \frac{\sum_{i=1}^{n} i \sum_{j=1}^{n} j - \sum_{i=1}^{n} i^2}{2}$$

$$= \frac{\frac{n(n+1)}{2}\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}}{2}$$

When $f(i)$ and $f(j)$ are not symmetrical, we find the sum by listing all the terms.

Consider the sum $\sum_{0\le i\le j\le n}\sum ij$.

In this sum, we have to find the sum of the upper triangle and the diagonal of the above square. Hence,

$$\sum_{0\le i\le j\le n}\sum ij = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} ij - \sum_{i=j}\sum ij}{2} + \sum_{i=j}\sum ij$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} ij + \sum_{i=j}\sum ij}{2}$$

Alternative method:

$(1 + 2 + 3 + 4 + \cdots + n)^2 = (1^2 + 2^2 + 3^2 + \cdots + n^2) + 2 \sum_{0\le i<j\le n}\sum ij$

$$\Rightarrow \left(\frac{n(n+1)}{2}\right)^2 = \frac{n(n+1)(2n+1)}{6} + 2\sum_{0\le i<j\le n}\sum ij$$

$$\Rightarrow \sum_{0\le i<j\le n}\sum ij = \frac{\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6}}{2}$$

Example 3.101 **Find the sum of the products of the ten numbers ±1, ±2, ±3, ±4, and ±5 taking two at a time.**

Sol. We have,

$(1 - 1 + 2 - 2 + \cdots + 5 - 5)^2 = 1^2 + 1^2 + 2^2 + 2^2 + \cdots + 5^2 + 5^2 + 2S$,

where S is the required sum. Hence,

$0 = 2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 2S$

$\Rightarrow S = -55$

Example 3.102 **Find the sum $\sum_{0\le i<j\le n}\sum 1$.**

Sol.
$$\sum_{0\le i<j\le n}\sum 1 = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} 1 - \sum_{i=j}\sum 1}{2}$$

$$= \frac{\left(\sum_{j=1}^{n} 1\right)\left(\sum_{j=1}^{n} 1\right) - \sum_{j=1}^{n} 1}{2}$$

$$= \frac{n^2 - n}{2}$$

$$= \frac{n(n-1)}{2}$$

Concept Application Exercise 3.5

1. Find the sum $\frac{3}{2} - \frac{5}{6} + \frac{7}{18} - \frac{9}{54} + \cdots \infty$.
2. Find the sum $\frac{1^2}{2} + \frac{3^2}{2^2} + \frac{5^2}{2^3} + \frac{7^2}{2^4} + \cdots \infty$.
3. If the sum to infinity of the series $3 + (3 + d)\frac{1}{4} + (3 + 2d)\frac{1}{4^2} + \cdots \infty$ is $\frac{44}{9}$, then find d.
4. Find the sum of the series $1^2 + 3^2 + 5^2 + \cdots$ to n terms.
5. Find the sum up to 20 terms. $1 + \frac{1}{2}(1+2) + \frac{1}{3}(1+2+3) + \frac{1}{4}(1+2+3+4) + \cdots$
6. If the sum of the squares of first n natural numbers exceeds their sum by 330, then find n.
7. Find the value of $11^2 + 12^2 + 13^2 + \cdots + 20^2$.
8. Find the sum $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots$ up to 22nd term.
9. Find the sum $2 + 5 + 10 + 17 + 26 + \cdots$.
10. Find the sum $1 + 4 + 13 + 40 + 121 + \cdots$.
11. If the set of natural numbers is partitioned into subsets $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$ and so on, then find the sum of the terms in S_{50}.
12. If $T_r = r(r^2 - 1)$, then find $\sum_{r=2}^{\infty} \frac{1}{T_r}$.
13. If $S = \frac{1}{1\times3\times5} + \frac{1}{3\times5\times7} + \frac{1}{5\times7\times9} + \cdots$ to infinity, then find the value of $[36S]$, where $[\cdot]$ represents the greatest integer function.
14. If $\sum_{r=1}^{n} t_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{t_r}$.
15. Find the sum of the series $1 + 2(1 - x) + 3(1 - x)(1 - 2x) + \cdots + n(1 - x)(1 - 2x)(1 - 3x) \cdots [1 - (n - 1)x]$

EXERCISES

Subjective Type

Solutions on page 3.43

1. For $a, x > 0$ prove that at the most one term of the G.P. $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}$ can be rational.
2. If the terms of the A.P. $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}, \ldots$ are all integers, where $a, x > 0$, then find the least composite value of a.
3. Find a three-digit number such that its digits are in increasing G.P. (from left to right) and the digits of the number obtained from it by subtracting 100 form an A.P.
4. Along a road lies an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones, he covered a distance of 3 km. Find the number of stones.
5. If the first and the n^{th} terms of a G.P. are a and b, respectively, and if P is the product of the first n terms, prove that $P^2 = (ab)^n$.
6. Let $x = 1 + 3a + 6a^2 + 10a^3 + \cdots, |a| < 1$.
   $y = 1 + 4b + 10b^2 + 20b^3 + \cdots, |b| < 1$.
   Find $S = 1 + 3(ab) + 5(ab)^2 + \cdots$ in terms of x and y.
7. If the sum of n terms of the series $\frac{2n+1}{2n-1} + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \cdots$ is 36, then find the value of n.
8. Find the sum $\frac{3}{1\times 2} \times \frac{1}{2} + \frac{4}{2\times 3} \times \left(\frac{1}{2}\right)^2 + \frac{5}{3\times 4} \times \left(\frac{1}{2}\right)^3 + \cdots$ to n terms.
9. If $S_1, S_2, S_3, \ldots, S_m$ are the sums of n terms of m A.P.'s whose first terms are $1, 2, 3, \ldots, m$ and common differences are $1, 3, 5, \ldots, (2m-1)$, respectively, show that
   $$S_1 + S_2 + \cdots + S_m = \frac{mn}{2}(mn+1)$$
10. If S_1, S_2 and S_3 be, respectively, the sum of n, $2n$ and $3n$ terms of a G.P., prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.
11. Find four numbers in a G.P. whose sum is 85 and product is 4096.
12. There are $(4n + 1)$ terms in a certain sequence of which the first $(2n + 1)$ terms form an A.P. of common difference 2 and the last $(2n + 1)$ terms are in G.P. of common ratio 1/2. If the middle term of both A.P. and G.P. be the same, then find the mid-term of this sequence.
13. Let there be $a_1, a_2, a_3, \ldots, a_n$ terms in G.P. whose common ratio is r. Let S_k denote the sum of first k terms of this G.P. Prove that $S_{m-1} S_m = \frac{r+1}{r} \sum_{i<j}^{m} a_i a_j$.
14. Find the sum of n terms of the series whose n^{th} term is
   $$T(n) = \tan\frac{x}{2^n} \times \sec\frac{x}{2^{n-1}}.$$
15. Find the value of $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\substack{k=0 \\ (i\neq j\neq k)}}^{\infty} \frac{1}{3^i 3^j 3^k}$.
16. Let $a_1, a_2, \ldots, a_n$ be real numbers such that
   $\sqrt{a_1} + \sqrt{a_2 - 1} + \sqrt{a_3 - 2} + \cdots + \sqrt{a_n - (n-1)} = \frac{1}{2}(a_1 + a_2 + \ldots + a_n) - \frac{n(n-3)}{4}$. Compute the value of $\sum_{i=1}^{100} a_i$.

Objective Type

Solutions on page 3.46

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $\log_2(5 \times 2^x + 1)$, $\log_4(2^{1-x} + 1)$ and 1 are in A.P., then x equals
   a. $\log_2 5$ **b.** $1 - \log_5 2$
   c. $\log_5 2$ **d.** none of these
2. If three positive real numbers a, b, c are in A.P. such that $abc = 4$, then the minimum value of b is
   a. $2^{1/3}$ **b.** $2^{2/3}$
   c. $2^{1/2}$ **d.** $2^{3/2}$
3. The maximum sum of the series $20 + 19\frac{1}{3} + 18\frac{2}{3} + \cdots$ is
   a. 310 **b.** 300
   c. 320 **d.** none of these
4. The largest term common to the sequences 1, 11, 21, 31, ... to 100 terms and 31, 36, 41, 46, ... to 100 terms is
   a. 381 **b.** 471
   c. 281 **d.** none of these
5. If the sum of m terms of an A.P. is the same as the sum of its n terms, then the sum of its $(m + n)$ terms is
   a. mn **b.** $-mn$
   c. $1/mn$ **d.** 0
6. If the sides of a right angled triangle are in A.P. then the sines of the acute angles are
   a. $\frac{3}{5}, \frac{4}{5}$ **b.** $\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}$
   c. $\frac{1}{2}, \frac{\sqrt{3}}{2}$ **d.** none of these
7. If the ratio of the sum to n terms of two A.P.'s is $(5n + 3):(3n + 4)$, then the ratio of their 17^{th} terms is
   a. 172:99 **b.** 168:103
   c. 175:99 **d.** 171:103
8. 150 workers were engaged to finish a piece of work in a certain number of days. Four workers stopped working on the second day, four more workers stopped their work on the third day and so on. It took 8 more days to finish the work. Then the number of days in which the work was completed is
   a. 29 days **b.** 24 days
   c. 25 days **d.** none of these
9. In an A.P. of which a is the first term, if the sum of the first p terms is zero, then the sum of the next q terms is
   a. $-\frac{a(p+q)p}{q+1}$ **b.** $\frac{a(p+q)p}{p+1}$

c. $-\dfrac{a(p+q)q}{p-1}$ d. none of these

10. If S_n denotes the sum of first 'n' terms of an A.P. and $\dfrac{S_{3n}-S_{n-1}}{S_{2n}-S_{2n-1}}=31$, then the value of n is

a. 21 b. 15
c. 16 d. 19

11. If a, b, and c are in A.P. then $a^3+c^3-8b^3$ is equal to

a. 2 abc b. 6 abc
c. 4 abc d. none of these

12. The number of terms of an A.P. is even; the sum of the odd terms is 24, and of the even terms is 30, and the last term exceeds the first by 10/2, then the number of terms in the series is

a. 8 b. 4
c. 6 d. 10

13. If $a, \dfrac{1}{b}, c$ and $\dfrac{1}{p}, q, \dfrac{1}{r}$ form two arithmetic progressions of the same common difference, then a, q, c are in A.P. if

a. p, b, r are in A.P. b. $\dfrac{1}{p}, \dfrac{1}{b}, \dfrac{1}{r}$ are in A.P.
c. p, b, r are in G.P. d. none of these

14. Suppose that $F(n+1)=\dfrac{2F(n)+1}{2}$ for $n = 1, 2, 3, \ldots$ and $F(1) = 2$. Then, $F(101)$ equals

a. 50 b. 52
c. 54 d. none of these

15. If the sum of n terms of an A.P. is $cn(n-1)$, where $c \neq 0$, then sum of the squares of these terms is

a. $c^2n(n+1)^2$ b. $\dfrac{2}{3}c^2n(n-1)(2n-1)$
c. $\dfrac{2c^2}{3}n(n+1)(2n+1)$ d. none of these

16. Consider an A.P. $a_1, a_2, a_3, \ldots$ such that $a_3+a_5+a_8=11$ and $a_4+a_2=-2$, then the value of $a_1+a_6+a_7$ is

a. –8 b. 5
c. 7 d. 9

17. If $a_1, a_2, a_3, \ldots$ are in A.P., then a_p, a_q, a_r are in A.P. if p, q, r are in

a. A.P. b. G.P.
c. H.P. d. none of these

18. Let $a_1, a_2, a_3, \ldots$ be terms of an A.P. If $\dfrac{a_1+a_2+\cdots+a_p}{a_1+a_2+\cdots+a_q}=\dfrac{p^2}{q^2}$, $p \neq q$, then $\dfrac{a_6}{a_{21}}$ equals

a. 41/11 b. 7/2
c. 2/7 d. 11/41

19. If S_n denotes the sum of first n terms of an A.P. whose first term is a and $\dfrac{S_{nx}}{S_x}$ is independent of x, then $S_p=$

a. p^3 b. p^2a
c. pa^2 d. a^3

20. Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of the G.P. is

a. $2-\sqrt{3}$ b. $2+\sqrt{3}$
c. $\sqrt{3}-2$ d. $3+\sqrt{2}$

21. If a_1, a_2, a_3 ($a_1 > 0$) are three successive terms of a G.P. with common ratio r, the value of r for which $a_3 > 4a_2 - 3a_1$ holds is given by

a. $1<r<3$ b. $-3<r<-1$
c. $r>3$ or $r<1$ d. none of these

22. Let $S \subset (0, \pi)$ denote the set of values of x satisfying the equation $8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\cdots \text{ to } \infty}=4^3$. Then, $S=$

a. $\{\pi/3\}$ b. $\{\pi/3, -2\pi/3\}$
c. $\{-\pi/3, 2\pi/3\}$ d. $\{\pi/3, 2\pi/3\}$

23. If $|a|<1$ and $|b|<1$, then the sum of the series $1+(1+a)b+(1+a+a^2)b^2+(1+a+a^2+a^3)b^3+\cdots$ is

a. $\dfrac{1}{(1-a)(1-b)}$ b. $\dfrac{1}{(1-a)(1-ab)}$
c. $\dfrac{1}{(1-b)(1-ab)}$ d. $\dfrac{1}{(1-a)(1-b)(1-ab)}$

24. If $(p+q)^{\text{th}}$ term of a G.P. is 'a' and its $(p-q)^{\text{th}}$ term is 'b' where $a, b \in R^+$, then its p^{th} term is

a. $\sqrt{\dfrac{a^3}{b}}$ b. $\sqrt{\dfrac{b^3}{a}}$
c. $\sqrt{ab}$ d. none of these

25. If the sides of a triangle are in G.P., and its largest angle is twice the smallest, then the common ratio r satisfies the inequality

a. $0<r<\sqrt{2}$ b. $1<r<\sqrt{2}$
c. $1<r<2$ d. none of these

26. The value of $0.2^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)}$ is

a. 4 b. log 4
c. log 2 d. none of these

27. If $(1+x)(1+x^2)(1+x^4)\cdots(1+x^{128})=\sum_{r=0}^{n} x^r$, then n is equal to

a. 256 b. 255
c. 254 d. none of these

28. If x, y, z are in G.P. and $a^x = b^y = c^z$, then

a. $\log_b a = \log_a c$ b. $\log_c b = \log_a c$
c. $\log_b a = \log_c b$ d. none of these

29. The geometric mean between –9 and –16 is

a. 12 b. –12
c. –13 d. none of these

30. If S denotes the sum to infinity and S_n the sum of n terms of the series $1+\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\cdots$, such that $S-S_n<\dfrac{1}{1000}$, then the least value of n is

a. 8 b. 9
c. 10 d. 11

31. The first term of an infinite geometric series is 21. The second term and the sum of the series are both positive integers. Then which of the following is not the possible value of the second term

a. 12 b. 14
c. 18 d. None of these

32. The number of terms common between the series 1 + 2 + 4 + 8 + ... to 100 terms and 1 + 4 + 7 + 10 + ... to 100 terms is

a. 6 b. 4
c. 5 d. none of these

33. After striking the floor, a certain ball rebounds $(4/5)^{\text{th}}$ of height from which it has fallen. Then the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 m is

a. 1260 m b. 600 m
c. 1080 m d. none of these

34. The sum of an infinite G.P. is 57 and the sum of their cubes is 9747, then common ratio of the G.P. is

a. 1/3 b. 2/3
c. 1/6 d. none of these

35. If $a^2 + b^2$, $ab + bc$ and $b^2 + c^2$ are in G.P., then a, b, c are in

a. A.P. b. G.P.
c. H.P. d. none of these

36. Consider the ten numbers $ar, ar^2, ar^3, \ldots, ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6 then the product of these ten numbers, is

a. 81 b. 243
c. 343 d. 324

37. Let $a = 1\ 1\ 1 \ldots 1$ (55 digits), $b = 1 + 10 + 10^2 + \cdots + 10^4$, $c = 1 + 10^5 + 10^{10} + 10^{15} + \cdots + 10^{50}$, then

a. $a = b + c$ b. $a = bc$
c. $b = ac$ d. $c = ab$

38. Let a_n be the n^{th} term of a G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$, then the common ratio is

a. α/β b. β/α
c. $\sqrt{\alpha/\beta}$ d. $\sqrt{\beta/\alpha}$

39. The sum of 20 terms of a series of which every even term is 2 times the term before it, and every odd term is 3 times the term before it, the first term being unity is

a. $\left(\frac{2}{7}\right)(6^{10}-1)$ b. $\left(\frac{3}{7}\right)(6^{10}-1)$
c. $\left(\frac{3}{5}\right)(6^{10}-1)$ d. none of these

40. In a G.P. the first, third and fifth terms may be considered as the first, fourth and sixteenth terms of an A.P. Then the fourth term of the A.P., knowing that its first term is 5 is

a. 10 b. 12
c. 16 d. 20

41. If the p^{th}, q^{th}, and r^{th} terms of an A.P. are in G.P., then common ratio of the G.P. is

a. $\frac{pr}{q^2}$ b. $\frac{r}{p}$
c. $\frac{q+r}{p+q}$ d. $\frac{q-r}{p-q}$

42. If the p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then $p - q, q - r, r - s$ are in

a. A.P. b. G.P.
c. H.P. d. none of these

43. If a, b, c, d are in G.P., then $(b-c)^2 + (c-a)^2 + (d-b)^2$ in equal to

a. $(a-d)^2$ b. $(ad)^2$
c. $(a+d)^2$ d. $(a/d)^2$

44. If a, b, c are digits, then the rational number represented by $0.cababab \ldots$ is

a. $cab/990$ b. $(99c + ba)/990$
c. $(99c + 10a + b)/99$ d. $(99c + 10a + b)/990$

45. The sum of an infinite geometric series is 162 and the sum of its first n terms is 160. If the inverse of its common ratio is an integer, then which of the following is not a possible first term?

a. 108 b. 144
c. 160 d. none of these

46. Let $f(x) = 2x + 1$. Then the number of real number of real values of x for which the three unequal numbers $f(x)$, $f(2x)$, $f(4x)$ are in G.P. is

a. 1 b. 2
c. 0 d. none of these

47. Concentric circles of radii 1, 2, 3, ..., 100 cm are drawn. The interior of the smallest circle is coloured red and the angular regions are coloured alternately green and red, so that no two adjacent regions are of the same colour. Then, the total area of the green regions in sq. cm is equal to

a. 1000π b. 5050π
c. 4950π d. 5151π

48. Let $\{t_n\}$ be a sequence of integers in G.P. in which $t_4:t_6 = 1:4$ and $t_2 + t_5 = 216$. Then t_1 is

a. 12 b. 14
c. 16 d. none of these

49. If x, $2y$, $3z$ are in A.P., where the distinct numbers x, y, z are in G.P., then the common ratio of the G.P. is

a. 3 b. $\frac{1}{3}$
c. 2 d. $\frac{1}{2}$

50. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \cdots$ to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - r^{3p} + \cdots$ to ∞, $|r| < 1$, then $S_p + s_p$ in terms of S_{2p} is

a. $2S_{2p}$ b. 0
c. $\frac{1}{2}S_{2p}$ d. $-\frac{1}{2}S_{2p}$

51. If x, y, z are real and $4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8zx = 0$, then x, y, z are in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

52. If $a_1, a_2, \ldots, a_n$ are in H.P., then

$$\frac{a_1}{a_2 + a_3 + \cdots + a_n}, \frac{a_2}{a_1 + a_3 + \cdots + a_n}, \ldots, \frac{a_n}{a_1 + a_2 + \cdots + a_{n-1}} \text{ are in}$$

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

53. If $H_1, H_2, \ldots, H_{20}$ be 20 harmonic means between 2 and 3, then $\frac{H_1 + 2}{H_1 - 2} + \frac{H_{20} + 3}{H_{20} - 3} =$

a. 20 **b.** 21
c. 40 **d.** 38

54. Let a_1, a_2, a_3, a_4 and a_5 be such that a_1, a_2 and a_3 are in A.P., a_2, a_3 and a_4 are in G.P., and a_3, a_4 and a_5 are in H.P. Then $\log_e a_1$, $\log_e a_3$ and $\log_e a_5$ are in

a. G.P. **b.** A.P.
c. H.P. **d.** none of these

55. If a, b, c are in A.P., then $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ will be in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

56. If x, $2x + 2$, and $3x + 3$ are first three terms of a G.P., then the fourth term is

a. 27 **b.** –27
c. 13.5 **d.** –13.5

57. Sum of three numbers in G.P. be 14. If one is added to first and second and 1 is subtracted from the third, the new numbers are in A.P. The smallest of them is

a. 2 **b.** 4
c. 6 **d.** 10

58. If a, b, and c are in A.P., p, q, and r are in H.P. and ap, bq, and cr are in G.P., then $\frac{p}{r} + \frac{r}{p}$ is equal to

a. $\frac{a}{c} - \frac{c}{a}$ **b.** $\frac{a}{c} + \frac{c}{a}$
c. $\frac{b}{q} + \frac{q}{b}$ **d.** $\frac{b}{q} - \frac{q}{b}$

59. If a, b, and c are in A.P. and $b - a, c - b$ and a are in G.P., then $a:b:c$ is

a. 1:2:3 **b.** 1:3:5
c. 2:3:4 **d.** 1:2:4

60. Let $a \in (0, 1]$ satisfies the equation $a^{2008} - 2a + 1 = 0$ and $S = 1 + a + a^2 + \ldots + a^{2007}$. Sum of all possible value(s) of S, is

a. 2010 **b.** 2009
c. 2008 **d.** 2

61. Let $\alpha, \beta \in R$. If α, β^2 be the roots of quadratic equation $x^2 - px + 1 = 0$ and α^2, β be the roots of quadratic equation $x^2 - qx + 8 = 0$, then the value of 'r' if $\frac{r}{8}$ be arithmetic mean of p and q, is

a. $\frac{83}{2}$ **b.** 83
c. $\frac{83}{8}$ **d.** $\frac{83}{4}$

62. $a, b, c, d \in R^+$ such that a, b, and c are in A.P. and b, c and, d are in H.P., then

a. $ab = cd$ **b.** $ac = bd$
c. $bc = ad$ **d.** none of these

63. If in a progression $a_1, a_2, a_3, \ldots$, etc., $(a_r - a_{r+1})$ bears a constant ratio with $a_r \times a_{r+1}$, then the terms of the progression are in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

64. If a, b, and c are in G.P., then $a + b$, $2b$, and $b + c$ are in

a. A.P. **b.** G.P.
c. H.P. **d.** none of these

65. If a, x, and b are in A.P., a, y, and b are in G.P. and a, z, b are in H.P. such that $x = 9z$ and $a > 0, b > 0$, then

a. $|y| = 3z$ **b.** $x = 3|y|$
c. $2y = x + z$ **d.** none of these

66. Let $n \in N, n > 25$. Let A, G, H denote the arithmetic mean, geometric mean and harmonic mean of 25 and n. The least value of n for which $A, G, H \in \{25, 26, \ldots, n\}$ is

a. 49 **b.** 81
c. 169 **d.** 225

67. The 15th term of the series $2\frac{1}{2} + 1\frac{7}{13} + 1\frac{1}{9} + \frac{20}{23} + \cdots$ is

a. $\frac{10}{39}$ **b.** $\frac{10}{21}$
c. $\frac{10}{23}$ **d.** none of these

68. If a, b, and c are in G.P. and x, y, respectively, be arithmetic means between a, b and b, c, then the value of $\frac{a}{x} + \frac{c}{y}$ is

a. 1 **b.** 2
c. 1/2 **d.** none of these

69. If a, b, and c are in A.P. and p, p' are, respectively, A.M. and G.M. between a and b while q, q' are, respectively, the A.M. and G.M. between b and c, then

a. $p^2 + q^2 = p'^2 + q'^2$ **b.** $pq = p'q'$
c. $p^2 - q^2 = p'^2 - q'^2$ **d.** none of these

70. If $a_1, a_2, \ldots, a_n$ are in A.P. with common difference $d \neq 0$, then sum of the series $\sin d[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \cdots + \sec a_{n-1} \sec a_n]$ is

a. $\operatorname{cosec} a_n - \operatorname{cosec} a$ **b.** $\cot a_n - \cot a$
c. $\sec a_n - \sec a_1$ **d.** $\tan a_n - \tan a_1$

71. The sum of the series $a - (a + d) + (a + 2d) - (a + 3d) + \cdots$ up to $(2n + 1)$ terms is

a. $-nd$ **b.** $a + 2nd$
c. $a + nd$ **d.** $2nd$

72. The sum to 50 terms of the series $1 + 2\left(1 + \frac{1}{50}\right) + 3\left(1 + \frac{1}{50}\right)^2 + \cdots$ is given by

a. 2500 **b.** 2550
c. 2450 **d.** none of these

73. The sum to 50 terms of the series $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \cdots$ is

a. $\frac{100}{17}$ b. $\frac{150}{17}$

c. $\frac{200}{51}$ d. $\frac{50}{17}$

74. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ to $\infty = \frac{\pi^2}{6}$, then $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ equals

a. $\pi^2/8$ b. $\pi^2/12$

c. $\pi^2/3$ d. $\pi^2/2$

75. Coefficient of x^{18} in $(1 + x + 2x^2 + 3x^3 + \cdots + 18x^{18})^2$ is equal to

a. 995 b. 1005

c. 1235 d. none of these

76. $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{1\times3\times5\times7\times9\times\cdots\times(2r+1)}$ is equal to

a. $\frac{1}{3}$ b. $\frac{3}{2}$

c. $\frac{1}{2}$ d. none of these

77. Greatest integer by which $1 + \sum_{r=1}^{30} r \times r!$ is divisible is

a. composite number b. odd number

c. divisible by 3 d. none of these

78. If $\sum_{r=1}^{n} r^4 = I(n)$, then $\sum_{r=1}^{n} (2r-1)^4$ is equal to

a. $I(2n) - I(n)$ b. $I(2n) - 16I(n)$

c. $I(2n) - 8I(n)$ d. $I(2n) - 4I(n)$

79. Value of $\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\infty$ is equal to

a. 3 b. $\frac{6}{5}$

c. $\frac{3}{2}$ d. none of these

80. If $x_1, x_2, \ldots, x_{20}$ are in H.P. and $x_1, 2, x_{20}$ are in G.P., then $\sum_{r=1}^{19} x_r x_{r+1} =$

a. 76 b. 80

c. 84 d. none of these

81. The value of $\sum_{r=0}^{n} (a + r + ar)(-a)^r$ is equal to

a. $(-1)^n [(n+1)a^{n+1} - a]$ b. $(-1)^n (n+1)a^{n+1}$

c. $(-1)^n \frac{(n+2)a^{n+1}}{2}$ d. $(-1)^n \frac{na^n}{2}$

82. If $b_i = 1 - a_i$, $na = \sum_{i=1}^{n} a_i$, $nb = \sum_{i=1}^{n} b_i$, then $\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} (a_i - a)^2 =$

a. ab b. $-nab$ c. $(n+1)ab$ d. nab

83. The sum of the series $\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \cdots$ to infinite terms, if $|x| < 1$, is

a. $\frac{x}{1-x}$ b. $\frac{1}{1-x}$

c. $\frac{1+x}{1-x}$ d. 1

84. If $a_1, a_2, a_3, \ldots, a_{2n+1}$ are in A.P., then $\frac{a_{2n+1}-a_1}{a_{2n+1}+a_1} + \frac{a_{2n}-a_2}{a_{2n}+a_2} + \cdots + \frac{a_{n+2}-a_n}{a_{n+2}+a_n}$ is equal to

a. $\frac{n(n+1)}{2} \times \frac{a_2-a_1}{a_{n+1}}$ b. $\frac{n(n+1)}{2}$

c. $(n+1)(a_2 - a_1)$ d. none of these

85. The sum of $i - 2 - 3i + 4 \cdots$ up to 100 terms, where $i = \sqrt{-1}$ is

a. $50(1 - i)$ b. $25i$

c. $25(1 + i)$ d. $100(1 - i)$

86. Let $S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \cdots$ up to ∞. Then S is equal to

a. 40/9 b. 38/81

c. 36/171 d. none of these

87. If $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, then value of $S_n = 1 + \frac{3}{2} + \frac{5}{3} + \cdots + \frac{99}{50}$ is

a. $H_{50} + 50$ b. $100 - H_{50}$

c. $49 + H_{50}$ d. $H_{50} + 100$

88. If the sum to infinity of the series $1 + 2r + 3r^2 + 4r^3 + \cdots$ is 9/4, then value of r is

a. 1/2 b. 1/3

c. 1/4 d. none of these

89. The sum of series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots \infty$ is

a. 7/16 b. 5/16

d. 105/64 d. 35/16

90. The sum $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$ to 16 terms is

a. 246 b. 646

c. 446 d. 746

91. The sum $1 + 3 + 7 + 15 + 31 + \cdots$ to 100 terms is

a. $2^{100} - 102$ b. $2^{99} - 101$

c. $2^{101} - 102$ d. none of these

92. In a sequence of $(4n + 1)$ terms the first $(2n + 1)$ terms are in AP whose common difference is 2, and the last $(2n + 1)$ terms are in GP whose common ratio is 0.5 if the middle terms of the AP and GP are equal then the middle term of the sequence is

a. $\frac{n \cdot 2^{n+1}}{2^n - 1}$ b. $\frac{n \cdot 2^{n+1}}{2^{2n} - 1}$

c. $n \cdot 2^n$ d. none of these

93. The coefficient of x^{49} in the product $(x - 1)(x - 3) \cdots (x - 99)$ is

a. -99^2 b. 1

c. -2500 d. none of these

94. The sum of 20 terms of the series whose r^{th} term is given by $T(n) = (-1)^n \frac{n^2+n+1}{n!}$ is

a. $\frac{20}{19!} - 2$ **b.** $\frac{21}{20!} - 1$

c. $\frac{21}{20!}$ **d.** none of these

95. Consider the sequence 1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, ... Then 1025^{th} term will be

a. 2^9 **b.** 2^{11}

c. 2^{10} **d.** 2^{12}

96. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, \ldots$, then

$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \cdots + \frac{1}{t_{2003}} =$

a. $\frac{4006}{3006}$ **b.** $\frac{4003}{3007}$

c. $\frac{4006}{3008}$ **d.** $\frac{4006}{3009}$

97. The sum of $0.2 + 0.004 + 0.00006 + 0.0000008 + \cdots$ to ∞ is

a. $\frac{200}{891}$ **b.** $\frac{2000}{9801}$

c. $\frac{1000}{9801}$ **d.** none of these

98. The value of $\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 = 220$, then the value of n equals

a. 11 **b.** 12

c. 10 **d.** 9

99. If $1^2 + 2^2 + 3^2 + \cdots + 2003^2 = (2003)(4007)(334)$ and $(1)(2003) + (2)(2002) + (3)(2001) + \cdots + (2003)(1) = (2003)(334)(x)$, then x equals

a. 2005 **b.** 2004

c. 2003 **d.** 2001

100. If t_n denotes the n^{th} term of the series $2 + 3 + 6 + 11 + 18 + \cdots$ then t_{50} is

a. $49^2 - 1$ **b.** 49^2

c. $50^2 + 1$ **d.** $49^2 + 2$

101. The positive integer n for which $2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + \cdots + n \times 2^n = 2^{n+10}$ is

a. 510 **b.** 511

c. 512 **d.** 513

102. If $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$, then value of $\frac{1}{1\times3} + \frac{1}{5\times7} + \frac{1}{9\times11} + \cdots$ is

a. $\pi/8$ **b.** $\pi/6$

c. $\pi/4$ **d.** $\pi/36$

103. The coefficient of x^{19} in the polynomial $(x-1)(x-2)(x-2^2)\cdots(x-2^{19})$ is

a. $2^{20} - 2^{19}$ **b.** $1 - 2^{20}$

c. 2^{20} **d.** none of these

104. If $b_{n+1} = \frac{1}{1-b_n}$ for $n \geq 1$ and $b_1 = b_3$, then $\sum_{r=1}^{2001} b_r^{2001}$ is equal to

a. 2001 **b.** –2001

c. 0 **d.** none of these

105. If $(1^2 - t_1) + (2^2 - t_2) + \cdots + (n^2 - t_n) = \frac{n(n^2-1)}{3}$, then t_n is equal to

a. n^2 **b.** $2n$

c. $n^2 - 2n$ **d.** none of these

106. If $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$, then a, b, c, d are in

a. A.P. **b.** G.P.

c. H.P. **d.** none of these

107. If $(1 + 3 + 5 + \cdots + p) + (1 + 3 + 5 + \cdots + q) = (1 + 3 + 5 + \cdots + r)$ where each set of parentheses contains the sum of consecutive odd integers as shown, the smallest possible value of $p + q + r$ (where $p > 6$) is

a. 12 **b.** 21

c. 45 **d.** 54

108. In a geometric series, the first term is a and common ratio is r. If S_n denotes the sum of the n terms and $U_n = \sum_{n=1}^{n} S_n$ then $rS_n + (1-r)U_n$ equals

a. 0 **b.** n

c. na **d.** nar

109. The line $x + y = 1$ meets x-axis at A and y-axis at B, P is the mid-point of AB;
P_1 is the foot of the perpendicular from P to OA;
M_1 is that of P_1 from OP;
P_2 is that of M_1 from OA;
M_2 is that of P_2 from OP;
P_3 is that of M_2 from OA; and so on.
If P_n denotes the n^{th} foot of the perpendicular on OA;
then OP_n is

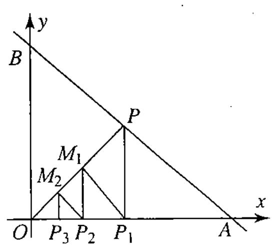

Fig. 3.3

a. $\left(\frac{1}{2}\right)^{n-1}$ **b.** $\left(\frac{1}{2}\right)^{n}$

c. $\left(\frac{1}{2}\right)^{n+1}$ **d.** none of these

110. If $(1-p)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - p^6$, $p \neq 1$, then the value of $\frac{p}{x}$ is

a. $\frac{1}{3}$ **b.** 3

c. $\frac{1}{2}$ **d.** 2

111. ABC is a right-angled triangle in which $\angle B = 90°$ and $BC = a$. If n points $L_1, L_2, \ldots, L_n$ on AB is divided in $n + 1$ equal parts and $L_1M_1, L_2M_2, \ldots, L_nM_n$ are line segments parallel to BC and $M_1, M_2, \ldots, M_n$ are on AC, then the sum of the lengths of $L_1M_1, L_2M_2, \ldots, L_nM_n$ is

a. $\frac{a(n+1)}{2}$ **b.** $\frac{a(n-1)}{2}$

c. $\frac{an}{2}$ d. none of these

112. Let T_r and S_r be the r^{th} term and sum up to r^{th} term of a series respectively. If for an odd number n, $S_n = n$ and $T_n = \frac{T_{n-1}}{n^2}$, then T_m (m being even) is

a. $\frac{2}{1+m^2}$ b. $\frac{2m^2}{1+m^2}$

c. $\frac{(m+1)^2}{2+(m+1)^2}$ d. $\frac{2(m+1)^2}{1+(m+1)^2}$

113. $ABCD$ is a square of length a, $a \in N$, $a > 1$. Let $L_1, L_2, L_3, \ldots$ be points on BC such that $BL_1 = L_1L_2 = L_2L_3 = \cdots = 1$ and $M_1, M_2, M_3, \ldots$ be points on CD such that $CM_1 = M_1M_2 = M_2M_3 = \cdots = 1$. Then $\sum_{n=1}^{a-1}(AL_n^2 + L_nM_n^2)$ is equal to

a. $\frac{1}{2}a(a-1)^2$ b. $\frac{1}{2}(a-1)(2a-1)(4a-1)$

c. $\frac{1}{2}a(a-1)(4a-1)$ d. none of these

114. If x, y, and z are in G.P. and $x+3$, $y+3$, and $z+3$ are in H.P., then

a. $y = 2$ b. $y = 3$

c. $y = 1$ d. $y = 0$

115. If x, y, and z are distinct prime numbers, then

a. x, y, and z may be in A.P. but not in G.P.

b. x, y, and z may be in G.P. but not in A.P.

c. x, y, and z can neither be in A.P. nor in G.P.

d. none of these

Multiple Correct Answers Type

Solutions on page 3.59

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. For the series, $S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \cdots$

a. 7^{th} term is 16

b. 7^{th} term is 18

c. sum of first 10 terms is $\frac{505}{4}$

d. sum of first 10 terms is $\frac{405}{4}$

2. If sum of an infinite G.P. $p, 1, 1/p, 1/p^2, \ldots$ is 9/2, then value of p is

a. 2 b. 3/2

c. 3 d. 9/2

3. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

a. $a - b = d - c$

b. $e = 0$

c. $a, b - 2/3, c - 1$ are in A.P.

d. $(b+d)/a$ is an integer

4. The terms of an infinitely decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is 32/81, then

a. $r = 1/3$ b. $r = 2\sqrt{2}/3$

c. $S_\infty = 6$ d. none of these

5. The consecutive digits of a three digit number are in G.P. If the middle digit be increased by 2, then they form an A.P. If 792 is subtracted from this, then we get the number constituting of same three digits but in reverse order. Then number is divisible by

a. 7 b. 49

c. 19 d. none of these

6. If $S_n = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots$, then

a. $S_{40} = -820$ b. $S_{2n} > S_{2n+2}$

c. $S_{51} = 1326$ d. $S_{2n+1} > S_{2n-1}$

7. Given that $x + y + z = 15$ when a, x, y, z, b are in A.P. and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$ when a, x, y, z, b are in H.P. Then

a. G.M. of a and b is 3

b. one possible value of $a + 2b$ is 11

c. A.M. of a and b is 6

d. greatest value of $a - b$ is 8

8. Let $a_1, a_2, a_3, \ldots, a_n$ be in G.P. such that $3a_1 + 7a_2 + 3a_3 - 4a_5 = 0$. Then common ratio of G.P. can be

a. 2 b. $\frac{3}{2}$

c. $\frac{5}{2}$ d. $-\frac{1}{2}$

9. $\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots n$ terms, is equal to

a. $\frac{\sqrt{3n+2}-\sqrt{2}}{3}$ b. $\frac{n}{\sqrt{2+3n}+\sqrt{2}}$

c. less than n d. less than $\sqrt{\frac{n}{3}}$

10. If a, b, and c are in H.P. then the value of $\frac{(ac+ab-bc)(ab+bc-ac)}{(abc)^2}$ is

a. $\frac{(a+c)(3a-c)}{4a^2c^2}$ b. $\frac{2}{bc} - \frac{1}{b^2}$

c. $\frac{2}{bc} - \frac{1}{a^2}$ d. $\frac{(a-c)(3a+c)}{4a^2c^2}$

11. If $p(x) = \frac{1+x^2+x^4+\cdots+x^{2n-2}}{1+x+x^2+\cdots+x^{n-1}}$ is a polynomial in x, then n can be

a. 5 b. 10

c. 20 d. 17

12. For an increasing A.P. $a_1, a_2, \ldots, a_n$ if $a_1 + a_3 + a_5 = -12$ and $a_1a_3a_5 = 80$, then which of the following is/are true?

a. $a_1 = -10$ **b.** $a_2 = -1$
c. $a_3 = -4$ **d.** $a_5 = +2$

13. If $n > 1$, the values of the positive integer m for which $n^m + 1$ divides $a = 1 + n + n^2 + \cdots + n^{63}$ is/are

a. 8 **b.** 16
c. 32 **d.** 64

14. If p, q, and r are in A.P. then which of the following is/are true?

a. p^{th}, q^{th} and r^{th} terms of A.P, are in A.P.
b. p^{th}, q^{th} and r^{th} terms of G.P. are in G.P.
c. p^{th}, q^{th} and r^{th} terms of H.P, are in H.P.
d. none of these

15. If $1 + 2x + 3x^2 + 4x^3 + \cdots \infty \geq 4$, then

a. least value of x is 1/2
b. greatest value of x is 4/3
c. least value of is x 2/3
d. greatest value of x does not exists

16. Let $E = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$. Then,

a. $E < 3$ **b.** $E > 3/2$ **c.** $E > 2$ **d.** $E < 2$

17. If the sum of n terms of an A.P. is given by $S_n = a + bn + cn^2$, where a, b, c are independent of n, then

a. $a = 0$
b. common difference of A.P. must be $2b$
c. common difference of A.P. must be $2c$
d. first term of A.P. is $b + c$

18. If $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$, then

a. x, y, and z are in H.P. **b.** $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

c. x, y, z are in G.P. **d.** $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in G.P.

19. If a, b, c, and d are four unequal positive numbers which are in A.P., then

a. $\frac{1}{a} + \frac{1}{d} > \frac{1}{b} + \frac{1}{c}$ **b.** $\frac{1}{a} + \frac{1}{d} < \frac{1}{b} + \frac{1}{c}$

c. $\frac{1}{b} + \frac{1}{c} > \frac{4}{a+d}$ **d.** $\frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c}$

20. The next term of the G.P. x, $x^2 + 2$, and $x^3 + 10$ is

a. $\frac{729}{16}$ **b.** 6
c. 0 **d.** 54

21. In the 20^{th} row of the triangle

1
2 3
4 5 6
7 8 9 10
. . . .

a. last term = 210 **b.** first term = 191
c. sum = 4010 **d.** sum = 4200

22. If A_1, A_2; G_1, G_2; and H_1, H_2 are two arithmetic, geometric and harmonic means respectively, between two quantities a and b, then ab is equal to

a. A_1H_2 **b.** A_2H_1
c. $G_1 G_2$ **d.** none of these

23. Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm?

a. 7 **b.** 8
c. 9 **d.** 10

24. If $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$, then

a. a, b, and c are in H.P.
b. a, b, and c are in A.P.
c. $b = a + c$
d. $3a = b + c$

25. If a, b, and c are in G.P. and x and y, respectively, be arithmetic means between a, b and b, c, then

a. $\frac{a}{x} + \frac{c}{y} = 2$ **b.** $\frac{a}{x} + \frac{c}{y} = \frac{c}{a}$

c. $\frac{1}{x} + \frac{1}{y} = \frac{2}{b}$ **d.** $\frac{1}{x} + \frac{1}{y} = \frac{2}{ac}$

26. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \frac{a_{n-1}^2}{a_{n-2}}$ for all $n \geq 3$, terms of the sequence being distinct. Given that a_2 and a_5 are positive integers and $a_5 \leq 162$ then the possible value(s) of a_5 can be

a. 162 **b.** 64
c. 32 **d.** 2

27. Which of the following can be terms (not necessarily consecutive) of any A.P.

a. 1, 6, 19 **b.** $\sqrt{2}, \sqrt{50}, \sqrt{98}$
c. log2, log16, log128 **d.** $\sqrt{2}, \sqrt{3}, \sqrt{7}$

28. The numbers 1, 4, 16 can be three terms (not necessarily consecutive) of

a. no AP **b.** only one GP
c. infinite number of APs **d.** infinite number of GPs

Reasoning Type

Solutions on page 3.63

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be the terms of a single A.P. with non-zero common difference.
   Statement 2: If p, q, r ($p \neq q$) are terms (not necessarily consecutive) of an A.P., then there exists a rational number k such that $(r-q)/(q-p)=k$.
2. **Statement 1:** In a G.P. if the $(m+n)^{\text{th}}$ term be p and $(m-n)^{\text{th}}$ term be q, then its m^{th} term is $\sqrt{pq}$.
   Statement 2: T_{m+n}, T_m, T_{m-n} are in G.P.
3. **Statement 1:** There are infinite geometric progressions for which 27, 8 and 12 are three of its terms (not necessarily consecutive).
   Statement 2: Given terms are integers.
4. **Statement 1:** If $x^2+9y^2+25z^2 = xyz\left(\frac{15}{x}+\frac{5}{y}+\frac{3}{z}\right)$, then x, y, z are in H.P.
   Statement 2: If $a_1^{\,2}+a_2^{\,2}+\cdots+a_n^{\,2}=0$, then $a_1=a_2=a_3=\cdots=a_n=0$.
5. **Statement 1:** Coefficient of x^{14} in $(1+2x+3x^2+\cdots+16x^{15})^2$ is 560.
   Statement 2: $\sum_{r=1}^{n} r(n-r) = \frac{n(n^2-1)}{6}$.
6. **Statement 1:** $x = 1111\cdots91$ times is composite number.
   Statement 2: 91 is composite number.
7. Let $a, r \in R - \{0, 1, -1\}$ and n be an even number.
   Statement 1: $a \times ar \times ar^2 \cdots ar^{n-1} = (a^2 r^{n-1})^{n/2}$.
   Statement 2: Product of i^{th} term from the beginning and from the end in a G.P. is independent of i.
8. **Statement 1:** Sum of the series $1^3-2^3+3^3-4^3+\cdots+11^3 = 378$.
   Statement 2: For any odd integer $n \geq 1$, $n^3-(n-1)^3+\cdots+(-1)^{n-1}1^3 = \frac{1}{4}(2n-1)(n+1)^2$.
9. **Statement 1:** If an infinite G.P. has 2nd term x and its sum is 4, then x belongs to $(-8, 1)$.
   Statement 2: Sum of an infinite G.P. is finite if for its common ratio r, $0<|r|<1$.
10. **Statement 1:** Let $p_1, p_2, \ldots, p_n$ and x be distinct real number such that $\left(\sum_{r=1}^{n-1} p_r^2\right)x^2 + 2\left(\sum_{r=1}^{n-1} p_r p_{r+1}\right)x + \sum_{r=2}^{n} p_r^2 \leq 0$, then $p_1, p_2, \ldots, p_n$ are in G.P. and when $a_1^2+a_2^2+a_3^2+\cdots+a_n^2=0$, $a_1=a_2=a_3=\cdots=a_n=0$
    Statement 2: If $\frac{p_2}{p_1}=\frac{p_3}{p_2}=\cdots=\frac{p_n}{p_{n-1}}$, then $p_1, p_2, \ldots, p_n$ are in G.P.
11. **Statement 1:** $1^{99}+2^{99}+\cdots+100^{99}$ is divisible by 10100.
    Statement 2: a^n+b^n is divisible by $a+b$ if n is odd.
12. **Statement 1:** If the arithmetic mean of two numbers is 5/2, geometric mean of the numbers is 2, then the harmonic mean will be 8/5.
    Statement 2: For a group of positive numbers $(\text{G.M.})^2 = (\text{A.M.}) \times (\text{H.M.})$.

Linked Comprehension Type

Solutions on page 3.64

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Sum of certain consecutive odd positive integers is 57^2-13^2.

1. Number of integers are
   a. 40 **b.** 37
   c. 44 **d.** 51
2. The least value of an integer is
   a. 22 **b.** 27
   c. 31 **d.** 43
3. The greatest integer is
   a. divisible by 7 **b.** divisible by 11
   c. divisible by 9 **d.** none of these

For Problems 4–6

Consider three distinct real numbers a, b, c in a G.P. with $a^2+b^2+c^2=t^2$ and $a+b+c=\alpha t$. Sum of the common ratio and its reciprocal is denoted by S.

4. Complete set of α^2 is
   a. $\left(\frac{1}{3}, 3\right)$ **b.** $\left[\frac{1}{3}, 3\right]$
   c. $\left(\frac{1}{3}, 3\right) - \{1\}$ **d.** $\left(-\infty, \frac{1}{3}\right) \cup (3, \infty)$
5. Complete set of S is
   a. $(-2, 2)$ **b.** $(-\infty, -2) \cup (2, \infty)$
   c. $(-1, 1)$ **d.** $(-\infty, -1) \cup (1, \infty)$
6. If a, b, and c also represent the sides of a triangle, then the complete set of α^2 is
   a. $\left(\frac{1}{3}, 3\right)$ **b.** $(2, 3)$
   c. $\left[\frac{1}{3}, 2\right]$ **d.** $\left(\frac{\sqrt{5}+3}{2}, 3\right)$

For Problems 7–9

In a G.P., the sum of the first and last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126.

7. If an increasing G.P. is considered, then the number of terms in G.P. is
   a. 9 **b.** 8
   c. 12 **d.** 6
8. If the decreasing G.P. is considered, then the sum of infinite terms is
   a. 64 **b.** 128
   c. 256 **d.** 729
9. In any case, the difference of the least and greatest term is
   a. 78 **b.** 126
   c. 126 **d.** none of these

For Problems 10–12

Four different integers form an increasing A.P. One of these numbers is equal to the sum of the squares of the other three numbers. Then

10. The product of all numbers is

a. –2 **b.** 1 **c.** 0 **d.** 2

11. The sum of all the four numbers is

a. 3 **b.** 0 **c.** 4 **d** 2

12. The common difference of the four numbers is

a. 1 **b.** 3 **c.** 2 **d.** – 2

For Problems 13–15

Consider the sequence in the form of groups (1), (2, 2), (3, 3, 3), (4, 4, 4, 4), (5, 5, 5, 5, 5), ...

13. The 2000[th] term of the sequence is not divisible by

a. 3 **b.** 9 **c.** 7 **d.** none of these

14. The sum of first 2000 terms is

a. 84336 **b.** 96324 **c.** 78466 **d.** none of these

15. The sum of the remaining terms in the group after 2000[th] term in which 2000[th] term lies is

a. 1088 **b.** 1008 **c.** 1040 **d.** none of these

For Problems 16–18

There are two sets A and B each of which consists of three numbers in A.P. whose sum is 15 and where D and d are the common differences such that $D - d = 1$. If $\frac{p}{q} = \frac{7}{8}$, where p and q are the product of the numbers, respectively, and $d > 0$ in the two sets.

16. Sum of the product of the numbers in set A taken two at a time is

a. 51 **b.** 71 **c.** 74 **d.** 86

17. Sum of the product of the numbers in set B taken two at a time is

a. 74 **b.** 64 **c.** 73 **d.** 81

18. Value of $q - p$ is

a. 20 **b.** 30 **c.** 15 **d.** 25

For Problems 19–21

Let $A_1, A_2, A_3, \ldots, A_m$ be the arithmetic means between –2 and 1027 and $G_1, G_2, G_3, \ldots, G_n$ be the geometric means between 1 and 1024. The product of geometric means is 2^{45} and sum of arithmetic means is 1025×171.

19. The value of $\sum_{r=1}^{n} G_r$ is

a. 512 **b.** 2046 **c.** 1022 **d.** none of these

20. The number of arithmetic means is

a. 442 **b.** 342 **c.** 378 **d.** none of these

21. The numbers $2A_{171}, G_5^2 + 1, 2A_{172}$ are in

a. A. P. **b.** G.P. **c.** H.P. **d.** none of these

For Problems 22–24

Two consecutive numbers from 1, 2, 3, ..., n are removed. The arit metic mean of the remaining numbers is 105/4.

22. The value of n lies in

a. [45, 55] **b.** [52, 60] **c.** [41, 49] **d.** none of these

23. The removed numbers

a. lie between 10 and 20 **b.** are greater than 10 **c.** are less than 15 **d.** none of these

24. Sum of all numbers

a. exceeds 1600 **b.** is less than 1500 **c.** lies between 1300 and 1500 **d.** none of these

For Problems 25–27

Two arithmetic progressions have the same numbers. The ratio of the last term of the first progression to first term of the second progression is equal to the ratio of the last term of the second progression to the first term of the first progression and is equal to 4, the ratio of the sum of the n terms of the first progression to the sum of the n terms of the second progression is equal to 2.

25. The ratio of their common difference is

a. 12 **b.** 24 **c.** 26 **d.** 9

26. The ratio of their nth term is

a. 6/5 **b.** 7/2 **c.** 9/5 **d.** none of these

27. Ratio of their first term is

a. 2/7 **b.** 3/5 **c.** 4/7 **d.** 2/5

For Problems 28–30

The numbers a, b, and c are between 2 and 18, such that

(i) their sum is 25

(ii) the numbers 2, a, and b are consecutive terms of an A.P.

(iii) the numbers b, c, 18 are consecutive terms of a G.P.

28. The value of abc is

a. 500 **b.** 450 **c.** 720 **d.** none of these

29. Roots of the equation $ax^2 + bx + c = 0$ are

a. real and positive

b. real and negative

c. imaginary

d. real and of opposite sign

30. If a, b, and c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then the value of r is

a. 184 **b.** 196 **c.** 224 **d.** none of these

For Problems 31–33

Let $T_1, T_2, T_3, \ldots, T_n$ be the terms of a sequence and let $(T_2 - T_1) = T'_1$, $(T_3 - T_2) = T'_2, \ldots, (T_n - T_{n-1}) = T'_{n-1}$.

Case I:

If $T'_1, T'_2, \ldots, T'_{n-1}$ are in A.P., then T_n is quadratic in 'n'. If $T'_1 - T'_2, T'_2 - T'_3, \ldots$, are in A.P., then T_n is cubic in n.

Case II:

If $T'_1, T'_2, \ldots, T'_{n-1}$ are not in A.P., but in G.P., then $T_n = ar^n + b$, where r is the common ratio of the G.P. $T'_1, T'_2, T'_3, \ldots$ and $a, b \in R$.
Again, if $T'_1, T'_2, \ldots, T'_{n-1}$ are not in G.P. but $T'_2 - T'_1, T'_3 - T'_2, \ldots, T'_{n-2}$ are in G.P., then T_n is of the form $ar^n + bn + c$ and r is the common ratio of the G.P. $T'_2 - T'_1, T'_3 - T'_2, T'_4 - T_3, \ldots$ and $a, b, c \in R$.

31. The sum of 20 terms of the series $3 + 7 + 14 + 24 + 37 + \cdots$ is

a. 4010 **b.** 3860
c. 4240 **d.** none of these

32. The 100th term of the series $3 + 8 + 22 + 72 + 266 + 1036 + \cdots$ is divisible by 2^n, then maximum value of n is

a. 4 **b.** 2
c. 3 **d.** 5

33. For the series $2 + 12 + 36 + 80 + 150 + 252 + \cdots$, the value of $\lim_{n\to\infty} \frac{T_n}{n^3}$ is (where T_n is n^{th} term)

a. 2 **b.** 1/2
c. 1 **d.** none of these

Matrix-Match Type

Solutions on page 3.67

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, a→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1.

Column I	Column II
a. If a, b, c are in G.P., then $\log_a 10$, $\log_b 10$, $\log_c 10$ are in	**p.** A.P.
b. If $\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} = \frac{c+de^x}{c-de^x}$, then a, b, c, d are in	**q.** H.P.
c. If a, b, c are in A.P.; a, x, b are in G.P. and b, y, c are in G.P., then x^2, b^2, y^2 are in	**r.** G.P.
d. If x, y, z are in G.P., $a^x = b^y = c^z$, then $\log a, \log b, \log c$ are in	**s.** none of these

2.

Column I	Column II
a. If $\Sigma n = 210$, then Σn^2 is divisible by the greatest prime number which is greater than	**p.** 16
b. Between 4 and 2916 is inserted odd number $(2n + 1)$ G.M's. Then the $(n + 1)$th G.M. is divisible by greatest odd integer which is less than	**q.** 10
c. In a certain progression, four consecutive terms are 40, 30, 24, 20. Then the integral part of the next term of the progression is more than	**r.** 34
d. $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$ to $\infty = \frac{a}{b}$, where H.C.F.$(a, b) = 1$, then $a - b$ is less than	**s.** 30

Integer Type

Solutions on page 3.68

1. If the roots of $10x^3 - nx^2 - 54x - 27 = 0$ are in harmonic progression, then 'n' equals.

2. The difference between the sum of the first k terms of the series $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$ and the sum of the first k terms of $1 + 2 + 3 + \ldots\ldots + n$ is 1980. The value of k is.

3. The value of the $\sum_{n=0}^{\infty} \frac{2n+3}{3^n}$ is equal to.

4. Let $a_1, a_2, a_3, \ldots, a_{101}$ are in G.P. with $a_{101} = 25$ and $\sum_{i=1}^{201} a_i = 625$. Then the value of $\sum_{i=1}^{201} \frac{1}{a_i}$ equals.

5. Let $S = \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt[4]{n}+\sqrt[4]{n+1}\right)}$, then S equals.

6. The 5th and 8th terms of a geometric sequence of real numbers are 7! and 8! respectively. If the sum to first n terms of the G.P. is 2205, then n equals.

7. Let a, b, c, d be four distinct real numbers in A.P. Then half of the smallest positive value of k satisfying $2(a - b) + k(b - c)^2 + (c - a)^3 = 2(a - d) + (b - d)^2 + (c - d)^3$ is.

8. Number of positive integral ordered pairs of (a, b) such that $6, a, b$ are in harmonic progression is.

9. For $a, b > 0$, let $5a - b, 2a + b, a + 2b$ be in A.P. and $(b + 1)^2, ab + 1, (a - 1)^2$ are in G.P., then the value of $(a^{-1} + b^{-1})$ is.

10. The coefficient of the quadratic equation $ax^2 + (a + d)x + (a + 2d) = 0$ are consecutive terms of a positively valued, increasing arithmetic sequence. Then the least integral value of $\frac{d}{a}$ such that the equation has real solutions is.

11. Let $a + ar_1 + ar_1^2 + \ldots + \infty$ and $a + ar_2 + ar_2^2 + \ldots + \infty$ be two infinite series of positive numbers with the same first term. The sum of the first series is r_1 and the sum of the second series is r_2. Then the value of $(r_1 + r_2)$ is.

12. If the equation $x^3 + ax^2 + bx + 216 = 0$ has three real roots in G.P., then b/a has the value equal to.

13. Let $a_n = 16, 4, 1, \ldots$ be a geometric sequence. Define P_n as the product of the first n terms. Then the value of $\frac{1}{4}\sum_{n=1}^{\infty} \sqrt[n]{P_n}$ is.

14. The terms a_1, a_2, a_3 form an arithmetic sequence whose sum is 18. The terms $a_1 + 1$, a_2 $a_3 + 2$, in that order, form a geometric sequence. Then the absolute value of the sum of all possible common difference of the A.P. is.

15. Given a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P.If $a = 2$ and $e = 18$, then the sum of all possible value of 'c' is.

16. Let sum of first three terms of G.P. with real terms is $\frac{13}{12}$ and their product is -1. If the absolute value of the sum of their infinite terms is S, then the value of $7S$ is.

17. Let S denote sum of the series $\frac{3}{2^3}+\frac{4}{2^4\cdot 3}+\frac{5}{2^6\cdot 3}+\frac{6}{2^7\cdot 5}+\cdots\infty$ Then the value of S^{-1} is.

18. The first term of an arithmetic progression is 1 and the sum of the first nine terms equal to 369. The first and the ninth term of a geometric progression coincide with the first and the ninth term of the arithmetic progression. If the seventh term of the geometric progression is T_7, then the value of $T_7/9$ is.

Archives

Solutions on page 3.70

Subjective Type

1. The harmonic mean of two numbers is 4. Their arithmetic mean A and the geometric mean G satisfy the relation $2A + G^2 = 27$. Find the two numbers. **(IIT-JEE, 1997)**

2. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°. Find the number of sides of the polygon. **(IIT-JEE, 1980)**

3. If $a_1, a_2, \cdots, a_n$ are in arithmetic progression, where $a_i > 0$ for all i. Show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

(IIT-JEE, 1982)

4. Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exists, how many such progressions are possible? **(IIT-JEE, 1982)**

5. Find three numbers a, b, and c, between 2 and 18, such that
   (i) their sum is 25
   (ii) the numbers 2, a, b, are consecutive terms of an A.P. and
   (iii) the numbers b, c, and 18 are consecutive terms of a G.P. **(IIT-JEE, 1983)**

6. The sum of the squares of three distinct real numbers, which are in G.P., is S^2. If their sum is aS, show that $a^2 \in \left(\frac{1}{3}, 1\right) \cup (1,3)$. **(IIT-JEE, 1986)**

7. If $\log_3 2$, $\log_3 (2^x - 5)$, and $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, determine the value of x. **(IIT-JEE, 1990)**

8. Let p be the first of the n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$. **(IIT-JEE, 1991)**

9. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of an infinite geometric series whose first terms are $1, 2, 3, \ldots, n$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$, respectively, then find the value of $S_1^2 + S_2^2 + S_3^2 + \cdots + S_{2n-1}^2$. **(IIT-JEE, 1991)**

10. The real number x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie. **(IIT-JEE, 1996)**

11. Let a, b, c, and d be real numbers in a G.P. u, v, w, satisfy the system of equations
$u + 2v + 3w = 6$
$4u + 5v + 6w = 12$
$6u + 9v = 4$
Show that the roots of the equation $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + u + v + w = 0$ and $20x^2 + 10(a-d)^2 x - 9 = 0$ are reciprocals of each other. **(IIT-JEE, 1999)**

12. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive of it. Prove that the resulting sum is the squares of an integer. **(IIT-JEE, 2000)**

13. Let $a_1, a_2, \ldots$ be positive real numbers in a geometric progression. For each n, let A_n, G_n, H_n be, respectively, the arithmetic mean, geometric mean and harmonic mean of $a_1, a_2, \ldots, a_n$. Find an expression for the geometric mean of $G_1, G_2, \ldots, G_n$ in terms of $A_1, A_2, \ldots, A_n, H_1, H_2, \ldots, H_n$. **(IIT-JEE, 2005)**

14. Let a, b be positive real numbers. If a A_1, A_2, b be are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$$ **(IIT-JEE, 2002)**

15. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then prove that either $a = b = c$ or a, b, $-\frac{c}{2}$ form a G.P. **(IIT-JEE, 2003)**

16. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the least natural number n_0 such that $b_n > a_n \; \forall \; n \geq n_0$. **(IIT-JEE, 2006)**

Objective Type

Fill in the blanks

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is ______________. **(IIT-JEE, 1984)**

2. The sum of the first n terms of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \cdots$ is $n(n+1)^2/2$, when n is even. When n is odd, the sum is ______________. **(IIT-JEE, 1988)**

3. Let the harmonic mean and geometric mean of two positive numbers be in the ratio 4 : 5. Then the two numbers are in the ratio ______________. **(IIT-JEE, 1992)**

4. For any odd integer $n \geq 1$, $n^3 - (n-1)^3 + \cdots + (-1)^{n-1} 1^3$ = ______________. **(IIT-JEE, 1996)**

5. Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. Then $\frac{y^3+z^3}{xyz}$ = ______________. **(IIT-JEE, 1997)**

6. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then A = ______________. **(IIT-JEE, 1997)**

Multiple choice questions with one correct answer

1. If x, y and z are p^{th}, q^{th} and r^{th} terms respectively of an A.P. and also of a G.P., then $x^{y-z}\, y^{z-x}\, z^{x-y}$ is equal to

a. xyz **b.** 0
c. 1 **d.** none of these

2. The third term of a geometric progression is 4. The product of the first five terms is

a. 4^3 **b.** 4^5
c. 4^4 **d.** none of these
(IIT-JEE, 1982)

3. The rational number which equals the number $2.\overline{357}$ with recurring decimal is

a. $\frac{2355}{1001}$ **b.** $\frac{2379}{997}$
c. $\frac{2355}{999}$ **d.** none of these
(IIT-JEE, 1983)

4. If a, b, and c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

a. A.P. **b.** G.P.
c. H. P. **d.** none of these
(IIT-JEE, 1985)

5. Sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots$ is equal to

a. $2^n - n - 1$ **b.** $1 - 2^{-n}$
c. $n + 2^{-n} - 1$ **d.** $2^n + 1$ **(IIT-JEE, 1988)**

6. Find the sum $(x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2 + \cdots + (x+1)^{n-1}$

a. $(x+2)^{n-2} - (x+1)^n$ **b.** $(x+2)^{n-1} - (x+1)^{n-1}$
c. $(x+2)^n - (x+1)^n$ **d.** none of these
(IIT-JEE, 1990)

7. If $\ln(a + c)$, $\ln(a - c)$, and $\ln(a - 2b + c)$ are in A.P., then

a. a, b, c are in A.P. **b.** a^2, b^2, c^2 are in A.P.
c. a, b, c are in G.P. **d.** a, b, c are in H.P.
(IIT-JEE, 1994)

8. Let $a_1, a_2, \ldots, a_{10}$ be in A.P. and $h_1, h_2, \ldots, h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is

a. 2 **b.** 3
c. 5 **d.** 6 **(IIT-JEE, 1999)**

9. The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 8 + 2\sqrt{5} = 0$ is

a. 2 **b.** 4
c. 6 **d.** 8 **(IIT-JEE, 1999)**

10. Let the positive numbers a, b, c, and d be in A.P. Then abc, abd, acd, and bcd are

a. not in A.P./G.P./H.P. **b.** in A.P.
c. in G.P. **d.** in H.P. **(IIT-JEE, 2001)**

11. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then

a. $a = \frac{4}{7}, r = \frac{3}{7}$ **b.** $a = 2, r = \frac{3}{8}$
c. $a = \frac{3}{2}, r = \frac{1}{2}$ **d.** $a = 3, r = \frac{1}{4}$
(IIT-JEE, 2001)

12. Let α, and β be the roots of $x^2 - x + p = 0$ and γ and δ be the root of $x^2 - 4x + q = 0$. If α, β, and γ, δ are in G.P., then the integral values of p and q, respectively, are

a. $-2, -32$ **b.** $-2, 3$
c. $-6, 3$ **d.** $-6, -32$

13. If the sum of the first $2n$ terms of the A.P. 2, 5, 8, …, is equal to the sum of the first n terms of A.P. 57, 59, 61, …, then n equals

a. 10 **b.** 12
c. 11 **d.** 13

14. Suppose a, b, and c are in A.P. and a^2, b^2, and c^2 are in G.P., if $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is

a. $\frac{1}{2\sqrt{2}}$ **b.** $\frac{1}{2\sqrt{3}}$
c. $\frac{1}{2} - \frac{1}{\sqrt{3}}$ **d.** $\frac{1}{2} - \frac{1}{\sqrt{2}}$ **(IIT-JEE, 2002)**

15. An infinite G.P. has first term as a and sum 5, then

a. $a < -10$ **b.** $-10 < a < 10$
c. $0 < a < 10$ and $a \neq 5$ **d.** $a > 10$ **(IIT-JEE, 2004)**

16. In the quadratic equation $ax^2 + bx + c = 0$, $\Delta = b^2 - 4ac$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P. where α, β are the roots of $ax^2 + bx + c = 0$, then

a. $\Delta \neq 0$ **b.** $b\Delta = 0$
c. $c\,\Delta = 0$ **d.** $\Delta = 0$ **(IIT-JEE, 2005)**

Multiple choice questions with one or more than one correct answer

1. If the first and the $(2n - 1)^{st}$ terms of an A.P., a G.P. and a H.P. are equal and their n^{th} terms are a, b and c respectively, then

a. $a = b = c$ **b.** $a \geq b \geq c$
c. $a + b = b$ **d.** $ac - b^2 = 0$
(IIT-JEE, 1988)

2. For $0 < \phi < \pi/2$, if

$$x = \sum_{n=0}^{\infty} \cos^{2n}\phi,\ y = \sum_{n=0}^{\infty} \sin^{2n}\phi, \text{ and } z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi, \text{ then}$$

a. $xyz = xz + y$ **b.** $xyz = xy + z$
c. $xyz = x + y + z$ **d.** $xyz = yz + x$
(IIT-JEE, 1993)

3. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^{n} b_r \sin^r \theta$, for every value of θ, then
   a. $b_0 = 1, b_1 = 3$
   b. $b_0 = 0, b_1 = n$
   c. $b_0 = -1, b_1 = n$
   d. $b_0 = 0, b_1 = n^2 - 3n + 3$ **(IIT-JEE, 1998)**

4. Let T_r be the r^{th} term of an A.P., for $r = 1, 2, 3, \ldots$. If for some positive integers m, n, we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals
   a. $\frac{1}{mn}$ b. $\frac{1}{m} + \frac{1}{n}$
   c. 1 d. 0 **(IIT-JEE, 1998)**

5. If $x > 1, y > 1$, and $z > 1$ are in G.P., then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}$, and $\frac{1}{1+\ln z}$ are in
   a. A.P. b. H.P.
   c. G.P. d. none of these **(IIT-JEE, 1998)**

6. For a positive integer n, let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{(2^n) - 1}$. Then,
   a. $a(100) \le 100$ b. $a(100) > 100$
   c. $a(200) \le 100$ d. $a(200) > 100$ **(IIT-JEE, 1999)**

Comprehension

For Problems 1–3

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $2r - 1$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$. **(IIT-JEE, 2007)**

1. The sum $V_1 + V_2 + \cdots + V_n$ is
   a. $\frac{1}{12} n(n+1)(3n^2 - n + 1)$
   b. $\frac{1}{12} n(n+1)(3n^2 + n + 2)$
   c. $\frac{1}{2} n(2n^2 - n + 1)$
   d. $\frac{1}{3}(2n^3 - 2n + 3)$

2. T_r is always
   a. an odd number b. an even number
   c. a prime number d. a composite number

3. Which one of the following is a correct statement?
   a. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 5
   b. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 6
   c. $Q_1, Q_2, Q_3, \ldots$ are in A. P. with common difference 11
   d. $Q_1 = Q_2 = Q_3 = \cdots$

For Problems 4–6

Let A_1, G_1, and H_1 denote the arithmetic geometric and harmonic means, respectively, of two distinct positive numbers. For $n \ge 2$, let A_{n-1} and H_{n-1} have arithmetic and harmonic means as A_n, G_n, H_n, respectively. **(IIT-JEE, 2007)**

4. Which one of the following statements is correct?
   a. $G_1 > G_2 > G_3 > \cdots$
   b. $G_1 < G_2 < G_3 < \cdots$
   c. $G_1 = G_2 = G_3 = \cdots$
   d. $G_1 < G_3 < G_5 < \cdots$ and $G_2 > G_4 > G_6 > \cdots$

5. Which one of the following statements is correct?
   a. $A_1 > A_2 > A_3 > \cdots$
   b. $A_1 < A_2 < A_3 < \cdots$
   c. $A_1 > A_3 > A_5 > \cdots$ and $A_2 < A_4 < A_6 >$
   d. $A_1 < A_3 < A_5 < \cdots$ and $A_2 > A_4 > A_6 > \cdots$

6. Which one of the following statements is correct?
   a. $H_1 > H_2 > H_3 > \cdots$
   b. $H_1 < H_2 < H_3 < \cdots$
   c. $H_1 > H_3 > H_5 > \cdots$ and $H_2 < H_4 < H_6 < \cdots$
   d. $H_1 < H_3 < H_6 < \cdots$ and $H_2 > H_4 > H_6 > \cdots$

Integer type

1. Let S_k, $k = 1, 2, \ldots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$, then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} (k^2 - 3k + 1) S_k$ is. **(IIT-JEE, 2010)**

2. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \cdots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \cdots + a_{11}}{11}$ is equal to. **(IIT-JEE, 2010)**

3. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is. **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}$ are in G. P.

$\Rightarrow \quad x = \sqrt{a^2-x^2}$

$\Rightarrow \quad x^2 = a^2 - x^2$

$\Rightarrow \quad x = \dfrac{a}{\sqrt{2}} \qquad (\because \; a, x > 0)$

Let $\sqrt{x}$ be rational. Then,

$\sqrt{x} = \dfrac{p}{q}, p, q \in Z, q \neq 0$

$\Rightarrow \quad x = \dfrac{p^2}{q^2}$

$\Rightarrow \quad \sqrt{a-x} = \sqrt{\sqrt{2}x - x}$

$= \sqrt{\sqrt{2}-1}\sqrt{x}$

$= \sqrt{\sqrt{2}-1}\dfrac{p}{q}$ which is irrational.

Similarly, $\sqrt{a+x} = \sqrt{\sqrt{2}+1}\dfrac{p}{q}$

2. $\sqrt{a-x}$, $\sqrt{x}$, $\sqrt{a+x}$ are in A.P.

$\Rightarrow \quad 2\sqrt{x} = \sqrt{a-x} + \sqrt{a+x}$

$\Rightarrow \quad 4x = a - x + a + x + 2\sqrt{a^2-x^2}$ (squaring both sides)

$\Rightarrow \quad 2x - a = \sqrt{a^2-x^2}$

$\Rightarrow \quad 4x^2 - 4ax + a^2 = a^2 - x^2$

$\Rightarrow \quad 5x^2 = 4ax$

$\Rightarrow \quad a = \dfrac{5x}{4}$ (as $a, x > 0$)

Now, x must be a perfect square as $\sqrt{x}$ is an integer. Hence, $x = 1, 4, 9, 16, \ldots$, etc. For $x = 1$, $a = 5/4$ (rational number). For $x = 4$, $a = 5$ (prime number). For $x = 9$, $a = 45/4$ (rational number).For $x = 16$, $a = 20$ (composite number). Hence, the least composite value of a is 20.

3. Let the three digits be a, ar and ar^2. Each of the three quantities must lie between 1 and 9, and r must be rational. The three-digit number so formed can be written as $100a + 10ar + ar^2$. Now, from the given condition, the digits of the number $100a + 10ar + ar^2 - 100 = 100(a-1) + 10ar + ar^2$, i.e., $a-1$, ar, ar^2 are in A.P. Therefore,

$2ar = a - 1 + ar^2$

$\Rightarrow \quad a(r^2 + 1 - 2r) = 1$

$\Rightarrow \quad a(r-1)^2 = 1$

$\Rightarrow \quad r - 1 = \pm\dfrac{1}{\sqrt{a}}$

Since $(r-1)$ is rational, $\pm 1/\sqrt{a}$ must also be rational.Furthermore, $1 \le a - 1 \le 9$ so that $2 \le a \le 10$. Since $a \le 9$, we get $2 \le a \le 9$.

The only integer a between 2 and 9 such that $1/\sqrt{a}$ is rational is 4 or 9.

$\therefore \quad r - 1 = \pm\dfrac{1}{2}$ or $r - 1 = \pm\dfrac{1}{3}$

$\Rightarrow \quad r = \dfrac{3}{2}, \dfrac{1}{2}$ or $r = \dfrac{4}{3}, \dfrac{2}{3}$

Thus, $r = 3/2$ (rejecting $r = 1/2, 2/3, 4/3$). Hence, the required digits are 4, 6 and 9 forming the number 469.

4. Let there be $2n + 1$ stones. Clearly, one stone lies in the middle and n stones on each side of it in a row. Let P be the mid-stone and let A and B be the end stones on the left and right of P, respectively. Clearly, there are n intervals, each of length 10 m on both the sides of P. Now, suppose the man starts from A. He picks up the end stone on the left of mid-stone and goes to the mid-stone, drops it and goes to $(n-1)^{\text{th}}$ stone on left, picks it up, goes to the mid-stone and drops it. This process is repeated till he collects all stones on the left of the mid-stone at the mid-stone. So, the distance covered in collecting stones on the left of the mid-stones is

$$10 \times n + 2[10 \times (n-1) + 10 \times (n-2) + \cdots + 10 \times 2 + 10 \times 1]$$

After collecting all the stones on left of the mid-stone, the man goes to the stone B on the right side of the mid-stone, picks it up, goes to the mid-stone and drops it.

Then, he goes to n^{th} stone on the right and the process is repeated till he collects all stones at the mid-stone.

Distance covered in collecting the stones on the right side of the mid-stone is

$$2[10 \times n + 10 \times (n-1) + 10 \times (n-2) + \cdots + 10 \times 2 + 10 \times 1]$$

Therefore, total distance covered is

$$10 \times n + 2[10 \times (n-1) + 10 \times (n-2) + \cdots + 10 \times 2 + 10 \times 1] + 2[10 \times n + 10 \times (n-1) + \cdots + 10 \times 2 + 10 \times 1]$$

$= 4[10 \times n + 10 \times (n-1) + \cdots + 10 \times 2 + 10 \times 1] - 10 \times n$

$= 40[1 + 2 + 3 + \cdots + n] - 10n$

$= 40\left[\dfrac{n}{2}(1+n)\right] - 10n$

$= 20n(n+1) - 10n$

$= 20n^2 + 10n$

But the total distance covered is 3 km, i.e., 3000 m.

$\therefore \quad 20n^2 + 10n = 3000$

$\Rightarrow \quad 2n^2 + n - 300 = 0$

$\Rightarrow \quad (n-12)(2n+25) = 0$

$\Rightarrow \quad n = 12$

Hence, the number of stones is $2n + 1 = 25$.

5. Let r be the common ratio of the given G.P. Then,

$$b = n^{\text{th}} \text{ term} = ar^{n-1} \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$$

Now, product of the first n terms is

$$P = a \times ar \times ar^2 \cdots ar^{n-1}$$
$$= a^n r^{1+2+3+\cdots+(n-1)}$$
$$= a^n r^{\frac{n(n-1)}{2}} \quad \left[\because 1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}\right]$$
$$= a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right\}^{\frac{n(n-1)}{2}}$$
$$= a^n \left(\frac{b}{a}\right)^{n/2}$$
$$= a^{n/2} b^{n/2}$$
$$= (ab)^{n/2}$$
$$\therefore \quad P^2 = [(ab)^{n/2}]^2 = (ab)^n$$

6. $x = 1 + 3a + 6a^2 + 10a^3 + \cdots$ (1)

$\Rightarrow \quad ax = a + 3a^2 + 6a^3 + \cdots$ (2)

Subtracting (2) from (1), we have

$x(1-a) = 1 + 2a + 3a^2 + 4a^3 + \cdots$ (3)

Equation (3) is an arithmetico-geometric series. Therefore,

$$S_\infty = \frac{A}{1-R} + \frac{dR}{(1-R)^2}$$
$$\Rightarrow \quad x(1-a) = \frac{1}{1-a} + \frac{a}{(1-a)^2} = \frac{1}{(1-a)^2}$$
$$\Rightarrow \quad x = \frac{1}{(1-a)^3}$$
$$\Rightarrow \quad (1-a)^3 = x^{-1}$$
$$\Rightarrow \quad a = 1 - x^{-1/3}$$

Similarly,

$$b = 1 - y^{-1/4}$$

Now,

$$S = 1 + 3(ab) + 5(ab)^2 + \cdots \text{ infinity}$$
$$= \frac{1}{1-ab} + \frac{2ab}{(1-ab)^2}$$
$$= \frac{1+ab}{(1-ab)^2}$$
$$= \frac{1+\left(1-\frac{1}{x^{1/3}}\right)\left(1-\frac{1}{y^{1/4}}\right)}{\left(1-\left(1-\frac{1}{x^{1/3}}\right)\left(1-\frac{1}{y^{1/4}}\right)\right)^2}$$

7. Let $\frac{2n+1}{2n-1} = r$. Then, the given series is

$S = r + 3r^2 + 5r^3 + 7r^4 + \cdots + (2n-1)r^n$ (1)

$rS = r^2 + 3r^3 + 5r^4 + \cdots + (2n-3)r^n + (2n-1)r^{n+1}$ (2)

Subtracting (2) from (1), we get

$$(1-r)S = r + 2r^2 + 2r^3 + \cdots + 2r^n - (2n-1)r^{n+1}$$
$$\Rightarrow \quad 36(1-r) = r + \frac{2r^2(1-r^{n-1})}{1-r} - (2n-1)r^{n+1}$$
$$\Rightarrow \quad 36(1-r)^2 = r - r^2 + 2r^2 - 2r^{n+1} - (2n-1)r^{n+1} + (2n-1)r^{n+2}$$
$$= r + r^2 - (2n+1)r^{n+1} + (2n-1)r^{n+2}$$
$$= r + r^2 - (2n-1)\left[\frac{2n+1}{2n-1}r^{n+1} - r^{n+2}\right]$$
$$= r + r^2 - (2n-1)\left[r\, r^{n+1} - r^{n+2}\right]$$
$$= r(1+r)$$
$$\Rightarrow \quad 36\left[1-\frac{2n+1}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[1+\frac{2n+1}{2n-1}\right]$$
$$\Rightarrow \quad 36\left[\frac{-2}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[\frac{4n}{2n-1}\right]$$
$$\Rightarrow \quad 36 = n(2n+1)$$
$$\Rightarrow \quad n = 4$$

8.
$$t_n = \frac{n+2}{n(n+1)} \times \left(\frac{1}{2}\right)^n$$
$$= \frac{2(n+1)-n}{n(n+1)} \times \left(\frac{1}{2}\right)^n$$
$$= \frac{1}{n}\left(\frac{1}{2}\right)^{n-1} - \frac{1}{n+1} \times \left(\frac{1}{2}\right)^n$$
$$S_n = \sum_{n=}^{n} t_n$$
$$= \left\{\frac{1}{1}\left(\frac{1}{2}\right)^0 - \frac{1}{2}\left(\frac{1}{2}\right)^1\right\} + \left[\frac{1}{2}\left(\frac{1}{2}\right)^1 - \frac{1}{3}\left(\frac{1}{2}\right)^2\right] + \cdots + \left\{\frac{1}{n}\left(\frac{1}{2}\right)^{n-1} - \frac{1}{n+1}\left(\frac{1}{2}\right)^n\right\}$$
$$= 1 - \frac{1}{(n+1)\,2^n}$$

9. We have the following:

First term	Common difference	Sums of n terms
1	1	$S_1 = \frac{n}{2}[2 \times 1 + (n-1) \times 1]$
2	3	$S_2 = \frac{n}{2}[2 \times 2 + (n-1) \times 3$
3	5	$S_3 = \frac{n}{2}[2 \times 3 + (n-1) \times 5]$
⋮	⋮	⋮
m	$2m-1$	$S_m = \frac{n}{2}[2m + (n-1)(2m-1)]$

Hence, $S_1 + S_2 + \cdots + S_m$

$$= \frac{n}{2}[2\times 1+(n-1)\times 1]+\frac{n}{2}[2\times 2+(n-1)\times 3]+\cdots$$

$$+\frac{n}{2}[2m+(n-1)(2m-1)]$$

$$= \frac{n}{2}[2\times(1+2+3+\cdots+m)+(n-1)(1+3+5+\cdots+(2m-1)]$$

$$= \frac{n}{2}[2\times\frac{m}{2}(1+m)+(n-1)\frac{m}{2}(1+(2m-1))]$$

$$= \frac{n}{2}[m(m+1)+m^2(n-1)]$$

$$= \frac{mn}{2}(mn+1)$$

10. Let a be the first term and r the common ratio of the G.P. Then,

$$S_1 = a\left(\frac{r^n-1}{r-1}\right),\ S_2 = a\left(\frac{r^{2n}-1}{r-1}\right) \text{ and } S_3 = a\left(\frac{r^{3n}-1}{r-1}\right)$$

Now,

$$S_1(S_3-S_2) = a\left(\frac{r^n-1}{r-1}\right)\left\{a\left(\frac{r^{3n}-1}{r-1}\right)-a\left(\frac{r^{2n}-1}{r-1}\right)\right\}$$

$$= \frac{a^2}{(r-1)^2}(r^n-1)\{(r^{3n}-1)-(r^{2n}-1)\}$$

$$= \frac{a^2}{(r-1)^2}(r^n-1)(r^{3n}-r^{2n})$$

$$= \frac{a^2}{(r-1)^2}(r^n-1)r^{2n}(r^n-1)$$

$$= \left[ar^n\left(\frac{r^n-1}{r-1}\right)\right]^2$$

$$(S_2-S_1)^2 = \left[a\left(\frac{r^{2n}-1}{r-1}\right)-a\left(\frac{r^n-1}{r-1}\right)\right]^2$$

$$= \frac{a^2}{(r-1)^2}\{(r^{2n}-1)-(r^n-1)\}^2$$

$$= \frac{a^2}{(r-1)^2}\{r^n(r^n-1)\}^2$$

$$= \left[ar^n\left(\frac{r^n-1}{r-1}\right)\right]^2$$

$$\therefore \quad S_1(S_3-S_1) = (S_2-S_1)^2$$

11. Let the four numbers in G.P. be $\frac{a}{r^3}, \frac{a}{r}, ar,$ and ar^3. The product is

$a^4 = 4096 = 8^4$

$\Rightarrow a = 8$

The sum is

$$8\left(\frac{1}{r^3}+\frac{1}{r}+r+r^3\right)=85$$

$$\Rightarrow 8\left(r^3+\frac{1}{r^3}\right)+8\left(r+\frac{1}{r}\right)-85=0$$

$$\Rightarrow 8\left[\left(r+\frac{1}{r}\right)^3-3\left(r+\frac{1}{r}\right)\right]+8\left(r+\frac{1}{r}\right)-85=0 \quad (1)$$

Let $r + 1/r = t$. Hence, (1) becomes

$$8t^3 - 16t - 85 = 0 \quad (2)$$

Putting $2t = y$, we have

$y^3 - 8y - 85 = 0$

$\Rightarrow (y-5)(y^2+5y+17) = 0$

$\Rightarrow y = 2t = 5$

$$\Rightarrow 2\left(r+\frac{1}{r}\right)=5 \quad (3)$$

The other factor gives imaginary values. From (3),

$2r^2 - 5r + 2 = 0$

$\Rightarrow (r-2)(2r-1) = 0$

$\Rightarrow r = 2, \frac{1}{2}$ and $a = 8$

Hence, the four numbers are 1, 4, 16, 64 or 64, 16, 4, 1.

12. $d = 2, r = 1/2$

There are $4n + 1$ terms. Then the mid-term is $(2n + 1)^{\text{th}}$ term. T_{n+1} and t_{n+1} are mid-terms of A.P. and G.P.

$T_{n+1} = a + nd = a + 2n$

$$t_{n+1} = AR^n = T_{2n+1}\times\left(\frac{1}{2}\right)^n = (a+4n)\left(\frac{1}{2}\right)^n$$

By given condition,

$T_{n+1} = t_{n+1}$

$$\Rightarrow a + 2n = (a+4n)\frac{1}{2^n}$$

$\Rightarrow (2^n - 1)a = 4n - 2n\times 2^n$

$$\Rightarrow a = \frac{4n - n\times 2^{n+1}}{2^n-1}$$

Hence, the mid-term of the sequence is

$$a + 4n = \frac{4n - n\times 2^{n+1}}{2^n-1}+4n$$

$$= \frac{-n\times 2^{n+1}+2n\times 2^{n+1}}{2^n-1}$$

$$= \frac{n\times 2^{n+1}}{2^n-1}$$

13. We have,

$(a_1 + a_2 + \cdots + a_m)^2 = a_1^2 + a_2^2 + \cdots + a_m^2 + 2(a_1a_2 + a_2a_3 + \cdots)$

or

$$\left[\frac{a_1(1-r^m)}{1-r}\right]^2 = \frac{a_1^2(1-r^{2m})}{1-r^2}+2\sum_{i<j}^{m}a_ia_j$$

$$\Rightarrow 2\sum_{i<j}^{m}a_ia_j = \frac{a_1^2(1-r^m)^2}{(1-r)^2}-\frac{a_1^2(1-r^{2m})}{1-r^2}$$

$$= \frac{2a_1^2}{(1-r)^2(1+r)}[r-r^m-r^{m+1}+r^{2m}]$$

$$= \frac{2r}{1+r}\left\{a_1\times\frac{(1-r^{m-1})}{1-r}\right\}\left\{\frac{a_1(1-r^m)}{1-r}\right\}$$

$$\Rightarrow \frac{r+1}{r}\sum_{i<j}^{m}a_ia_j = S_{m-1}\times S_m$$

14. $\mathcal{T}_r = \tan\frac{x}{2^r}\sec\frac{x}{2^{r-1}}$

$$= \frac{\sin\frac{x}{2^r}}{\cos\frac{x}{2^r}\cos\frac{x}{2^{r-1}}}$$

$$= \frac{\sin\left(\frac{1}{2^{r-1}} - \frac{1}{2^r}\right)x}{\cos\frac{x}{2^r}\times\cos\frac{x}{2^{r-1}}}$$

$$= \frac{\sin\frac{x}{2^{r-1}}\cos\frac{x}{2^r} - \sin\frac{x}{2^r}\times\cos\frac{x}{2^{r-1}}}{\cos\frac{x}{2^r}\cos\frac{x}{2^{r-1}}}$$

$$= \tan\frac{x}{2^{r-1}} - \tan\frac{x}{2^r}$$

$$\Rightarrow \sum_{r=1}^{n} T_r = \tan\frac{x}{2^0} - \tan\frac{x}{2^n} = \tan x - \tan\frac{x}{2^n}$$

15. $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k}$ $(i\neq j\neq k)$

= sum when i, j, k are independent

− 3 × (sum when any two of i, j, k are equal)

+ 2 × (sum when $i = j = k$)

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} - 3\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{9^i3^k} + 2\sum_{i=0}^{\infty}\frac{1}{27^i}$$

$$= \left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)^3 - 3\left(\sum_{i=0}^{\infty}\frac{1}{9^i}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right) + 2\left(\sum_{i=0}^{\infty}\frac{1}{27^i}\right)$$

$$= \left(\frac{3}{2}\right)^3 - 3\left(\frac{9}{8}\right)\left(\frac{3}{2}\right) + 2\left(\frac{27}{26}\right)$$

$$= \frac{81}{208}$$

16. Let $\sqrt{a_1} = b_1$;

$\sqrt{a_2 - 1} = b_2$;

$\sqrt{a_3 - 2} = b_3$;

....

$\sqrt{a_n - (n-1)} = b_n$

$\therefore\ b_1 + b_2 + ... + b_n =$

$$\frac{1}{2}\left[b_1^2 + (b_2^2+1) + (b_3^2+2) + \cdots + (b_n^2 + (n-1))\right] - \frac{n(n-3)}{4}$$

$$\therefore\ \sum b_i = \frac{1}{2}[(b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2) +$$

$$(1 + 2 + 3 + \cdots + (n-1)] - \frac{n(n-3)}{4}$$

$$\Rightarrow\ 2\sum b_i = \sum b_i^2 + \frac{n(n-1)}{2} - \frac{n(n-3)}{2}$$

$$\Rightarrow\ 2\sum b_i = \sum b_i^2 + n$$

$$\therefore\ \sum b_i^2 - 2\sum b_i + \sum 1 = 0$$

$$\Rightarrow\ \sum_{i=1}^{n}(b_i - 1)^2 = 0$$

$b_1 - 1 = 0 \Rightarrow b_1^2 = a_1 = 1$

$b_2 - 1 = 0 \Rightarrow b_2^2 = a_2 - 1 = 1 \Rightarrow a_2 = 2$

b3 − 1 = 0 $\Rightarrow b_3^2 =$ a3 − 2 = 1 $\Rightarrow a_3 = 3$ and so on

Hence $a_n = n$

$$\therefore\ \sum_{i=1}^{100} a_i = 1 + 2 + 3 + ... + 100 = 5050$$

Objective Type

1. d. The given numbers are in A.P. Therefore,

$2\log_4(2^{1-x} + 1) = \log_2(5\times 2^x + 1) + 1$

$\Rightarrow\ 2\log_{2^2}\left(\frac{2}{2^x}+1\right) = \log_2(5\times 2^x + 1) + \log_2 2$

$\Rightarrow\ \frac{2}{2}\log_2\left(\frac{2}{2^x}+1\right) = \log_2(5\times 2^x + 1)2$

$\Rightarrow\ \log_2\left(\frac{2}{2^x}+1\right) = \log_2(10\times 2^x + 2)$

$\Rightarrow\ \frac{2}{2^x}+1 = 10\times 2^x + 2$

$\Rightarrow\ \frac{2}{y}+1 = 10y + 2$, where $2^x = y$

$\Rightarrow\ 10y^2 + y - 2 = 0$

$\Rightarrow\ (5y - 2)(2y + 1) = 0$

$\Rightarrow\ y = 2/5$ or $y = -1/2$

$\Rightarrow\ 2^x = 2/5$ or $2^x = -1/2$

$\Rightarrow\ x = \log_2(2/5)$ [$\because 2^x$ cannot be negative]

$\Rightarrow\ x = \log_2 2 - \log_2 5$

$\Rightarrow\ x = 1 - \log_2 5$

2. b. Since a, b, c are in A.P., therefore, $b - a = d$ and $c - b = d$, where d is the common difference of the A.P.

$\therefore\ a = b - d$ and $c = b + d$

Now,

$abc = 4$

$\Rightarrow\ (b-d)\,b\,(b+d) = 4$

$\Rightarrow\ b(b^2 - d^2) = 4$

But,

$b(b^2 - d^2) < b\times b^2$

$\Rightarrow\ b(b^2 - d^2) < b^3$

$\Rightarrow\ 4 < b^3$

$\Rightarrow\ b^3 > 4$

$\Rightarrow\ b > 2^{2/3}$

Hence, the minimum value of b is $2^{2/3}$.

3. a. n^{th} term of the series is $20 + (n-1)(-2/3)$.

For the sum to be maximum,

n^{th} term ≥ 0

$\Rightarrow\ 20 + (n-1)\left(-\frac{2}{3}\right) \geq 0$

$\Rightarrow\ n \leq 31$

Thus, the sum of 31 terms is maximum and is

$$\frac{31}{2}\left[40 + 30\times\left(-\frac{2}{3}\right)\right] = 310$$

4. d. 100^{th} term of 1, 11, 21, 31, ... is $1 + (100-1)10 = 991$.

100^{th} term of 31, 36, 41, 46, ... is $31 + (100-1)5 = 526$.

Let the largest common term be 526. Then,

$526 = 31 + (n-1)10$

$\Rightarrow\ n = 50.5$

But n is an integer; hence $n = 50$. Hence, the largest common term is $31 + (50-1)10 = 521$.

5. d. Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = S_n \Rightarrow \frac{m}{2}[2a + (m-1)d] = \frac{n}{2}[2a + (n-1)d]$$

$\Rightarrow\ 2a(m-n) + \{m(m-1) - n(n-1)\}\,d = 0$

$\Rightarrow\ 2a(m-n) + \{(m^2 - n^2) - (m-n)\}d = 0$

$\Rightarrow\ (m-n)\,[2a + (m+n-1)d] = 0$

$\Rightarrow\ 2a + (m+n-1)d = 0\quad [\because m - n \neq 0]\qquad (1)$

Now,

$$S_{m+n} = \frac{m+n}{2}[2a + (m+n-1)d] = \frac{m+n}{2}\times 0 = 0\qquad \text{[Using (1)]}$$

6. a. Let $\angle C = 90°$ being greatest and $B = 90° - A$.

The sides are $a - d$, a and $a + d$

We have $(a+d)^2 = (a-d)^2 + a^2$ (using Pythagoras Theorem)

$\therefore\ 4ad - a^2 = 0 \Rightarrow a = 4d$

Hence the sides are $3d$, $4d$, $5d$

Clearly, $\sin A = \frac{BC}{AB} = \frac{a-d}{a+d} = \frac{3d}{5d} = \frac{3}{5}$

$\sin B = \frac{AC}{AB} = \frac{a}{a+d} = \frac{4d}{5d} = \frac{4}{5}$

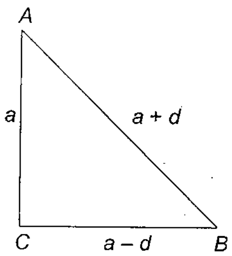

Fig. 3.4

7. b. $$\frac{\frac{n}{2}(2a + (n-1)d)}{\frac{n}{2}(2a' + (n-1)d')} = \frac{5n+3}{3n+4}$$

$$\Rightarrow\ \frac{(2a + (2n-2)d)}{(2a' + (2n-2)d')} = \frac{5(2n-1)+3}{3(2n-1)+4}\quad \text{(replace } n \text{ by } 2n-1)$$

$$\Rightarrow\ \frac{(a + (n-1)d)}{(a' + (n-1)d')} = \frac{10n-2}{6n+1}$$

$$\Rightarrow\ \frac{(a + (17-1)d)}{(a' + (17-1)d')} = \frac{168}{103}$$

8. c. Suppose the work is completed in n days when the workers stopped working. Since four workers stopped working every day except the first day. Therefore, the total number of workers who worked all the n days is the sum of n terms of an A.P. with first term 150 and common difference -4, i.e.,

$$\frac{n}{2}[2\times 150 + (n-1)\times -4] = n(152 - 2n)$$

Had the workers not stopped working, then the work would have finished in $(n-8)$ days with 150 workers working on each day. Therefore, the total number of workers who would have worked all the n days is $150(n-8)$.

$\therefore\ n(152 - 2n) = 150(n-8)$

$\Rightarrow\ n^2 - n - 600 = 0$

$\Rightarrow\ (n-25)(n+24) = 0$

$\Rightarrow\ n = 25$

Thus, the work is completed in 25 days.

9. c. Given, $S_p = 0$. Therefore,

$$\frac{p}{2}[2a + (p-1)d] = 0 \Rightarrow d = \frac{-2a}{p-1}\qquad (1)$$

Sum of next q terms is sum of an A.P. whose first term will be

$T_{p+1} = a + pd.$

$$\therefore\ S = \frac{q}{2}[2(a+pd) + (q-1)d]$$

$$= \frac{q}{2}[2a + (p-1)d + (p+q)d]$$

$$= \frac{q}{2}\left[0 - (p+q)\frac{2a}{p-1}\right]$$

$$= -a\,\frac{(p+q)q}{p-1}\qquad \text{[Using (1)]}$$

10. b. $$S_{3n} = \frac{3n}{2}\left[2a + (3n-1)d\right]$$

$$S_{n-1} = \frac{n-1}{2}\left[2a + (n-2)d\right]$$

$$\Rightarrow\ S_{3n} - S_{n-1} = \frac{1}{2}\left[2a(3n - n + 1)\right] + \frac{d}{2}[3n(3n-1) - (n-1)(n-2)]$$

$$= \frac{1}{2}\left[2a(2n+1) + d(8n^2 - 2)\right]$$

$$= a(2n+1) + d(4n^2 - 1)$$

$$= (2n+1)\,[a + (2n-1)d]$$

$S_{2n} - S_{2n-1} = T_{2n} = a + (2n-1)\,d$

$$\Rightarrow\ \frac{S_{3n} - S_{n-1}}{S_{2n} - S_{2n-1}} = (2n+1)$$

Given,

$$\frac{S_{3n} - S_{n-1}}{S_{2n} - S_{2n-1}} = 31 \Rightarrow n = 15$$

11. d. $2b = a + c$

$\Rightarrow\ 8b^3 = (a+c)^3 = a^3 + c^3 + 3ac\,(a+c)$

$\Rightarrow\ 8b^3 = a^3 + c^3 + 3ac(2b)$

$\Rightarrow\ a^3 + c^3 - 8b^3 = -6abc$

12. b. Let the series have $2n$ terms and the series is $a, a+d, a+2d, \ldots, a + (2n-1)d$.

According to the given conditions, we have

$[a + (a+2d) + (a+4d) + \cdots + (a + (2n-2)d)] = 24$

$\Rightarrow \quad \frac{n}{2}\left[2a+(n-1)2d\right]=24$

$\Rightarrow \quad n[a+(n-1)d]=24$ (1)

Also,

$[(a+d)+(a+3d)+\cdots+(a+(2n-1)d)]=30$

$\Rightarrow \quad \frac{n}{2}\left[2(a+d)+(n-1)2d\right]=30$

$\Rightarrow \quad n[(a+d)+(n-1)d]=30$ (2)

Also, the last term exceeds the first by 21/2. Therefore,

$a+(2n-1)d-a=21/2$

$\Rightarrow \quad (2n-1)d=21/2$ (3)

Now, subtracting (1) from (2),

$nd=6$ (4)

Dividing (3) by (4), we get

$$\frac{2n-1}{n}=\frac{21}{12}$$

$\Rightarrow \quad n=4$

13. b. Since a, q and c are in A.P., so

$2q=a+c$

$\Rightarrow \quad \frac{1}{p}+\frac{1}{r}=\frac{2}{b}$

$\Rightarrow \quad \frac{1}{p},\frac{1}{b},\frac{1}{r}$ are in A.P.

14. b. Given,

$$F(n+1)=\frac{2F(n)+1}{2}$$

$\Rightarrow \quad F(n+1)-F(n)=1/2$

Hence, the given series is an A.P. with common difference 1/2 and first term being 2. $F(101)$ is 101[st] term of A.P. given by $2+(101-1)(1/2)=52$.

15. b. If t_r be the r^{th} term of the A.P., then

$t_r=S_r-S_{r-1}$

$=cr(r-1)-c(r-1)(r-2)$

$=c(r-1)(r-r+2)=2c(r-1)$

We have,

$$t_1^2+t_2^2+\cdots+t_n^2=4c^2(0^2+1^2+2^2+\cdots+(n-1)^2)$$

$$=4c^2\frac{(n-1)n(2n-1)}{6}$$

$$=\frac{2}{3}c^2n(n-1)(2n-1)$$

16. c. Given that

$a_3+a_5+a_8=11$

$\Rightarrow \quad a+2d+a+4d+a+7d=11$

$\Rightarrow \quad 3a+13d=11$ (1)

Given,

$a_4+a_2=-2$

$\Rightarrow \quad a+3d+a+d=-2$

$\Rightarrow \quad a=-1-2d$ (2)

Putting value of a from (2) in (1), we get

$3(-1-2d)+13d=11 \Rightarrow 7d=14 \Rightarrow d=2$ and $a=-5$

$\Rightarrow \quad a_1+a_6+a_7=7$

17. a. If p, q, r are in A.P., then in an A.P. or G.P. or an H.P. $a_1, a_2, a_3, \ldots$, etc., the terms a_p, a_q, a_r are in A.P., G.P. or H.P., respectively.

18. d. Given, $a_1, a_2, a_3, \ldots$ are terms of A.P.

$$\therefore \quad \frac{a_1+a_2+\cdots+a_p}{a_1+a_2+\cdots+a_q}=\frac{p^2}{q^2}$$

$$\Rightarrow \quad \frac{\frac{p}{2}[2a_1+(p-1)d]}{\frac{q}{2}[2a_1+(q-1)d]}=\frac{p^2}{q^2}$$

$$\Rightarrow \quad \frac{2a_1+(p-1)d}{2a_1+(q-1)d}=\frac{p}{q}$$

$\Rightarrow \quad [2a_1+(p-1)d]\,q=[2a_1+(q-1)d]\,p$

$\Rightarrow \quad 2a_1(q-p)=d[(q-1)p-(p-1)q]$

$\Rightarrow \quad 2a_1(q-p)=d(q-p)$

$\Rightarrow \quad 2a_1=d$

$$\therefore \quad \frac{a_6}{a_{21}}=\frac{a_1+5d}{a_1+20d}=\frac{a_1+10a_1}{a_1+40a_1}=\frac{11}{41}$$

19. b. $$\frac{S_{nx}}{S_x}=\frac{\frac{nx}{2}[2a+(nx-1)d]}{\frac{x}{2}[2a+(x-1)d]}$$

$$=\frac{n[(2a-d)+nxd]}{(2a-d)+xd}$$

For $\frac{S_{xn}}{S_x}$ to be independent of x,

$2a-d=0 \Rightarrow 2a=d$

Now,

$$S_p=\frac{p}{2}[2a+(p-1)d]=p^2a$$

20. b. Let the three numbers be a/r, a, ar. As the numbers form an increasing G.P., so, $r>1$. It is given that a/r, $2a$, ar are in A.P. Hence,

$4a=\frac{a}{r}+ar$

$\Rightarrow \quad r^2-4r+1=0$

$\Rightarrow \quad r=2\pm\sqrt{3}$

$=2+\sqrt{3} \quad [\because r>1]$

21. c. Let a_1, a_2, and a_3 be first three consecutive terms of G.P. with common ratio r. Then,

$a_2=a_1r$ and $a_3=a_1r^2$

Now,

$a_3>4a_2-3a_1$

$\Rightarrow \quad a_1r^2>4a_1r-3a_1$

$\Rightarrow \quad r^2>4r-3$

$\Rightarrow \quad r^2-4r+3>0$

$\Rightarrow \quad (r-1)(r-3)>0$

$\Rightarrow \quad r<1$ or $r>3$

22. d. We know that

$-1\le\cos x\le 1$

$\Rightarrow \quad |\cos x|\le 1$

But,

$x \in S \Rightarrow x \in (0, \pi) \Rightarrow |\cos x| < 1$

Now,

$$8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\cdots \text{ to } \infty} = 4^3$$

$\Rightarrow$ $8^{1/(1-|\cos x|)} = 8^2$

$\Rightarrow$ $\dfrac{1}{1-|\cos x|} = 2$

$\Rightarrow$ $|\cos x| = \dfrac{1}{2}$

$\Rightarrow$ $\cos x = \pm\dfrac{1}{2}$

$\Rightarrow$ $x = \pi/3, 2\pi/3$

$\Rightarrow$ $S = \{\pi/3, 2\pi/3\}$

23. c. We have,

$1 + (1 + a)b + (1 + a + a^2)b^2 + (1 + a + a^2 + a^3)b^3 + \cdots \infty$

$$= \sum_{n=1}^{\infty}(1+a+a^2+\cdots+a^{n-1})b^{n-1}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1-a^n}{1-a}\right)b^{n-1}$$

$$= \sum_{n=1}^{\infty}\frac{b^{n-1}}{1-a} - \sum_{n=1}^{\infty}\frac{a^n b^{n-1}}{1-a}$$

$$= \frac{1}{1-a}\sum_{n=1}^{\infty} b^{n-1} - \frac{a}{1-a}\sum_{n=1}^{\infty}(ab)^{n-1}$$

$$= \frac{1}{1-a}[1+b+b^2+\cdots\infty] - \frac{a}{1-a}[1+ab+(ab)^2+\cdots\infty]$$

$$= \frac{1}{1-a}\times\frac{1}{1-b} - \frac{a}{(1-a)(1-ab)}$$

$$= \frac{1}{(1-ab)(1-b)}$$

24. c. Let 'A' be first term and 'r' be the common ratio. We have,

$a = Ar^{p+q-1}$, $b = Ar^{p-q-1}$

$\Rightarrow$ $ab = A^2 \times r^{2p-2}$

$\Rightarrow$ $\sqrt{ab} = Ar^{p-1} = p^{\text{th}}$ term

25. b. Let the sides of the triangle be a/r, a and ar, with $a > 0$ and $r > 1$. Let α be the smallest angle, so that the largest angle is 2α. Then α is opposite to the side a/r, and 2α is positive to the side ar. Applying sine rule, we get

$$\frac{a/r}{\sin\alpha} = \frac{ar}{\sin 2\alpha}$$

$\Rightarrow$ $\dfrac{\sin 2\alpha}{\sin\alpha} = r^2$

$\Rightarrow$ $2\cos\alpha = r^2$

$\Rightarrow$ $r^2 < 2$

$\Rightarrow$ $r < \sqrt{2}$

Hence, $1 < r < \sqrt{2}$.

26. a. We have,

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \frac{1/4}{1-1/2} = \frac{1}{2}$$

Hence,

$$0.2^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)} = 0.2^{\log_{\sqrt{5}}\frac{1}{2}}$$

$$= \left(\frac{1}{5}\right)^{\log_{\sqrt{5}}\frac{1}{2}}$$

$$= (5^{-1})^{2\log_5\frac{1}{2}}$$

$$= (5)^{-2\log_5\frac{1}{2}}$$

$$= (5)^{\log_5 4}$$

$$= 4$$

27. b. Degree of x on L.H.S. is

$1 + 2 + 4 + \cdots + 128$

$= 1 + 2 + 2^2 + \cdots + 2^7$

$= \dfrac{2^8-1}{2-1}$

$= 255$

28. c. x, y, and z are in G.P. Hence,

$y^2 = xz$ (1)

We have,

$a^x = b^y = c^z = \lambda$ (say)

$\Rightarrow$ $x\log a = y\log b = z\log c = \log\lambda$

$\Rightarrow$ $x = \dfrac{\log\lambda}{\log a}, y = \dfrac{\log\lambda}{\log b}, z = \dfrac{\log\lambda}{\log c}$

Putting the values of x, y and z in (1), we get

$$\left(\frac{\log\lambda}{\log b}\right)^2 = \frac{\log\lambda}{\log a}\frac{\log\lambda}{\log c}$$

$\Rightarrow$ $(\log b)^2 = \log a \log c$

$\Rightarrow$ $\log_b a = \log_c b$

29. b. Required G.M. is $-\sqrt{-9\times-16} = -12$.

30. d. We have,

$$S = \frac{1}{1-\frac{1}{2}} = 2$$

$$S_n = \frac{(1-1/2^n)}{(1-1/2)} = 2\left(1-\frac{1}{2^n}\right) = 2-\frac{1}{2^{n-1}}$$

$\therefore$ $S - S_n < \dfrac{1}{1000} \Rightarrow \dfrac{1}{2^{n-1}} < \dfrac{1}{1000}$

$\Rightarrow$ $2^{n-1} > 1000$

$\Rightarrow$ $n - 1 \geq 10$

$\Rightarrow$ $n \geq 11$

Hence, the least value of n is 11.

31. d. Let the series be $21, 21r, 21r^2, \ldots$

Sum $= \dfrac{21}{1-r}$ is a positive integer

also $21r$ is a positive integer

$S = \dfrac{(21)(21)}{21-21r}$ as $21r \in N$ hence $21 - 21r$ must be an integer

also $21r < 21$

hence $21 - 21r$ may be equal to 1, 3, 7 or 9

i.e.must be a divisor of (21)(21)

hence $21 - 21r = 1$ or 3 or 7 or 9

$21r = 20, 18, 14$ or 12

32. c. For G.P., $t_n = 2^{n-1}$; for A.P., $T_m = 1 + (m - 1)3 = 3m - 2$. They are common if $2^{n-1} = 3m - 2$. For G.P. 100th term is 2^{99}. For A.P. 100th term is $1 + (100 - 1)3 = 298$. Now we must choose value of m such that $3m - 2$ is of type 2^{n-1}. Hence, $3m - 2 = 1, 2, 4, 8, 16, 32, 64, 128, 256$ for which $m = 1, 4/3, 2, 10/3, 6, 34/2, 22, 130/3, 86$. Hence, possible values of m are 1, 2, 6, 22, 86. Hence, there are five common terms.

33. c. Initially the ball falls from a height of 120 m. After striking the floor, it rebounds and goes to a height of $\frac{4}{5} \times (120)$ m. Now, it falls from a height of $\frac{4}{5} \times (120)$ m and after rebounding goes to a height of $\frac{4}{5}\left(\frac{4}{5}(120)\right)$ m. This process is continued till the ball comes to rest.

Hence, the total distance travelled is

$$120 + 2\left[\frac{4}{5}(120) + \left(\frac{4}{5}\right)^2(120) + \cdots \infty\right]$$

$$= 120 + 2\left[\frac{\frac{4}{5}(120)}{1-\frac{4}{5}}\right] = 1080 \text{ m}$$

34. b. Let a be the first term and r the common ratio of the G.P. Then, the sum is given by

$$\frac{a}{1-r} = 57 \qquad (1)$$

Sum of the cubes is 9747. Hence,

$$a^3 + a^3r^3 + a^3r^6 + \cdots = 9747$$

$$\Rightarrow \frac{a^3}{1-r^3} = 9747 \qquad (2)$$

Dividing the cube of (1) by (2), we get

$$\frac{a^3}{(1-r)^3} \cdot \frac{(1-r^3)}{a^3} = \frac{(57)^3}{9747}$$

$$\Rightarrow \frac{1-r^3}{(1-r)^3} = 19$$

$$\Rightarrow \frac{1+r+r^2}{(1-r)^2} = 19$$

$\Rightarrow 18r^2 - 39r + 18 = 0$

$\Rightarrow (3r-2)(6r-9) = 0$

$\Rightarrow r = 2/3$ or $r = 3/2$

$= 2/3$ [$\because r \neq 3/2$, because $0 < |r| < 1$ for an infinite G.P.]

35. b. $a^2 + b^2, ab + bc, b^2 + c^2$ are in G.P.

$\Rightarrow (ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$

$\Rightarrow a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^2c^2 + b^4$

$\Rightarrow b^4 + a^2c^2 - 2ab^2c = 0$

$\Rightarrow (b^2 - ac)^2 = 0$

$\Rightarrow b^2 = ac$

$\Rightarrow a, b,$ and c are in G.P.

36. b. Given $\dfrac{ar(r^{10}-1)}{r-1} = 18 \qquad (1)$

Also $\dfrac{\frac{1}{ar}\left(1-\frac{1}{r^{10}}\right)}{1-\frac{1}{r}} = 6$

$$\Rightarrow \frac{1}{ar^{11}} \cdot \frac{(r^{10}-1)r}{r-1} = 6$$

$$\Rightarrow \frac{1}{a^2r^{11}} \cdot \frac{ar(r^{10}-1)}{r-1} = 6 \qquad (2)$$

From (1) and (2),

$$\frac{1}{a^2r^{11}} \cdot 18 = 6$$

$\Rightarrow a^2r^{11} = 3$

Now $P = a^{10}r^{55} = (a^2r^{11})^5 = 3^5 = 243$

37. b. $a = 1 + 10 + 10^2 + \cdots + 10^{54}$

$$= \frac{10^{55}-1}{10-1} = \frac{10^{55}-1}{10^5-1} \times \frac{10^5-1}{10-1} = bc$$

38. a. Let a be the first term and r be the common ratio of the given G.P. Then,

$\alpha = \sum_{n=1}^{100} a_{2n} \Rightarrow \alpha = a_2 + a_4 + \cdots + a_{200}$

$= ar + ar^3 + \cdots + ar^{199}$

$= ar(1 + r^2 + r^4 + \cdots + r^{198})$

$\beta = \sum_{n=1}^{100} a_{2n-1} \Rightarrow \beta = a_1 + a_3 + \cdots + a_{199}$

$= a + ar^2 + \cdots + ar^{198}$

$= a(1 + r^2 + \cdots + r^{198})$

Clearly, $\alpha/\beta = r$.

39. c. The series is

$1 + 2 + 2 \times 3 + 2^2 \times 3 + 2^2 \times 3^2 + 2^3 \times 3^2 + \cdots$ to 20 terms

$= (1 + 2 \times 3 + 2^2 \times 3^2 + \cdots$ to 10 terms$)$

$+ (2 + 2^2 \times 3 + 2^3 \times 3^2 + \cdots$ to 10 terms$)$

$$= \frac{1(2^{10}3^{10}-1)}{6-1} + \frac{2(2^{10}3^{10}-1)}{6-1}$$

$$= \left(\frac{3}{5}\right)(6^{10}-1)$$

40. d. $a = 5, ar^2 = a + 3d, ar^4 = a + 15d$

$\therefore 5r^2 = 5 + 3d, 5r^4 = 5 + 15d$

$\Rightarrow r^4 = 1 + 3d$

$\Rightarrow 25r^4 = 25 + 75d$

$\Rightarrow (5 + 3d)^2 = 25 + 75d$

$\Rightarrow 25 + 30d + 9d^2 = 25 + 75d$

$\Rightarrow 9d^2 - 45d = 0$

$\Rightarrow d = 5, 0$

$\Rightarrow$ $T_4 = a + 3d = 5 + 15 = 20$

41. d. p^{th}, q^{th}, r^{th} terms of A.P. are

$a + (p-1)d = x$ (1)

$a + (q-1)d = xR$ (2)

$a + (r-1)d = xR^2$ (3)

where R is common ratio of G.P.

Subtracting (2) from (3) and (1) from (2) and then dividing the former by the later, we have

$$\frac{q-r}{p-q} = \frac{xR^2 - xR}{xR - x} = R$$

42. b. Given that

$a + (p-1)d = A$

$a + (q-1)d = AR$

$a + (r-1)d = AR^2$

$a + (s-1)d = AR^3$

where R is common ratio of G.P. Now,

$$p - q = \frac{A - AR}{d}, q - r = R\left(\frac{A-AR}{d}\right),$$

$$r - s = R^2\left(\frac{A-AR}{d}\right)$$

Clearly, $p-q$, $q-r$, $r-s$ are in G.P.

43. a. Let r be the common ratio of the G.P., a, b, c, d. Then,

$b = ar$, $c = ar^2$ and $d = ar^3$

$\therefore$ $(b-c)^2 + (c-a)^2 + (d-b)^2$

$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= a^2r^2(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$

$= a^2(r^6 - 2r^3 + 1)$

$= a^2(1-r^3)^2$

$= (a - ar^3)^2$

$= (a-d)^2$

44. d. Let $P = 0.cababab\ldots$

$\Rightarrow 10P = c.ababab\ldots$ (1)

and

$1000P = cab.ababab\ldots$ (2)

Subtracting Eq. (1) from Eq. (2), we have

$990P = cab - c$

or

$$P = \frac{100c + 10a + b - c}{990} = \frac{99c + 10a + b}{990}$$

45. c. $S_\infty = \dfrac{a}{1-r} = 162$

$$S_n = \frac{a(1-r^n)}{1-r} = 160$$

Dividing,

$$1 - r^n = \frac{160}{162} = \frac{80}{81}$$

$$\Rightarrow 1 - \frac{80}{81} = r^n$$

$$\Rightarrow r^n = \frac{1}{81} \text{ or } \left(\frac{1}{r}\right)^n = 81 \quad (1)$$

Now, it is given that $1/r$ is an integer and n is also an integer.

Hence, the relation (1) implies that $1/r = 3, 9$ or 81 so that $n = 4, 2$ or 1.

$$\therefore \quad a = 162\left(1-\frac{1}{3}\right) \text{ or } 162\left(1-\frac{1}{9}\right) \text{ or } 162\left(1-\frac{1}{81}\right)$$

$= 108$ or 144 or 160

46. d. $f(x) = 2x + 1$

$\Rightarrow$ $f(2x) = 2(2x) + 1 = 4x + 1$ and $f(4x) = 2(4x) + 1 = 8x + 1$

Now, $f(x), f(2x), f(4x)$ are in G.P. Hence,

$(4x+1)^2 = (2x+1)(8x+1)$

$\Rightarrow$ $2x = 0$

Hence, $f(x), f(2x)$, and $f(4x)$ is equal to 1 which contradicts the given condition. Hence no such x exists.

47. b.

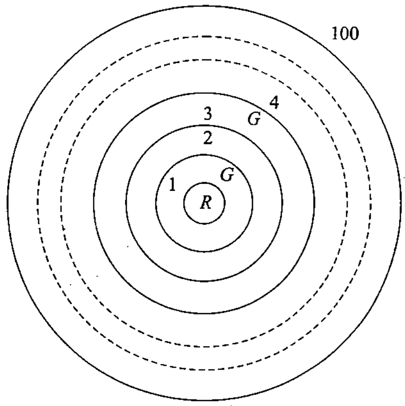

Fig. 3.5

$$\pi\left[(r_2^2 - r_1^2) + (r_4^2 - r_3^2) + \cdots + (r_{100}^2 - r_{99}^2)\right]$$

$= \pi\,[r_1 + r_2 + r_3 + r_4 + \cdots + r_{100}]$ $(\because r_2 - r_1 = r_4 - r_3 = \cdots = r_{100} - r_{99} = 1)$

$= \pi\,[1 + 2 + 3 + \cdots + r_{100}]$

$= 5050\pi$ sq. cm

48. a. $\dfrac{t_4}{t_6} = \dfrac{1}{4} \Rightarrow \dfrac{ar^3}{ar^5} = \dfrac{1}{4} \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

Also,

$t_2 + t_5 = 216$

$\Rightarrow$ $ar + ar^4 = 216$

$\Rightarrow$ $a + 8a = 108$

$\Rightarrow$ $a = 12$ (when $r = 2$)

49. b. x, y, and z are in G.P. Hence,

$y = xr$, $z = xr^2$

Also, x, $2y$, and $3z$ are in A.P. Hence,

$4y = x + 3z$

$\Rightarrow$ $4xr = x + 3xr^2$

$\Rightarrow$ $3r^2 - 4r + 1 = 0$

$\Rightarrow$ $(3r-1)(r-1) = 0$

$\Rightarrow$ $r = 1/3$ ($r \neq 1$ is not possible as x, y, z are distinct)

50. a. $S_p = \dfrac{1}{1-r^p}$, $s_p = \dfrac{1}{1+r^p}$, $S_{2p} = \dfrac{1}{1-r^{2p}}$

Clearly,

$$S_p + s_p = \frac{2}{1-r^{2p}} = 2S_{2p}$$

51. c. Multiplying the given expression by 2 and rewriting it, we have

$(2x-3y)^2+(3y-4z)^2+(4z-2x)^2=0$

$\Rightarrow$ $2x=3y=4z$

$\Rightarrow$ $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in A.P.

$\Rightarrow$ x, y, z are in H.P.

52. c. $a_1, a_2, \ldots, a_n$ are in H.P.

$\Rightarrow$ $\frac{1}{a_1},\frac{1}{a_2},\cdots,\frac{1}{a_n}$ are in A.P.

$\Rightarrow$ $\frac{a_1+a_2+a_3+\cdots+a_n}{a_1}, \frac{a_1+a_2+a_3+\cdots+a_n}{a_2}, \ldots, \frac{a_1+a_2+a_3+\cdots+a_n}{a_n}$ are in A.P.

$\Rightarrow$ $1+\frac{a_2+a_3+\cdots+a_n}{a_1}, 1+\frac{a_1+a_3+\cdots+a_n}{a_2}, \ldots, 1+\frac{a_1+a_2+\cdots+a_{n-1}}{a_n}$ are in A.P.

$\Rightarrow$ $\frac{a_2+a_3+\cdots+a_n}{a_1}, \frac{a_1+a_3+\cdots+a_n}{a_2}, \ldots, \frac{a_1+a_2+\cdots+a_{n-1}}{a_n}$ are in A.P.

$\Rightarrow$ $\frac{a_1}{a_2+a_3+\cdots+a_n}, \frac{a_2}{a_1+a_3+\cdots+a_n}, \ldots, \frac{a_n}{a_1+a_2+\cdots a_{n-1}}$ are in H.P.

53. c.

$$\frac{H_1+2}{H_1-2}+\frac{H_{20}+3}{H_{20}-3}=\frac{\frac{1}{2}+\frac{1}{H_1}}{\frac{1}{2}-\frac{1}{H_1}}+\frac{\frac{1}{3}+\frac{1}{H_{20}}}{\frac{1}{3}-\frac{1}{H_{20}}}$$

$$=\frac{\frac{1}{2}+\frac{1}{2}+d}{\frac{1}{2}-d-\frac{1}{2}}+\frac{\frac{1}{3}+\frac{1}{3}-d}{\frac{1}{3}+d-\frac{1}{3}}$$

$$=\frac{\frac{2}{2}+d}{-d}+\frac{\frac{2}{3}-d}{d}$$

$$=\frac{\frac{2}{3}-1}{d}-2$$

$$=2\times 21-2 \quad [\text{as also, } \frac{1}{3}=\frac{1}{2}+21d]$$

$$=40$$

54. b. We have,

a_1, a_2, a_3 are in A.P. $\Rightarrow 2a_2=a_1+a_3$ (1)

a_2, a_3, a_4 are in G.P. $\Rightarrow a_3^2=a_2a_4$ (2)

a_3, a_4, a_5 are in H.P. $\Rightarrow a_4=\frac{2a_3a_5}{a_3+a_5}$ (3)

Putting $a_2=\frac{a_1+a_3}{2}$ and $a_4=\frac{2a_3a_5}{a_3+a_5}$ in (2), we get

$$a_3^2=\frac{a_1+a_3}{2}\times\frac{2a_3a_5}{a_3+a_5}$$

$\Rightarrow$ $a_3^2=a_1a_5$

Hence, a_1, a_3, and a_5 are in G.P. So, $\log_e a_1$, $\log_e a_3$ and $\log_e a_5$ are in A.P.

55. d. a, b, and c are in A.P. Hence,

$2b=a+c$ (1)

$\frac{a}{bc}+\frac{2}{b}=\frac{a+2c}{bc}\neq\frac{2}{c}$

$\Rightarrow$ $\frac{a}{bc},\frac{1}{c},\frac{2}{b}$ are not in A.P.

$\frac{bc}{a}+\frac{b}{2}=\frac{2bc+ab}{2a}\neq c$

Hence, the given numbers are not in H.P. Again,

$\frac{a}{bc}\frac{2}{b}=\frac{2a}{b^2c}\neq\frac{1}{c^2}$

Therefore, the given numbers are not in G.P.

56. d. x, $2x+2$, $3x+3$ are in G.P. Hence,

$(2x+2)^2=x(3x+3)$

$\Rightarrow$ $4x^2+8x+4=3x^2+3x$

$\Rightarrow$ $x^2+5x+4=0$

$\Rightarrow$ $x=-1,-4$

So, the G.P. is $-4, -6, -9, \ldots$ (considering $x=-4$, as for $x=-1$, $2x+2=0$). Hence, the fourth term is $-9\times1.5=-13.5$.

57. a. Let the numbers be a, ar, ar^2. Then,

$a+ar+ar^2=14$ (given) (1)

Now,

$a+1$, $ar+1$, ar^2-1 are in A.P.

$\Rightarrow$ $2(ar+1)=a+1+ar^2-1$

$\Rightarrow$ $2ar+2=a+ar^2$ (2)

From (1) and (2),

$2ar+2=14-ar$

$\Rightarrow$ $3ar=12$

$\Rightarrow$ $ar=4$ (3)

From (1),

$a+4+4r=14$

$\Rightarrow$ $a+4r=10$ (4)

From (3) and (4),

$a+\frac{16}{a}=10\Rightarrow a=2, 8$

Hence, the smallest number is 2.

58. b. $\frac{p}{r}+\frac{r}{p}=\frac{p^2+r^2}{pr}=\frac{(p+r)^2-2pr}{pr}$

$$=\frac{\frac{4p^2r^2}{q^2}-2pr}{pr} \quad \left[\because p, q, r \text{ are in H.P.} \;\therefore q=\frac{2pr}{p+r}\right]$$

$$=\frac{4pr}{q^2}-2=\frac{4b^2}{ac}-2$$

[$\because$ ap, bq, cr are in A.P. $\Rightarrow b^2q^2=acpr$]

$$= \frac{(a+c)^2}{ac} - 2 \qquad [a, b, c \text{ are in A.P.} \Rightarrow 2b = a + c]$$

$$= \frac{a}{c} + \frac{c}{a}$$

59. a. We have,

$2b = a + c$

$(c - b)^2 = (b - a)a$

$\Rightarrow$ $(b - a)^2 = (b - a)a$ $[2b = a + c \Rightarrow b - a = c - b]$

$\Rightarrow$ $b = 2a$

$\Rightarrow$ $c = 3a$ [Using $2b = a + c$]

$\Rightarrow$ $a{:}b{:}c = 1{:}2{:}3$

60. a. Let $a = 1$, then $S_1 = 2008$

If $a \neq 1$ then $S = \dfrac{a^{2008} - 1}{a - 1}$

but $a^{2008} = 2a - 1$, therefore, $S_2 = \dfrac{2(a-1)}{a-1} = 2$

$\therefore$ $S = S_1 + S_2 = 2010$

61. b. For the equation $x^2 - px + 1 = 0$,

the product of roots, $\alpha\beta^2 = 1$

and for the equation $x^2 - qx + 8 = 0$,

the product of roots $\alpha^2\beta = 8$

Hence, $(\alpha\beta^2)(\alpha^2\beta) = 8$

$\Rightarrow$ $\alpha^3\beta^3 = 8 \Rightarrow \alpha\beta = 2$

$\therefore$ From $\alpha\beta^2 = 1$, we have $\beta = \dfrac{1}{2}$ and from $\alpha^2 . \beta = 8$, we have $\alpha = 4$

Hence, from sum of roots $= -\dfrac{b}{a}$, we have

$p = \alpha + \beta^2 = 4 + \dfrac{1}{4} = \dfrac{17}{4}$ and $q = \alpha^2 + \beta = 16 + \dfrac{1}{2} = \dfrac{33}{2}$

$\dfrac{r}{8}$ is arithmetic mean of p and q

$\therefore$ $\dfrac{r}{8} = \dfrac{p+q}{2}$

$\Rightarrow$ $r = 4(p+q) = 4\left(\dfrac{17}{4} + \dfrac{33}{2}\right) = 17 + 66 = 83$

62. c. $2b = a + c$, $c = \dfrac{2bd}{b+d}$

$\Rightarrow$ $2bd = c(b + d)$

$\Rightarrow$ $(a + c)d = c(b + d)$ [as $2b = a + c$]

$\Rightarrow$ $ad + cd = bc + cd$

$\Rightarrow$ $bc = ad$

63. c. $\dfrac{a_r - a_{r+1}}{a_r\, a_{r+1}} = k$ (constant)

$\Rightarrow$ $\dfrac{1}{a_{r+1}} - \dfrac{1}{a_r} = k$

$\Rightarrow$ $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \ldots, \dfrac{1}{a_n}$ are in A.P.

$\Rightarrow$ $a_1, a_2, a_3, \ldots,$ are in H.P.

64. c. Let $a = 1, b = 2, c = 4$ Then,

$a + b = 3, 2b = 4, b + c = 6$

$\Rightarrow$ $\dfrac{1}{4} - \dfrac{1}{3} = -\dfrac{1}{12}$ and $\dfrac{1}{6} - \dfrac{1}{4} = -\dfrac{1}{12}$

Hence, $a + b$, $2b$, $b + c$ are in H.P.

65. b. x is A.M. of a and b, y is G.M. of a and b, z is H.M. of a and b.

$y^2 = xz$

Also given,

$x = 9z$

$\Rightarrow$ $x = 9y^2/x \Rightarrow 9y^2 = x^2 \Rightarrow x = 3|y|$

66. d. $A = \dfrac{25+n}{2}, G = 5\sqrt{n}, H = \dfrac{50n}{25+n}$

As A, G, H are natural numbers, n must be odd perfect square. Now, H will be a natural number, if we take $n = 225$.

67. a. Reciprocals of the terms of the series are 2/5, 13/20, 9/10, 23/20, ... or 8/20, 13/20, 18/20, 23/20, ... Its n^{th} term is

$$\frac{8+(n-1)5}{20} = \frac{5n+3}{20}$$

Therefore, the 15th term is $\dfrac{20}{78} = \dfrac{10}{39}$.

68. b. Given, $b^2 = ac$ and $x = \dfrac{a+b}{2}$, $y = \dfrac{b+c}{2}$. Therefore,

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$= \frac{2a(b+c) + 2c(a+b)}{(a+b)(b+c)}$$

$$= 2\frac{2ac + ab + bc}{ab + ac + b^2 + bc}$$

$$= 2\frac{2ac + ab + bc}{2ac + ab + bc}$$

$$= 2$$

69. c. $2b = a + c$

a, p, b, q, c are in A.P. Hence,

$p = \dfrac{a+b}{2}$ and $q = \dfrac{b+c}{2}$

Again, a, p', b, q', and c are in G.P. Hence,

$p' = \sqrt{ab}$ and $q' = \sqrt{bc}$

$\Rightarrow$ $p^2 - q^2 = \dfrac{(a-c)(a+c+2b)}{4}$

$= (a - c)b$

$= ab - bc$

$= p'^2 - q'^2$

70. d. As $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ are in A.P., hence

$d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$

$\sin d[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \cdots + \sec a_{n-1} \sec a_n]$

$$= \frac{\sin(a_2 - a_1)}{\cos a_1 \cos a_2} + \frac{\sin(a_3 - a_2)}{\cos a_2 \cos a_3} + \cdots + \frac{\sin(a_n - a_{n-1})}{\cos a_{n-1} \cos a_n}$$

$= (\tan a_2 - \tan a_1) + (\tan a_3 - \tan a_2) + \cdots + (\tan a_n - \tan a_{n-1})$

$= \tan a_n - \tan a_1$

71. c. $S = [a - (a + d)] + [(a + 2d) - (a + 3d)] + \cdots$

$+ [(a + (2n - 2)d)] - a + (2n - 1)d] + (a + 2nd)$

$= [(-d) + (-d) + \cdots + n \text{ times}] + a + 2nd$

$= -nd + a + 2nd$

$= a + nd$

72. **a.** Let $1 + 1/50 = x$. Let S be the sum of 50 terms of the given series. Then,

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots + 49x^{48} + 50x^{49} \quad (1)$$
$$xS = \quad x + 2x^2 + 3x^3 + \cdots \quad + 49x^{49} + 50x^{50} \quad (2)$$
$$(1 - x)S = 1 + x + x^2 + x^3 + \cdots + x^{49} - 50x^{50}$$

[Subtracting (2) from (1)]

$$\Rightarrow \quad S(1 - x) = \frac{1 - x^{50}}{1 - x} - 50x^{50}$$
$$\Rightarrow \quad S(-1/50) = -50(1 - x^{50}) - 50x^n$$
$$\Rightarrow \quad \frac{1}{50}S = 50$$
$$\Rightarrow \quad S = 2500$$

73. **a.** Let T_r be the r^{th} term of the given series. Then,

$$T_r = \frac{2r+1}{1^2 + 2^2 + \cdots + r^2}$$
$$= \frac{6(2r+1)}{(r)(r+1)(2r+1)}$$
$$= 6\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

So, sum is given by

$$\sum_{r=1}^{50} T_r = 6\sum_{r=1}^{50}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$
$$= 6\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{50} - \frac{1}{51}\right)\right]$$
$$= 6\left[1 - \frac{1}{51}\right]$$
$$= \frac{100}{17}$$

74. **a.** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right)$$
$$= \frac{\pi^2}{6} - \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$
$$= \frac{\pi^2}{6} - \frac{1}{4}\left(\frac{\pi^2}{6}\right)$$
$$= \frac{\pi^2}{8}$$

75. **b.** Coefficient of x^{18} in $(1 + x + 2x^2 + 3x^3 + \cdots + 18x^{18})^2$

= Coefficient of x^{18} in $(1 + x + 2x^2 + 3x^3 + \cdots + 18x^{18}) \times (1 + x + 2x^2 + 3x^3 + \cdots + 18x^{18})$

$$= 1 \times 18 + 1 \times 17 + 2 \times 16 + \cdots + 17 \times 1 + 18 \times 1$$
$$= 36 + \sum_{r=1}^{17} r(18 - r)$$
$$= 36 + 18\sum_{r=1}^{17} r - \sum_{r=1}^{17} r^2$$
$$= 1005$$

76. **c.** $T(r) = \dfrac{r}{1 \times 3 \times 5 \times \cdots \times (2r+1)}$

$$= \frac{2r+1-1}{2(1 \times 3 \times 5 \cdots (2r+1))}$$
$$= \frac{1}{2}\left(\frac{1}{1 \times 3 \times 5 \cdots (2r-1)} - \frac{1}{1 \times 3 \times 5 \cdots (2r+1)}\right)$$
$$= -\frac{1}{2}[V(r) - V(r-1)]$$
$$\Rightarrow \quad \sum_{r=1}^{n} T(r) = -\frac{1}{2}(V(n) - V(0))$$
$$= \frac{1}{2}\left(1 - \frac{1}{1 \times 3 \times 5 \times \cdots \times (2n+1)}\right)$$
$$\Rightarrow \quad \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{1 \times 3 \times 5 \times 7 \times 9 \times \cdots \times (2r+1)}$$
$$= \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{1 \times 3 \times 5 \times \cdots \times (2n+1)}\right) = \frac{1}{2}$$

77. **d.** $r \times r! = (r + 1 - 1) \times r!$

$$= (r + 1)! - r!$$
$$= V(r) - V(r - 1)$$
$$\Rightarrow \quad \sum_{r=1}^{30} r(r!) = V(31) - V(0)$$
$$= (31)! - 1$$
$$\Rightarrow \quad 1 + \sum_{r=1}^{30} r(r!) = 31!$$

which is divisible by 31 consecutive integers which is a prime number.

78. **b.** $I(2n) = 1^4 + 2^4 + 3^4 + \cdots + (2n - 1)^4 + (2n)^4$

$$= [(1^4 + 3^4 + 5^4 + \cdots + (2n - 1)^4] + 2^4(1^4 + 2^4 + 3^4 + 4^4 + \cdots n^4)$$
$$= \sum_{r=1}^{n} (2r-1)^4 + 16 \times I(n)$$
$$\Rightarrow \quad \sum_{r=1}^{n} (2r-1)^4 = I(2n) - 16I(n)$$

79. **c.** Consider the first product,

$$P = \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{3^4}\right)\left(1 + \frac{1}{3^8}\right)\cdots\left(1 + \frac{1}{3^{2^n}}\right)$$
$$= \frac{\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{3^4}\right)\left(1 + \frac{1}{3^8}\right)\cdots\left(1 + \frac{1}{3^{2^n}}\right)}{\left(1 - \frac{1}{3}\right)}$$
$$= \frac{\left(1 - \frac{1}{3^2}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{3^4}\right)\left(1 + \frac{1}{3^8}\right)\cdots\left(1 + \frac{1}{3^{2^n}}\right)}{\left(1 - \frac{1}{3}\right)}$$

$$= \frac{\left(1-\frac{1}{3^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{1}{\left(1-\frac{1}{3}\right)}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$= \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$\Rightarrow \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\ldots \text{ infinity}$$

$$= \lim_{n\to\infty}\frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$= \frac{3}{2}$$

80. a. Clearly, $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_{20}}$ will be in A.P. Hence,

$$\frac{1}{x_2}-\frac{1}{x_1} = \frac{1}{x_3}-\frac{1}{x_2} = \cdots = \frac{1}{x_{r+1}}-\frac{1}{x_r} = \cdots = \lambda \text{ (say)}$$

$$\Rightarrow \frac{x_r - x_{r+1}}{x_r x_{r+1}} = \lambda$$

$$\Rightarrow x_r x_{r+1} = -\frac{1}{\lambda}(x_{r+1} - x_r)$$

$$\Rightarrow \sum_{r=1}^{19} x_r x_{r+1} = -\frac{1}{\lambda}\sum_{r=1}^{19}(x_{r+1} - x_r)$$

$$= -\frac{1}{\lambda}(x_{20} - x_1)$$

Now,

$$\frac{1}{x_{20}} = \frac{1}{x_1} + 19\lambda$$

$$\Rightarrow \frac{x_1 - x_{20}}{x_1 x_{20}} = 19\lambda$$

$$\Rightarrow \sum_{r=1}^{19} x_r x_{r+1} = 19x_1x_{20} = 19 \times 4 = 76$$

($\because$ x_1, 2, x_{20} are in G.P., then $x_1x_{20} = 4$)

81. b. $T_r = r(-a)^r + (r+1)\,a(-a)^r$

$= r(-a)^r - (r+1)(-a)^{r+1}$

$= v_r - v_{r+1}$ (say)

So,

$$\sum_{r=0}^{n} T_r = \sum_{r=0}^{n}(v_r - v_{r+1})$$

$= v_0 - v_{n+1}$

$= -(n+1)(-a)^{n+1}$

82. d. $\Sigma a_i b_i = \Sigma a_i(1 - a_i)$

$= na - \Sigma a_i^2$

$= na - \Sigma(a_i - a + a)^2$

$= na - \Sigma[(a_i - a)^2 + a^2 + 2a(a_i - a)]$

$= na - \Sigma(a_i - a)^2 - \Sigma a^2 - 2a\,\Sigma(a_i - a)$

$\Rightarrow \Sigma a_i b_i + \Sigma(a_i - a)^2 = na - na^2 - 2a(na - na)$

$= na(1 - a) = nab$ $\left[\because \Sigma b_i = \Sigma 1 - \Sigma a_i \;\therefore nb = n - na \text{ or } a + b = 1\right]$

83. a. The general term of the given series is

$$t_n = \frac{x^{2^{n-1}}}{1-x^{2^n}} = \frac{1+x^{2^{n-1}}-1}{\left(1+x^{2^{n-1}}\right)\left(1-x^{2^{n-1}}\right)}$$

$$\Rightarrow t_n = \frac{1}{1-x^{2^{n-1}}} - \frac{1}{1-x^{2^n}}$$

Now,

$$S_n = \sum_{n=1}^{n} t_n$$

$$= \left[\left\{\frac{1}{1-x} - \frac{1}{1-x^2}\right\} + \left\{\frac{1}{1-x^2} - \frac{1}{1-x^4}\right\} + \cdots + \left\{\frac{1}{1-x^{2^{n-1}}} - \frac{1}{1-x^{2^n}}\right\}\right]$$

$$= \frac{1}{1-x} - \frac{1}{1-x^{2^n}}$$

Therefore, the sum to infinite terms is

$$\lim_{n\to\infty} S_n = \frac{1}{1-x} - 1$$

$$= \frac{x}{1-x} \quad [\because \lim_{n\to\infty} x^{2n} = 0, \text{ as } |x| < 1]$$

84. a. The general term can be given by

$$t_{r+1} = \frac{a_{2n+1-r} - a_{r+1}}{a_{2n+1-r} + a_{r+1}}, \; r = 0, 1, 2, \ldots, n-1$$

$$= \frac{a_1 + (2n-r)d - \{a_1 + rd\}}{a_1 + (2n-r)d + \{a_1 + rd\}}$$

$$= \frac{(n-r)d}{a_1 + nd}$$

Therefore, the required sum is

$$S_n = \sum_{r=0}^{n-1} t_{r+1}$$

$$= \sum_{r=0}^{n-1} \frac{(n-r)\,d}{a_1 + nd}$$

$$= \left[\frac{n + (n-1) + (n-2) + \cdots + 1}{a_1 + nd}\right]d$$

$$= \frac{n(n+1)\,d}{2a_{n+1}}$$

$$= \frac{n(n+1)}{2}\,\frac{a_2 - a_1}{a_{n+1}} \quad [\because d = a_2 - a_1]$$

85. a. Let,

$S = i - 2 - 3i + 4 + 5i + \cdots + 100i^{100}$

$= i + 2i^2 + 3i^3 + 4i^4 + 5i^5 + \cdots + 100i^{100}$

$\Rightarrow iS = i^2 + 2i^3 + 3i^4 + \cdots + 99i^{100} + 100i^{101}$

$\Rightarrow S - iS = [i + i^2 + i^3 + i^4 + \cdots + i^{100}] - 100i^{101}$

$$\Rightarrow S(1-i) = \frac{i(i^{100}-1)}{i-1} - 100i^{101}$$

$= -100i^{101}$

$\Rightarrow \quad S = \dfrac{-100i}{1-i} = -50i(1+i) = -50(i-1) = 50(1-i)$

86. b. $S = \dfrac{4}{19} + \dfrac{44}{19^2} + \dfrac{444}{19^3} + \cdots \qquad (1)$

$\Rightarrow \quad \dfrac{1}{19}S = \dfrac{4}{19^2} + \dfrac{44}{19^3} + \cdots \qquad (2)$

Subtracting (2) from (1), we get

$\dfrac{18}{19}S = \dfrac{4}{19} + \dfrac{40}{19^2} + \dfrac{400}{19^3} + \cdots$

$= \dfrac{\frac{4}{19}}{1 - \frac{10}{19}}$

$= 4/9$

$\Rightarrow \quad S = 38/81$

87. b. $S_n = 1 + \dfrac{3}{2} + \dfrac{5}{3} + \cdots + \dfrac{99}{50}$

$= (2-1) + \left(2 - \dfrac{1}{2}\right) + \left(2 - \dfrac{1}{3}\right) + \cdots + \left(2 - \dfrac{1}{50}\right)$

$= 100 - H_{50}$

88. b. $S = 1 + 2r + 3r^2 + 4r^3 + \cdots$

$rS = r + 2r^2 + 3r^3 + 4r^4 + \cdots$

$\Rightarrow \quad (1-r)S = 1 + r + r^2 + r^3 + \cdots$

$= \dfrac{1}{1-r}$

$\Rightarrow \quad S = \dfrac{1}{(1-r)^2}$

Given, $S = 9/4 \Rightarrow \dfrac{1}{(1-r)^2} = 9/4$

$\Rightarrow \quad 1 - r = \pm\dfrac{2}{3}$

$\Rightarrow \quad r = 1/3$ or $5/3$

Hence, $r = 1/3$ as $0 < |r| < 1$.

89. d. Let,

$S = 1 + \dfrac{4}{5} + \dfrac{7}{5^2} + \dfrac{10}{5^3} + \cdots\infty$

Then,

$\dfrac{1}{5}S = \dfrac{1}{5} + \dfrac{4}{5^2} + \dfrac{7}{5^3} + \cdots\infty$

$\Rightarrow \quad S\left(1 - \dfrac{1}{5}\right) = 1 + 3\left[\dfrac{1}{5} + \dfrac{1}{5^2} + \dfrac{1}{5^3} + \cdots\infty\right]$

$\Rightarrow \quad \dfrac{4}{5}S = 1 + 3 \times \dfrac{1/5}{1-(1/5)} = 1 + \dfrac{3}{4} = \dfrac{7}{4}$

$\Rightarrow \quad S = \dfrac{35}{16}$

90. c. $T_n = \dfrac{1^3 + 2^3 + 3^3 + \cdots + n^3}{1 + 3 + 5 + \cdots n \text{ terms}} = \dfrac{\Sigma n^3}{\frac{n}{2}[2\times 1 + (n-1)2]}$

$= \dfrac{1}{4} \times \dfrac{n^2(n+1)^2}{n^2} = \dfrac{1}{4}(n^2 + 2n + 1) \qquad (1)$

Now,

$S_n = \dfrac{1}{4}(\Sigma n^2 + 2\Sigma n + n)$

$= \dfrac{1}{4}\left[\dfrac{n(n+1)(2n+1)}{6} + 2 \times \dfrac{n(n+1)}{2} + n\right]$

$= \dfrac{n}{24}[2n^2 + 3n + 1 + 6n + 6 + 6]$

$= \dfrac{n}{24}[2n^2 + 9n + 13]$

Putting $n = 16$, we get

$S_{16} = \dfrac{16}{24}[2(256) + 144 + 13]$

$= \dfrac{2}{3}(669) = 446$

91. c. Here the successive differences are 2, 4, 8, 16, ... which are in G.P.

$S = 1 + 3 + 7 + 15 + 31 + \cdots + T_{100}$

$S = (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{100} - 1)$

$= (2 + 2^2 + 2^3 + \cdots + 2^{100}) - 100$

$= 2\left(\dfrac{2^{100} - 1}{2 - 1}\right) - 100$

$= 2^{101} - 102$

92. a. Series is $a, a+2, a+4, \ldots + a + 4n, (a+4n)0.5, (a+4n)(0.5)^2, \ldots, (a+4n)(0.5)^{2n-1}$

The middle term of A.P. and G.P. are equal

$\Rightarrow \quad a + 2n = (a + 4n)(0.5)^n$

$\Rightarrow \quad a.2^n + 2^{n+1}n = a + 4n$

$\Rightarrow \quad a = \dfrac{4n - n2^{n+1}}{2^n - 1}$

$\Rightarrow$ The middle term of entire sequence

$= (a + 4n)0.5 = \left(\dfrac{4n - n2^{n+1}}{2^n - 1} + 4n\right)\dfrac{1}{2} = \dfrac{n.2^{n+1}}{2^n - 1}$

93. c. Here, number of factors is 50. Therefore, the coefficient of x^{49} is

$-1 - 3 - 5 - \cdots - 99 = -\dfrac{50}{2}(1 + 99) = -2500$

94. b. $T_r = (-1)^r \dfrac{r^2 + r + 1}{r!}$

$= (-1)^r\left[\dfrac{r}{(r-1)!} + \dfrac{1}{(r-1)!} + \dfrac{1}{r!}\right]$

$= (-1)^r\left[\dfrac{1}{(r-2)!} + \dfrac{1}{(r-1)!} + \dfrac{1}{(r-1)!} + \dfrac{1}{r!}\right]$

$= \left[\dfrac{(-1)^r}{r!} + \dfrac{(-1)^r}{(r-1)!}\right] + \left[\dfrac{(-1)^r}{(r-1)!} + \dfrac{(-1)^r}{(r-2)!}\right]$

$= \left[\dfrac{(-1)^r}{r!} - \dfrac{(-1)^{r-1}}{(r-1)!}\right] - \left[\dfrac{(-1)^{r-1}}{(r-1)!} - \dfrac{(-1)^{r-2}}{(r-2)!}\right]$

$= V(r) - V(r-1)$

$\therefore \quad \sum_{r=1}^{n} T_r = V(n) - V(0) = \left[\dfrac{(-1)^n}{n!} - \dfrac{(-1)^{n-1}}{(n-1)!}\right] - 1$

Therefore the sum of 20 terms is

$\left[\dfrac{1}{20!} - \dfrac{-1}{19!}\right] - 1 = \dfrac{21}{20!} - 1$

95. c. Let the 1025^{th} term fall is in the n^{th} group. Then

$1+2+4+\cdots+2^{n-1}<1025\le 1+2+4+\cdots+2^n$

$\Rightarrow\ 2^{n-1}<1026\le 2^{n+1}$

$\Rightarrow\ n=10$

$\Rightarrow\ 1025^{th}$ term is 2^{10}

96. d. Let $t_n=\dfrac{1}{4(n+2)(n+3)}$. Then

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\cdots+\frac{1}{t_{2003}}$$

$$=4\left[\frac{1}{3\times4}+\frac{1}{4\times5}+\frac{1}{5\times6}+\cdots+\frac{1}{2005\times2006}\right]$$

$$=4\left[\frac{1}{3}-\frac{1}{2006}\right]$$

$$=4\times\frac{2003}{3\,(2006)}=\frac{4006}{3009}$$

97. d. $S=\dfrac{2}{10}+\dfrac{4}{10^3}+\dfrac{6}{10^5}+\dfrac{8}{10^7}+\cdots$ to ∞ (1)

$$=\frac{\frac{2}{10}}{1-\frac{1}{10^2}}+\frac{2\times\left(\frac{1}{10^2}\right)}{\left(1-\frac{1}{10^2}\right)^2}$$

$$=\frac{20}{99}+\frac{200}{9801}$$

$$=\frac{2180}{9801}$$

98. c. The sum equals $\dfrac{n(n+1)(n+2)}{6}=220$

which is true for $n=10$

99. a. $S=(1)(2003)+(2)(2002)+(3)(2001)+\cdots+(2003)(1)$

$$=\sum_{r=1}^{2003}r(2003-(r-1))$$

$$=\sum_{r=1}^{2003}r(2004-r)$$

$$=\sum_{r=1}^{2003}2004r-\sum_{r=1}^{2003}r^2$$

$$=\frac{2004\times2003\times2004}{2}-2003\times4007\times334$$

$$=2003\times334\times(6012-4007)$$

$$=2003\times334\times2005$$

Hence, $x=2005$.

100. d. $2+3+6+11+18+\cdots=(0^2+2)+(1^2+2)+(2^2+2)+(3^2+2)+\cdots$

Hence, $t_{50}=49^2+2$.

101. d. We have,

$2^{n+10}=2\times2^2+3\times2^3+4\times2^4+\cdots+n\times2^n$

$\Rightarrow\ 2(2^{n+10})=2\times2^3+3\times2^4+\cdots+(n-1)\times2^n+n\times2^{n+1}$

Subtracting, we get

$-2^{n+10}=2\times2^2+2^3+2^4+\cdots+2^n-n\times2^{n+1}$

$$=8+\frac{8(2^{n-2}-1)}{2-1}-n\cdot2^{n+1}$$

$$=8+2^{n+1}-8-n\times2^{n+1}=2^{n+1}-(n)\,2^{n+1}$$

$\Rightarrow\ 2^{10}=2n-2\Rightarrow n=513$

102. a. We have,

$$\frac{\pi}{4}=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots$$

$$=\frac{2}{1\times3}+\frac{2}{5\times7}+\frac{2}{9\times1}+\cdots$$

$$\Rightarrow\ \frac{1}{1\times3}+\frac{1}{5\times7}+\frac{1}{9\times11}+\cdots=\frac{\pi}{8}$$

103. b. The coefficient of x^{19} in the polynomial $(x-1)(x-2)(x-2^2)\cdots(x-2^{19})$ is

$$-(1+2+2^2+\cdots+2^{19})=-1\left(\frac{2^{20}-1}{2-1}\right)$$

$$=1-2^{20}$$

104. b $b_2=\dfrac{1}{1-b_1}$

$$b_3=\frac{1}{1-b_2}=\frac{1}{1-\frac{1}{1-b_1}}=\frac{1-b_1}{-b_1}=\frac{b_1-1}{b_1}$$

$$b_1=b_3\Rightarrow b_1^2-b_1+1=0$$

$\Rightarrow\ b_1=-\omega$ or $\omega^2\Rightarrow b_2=\dfrac{1}{1+\omega}=-\omega$ or ω^2

$$\sum_{r=1}^{2001}b_r^{2001}=\sum_{r=1}^{2001}(-\omega)^{2001}$$

$$=-\sum_{r=1}^{2001}1$$

$$=-2001$$

105. d. $(1^2-t_1)+(2^2-t_2)+\cdots+(n^2-t_n)=\dfrac{1}{3}n(n^2-1)$

$\Rightarrow\ 1^2+2^2+3^2+\cdots+n^2$

$-\{t_1+t_2+\cdots+t_n\}=\dfrac{1}{3}n(n^2-1)$

$$\Rightarrow\ \frac{n(n+1)(2n+1)}{6}-S_n=\frac{1}{3}n(n^2-1)$$

$$\Rightarrow\ S_n=\frac{n(n+1)}{6}[2n+1-2(n-1)]$$

$$=\frac{n(n+1)}{6}[2n+1-2n+2]$$

$$=\frac{n(n+1)}{2}$$

$$\Rightarrow\ S_{n-1}=\frac{n(n-1)}{2}$$

$\Rightarrow \quad T_n = S_n - S_{n-1} = n$

106. d. Since $ax^3 + bx^2 + cx + d$ is divisible by $ax^2 + c$, therefore, when $ax^3 + bx^2 + cx + d$ is divided by $ax^2 + c$ the remainder should be zero. Now when $ax^3 + bx^2 + cx + d$ is divided by $ax^2 + c$, then the remainder is $(bc/a) - d$.

$$\therefore \quad \frac{bc}{a} - d = 0$$

$$\Rightarrow \quad bc = ad$$

$$\Rightarrow \quad \frac{b}{a} = \frac{d}{c}$$

Hence, from this, a, b, c, d are not necessarily in G.P.

107. b. We know that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Thus, the given equation can be written as

$$\left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = \left(\frac{r+1}{2}\right)^2$$

$$\Rightarrow \quad (p + 1)^2 + (q + 1)^2 = (r + 1)^2$$

As $p > 6$, $p + 1 > 7$, we may take $p + 1 = 8$, $q + 1 = 6$, $r + 1 = 10$. Hence,

$$p + q + r = 21$$

108. c. $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$\Rightarrow \quad U_n = \sum_{n=1}^{n} \frac{a(r^n - 1)}{r - 1} = \frac{a}{r-1} \sum_{n=1}^{n} (r^n - 1)$$

$$\Rightarrow \quad U_n = \frac{a}{r-1}\{r + r^2 + \cdots + r^n - n\}$$

$$= \frac{a}{r-1}\left\{\frac{r(r^n - 1)}{r-1} - n\right\}$$

$$\Rightarrow \quad (r - 1)\, U_n = \frac{ar(r^n - 1)}{r - 1} - an$$

$$\Rightarrow \quad (r - 1)\, U_n = rS_n - an$$

$$\Rightarrow \quad rS_n + (1 - r)\, U_n = an$$

109. b. We have,

$$(OM_{n-1})^2 = (OP_n)^2 + (P_nM_{n-1})^2$$

$$= 2(OP_n)^2$$

$$= 2\alpha_n^2 \text{ (say)}$$

Also,

$$(OP_{n-1})^2 = (OM_{n-1})^2 + (P_{n-1}M_{n-1})^2$$

$$\Rightarrow \quad \alpha_{n-1}^2 = 2\alpha_n^2 + \frac{1}{2}\alpha_{n-1}^2$$

$$\Rightarrow \quad \alpha_n = \frac{1}{2}\alpha_{n-1}$$

$$\Rightarrow \quad OP_n = \alpha_n = \frac{1}{2}\alpha_{n-1} = \frac{1}{2^2}\alpha_{n-1} = \cdots = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

110. b. $(1 - p)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - p^6$

$$\Rightarrow \quad 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 = \frac{1 - p^6}{1 - p}$$

$$\Rightarrow \quad 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 = 1 + p + p^2 + p^3 + p^4 + p^5$$

Comparing, we get $p = 3x$ or $p/x = 3$.

111. c.

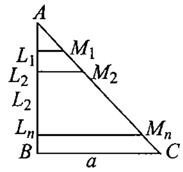

Fig. 3.6

$$\frac{AL_1}{AB} = \frac{L_1M_1}{BC}$$

$$\Rightarrow \quad \frac{1}{n+1} = \frac{L_1M_1}{a}$$

$$\Rightarrow \quad L_1M_1 = \frac{a}{n+1}$$

$$\frac{AL_2}{AB} = \frac{L_2M_2}{BC}$$

$$\Rightarrow \quad \frac{2}{n+1} = \frac{L_2M_2}{a} \Rightarrow L_2M_2 = \frac{2a}{n+1}, \quad \text{etc.}$$

Hence, the required sum is

$$\frac{a}{n+1} + \frac{2a}{n+1} + \frac{3a}{n+1} + \cdots + \frac{na}{n+1}$$

$$= \frac{a}{n+1} \frac{n(n+1)}{2} = \frac{an}{2}$$

112. d. $S_n - S_{n-2} = 2$

$$\Rightarrow \quad T_n + T_{n-1} = 2$$

Also,

$$T_n + T_{n-1} = \left(\frac{1}{n^2} + 1\right) T_{n-1} = 2$$

$$\Rightarrow \quad T_{n-1} = \frac{2}{1 + \frac{1}{n^2}} = \frac{2n^2}{1 + n^2}$$

So,

$$T_m = \frac{2(m+1)^2}{1 + (m+1)^2}$$

113. c.

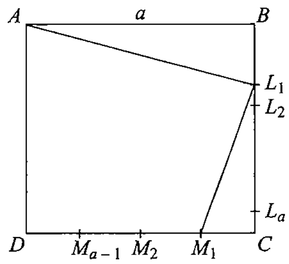

Fig. 3.7

$$(AL_1)^2 + (L_1M_1)^2 = (a^2 + 1^2) + \{(a - 1)^2 + 1^2\}$$

$$(AL_2)^2 + (L_2M_2)^2 = (a^2 + 2^2) + \{(a - 2)^2 + 2^2\}$$

$$\vdots$$

$$(AL_{a-1})^2 + (L_{a-1}M_{a-1})^2 = a^2 + (a - 1^2) + \{1^2 + (a - 1)^2\}$$

Therefore, the required sum is

$(a-1)a^2 + \{1^2 + 2^2 + \cdots + (a-1)^2\} + 2\{1^2 + 2^2 + \cdots + (a-1)^2\}$

$$= (a-1)a^2 + 3\frac{(a-1)a(2a-1)}{6}$$

$$= a(a-1)\left(a + \frac{2a-1}{2}\right)$$

$$= \frac{1}{2}(a-1)(4a-1)$$

114. b. x, y, z are in G.P. Hence,

$$y^2 = xz$$

Now, $x+3, y+3, z+3$ are in H.P. Hence,

$$y+3 = \frac{2(x+3)(z+3)}{(x+3)+(z+3)}$$

$$= \frac{2[xz + 3(x+z) + 9]}{[(x+z)+6]}$$

$$= \frac{2[y^2 + 3(x+z) + 9]}{[x+z+6]}$$

Obviously, $y = 3$ satisfies it.

115. a. x, y, z are in G.P.

$\Leftrightarrow y^2 = xz$

$\Leftrightarrow x$ is a factor of y (not possible)

Taking $x = 3, y = 5, z = 7$, we have x, y, z are in A.P. Thus x, y, z may be in A.P. but not in G.P.

Multiple Correct Answers Type

1. a, c.

$$S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \cdots$$

The r^{th} term is given by

$$T_r = \frac{1}{r^2}(1+2+\cdots+r)^2$$

$$= \frac{1}{r^2}\left\{\frac{r(r+1)}{2}\right\}^2$$

$$= \frac{r^2+2r+1}{4}$$

$\therefore\ T_7 = 16$ and $S_{10} = \sum_{r=1}^{10} T_r$

$$= \frac{1}{4}\left\{\frac{(10)(10+1)(20+1)}{6} + (10)(10+1) + 10\right\} = \frac{505}{4}$$

2. b, c.

We have,

$$\frac{p}{1-1/p} = \frac{9}{2}$$

$\Rightarrow\ 2p^2 - 9p + 9 = 0$

$\Rightarrow\ p = 3/2, 3$

3. a, b, c, d.

$$an^4 + bn^3 + cn^2 + dn + e$$

$$= 2\sum_{r=1}^{n} r(r+1)(r+2) - \sum_{r=1}^{n} r(r+1)$$

$$= \frac{2}{4}n(n+1)(n+2)(n+3) - \frac{1}{3}n(n+1)(n+2)$$

$$= \frac{1}{6}(3n^4 + 16n^3 + 27n^2 + 14n)$$

4. a, b, c.

Given that $a = 4$, $T_3 - T_5 = 32/81$. Hence,

$$a(r^2 - r^4) = 32/81$$

or

$$r^4 - r^2 + 8/81 = 0$$

or

$$81r^4 - 81r^2 + 8 = 0$$

or

$$(9r^2 - 8)(9r^2 - 1) = 0$$

$\therefore\ r^2 = 8/9, 1/9$

Therefore, the value of r is to be +ve since all the terms are +ve.

For $r = 1/3$,

$$S_\infty = \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4\times 3}{2} = 6$$

Similarly, we can find S_∞ when $r = 2\sqrt{2}/3$.

5. a, b, c.

Let the three-digit number be xyz. According to given condition, we have

$y^2 = xz$ (1)

$2(y+2) = x + z$ (2)

$100x + 10y + z - 792 = 100z + 10y + x$

$\Rightarrow\ x - z = 8$ (3)

Squaring (2) and (3), and subtracting, we have

$4xz = 4(y+2)^2 - 64$ (4)

$\Rightarrow\ y^2 = (y+2)^2 - 16$ [Using (1)]

$\Rightarrow\ y = 3$

$\Rightarrow\ x + z = 10$ [Using (2)]

$\Rightarrow\ x = 9, z = 1$

Hence, the number is $931 = 7^2 \times 19$.

6. a, b, c, d.

Clearly, n^{th} term of the given series is negative or positive accordingly as n is even or odd, respectively.

Case I: When n is even: In this case, the given series is

$S_n = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (n-1)^2 - n^2$

$= (1^2 - 2^2) + (3^2 - 4^2) + \cdots + ((n-1)^2 - n^2)$

$= (1-2)(1+2) + (3-4)(3+4) + \cdots + ((n-1) - (n))(n-1+n)$

$= -(1 + 2 + 3 + 4 + \cdots + (n-1) + n)$

$= -\frac{n(n+1)}{2}$ (1)

Case II: When n is odd: In this case, the given series is

$S_n = (1^2 - 2^2) + (3^2 - 4^2) + \cdots + \{(n-2)^2 - (n-1)^2\} + n^2$

$= (1-2)(1+2) + (3-4)(3+4) + \cdots + ((n-2) - (n-1)) \times ((n-2) + (n-1)) + n^2$

$= -(1 + 2 + 3 + 4 + \cdots + (n-2) + (n-1)) + n^2$

$= -\frac{(n-1)(n-1+1)}{2} + n^2 = \frac{n(n+1)}{2}$ (2)

$\Rightarrow\ S_{40} = -820$ [Using (1)]

$S_{51} = 1326$ [Using (2)]

Also,

$S_{2n} > S_{2n+2}$ [From (1)]
$S_{2n+1} > S_{2n-1}$ [From (2)]

7. a, b, d.

$$x + y + z = 3\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \quad 15 = 3\frac{(a+b)}{2}$$

$$\Rightarrow \quad a + b = 10 \qquad (1)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{3\left(\frac{1}{a}+\frac{1}{b}\right)}{2}$$

$$\Rightarrow \quad \frac{5}{3} = \frac{3(a+b)}{2ab} = \frac{3\times 10}{2ab}$$

$$\Rightarrow \quad ab = 9 \qquad (2)$$

From (1) and (2), $a = 9$, $b = 1$ or $a = 1$ and $b = 9$. Hence, G.M. $= \sqrt{ab} = 3$, $a + 2b = 11$ or 19.

8. b, d.

Given,

$$3a_1 + 7a_2 + 3a_3 - 4a_5 = 0$$
$$\Rightarrow \quad 7(a_1 + a_2 + a_3) = 4(a_1 + a_3 + a_5)$$
$$\Rightarrow \quad 7(1 + r + r^2) = 4(1 + r^2 + r^4)$$
$$\Rightarrow \quad 7 = 4(r^2 - r + 1)$$
$$\Rightarrow \quad 4r^2 - 4r + 1 = 4$$
$$\Rightarrow \quad (2r - 1)^2 = 4$$
$$\Rightarrow \quad 2r - 1 = \pm 2$$
$$\Rightarrow \quad r = 3/2, -1/2$$

9. a, b, c.

$$\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots n \text{ terms}$$

$$= \frac{\sqrt{5}-\sqrt{2}}{3} + \frac{\sqrt{8}-\sqrt{5}}{3} + \cdots + \frac{\sqrt{5+(n-1)3} - \sqrt{2+(n-1)3}}{3}$$

$$= \frac{\sqrt{3n+2}-\sqrt{2}}{3}$$

$$= \frac{3n+2-2}{3(\sqrt{3n+2}+\sqrt{2})}$$

$$= \frac{n}{\sqrt{3n+2}+\sqrt{2}}$$

$$= \frac{n}{\sqrt{2+3n}+\sqrt{2}} < \frac{n}{\sqrt{3n}} < n$$

10. a, b.

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right) = \left(\frac{1}{b}+\frac{1}{c}-\frac{2}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{c}\right)$$

$$= \left(\frac{2}{c}-\frac{1}{b}\right)\frac{1}{b} = \frac{2}{bc} - \frac{1}{b^2}$$

Also by eliminating b, we get the given expression $\dfrac{(a+c)(3a-c)}{4a^2c^2}$.

11. a, d.

$$p(x) = \left(\frac{1-x^{2n}}{1-x^2}\right)\left(\frac{1-x}{1-x^n}\right) = \frac{1+x^n}{1+x}$$

As $p(x)$ is a polynomial, $x = -1$ must be a zero of $1 + x^n$. Hence, $1 + (-1)^n = 0$. So, n must be odd.

12. a, c, d.

$$a_1 + a_3 + a_5 = -12$$
$$a + a + 2d + a + 4d = -12 \quad (d > 0)$$
$$a + 2d = -4 \qquad (1)$$
$$a_1a_3a_5 = 80$$
$$a(a + 2d)(a + 4d) = 80$$

or

$$(-4 - 2d)(-4 + 2d) = -20 \Rightarrow d = \pm 3$$

Since A.P. is increasing, so $d = +3$; $a = -10$. Hence,

$$\left.\begin{aligned} a_1 &= -10;\ a_2 = -7 \\ a_3 &= a + 2d = -10 + 6 = -4 \\ a_5 &= a + 4d = -10 + 12 = 2 \end{aligned}\right\}$$

13. a, b, c.

$$a = \frac{n^{64}-1}{n-1}$$
$$= (n + 1)(n^2 + 1)(n^4 + 1)(n^8 + 1)(n^{16} + 1)(n^{32} + 1)$$

14. a, b, c.

If p, q, r are in A.P., then p^{th}, q^{th} and r^{th} terms are equal distant terms which are always in the same series of which they are terms.

15. a, d.

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots\infty$$
$$\Rightarrow \quad xS = x + 2x^2 + 3x^3 + 4x^4 + \cdots\infty$$
$$\Rightarrow \quad (1 - x)S = 1 + x + x^2 + \cdots \infty = \frac{1}{1-x}$$
$$\Rightarrow \quad S = \frac{1}{(1-x)^2}$$

Now,

$$S \geq 4 \Rightarrow \frac{1}{(1-x)^2} > 4$$
$$\Rightarrow \quad (x-1)^2 \leq \frac{1}{4}$$
$$\Rightarrow \quad -\frac{1}{2} \leq x-1 \leq \frac{1}{2}$$
$$\Rightarrow \quad \frac{1}{2} \leq x \leq \frac{3}{2}. \text{ Also } 0 < |x| < 1$$
$$\Rightarrow \quad \frac{1}{2} \leq x < 1$$

16. a, b, d.

$$E < 1 + \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \cdots$$
$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots = 2$$
$$E > 1 + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \cdots$$
$$= 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = \frac{3}{2}$$

17. a, c, d.

$$S_n = \frac{n}{2}\left[2a' + (n-1)d\right] = a + bn + cn^2$$

$$\Rightarrow \quad na' + \frac{n(n-1)}{2}d = a + bn + cn^2$$

$$\Rightarrow \quad \left(a' - \frac{d}{2}\right)n + \frac{n^2 d}{2} = a + bn + cn^2$$

On comparing,

$$a = 0,\ b = a' - \frac{d}{2},\ c = \frac{d}{2} \quad \Rightarrow d = 2c$$

18. a, b.

$$x^2 + 9y^2 + 25z^2 = 15yz + 5zx + 3xy$$

$$\Rightarrow \quad (x)^2 + (3y)^2 + (5z)^2 - (x)(3y) - (3y)(5z) - (x)(5z) = 0$$

$$\Rightarrow \quad \frac{1}{2}[(x-3y)^2 + (3y-5z)^2 + (x-5z)^2] = 0$$

$$\Rightarrow \quad x - 3y = 0,\ 3y - 5z = 0,\ x - 5z = 0$$

$$x = 34 = 5z$$

$$\Rightarrow \quad x : y : z = \frac{1}{1} : \frac{1}{3} : \frac{1}{5}$$

Therefore, $1/x$, $1/y$, and $1/z$ are in A.P. and x, y, and z are in H.P.

19. a, c.

Let $b = a + p$, $c = a + 2p$, $d = a + 3p$ (where p is common difference). Then,

$$\frac{\frac{1}{a}+\frac{1}{d}}{\frac{1}{b}+\frac{1}{c}} = \frac{\frac{1}{a}+\frac{1}{a+3p}}{\frac{1}{a+p}+\frac{1}{a+2p}}$$

$$= \frac{(a+p)(a+2p)}{a(a+3p)}$$

$$= \frac{a^2+3ap+2p^2}{a^2+3ap} > 1$$

$$\therefore \quad \frac{1}{a}+\frac{1}{d} > \frac{1}{b}+\frac{1}{c}$$

$$\left(\frac{1}{b}+\frac{1}{c}\right)(a+d) = \left(\frac{1}{a+p}+\frac{1}{a+2p}\right)(a+a+3p)$$

$$= \frac{(2a+3p)^2}{a^2+3ap+2p^2}$$

$$= 4 + \frac{p^2}{a^2+3ap+2p^2} > 4$$

20. a, d.

x, x^2+2, x^3+10 are in G.P. Hence,

$$x(x^3+10) = (x^2+2)^2 = x^4 + 4x^2 + 4$$

$$\Rightarrow \quad 4x^2 - 10x + 4 = 0$$

$$\Rightarrow \quad 2x^2 - 5x + 2 = 0$$

$$\Rightarrow \quad x = 2,\ \frac{1}{2}$$

The 4th term of G.P. is

$$(x^3+10)r = (x^3+10)\left(\frac{x^2+2}{x}\right)$$

$$= \begin{cases} 54 \text{ when } x = 2 \\ \frac{729}{16} \text{ when } x = \frac{1}{2} \end{cases}$$

21. a, b, c.

Last term in n^{th} row is

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1) \qquad (1)$$

As terms in the n^{th} row forms an A.P. with common difference 1, so

First term = Last term – $(n-1)(1)$

$$= \frac{1}{2}n(n+1) - n + 1$$

$$= \frac{1}{2}(n^2 - n + 2) \qquad (2)$$

$$\text{Sum of terms} = \frac{1}{2}n\left[\frac{1}{2}(n^2-n+2) + \frac{1}{2}(n^2+n)\right]$$

$$= \frac{1}{2}n(n^2+1) \qquad (3)$$

Now, put $n = 20$ in (1), (2), (3) to get required answers.

22. a, b, c.

Since A_1, A_2 are two arithmetic means between a and b, therefore, a, A_1, A_2 b are in A.P. with common difference d given by

$$d = \frac{b-a}{2+1} = \frac{b-a}{3} \quad \left[\text{using } d = \frac{b-a}{n+1}\right]$$

Now,

$$A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}$$

and

$$A_2 = a + 2d = a + 2\left(\frac{b-a}{3}\right) = \frac{a+2b}{3}$$

It is given that $G_1 G_2$ are two geometric means between a and b. Therefore, a, G_1, G_2, b are in G.P. with common ratio r given by

$$r = \left(\frac{b}{a}\right)^{\frac{1}{2+1}} = \left(\frac{b}{a}\right)^{1/3} \quad \left[\because r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]$$

Now,

$$G_1 = ar = a\left(\frac{b}{a}\right)^{1/3} = a^{2/3}\, b^{1/3}$$

and

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/3} = a^{1/3}\, b^{2/3}$$

It is also given that H_1, H_2 are two harmonic means between a and b, therefore, a, H_1, H_2, b are in H.P. Hence, $1/a$, $1/H_1$, $1/H_2$, $1/b$, are in A.P. with common difference D given by

$$D = \frac{a-b}{(2+1)ab} = \frac{a-b}{3ab} \quad \left[\because D = \frac{a-b}{(n+1)ab}\right]$$

Now,

$$\frac{1}{H_1} = \frac{1}{a} + D = \frac{1}{a} + \frac{a-b}{3ab} = \frac{a+2b}{3ab}$$

$\Rightarrow \quad H_1 = \dfrac{3ab}{a+2b}$

$\dfrac{1}{H_2} = \dfrac{1}{a} + 2D$

$= \dfrac{1}{a} + \dfrac{2(a-b)}{3ab}$

$= \dfrac{2a+b}{3ab}$

$\Rightarrow \quad H_2 = \dfrac{3ab}{2a+b}$

We have,

$A_1H_2 = \dfrac{2a+b}{3} \times \dfrac{3ab}{2a+b} = ab,$

$A_2H_1 = \dfrac{a+2b}{3} \times \dfrac{3ab}{a+2b} = ab,$

$G_1G_2 = (a^{2/3}b^{1/3})(a^{1/3}b^{2/3}) = ab$

$\therefore \quad A_1H_2 = A_2H_1 = G_1G_2 = ab$

23. b, c, d.

We have, length of a side of S_n is equal to the length of a diagonal of S_{n+1}. Hence,

Length of a side of $S_n = \sqrt{2}$(Length of a side of S_{n+1})

$\Rightarrow \dfrac{\text{Length of a side of } S_{n+1}}{\text{Length of side of } S_n} = \dfrac{1}{\sqrt{2}}$, for all $n \ge 1$

Hence, sides of $S_1, S_2, \ldots, S_n$ form a G.P. with common ratio $1/\sqrt{2}$ and first term 10.

$\therefore \quad$ Side of $S_n = 10\left(\dfrac{1}{\sqrt{2}}\right)^{n-1} = \dfrac{10}{2^{\frac{n-1}{2}}}$

$\Rightarrow \quad$ Area of $S_n = (\text{side})^2 = \left(\dfrac{10}{2^{\frac{n-1}{2}}}\right)^2 = \dfrac{100}{2^{n-1}}$

Now, area of $S_n < 1 \Rightarrow n = b, c, d$.

24. a, c.

$\dfrac{1}{b-a} + \dfrac{1}{b-c} = \dfrac{1}{a} + \dfrac{1}{c}$

$\Rightarrow \dfrac{1}{b-a} - \dfrac{1}{c} = \dfrac{1}{a} - \dfrac{1}{b-c}$

$\Rightarrow \dfrac{c-b+a}{c(b-a)} = \dfrac{b-c-a}{a(b-c)}$

$\Rightarrow c-b+a = 0$ or $\dfrac{1}{c(b-a)} = \dfrac{1}{a(c-b)}$

$\Rightarrow b = a+c$ or $bc - ac = ac - ab$

$\Rightarrow b = a+c$ or $b = \dfrac{2ac}{a+c}$

25. a, c.

a, b, c are in G.P. Hence,

$b^2 = ac$ (1)

x is A.M. of a and b. Hence,

$2x = a+b$ (2)

y is A.M. of b and c. Hence,

$2y = b+c$ (3)

$\therefore \quad \dfrac{a}{x} + \dfrac{c}{y} = a \times \dfrac{2}{a+b} + c \times \dfrac{2}{b+c}$ [Using (2) and (3)]

$= 2\left[\dfrac{ab+ac+ac+bc}{ab+ac+b^2+bc}\right]$

$= 2$ [Using (i)]

Again,

$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2}{a+b} + \dfrac{2}{b+c}$

$= \dfrac{2(a+c+2b)}{ab+ac+b^2+bc}$

$= \dfrac{2(a+c+2b)}{ab+2b^2+bc}$ $(\because b^2 = ac)$

$= \dfrac{2(a+c+2b)}{b(a+c+2b)}$

$= \dfrac{2}{b}$

26. a, c. Given $a_1 = 2$; $\dfrac{a_n}{a_{n-1}} = \dfrac{a_{n-1}}{a_{n-2}}$

$\Rightarrow a_1, a_2, a_3, a_4, a_5, \ldots$ in G.P.

Let $a_2 = x$ then for $n = 3$ we have

$\dfrac{a_3}{a_2} = \dfrac{a_2}{a_1} = \dfrac{x^2}{2}$

$\Rightarrow a_1^2 = a_1a_3$

$\Rightarrow a_3 = \dfrac{x^2}{2}$

i.e. $2, x, \dfrac{x^2}{2}, \dfrac{x^3}{4}, \dfrac{x^4}{8}, \ldots$ with common ratio $r = \dfrac{x}{2}$

given $\dfrac{x^4}{8} \le 162$

$\Rightarrow x^4 \le 1296 \le \; x \le 6$

Also x $\dfrac{x^4}{8}$ and are integers

$\Rightarrow x$ must be even then only $\dfrac{x^4}{8}$ will be an integer.

hence possible values of x is 4 and 6. ($x \ne 2$ as terms are distinct)

hence possible value of $a_5 = \dfrac{x^4}{8}$ is $\dfrac{4^4}{8}, \dfrac{6^4}{8}$

27. a, b, c. Let a, b, c are pth, qth and rth terms of A.P. then $a = A + (p-1)D$, $b = A + (q-1)D$, $c = A + (r-1)D$

$\Rightarrow \dfrac{r-q}{q-p} = \dfrac{c-b}{b-a}$ is rational number.

Now for 1, 6, 19 $\dfrac{r-q}{q-p} = \dfrac{19-6}{6-1}$ is rational number.

For $\sqrt{2}, \sqrt{50}, \sqrt{98}$, $\dfrac{r-q}{q-p} = \dfrac{\sqrt{98}-\sqrt{50}}{\sqrt{50}-\sqrt{2}} = \dfrac{7\sqrt{2}-5\sqrt{2}}{5\sqrt{2}-\sqrt{2}}$

$= \dfrac{1}{2}$ is rational number.

For log2, log16, log128,

$\dfrac{r-q}{q-p} = \dfrac{\log 128 - \log 16}{\log 16 - \log 2} = \dfrac{7\log 2 - 4\log 2}{4\log 2 - \log 2} = 1$ is rational number.

But for $\sqrt{2}, \sqrt{3}, \sqrt{7}$, $\frac{r-q}{q-p}$ is not rational number.

28. c, d. $4 = 1 + (n-1)d$, $16 = 1 + (m-1)d \Rightarrow \frac{15}{3} = \frac{m-1}{n-1}$ or

$\frac{n-1}{1} = \frac{m-1}{5} = p$ = positive integer.

$\therefore$ $n = p + 1, m = 5p + 1$. So, n m have infinite pairs of values. Also, $4 = 1 . r^n$, $16 = 1 . r^m \Rightarrow rm^{-n} = 4 = r^n$. So, $m - n = n$

$\therefore$ $\frac{m}{2} = \frac{n}{1} = q$ = positive integer. So, m, m have infinite pairs of values.

Reasoning Type

1. a. Let p, q, r be the l^{th}, m^{th} and n^{th} terms of an A.P. Then

$p = (a + (l-1)d, q = a + (m-1)d$ and $r = a + (n-1)d$

Hence, $r - q = (n - m)d$ and $q - p = (m - l)d$, so that

$$\frac{r-q}{p-q} = \frac{(n-m)d}{(m-l)d} = \frac{n-m}{m-l} \quad (\because d \neq 0) \qquad (1)$$

Since l, m, n are positive integers and $m \neq l$, $(n-m)/(m-l)$ is a rational number. From (1), using $p = \sqrt{2}, q = \sqrt{3}, r = \sqrt{5}$, we have

$$\frac{\sqrt{5}-\sqrt{3}}{\sqrt{3}-\sqrt{2}} = \frac{n-m}{m-l} \text{ (which is not possible.)}$$

Hence, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be the terms of an A.P.

2. a. Statement 2 is true as it is a property of sequence in G.P.

Now T_{m-n}, T_m and T_{m+n} are in G.P. ($\because T_m$ from T_{m-n} and T_{m+n} from T_m are at same distance)

$\therefore$ $T_m^2 = T_{m-n} T_{m+n}$

$\Rightarrow$ $T_m = \sqrt{pq}$

3. b. Let, if possible, 8 be the first term and 12 and 27 be m^{th} and n^{th} terms, respectively. Then,

$12 = ar^{m-1} = 8r^{m-1}$, $27 = 8r^{n-1}$

$\Rightarrow$ $\frac{3}{2} = r^{m-1}, \left(\frac{3}{2}\right)^3 = r^{n-1} = r^{3(m-1)}$

$\Rightarrow$ $n - 1 = 3m - 3$ or $3m = n + 2$

$\Rightarrow$ $\frac{m}{1} = \frac{n+2}{3} = k$ (say)

$\therefore$ $m = k, n = 3k - 2$

By giving k different values, we get the integral values of m and n.

Hence there can be infinite number of G.P.'s whose any three terms will be 8, 12, 27 (not consecutive). Obviously, statement 2 is not a correct explanation of statement 1.

4. a. $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$

$\Rightarrow$ $x^2 + 9y^2 + 25z^2 - 15yz - 5xz - 3xy = 0$

$\Rightarrow$ $2x^2 + 18y^2 + 50z^2 - 30yz - 10xz - 6xy = 0$

$\Rightarrow$ $(x - 3y)^2 + (3y - 5z)^2 + (5z - x)^2 = 0$

$\Rightarrow$ $x - 3y = 0, 3y - 5z = 0, 5z - x = 0$

$\Rightarrow$ $x = 3y = 5z = k$ (say)

$\Rightarrow$ $x = k, y = k/3, z = k/5$

Hence, x, y, z are in H.P. Hence option (a) is correct.

5. a. Coefficient of x^{14} in $(1 + 2x + 3x^2 + \cdots + 16x^{15})^2$

= Coefficient of x^{14} in $(1 + 2x + 3x^2 + \cdots + 16x^{15})(1 + 2x + 3x^2 + \cdots + 16x^{15})$

$= 1 \times 15 + 2 \times 14 + \cdots + 15 \times 1$

$= \sum_{r=1}^{15} r(16-r)$

Also,

$$\sum_{r=1}^{n-1} r(n-r) = \sum_{r=1}^{n-1} nr - \sum_{r=1}^{n-1} r^2$$

$$= n\frac{n(n-1)}{2} - \frac{n(n-1)(2n-1)}{6}$$

$$= \frac{n(n-1)}{6}(3n-(2n-1))$$

$$= \frac{n(n^2-1)}{6}$$

$$\Rightarrow \sum_{r=1}^{15} r(16-r) = \frac{15(15^2-1)}{6} = 560$$

Hence option (a) is correct.

6. b. $x = 1111\cdots91$ times

$= 1 + 10 + 10^2 + 10^3 + \cdots + 10^{90}$

$= \frac{1(10^{91}-1)}{10-1}$

$= \frac{(10^{13\times 7}-1)}{10-1}$

$= \frac{((10^{13})^7-1)}{10^{13}-1} \times \frac{(10^{13}-1)}{10-1}$

$= (1 + 10^{13} + 10^{26} + \cdots + 10^{78}) \times (1 + 10 + 10^2 + \cdots + 10^{12})$

= composite numbers

But statement 2 is not a correct explanation of statement 1 as 111 has 1 digit 3 times, and 3 is a prime number but $111 = 3 \times 37$ is a composite number. Hence (b) is the correct option.

7. a. We have,

$$a \times ar \times \cdots \times ar^{n-1} = a^n \times r^{1+2+\cdots+(n-1)} = a^n r^{\frac{n(n-1)}{2}} = (a^2 r^{n-1})^{n/2}$$

Hence, statement 1 is true.

Also, $(a \times r^{i-1})(a \times r^{n-ik}) = a^2 \times r^{n-1}$, which is independent of k. Hence, statement 2 is a correct explanation for statement 1, as in the product of $a, ar, ar^2, \ldots ar^{n-1}$, there are $n/2$ groups of numbers, whose product is $a^2 r^{n-1}$. Hence (a) is the correct option.

8. d. For odd integer n, we have

$S_n = n^3 - (n-1)^3 + \cdots + (-1)^{n-1} 1^3$

$= 1^3 - 2^3 + 3^3 - 4^3 + \cdots + n^3$

$= [1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3] - 2[2^3 + 4^3 + 6^3 + \cdots + (n-1)^3]$

$$= \frac{n^2(n+1)^2}{4} - 2 \times 2^3\left[1^3 + 2^3 + \cdots + \left(\frac{n-1}{2}\right)^3\right]$$

$$= \frac{n^2(n+1)^2}{4} - 2^4 \frac{\left(\frac{n-1}{2}\right)^2\left(\frac{n-1}{2}+1\right)^2}{4}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2(n+1)^2}{4}$$

$$= \frac{(n+1)^2}{4}\left[n^2 - (n-1)^2\right]$$

$$= \frac{1}{4}(2n-1)(n+1)^2$$

Now, putting $n = 11$ in above formula, $S_{11} = 756$. Hence statement 1 is false and statement 2 is correct.

9. d. Sum $= \frac{x/r}{1-r} = 4$ (where r is common ratio)

$$x = 4r(1-r) = 4(r-r^2)$$

For $r \in (-1, 1) - \{0\}$

$$r - r^2 \in \left(-2, \frac{1}{4}\right) - \{0\}$$

$\Rightarrow$ $x \in (-8, 1) - \{0\}$

10. b. The given inequality is

$(p_1^2 + p_2^2 + \cdots + p_{n-1}^2)x^2 + 2(p_1p_2 + p_2p_3 + \cdots + p_{n-1}p_n)x + (p_2^2 + \cdots + p_n^2) \le 0$

$\Rightarrow$ $(p_1x + p_2)^2 + (p_2x + p_3)^2 + \cdots + (p_{n-1}x + p_n)^2 \le 0$ (1)

But each one of the terms on the L.H.S. is a perfect square and hence is positive or zero.

Therefore (1) holds only if

$$p_1x + p_2 = 0 = p_2x + p_3 = p_3x + p_4 = \cdots = p_{n-1}x + p_n$$

$\Rightarrow$ $-x = \frac{p_2}{p_1} = \frac{p_3}{p_2} = \cdots = \frac{p_n}{p_{n-1}}$

Hence, $p_1, p_2, \ldots, p_n$ are in G.P.

11. a. Statement 2 is true as

$$\frac{a^n + b^n}{a+b} = \frac{a^n - (-b)^n}{a-(-b)} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots (-1)^{n-1}b^{n-1}$$

Now,

$1^{99} + 2^{99} + \cdots + 100^{99} = (1^{99} + 100^{99}) + (2^{99} + 99^{99}) + \cdots + (50^{99} + 51^{99})$

Each bracket is divisible by 101; hence the sum is divisible by 101. Also,

$1^{99} + 2^{99} + \text{L} \cdots 100^{99} = (1^{99} + 99^{99}) + (2^{99} + 98^{99}) + \cdots + (49^{99} + 51^{99}) + 50^{99} + 100^{99}$

Here, each bracket and 50^{99} and 100^{99} are divisible by 100. Hence sum is divisible by 100. Hence sum is divisible by $101 \times 100 = 10100$.

12. a. For two positive numbers $(\text{GM})^2 = (\text{AM}) \times (\text{HM})$.

Linked Comprehension Type

For Problems 1–3

1. c, 2. b, 3. d.

Sol. Let the odd integers be $2m + 1, 2m + 3, 2m + 5, \ldots$ and let their number be n. Then,

$$57^2 - 13^2 = (n/2)[2(2m+1) + (n-1) \times 2]$$
$$= n(2m + n)$$
$$= 2mn + n^2$$

$\Rightarrow$ $57^2 - 13^2 = (n+m)^2 - m^2$

$\Rightarrow$ $m = 13$ and $n + m = 57$

$\Rightarrow$ $n = 57 - 13 = 44$

Hence, the required odd integers are 27, 29, 31, …, 113.

For Problems 4–6

4. c, 5. c, 6. d.

Sol. 4. a, b, c are in G.P. Hence, a, ar, ar^2 are in G.P. So,

$$\frac{a^2 + a^2r^2 + a^2r^4}{(a + ar + ar^2)^2} = \frac{t^2}{\alpha^2 t^2} = \frac{1}{\alpha^2}$$

$$\alpha^2 = \frac{r^2 + r + 1}{r^2 - r + 1}$$

Let $\alpha^2 = y$.

$\therefore$ $y = \frac{r^2 + r + 1}{r^2 - r + 1}$

$$(y-1)r^2 - r(y+1) + (y-1) = 0$$

For real r,

$$(y+1)^2 - 4(y-1)^2 \ge 0$$

$\Rightarrow$ $\frac{1}{3} \le y \le 3$

But $y \ne 1/3, 1, 3$ $(\because r \ne 1, -1, 0)$

$\therefore$ $\frac{1}{3} < y < 3$ and $y \ne 1$

$$\alpha^2 \in \left(\frac{1}{3}, 3\right) - \{1\}$$

5. $S = r + \frac{1}{r}$

$$S \in (-\infty, -2) \cup (2, \infty)$$

6. Let $b = ar$, $c = ar^2$ and $r > 0$.

As sum of two sides is more than the third side, we have,

$$r \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right) - \{1\}$$

$\Rightarrow$ $r + \frac{1}{r} - 1 \in (1, \sqrt{5} - 1)$

As $\alpha^2 = \frac{r^2 + r + 1}{r^2 - r + 1} = 1 + \frac{2}{r + \frac{1}{r} - 1}$

$\therefore$ $\alpha^2 \in \left(\frac{\sqrt{5}+3}{2}, 3\right)$

For Problems 7–9

7. d, 8. b, 9. d.

Sol. Let a be the first term and r the common ratio of the given G.P. Further, let there be n terms in the given G.P. Then,

$a_1 + a_n = 66 \Rightarrow a + ar^{n-1} = 66$ (i)

$a_2 \times a_{n-1} = 128$

$\Rightarrow$ $ar \times ar^{n-2} = 128$

$\Rightarrow$ $a^2 r^{n-1} = 128$

$\Rightarrow$ $a \times (ar^{n-1}) = 128 \Rightarrow ar^{n-1} = \frac{128}{a}$

Putting this value of ar^{n-1} in (i), we get

$$a + \frac{128}{a} = 66$$

$\Rightarrow$ $a^2 - 66a + 128 = 0$

$\Rightarrow$ $(a-2)(a-64) = 0$

$\Rightarrow$ $a = 2, 64$

Putting $a = 2$ in (1), we get

$$2 + 2 \times r^{n-1} = 66 \Rightarrow r^{n-1} = 32$$

Putting $a = 64$ in (1), we get

$$64 + 64r^{n-1} = 66 \Rightarrow r^{n-1} = \frac{1}{32}$$

For an increasing G.P., $r > 1$. Now,

$$S_n = 126$$

$$\Rightarrow \quad 2\left(\frac{r^n - 1}{r - 1}\right) = 126$$

$$\Rightarrow \quad \frac{r^n - 1}{r - 1} = 63$$

$$\Rightarrow \quad \frac{r^{n-1} \times r - 1}{r - 1} = 63$$

$$\Rightarrow \quad \frac{32r - 1}{r - 1} = 63$$

$$\Rightarrow \quad r = 2$$

$$\therefore \quad r^{n-1} = 32 \Rightarrow 2^{n-1} = 32 = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$$

For decreasing G.P., $a = 64$ and $r = 1/2$. Hence, the sum of infinite terms is $64/\{1 - (1/2)\} = 128$.

For $a = 2$, $r = 2$, terms are 2, 4, 8, 16, 32, 64. For $a = 64$, $r = 1/2$ terms are 64, 32, 16, 8, 4, 2 Hence difference is 62.

For Problems 10–12

10. c, 11. d, 12. a.

Sol. Let the four integers be $a - d$, a, $a + d$ and $a + 2d$, where a and d are integers and $d > 0$. Now,

$$a + 2d = (a - d)^2 + a^2 + (a + d)^2$$

$$\Rightarrow \quad 2d^2 - 2d + 3a^2 - a = 0 \qquad (1)$$

$$\therefore \quad d = \frac{1}{2}\left[1 \pm \sqrt{1 + 2a - 6a^2}\right] \qquad (2)$$

Since d is a positive integer, so

$$1 + 2a - 6a^2 > 0$$

$$\Rightarrow \quad 6a^2 - 2a - 1 < 0$$

$$\Rightarrow \quad \frac{1 - \sqrt{7}}{6} < a < \frac{1 + \sqrt{7}}{6}$$

$$\Rightarrow \quad a = 0 \qquad (\because a \text{ is an integer})$$

Hence from (2),

$$d = 1 \text{ or } 0$$

But since $d > 0$,

$$\therefore \quad d = 1$$

Hence, the four numbers are $-1, 0, 1, 2$.

For Problems 13–15

13. d, 14. a, 15. b.

Sol. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …

Let us write the terms in the groups as follows: 1, (2, 2), (3, 3, 3), (4, 4, 4, 4), … consisting of 1, 2, 3, 4, … terms. Let 2000[th] term fall in n^{th} group. Then,

$$\frac{(n-1)n}{2} < 2000 \le \frac{n(n+1)}{2}$$

$$\Rightarrow \quad n(n - 1) < 4000 \le n(n + 1)$$

Let us consider,

$$n(n - 1) < 4000$$

$$\Rightarrow \quad n^2 - n - 4000 < 0$$

$$\Rightarrow \quad n < \frac{1 + \sqrt{16001}}{2} \Rightarrow n < 64$$

We have,

$$n(n + 1) \ge 4000 \Rightarrow n^2 + n - 4000 \ge 0 \Rightarrow n \ge 63$$

That means 2000[th] term falls in 63[rd] group. That also means that the 2000[th] term is 63. Now, total number of terms up to 62[nd] group is $(62 \times 63)/2 = 1953$. Hence, sum of first 2000 terms is

$$1^2 + 2^2 + \cdots + 62^2 + 63(2000 - 1953) = \frac{62(63)125}{6} + 63 \times 47$$
$$= 84336$$

Sum of the remaining terms is $63 \times 16 = 1008$.

For Problems 16–18

16. b, 17. a, 18. c.

Sol. Let numbers in set A be $a - D, a, a + D$ and these in set B be $b - d, b, b + d$. Now,

$$3a = 3b = 15$$

$$\Rightarrow \quad a = b = 5$$

$$\text{Set } A = \{5 - D, 5, 5 + D\}$$

$$\text{Set } B = \{5 - d, 5, 5 + d\}$$

where $D = d + 1$

Also,

$$\frac{p}{q} = \frac{5(25 - D^2)}{5(25 - d^2)} = \frac{7}{8}$$

$$\Rightarrow \quad 25(8 - 7) = 8(d + 1)^2 - 7d^2$$

$$\Rightarrow \quad d = -17, 1 \text{ but } d > 0 \Rightarrow d = 1$$

So, the numbers in set A are 3, 5, 7 and the numbers in set B are 4, 5, 6.

Now, sum of product of numbers in set A taken two at a time is $3 \times 5 + 3 \times 7 + 5 \times 7 = 71$. The sum of product of numbers in set B taken two at a time is $4 \times 5 + 5 \times 6 + 6 \times 4 = 74$. Also,

$$p = 3 \times 5 \times 7 = 105 \text{ and } q = 4 \times 5 \times 6 = 120$$

$$\Rightarrow \quad q - p = 15$$

For Problems 19–21

19. c, 20. b, 21. a.

Sol. 19. $G_1 G_2 \cdots G_n = (\sqrt{1 \times 1024})^n = 2^{5n}$

Given,

$$2^{5n} = 2^{45} \Rightarrow n = 9$$

Hence,

$$r = (1024)^{\frac{1}{9+1}} = 2$$

$$\Rightarrow \quad G_1 = 2, r = 2$$

$$\Rightarrow \quad G_1 + G_2 + \cdots + G_9 = \frac{2 \times (2^9 - 1)}{2 - 1} = 1024 - 2 = 1022$$

20. b. $A_1 + A_2 + A_3 + \cdots + A_{m-1} + A_m = 1025 \times 771$

$$\Rightarrow m\left(\frac{-2 + 1027}{2}\right) = 1025 \times 171$$

$$\Rightarrow m = 342$$

21. a. We have,

$$A_{171} + A_{172} = -2 + 1027 = 1025$$

$$\Rightarrow \quad \frac{2A_{171} + 2A_{172}}{2} = 1025$$

Also,

$$G_5 = 1 \times 2^5 = 32$$

$\Rightarrow \quad G_5^2 = 1024$

$\Rightarrow \quad G_5^2 + 1 = 1025$

$\Rightarrow \quad 2A_{171}, G_5^2 + 1, 2A_{172}$ are in A.P.

For Problems 22–24

22. a, 23. c, 24. b.

Sol. Let m and $(m + 1)$ be the removed numbers from 1, 2, ..., n.

Then, sum of the remaining numbers is $n(n + 1)/2 - (2m + 1)$. From given condition,

$$\frac{105}{4} = \frac{\frac{n(n+1)}{2} - (2m+1)}{(n-2)}$$

$\Rightarrow \quad 2n^2 - 103n - 8m + 206 = 0$

Since n and m are integers, so n must be even. Let $n = 2k$. Then,

$$m = \frac{4k^2 + 103(1-k)}{4}$$

Since m is an integer, then $1 - k$ must be divisible by 4. Let $k = 1 + 4t$. Then we get $n = 8t + 2$ and $m = 16t^2 - 95t + 1$. Now,

$1 \le m < n$

$\Rightarrow \quad 1 \le 16t^2 - 95t + 1 < 8t + 2$

Solving, we get $t = 6$. Hence,

$n = 50$ and $m = 7$

Hence, the removed numbers are 7 and 8. Also, sum of all numbers is $50(50 + 1)/2 = 1275$.

For Problems 25–27

25. c, 26. b, 27. a.

Sol. Let the first term a and common difference d of the first A.P. and the first term b and common difference e of the second A.P. and let the number of terms be n. Then,

$$\frac{a + (n-1)d}{b} = \frac{b + (n-1)e}{a} = 4 \qquad (1)$$

$$\frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2b + (n-1)e]} = 2 \qquad (2)$$

From (1) and (2), we get

$a - 4b + (n - 1)d = 0 \qquad (3)$

$b - 4a + (n - 1)e = 0 \qquad (4)$

$2a - 4b + (n - 1)d - 2(n - 1)e = 0 \qquad (5)$

$4 \times (3) + (4)$ gives

$-15b + 4(n - 1)d + (n - 1)e = 0 \qquad (6)$

$(4) + 2 \times (5)$ gives

$-7b + 2(n - 1)d - 3(n - 1)e = 0 \qquad (7)$

Further, $15 \times (7) - 7 \times (6)$ gives

$2(n - 1)d - 52(n - 1)e = 0$

or

$d = 26e \quad (\because n > 1)$

$\therefore \quad d/e = 26$

Putting $d = 26e$ in (3) and solving it with (4), we get

$a = 2(n - 1)e, b = 7(n - 1)e$

Then, the ratio of their n^{th} terms is

$$\frac{2(n-1)e + (n-1)26e}{7(n-1)e + (n-1)e} = \frac{7}{2}$$

For Problems 28–30

28. d, 29. c, 30. b.

Sol. We have,

$a + b + c = 25 \qquad (1)$

$2a = b + 2 \qquad (2)$

$c^2 = 18b \qquad (3)$

Eliminating a from (1) and (2), we have

$$b = 16 - \frac{2c}{3}$$

Then from (3),

$$c^2 = 18\left(16 - \frac{2c}{3}\right)$$

$\Rightarrow \quad c^2 + 12c - 18 \times 16 = 0$

$\Rightarrow \quad (c - 12)(c + 24) = 0$

Now, $c = -24$ is not possible since it does not lie between 2 and 18. Hence, $c = 12$. Then from (3), $b = 8$ and finally from (2), $a = 5$.

Thus, $a = 5$, $b = 8$ and $c = 12$. Hence, $abc = 5 \times 8 \times 12 = 480$.

Also, equation $ax^2 + bx + c = 0$ is $5x^2 + 8x + 12 = 0$, which has imaginary roots.

If a, b, c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then sum of product of roots taken two at a time is $r = 5 \times 8 + 5 \times 12 + 8 \times 12 = 196$.

For Problems 31–33

31. c, 32. c, 33. c.

Sol. 31. Clearly, here the differences between the successive terms are $7 - 3, 14 - 7, 24 - 14, \ldots$, i.e., 4, 7, 10, ... which are in A.P.

$\therefore \quad T_n = an^2 + bn + c$

Thus, we have

$3 = a + b + c$

$7 = 4a + 2b + c$

$14 = 9a + 3b + c$

Solving, we get $a = 3/2, b = -1/2, c = 2$. Hence,

$$T_n = \frac{1}{2}(3n^2 - n + 4)$$

$$\therefore \quad S_n = \frac{1}{2}\left[3\Sigma n^2 - \Sigma n + 4n\right]$$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$

$$= \frac{n}{2}(n^2 + n + 4)$$

$\Rightarrow \quad S_{20} = 4240$

32. The first differences are 5, 14, 50 194, 770, ...

The second differences are 9, 36, 144, 576, ...

They are in G.P. whose n^{th} term is $ar^{n-1} = a4^{n-1}$.

Therefore, T_n of the given series will be of the form

$T_n = a4^{n-1} + bn + c$

$T_1 = a + b + c = 3$

$T_2 = 4a + 2b + c = 8$

$T_3 = 16a + 3b + c = 22$

Solving, we have $a = 1, b = 2, c = 0$.

$\therefore \quad T_n = 4^{n-1} + 2n$

$\Rightarrow \quad T_{100} = 4^{99} + 200$

$= 2^{198} + 2^3 \times 25$

$= 8(2^{195} + 25)$ (which is divisible by 8)

33. Given series is $2 + 12 + 36 + 80 + 150 + 252 + \cdots$

The first differences are 10, 24, 44, 70, 102, ...

The second differences are 14, 20, 26, 32, ... which are in A.P.

Hence, general term of the series is

$T_n = an^3 + bn^2 + cn + d$

$\Rightarrow \quad 2 = a + b + c + d$

$12 = 8a + 4b + 2c + d$

$36 = 27a + 9b + 3c + d$

$80 = 64a + 16b + 4c + d$

Solving for a, we get $a = 1$.

$$\therefore \quad \lim_{n\to\infty} \frac{T_n}{n^3} = \lim_{n\to\infty}\left(1 + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3}\right) = 1$$

Matrix-Match Type

1. a → q; b → r; c → p; d → r.

a. a, b, c are in G.P. Hence,

$b^2 = ac$

$\Rightarrow \quad 2\log_{10} b = \log_{10} a + \log_{10} c$

$$\Rightarrow \quad \frac{2}{\log_b 10} = \frac{1}{\log_a 10} + \frac{1}{\log_c 10}$$

$$\Rightarrow \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

Hence, x, y, z are in H.P.

b. $$\frac{a+be^x}{a-be^x} = \frac{b+ce^x}{b-ce^x} = \frac{c+de^x}{c-de^x}$$

$$\Rightarrow \quad \frac{2a}{a-be^x} - 1 = \frac{2b}{b-ce^x} - 1 = \frac{2c}{c-de^x} - 1$$

$$\Rightarrow \quad \frac{a-be^x}{a} = \frac{b-ce^x}{b} = \frac{c-de^x}{c}$$

$$\Rightarrow \quad 1 - \frac{b}{a}e^x = 1 - \frac{c}{b}e^x = 1 - \frac{d}{e}e^x$$

$$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

Hence, a, b, c, d are in G.P.

c. Given, $2b = a + c$, $x^2 = ab$, $y^2 = bc$. Now,

$x^2 + y^2 = b(a + c) = b\,2b = 2b^2$

$\Rightarrow \quad x^2 + y^2 = 2b^2$

Hence, x^2, b^2, y^2 are in A.P.

d. $x \log a = y \log b = z \log c = k$ (say)

Also,

$y^2 = xz$

$$\Rightarrow \quad \frac{k^2}{(\log b)^2} = \frac{k^2}{\log a \log c}$$

Hence, $\log a$, $\log b$, $\log c$ are in G.P.

2. a → p, q, r, s; b → r, s; c → p, q; d → r, s.

a. $\Sigma n = 210$

$\Rightarrow \quad n(n + 1) = 420$

$\Rightarrow \quad (n - 20)(n + 21) = 0$

$\Rightarrow \quad n = 20$

Hence,

$$\Sigma n^2 = \frac{n}{6}(n+1)(2n+1)$$

$$= \frac{20}{6}(21)(41)$$

$= (10)(7)(41)$

Hence, the greatest prime number by which Σn^2 is divisible is 41.

b. $4, G_1, G_2, \ldots, G_{n+1}, \ldots, G_{2n}, G_{2n+1}, 2916$

G_{n+1} will be the middle mean of $(2n + 1)$ odd means and it will be equidistant from the first and last terms. Hence,

$4, G_{n+1}, 2916$ will also be in G.P. So,

$\Rightarrow \quad G_{n+1}^2 = 4 \times 2916$

$= 4 \times 9 \times 324$

$= 4 \times 9 \times 4 \times 81$

$\Rightarrow \quad G_{n+1} = 2 \times 3 \times 2 \times 9 = 108$

Hence, the greatest odd number by which G_{n+1} is divisible is 27.

c. Terms are 40, 30, 24, 20. Now,

$$\frac{1}{30} - \frac{1}{40} = \frac{1}{120}$$

$$\Rightarrow \quad \frac{1}{24} - \frac{1}{30} = \frac{6}{24 \times 30} = \frac{1}{120}$$

and

$$\frac{1}{20} - \frac{1}{24} = \frac{4}{20 \times 24} = \frac{1}{120}$$

Hence, 1/30, 1/24, 1/20 are in A.P. with common difference d = 1/120. Hence, the next term is 1/20 + 1/120 = 7/120. Therefore, the next term of given series is $\frac{120}{7} = 17\frac{1}{7}$. Hence, the integral part of $17\frac{1}{7}$ is 17.

d. $$S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots \infty$$

$$\Rightarrow \quad \frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots$$

$$\Rightarrow \quad S\left(1 - \frac{1}{5}\right) = 1 + 3\left[\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots \infty\right]$$

$$\Rightarrow \quad \frac{4}{5}S = 1 + 3\left[\frac{1/5}{1-1/5}\right] = 1 + \frac{3}{4} = \frac{7}{4}$$

$$\Rightarrow \quad S = \frac{35}{16}$$

$\Rightarrow \quad a = 35$ and $b = 16$

$\Rightarrow \quad a - b = 19$

Integer Type

1.(0) $10x^3 - nx^2 - 54x - 27 = 0$ has roots in H.P.

put $x = 1/t$

$27t^3 + 54t^2 + nt - 10 = 0$

This equation has roots in A.P., let the roots are $a - d$, a and $a + d$

$\therefore 3a = -\frac{54}{27} \Rightarrow a = -\frac{2}{3}$

Also $(a - d)\, a\, (a + d) = \frac{10}{27}$

$\therefore \frac{2}{3}\left(\frac{4}{9} - d^2\right) = -\frac{10}{27} \Rightarrow \left(\frac{4}{9} - d^2\right) = -\frac{5}{9}$

$\therefore d^2 = 1 \Rightarrow d = \pm 1$

For $d = 1$, roots are $-\frac{2}{3}+1, -\frac{2}{3}, -\frac{2}{3}-1 \Rightarrow \frac{1}{3}, -\frac{2}{3}, -\frac{5}{3}$,

for $d = -\frac{2}{3}, -1-\frac{2}{3}, -\frac{2}{3}+1 \Rightarrow -\frac{5}{3}, -\frac{2}{3}, \frac{1}{3}$

$\therefore \frac{n}{27} = \frac{10}{9} - \frac{5}{9} - \frac{2}{9} \Rightarrow \frac{n}{27} = \frac{3}{9}$

$\Rightarrow n = 9$

2.(9) $\left[\frac{k(k+1)}{2}\right]^2 - \frac{k(k+1)}{2} = 1980$

$\Rightarrow \frac{k(k+1)}{2}\left[\frac{k(k+1)}{2} - 1\right] = 1980$

$\Rightarrow k(k + 1)(k^2 + k - 2) = 1980 \times 4$

$\Rightarrow (k - 1)k(k + 1)(k + 2) = 8.\,9.\,10.\,11$

$\therefore k - 1 = 8 \Rightarrow k = 9$

3.(6) We have $S = 3 + \sum_{n=1}^{\infty} \frac{2n+3}{3^n} = 3 + \underbrace{\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}}_{S_1} + \underbrace{\sum_{n=1}^{\infty} \frac{2n}{3^n}}_{S_2}$

Now $S_1 = \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \infty$

$\therefore S_1 = \frac{1}{1-(1/3)} = \frac{3}{2}$

$S_2 = \sum_{n=1}^{\infty} \frac{2n}{3^n} = \frac{2}{3} + \frac{4}{3^2} + \frac{6}{3^3} + \frac{8}{3^4} + \cdots \infty$

$S_2 = \frac{2}{3} + \frac{4}{3^2} + \frac{6}{3^3} + \frac{8}{3^4} + \cdots \infty$

Now, $\frac{S_2}{3} = \quad + \frac{2}{3^2} + \frac{4}{3^3} + \frac{6}{3^4} + \cdots \infty$

$\frac{2S_2}{3} = \frac{2}{3}\left[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \infty\right]$ [On subtracting]

$\therefore S_2 = \frac{1}{1-(1/3)} = \frac{3}{2}$

Hence, $S = 3 + \frac{3}{2} + \frac{3}{2} = 6$.

4.(1) Let a be the first term r be the common ratio of G.P.

$\therefore \frac{a(1 - r^{201})}{1 - r} = 625 \quad (1)$

Now $\sum_{i=1}^{201} \frac{1}{a_i} = \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{201}}\right)$

$= \frac{1}{a} + \frac{1}{ar} + \cdots + \frac{1}{ar^{200}}$

$= \dfrac{\frac{1}{a}\left(\left(\frac{1}{r}\right)^{201} - 1\right)}{\left(\frac{1}{r} - 1\right)}$

$= \frac{1}{a}\left(\frac{1 - r^{201}}{1 - r}\right)\frac{1}{r^{200}}$

$= \frac{1}{a} \times \frac{625}{a} \times \frac{1}{r^{200}}$ [from (1)]

$= \frac{625}{(ar^{100})^2} = \frac{625}{(a_{101})^2} = \frac{625}{625} = 1$

5.(9) Given $S = \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n} + \sqrt{n+1}\right)\left(\sqrt[4]{n} + \sqrt[4]{n+1}\right)}$

$= \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n} + \sqrt{n+1}\right)\left(\sqrt[4]{n} + \sqrt[4]{n+1}\right)} \left(\frac{\sqrt[4]{n} - \sqrt[4]{n+1}}{\sqrt[4]{n} - \sqrt[4]{n+1}}\right)$

$= \sum_{n=1}^{9999} \left((n+1)^{1/4} - n^{1/4}\right)$

$= \left(\left(2^{\frac{1}{4}} - 1\right) + \left(3^{\frac{1}{4}} - 2^{\frac{1}{4}}\right) + \left(4^{\frac{1}{4}} - 3^{\frac{1}{4}}\right) + \cdots + \left((9999+1)^{\frac{1}{4}} - (9999)^{\frac{1}{4}}\right)\right)$

$= (10^4)^{\frac{1}{4}} - 1 = 9$

6.(3) Let $a, ar, ar^2, ar^3, \ldots$ are in G.P.

Now $ar^4 = 7!$ and $ar^7 = 8!$

$\therefore$ On dividing, we get $r^3 = 8 \Rightarrow r = 2$

Hence, $a\,.\,2^4 = 5040$

$\therefore a = \frac{5040}{16} = 315$

So 315, 630, 1260, ... are in G.P.

$\therefore S_3 = 2205 \quad \Rightarrow \quad n = 3$

7.(8) Since a, b, c, d are in A.P.

$\therefore b - a = c - b = d - c = D$ (let common difference)

$\Rightarrow d = a + 3D$

$\Rightarrow a - d = -3D$ and $d = b + 2D$

$\Rightarrow b - d = -2D$

Also $c = a + 2D \quad \Rightarrow \quad c - a = 2D$

$\therefore$ Given equation $2(a - b) + k(b - c)^2 + (c - a)^3 = 2(a - d) + (b - d)^2 + (c - d)^3$

becomes $-2D + kD^2 + (2D)^3 = -6D + 4D^2 - D^3$

$\Rightarrow 9D^2 + (k-4)D + 4 = 0$

Since D is real $\Rightarrow (k-4)^2 - 4(4)(9) \geq 0$

$\Rightarrow k^2 - 8k - 128 \geq 0 \Rightarrow (k-16)(k+8) \geq 0$

$\therefore k \in (-\infty, -8] \cup [16, \infty)$

Hence, the smallest positive value of $k = 16$.

8.(7) 6, a, b in H.P.

$\Rightarrow \frac{1}{6}, \frac{1}{a}, \frac{1}{b}$ are in A.P.

$\Rightarrow \frac{2}{a} = \frac{1}{6} + \frac{1}{b}$

$\Rightarrow \frac{1}{b} = \frac{2}{a} - \frac{1}{6}$

$\Rightarrow \frac{1}{b} = \frac{12-a}{6a}$

$\Rightarrow b = \frac{6a}{12-a}$

$a \in \{3, 4, 6, 8, 9, 10, 11\}$

9.(6) 10 For the given A.P., we have $2(2a + b) = (5a - b) + (a + 2b)$

$\Rightarrow b = 2a$ (i)

Also for the given G.P., we have $(ab + 1)^2 = (a-1)^2 (b+1)^2$ (ii)

$\therefore$ Putting $b = 2a$ from (i) in (ii), we get $a = 0, -2$ or $\frac{1}{4}$

But $a > 0$, so $a = \frac{1}{4}$ and $b = 2a = \frac{1}{2}$

Hence, $(a^{-1} + b^{-1}) = 2 + 4 = 6$.

10.(7) $ax^2 + (a+d)x + (a+2d) = 0$

$a, a+d, a+2d$ are in increasing A.P. $(d > 0)$

for real roots $D \geq 0$

$\Rightarrow (a+d)^2 - 4a(a+2d) \geq 0$

$\Rightarrow d^2 - 3a^2 - 6ad \geq 0$

$\Rightarrow (d-3a)^2 - 12a^2 \geq 0$

$\Rightarrow (d - 3a - \sqrt{12}\,a)(d - 3a + \sqrt{12}\,a) \geq 0$

$\Rightarrow \left[\frac{d}{a} - (3+2\sqrt{3})\right]\left[\frac{d}{a} - (3-2\sqrt{3})\right] \geq 0$

$\therefore \left.\frac{d}{a}\right|_{\text{Min}} = 3 + 2\sqrt{3}$

$\Rightarrow$ least integral value = 7

11.(1) $\frac{a}{1-r_1} = r_1$ and $\frac{a}{1-r_2} = r_2$

Hence, r_1 and r_2 are the roots of $\frac{a}{1-r} = r$

$\Rightarrow r^2 - r + a = 0$

$\Rightarrow r_1 + r_2 = 1$

12.(6) Let $\frac{\alpha}{r}, \alpha, \alpha r$ be the roots.

$\therefore \alpha^3 = -216$ (1)

Again $\frac{\alpha^2}{r} + \alpha^2 r + \alpha^2 = b$

$\alpha^2\left(1 + r + \frac{1}{r}\right) = b$ (2)

and $\alpha\left(1 + r + \frac{1}{r}\right) = -a$ (3)

On dividing (2) by (3), we get

$\Rightarrow \alpha = -\frac{b}{a}$

$\Rightarrow \alpha^3 = -\frac{b^3}{a^3}$ (4)

From (1) and (4), $\left(\frac{b}{a}\right)^3 = 216$

$\Rightarrow \frac{b}{a} = 6$

13.(8) For the G.P. $a, ar, ar^2, \ldots$

$P_n = a(ar)(ar^2)\ldots(ar^{n-1}) = a^n.r^{n(n-1)/2}$

$\therefore S = \sum_{n=1}^{\infty} \sqrt[n]{P_n} = \sum_{n=1}^{\infty} ar^{(n-1)/2}$

Now, $\sum_{n=1}^{\infty} ar^{(n-1)/2} = a[1 + \sqrt{r} + r + r\sqrt{r} + \cdots + \infty] = \frac{a}{1-\sqrt{r}}$

Given $a = 16$ and $r = 1/4$

$\therefore S = \frac{16}{1-(1/2)} = 32$

14.(1) Let $a_1 = a - d$; $a_2 = a$; $a_3 = a + d$

$\therefore 3a = 18 \Rightarrow a = 6$

Hence, the number in A.P.

$6 - d, d, 6 + d$

$a_1 + 1, a_2, a_3 + 2$ in G.P.

i.e. $7 - d$, 6 $8 + d$ in G.P.

$\therefore 36 = (7-d)(8+d)$

$36 = 56 - d - d^2$

$d^2 + d - 20 = 0$

Hence, the sum of all possible common different is -1.

15.(0) a, b, c are in A.P. $\Rightarrow b = \frac{a+c}{2}$ (1)

b, c, d are in G.P. $\Rightarrow c^2 = bd$ (2)

and c, d, e are in H.P. $\Rightarrow d = \frac{2ce}{c+e}$ (3)

Now $c^2 = bd \Rightarrow c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$ [using (1) and (3)]

$\therefore c^2 + ce = ae + ce$

$\Rightarrow c^2 = ae$

Now given $a = 2$ and $e = 18$

$\therefore c^2 = ae \Rightarrow c^2 = 2 \times 18 = 36 \Rightarrow c = 6$ or -6

16.(4) Let $\frac{a}{r}, a, ar$ be three terms in G.P.

$\therefore$ Product of terms $= a^3 = -1$ (Given)

$\Rightarrow a = -1$

Now, sum of terms $= \frac{a}{r} + a + ar = \frac{13}{12}$ (Given)

$\Rightarrow \frac{-1}{r} - 1 - r = \frac{13}{12}$

$\Rightarrow 12r^2 + 25r + 12 = 0$

$\therefore (3r+4)(4r+3) = 0$

$\Rightarrow r = \frac{-4}{3}, \frac{-3}{4}$

But $r \neq \frac{-4}{3}$

$\therefore |S| = \left|\frac{a}{1-r}\right| = \left|\frac{-1}{1-\left(\frac{-3}{4}\right)}\right| = \left|\frac{-1}{1+\frac{3}{4}}\right| = \left|\frac{-4}{7}\right| = \frac{4}{7}$

17.(2) Let $S = \sum_{r=1}^{\infty} \frac{r+2}{2^{r+1} \cdot r \cdot (r+1)}$

$= \sum_{r=1}^{\infty} \frac{2(r+1) - r}{2^{r+1} \cdot r \cdot (r+1)}$

$= \sum_{r=1}^{\infty} \frac{1}{2^{r+1}}\left(\frac{2}{r} - \frac{1}{r+1}\right)$

$= \sum_{r=1}^{\infty}\left(\frac{1}{2^r \cdot r} - \frac{1}{2^{r+1}(r+1)}\right)$

$= \lim_{n\to\infty}\left[\left(\frac{1}{2^1 \cdot 1} - \frac{1}{2^2 \cdot 2}\right) + \left(\frac{1}{2^2 \cdot 2} - \frac{1}{2^3 \cdot 3}\right)\right] + \left(\frac{1}{2^3 \cdot 3} - \frac{1}{2^4 \cdot 4}\right)$

$= + \cdots + \left(\frac{1}{2^n \cdot n} - \frac{1}{2^{n+1} \cdot (n+1)}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^{n+1}(n+1)}\right)$

$\therefore \quad S = \frac{1}{2}$

Hence, $\quad S^{-1} = 2$.

18.(3) $369 = \frac{9}{2}[2 + (9-1)d]$

$\Rightarrow 82 = 2 + 8d$

$\Rightarrow d = 10$

Now $ar^8 = a + 8d = 1 + 8 \times 80 = 81$

$\Rightarrow r^8 = 81$

$\Rightarrow r = \sqrt{3}$

$\Rightarrow ar^{(7-1)} = 1 \times (\sqrt{3})^6 = 27$

Archives

Subjective Type

1. Let a and b be the two numbers and let H be the harmonic mean between them. Then, $H = 4$ (given). Since A, G, H are in G.P., therefore,

$G^2 = AH$

$\Rightarrow \quad G^2 = 4A$

But

$2A + G^2 = 27$ (given)

$\therefore \quad 6A = 27 \quad [\because G^2 = 4A]$

$\Rightarrow \quad A = \frac{9}{2}$

$\Rightarrow \quad \frac{a+b}{2} = \frac{9}{2}$

$\Rightarrow \quad a + b = 9$

Now, $G^2 = 4A$ and $A = 9/2 \Rightarrow G^2 = 18 \Rightarrow ab = 18$. The quadratic equation having a, b as its roots is

$x^2 - (a+b)x + ab = 0$

$\Rightarrow \quad x^2 - 9x + 18 = 0 \quad [\because a + b = 9 \text{ and } ab = 18]$

$\Rightarrow \quad x = 3, 6$

Hence, the two numbers are 3 and 6.

2. Let there be n sides in the polygon. Then by geometry, sum of all n interior angles of polygon is $(n-2) \times 180°$. Also the angles are in A.P. with the smallest angle 120° and common difference 5°. Therefore, sum of all interior angles of polygon is

$\frac{n}{2}[2 \times 120 + (n-1) \times 5]$

Thus, we must have

$\frac{n}{2}[2 \times 120 + (n-1)5] = (n-2) \times 180$

$\Rightarrow \quad \frac{n}{2}[5n + 235] = (n-2) \times 180$

$\Rightarrow \quad 5n^2 + 235n = 360n - 720$

$\Rightarrow \quad 5n^2 - 125n + 720 = 0$

$\Rightarrow \quad n^2 - 25n + 144 = 0$

$\Rightarrow \quad (n-16)(n-9) = 0$

$\Rightarrow \quad n = 16, 9$

But if $n = 16$, then 16th angle $= 120 + 15 \times 5 = 195 > 180°$ which is not possible. Hence $n = 9$.

3. Given, $a_1, a_2, \ldots, a_n$ are in A.P., $\forall\, a_i > 0$.

$\therefore \quad a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = d$ (a constant)

Now,

$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \cdots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}}$

$= \frac{\sqrt{a_1} - \sqrt{a_2}}{a_1 - a_2} + \frac{\sqrt{a_2} - \sqrt{a_3}}{a_2 - a_3} + \cdots + \frac{\sqrt{a_{n-1}} - \sqrt{a_n}}{a_{n-1} - a_n}$

$= \frac{1}{-d}\left[\sqrt{a_1} - \sqrt{a_n}\right]$

$= \frac{a_1 - a_n}{-d(\sqrt{a_1} + \sqrt{a_n})}$

$= \frac{(n-1)d}{d(\sqrt{a_1} + \sqrt{a_n})}$

$= \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$

4. See problem 3 in Reasoning Type section.
5. See problems 28–30 in the Linked Comprehension Type section.
6. Let the three distinct real numbers be $\alpha/r, \alpha, \alpha r$. As sum of squares of three numbers is S^2,

$\therefore \quad \frac{\alpha^2}{r^2} + \alpha^2 + \alpha^2 r^2 = S^2$

$$\Rightarrow \quad \frac{\alpha^2(1+r^2+r^4)}{r^2}=S^2 \qquad (1)$$

Sum of numbers is aS. Hence,

$$\frac{\alpha}{r}+\alpha+\alpha r=aS$$

$$\Rightarrow \quad \frac{\alpha(1+r+r^2)}{r}=aS \qquad (2)$$

Dividing Eq. (1) by the square of Eq. (2), we get

$$\frac{\alpha^2(1+r^2+r^4)}{r^2}\times\frac{r^2}{\alpha^2(1+r+r^2)^2}=\frac{S^2}{a^2S^2}$$

$$\Rightarrow \quad \frac{(1+2r^2+r^4)-r^2}{(1+r+r^2)^2}=\frac{1}{a^2}$$

$$\Rightarrow \quad \frac{(1+r+r^2)(1-r+r^2)}{(1+r+r^2)^2}=\frac{1}{a^2}$$

$\Rightarrow \quad a^2r^2-a^2r+a^2=1+r+r^2$

$\Rightarrow \quad (a^2-1)r^2-(a^2+1)r+(a^2-1)=0$

$$\Rightarrow \quad r^2+\left(\frac{1+a^2}{1-a^2}\right)r+1=0 \qquad (3)$$

For real values of r,

$D\geq 0$

$$\Rightarrow \quad \left(\frac{1+a^2}{1-a^2}\right)^2-4\geq 0$$

$\Rightarrow \quad 1+2a^2+a^4-4+8a^2-4a^4\geq 0$

$\Rightarrow \quad 3a^4-10a^2+3\leq 0$

$\Rightarrow \quad (3a^2-1)(a^2-3)\leq 0$

$$\Rightarrow \quad \left(a^2-\frac{1}{3}\right)(a^2-3)\leq 0$$

Clearly, the above inequality holds for $1/3\leq a^2\leq 3$.

But from Eq. (3), $a\neq 1$.

$$\therefore \quad a^2\in\left(\frac{1}{3},1\right)\cup(1,3)$$

7. Given that $\log_3 2$, $\log_3(2^x-5)$, $\log_3(2^x-7/2)$ are in A.P. Hence,

$$2\log_3(2^x-5)=\log_3\left(2^x-\frac{7}{2}\right)+\log_3 2$$

$$\Rightarrow \quad (2^x-5)^2=2\left(2^x-\frac{7}{2}\right)$$

$\Rightarrow \quad (2^x)^2-10\times 2^x+25-2\times 2^x+7=0$

$\Rightarrow \quad (2^x)^2-12\times 2^x+32=0$

Let $2^x=y$. Then we get

$y^2-12y+32=0$

$\Rightarrow \quad (y-4)(y-8)=0$

$\Rightarrow \quad y=4$ or 8

$\Rightarrow \quad 2^x=2^2$ or 2^3

$\Rightarrow \quad x=2$ or 3

But for $\log_3(2^x-5)$ and $\log_3(2^x-7/2)$ to be defined,

$$2^x-5>0 \text{ and } 2^x-\frac{7}{2}>0$$

$$\Rightarrow \quad 2^x>5 \text{ and } 2^x>\frac{7}{2}$$

$\Rightarrow \quad 2^x>5$

$\Rightarrow \quad x\neq 2$

and therefore, $x=3$.

8. Let a and b be two numbers and $A_1, A_2, A_3, \ldots, A_n$ be n A.M.'s between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ are in A.P. There are $n+2$ terms in the series. Now,

$$a+(n+1)d=b\Rightarrow d=\frac{b-a}{n+1}$$

$$\Rightarrow \quad A_1=p=a+\frac{b-a}{n+1}=\frac{an+b}{n+1} \qquad (1)$$

The first H.M. between a and b, when n H.M.'s are inserted between a and b can be obtained by replacing a by $1/a$ and b by $1/b$ in Eq. (1) and then taking its reciprocal. Therefore,

$$q=\frac{1}{\dfrac{\left(\dfrac{1}{a}\right)n+\dfrac{1}{b}}{n+1}}=\frac{(n+1)ab}{bn+a} \qquad (2)$$

Substituting $b=p(n+1)-an$ [from (1)] in Eq. (2), we get

$aq+nq[p(n+1)-an]=(n+1)a[p(n+1)-an]$

$\Rightarrow \quad a^2n(n+1)+a[q(1-n^2)-p(n+1)^2]+npq(n+1)=0$

$\Rightarrow \quad na^2-[(n+1)p+(n-1)q]a+npq=0$

$\Rightarrow \quad D\geq 0$ ($\because$ a is real)

$\Rightarrow \quad [(n+1)p+(n-1)q]^2-4n^2pq\geq 0$

$\Rightarrow \quad (n-1)^2q^2+\{2(n^2-1)-4n^2\}pq+(n+1)^2p^2\geq 0$

$$\Rightarrow \quad q^2-2\frac{n^2+1}{(n-1)^2}pq+\left(\frac{n+1}{n-1}\right)^2p^2\geq 0$$

$$\Rightarrow \quad \left[q-p\left(\frac{n+1}{n-1}\right)^2\right][q-p]\geq 0$$

[On factorizing by discriminent method]

Hence, q cannot lie between p and $p\left(\dfrac{n+1}{n-1}\right)^2$.

9. According to the question, we have

$$S_1=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots\infty=\frac{1}{1-\frac{1}{2}}=2$$

$$S_2=2+2\times\frac{1}{3}+2\times\left(\frac{1}{3}\right)^2+\cdots\infty=\frac{2}{1-\frac{1}{3}}=3$$

$$S_3=3+3\times\frac{1}{4}+3\times\left(\frac{1}{4}\right)^2+\cdots\infty=\frac{3}{1-\frac{1}{4}}=4$$

$$S_n=n+n\times\frac{1}{n+1}+n\times\left(\frac{1}{n+1}\right)^2+\cdots\infty=\frac{n}{1-\frac{1}{n+1}}=(n+1)$$

$$\therefore \quad S_1^2+S_2^2+S_3^2+\cdots+S_{2n-1}^2$$

$$=2^2+3^2+4^2+\cdots+(n+1)^2+\cdots+(2n)^2$$

$$= \left(\sum_{r=1}^{2n} r^2\right) - 1$$

$$= \frac{2n(2n+1)(4n+1)}{6} - 1^2$$

$$= \frac{n(2n+1)(4n+1) - 3}{3}$$

10. Since x_1, x_2, x_3 are in A.P., let $x_1 = a - d$, $x_2 = a$ and $x_3 = a + d$. And x_1, x_2, x_3 are the roots of $x^3 - x^2 + \beta x + \gamma = 0$. Now, the sum of roots is

$$\Sigma\alpha = a - d + a + a + d = 1 \quad (1)$$

Sum of product of roots taken two at a time is

$$\Sigma\alpha\beta = (a-d)a + a(a+d) + (a-d)(a+d)$$
$$= \beta \quad (2)$$

Product of roots is $\alpha\beta\gamma = (a-d)a(a+d)$

$$= -\gamma \quad (3)$$

From (1), we get

$$3a = 1 \Rightarrow a = 1/3$$

From (2), we get

$$3a^2 - d^2 = \beta$$

$$\Rightarrow 3(1/3)^2 - d^2 = \beta \Rightarrow 1/3 - \beta = d^2$$

$$\Rightarrow \frac{1}{3} - \beta \geq 0 \ (\because d^2 \geq 0)$$

$$\Rightarrow \beta \leq \frac{1}{3}$$

$$\Rightarrow \beta \in (-\infty, 1/3]$$

From (3),

$$a(a^2 - d^2) = -\gamma$$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{9} - d^2\right) = -\gamma$$

$$\Rightarrow \frac{1}{27} - \frac{1}{3}d^2 = -\gamma$$

$$\Rightarrow \gamma + \frac{1}{27} = \frac{1}{3}d^2$$

$$\Rightarrow \gamma + \frac{1}{27} \geq 0$$

$$\Rightarrow \gamma \geq -\frac{1}{27}$$

$$\Rightarrow \gamma \in \left[-\frac{1}{27}, \infty\right)$$

Hence, $\beta \in (-\infty, 1/3]$ and $\gamma \in [-1/27, \infty)$.

11. Solving the system of equations, $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$ and $6u + 9v = 4$, we get

$$u = -1/3,\ v = 2/3,\ w = 5/3$$

$$\therefore\ u + v + w = 2 \text{ and } \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = -\frac{9}{10}$$

Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$, $d = ar^3$. Then the first equation

$$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b-c)^2 + (c-a)^2 + (d-b)^2]x + (u+v+w) = 0 \text{ becomes}$$

$$-\frac{9}{10}x^2 + [(ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2]x + 2 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1-r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)x - 20 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1-r)^2(1 + r + r^2)^2 x - 20 = 0$$

$$\Rightarrow 9x^2 - 10a^2(1 - r^3)^2 x - 20 = 0 \quad (1)$$

The second equation is

$$20x^2 + 10(a - ar^3)^2 x - 9 = 0$$

$$\Rightarrow 20x^2 + 10a^2(1 - r^3)^2 x - 9 = 0 \quad (2)$$

Since (2) can be obtained by changing x to $1/x$, so Eqs. (1) and (2) have reciprocal roots.

12. Let $a - 3d$, $a - d$, $a + d$ and $a + 3d$ be any consecutive terms of an A.P. with common difference $2d$. Hence,

$$P = (2d)^4 + (a-3d)(a-d)(a+d)(a+3d)$$
$$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2)$$
$$= (a^2 - 5d^2)^2$$

which is an integer.

13. $G_k = (a_1 a_2 \cdots a_k)^{1/k}$

$$= a_1 (r^{1+2+\cdots+(k-1)})^{1/k}$$

$$= a_1 r^{\frac{k-1}{2}} \quad (1)$$

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}$$

$$= \frac{a_1(1 + r + \cdots + r^{k-1})}{k}$$

$$= \frac{a_1(r^k - 1)}{(r-1) \times k} \quad (2)$$

$$H_k = \frac{k}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}}$$

$$= \frac{a_1 k}{\left(1 + \frac{1}{r} + \cdots + \frac{1}{r^{k-1}}\right)}$$

$$= \frac{a_1 k (r-1) r^{k-1}}{r^k - 1} \quad (3)$$

From (1), (2) and (3), we get

$$G_k = (A_k H_k)^{1/2}$$

$$\Rightarrow \prod_{k=1}^{n} G_k = \prod_{k=1}^{n} (A_k H_k)^{1/2}$$

$$\Rightarrow \left(\prod_{k=1}^{n} G_k\right)^{1/n} = (A_1 A_2 \cdots A_n \times H_1 H_2 \cdots H_n)^{1/2n}$$

14. Clearly, $A_1 + A_2 = a + b$. Now,

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1 H_2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1 G_2}$$

$$\Rightarrow \quad \frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

Also,

$$\frac{1}{H_1} = \frac{1}{a} + \frac{1}{3}\left(\frac{1}{b} - \frac{1}{a}\right) \Rightarrow H_1 = \frac{3ab}{2b+a}$$

and

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2}{3}\left(\frac{1}{b} - \frac{1}{a}\right) \Rightarrow H_2 = \frac{3ab}{2a+b}$$

$$\Rightarrow \quad \frac{A_1 + A_2}{H_1 + H_2} = \frac{a+b}{3ab\left(\dfrac{1}{2b+a} + \dfrac{1}{2a+b}\right)}$$

$$= \frac{(2b+a)(2a+b)}{9ab}$$

15. Given that a, b, and c are in A.P. Hence,

$2b = a + c$ (1)

a^2, b^2, c^2 are in H.P. Hence,

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{c^2} - \frac{1}{b^2}$$

$$\Rightarrow \quad \frac{(a-b)(a+b)}{b^2a^2} = \frac{(b-c)(b+c)}{b^2c^2}$$

$\Rightarrow$ $ac^2 + bc^2 = a^2b + a^2c$ [$\because a - b = b - c$]

$\Rightarrow$ $ac(c - a) + b(c - a)(c + a) = 0$

$\Rightarrow$ $(c - a)(ab + bc + ca) = 0$

$\Rightarrow$ $c - a = 0$ or $ab + bc + ca = 0$

For $c = a$, from (1), $a = b = c$. For $(a + c)b + ca = 0$, from (1),

$2b^2 + ca = 0$

$$\Rightarrow \quad b^2 = a\left(\frac{-c}{2}\right)$$

Hence, a, b, $-c/2$ are in G.P.

16. $$a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$$

$$= \frac{\frac{3}{4}\left(1 - \left(-\frac{3}{4}\right)^n\right)}{1 - \left(-\frac{3}{4}\right)}$$

$$= \frac{3}{7}\left(1 - \left(-\frac{3}{4}\right)^n\right)$$

Now, $b_n = 1 - a_n$ and $b_n > a_n$ for $n \geq n_0$.

$\therefore$ $1 - a_n > a_n \Rightarrow 2a_n < 1$

$$\Rightarrow \quad \frac{6}{7}\left[1 - \left(-\frac{3}{4}\right)^n\right] < 1$$

$$\Rightarrow \quad -\left(-\frac{3}{4}\right)^n < \frac{1}{6}$$

$$\Rightarrow \quad (-3)^{n+1} < 2^{2n-1}$$

For n to be even, inequality always holds. For n to be odd, it holds for $n \geq 7$. Therefore,the least natural number for which it holds is 6.

Objective Type

Fill in the blanks

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

S = sum of integers from 1 to 100 divisible by 2

+ sum of integers from 1 to 100 divisible by 5

− sum of integers from 1 to 100 divisible by 10

$= (2 + 4 + 6 + \cdots + 100) + (5 + 10 + 15 + \cdots + 100) - (10 + 20 + \cdots + 100)$

$$= \frac{50}{2}[2\times2+49\times2] + \frac{20}{2}[2\times5+19\times5] - \frac{10}{2}[2\times10+9\times10]$$

$= 2550 + 1050 - 550 = 3050$

2. When n is odd, last term is n^2. Hence, the required sum is

$$S = [1^2 + 2\times2^2 + 3^2 + 2\times4^2 + \cdots + 2\times(n-1)^2] + n^2$$

$$= \frac{(n-1)n^2}{2} + n^2 \qquad \text{[Using sum for } (n-1) \text{ to be even]}$$

$$= \frac{n^2(n+1)}{2}$$

3. Let a and b be two positive numbers. Then, H.M. $= \dfrac{2ab}{a+b}$ and G.M. $= \sqrt{ab}$. According to question, H.M.:G.M. = 4:5

$$\therefore \quad \frac{2ab}{(a+b)\sqrt{ab}} = \frac{4}{5}$$

$$\Rightarrow \quad \frac{2\sqrt{ab}}{a+b} = \frac{4}{5}$$

$$\Rightarrow \quad \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{5+4}{5-4}$$

$$\Rightarrow \quad \left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2 = 9$$

$$\Rightarrow \quad \frac{(\sqrt{a}+\sqrt{b})}{\sqrt{a}-\sqrt{b}} = 3, -3$$

$$\Rightarrow \quad \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{3+1}{3-1}, \frac{-3+1}{-3-1}$$

$$\Rightarrow \quad \frac{\sqrt{a}}{\sqrt{b}} = 2, \frac{1}{2} \Rightarrow \frac{a}{b} = 4, \frac{1}{4}$$

$\Rightarrow$ $a{:}b = 4{:}1$ or $1{:}4$

4. Since n is an odd integer, $(-1)^{n-1} = 1$ and $n-1, n-3, n-5, \ldots$ are even integers. The given series is

$n^3 - (n-1)^3 + (n-2)^3 - (n-3)^3 + \cdots + (-1)^{n-1}1^3$

$= [n^3 + (n-1)^3 + (n-2)^3 + \cdots + 1^3] - 2[(n-1)^3 + (n-3)^3 + \cdots + 2^3]$

$$= \left[\frac{n(n+1)}{2}\right]^2 - 16\left[\left\{\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\right\}\right]^2$$

$$= \frac{1}{4} n^2 (n+1)^2 - 16 \frac{(n-1)^2 (n+1)^2}{16 \times 4}$$

$$= \frac{1}{4} (n+1)^2 [n^2 - (n-1)^2]$$

$$= \frac{1}{4} (n+1)^2 (2n-1)$$

5. Given that x is the A.M. between a and b and y, and z are the G.M.'s between a and b where a and b are positive. Then a, x, b are in A.P. So,

$$x = \frac{a+b}{2}$$

a, y, z, b are in G.P. So,

$y = ar$, and $z = ar^2$, where $r = \sqrt[3]{\frac{b}{a}}$. Also, $yz = ab$. Now,

$$\frac{y^3 + z^3}{xyz} = \frac{a^3 r^3 + a^3 r^6}{ab\left(\frac{a+b}{2}\right)}$$

$$= \frac{a^3 \times \frac{b}{a} + a^3 \times \frac{b^2}{a^2}}{ab\left(\frac{a+b}{2}\right)}$$

$$= \frac{2(a^2 b + ab^2)}{a^2 b + ab^2}$$

$$= 2$$

6. Let p and q be roots of the equation $x^2 - 2x + A = 0$. Then,

$$p + q = 2, pq = A$$

Let r and s be the roots of the equation $x^2 - 18x + B = 0$. Then,

$$r + s = 18, rs = B$$

And it is given that p, q, r, and s are in A.P. Let $p = a - 3b$, $q = a - d$, $r = a + d$ and $s = a + 3d$. As $p < q < r < s$, we have $d > 0$. Now,

$$2 = p + q = a - 3d + a - d = 2a - 4d$$

$$\Rightarrow \quad a - 2d = 1 \quad (1)$$

and

$$18 = r + s = a + d + a + 3d$$

$$\Rightarrow \quad a + 2d = 9 \quad (2)$$

Solving (1) and (2), $a = 5, d = 2$

$\therefore \quad p = -1, q = 3, r = 7, s = 11$

Therefore, $A = pq = -3$ and $B = rs = 77$.

Multiple choice questions with one correct answer

1. c. Given that x, y, z are the p^{th}, q^{th} and r^{th} terms of an A.P.

$$\therefore \quad x = A + (p-1) D$$
$$y = A + (q-1) D$$
$$z = A + (r-1) D$$

$$\Rightarrow \quad x - y = (p-q) D$$
$$y - z = (q-r) D$$
$$z - x = (r-p) D$$

where A is the first term and D is the common difference. Also x, y, z are the p^{th}, q^{th} and r^{th} terms of a G.P.

$$\therefore \quad x = aR^{p-1}, y = aR^{q-1}, z = aR^{r-1}$$

$$\therefore \quad x^{y-z} y^{z-x} z^{x-y} = (aR^{p-1})^{y-z} (aR^{q-1})^{z-x} (aR^{r-1})^{x-y}$$

$$= a^{y-z+z-x+x-y} R^{(p-1)(y-z)+(q-1)(z-x)+(r-1)(x-y)}$$

$$= A^\circ R^{(p-1)(q-r)D+(q-1)(r-p)D+(r-1)(p-q)D}$$

$$= A^\circ R^\circ = 1$$

2. b. Given

$$ar^2 = 4$$

$$\Rightarrow \quad a \times ar \times ar^2 \times ar^3 \times ar^4 = a^5 r^{10} = (ar^2)^5 = 4^5$$

3. c. $2.\overline{357} = 2 + 0.357 + 0.000357 + \cdots \infty$

$$= 2 + \frac{357}{10^3} + \frac{357}{10^6} + \cdots \infty$$

$$= 2 + \frac{\frac{357}{10^3}}{1 - \frac{1}{10^3}}$$

$$= 2 + \frac{357}{999} = \frac{2355}{999}$$

Alternative solution:

Let,

$$x = 2.\overline{357}$$

$$\Rightarrow \quad 1000x = 2357.\overline{357}$$

On subtracting, we get

$$999x = 2355 \Rightarrow x = \frac{2355}{999}$$

4. a. For first equation $D = 4b^2 - 4ac = 0$ (as given a, b, c are in G.P.)

$\Rightarrow$ equation has equal roots which are equal to $-\frac{b}{a}$ each.

Thus it should also be the root of the second equation.

$$\text{Thus, } d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$$

$$\Rightarrow \quad d\frac{b^2}{a^2} - 2\frac{be}{a} + f = 0$$

$$\Rightarrow d\frac{ac}{a^2} - 2\frac{be}{a} + f = 0 \text{ (as } b^2 = ac)$$

$$\Rightarrow \frac{d}{a} + \frac{f}{c} = 2\frac{eb}{ac} = 2\frac{e}{b}$$

5. c. Let,

$$S = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots n \text{ terms}$$

$$= \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right) + \cdots n \text{ terms}$$

$$= (1 + 1 + 1 + \cdots n \text{ times}) - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n}\right)$$

$$= n - \left[\frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}\right] = n - 1 + 2^{-n}$$

6. c. We have,

$$\frac{(x+2)^n - (x+1)^n}{(x+2) - (x+1)} = (x+2)^{n-1} + (x+2)^{n-2}(x+1)$$

$$+ (x+2)^{n-3}(x+1)^2 + \cdots + (x+1)^{n-1}$$

Hence, the required sum is

$(x+2)^n-(x+1)^n$ [$\because (x+2)-(x+1)=1$]

7. **d.** $\ln(a+c)$, $\ln(c-a)$, $\ln(a-2b+c)$ are in A.P. Hence, $a+c$, $c-a$, $a-2b+c$ are in G.P. Therefore,

$$(c-a)^2=(a+c)(a-2b+c)$$
$$\Rightarrow (c-a)^2=(a+c)^2-2b(a+c)$$
$$\Rightarrow 2b(a+c)=(a+c)^2-(c-a)^2$$
$$\Rightarrow 2b(a+c)=4ac$$
$$\Rightarrow b=\frac{2ac}{a+c}$$

Hence, a, b, and c are in H.P.

8. **d.** $a_1=h_1=2$, $a_{10}=h_{10}=3$

$$3=a_{10}=2+9d\Rightarrow d=1/9$$
$$\therefore a_4=2+3d=7/3$$

Also,

$$3=h_{10}\Rightarrow\frac{1}{3}=\frac{1}{h_{10}}=\frac{1}{2}+9D$$
$$\Rightarrow D=-\frac{1}{54}$$
$$\Rightarrow \frac{1}{h_7}=\frac{1}{2}+6D=\frac{1}{2}-\frac{1}{9}=\frac{7}{18}$$
$$\therefore a_4h_7=\frac{7}{3}\times\frac{18}{7}=6$$

9. **b.** Harmonic mean H of roots α and β is

$$H=\frac{2\alpha\beta}{\alpha+\beta}=\frac{2\dfrac{8+2\sqrt{5}}{5+\sqrt{2}}}{\dfrac{4+\sqrt{5}}{5+\sqrt{2}}}=4$$

10. **d.** a, b, and c, d are in A.P. Therefore, d, c, b and a are also in A.P. Hence,

$$\frac{d}{abcd},\frac{c}{abcd},\frac{b}{abcd},\frac{a}{abcd}\text{ are also in A.P.}$$
$$\Rightarrow \frac{1}{abc},\frac{1}{abd},\frac{1}{acd},\frac{1}{bcd}\text{ are in A.P.}$$

$\Rightarrow$ abc, abd, acd, bcd are in H.P.

11. **d.** Sum is 4 and second term is 3/4. It is given that first term is a and common ratio is r. Hence,

$$\frac{a}{1-r}=4\text{ and }ar=3/4\Rightarrow r=\frac{3}{4a}$$

Therefore,

$$\frac{a}{1-\dfrac{3}{4a}}=4\Rightarrow\frac{4a^2}{4a-3}=4$$
$$\Rightarrow a^2-4a+3=0$$
$$\Rightarrow (a-1)(a-3)=0$$
$$\Rightarrow a=1\text{ or }3$$

When $a=1$, $r=3/4$ and when $a=3$, $r=1/4$.

12. **a.** α,β are the roots of $x^2-x+p=0$. Hence,

$$\alpha+\beta=1 \quad (1)$$
$$\alpha\beta=p \quad (2)$$

γ,δ are the roots of $x^2-4x+q=0$. Hence,

$$\therefore \gamma+\delta=4 \quad (3)$$
$$\gamma\delta=q \quad (4)$$

$\alpha,\beta,\gamma,\delta$ are in G.P. Let $\alpha=a$, $\beta=ar$, $\gamma=ar^2$, $\delta=ar^3$. Substituting these values in Eqs. (1), (2), (3) and (4), we get

$$a+ar=1 \quad (5)$$
$$a^2r=p \quad (6)$$
$$ar^2+ar^3=4 \quad (7)$$
$$a^2r^5=q \quad (8)$$

Dividing (7) by (5), we get

$$\frac{ar^2(1+r)}{a(1+r)}=\frac{4}{1}\Rightarrow r^2=4\Rightarrow r=2,-2$$
$$(5)\Rightarrow a=\frac{1}{1+r}=\frac{1}{1+2}\text{ or }\frac{1}{1-2}=\frac{1}{3}\text{ or }-1$$

As p is an integer (given), r is also an integer (2 or -2). Therefore, from (6), $a\neq 1/3$. Hence, $a=-1$ and $r=-2$.

$$\therefore p=(-1)^2\times(-2)=-2$$
$$q=(-1)^2\times(-2)^5=-32$$

13. **c.** Given,

$$2+5+8+\cdots 2n\text{ terms}=57+59+61+\cdots n\text{ terms}$$
$$\Rightarrow \frac{2n}{2}[4+(2n-1)3]=\frac{n}{2}[114+(n-1)2]$$
$$\Rightarrow 6n+1=n+56$$
$$\Rightarrow 5n=55$$
$$\Rightarrow n=11$$

14. **d.** Given that a, b, and c are in A.P. Hence,

$$2b=a+c$$

But given,

$$a+b+c=3/2$$
$$\Rightarrow 3b=3/2$$
$$\Rightarrow b=1/2$$

Hence,

$$a+c=1$$

Again, a^2, b^2, c^2 are in G.P. Hence,

$$b^4=a^2c^2$$
$$\Rightarrow b^2=\pm ac$$
$$\Rightarrow ac=\frac{1}{4}\text{ or }-\frac{1}{4}\text{ and }a+c=1 \quad (1)$$

Now,

$$a+c=1\text{ and }ac=\frac{1}{4}$$
$$\Rightarrow (a-c)^2=(a+c)^2-4ac=1-1=0$$
$$\Rightarrow a=c$$

But $a\neq c$ as given that $a<b<c$. We consider $a+c=1$ and $ac=-1/4$. Hence,

$$(a-c)^2=1+1=2$$
$$\Rightarrow a-c=\pm\sqrt{2}$$

But

$$a<c\Rightarrow a-c=-\sqrt{2} \quad (2)$$

Solving (1) and (2), we get

$$a=\frac{1}{2}-\frac{1}{\sqrt{2}}$$

15. **c.** $S_\infty=\dfrac{a}{1-r}=5$ (given)

$$\Rightarrow r=\frac{5-a}{5}$$

But

$$0<|r|<1$$

$$\Rightarrow \quad 0<\left|\frac{5-a}{5}\right|<1$$

$$\Rightarrow \quad -1<\frac{5-a}{5}<1 \text{ and } a \neq 5$$

$$\Rightarrow \quad -5<5-a<5 \text{ and } a \neq 5$$

$$\Rightarrow \quad -10<-a<0 \text{ and } a \neq 5$$

$$\Rightarrow \quad 10>a>0 \text{ and } a \neq 5$$

$$\Rightarrow \quad 0<a<10 \text{ and } a \neq 5$$

16. c. $\alpha+\beta, \alpha^2+\beta^2, \alpha^3+\beta^3$ are in G.P. Hence,

$$(\alpha^2+\beta^2)^2=(\alpha+\beta)(\alpha^3+\beta^3)$$

$$\Rightarrow \alpha^4+\beta^4+2\alpha^2\beta^2=\alpha^4+\beta^4+\alpha\beta^3+\beta\alpha^3$$

$$\Rightarrow \quad \alpha\beta(\alpha^2+\beta^2-2\alpha\beta)=0$$

$$\Rightarrow \quad \alpha\beta(\alpha-\beta)^2=0$$

$$\Rightarrow \quad \alpha=0 \text{ or } \alpha-\beta=0$$

$$\Rightarrow \quad c/a=0 \text{ or } \alpha=\beta$$

$$\Rightarrow \quad c=0 \text{ or } \Delta=0 \text{ (equal roots)}$$

$$\therefore \quad c\Delta=0$$

Multiple choice questions with one or more than one correct answer

1. b, d. Let x be the first term and y be the $(2n-1)^{\text{th}}$ term of A.P., G.P. and H.P. whose n^{th} terms are a, b, c, respectively. Now according to the property of A.P., G.P. and H.P., x, a, y are in A.P.; x, b, y are in G.P. and x, c, y are in H.P. Hence,

$$a=\frac{x+y}{2}=\text{A.M.}$$

$$b=\sqrt{xy}=\text{G.M.}$$

$$c=\frac{2xy}{x+y}=\text{H.M.}$$

Now, A.M, G.M. and H.M. are in G.P. Hence,

$$b^2=ac$$

Also, A.M. ≥ G.M. ≥ H.M. Hence,

$$a \geq b \geq c$$

2. b, c. We have, for $0<\phi<\pi/2$

$$x=\sum_{n=0}^{\infty}\cos^{2n}\phi$$

$$=1+\cos^2\phi+\cos^4\phi+\cdots\infty$$

$$=\frac{1}{1-\cos^2\phi}$$

$$=\frac{1}{\sin^2\phi} \qquad (1)$$

$$y=\sum_{n=0}^{\infty}\sin^{2n}\phi$$

$$=1+\sin^2\phi+\sin^4\phi+\cdots\infty$$

$$=\frac{1}{1-\sin^2\phi}$$

$$=\frac{1}{\cos^2\phi} \qquad (2)$$

$$z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$$

$$=1+\cos^2\phi\sin^2\phi+\cos^4\phi\sin^4\phi+\cdots\infty$$

$$=\frac{1}{1-\cos^2\phi\sin^2\phi} \qquad (3)$$

Substituting the values of $\cos^2\phi$ and $\sin^2\phi$ in (3), from (1) and (2), we get

$$z=\frac{1}{1-\frac{1}{x}\frac{1}{y}}$$

$$\Rightarrow \quad z=\frac{xy}{xy-1}$$

$$\Rightarrow \quad xyz-z=xy$$

$$\Rightarrow \quad xyz=xy+z$$

Also,

$$x+y+z=\frac{1}{\cos^2\phi}+\frac{1}{\sin^2\phi}+\frac{1}{1-\cos^2\phi\sin^2\phi}$$

$$=\frac{\sin^2\phi(1-\cos^2\phi\sin^2\phi)+\cos^2\phi(1-\cos^2\phi\sin^2\phi)+\cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)}$$

$$=\frac{(\sin^2\phi+\cos^2\phi)(1-\cos^2\phi\sin^2\phi)+\cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)}$$

$$=\frac{1}{\cos^2\phi\sin^2\phi(1-\cos^2\phi\sin^2\phi)}=xyz$$

Thus, (b) and (c) both are correct.

3. b. Putting $\theta=0$, we get $b_0=0$.

$$\therefore \quad \sin n\theta=\sum_{r=1}^{n}b_r\sin^r\theta$$

$$\Rightarrow \quad \frac{\sin n\theta}{\sin\theta}=\sum_{r=1}^{n}b_r(\sin\theta)^{r-1}$$

$$=b_1+b_2\sin\theta+b_3\sin^2\theta+\cdots+b_n\sin^{n-1}\theta$$

Taking limit as $\theta \to 0$, we obtain

$$\lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta}=b_1 \Rightarrow b_1=n$$

4. c. $T_m=a+(m-1)d=1/n$

$$T_n=a+(n-1)d=1/m$$

$$\Rightarrow \quad (m-n)d=1/n-1/m=(m-n)/mn$$

$$\Rightarrow \quad d=1/mn$$

$$\Rightarrow \quad a=\frac{1}{mn}$$

$$\therefore \quad T_{mn}=a+(mn-1)d$$

$$=\frac{1}{mn}+(mn-1)\frac{1}{mn}$$

$$=\frac{1}{mn}+1-\frac{1}{mn}$$

$$=1$$

5. b. If x, y, and z are in G.P. ($x, y, z>1$), then $\log x$, $\log y$, $\log z$ are in A.P. Hence,

$1+\log x$, $1+\log y$, $1+\log z$ will also be in A.P.

$$\Rightarrow \quad \frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z} \text{ will be in H.P.}$$

6. a, d. We have

$$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{2^2 - 1}\right) + \left(\frac{1}{2^2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{2^3 - 1}\right) + \left(\frac{1}{2^3} + \cdots + \frac{1}{2^4 - 1}\right) + \cdots$$

$$< 1 + 1 + \cdots + 1 = n$$

Thus,

$$a(100) < 100$$

Also,

$$a(n) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2^1 + 1} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2 + 1} + \frac{1}{2^3}\right) + \cdots + \left(\frac{1}{2^{n-1} + 1}\right)$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}\right) - \frac{1}{2^n}$$

$$= 1 + \frac{n}{2} - \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right) + \frac{n}{2}$$

Thus,

$$a > \left(1 - \frac{1}{2^{200}}\right) + \frac{200}{2} > 100$$

i.e.,

$$a(200) > 100$$

Comprehension

1. b. $V_1 + V_2 + \cdots + V_n$

$$= \sum_{r=1}^{n} V_r$$

$$= \sum_{r=1}^{n} \left(\frac{r}{2}(2r + (r-1)(2r-1))\right)$$

$$= \sum_{r=1}^{n} \left(r^3 - \frac{r^2}{2} + \frac{r}{2}\right)$$

$$= \Sigma n^3 - \frac{\Sigma n^2}{2} + \frac{\Sigma n}{2}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4}\left[n(n+1) - \frac{2n+1}{3} + 1\right]$$

$$= \frac{n(n+1)(3n^2 + n + 2)}{12}$$

2.d. $T_r = r + (r-1)(2r-1)$

$$= (r+1)(3r-1)$$

For each r, T_r has two different factors other than 1 and itself. Therefore, T_r is always a composite number.

3. b. Since $Q_{r+1} - Q_r = 6(r+1) + 5 - 6r - 5 = 6$ (constant), therefore, $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 6.

4. c. Given,

$$A_1 = \frac{a+b}{2},\ G_1 = \sqrt{ab},\ H_1 = \frac{2ab}{a+b}$$

Also,

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}$$

$$G_n = \sqrt{A_{n-1} H_{n-1}}$$

$$H_n = \frac{2 A_{n-1} \times H_{n-1}}{A_{n-1} + H_{n-1}}$$

$\Rightarrow$ $G_n^2 = A_n H_n \Rightarrow A_n H_n = A_{n-1} H_{n-1}$

Similarly, we can prove

$$A_n H_n = A_{n-1} H_{n-1} = A_{n-2} H_{n-2} = \cdots = A_1 H_1$$

$\Rightarrow$ $A_n H_n = ab$

$\Rightarrow$ $G_1^2 = G_2^2 = G_3^2 = \cdots = G_n^2 = ab$

$\Rightarrow$ $G_1 = G_2 = G_3 = \cdots = \sqrt{ab}$

5. a. We have,

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}$$

$\therefore$ $A_n - A_{n-1} = \frac{A_{n-1} + H_{n-1}}{2} - A_{n-1}$

$$= \frac{H_{n-1} - A_{n-1}}{2} < 0\ (\because A_{n-1} > H_{n-1})$$

$\Rightarrow$ $A_n < A_{n-1}$ or $A_{n-1} > A_n$

Hence, we can conclude that $A_1 > A_2 > A_3 > \cdots$

6. b. We have,

$$A_n H_n = ab \Rightarrow H_n = \frac{ab}{A_n}$$

$$\frac{1}{A_{n-1}} < \frac{1}{A_i} \Rightarrow H_{n-1} < H_n$$

$\therefore$ $H_1 < H_2 < H_3 < \cdots$

Integer type

1.(3) $S_k = \dfrac{\dfrac{k-1}{k!}}{1 - \dfrac{1}{k}} = \dfrac{1}{(k-1)!}$

$$\sum_{k=2}^{100} \left|(k^2 - 3k + 1)\frac{1}{(k-1)!}\right|$$

$$= \sum_{k=2}^{100} \left|\frac{(k-1)^2 - k}{(k-1)!}\right|$$

$$= \Sigma\left|\frac{k-1}{(k-2)!} - \frac{k}{(k-1)!}\right|$$

$$= \left|\frac{1}{0!} - \frac{2}{1!}\right|\left|\frac{2}{1!} - \frac{3}{2!}\right| + \left|\frac{3}{2!} - \frac{4}{3!}\right| + \cdots$$

$$= \frac{2}{1!} - \frac{1}{0!} + \frac{2}{1!} - \frac{3}{2!} + \frac{3}{2!} - \frac{4}{3!} + \cdots + \frac{99}{98!} - \frac{100}{99!}$$

$$= 3 - \frac{100}{99!}$$

2.(0) $a_k = 2a_{k-1} - a_{k-2} \Rightarrow a_1, a_2, \ldots, a_{11}$ are in A.P.

$$\therefore \quad \frac{a_1^2 + a_2^2 + \cdots + a_{11}^2}{11} = \frac{11a^2 + 35 \times 11d^2 + 10ad}{11} = 90$$

$\Rightarrow$ $225 + 35d^2 + 150d = 90$

$35d^2 + 150d + 135 = 0 \quad \Rightarrow \quad -3, -9/7$

Given $a_2 < \frac{27}{2}$, we get $d = -3$ and $d = -9/7$

$$\Rightarrow \quad \frac{a_1 + a_2 + \cdots + a_{11}}{11} = \frac{11}{2}[30 - 10 \times 3] = 0$$

3.(6) $a_1, a_2, a_3, \ldots, a_{100}$ is an A.P.

$$a_1 = 3, S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$$

$$\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}(6 + (5n - 1)d)}{\frac{n}{2}(6 - d + nd)}$$

$\frac{S_m}{S_n}$ is independent of n of $6 - d = 0 \quad \Rightarrow \quad d = 6.$