Exercise 14.6

Answer 1E.

Consider the figure given in the text book page 956.

The figure is showing the level curves for barometric pressure.

Let P be the pressure function.

Find the directional derivative $D_u P$ at K in the direction of S.

The initial point of the line joining the points K and S is lies on the level curve which representing the 1000 mb of pressure.

Take, $x_1 = 1000 \text{ mb.}$

The terminal point of the line joining the points K and S is lies on the level curve which representing the 976 mb of pressure.

Take, $x_2 = 976 \text{ mb.}$

The distance between the points K and S is $d = 300 \text{ km.}$

The directional derivative $D_u P$ at K in the direction of S is approximated by finding the average rate of change of the pressure between the points K and S.

$$D_u P \approx \frac{x_2 - x_1}{d}$$
$$\approx \frac{976 - 1000}{300}$$
$$\approx \frac{-24}{300}$$
$$\approx \frac{-2}{25}$$
$$\approx -0.08$$

Here, the units of pressure is millibars and the units of the distance is kilometer.

So the units of directional derivative should be millibars per kilometer.

Therefore, the directional derivative $D_u P$ at K in the direction of S is approximately $\boxed{-0.08 \text{ mb/km}}$.

Answer 2E.

Take the temperature at Sydney, 24 degrees C minus the temperature at Dubbo,

30 degrees C. This gives -6 degrees C.

The scale of the map shows that 300 km is 19 mm. The distance on the map from Sydney to Dubbo is 21 mm. Divide 300 km by 19mm and multiply the result by 21 mm. This gives us a distance of 331.6 km from Dubbo to Sydney.

Dividing -6 degrees C by 331.6 km gives -0.018 degrees C / km

Answer 3E.

We know

$$f_T(-20,30) = \lim_{h\to 0} \frac{f(-20+h,30) - f(-20,30)}{h}$$

Using table taking $h = 5$, -5 we have

$$f_T(-20,30) = \frac{f(-15,30) - f(-20,30)}{5}$$
$$= \frac{-26+33}{5}$$
$$= 1.4$$

And $$f_T(-20,30) = \frac{f(-25,30) - f(-20,30)}{-5}$$
$$= \frac{-39+33}{-5}$$
$$= 1.2$$

On averaging these values we find

$$f_T(-20,30) = 1.3$$

And $$f_v(-20,30) = \lim_{h\to 0} \frac{f(-20,30+h) - f(-20,30)}{h}$$

Using table taking $h = 10$, -10 we have

$$f_v(-20,30) = \frac{f(-20,40) - f(-20,30)}{10}$$
$$= \frac{-34+33}{10}$$
$$= -0.1$$

And $f_y(-20,30) = \dfrac{f(-20,20) - f(-20,30)}{-10}$

$$= \frac{-30+33}{-10}$$

$$= -0.3$$

On averaging we find $f_y(-20,30) = -0.2$

We need to find $D_u f(-20,30)$ where $u = (\hat{i} + \hat{j})/\sqrt{2}$

i.e. $u = \langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \rangle$

As $D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$

Then $D_u f(-20,30) = f_x(-20,30)\left(\dfrac{1}{\sqrt{2}}\right) + f_y(-20,30)\left(\dfrac{1}{\sqrt{2}}\right)$

$$= 1.3 \times \frac{1}{\sqrt{2}} - 0.2 \times \frac{1}{\sqrt{2}}$$

$$= (1.3 - 0.2)/\sqrt{2}$$

$$= 1.1/\sqrt{2}$$

$$= 0.778$$

Hence $\boxed{D_u f(-20,30) = 0.778}$

Answer 4E.

If f is a differentiable function of two variables x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = (\cos\theta, \sin\theta)$

and $D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$.

Find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = \frac{\partial}{\partial x}(x^3y^4 + x^4y^3)$$
$$= 3x^2y^4 + 4x^3y^3$$

$$f_y(x, y) = \frac{\partial}{\partial y}(x^3y^4 + x^4y^3)$$
$$= 4x^3y^3 + 3x^4y^2$$

Now, find $f_x(1, 1)$ and $f_y(1, 1)$.

$$f_x(1, 1) = 3(1)^2(1)^4 + 4(1)^3(1)^3$$
$$= 7$$

$$f_y(1, 1) = 4(1)^3(1)^3 + 3(1)^4(1)^2$$
$$= 7$$

Substitute the known values in $D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$.

$$D_u f(1, 1) = (7)\cos\left(\frac{\pi}{6}\right) + (7)\sin\left(\frac{\pi}{6}\right)$$

$$= (7)\left(\frac{\sqrt{3}}{2}\right) + (7)\left(\frac{1}{2}\right)$$

$$= \frac{7}{2}\left(1 + \sqrt{3}\right)$$

Thus, we get $\boxed{D_u f(1, 1) = \frac{7}{2}\left(1 + \sqrt{3}\right)}$.

Answer 5E.

Given that $f(x, y) = ye^{-x}$ and $\vec{u}$ is the unit vector given by angle $\theta = \frac{2\pi}{3}$.

We want $D_{\vec{u}} f(0, 4)$.

If the unit vector $\vec{u}$ makes an angle θ with the positive x-axis (as in our case), then we can write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula becomes

$$D_{\vec{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

This formula gives

$$D_{\vec{u}} f(x, y) = f_x(x, y)\cos\frac{2\pi}{3} + f_y(x, y)\sin\frac{2\pi}{3}$$

$$= \left(-ye^{-x}\right)\frac{-1}{2} + \left(e^{-x}\right)\frac{\sqrt{3}}{2}$$

$$= \frac{1}{2}\left[ye^{-x} + \sqrt{3}\, e^{-x}\right]$$

Therefore

$$D_{\vec{u}} f(0, 4) = \frac{1}{2}\left[4e^{0} + \sqrt{3}\, e^{0}\right] = \frac{1}{2}\left[4 + \sqrt{3}\right] = 2 + \frac{\sqrt{3}}{2}$$

Answer 6E.

If f is a differentiable function of two variables x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$

and $D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$.

Find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = \frac{\partial}{\partial x}(e^x \cos y)$$
$$= e^x \cos y$$

$$f_y(x, y) = \frac{\partial}{\partial y}(e^x \cos y)$$
$$= -e^x \sin y$$

Now, find $f_x(0, 0)$ and $f_y(0, 0)$.

$$f_x(0, 0) = e^{(0)} \cos(0)$$
$$= 1$$
$$f_y(0, 0) = -e^{(0)} \sin(0)$$
$$= 0$$

Substitute the known values in $D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$.

$$D_{\mathbf{u}} f(0, 0) = (1)\cos\frac{\pi}{4} + (0)\sin\frac{\pi}{4}$$
$$= (1)\left(\frac{1}{\sqrt{2}}\right) + (0)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}$$

Thus, we get $\boxed{D_{\mathbf{u}} f(0, 0) = \frac{1}{\sqrt{2}}}$.

Answer 7E.

Given function is $f(x, y) = \sin(2x + 3y)$ and P(-6,4),u = $\frac{1}{2}\left(\sqrt{3}\,i - j\right)$ *******

(a.)

$$\nabla f(x, y) = \langle 2\cos(2x + 3y)\ ,\ 3\cos(2x + 3y)\rangle$$

and

(b.)

$$\nabla f(-6, 4) = \langle 2\cos(-12 + 12), 3\cos(-12 + 12)\rangle = \langle 2\cos 0, 3\cos 0\rangle = \langle 2\ ,\ 3\rangle$$

(c.)

The unit vector is $\vec{u} = \frac{1}{2}\left(\sqrt{3}\,i - j\right)$, so the rate of change of f at P in the direction of the vector $\vec{u}$ is

$$D_{\vec{u}} f\left(-6, 4\right) = \nabla f\left(-6, 4\right)\cdot\vec{u} = \left\langle 2, 3\right\rangle\cdot\left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2}\right\rangle$$

$$= 2\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{-1}{2}\right) = \sqrt{3} - \frac{3}{2}$$

Answer 8E.

(a)

We are given that a function

$$f\left(x, y\right) = \frac{y^2}{x}$$

The gardient of f is

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$= \left\langle \frac{-y^2}{x^2}, \frac{2y}{x} \right\rangle$$

$$\nabla f(x, y) = \left\langle \frac{-y^2}{x^2}, \frac{2y}{x} \right\rangle$$

(b)

We have to find the gradient at the point $P(1, 2)$

$$\nabla f(1, 2) = \left\langle \frac{(-2)^2}{(1)^2}, \frac{2(2)}{1} \right\rangle$$

$$= \left\langle \frac{-4}{1}, \frac{4}{1} \right\rangle$$

$$= \langle -4, 4 \rangle$$

$$\Rightarrow \nabla f(1, 2) = \langle -4, 4 \rangle$$

(c)

Here,

$$u = \frac{1}{3}\left\langle 2i + \sqrt{5}j \right\rangle$$

$$\Rightarrow u = \left\langle \frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle$$ **(It is a unit vector)**

Then the rate of change of f at the point $P(1, 2)$ in the direction of u is

$$D_u f(x, y) = \nabla f(x, y) u$$

$$D_u f(1, 2) = \nabla f_u(1, 2) = \left\langle -4, 4 \right\rangle \cdot \left\langle \frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle$$

$$= \frac{-8}{3} + \frac{4\sqrt{5}}{3}$$

$$= \frac{4\sqrt{5} - 8}{3}$$

$$\Rightarrow D_u f\left(1, 2\right) = \frac{4\sqrt{5} - 8}{3}$$

Answer 9E.

(a) If f is a function of two variables x and y, then the gradient of f is the vector function ∇f is defined by $\nabla f(x, y) = \left\langle f_x(x, y), f_y(x, y) \right\rangle$

or $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2yz - xyz^3\right]$$

$$= 2xyz - yz^3$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2yz - xyz^3\right]$$
$$= x^2z - xz^3$$
$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x^2yz - xyz^3\right]$$
$$= x^2y - 3xyz^2$$

Thus, we get $\boxed{\nabla f(x, y) = \left\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 \right\rangle}$.

(b) Replace x with 2, y with -1, and z with 1.

$$\nabla f(2, -1, 1) = \left\langle \left[2(2)(-1)(1) - (-1)(1)^3\right], \left[(2)^2(1) - (2)(1)^3\right], \left[(2)^2(-1) - 3(2)(-1)(1)^2\right] \right\rangle$$
$$= \left\langle (-4 + 1), (4 - 2), (-4 + 6) \right\rangle$$
$$= \left\langle -3, 2, 2 \right\rangle$$

Thus, we get $\boxed{\nabla f(2, -1, 1) = \left\langle -3, 2, 2 \right\rangle}$.

(c) Find $\nabla f(2, -1, 1) \cdot \mathbf{u}$.

$$\nabla f(2, -1, 1) \cdot \mathbf{u} = \left\langle -3, 2, 2 \right\rangle \cdot \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$$
$$= (-3)(0) + (2)\left(\frac{4}{5}\right) - (2)\left(\frac{3}{5}\right)$$
$$= 0 + \frac{8}{5} - \frac{6}{5}$$
$$= \boxed{\frac{2}{5}}$$

Thus, the rate of change of f at P in the direction of the vector $\mathbf{u}$ is obtained as $\boxed{\frac{2}{5}}$.

(c) Find $\nabla f(2, -1, 1) \cdot \mathbf{u}$.

$$
\begin{aligned}
\nabla f(2, -1, 1) \cdot \mathbf{u} &= \langle -3, 2, 2 \rangle \cdot \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle \\
&= (-3)(0) + (2)\left(\frac{4}{5}\right) - (2)\left(\frac{3}{5}\right) \\
&= 0 + \frac{8}{5} - \frac{6}{5} \\
&= \boxed{\frac{2}{5}}
\end{aligned}
$$

Thus, the rate of change of f at P in the direction of the vector $\mathbf{u}$ is obtained as $\boxed{\frac{2}{5}}$.

Answer 10E.

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[y^2 e^{xyz}\right] \\
&= 2ye^{xyz} + xy^2 z e^{xyz} \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[y^2 e^{xyz}\right] \\
&= xy^3 e^{xyz}
\end{aligned}
$$

Thus, we get $\boxed{\nabla f(x, y) = \left\langle y^3 z e^{xyz}, 2ye^{xyz} + xy^2 z e^{xyz}, xy^3 e^{xyz} \right\rangle}$.

(b) Replace x with 0, y with 1, and z with -1.

$$
\begin{aligned}
\nabla f(0, 1, -1) &= \left\langle (1)^3(-1)e^{(0)(1)(-1)}, 2(1)e^{(0)(1)(-1)} + (0)(1)^2(-1)e^{(0)(1)(-1)}, (0)(1)^3 e^{(0)(1)(-1)} \right\rangle \\
&= \langle -1, 2, 0 \rangle
\end{aligned}
$$

Thus, we get $\nabla f(0, 1, -1) = \boxed{\langle -1, 2, 0 \rangle}$.

(c) Find $\nabla f(0, 1, -1) \cdot \mathbf{u}$.

$$\begin{aligned}\nabla f(0, 1, -1) \cdot \mathbf{u} &= \langle -1, 2, 0 \rangle \cdot \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle \\ &= (-1)\left(\frac{3}{13}\right) + (2)\left(\frac{4}{13}\right) + (0)\left(\frac{12}{13}\right) \\ &= -\frac{3}{13} + \frac{8}{13} + 0 \\ &= \boxed{\frac{5}{13}}\end{aligned}$$

Thus, the rate of change of f at P in the direction of the vector $\mathbf{u}$ is obtained as $\boxed{\boxed{\frac{5}{13}}}$.

Answer 11E.

We know that $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$, where $\mathbf{u}$ is the unit vector and $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.

Find $f_x(x, y)$ and $f_y(x, y)$.

$$\begin{aligned}f_x(x, y) &= \frac{\partial}{\partial x}(e^x \sin y) \\ &= e^x \sin y\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= \frac{\partial}{\partial y}(e^x \sin y) \\ &= e^x \cos y\end{aligned}$$

Now, compute $f_x\left(0, \frac{\pi}{3}\right)$ and $f_y\left(0, \frac{\pi}{3}\right)$.

$$\begin{aligned}f_x\left(0, \frac{\pi}{3}\right) &= e^0 \sin \frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$f_y\left(0, \frac{\pi}{3}\right) = e^0 \cos\frac{\pi}{3}$$
$$= \frac{1}{2}$$

Then, $\nabla f\left(0, \frac{\pi}{3}\right) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$.

We have to find the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ given by $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

$$\mathbf{u} = \frac{\langle -6, 8 \rangle}{\sqrt{6^2 + 8^2}}$$
$$= \frac{\langle -6, 8 \rangle}{\sqrt{100}}$$
$$= \frac{\langle -6, 8 \rangle}{10}$$
$$= \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$$

Substitute the known values in $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$.

$$D_{\mathbf{u}} f\left(0, \frac{\pi}{3}\right) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$$
$$= -\frac{3\sqrt{3}}{10} + \frac{4}{10}$$
$$= \frac{4 - 3\sqrt{3}}{10}$$

Thus, we get $\boxed{D_{\mathbf{u}} f\left(0, \frac{\pi}{3}\right) = \frac{4 - 3\sqrt{3}}{10}}$.

Answer 12E.

We know that $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$, where $\mathbf{u}$ is the unit vector and $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.

Find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right)$$
$$= -\frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right)$$
$$= -\frac{2xy}{(x^2 + y^2)^2}$$

Now, compute $f_x(1, 2)$ and $f_y(1, 2)$

$$f_x(1, 2) = -\frac{1^2 - 2^2}{(1^2 + 2^2)^2}$$
$$= \frac{3}{25}$$

$$f_y(1, 2) = -\frac{2(1)(2)}{(1^2 + 2^2)^2}$$
$$= -\frac{4}{25}$$

Then, $\nabla f(1, 2) = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle$.

We have to find the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ given by $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

$$\mathbf{u} = \frac{\langle 3, 5 \rangle}{\sqrt{3^2 + 5^2}}$$
$$= \frac{\langle 3, 5 \rangle}{\sqrt{34}}$$
$$= \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

Substitute the known values in $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$.

$$D_{\mathbf{u}}f(1, 2) = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$
$$= \frac{9}{25\sqrt{34}} - \frac{20}{25\sqrt{34}}$$
$$= -\frac{11}{25\sqrt{34}}$$

Thus, we get $D_{\mathbf{u}}f(1, 2) = \boxed{-\frac{11}{25\sqrt{34}}}$.

Answer 13E.

Given function is $g(p, q) = p^4 - p^2 q^3$

We first need to compute the gradient vector at (2,1):

$$\nabla g(p, q) = \langle 4p^3 - 2pq^3, \ -3p^2 q^2 \rangle$$
$$\nabla g(2, 1) = \langle 4(2)^3 - 2(2)(1)^3, \ -3(2)^2(1)^2 \rangle = \langle 32 - 4, \ -12(1) \rangle = \langle 28, \ -12$$

Note that $\vec{v}$ is not a unit vector, but since $|\vec{v}| = \sqrt{10}$, the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$ Therefore, by the equation $D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$, we have

$$D_{\vec{u}} g(p, q) = \nabla g(2, 1) \cdot \vec{u} = \langle 28, \ -12 \rangle \cdot \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$
$$= \frac{28 \cdot 1 - 12 \cdot 3}{\sqrt{10}} = \frac{-8}{\sqrt{10}}$$

Answer 14E.

The given function is

$$g\left(r, s\right) = \tan^{-1}\left(\text{rs}\right)$$

We first find need to compute the gradient vector at $(1, 2)$

$$\nabla g(r, s) = \left\langle \frac{1(s)}{1+r^2s^2}, \frac{1(r)}{1+r^2s^2} \right\rangle$$

$$= \left\langle \frac{s}{1+r^2s^2}, \frac{r}{1+r^2s^2} \right\rangle$$

$$\Rightarrow \nabla g(r, s) = \left\langle \frac{s}{1+r^2s^2}, \frac{r}{1+r^2s^2} \right\rangle$$

Then,

$$\nabla f(1, 2) = \left\langle \frac{2}{1+(1)^2(2)^2}, \frac{1}{1+(1)^2(2)^2} \right\rangle$$

$$= \left\langle \frac{2}{1+4}, \frac{1}{1+4} \right\rangle$$

$$= \left\langle \frac{2}{5}, \frac{1}{5} \right\rangle$$

$$\nabla g(1, 2) = \left\langle \frac{2}{5}, \frac{1}{5} \right\rangle$$

Note that $v = 5i + 10j$ is not a unit vector. The unit vector in the direction of $v = 5i + 10j$ is

$$u = \frac{1}{\sqrt{125}}\left(5i + 10j\right)$$

$$= \frac{1}{5\sqrt{5}}\left(5i + 10j\right)$$

$$= \frac{1}{\sqrt{5}}\left(i + 2j\right)$$

$$\Rightarrow u = \frac{1}{\sqrt{5}}\left(i + 2j\right)$$

$$\Rightarrow u = \frac{v}{|v|} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

Therefore, by the equation

$$D_u g(1, 2) = \nabla g(1, 2) \cdot u$$

$$= \left\langle \frac{2}{5}, \frac{1}{5} \right\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$= \frac{2}{5\sqrt{5}} + \frac{2}{5\sqrt{5}}$$

$$= \frac{4}{5\sqrt{5}}$$

$$\Rightarrow D_u g\left(1, 2\right) = \frac{4}{5\sqrt{5}}$$

So, the required solution is $D_u g\left(1, 2\right) = \frac{4}{5\sqrt{5}}$

Answer 15E.

Given that $f\left(x, y, z\right) = xe^{y} + ye^{z} + ze^{x}$

We first need to compute the gradient vector at (0,0,0):

$$\nabla f\left(x, y, z\right) = \left\langle e^{y} + ze^{x},\ xe^{y} + e^{z},\ ye^{z} + e^{x}\right\rangle$$

$$\nabla f\left(0, 0, 0\right) = \left\langle e^{0} + 0(e)^{0},\ 0(e)^{0} + e^{0},\ 0(e)^{0} + e^{0}\right\rangle = \left\langle 1 + 0,\ 0 + 1,\ 0 + 1\right\rangle = \langle$$

And also given that $V = \ <5, 1, \ -2>$ ******

Note that $\vec{v}$ is not a unit vector, but since $\left|\vec{v}\right| = \sqrt{30}$, the unit vector in the direction of $\vec{v}$ is $\vec{u} = \frac{\vec{v}}{\left|\vec{v}\right|} = \left\langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}} \right\rangle$

Therefore , by the equation $D_{\vec{u}}f\left(x, y, z\right) = \ \nabla f\left(x, y, z\right)\cdot\vec{u}$, we have

$$D_{\vec{u}}f\left(x, y, z\right) = \ \nabla f\left(0, 0, 0\right)\cdot\vec{u} = \left\langle 1, 1, 1\right\rangle\cdot\left\langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}}\right\rangle$$

$$= \frac{1\cdot5 + 1\cdot1 - 1\cdot2}{\sqrt{30}} = \frac{4}{\sqrt{30}}$$

Answer 16E.

We are given that

$$f(x, y, z) = \sqrt{xyz}$$

We first need to compute the gradient vector at the point $(3, 2, 6)$

$$\nabla f\left(x, y, z\right) = \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}} \right\rangle$$

$$\Rightarrow \nabla f\left(3, 2, 6\right) = \left\langle \frac{(2)(6)}{2\sqrt{3\cdot 2\cdot 6}}, \frac{(3)(6)}{2\sqrt{3\cdot 2\cdot 6}}, \frac{(3)(2)}{2\sqrt{3\cdot 2\cdot 6}} \right\rangle$$

$$= \left\langle \frac{6}{\sqrt{36}}, \frac{9}{\sqrt{36}}, \frac{3}{\sqrt{36}} \right\rangle$$

$$= \left\langle 1, \frac{3}{2}, \frac{1}{2} \right\rangle$$

$$\Rightarrow \nabla f\left(3, 2, 6\right) = \left\langle 1, \frac{3}{2}, \frac{1}{2} \right\rangle$$

Note that v is not a unit vector but since $|v| = \sqrt{1+4+4} = 3$

The unit vector in the direction of $v = -i - 2j + 2k$ is

$$u = \frac{1}{3}\left(-i - 2j + 2k\right) = \left\langle \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3} \right\rangle$$

Therefore, by the equation

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot u$$

$$\Rightarrow D_u f(3, 2, 6) = \nabla f(3, 2, 6) \cdot u$$

$$= \left\langle 1, \frac{3}{2}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{-1}{3}, \frac{-2}{3}, \frac{2}{3} \right\rangle$$

$$= \frac{-1}{3} - 1 + \frac{1}{3}$$

$$= -1$$

$$\Rightarrow D_u f(3, 2, 6) = -1$$

The solution is $D_u f(3, 2, 6) = -1$

Answer 17E.

We know that $D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$, where $\mathbf{u}$ is the unit vector and $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.

Find $h_r(r, s, t)$, $h_s(r, s, t)$, and $h_t(r, s, t)$.

$$h_r(r, s, t) = \frac{\partial}{\partial r}\left[\ln(3r + 6s + 9t)\right]$$

$$= \frac{3}{3r + 6s + 9t}$$

$$h_s(r, s, t) = \frac{\partial}{\partial s}\left[\ln(3r + 6s + 9t)\right]$$
$$= \frac{6}{3r + 6s + 9t}$$
$$h_t(r, s, t) = \frac{\partial}{\partial s}\left[\ln(3r + 6s + 9t)\right]$$
$$= \frac{9}{3r + 6s + 9t}$$

Now, compute $h_r(1, 1, 1)$, $h_s(1, 1, 1)$, and $h_t(1, 1, 1)$

$$h_r(1, 1, 1) = \frac{3}{3(1) + 6(1) + 9(1)}$$
$$= \frac{1}{6}$$

$$h_s(1, 1, 1) = \frac{6}{3r + 6s + 9t}$$
$$= \frac{1}{3}$$
$$h_t(1, 1, 1) = \frac{9}{3r + 6s + 9t}$$
$$= \frac{1}{2}$$

Then, $\nabla f(1, 1, 1) = \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle$.

We have to find the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ given by $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$.

$$\mathbf{u} = \frac{\langle 4, 12, 6 \rangle}{\sqrt{4^2 + 12^2 + 6^2}}$$
$$= \frac{\langle 4, 12, 6 \rangle}{\sqrt{16 + 144 + 36}}$$
$$= \frac{\langle 4, 12, 6 \rangle}{14}$$
$$= \left\langle \frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle$$

Substitute the known values in $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$.

$$\begin{aligned} D_{\mathbf{u}}f(1, 1, 1) &= \left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle \\ &= \left(\frac{1}{6}\right)\left(\frac{2}{7}\right) + \left(\frac{1}{3}\right)\left(\frac{6}{7}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{7}\right) \\ &= \frac{1}{21} + \frac{2}{7} + \frac{3}{14} \\ &= \frac{23}{42} \end{aligned}$$

Thus, we get $\boxed{D_{\mathbf{u}}f(1, 1, 1) = \frac{23}{42}}$.

Answer 18E.

If θ is the angle less than $180°$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ then

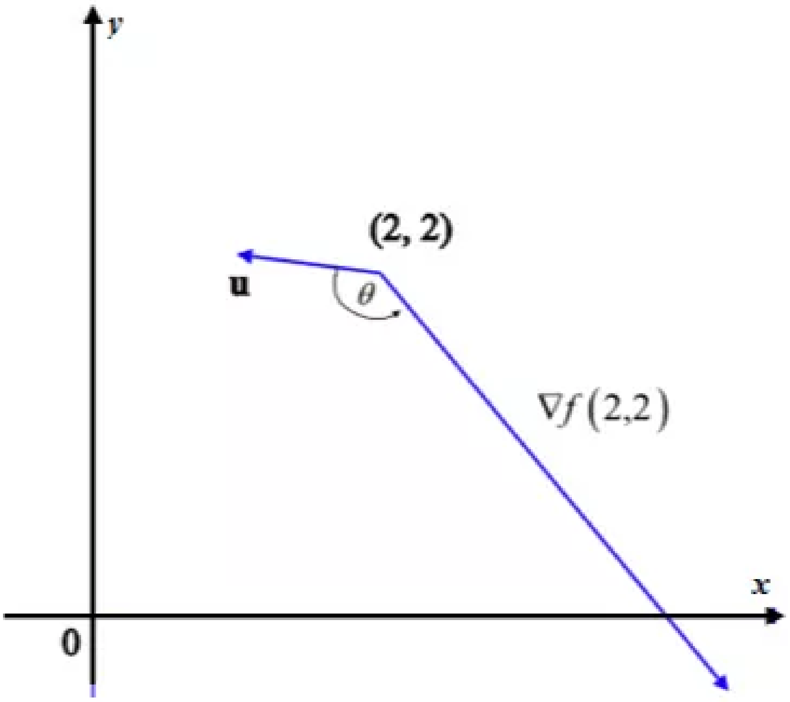
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Therefore, the directional derivative of *f* at the point $(2,2)$ in the direction of the unit vector $\mathbf{u}$ can be defined

$$\begin{aligned} D_u f(2,2) &= \nabla f(2,2) \cdot \mathbf{u} \\ &= |\nabla f(2,2)||\mathbf{u}|\cos\theta \\ &= |\nabla f(2,2)|\cos\theta \end{aligned}$$

where $\nabla f(2,2)$ is the gradient vector for *f* at the point $(2,2)$. Since $\mathbf{u}$ is a unit vector, its magnitude must be 1.

We reconstruct the diagram from the text and include the angle θ:



To estimate the directional derivative, we need to approximate both the magnitude of the gradient vector and the measure of the angle θ.

By using the length of the unit vector $\mathbf{u}$ as a basis, we estimate that the length of the gradient vector $\nabla f(2,2)$ is roughly 4 units. Now by estimation, we suppose that the angle θ measures approximately $135°$. By these approximations:

$$\begin{aligned}|\nabla f(2,2)|\cos\theta &= 4\cos 135° \\ &= 4\left(-\frac{\sqrt{2}}{2}\right) \\ &= -2\sqrt{2}\end{aligned}$$

Therefore, we estimate $\boxed{D_{\mathbf{u}} f(2,2) = -2\sqrt{2}}$

Answer 19E.

$$f(x,y) = \sqrt{xy}$$

Then $\quad f_x = \dfrac{y}{2\sqrt{xy}}$

And $\quad f_y = \dfrac{x}{2\sqrt{xy}}$

Then the gradient of f is

$$\vec{\nabla} f(x, y) = < f_x, f_y >$$

i.e. $$\vec{\nabla} f(x, y) = < \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} >$$

Then $$\vec{\nabla} f(2,8) = < \frac{8}{2\sqrt{16}}, \frac{2}{2\sqrt{16}} >$$

$$= < 1, \frac{1}{4} >$$

The vector in the direction of a(5, 4) is $\vec{v} = < 5-2, 4-8 >$

That is $\vec{v} = < 3, -4 >$ and the unit vector in this direction is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = < \frac{3}{5}, \frac{-4}{5} >$$

Then the directional derivative of f at P(2, 8) in the direction of Q(5, 4) is

$$D_u f(2,8) = \vec{\nabla} f(2,8) \vec{u}$$

$$= < 1, \frac{1}{4} > . < \frac{3}{5}, \frac{-4}{5} >$$

$$= \frac{3}{5} - \frac{1}{5}$$

i.e. $$\boxed{D_u f(2,8) = \frac{2}{5}}$$

Answer 20E.

We are given that a function

$$f(x, y, z) = xy + yz + zx$$

We first compute the gradient vector

$$\nabla f(x, y, z) = (f_x, f_y, f_z)$$

$$= \langle y + z, x + z, x + y \rangle$$

$$\Rightarrow \nabla f(x, y, z) = \langle y + z, x + z, x + y \rangle$$

We evaluate $\nabla f(x, y, z)$ at the point $P(1, -1, 3)$

$$\nabla f(1, -1, 3) = \langle -1+3, 1+3, 1-1 \rangle$$

$$= \langle 2, 4, 0 \rangle$$

$$\Rightarrow \nabla f(1, -1, 3) = \langle 2, 4, 0 \rangle$$

The vector in the direction of $Q(2, 4, 5)$

$$v = \langle 2-1, 4+1, 5-3 \rangle$$

$$= \langle 1, 5, 2 \rangle$$

$$\Rightarrow v = \langle 1, 5, 2 \rangle$$

The unit vector in the direction of $v = i + 5j + 2k$ is

$$u = \frac{v}{|v|} = \frac{i + 5j + 2k}{\sqrt{1 + 25 + 4}}$$

$$= \frac{1}{\sqrt{30}}\left(i + 5j + 2k\right)$$

$$\Rightarrow u = \left\langle \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}} \right\rangle$$

The directional derivative is given by

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot u$$

$$\Rightarrow D_u f(1, -1, 3) = \nabla f(1, -1, 3) \cdot u$$

$$= \left\langle 2, 4, 0 \right\rangle \cdot \left\langle \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}} \right\rangle$$

$$= \frac{2}{\sqrt{30}} + \frac{20}{\sqrt{30}} + \frac{0}{\sqrt{30}}$$

$$= \frac{22}{\sqrt{30}}$$

$$\Rightarrow D_u f\left(1, -1, 3\right) = \frac{22}{\sqrt{30}}$$

The solution is

$$D_u f\left(1, -1, 3\right) = \frac{22}{\sqrt{30}}$$

Answer 21E.

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f is defined by $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$ or $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[4y\sqrt{x}\right] = \frac{2y}{\sqrt{x}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[4y\sqrt{x}\right] = 4\sqrt{x}$$

Thus, we get $\boxed{\nabla f(x, y) = \left\langle \frac{2y}{\sqrt{x}}, 4\sqrt{x}\right\rangle}$.

Replace x with 4 and y with 1.

$$\nabla f(4, 1) = \left\langle \frac{2(1)}{\sqrt{4}}, 4\sqrt{4}\right\rangle = \langle 1, 8\rangle$$

Thus, we get $\nabla f(4, 1) = \boxed{\langle 1, 8\rangle}$.

Find the maximum rate of change of f.

$$|\nabla f(4, 1)| = \sqrt{1^2 + 8^2} = \sqrt{65}$$

Thus, the maximum rate of change of f at (4, 1) is $\boxed{\sqrt{65}}$.

Answer 22E.

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f is defined by $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$ or $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial}{\partial s}\left[te^{st}\right] \\ &= t^2e^{st}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}\left[te^{st}\right] \\ &= e^{st} + tse^{st}\end{aligned}$$

Then, $\nabla f(s, t) = \langle t^2e^{st}, e^{st} + tse^{st}\rangle$.

Replace s with 0 and t with 2.

$$\begin{aligned}\nabla f(0, 2) &= \langle 2^2e^{(0)(2)}, e^{(0)(2)} + (2)(0)e^{(0)(2)}\rangle \\ &= \langle 4, 1+0\rangle \\ &= \langle 4, 1\rangle\end{aligned}$$

Thus, we get $\boxed{\nabla f(0, 2) = \langle 4, 1\rangle}$.

Find the maximum rate of change of f.

$$\begin{aligned}|\nabla f(0, 2)| &= \sqrt{4^2 + 1^2} \\ &= \sqrt{17}\end{aligned}$$

Thus, the maximum rate of change of f at $(0, 2)$ is $\boxed{\sqrt{17}}$.

Answer 23E.

$$f(x,y) = \sin(xy)$$

Then $f_x = y\cos(xy)$

And $f_y = x\cos(xy)$

The gradient of f is

$$\vec{\nabla} f(x,y) = < f_x, f_y >$$

$$= < y\cos(xy), x\cos(xy) >$$

Then $\vec{\nabla} f(1,0) = <0,1>$

The maximum rate of change of f is

$$\left|\vec{\nabla} f(1,0)\right| = \sqrt{0+1^2}$$

$$= \boxed{1}$$

And it takes place in the direction of $\vec{\nabla} f$ that is in the direction of $< 0, 1>$

Answer 24E.

We are given that a function

$$f(x, y, z) = \frac{x+y}{z}$$

And the given point is $(1, 1, -1)$

We have to find the direction in which the maximum rate of change of f

We first compute the gradient vector:

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

$$= \left\langle \frac{1}{z}, \frac{1}{z}, \frac{-(x+y)}{z^2} \right\rangle$$

$$\Rightarrow \nabla f(x, y, z) = \left\langle \frac{1}{z}, \frac{1}{z}, \frac{-(x+y)}{z^2} \right\rangle$$

Now, we evalute $\nabla f(x, y, z)$ at the point $(1, 1, -1)$

$$\nabla f(1, 1, -1) = \left\langle \frac{1}{-1}, \frac{1}{-1}, \frac{-(1+1)}{(-1)^2} \right\rangle$$

$$= \langle -1,\ -1,\ -2 \rangle$$

$$\Rightarrow\ \nabla f(1,\ 1,\ -1) = \langle -1,\ -1,\ -2 \rangle$$

The function f increases fastest in the direction of the gradient vector

$$\nabla f(1,\ 1,\ -1) = \langle -1,\ -1,\ -2 \rangle$$

The maximum rate of change is

$$|\nabla f(1,\ 1,\ -1)| = |\langle -1,\ -1,\ -2 \rangle$$

$$= \sqrt{(-1)^2 + (-1)^2 + (-2)^2}$$

$$= \sqrt{1+1+4}$$

$$= \sqrt{6}$$

$$\Rightarrow |\nabla f(1,\ 1,\ -1)| = \sqrt{6}$$

The maximum rate of change is $\sqrt{6}$

Answer 25E.

Given function is

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

We first compute the gradient vector:

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$\nabla f(3, 6, -2) = \left\langle \frac{3}{\sqrt{49}}, \frac{6}{\sqrt{49}}, \frac{-2}{\sqrt{49}} \right\rangle = \left\langle \frac{3}{7}, \frac{6}{7}, \frac{-2}{7} \right\rangle$$

f increases fastest in the direction of the gradient vector

$\nabla f(3, 6, -2) = \left\langle \frac{3}{7}, \frac{6}{7}, \frac{-2}{7} \right\rangle$, which is $\langle 3, 6, -2 \rangle$.

The maximum rate of change is

$$|\nabla f(3, 6, -2)| = \left|\left\langle \frac{3}{7}, \frac{6}{7}, \frac{-2}{7} \right\rangle\right| = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = \frac{\sqrt{49}}{\sqrt{49}} = \frac{7}{7} = 1$$

Answer 26E.

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f is defined by $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ or $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

Find $\frac{\partial f}{\partial p}$, $\frac{\partial f}{\partial q}$, and $\frac{\partial f}{\partial r}$.

$$\begin{aligned}\frac{\partial f}{\partial p} &= \frac{\partial}{\partial p}\left[\arctan(pqr)\right] \\ &= \frac{qr}{1+p^2q^2r^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial q} &= \frac{\partial}{\partial q}\left[\arctan(pqr)\right] \\ &= \frac{pr}{1+p^2q^2r^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial}{\partial r}\left[\arctan(pqr)\right] \\ &= \frac{pq}{1+p^2q^2r^2}\end{aligned}$$

Then, $\nabla f(p, q, r) = \left\langle \frac{qr}{1+p^2q^2r^2}, \frac{pr}{1+p^2q^2r^2}, \frac{pq}{1+p^2q^2r^2} \right\rangle$.

Replace p with 1 q with 2, and r with 1.

$$\begin{aligned}\nabla f(1, 2, 1) &= \left\langle \frac{(2)(1)}{1+1^2 2^2 1^2}, \frac{(1)(1)}{1+1^2 2^2 1^2}, \frac{(1)(2)}{1+1^2 2^2 1^2} \right\rangle \\ &= \left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle\end{aligned}$$

Thus, we get $\boxed{\nabla f(1, 2, 1) = \left\langle \frac{2}{5}, \frac{1}{5}, \frac{2}{5} \right\rangle}$.

Find the maximum rate of change of f.

$$|\nabla f(1,2,1)| = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2}$$
$$= \sqrt{\frac{4}{25} + \frac{1}{25} + \frac{4}{25}}$$
$$= \sqrt{\frac{9}{25}}$$
$$= \frac{3}{5}$$

Thus, the maximum rate of change of f at (1, 2, 1) is $\boxed{\frac{3}{5}}$.

Answer 27E.

(A)

Now f is a differentiable function then, the directional derivative of f in the direction of the unit vector $\vec{u}$ is given by

$$D_u f = \vec{\nabla} f \, \vec{u}$$
$$= \left|\vec{\nabla} f\right| \left|\vec{u}\right| \cos\theta$$
$$= \left|\vec{\nabla} f\right| \cos\theta$$

As $\quad \left|\vec{u}\right| = 1$

Where θ is the angle between $\vec{\nabla} f$ and $\vec{u}$

The minimum value of $\cos\theta = -1$ and this occurs when $\theta = 180°$

Therefore we say that f decreases most rapidly in the direction of $-\vec{\nabla} f$ and minimum value is $-\left|\vec{\nabla} f\right|$

(B)

$$f(x,y) = x^4 y - x^2 y^3$$

Then $\quad f_x = 4x^3 y - 2xy^3$

And $\quad f_y = x^4 - 3x^2 y^2$

The gradient of f is

$$\vec{\nabla} f(x,y) = < f_x, f_y >$$

$$= < 4x^3y - 2xy^3, x^4 - 3x^2y^2 >$$

Then $\vec{\nabla} f(2,-3) = < 4(8)(-3) - 2(2)(-3)^3, (2)^4 - 3(2)^2(-3)^2 >$

$$= < -96 + 108, 16 - 108 >$$

$$= < 12, -92 >$$

By part (A), f decrease most rapidly in the direction of $-\vec{\nabla} f$

Hence f decreases most rapidly in the direction of vector $- < 12, -92 >$

i.e. $< -12, 92 >$

And the minimum value is

$$-\left|\vec{\nabla} f\right|$$

$$= -\sqrt{144 + 8464}$$

$$= -\sqrt{8608}$$

$$= -2\sqrt{2152}$$

Answer 28E.

Consider the function as follows,

$$f(x,y) = ye^{-xy}$$

To find the directions in which directional derivative of $f(x,y)$ at the point $(0,2)$ has the value 1.

Suppose that direction vector is,

$$u = x\mathbf{i} + y\mathbf{j}$$

This can be written as,

$$u = \langle x, y \rangle$$

First find the gradient of $f(x,y)$:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \langle -y^2e^{-xy}, e^{-xy} - xye^{-xy} \rangle$$

$$= \langle -y^2e^{-xy}, e^{-xy}(1 - xy) \rangle$$

At the point $(0,2)$, then

$$\nabla f(0,2) = \langle -2^2e^{-(0)(2)}, e^{-(0)(2)}(1 - 0\cdot 2) \rangle$$

$$= \langle -4, 1 \rangle$$

Therefore, by the definition of directional derivative, that is

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

Since At the point $(0,2)$ has the value 1, so

$$\begin{aligned} D_u f(0,2) &= \nabla f(0,2) \cdot (x\mathbf{i} + y\mathbf{j}) \\ 1 &= \langle -4,1 \rangle \cdot \langle x,y \rangle \\ 1 &= -4x + y \end{aligned}$$

That is,

$$y = 4x + 1$$

Since $\langle x,y \rangle$ is unit vector, so

$$x^2 + y^2 = 1$$

Substitute $y = 4x + 1$ into above equation,

$$\begin{aligned} x^2 + (4x+1)^2 &= 1 \\ x^2 + 16x^2 + 1 + 8x &= 1 \\ 17x^2 + 8x &= 0 \\ x(17x+8) &= 0 \end{aligned}$$

So, either $x = 0$ or $17x + 8 = 0$

This implies that,

$$x = -\frac{8}{17}$$

Substitute $x = 0$ and $x = -\frac{8}{17}$ into $y = 4x + 1$, to get

$$\begin{aligned} y &= 4(0) + 1 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

And,

$$\begin{aligned} y &= 4\left(-\frac{8}{17}\right) + 1 \\ &= -\frac{32}{17} + 1 \\ &= \frac{-32+17}{17} \\ &= -\frac{15}{17} \end{aligned}$$

Now substitute $x=0, y=1$ and $x=-\frac{8}{17}, y=-\frac{15}{17}$ into $u=x\mathbf{i}+y\mathbf{j}$.

$$u=(0)\mathbf{i}+(1)\mathbf{j}$$
$$=\mathbf{j}$$

And,

$$u=-\frac{8}{17}\mathbf{i}-\frac{15}{17}\mathbf{j}$$

Hence, two direction vectors are $\boxed{\mathbf{j}}$ and $\boxed{-\frac{8}{17}\mathbf{i}-\frac{15}{17}\mathbf{j}}$.

Answer 29E.

Consider the function in two variables as follows:

$$f(x,y)=x^2+y^2-2x-4y$$

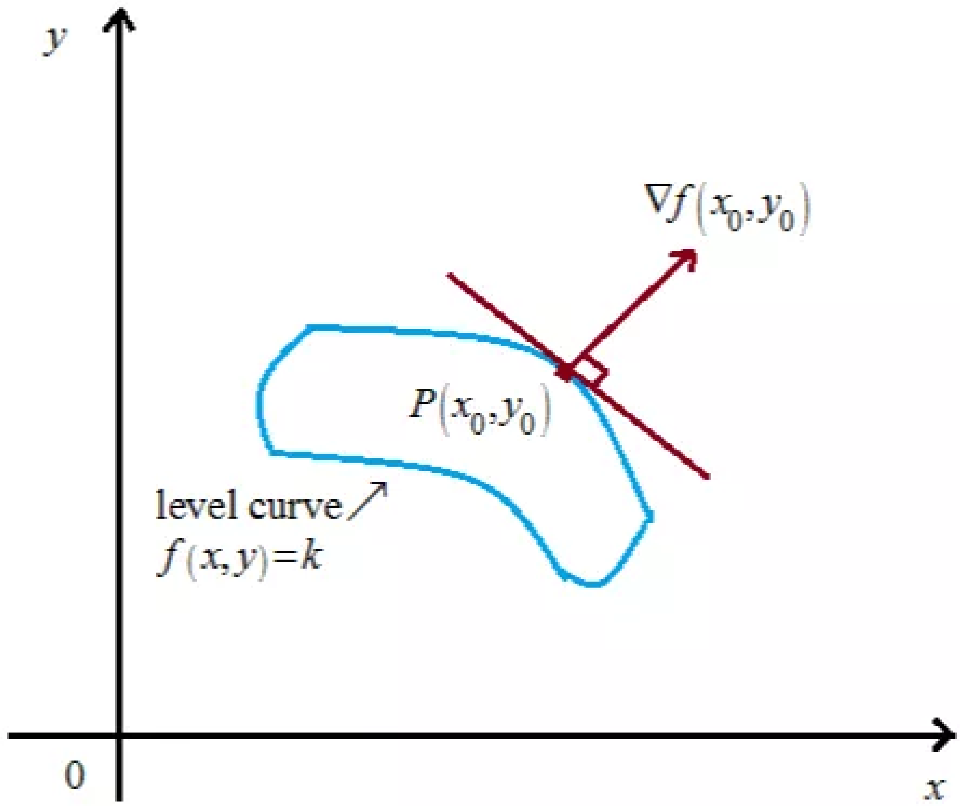
Find all points at which the direction of fastest change of the function f is $\mathbf{i}+\mathbf{j}$.

If $P(x_0,y_0)$ is a point in the domain of f, then the gradient vector $\nabla f(x_0,y_0)$ gives the direction of fastest increase of f.

And the gradient vector $\nabla f(x_0,y_0)$ is perpendicular to the level curve $f(x,y)=k$ that passes through P.

If we move along the curve, the values of f remain constant.

So we get the maximum increase of f if we move in the perpendicular direction.



The gradient vector $\nabla f(x,y) = \mathbf{i}f_x(x,y) + \mathbf{j}f_y(x,y)$.

Now, $f_x(x,y) = 2x-2$ and $f_y(x,y) = 2y-4$.

The gradient of f or the fastest change of the function f is as follows:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$
$$= \langle 2x-2,\ 2y-4 \rangle$$

But the fastest change of the function f is given as $\mathbf{i}+\mathbf{j}$ or $\langle 1,1 \rangle$.

Therefore,

$$\langle 2x-2,\ 2y-4 \rangle = \langle 1,1 \rangle$$
$$\Rightarrow 2x-2 = 1$$
$$2y-4 = 1$$

Using these two equations, get the following:

$$2x-2 = 2y-4$$
$$x-1 = y-2$$
$$x = y-1$$
$$y = x+1$$

Hence at all the points on line $y = x+1$, the direction of the fastest change of f is $\mathbf{i}+\mathbf{j}$.

Answer 30E.

$$z = 200 + 0.02x^2 - 0.001y^3$$

Then $z_x = 0.04x$

And $z_y = -0.003y^2$

The gradient of z is

$$\vec{\nabla} z = < z_x, z_y >$$
$$= < 0.04x, -0.003y^2 >$$

Then $\vec{\nabla} z(80,60) = < 0.04(80) - 0.003(60)^2 >$

$$= < 3.2, -10.8 >$$

We need to find the rate of change of z in the direction of the vector

$$\vec{v} = <0-80, 0-60>$$
$$= <-80, -60>$$

It is not a unit vector. Now $|\vec{v}| = \sqrt{(80)^2 + 60^2} = 100$

Then the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = <-0.8, -0.6>$$

The directional derivative of z in the direction of $\vec{u}$ is

$$D_u z(80,60) = \vec{\nabla} z(80,60).\vec{u}$$
$$= <3.2, -10.8>. <-0.8, -0.6>$$
$$= -2.56 + 6.48$$
$$= 3.92$$

Since the rate of change of z in the direction of buoy is 3.92 that is positive and hence we say that the water under the boat is getting deeper

Answer 31E.

According to the given condition

$$T(x,y,z) \;\alpha\; \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Or $$T(x,y,z) = \frac{k}{\sqrt{x^2+y^2+z^2}}$$

Where k is constant proportionality

Also $T(1,2,2) = 120^\circ$

Then $120 = \frac{k}{\sqrt{1+4+4}}$

i.e. $120 = \frac{k}{3}$

i.e. $k = 360$

Hence $$T(x,y,z) = \frac{360}{\sqrt{x^2+y^2+z^2}}$$

(A)

$$T_x = \frac{-360\,x}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$T_y = \frac{-360\,y}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$T_z = \frac{-360z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

The gradient of T is

$$\vec{\nabla}T(x,y,z) = <T_x, T_y, T_z>$$

$$= <\frac{-360x}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-360y}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-360z}{\left(x^2+y^2+z^2\right)^{3/2}}>$$

Then $\vec{\nabla}T(1,2,2) = <\frac{-360(1)}{(9)^{3/2}}, \frac{-360(2)}{(9)^{3/2}}, \frac{-360(2)}{(9)^{3/2}}>$

$$= <\frac{-40}{3}, \frac{-80}{3}, \frac{-80}{3}>$$

The gradient of T is

$$\vec{\nabla}T(x,y,z) = <T_x, T_y, T_z>$$

$$= <\frac{-360x}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-360y}{\left(x^2+y^2+z^2\right)^{3/2}}, \frac{-360z}{\left(x^2+y^2+z^2\right)^{3/2}}>$$

Then $\vec{\nabla}T(1,2,2) = <\frac{-360(1)}{(9)^{3/2}}, \frac{-360(2)}{(9)^{3/2}}, \frac{-360(2)}{(9)^{3/2}}>$

$$= <\frac{-40}{3}, \frac{-80}{3}, \frac{-80}{3}>$$

Then $D_u T(1,2,2) = \vec{\nabla}T(1,2,2)\,\vec{u}$

$$= <\frac{-40}{3}, \frac{-80}{3}, \frac{-80}{3}> \cdot <\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}>$$

$$= \frac{40}{3\sqrt{3}} + \frac{80}{3\sqrt{3}} - \frac{80}{3\sqrt{3}}$$

$$= \frac{-40}{3\sqrt{3}}$$

Hence the rate of change of T in the required direction is $\boxed{\frac{-40}{3\sqrt{3}}}$

(B)

Take an arbitrary point (x, y, z), then gradient of T at (x, y, z) is

$$\vec{\nabla}T(x,y,z) = <\frac{-360x}{(x^2+y^2+z^2)^{3/2}}, \frac{-360y}{(x^2+y^2+z^2)^{3/2}}, \frac{-360z}{(x^2+y^2+z^2)^{3/2}}>$$

We know the maximum rate of change of T is given by $\left|\vec{\nabla}T(x,y,z)\right|$ and it takes place in the direction of $\vec{\nabla}T(x,y,z)$ that is in the direction of vector

$$<\frac{-360x}{(x^2+y^2+z^2)^{3/2}}, \frac{-360y}{(x^2+y^2+z^2)^{3/2}}, \frac{-360z}{(x^2+y^2+z^2)^{3/2}}>$$

i.e. $$\frac{360}{(x^2+y^2+z^2)^{3/2}} <-x,-y,-z>$$

That is in the direction of vector

$$\frac{360}{x^2+y^2+z^2}<\frac{-x}{(x^2+y^2+z^2)^{1/2}}, \frac{-y}{(x^2+y^2+z^2)^{1/2}}, \frac{-z}{(x^2+y^2+z^2)^{1/2}}>$$

That is in the direction of vector

$$\frac{360}{(x^2+y^2+z^2)}<\frac{0-x}{(x^2+y^2+z^2)^{1/2}}, \frac{0-y}{(x^2+y^2+z^2)^{1/2}}, \frac{0-z}{(x^2+y^2+z^2)^{1/2}}>$$

Which is a vector that points towards the origin

Hence at any point in the ball the direction of greatest increase in temperature is given by a vector that point toward the origin

Answer 32E.

The temperature at a point (x,y,z) is given by

$T(x,y,z) = 200e^{-x^2-3y^2-9z^2}$, where T is measured in $°C$ and x,y,z in meters

(a) It is need to find the rate of change of temperature at the point $P(2,-1,2)$ in the direction toward the point $A=(3,-3,3)$.

The rate of change of temperature T at the point $P(x,y,z)$ in the direction of unit vector **u** is

$$D_{\mathbf{u}}T(x,y,z) = \nabla T(x,y,z)\cdot\mathbf{u}$$

Now the partial derivative of T with respect to x is,

$$\begin{aligned} T_x(x,y,z) &= \frac{\partial T}{\partial x} \\ &= 200e^{-x^2-3y^2-9z^2}\cdot\frac{\partial}{\partial x}(-x^2-3y^2-9z^2) \\ &= 200e^{-x^2-3y^2-9z^2}\cdot(-2x) \\ &= -400xe^{-x^2-3y^2-9z^2} \end{aligned}$$

Now the partial derivative of T with respect to y is,

$$T_y(x,y,z)=\frac{\partial T}{\partial y}$$
$$=200e^{-x^2-3y^2-9z^2}\cdot\frac{\partial}{\partial y}\left(-x^2-3y^2-9z^2\right)$$
$$=200e^{-x^2-3y^2-9z^2}\cdot(-6y)$$
$$=-1200\,y\,e^{-x^2-3y^2-9z^2}$$

Now the partial derivative of T with respect to z is,

$$T_z(x,y,z)=\frac{\partial T}{\partial z}$$
$$=200e^{-x^2-3y^2-9z^2}\cdot\frac{\partial}{\partial z}\left(-x^2-3y^2-9z^2\right)$$
$$=200e^{-x^2-3y^2-9z^2}\cdot(-18z)$$
$$=-3600\,z\,e^{-x^2-3y^2-9z^2}$$

The gradient of T is,

$$\nabla T(x,y,z)=\left\langle T_x,T_y,T_z\right\rangle$$
$$=\left\langle -400x\,e^{-x^2-3y^2-9z^2},-1200\,y\,e^{-x^2-3y^2-9z^2},-3600\,z\,e^{-x^2-3y^2-9z^2}\right\rangle$$

The gradient of T at $(2,-1,2)$ is,

$$\nabla T(2,-1,2)=\left\langle -400(2)e^{-43},-1200(-1)e^{-43},-3600(2)e^{-43}\right\rangle$$
$$=-400e^{-43}\langle 2,-3,18\rangle$$

We need to find the rate of change of T in the direction of the vector $\overrightarrow{PA}$.

Suppose $\overrightarrow{PA}=\mathbf{v}$

$$=\langle 3-2,\ -3+1,\ 3-2\rangle$$
$$=\langle 1,\ -2,\ 1\rangle$$

It is not a unit vector as $|\mathbf{v}|=\sqrt{6}$

So the unit vector **u** in the direction of $\overrightarrow{PA}=\mathbf{v}$ is,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left\langle \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

So the rate of change of temperature T at the point $P(2,-1,2)$ in the direction of **u** is,

$$D_{\mathbf{u}} T(2,-1,2) = \nabla T(2,-1,2) \cdot \mathbf{u}$$

$$= -400e^{-43} < \langle 2,-3,18 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$= -400e^{-43} \left(\frac{2}{\sqrt{6}} + \frac{6}{\sqrt{6}} + \frac{18}{\sqrt{6}} \right)$$

$$= -400e^{-43} \left(\frac{26}{\sqrt{6}} \right)$$

$$= \boxed{\frac{-10400e^{-43}}{\sqrt{6}}}$$

(b) To find that in which direction the temperature increase fastest at P, we use the following result.

Suppose T is differentiable function of two or three variables. Then the maximum value of the directional derivative $D_{\mathbf{u}}(\mathbf{x})$ is $|\nabla T(\mathbf{x})|$ and it occurs when **u** has the same direction as the gradient vector $\nabla T(\mathbf{x})$.

So the temperature T increase fastest in the direction of $\nabla T(2,-1,2)$ that is in the direction of vector (by part (a))

$$-400e^{-43} \langle 2,-3,18 \rangle$$

Or in the direction of vector $\langle 2,-3,18 \rangle$

Answer 33E.

$$V(x,y,z) = 5x^2 - 3xy + xyz$$

Then
$$V_x = 10x - 3y + yz$$
$$V_y = -3x + xz$$
$$V_z = xy$$

(A)

The gradient of V is

$$\vec{\nabla} V(x,y,z) = < V_x, V_y, V_z >$$
$$= < 10x - 3y + yz, -3x + xz, xy >$$

At P(3, 4, 5)

$$\vec{\nabla} V(3,4,5) = < 30 - 12 + 20, -9 + 15, 12 >$$
$$= < 38, 6, 12 >$$

We need to find the rate of change of V at P in the direction of vector

$$\vec{v} = \hat{i} + \hat{j} - \hat{k}$$

i.e. $\vec{v} = < 1, 1, -1 >$

It is not a unit vector. Now $|\vec{v}| = \sqrt{3}$

Then the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = < \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} >$$

The rate of change of V in the direction of $\vec{u}$ is

$$D_u V(3,4,5) = \vec{\nabla} V(3,4,5) \vec{u}$$
$$= < 38, 6, 12 > . < \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} >$$
$$= \frac{38}{\sqrt{3}} + \frac{6}{\sqrt{3}} - \frac{12}{\sqrt{3}} = \frac{32}{\sqrt{3}}$$

(B)

By theorem 15, we know the maximum increase of vat P takes place in the direction of $\vec{\nabla} v(P)$

That is in the direction of vector $\nabla v(3,4,5)$

That is in the direction of vector $< 38, 6, 12 >$

(C)

The maximum increase in the value of V at P is

$$|\vec{\nabla} V(3,4,5)| = \sqrt{(38)^2 + (6)^2 + (12)^2}$$
$$= \sqrt{1624}$$
$$= 2\sqrt{406}$$

Answer 34E.

(a) The directional derivative of f at the point $\mathbf{x}$ in the direction of the unit vector $\mathbf{u}$ is

$$D_u f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$$

where $\nabla f(\mathbf{x})$ is the gradient vector for f at the point $\mathbf{x}$. The directional derivative represents the rate of change of the function f at the point $\mathbf{x}$. Since the given function z measures height, the directional derivative tells us the change in height along any trajectory $\mathbf{u}$.

Due south is along the negative y-axis. A unit vector in this direction is $\mathbf{u} = \langle 0, -1 \rangle$.

The gradient's components are found from the partial derivatives of z. First we find z_x:

$$\begin{aligned} z_x &= \frac{\partial}{\partial x}\left(1000 - 0.005x^2 - 0.01y^2\right) \\ &= -0.005(2x) \\ &= -0.01x \end{aligned}$$

We evaluate at $(60, 40)$ to find the first component of the gradient vector:

$$\begin{aligned} z_x(60,40) &= -0.01(60) \\ &= -0.6 \end{aligned}$$

Now we find z_y:

$$\begin{aligned} z_y &= \frac{\partial}{\partial y}\left(1000 - 0.005x^2 - 0.01y^2\right) \\ &= -0.01(2y) \\ &= -0.02y \end{aligned}$$

We evaluate at $(60, 40)$ to find the second component of the gradient vector:

$$\begin{aligned} z_y(60,40) &= -0.02(40) \\ &= -0.8 \end{aligned}$$

Hence the gradient vector at $(60, 40)$ is $\nabla z(60,40) = \langle -0.6, -0.8 \rangle$. Therefore, the directional derivative of z at $(60, 40)$ heading in the direction $\langle 0, -1 \rangle$ is:

$$\begin{aligned} D_u z(60,40) &= \nabla z(60,40) \cdot \mathbf{u} \\ &= \langle -0.6, -0.8 \rangle \cdot \langle 0, -1 \rangle \\ &= -0.6(0) - 0.8(-1) \\ &= 0.8 \end{aligned}$$

Since the directional derivative is positive, it indicates we would be ascending. The change in height of the hill when heading due south is $\boxed{0.8 \text{ meters per meter}}$.

We evaluate at $(60,40)$ to find the second component of the gradient vector:

$$\begin{aligned} z_y(60,40) &= -0.02(40) \\ &= -0.8 \end{aligned}$$

Hence the gradient vector at $(60,40)$ is $\nabla z(60,40) = \langle -0.6, -0.8 \rangle$. Therefore, the directional derivative of *z* at $(60,40)$ heading in the direction $\langle 0, -1 \rangle$ is:

$$\begin{aligned} D_u z(60,40) &= \nabla z(60,40) \cdot \mathbf{u} \\ &= \langle -0.6, -0.8 \rangle \cdot \langle 0, -1 \rangle \\ &= -0.6(0) - 0.8(-1) \\ &= 0.8 \end{aligned}$$

Since the directional derivative is positive, it indicates we would be ascending. The change in height of the hill when heading due south is $\boxed{0.8 \text{ meters per meter}}$.

The directional derivative of *z* at $(60,40)$ heading in the direction $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ is:

$$\begin{aligned} D_u z(60,40) &= \nabla z(60,40) \cdot \mathbf{u} \\ &= \langle -0.6, -0.8 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{0.6}{\sqrt{2}} - \frac{0.8}{\sqrt{2}} \\ &= -\frac{0.2}{\sqrt{2}} \end{aligned}$$

$$= -\frac{1}{5\sqrt{2}}$$

Since the directional derivative is negative, it indicates we would be descending while heading northwest. The change in height of the hill when heading on this trajectory is approximately $\boxed{-0.14 \text{ meters per meter}}$.

(c) The slope is largest in the direction of the gradient vector $\nabla z(60,40) = \boxed{\langle -0.6, -0.8 \rangle}$. The magnitude of the gradient vector, and hence the maximum rate of change of *z* at $(60,40)$ is:

$$\begin{aligned} |\nabla z(60,40)| &= \sqrt{(-0.6)^2 + (-0.8)^2} \\ &= \sqrt{0.36 + 0.64} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Hence in the direction of the gradient, the rate of ascent is $\boxed{1 \text{ meter per meter}}$.

We want to determine the angle which the gradient vector makes with the positive *x*-axis. The reference angle θ for a vector $\langle x, y\rangle$ is found by evaluating $\theta = \tan^{-1}(y/x)$. Hence,

$$\begin{aligned}\theta &= \tan^{-1}(y/x)\\ &= \tan^{-1}(-0.8/-0.6)\\ &\approx 53.1^\circ\end{aligned}$$

The reference angle lies in quadrant 1. However, the vector $\langle -0.6, -0.8\rangle$ lies in quadrant 3. Hence the direction of the gradient is $180^\circ + 53.1^\circ = \boxed{233.1^\circ}$.

Answer 35E.

The given points are

$$A(1,3), B(3,3), C(1,7), D(6,15)$$

Then $\overrightarrow{AB} = <2,0>$ and $\left|\overrightarrow{AB}\right| = 2$

$\overrightarrow{AC} = <0,4>$ And $\left|\overrightarrow{AC}\right| = 4$

$\overrightarrow{AD} = <5,12>$ And $\left|\overrightarrow{AD}\right| = 13$

Then the unit vector in the direction of $\overrightarrow{AB}$ is

$$\overrightarrow{u_1} = <1,0>$$

In the direction of $\overrightarrow{AC}$ is

$$\overrightarrow{u_2} = <0,1>$$

In the direction of $\overrightarrow{AD}$ is: $\overrightarrow{u_3} = <\frac{5}{13}, \frac{12}{13}>$

Therefore $\vec{\nabla} f(1,3).\overrightarrow{u_1} = 3$

i.e. $<f_x(1,3), f_y(1,3)>. <1,0> = 3$

i.e. $f_x(1,3) = 3$

And $\vec{\nabla} f(1,3).\overrightarrow{u_2} = 26$

i.e. $<f_x(1,3), f_y(1,3)>. <0,1> = 26$

i.e. $f_y(1,3) = 26$

Therefore $\vec{\nabla} f(1,3) = < f_x(1,3), f_y(1,3) > = < 3, 26 >$

Hence the directional derivative of f at a in the direction of $\overrightarrow{AD}$ is

$$\vec{\nabla} f(1,3) \vec{u_3}$$

$$= < 3, 26 > . < \frac{5}{13}, \frac{12}{13} >$$

$$= \frac{15}{13} + \frac{312}{13}$$

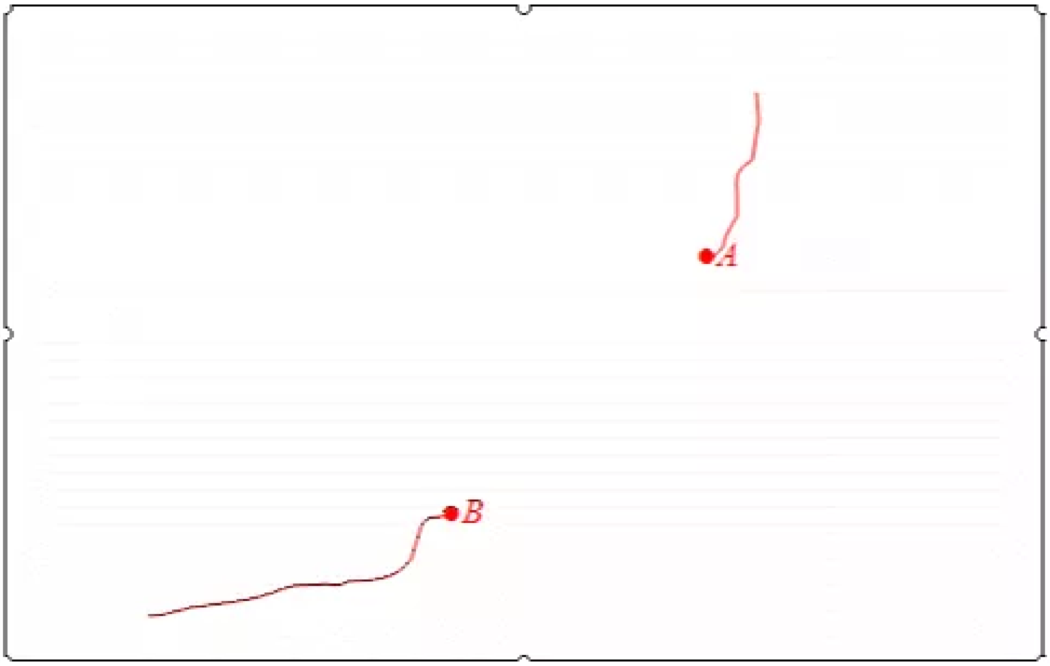
$$= \boxed{\frac{327}{13}}$$

Answer 36E.

The curves of steepest descent will descend through the contours most rapidly. The more rapidly the path goes from one contour to the next, the more rapidly it is dropping elevation, as each contour represents a constant change in elevation. Therefore, at any point the direction of steepest descent is the direction in which the next-lowest contour line is closest.

The closest path of approach from a point to a curve is a normal line from the point to the curve, so as we make each "step" downward, we will approach the following contour as perpendicularly as possible, looking for the closest line of approach. This will give the steepest path.

The curves of steepest descent on the map from A and B will look something like this:



The curve of steepest descent from A starts out by meeting the closest contour at the closest possible point and after that meets every subsequent contour perpendicularly. It trends down towards the lake. The curve of steepest descent from B also starts out towards the closest contour, to the left of B. It then meets every subsequent contour perpendicularly as it trends down towards the river.

Answer 37E.

(a)

Use the definition of gradient and compare each component.

$$\begin{aligned}\nabla(au+bv) &= (au_x+bv_x)\mathbf{i}+(au_y+bv_y)\mathbf{j}\\ &= au_x\mathbf{i}+bv_x\mathbf{i}+au_y\mathbf{j}+bv_y\mathbf{j}\end{aligned}$$

and

$$\begin{aligned}a\nabla u+b\nabla v &= a(u_x\mathbf{i}+u_y\mathbf{j})+b(v_x\mathbf{i}+v_y\mathbf{j})\\ &= au_x\mathbf{i}+au_y\mathbf{j}+bv_x\mathbf{i}+bv_y\mathbf{j}\end{aligned}$$

Clearly, each component is the same, and thus the gradient property is confirmed.

(b)

Use the definition of gradient and compare each component.

$$\begin{aligned}\nabla(uv) &= (uv)_x\mathbf{i}+(uv)_y\mathbf{j}\\ &= (uv_x+u_xv)\mathbf{i}+(uv_y+u_yv)\mathbf{j} && \text{Use the product rule each time.}\end{aligned}$$

$$\begin{aligned}u\nabla v+v\nabla u &= u(v_x\vec{i}+v_y\vec{j})+v(u_x\vec{i}+u_y\vec{j})\\ &= uv_x\vec{i}+uv_y\vec{j}+vu_x\vec{i}+vu_y\vec{j} && \text{Distribute.}\\ &= (uv_x+u_xv)\vec{i}+(uv_y+u_yv)\vec{j} && \text{Collect like terms.}\end{aligned}$$

Clearly, each component is the same, and thus the gradient property is confirmed.

(c)

Use the definition of gradient and compare each component.

$$\begin{aligned}\nabla\left(\frac{u}{v}\right) &= \left(\frac{u}{v}\right)_x\mathbf{i}+\left(\frac{u}{v}\right)_y\mathbf{j}\\ &= \left(\frac{vu_x-v_xu}{v^2}\right)\mathbf{i}+\left(\frac{vu_y-v_yu}{v^2}\right)\mathbf{j} && \text{Use the quotient rule each time.}\end{aligned}$$

$$\begin{aligned}\frac{v\nabla u-u\nabla v}{v^2} &= \frac{v(u_x\mathbf{i}+u_y\mathbf{j})-u(v_x\mathbf{i}+v_y\mathbf{j})}{v^2}\\ &= \frac{vu_x\mathbf{i}+vu_y\mathbf{j}-uv_x\mathbf{i}-uv_y\mathbf{j}}{v^2} && \text{Distribute through in the numerator.}\\ &= \frac{(vu_x-uv_x)\mathbf{i}+(vu_y-uv_y)\mathbf{j}}{v^2} && \text{Collect like components.}\end{aligned}$$

Clearly, each component is the same (break up the fraction in the second set of equations). Thus, the gradient property is confirmed.

(d)

Use the definition of gradient and regroup the components.

$$\nabla u^n = (u^n)_x \mathbf{i} + (u^n)_y \mathbf{j}$$

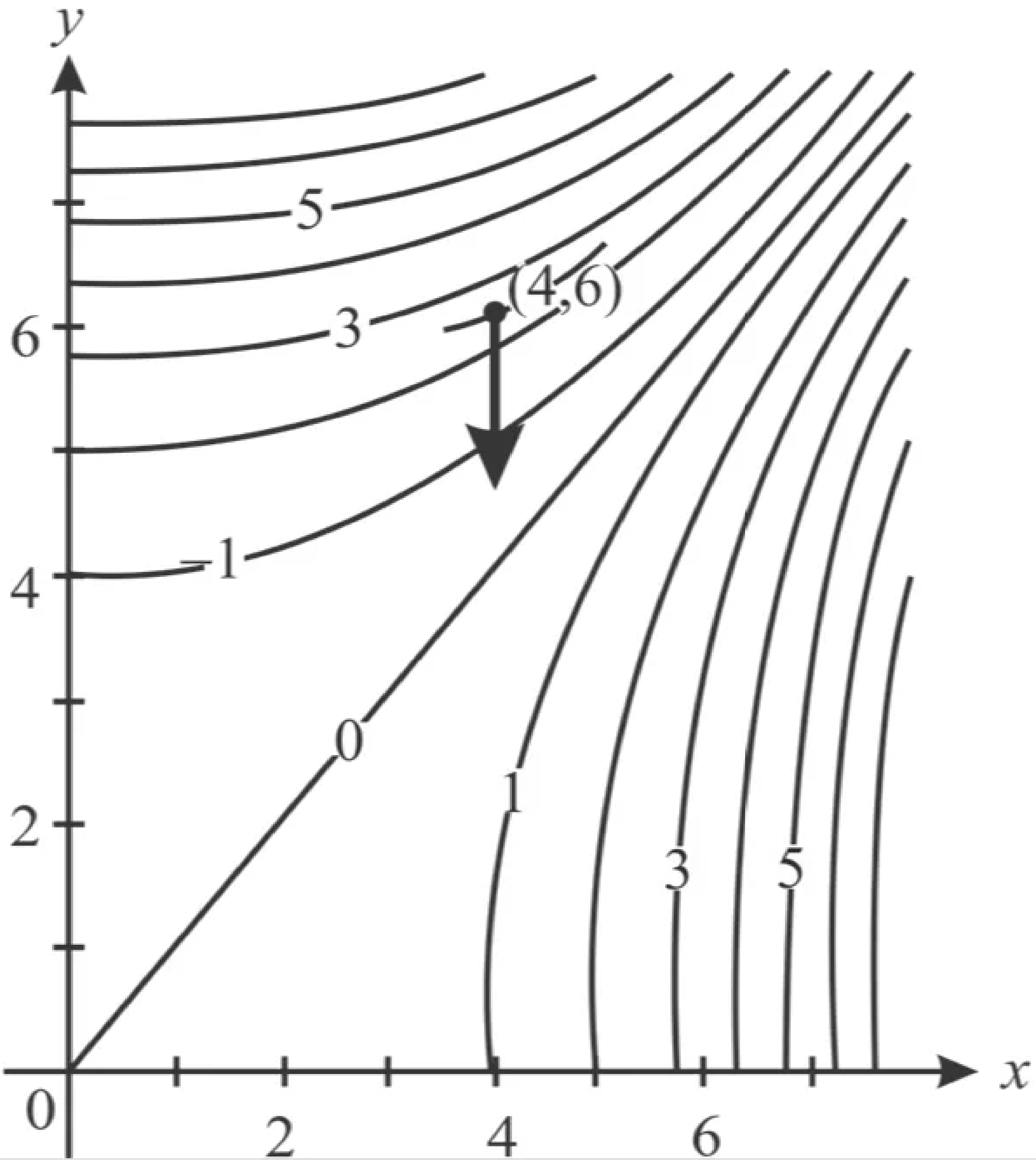
$$= (nu^{n-1}u_x)\mathbf{i} + (nu^{n-1}u_y)\mathbf{j}$$ Use the Chain Rule to differentiate each component.

$$= nu^{n-1}[u_x\mathbf{i} + u_y\mathbf{j}]$$ Factor out the common term.

$$= nu^{n-1}\nabla u$$ Use the definition of the gradient.

Answer 38E.

Consider the following graph which shows the level curves of the function f:

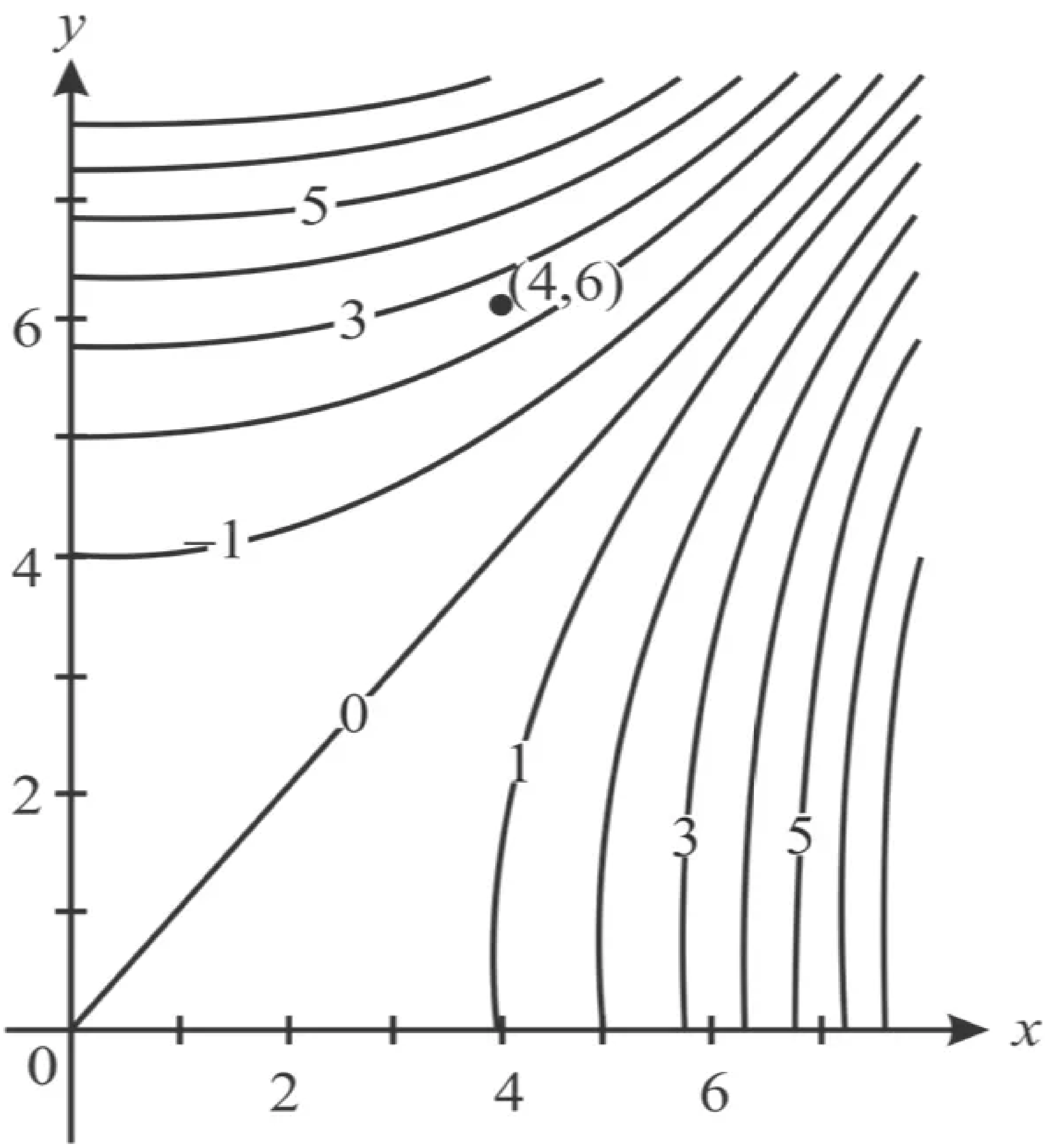


To sketch the gradient vector $\vec{\nabla} f(4,6)$ for the function, note that if the initial point of the gradient vector $\vec{\nabla} f(4,6)$ is placed at $(4,6)$, the vector is perpendicular to that level curve of f which includes the point $(4,6)$. So, first sketch a portion of the level curve through $(4,6)$, (use the nearby level curves as a guide line), and then draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, and will point in the **direction of increasing function** values also the length of the vector is equal to the maximum value of the directional derivative of f at $(4,6)$.

Now, to estimate the length of the vector, find the average rate of change in the direction of the gradient. Note that the line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus, the rate of change is approximately:

$$\frac{-2-(-3)}{0.5}=2$$

So, the gradient vector is of **length 2**.



Answer 39E.

Consider the function:

$$f(x,y)=x^3+5x^2y+y^3$$

Determine the expression for $f_x(x,y)$ and $f_y(x,y)$:

$$\begin{aligned}f_x(x,y)&=\frac{\partial}{\partial x}\left(x^3+5x^2y+y^3\right)\\&=3x^2+10xy\\f_y(x,y)&=\frac{\partial}{\partial y}\left(x^3+5x^2y+y^3\right)\\&=5x^2+3y^2\end{aligned}$$

Substitute the known values in $D_{\mathbf{u}}f(x,y)=f_x(x,y)a+f_y(x,y)b$:

$$\begin{aligned}D_{\mathbf{u}}f(x,y)&=\left(3x^2+10xy\right)a+\left(5x^2+3y^2\right)b\\&=3ax^2+10axy+5bx^2+3by^2\end{aligned}$$

Determine the partial derivative of $D_{\mathbf{u}}f(x,y)$ with respect to x and y:

$$\frac{\partial\left(D_{\mathbf{u}}f(x,y)\right)}{\partial x}=6ax+10ay+10bx$$

$$\frac{\partial\left(D_{\mathbf{u}}f(x,y)\right)}{\partial y}=10ax+6by$$

Consider the equation shown below:

$$D_{\mathbf{u}}\left[D_{\mathbf{u}}f(x,y)\right]=(6ax+10ay+10bx)a+(10ax+6by)b$$

Replace the variable 'a' with $\frac{3}{5}$, b with $\frac{4}{5}$, x with 2, and y with 1:

$$\begin{aligned}D_{\mathbf{u}}\left[D_{\mathbf{u}}f(2,1)\right]&=\left[6\left(\frac{3}{5}\right)(2)+10\left(\frac{3}{5}\right)(1)+10\left(\frac{4}{5}\right)(2)\right]\left(\frac{3}{5}\right)\\&\quad+\left[10\left(\frac{3}{5}\right)(2)+6\left(\frac{4}{5}\right)(1)\right]\left(\frac{4}{5}\right)\\D_{\mathbf{u}}^2f(2,1)&=\left(\frac{36}{5}+6+16\right)\left(\frac{3}{5}\right)+\left(12+\frac{24}{5}\right)\left(\frac{4}{5}\right)\\&=\frac{774}{25}\\&=30.96\end{aligned}$$

So, the final value is $\boxed{D_{\mathbf{u}}^2f(2,1)=30.96}$.

Answer 40E.

(a)We know that $D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$, where $\mathbf{u} = \langle a, b\rangle$.

Find $D_{\mathbf{u}}^2 f(x, y)$.

$$\begin{aligned} D_{\mathbf{u}}^2 f(x, y) &= a\left[f_{xx}(x, y)a + f_{xy}(x, y)b\right] + \\ &\quad b\left[f_{yx}(x, y)a + f_{yy}(x, y)b\right] \\ &= f_{xx}a^2 + 2f_{xy}(x, y)ab + f_{yy}(x, y)b^2 \end{aligned}$$

Thus, we get $D_{\mathbf{u}}^2 f(x, y) = f_{xx}a^2 + 2f_{xy}(x, y)ab + f_{yy}(x, y)b^2$.

(b)We have $f(x,y) = xe^{2y}$. Then, $f_x = e^{2y}$,

$$\begin{aligned} f_{xx} &= 0, \\ f_{xy} &= 2e^{2y}, \\ f_y &= 2xe^{2y}, \\ f_{yy} &= 4xe^{2y}. \end{aligned}$$

Then, $D_{\mathbf{u}}^2 f(x, y) = (0)a^2 + 4e^{2y}ab + 4xe^{2y}b^2$.

Since $\mathbf{v} = \langle 4, 6\rangle$, the unit vector in the direction of $\mathbf{v}$ is $\left\langle \frac{4}{\sqrt{52}}, \frac{6}{\sqrt{52}} \right\rangle$

Replace a with $\frac{4}{\sqrt{52}}$ and b with $\frac{6}{\sqrt{52}}$.

$$\begin{aligned} D_{\mathbf{u}}^2 f\left(\frac{4}{\sqrt{52}}, \frac{6}{\sqrt{52}}\right) &= 4e^{2y}\left(\frac{24}{52}\right) + 4xe^{2y}\left(\frac{36}{52}\right) \\ &= e^{2y}\left(\frac{24}{13} + \frac{36}{13}x\right) \end{aligned}$$

Thus, $\boxed{D_{\mathbf{u}}^2 f\left(\frac{4}{\sqrt{52}}, \frac{6}{\sqrt{52}}\right) = e^{2y}\left(\frac{24}{13} + \frac{36}{13}x\right)}$.

Answer 41E.

Given surface is $2(x-2)^2+(y-1)^2+(z-3)^2=10$

The sphere is the level surface (with $k=10$) of the function

$$F(x, y, z)= 2(x-2)^2+(y-1)^2+(z-3)^2$$

Therefore we have

$$F_x(x, y, z)= 4(x-2)\; F_y = 2(y-1)\; F_z = 2(z-3)$$

$$F_x(3, 3, 5)=4\; F_y(3, 3, 5)=4\; F_z(3, 3, 5)=4$$

(a.)Using the equation of a tangent plane

$$[F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(z-z_0)]$$

gives the equation of the tangent plane at $(3, 3, 5)$ as

$$4(x-3)+4(y-3)+4(z-5)=0$$

which simplifies (when dividing both sides by 4) to $x-3+y-3+z-5=0$

$$=x+y+z=11$$

(b.) Using the equation of the symmetric equations of a normal line

$$\left[\frac{x-x_0}{F_x(x_0, y_0, z_0)}=\frac{y-y_0}{F_y(x_0, y_0, z_0)}=\frac{z-z_0}{F_z(x_0, y_0, z_0}\right]$$

gives the symmetric equations of the normal line are

$$x-3=y-3=z-5$$

Answer 42E.

a)

Consider the following equation of a surface.

$$y = x^2 - z^2$$

The objective is to find the equation of tangent plane to the surface at the point $(4,7,3)$.

Rewrite the equation in the form $F(x,y,z) = k$ where k is a constant:

$$\begin{aligned} y &= x^2 - z^2 \\ -x^2 + y + z^2 &= 0 \end{aligned}$$

The equation of the tangent plane to the surface F at (x_0, y_0, z_0) is:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

Find the partial derivatives of F at $(4,7,3)$.

First find F_x:

$$\begin{aligned} F_x &= \frac{\partial}{\partial x}\left(-x^2 + y + z^2\right) \\ &= -2x + 0 + 0 \\ &= -2x \end{aligned}$$

Evaluate the partial derivative at $(4,7,3)$:

$$\begin{aligned} F_x(4,7,3) &= -2(4) \\ &= -8 \end{aligned}$$

Find F_y:

$$\begin{aligned} F_y &= \frac{\partial}{\partial y}\left(-x^2 + y + z^2\right) \\ &= 0 + 1 + 0 \\ &= 1 \end{aligned}$$

Hence, $F_y(4,7,3) = 1$.

Find F_z:

$$\begin{aligned} F_z &= \frac{\partial}{\partial z}\left(-x^2 + y + z^2\right) \\ &= 0 + 0 + 2z \\ &= 2z \end{aligned}$$

Evaluate the partial derivative at $(4,7,3)$:

$$\begin{aligned} F_z(4,7,3) &= 2(3) \\ &= 6 \end{aligned}$$

Answer 43E.

(a) The standard equation of the tangent plane to the level surface $F(x, y, z)$ at $P(x_0, y_0, z_0)$ is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

We have $F(x, y, z) = xyz^2 - 6$.

Find $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$.

$$\begin{aligned} F_x(x, y, z) &= \frac{\partial}{\partial x}(xyz^2 - 6) \\ &= yz^2 \end{aligned}$$

$$\begin{aligned} F_y(x, y, z) &= \frac{\partial}{\partial y}(xyz^2 - 6) \\ &= xz^2 \end{aligned}$$

$$\begin{aligned} F_z(x, y, z) &= \frac{\partial}{\partial z}(xyz^2 - 6) \\ &= 2xyz \end{aligned}$$

Replace x with 3, y with 2, and z with 1 to find $F_x(3, 2, 1)$, $F_y(3, 2, 1)$, and $F_z(3, 2, 1)$.

$$\begin{aligned} F_x(3, 2, 1) &= (2)(1)^2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} F_y(3, 2, 1) &= (3)(1)^2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} F_z(3, 2, 1) &= 2(3)(2)(1) \\ &= 12 \end{aligned}$$

Substitute the known values in the standard equation of the tangent plane.

$$\begin{aligned} (2)(x - 3) + (3)(y - 2) + (12)(z - 1) &= 0 \\ 2x - 6 + 3y - 6 + 12z - 12 &= 0 \\ 2x + 3y + 12z - 24 &= 0 \end{aligned}$$

Thus, The equation of the tangent plane is obtained as $\boxed{2x + 3y + 12z - 24 = 0}$.

(b) The normal line to the tangent plane is given by the equation $\frac{(x-x_0)}{F_x(x_0,y_0,z_0)}=\frac{(y-y_0)}{F_y(x_0,y_0,z_0)}=\frac{(z-z_0)}{F_z(x_0,y_0,z_0)}$.

On replacing with the known values, we get $\frac{(x-3)}{2}=\frac{(y-2)}{3}=\frac{(z-1)}{12}$.

Thus, the equation of the normal line is obtained as $\boxed{\frac{x-3}{2}=\frac{y-2}{3}=\frac{z-1}{12}}$.

Answer 44E.

(a) The standard equation of the tangent plane to the level surface $F(x, y, z)$ at $P(x_0, y_0, z_0)$ is given by

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.$$

We have $F(x, y, z) = xy + yz + zx - 5$.

Find $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$.

$$\begin{aligned} F_x(x,y,z) &= \frac{\partial}{\partial x}(xy+yz+zx-5) \\ &= y+z \end{aligned}$$

$$\begin{aligned} F_y(x,y,z) &= \frac{\partial}{\partial y}(xy+yz+zx-5) \\ &= x+z \end{aligned}$$

$$\begin{aligned} F_z(x,y,z) &= \frac{\partial}{\partial z}(xy+yz+zx-5) \\ &= x+y \end{aligned}$$

Replace x with 1, y with 2, and z with 1 to find $F_x(1, 2, 1)$, $F_y(1, 2, 1)$, and $F_z(1, 2, 1)$.

$$\begin{aligned} F_x(1,2,1) &= 2+1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} F_y(1,2,1) &= 1+1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} F_z(1,2,1) &= 1+2 \\ &= 3 \end{aligned}$$

Substitute the known values in the standard equation of the tangent plane.

$$\begin{aligned}(3)(x-1)+(2)(y-2)+(3)(z-1) &= 0\\ 3x-3+2y-4+3z-3 &= 0\\ 3x+2y+3z-10 &= 0\end{aligned}$$

Thus, the equation of the tangent plane is obtained as $\boxed{3x+2y+3z-10=0}$.

Answer 45E.

(a) The standard equation of the tangent plane to the level surface $F(x, y, z)$ at $P(x_0, y_0, z_0)$ is given by

$$F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(z-z_0) = 0.$$

We have $F(x, y, z) = x + y + z - e^{xyz}$.

Find $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$.

$$\begin{aligned}F_x(x, y, z) &= \frac{\partial}{\partial x}\left(x+y+z-e^{xyz}\right)\\ &= 1-yze^{xyz}\end{aligned}$$

$$\begin{aligned}F_y(x, y, z) &= \frac{\partial}{\partial y}\left(x+y+z-e^{xyz}\right)\\ &= 1-xze^{xyz}\\ F_z(x, y, z) &= \frac{\partial}{\partial z}\left(x+y+z-e^{xyz}\right)\\ &= 1-xye^{xyz}\end{aligned}$$

Replace x with 0, y with 0, and z with 1 to find $F_x(0, 0, 1)$, $F_y(0, 0, 1)$, and $F_z(0, 0, 1)$.

$$\begin{aligned}F_x(0, 0, 1) &= 1-(0)(1)e^{(0)(0)(1)}\\ &= 1\\ F_y(0, 0, 1) &= 1-(0)(1)e^{(0)(0)(1)}\\ &= 1\\ F_z(0, 0, 1) &= 1-(0)(0)e^{(0)(0)(1)}\\ &= 1\end{aligned}$$

Substitute the known values in the standard equation of the tangent plane.

$$\begin{aligned}(1)(x-0)+(1)(y-0)+(1)(z-1) &= 0\\ x+y+z-1 &= 0\end{aligned}$$

Thus, the equation of the tangent plane is obtained as $\boxed{x+y+z=1}$.

(b) The normal line to the tangent plane is given by the equation

$$\frac{(x-x_0)}{F_x(x_0, y_0, z_0)}=\frac{(y-y_0)}{F_y(x_0, y_0, z_0)}=\frac{(z-z_0)}{F_z(x_0, y_0, z_0)}.$$

On replacing with the known values, we get $\frac{(x-0)}{1}=\frac{(y-0)}{1}=\frac{(z-1)}{1}$.

Thus, the equation of the normal line is obtained as $\boxed{x=y=z-1}$.

Answer 46E.

(a) The standard equation of the tangent plane to the level surface $F(x, y, z)$ at $P(x_0, y_0, z_0)$ is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

We have $F(x, y, z) = x^4 + y^4 + z^4 - 3x^2y^2z^2$.

Find $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$.

$$\begin{aligned} F_x(x, y, z) &= \frac{\partial}{\partial x}\left(x^4 + y^4 + z^4 - 3x^2y^2z^2\right) \\ &= 4x^3 - 6xy^2z^2 \end{aligned}$$

$$\begin{aligned} F_y(x, y, z) &= \frac{\partial}{\partial y}\left(x^4 + y^4 + z^4 - 3x^2y^2z^2\right) \\ &= 4y^3 - 6x^2yz^2 \end{aligned}$$

$$\begin{aligned} F_z(x, y, z) &= \frac{\partial}{\partial z}\left(x^4 + y^4 + z^4 - 3x^2y^2z^2\right) \\ &= 4z^3 - 6x^2y^2z \end{aligned}$$

Replace x with 1, y with 1, and z with 1 to find $F_x(1, 1, 1)$, $F_y(1, 1, 1)$, and $F_z(1, 1, 1)$.

$$\begin{aligned} F_x(1, 1, 1) &= 4(1)^3 - 6(1)(1)^2(1)^2 \\ &= -2 \end{aligned}$$

$$\begin{aligned} F_y(1, 1, 1) &= 4(1)^3 - 6(1)^2(1)(1)^2 \\ &= -2 \end{aligned}$$

$$\begin{aligned} F_z(1, 1, 1) &= 4(1)^3 - 6(1)^2(1)^2(1) \\ &= -2 \end{aligned}$$

Substitute the known values in the standard equation of the tangent plane.

$$\begin{aligned} (-2)(x - 1) + (-2)(y - 1) + (-2)(z - 1) &= 0 \\ -2x + 2 - 2y + 2 - 2z + 2 &= 0 \\ -x + 1 - y + 1 - z + 1 &= 0 \\ x + y + z - 3 &= 0 \end{aligned}$$

Thus, The equation of the tangent plane is obtained as $\boxed{x + y + z = 3}$.

(b) The normal line to the tangent plane is given by the equation $\frac{(x-x_0)}{F_x(x_0,y_0,z_0)} = \frac{(y-y_0)}{F_y(x_0,y_0,z_0)} = \frac{(z-z_0)}{F_z(x_0,y_0,z_0)}$.

On replacing with the known values, we get $\frac{(x-1)}{-2} = \frac{(y-1)}{-2} = \frac{(z-1)}{-2}$

or $\frac{(x-1)}{1} = \frac{(y-1)}{1} = \frac{(z-1)}{1}$.

Thus, the equation of the normal line is obtained as $\boxed{x-1=y-1=z-1}$.

Answer 47E.

Consider the equation $xy+yz+zx=3$

Rewrite the equation in the form $z=f(x,y)$:

$$\begin{aligned} xy+yz+zx&=3 \\ yz+zx&=3-xy \\ z(y+x)&=3-xy \\ z&=\frac{3-xy}{y+x} \end{aligned}$$

Use MATLAB to plot the function.

First create a meshgrid for the domain which must include the point $(1,1,1)$.

Choose the domain $0.1 \le x \le 2$ and $0.1 \le y \le 2$.

We are careful not to choose $(x,y)=(0,0)$ where the equation for z is undefined. The meshgrid is like a wireframe surface where intersection points are places where the function z will be evaluated.

The command for the meshgrid is

```
[X,Y]=meshgrid(0.1:0.1:2);
```

This meshgrid spaces each of the points by 0.1 units.

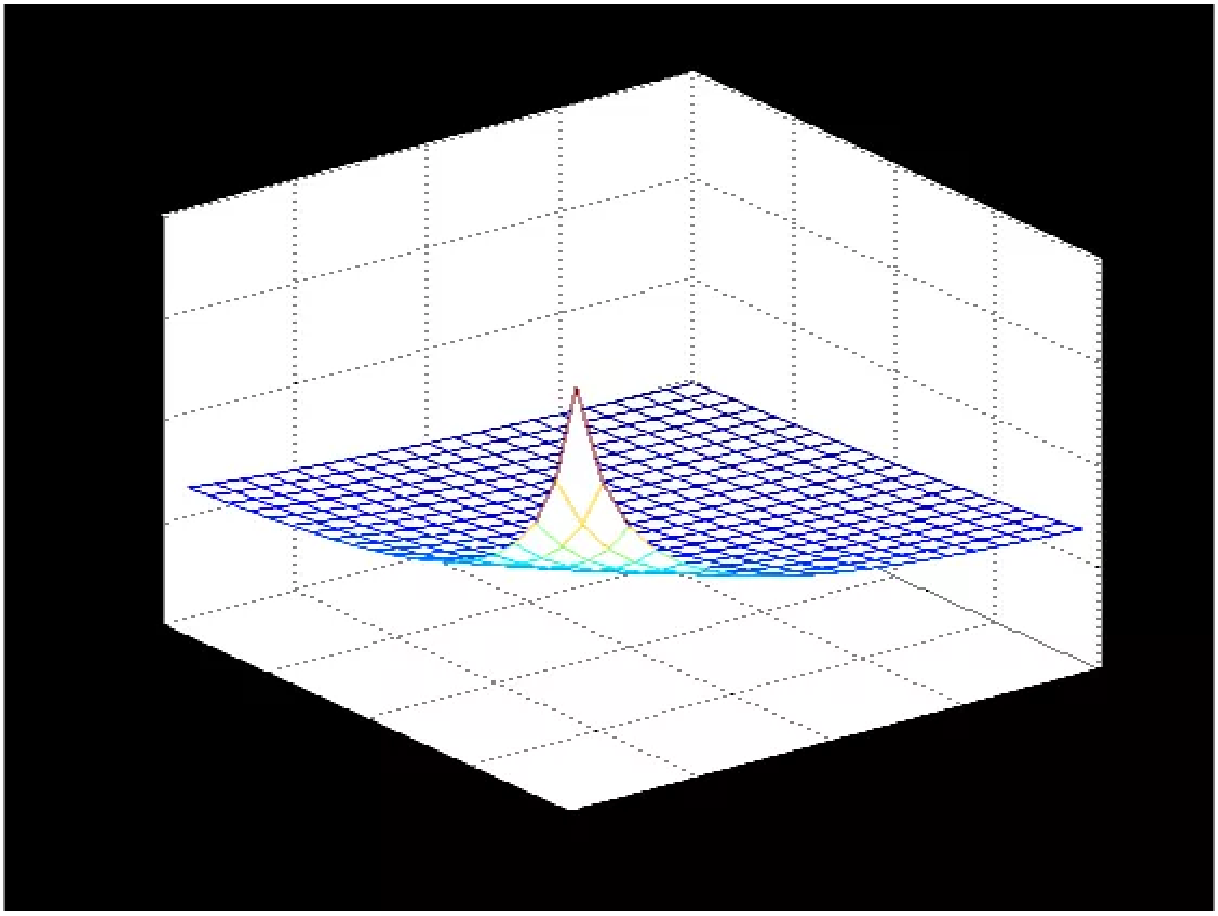
The next line of code defines the function z at a point (x,y):

```
Z=(3-X*Y)./(Y+X);
```

The next line uses the mesh function to plot *z* versus the meshgrid:

mesh(X,Y,Z)

Here is the result:



We hold the plot on the graph with the code: hold on.

To find the equation of the tangent plane to the surface at $(1,1,1)$, use the formula,

We have a differentiable function $F(x,y,z)=k$ where *k* is a constant. So long as the gradient of *F* at the given point (x_0,y_0,z_0) is non-zero, an equation of the tangent plane to *F* at (x_0,y_0,z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

To find the partial derivatives of F at $(1,1,1)$, first find F_x:

$$\begin{aligned}F_x &= \frac{\partial}{\partial x}(xy + yz + zx) \\ &= (1)y + 0 + z(1) \\ &= y + z\end{aligned}$$

Evaluate the partial derivative of F_x at $(1,1,1)$:

$$\begin{aligned}F_x(1,1,1) &= 1 + 1 \\ &= 2\end{aligned}$$

Now find F_y:

$$\begin{aligned}F_y &= \frac{\partial}{\partial y}(xy + yz + zx) \\ &= x(1) + (1)z + 0 \\ &= x + z\end{aligned}$$

We evaluate the partial derivative of F_y at $(1,1,1)$:

$$\begin{aligned}F_y(1,1,1) &= 1 + 1 \\ &= 2\end{aligned}$$

Finally find F_z:

$$\begin{aligned}F_z &= \frac{\partial}{\partial z}(xy + yz + zx) \\ &= 0 + y(1) + (1)x \\ &= y + x\end{aligned}$$

Evaluate the partial derivative of F_z at $(1,1,1)$:

$$\begin{aligned}F_z(1,1,1) &= 1 + 1 \\ &= 2\end{aligned}$$

Therefore, the equation of the tangent plane to F at $(1,1,1)$ is:

$$\begin{aligned}F_x(1,1,1)(x-1) + F_y(1,1,1)(y-1) + F_z(1,1,1)(z-1) &= 0 \\ 2(x-1) + 2(y-1) + 2(z-1) &= 0 \\ x - 1 + y - 1 + z - 1 &= 0 \\ z &= 3 - x - y\end{aligned}$$

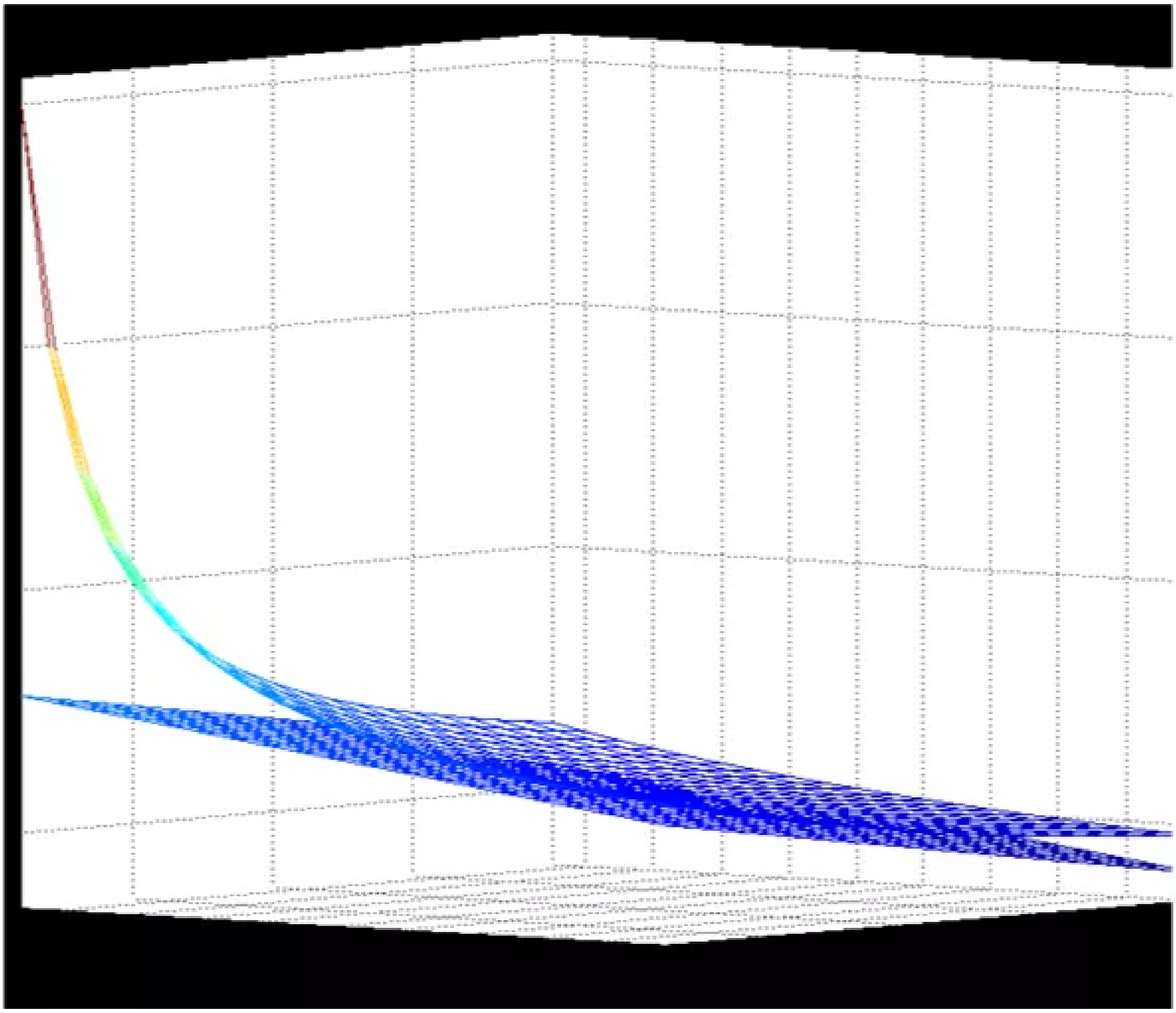
Define the function ztan for the tangent plane at a point (x, y):

ztan=3-X-Y

Plot this function over the meshgrid with the line of code:

mesh(X,Y,ztan)

We rotate the graph to view the surface and the plane:



The normal line to the surface defined by $F(x,y,z)=k$ at the given point (x_0,y_0,z_0) is given by the symmetric equations:

$$\frac{x-x_0}{F_x(x_0,y_0,z_o)}=\frac{y-y_0}{F_y(x_0,y_0,z_o)}=\frac{z-z_0}{F_z(x_0,y_0,z_o)}$$

Therefore, the symmetric equations of the normal line to the given surface defined by *F* at the given point $(1,1,1)$ are:

$$\frac{x-1}{F_x(1,1,1)}=\frac{y-1}{F_y(1,1,1)}=\frac{z-1}{F_z(1,1,1)}$$

$$\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-1}{2}$$

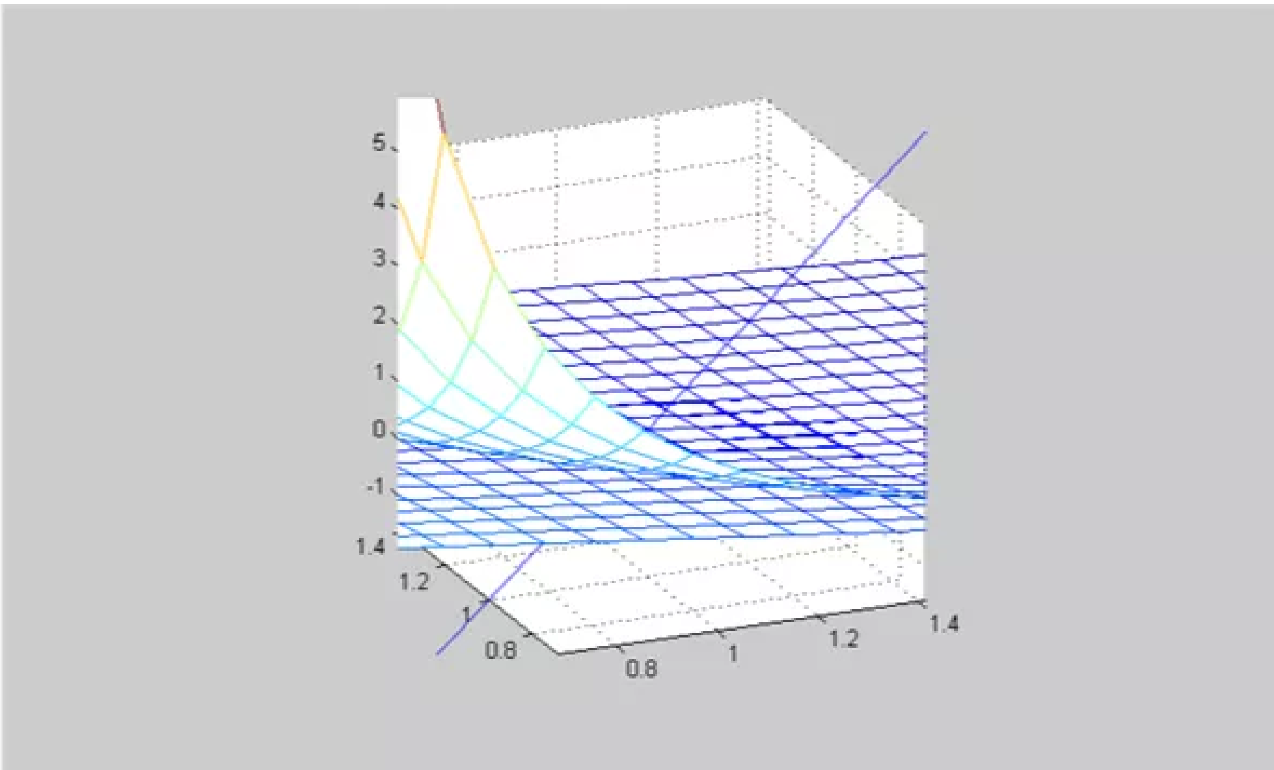
$$x-1=y-1=z-1$$

$$x=y=z$$

To plot the line, we first define the vector *x*: x=[0.1:0.1:2]. Now since $x=y$, we will also use this vector as *y* and *z* in the plot3 command. Plot3 will plot a 3 dimensional graph such as a line. The line of code to plot the line is:

plot3(x,x,x)

We display the results of plotting all three objects:



Answer 48E.

Consider the equation $xyz = 6$

Rewrite the equation is of the form $z = f(x, y)$:

$$xyz = 6$$
$$z = \frac{6}{xy}$$

Use MATLAB to plot the function. First we create a meshgrid for the domain which must include the point $(1,2,3)$.

Choose the domain $0.1 \le x \le 2$ and $1 \le y \le 3$. We are careful not to choose $x = 0$ where the equation for *z* is undefined.

The meshgrid is like a wireframe surface where intersection points are places where the function *z* will be evaluated.

The command for the meshgrid is:

```
[X,Y]=meshgrid(0.1:0.1:2, 1:0.1:3);
```

This meshgrid spaces each of the points by 0.1 units.

The next line of code defines the function *z* at a point (x, y):

```
Z=6./(X.*Y)
```

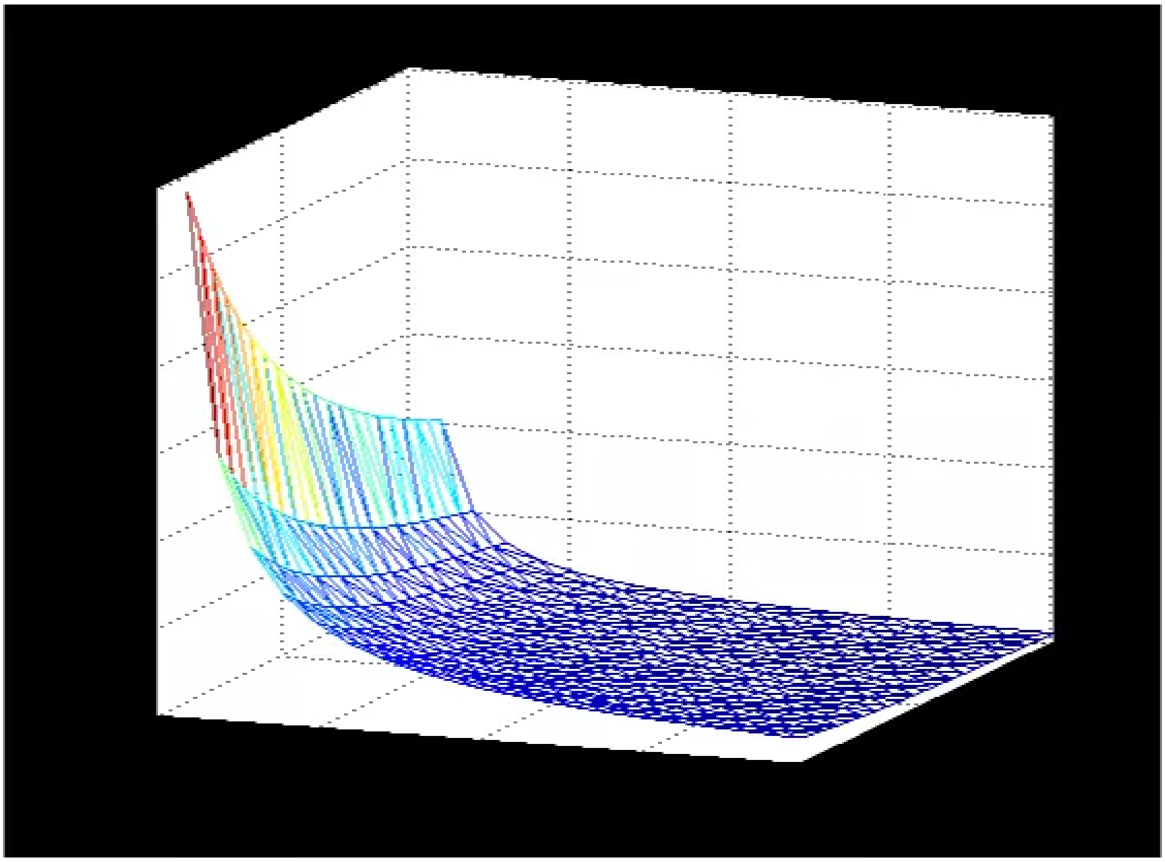
The next line uses the delaunay function to create a triangulation of the domain:

```
gridval=delaunay(X,Y)
```

Now we plot the function *z* versus the triangulated domain with the command:

```
trimesh(gridval,X,Y,Z)
```

Here is the result:



We hold the plot on the graph with the code: hold on.

Now we want the equation of the tangent plane to the surface at $(1,2,3)$. We have a differentiable function $F(x,y,z)=k$ where *k* is a constant. So long as the gradient of *F* at the given point (x_0,y_0,z_0) is non-zero, an equation of the tangent plane to *F* at (x_0,y_0,z_0) is:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

To find the partial derivatives of *F* at $(1,2,3)$. First we find F_x:

$$\begin{aligned} F_x &= \frac{\partial}{\partial x}(xyz) \\ &= yz \end{aligned}$$

Evaluate the partial derivative at $(1,2,3)$:

$$\begin{aligned} F_x(1,2,3) &= 2\cdot 3 \\ &= 6 \end{aligned}$$

To find F_y:

$$F_y = \frac{\partial}{\partial y}(xyz)$$
$$= xz$$

Evaluate the partial derivative at $(1,2,3)$:

$$F_y(1,2,3) = 1\cdot 3$$
$$= 3$$

To find F_z:

$$F_z = \frac{\partial}{\partial z}(xyz)$$
$$= xy$$

Evaluate the partial derivative at $(1,2,3)$:

$$F_z(1,2,3) = 1\cdot 2$$
$$= 2$$

Therefore, the equation of the tangent plane to *F* at $(1,2,3)$ is:

$$F_x(1,2,3)(x-1)+F_y(1,2,3)(y-2)+F_z(1,2,3)(z-3)=0$$
$$6(x-1)+3(y-2)+2(z-3)=0$$
$$6x-6+3y-6+2z-6=0$$
$$6x+3y+2z=18$$

Solving the equation for *z* we get:

$$2z = 18-6x-3y$$
$$z = 9-3x-1.5y$$

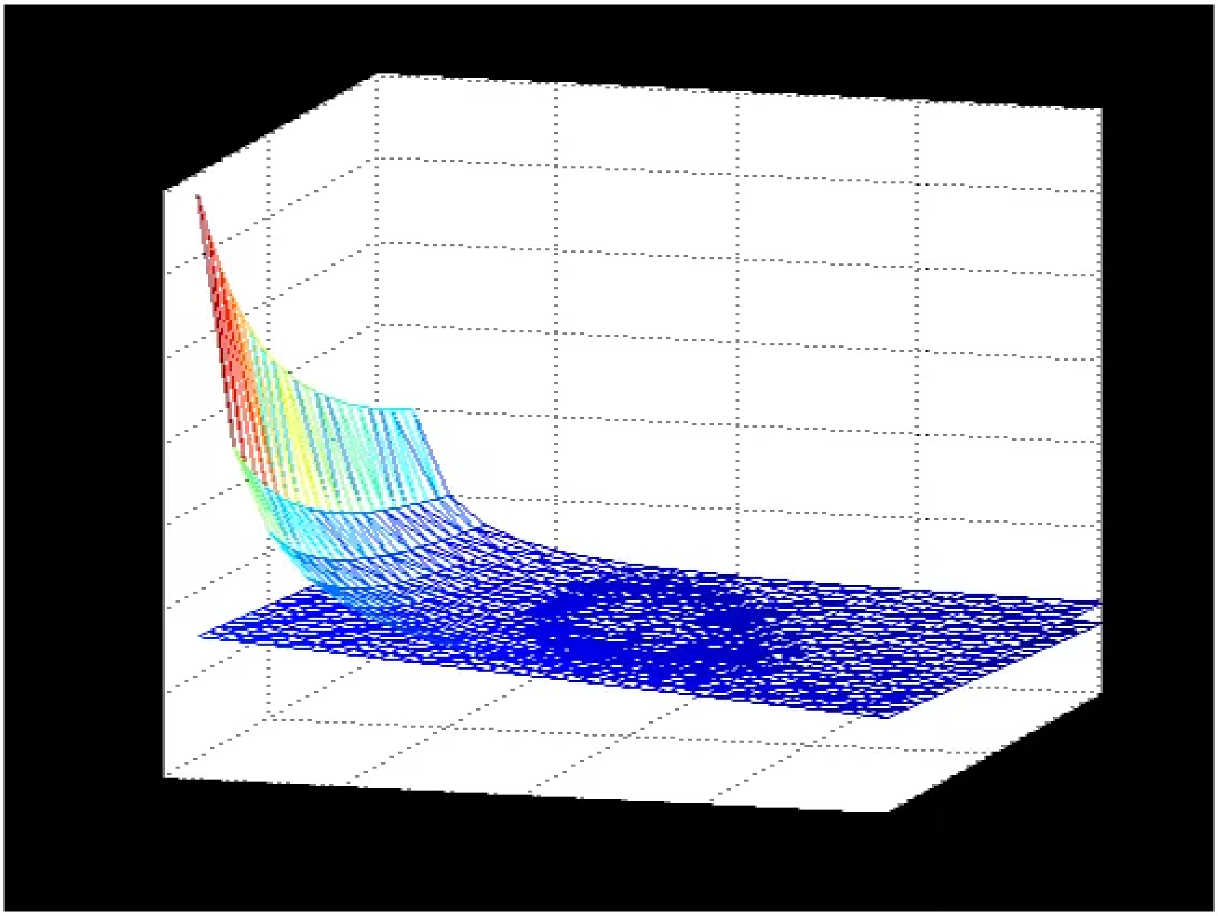
We define the function ztan for the tangent plane at a point (x, y):

ztan=9-3*X-1.5*Y

We plot this function over the meshgrid with the line of code:

trimesh(gridval,X,Y,ztan)

We rotate the graph to view the surface and the plane:



The normal line to the surface defined by $F(x,y,z)=k$ at the given point (x_0, y_0, z_0) is given by the symmetric equations:

$$\frac{x-x_0}{F_x(x_0,y_0,z_o)}=\frac{y-y_0}{F_y(x_0,y_0,z_o)}=\frac{z-z_0}{F_z(x_0,y_0,z_o)}$$

Therefore, the symmetric equations of the normal line to the given surface defined by *F* at the given point $(1,2,3)$ are:

$$\frac{x-1}{F_x(1,2,3)}=\frac{y-2}{F_y(1,2,3)}=\frac{z-3}{F_z(1,2,3)}$$

$$\frac{x-1}{6}=\frac{y-2}{3}=\frac{z-3}{2}$$

$$x-1=2y-4=3z-9$$

To plot the line, we first define the vector *x*: x = [0.1:0.1:2]. We solve for *y* in terms of *x*:

$$2y-4=x-1$$
$$2y=x+3$$
$$y=\frac{(x+3)}{2}$$

Hence we define the vector *y* : $y=\frac{(x+3)}{2}$

Similarly, we solve for *z* in terms of *x*:

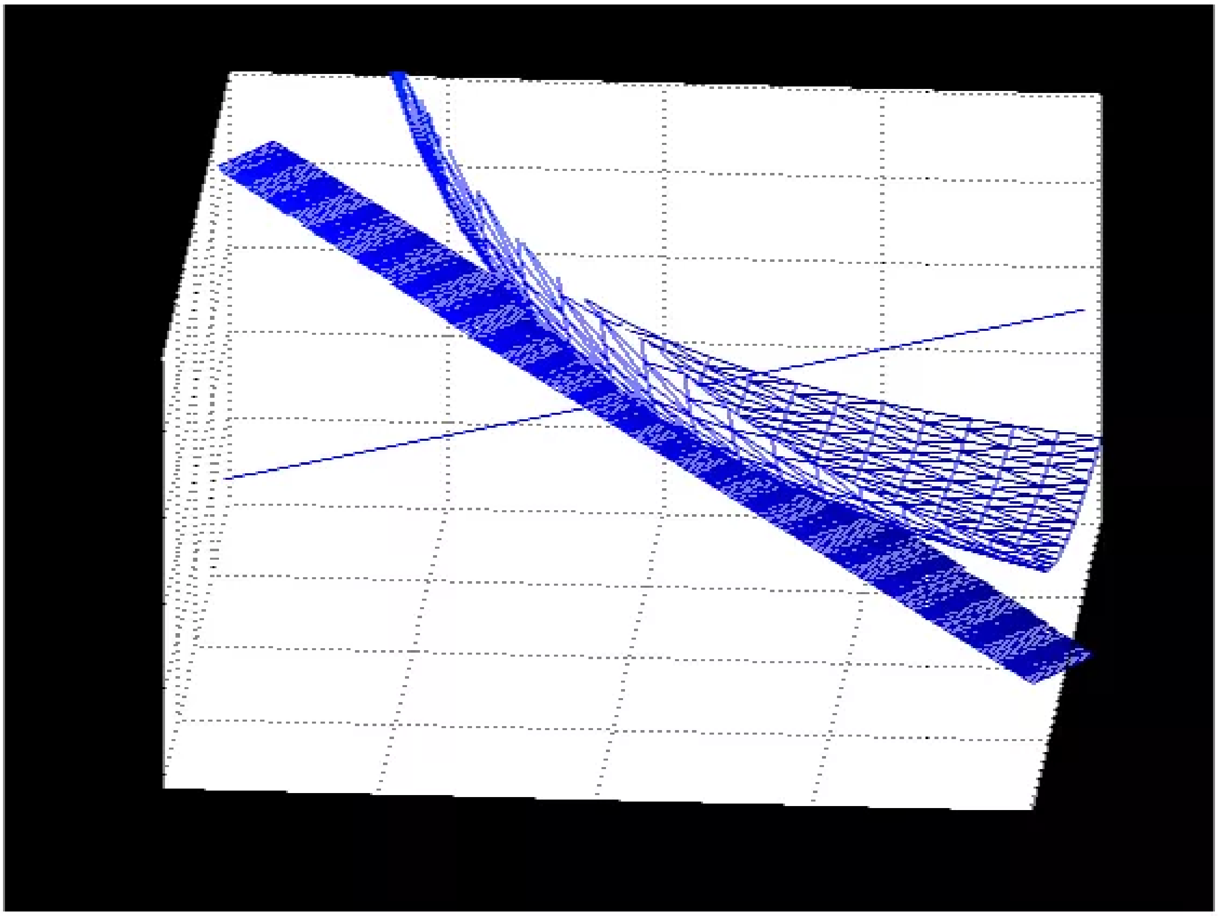
$$3y-9=x-1$$
$$3z=x+8$$
$$z=(x+8)/3$$

Hence we define the vector *z* : z = (x+8)./3

Plot3 will plot a 3 dimensional graph such as a line. The line of code to plot the line is:

plot3(x,y,z)

We display the results of plotting all three objects in an appropriate viewing window:



Answer 49E.

Given that $f(x, y) = \text{xy}$, find $\nabla f(3, 2)$

$$f_x = y,\ f_y = x$$

$$\nabla f(3, 2) = \langle y, x\rangle = \langle 2, 3\rangle$$

find the tangent line of the level curve f(x,y)=6 at the point (3,2)

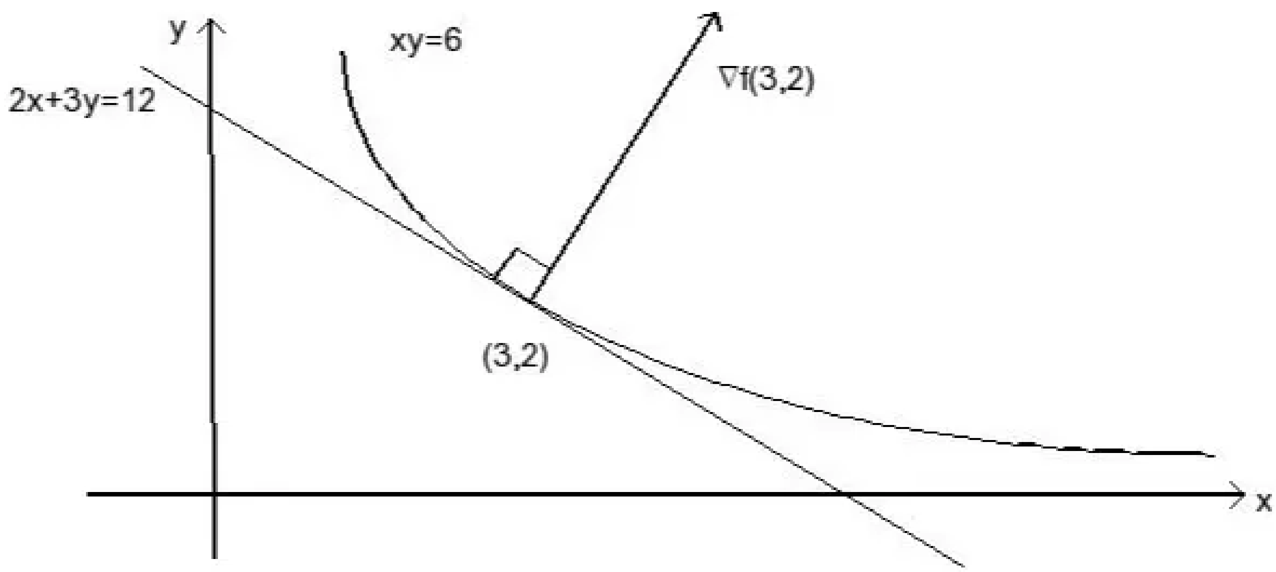
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

2(x-3) + 3(y-2) = 0

2x - 6 + 3y - 6 = 0

2x + 3y = 12

Sketch the level curve, the tangent line and the gradient vector.



Answer 50E.

$g(x, y) = x^2 + y^2 - 4x$, find $\nabla g(1, 2)$

$g_x = 2x - 4,\ g_y = 2y$

$\nabla g(1, 2) = \langle 2x - 4, 2y \rangle = \langle -2, 4 \rangle$

Find the tangent line to the level curve g(x,y) = 1 at the point (1,2)

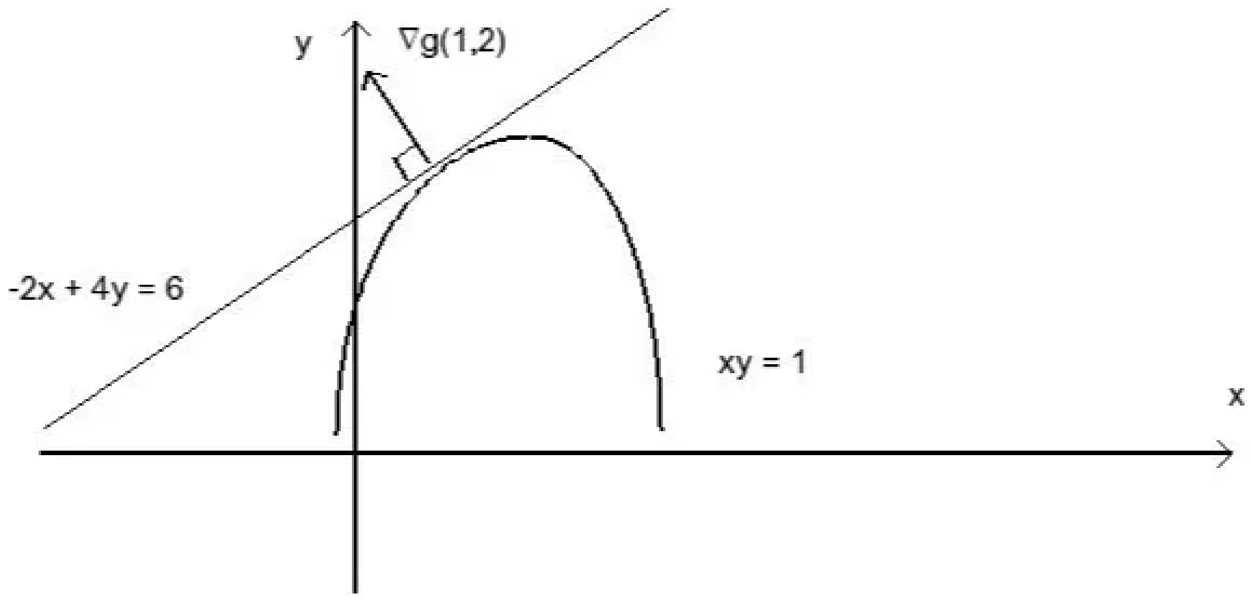
$$g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) = 0$$

-2(x - 1) + 4(y - 2) = 0

-2x + 2 + 4y - 8 = 0

-2x + 4y = 6

Sketch the level curve, the tangent line and the gradient vector.



Answer 51E.

The given equation of ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{------------ (1)}$$

Take $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$

Then $f_x(x,y,z)=\frac{2x}{a^2}$

$$f_y(x,y,z)=\frac{2y}{b^2}$$

$$f_z(x,y,z)=\frac{2z}{c^2}$$

The equation of tangent plane at (x_0,y_0,z_0) is

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$

i.e. $\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0$

i.e. $\frac{x\,x_0}{a^2}+\frac{y\,y_0}{b^2}+\frac{z\,z_0}{c^2}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}$

But $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1$ {as (x_0,y_0,z_0) lies on (1)}

Then the required equation of tangent plane is

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$$

Hence proved

Answer 52E.

The given equation of hyperbolic is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad \text{----------------- (1)}$$

Take $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}$

Then $f_x(x,y,z)=\frac{2x}{a^2}$

$$f_y(x,y,z)=\frac{2y}{b^2}$$

$$f_z(x,y,z)=-\frac{2z}{c^2}$$

The equation of tangent plane at (x_0, y_0, z_0) is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

i.e. $$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0$$

i.e. $$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} - \frac{2zz_0}{c^2} - 2\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 0$$

Or $$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}$$

But $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$ {as (x_0, y_0, z_0) lies on hyperboloid (1)

Then $$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 1$$

Which is the required equation of the tangent plane

Answer 53E.

The given equation of elliptic parabolic is

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Then $$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c}$$

Therefore $$F_x(x, y, z) = \frac{2x}{a^2}$$

$$F_y(x, y, z) = \frac{2y}{b^2}$$

$$F_z(x, y, z) = -\frac{1}{c}$$

The gradient $\vec{\nabla} F(x, y, z) = < F_x, F_y, F_z >$

Then $$\vec{\nabla} F(x_0, y_0, z_0) = < \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{-1}{c} >$$

The equation of the tangent plane is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)-\frac{1}{c}(z-z_0)=0$$

Or $$\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y-\frac{1}{c}z=\frac{2x_0^2}{a^2}+\frac{2y_0^2}{b^2}-\frac{z_0}{c}$$

Or $$\frac{2x_0}{a^2}x+\frac{2y_0}{b^2}y=\frac{z}{c}+2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)-\frac{z_0}{c}$$

But $$\frac{z_0}{c}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}$$

So, $$\frac{2x_0x}{a^2}+\frac{2y_0y}{b^2}=\frac{z}{c}+\frac{2z_0}{c}-\frac{z_0}{c}$$

Or $$\boxed{\frac{2x_0x}{a^2}+\frac{2y_0y}{b^2}=\frac{z+z_0}{c}}$$

Answer 54E.

Consider the equation of the paraboloid $y = x^2 + z^2$

Rewrite the equation in the form $F(x,y,z) = k$ where *k* is a constant:

$$y = x^2 + z^2$$
$$y - x^2 - z^2 = 0$$

The equation of the tangent plane of the surface $F(x,y,z)$ at (x_0, y_0, z_0) is

$$(x - x_0)F_x(x_0, y_0, z_0) + (y - y_0)F_y(x_0, y_0, z_0) + (z - z_0)F_z(x_0, y_0, z_0) = 0 \quad \text{...... (1)}$$

Take $F(x,y,z) = y - x^2 - z^2$

Find the partial derivatives of *F* at (x_0, y_0, z_0).

$$F_x = \frac{\partial}{\partial x}(y - x^2 - z^2)$$
$$= -2x$$
$$F_x(x_0, y_0, z_0) = -2x_0$$
$$F_y = \frac{\partial}{\partial y}(y - x^2 - z^2)$$
$$= 1$$
$$F_y(x_0, y_0, z_0) = 1.$$
$$F_z = \frac{\partial}{\partial z}(y - x^2 - z^2)$$
$$= -2z$$
$$F_z(x_0, y_0, z_0) = -2z_0$$

Therefore an equation of the tangent plane to *F* at (x_0, y_0, z_0) is:

$$-2x_0(x - x_0) + 1(y - y_0) - 2z_0(z - z_0) = 0$$

The direction for the normal vector to this plane is $\langle -2x_0, 1, -2z_0 \rangle$.

Since the tangent plane is parallel to the plane $x + 2y + 3z = 1$ at the point (x_0, y_0, z_0). so the normal lines of the 2 planes are parallel.

The gradients of parallel lines are scalar multiples of each other. The plane $x + 2y + 3z = 1$ has gradient vector $\langle 1, 2, 3 \rangle$.

So, from this we can write the equation as $k\langle -2x_0, 1, -2z_0 \rangle = \langle 1, 2, 3 \rangle$ where *k* is a constant.

Therefore,

$$\langle -2kx_0, 1k, -2kz_0 \rangle = \langle 1, 2, 3 \rangle$$

Solve the above equation, by equating the coefficients on both sides.

$-2kx_0 = 1, \qquad k = 2, \qquad -2kz_0 = 3$

Plug in $k = 2$ to solve for x_0 and z_0:

$$\begin{aligned} -2kx_0 &= 1 \\ -2(2)x_0 &= 1 \\ x_0 &= -\frac{1}{4} \end{aligned}$$

And

$$\begin{aligned} -2kz_0 &= 3 \\ -2(2)z_0 &= 3 \\ z_0 &= -\frac{3}{4} \end{aligned}$$

Solve for y_0 from the equation of the paraboloid:

$$\begin{aligned} y_0 &= x_0^2 + z_0^2 \\ y &= \left(-\frac{1}{4}\right)^2 + \left(-\frac{3}{4}\right)^2 \\ &= \frac{1}{16} + \frac{9}{16} \\ &= \frac{5}{8} \end{aligned}$$

Therefore, the point on the paraboloid whose tangent plane is parallel to $x + 2y + 3z = 1$ is

$$\boxed{\left(-\frac{1}{4}, -\frac{3}{4}, \frac{5}{8}\right)}.$$

Answer 55E.

Consider the following equation of the hyperboloid.

$x^2 - y^2 - z^2 = 1$.

The objective is to determine the existence of points on the hyperboloid where the tangent plane is parallel to the plane $z = x + y$.

Let a differentiable function $F(x, y, z) = k$ where *k* is a constant. So long as the gradient of *F* at the given point (x_0, y_0, z_0) is non-zero, an equation of the tangent plane to *F* at (x_0, y_0, z_0) is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Find the partial derivatives of F at (x_0, y_0, z_0). First find F_x:

$$F_x = \frac{\partial}{\partial x}(x^2 - y^2 - z^2)$$
$$= 2x$$

Evaluate the partial derivative at (x_0, y_0, z_0):

$$F_x(x_0, y_0, z_0) = 2x_0$$

Find F_y:

$$F_y = \frac{\partial}{\partial y}(x^2 - y^2 - z^2)$$
$$= -2y$$

Evaluate the partial derivative at (x_0, y_0, z_0):

$$F_y(x_0, y_0, z_0) = -2y_0$$

Find F_z:

$$F_z = \frac{\partial}{\partial z}(x^2 - y^2 - z^2)$$
$$= -2z$$

Evaluate the partial derivative at (x_0, y_0, z_0):

$$F_z(x_0, y_0, z_0) = -2z_0$$

Thus, an equation of the tangent plane to F at (x_0, y_0, z_0) is:

$$2x_0(x - x_0) - 2y_0(y - y_0) - 2z_0(z - z_0) = 0$$

Or similarly,

$$x_0(x - x_0) - y_0(y - y_0) - z_0(z - z_0) = 0$$

The gradient vector, and hence the direction for the normal line to this plane, is $\langle x_0, -y_0, -z_0 \rangle$.

The tangent plane is parallel to the plane $z = x + y$ at any point (x_0, y_0, z_0) where the normal lines of the 2 planes are parallel. The gradients of parallel lines are scalar multiples of each other. The plane $z = x + y$ can be rewritten as $-x - y + z = 0$.

Hence, we see its gradient must be $\langle -1, -1, 1 \rangle$.

Hence, we want to solve: $k\langle x_0, -y_0, -z_0 \rangle = \langle -1, -1, 1 \rangle$ where k is a constant. The solution which equates the gradients is $(x_0, y_0, z_0) = \left(-\frac{1}{k}, \frac{1}{k}, -\frac{1}{k}\right)$. To verify if this solution is valid for some k, we check to see that it is indeed a point on the hyperboloid:

$$x^2 - y^2 - z^2 = 1$$

$$\left(-\frac{1}{k}\right)^2 - \left(\frac{1}{k}\right)^2 - \left(-\frac{1}{k}\right)^2 = 1$$

$$\frac{1}{k^2} - \frac{1}{k^2} - \frac{1}{k^2} = 1$$

$$-\frac{1}{k^2} = 1$$

$$-1 = k^2$$

The equation is not satisfied since a square cannot be negative.

Hence, we conclude there are $\boxed{\text{no}}$ points on the hyperboloid where the tangent plane is parallel to the plane $z = x + y$.

Answer 56E.

The given equation of ellipsoid is

$$3x^2 + 2y^2 + z^2 = 9$$

The equation of tangent plane to ellipsoid at (x_0, y_0, z_0) is

$$3xx_0 + 2yy_0 + zz_0 = 9$$

Then the equation of tangent plane at (1, 1, 2) is

$$3x + 2y + 2z = 9 \qquad \text{--------------- (1)}$$

And the equation of sphere is

$$x^2 + y^2 + z^2 - 8x = 6y - 8z + 24 = 0$$

Then the equation of tangent plane at (1, 1, 2) is

$$x+y+2z-4(x+1)-3(y+1)-4(z+2)+24=0$$
$$x+y+2z-4x-4-3y-3-4z-8+24=0$$
$$-3x-2y-2z+9=0$$

Or $3x+2y+2z-9=0$

Or $3x+2y+2z=9$ ------------- (2)

From equation (1) and (2) we see that the tangent planes to given ellipsoid and sphere are same at (1, 1, 2) then the surfaces are tangent to each other at point $(1,1,2)$

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From equation (1) and (2) we see that the tangent planes to given ellipsoid and sphere are same at (1, 1, 2) then the surfaces are tangent to each other at point $(1,1,2)$

Answer 57E.

The equation cone is

$$x^2+y^2=z^2$$

$$f(x,y,z)=x^2+y^2-z^2$$

Then $f_x(x,y,z)=2x$

$$f_y(x,y,z)=2y$$
$$f_z(x,y,z)=-2z$$

Therefore $\vec{\nabla} f(x,y,z)=<2x,\ 2y,\ -2z>$

Let (x_0,y_0,z_0) be a point on the cone (other than the origin).

Then $\vec{\nabla} f(x_0,y_0,z_0)=<2x_0,\ 2y_0,-2z_0>$

The equation of tangent plane at (x_0,y_0,z_0) is

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$
$$2x_0(x-x_0)+2y_0(y-y_0)-2z_0(z-z_0)=0$$

Or $x_0(x-x_0)+y_0(y-y_0)-z_0(z-z_0)=0$

Or $xx_0+yy_0-zz_0=x_0^2+y_0^2-z_0^2$

But $x_0^2 + y_0^2 - z_0^2 = 0$

Then $xx_0 + yy_0 - zz_0 = 0$ is the equation of the tangent plane to the cone

This is the equation of the tangent plane which always contains the origin (0, 0, 0)

Hence proved

Answer 58E.

$$f(x,y,z) = x^2 + y^2 + z^2$$

Then
$$f_x(x,y,z) = 2x$$
$$f_y(x,y,z) = 2y$$
$$f_z(x,y,z) = 2z$$

Then the equation of the normal line to be sphere at (x_0, y_0, z_0) is

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

i.e. $$\frac{x - x_0}{2x_0} = \frac{y - y_0}{2y_0} = \frac{z - z_0}{2z_0}$$

Now the centre of the sphere is $(0,0,0)$

For the normal line to pass through $(0,0,0)$

$$\frac{0 - x_0}{2x_0} = \frac{0 - y_0}{2y_0} = \frac{0 - z_0}{2z_0}$$

Or $$\frac{-1}{2} = \frac{-1}{2} = \frac{-1}{2}$$

Or $1 = 1 = 1$, which is true

Hence every normal line to the sphere (1) passes through the centre of the sphere

Answer 59E.

Consider the equation of the paraboloid:

$$F(x,y,z) = x^2 + y^2 - z$$

Compute the gradient of F, by using the formula:

$$\nabla F(x,y,z) = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z) \rangle.$$

Thus,

$$\begin{aligned} \nabla F(x,y,z) &= \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,z) \rangle \\ &= \langle 2x, 2y, -1 \rangle \end{aligned}$$

At $(1,1,2)$,

$$\begin{aligned} \nabla F(x,y,z) &= \langle 2(1), 2(1), -1 \rangle \\ &= \langle 2, 2, -1 \rangle \end{aligned}$$

The equation of the tangent plane at $(1,1,2)$ is

$$2(x-1)+2(y-1)-(z-2)=0$$
$$2x-2+2y-2-z+2=0$$
$$2x+2y-z-2=0$$

The symmetric equations of the normal line are

$$\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-2}{-1}=t.$$

Then,

$x=2t+1, y=2t+1,$ and $z=-t+2.$

Replace the values of x, y, z in $z=x^2+y^2$ and then solve for t.

$$-t+2=(2t+1)^2+(2t+1)^2$$
$$-t+2=4t^2+4t+1+4t^2+4t+1$$
$$8t^2+9t=0$$
$$t(8t+9)=0$$
$$t=0,-\frac{9}{8}$$

Plug the value $t=-\frac{9}{8}$ to $x=2t+1, y=2t+1,$ and $z=-t+2.$

$$x=2\left(-\frac{9}{8}\right)+1, y=2\left(-\frac{9}{8}\right)+1, \text{ and } z=-\left(-\frac{9}{8}\right)+2$$
$$x=-\frac{9}{4}+1, \quad y=-\frac{9}{4}+1 \quad \text{and } z=\frac{9}{8}+2$$
$$x=-\frac{5}{4}, \quad y=-\frac{5}{4} \quad \text{and } z=\frac{25}{8}$$

Therefore, the normal intersects the paraboloid at $\boxed{\left(-\frac{5}{4},-\frac{5}{4},\frac{25}{8}\right)}$.

Answer 60E.

The symmetric equations for the normal line through the point $P(x_0, y_0, z_0)$ is given by

$$\frac{(x-x_0)}{F_x(x_0, y_0, z_0)} = \frac{(y-y_0)}{F_y(x_0, y_0, z_0)} = \frac{(z-z_0)}{F_z(x_0, y_0, z_0)}.$$

We have $F(x, y, z) = 4x^2 + y^2 + 4z^2 - 12 = 0.$

Find $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$.

$$\begin{aligned} F_x(x, y, z) &= \frac{\partial}{\partial x}\left(4x^2 + y^2 + 4z^2 - 12\right) \\ &= 8x \end{aligned}$$

$$\begin{aligned} F_y(x, y, z) &= \frac{\partial}{\partial y}\left(4x^2 + y^2 + 4z^2 - 12\right) \\ &= 2y \end{aligned}$$

$$\begin{aligned} F_z(x, y, z) &= \frac{\partial}{\partial z}\left(4x^2 + y^2 + 4z^2 - 12\right) \\ &= 8z \end{aligned}$$

Replace x with 1, y with 2, and z with 1 to find $F_x(1, 2, 1)$, $F_y(1, 2, 1)$, and $F_z(1, 2, 1)$.

$$\begin{aligned} F_x(1, 2, 1) &= 8(1) \\ &= 8 \end{aligned}$$

$$\begin{aligned} F_y(1, 2, 1) &= 2(2) \\ &= 4 \end{aligned}$$

$$\begin{aligned} F_z(1, 2, 1) &= 8(1) \\ &= 8 \end{aligned}$$

Substitute the known values in $\dfrac{(x-x_0)}{F_x(x_0, y_0, z_0)} = \dfrac{(y-y_0)}{F_y(x_0, y_0, z_0)} = \dfrac{(z-z_0)}{F_z(x_0, y_0, z_0)}.$

$$\frac{(x-1)}{8} = \frac{(y-2)}{4} = \frac{(z-1)}{8} = t$$

We thus, get $x = 8t + 1$, $y = 4t + 2$, and $z = 8t + 1$.

Replace x with $8t+1$, y with $4t+2$, and z with $8t+1$ in $x^2+y^2+z^2=102$ and simplify.

$$\begin{aligned}(8t+1)^2+(4t+2)^2+(8t+1)^2 &= 102\\ 2(8t+1)^2+(4t+2)^2 &= 102\\ 144t^2+48t+6 &= 102\\ 24t^2+8t-16 &= 0\\ 3t^2+t-2 &= 0\end{aligned}$$

Solve for t.

$$\begin{aligned}3t^2+t-2 &= 0\\ (t+1)(3t-2) &= 0\\ t &= -1, \frac{2}{3}\end{aligned}$$

Substitute -1 for t and $\frac{2}{3}$ for t in the parametric equation for x, y, and z.

$$x = 8(-1)+1 = -7 \qquad x = 8\left(\frac{2}{3}\right)+1 = \frac{19}{3}$$

$$y = 4(-1)+2 = -2 \qquad y = 4\left(\frac{2}{3}\right)+2 = \frac{14}{3}$$

$$z = 8(-1)+1 = -7 \qquad z = 8\left(\frac{2}{3}\right)+1 = \frac{19}{3}$$

Thus, we get the points of intersection as $\boxed{(-7, 2, -7) \text{ and } \left(\frac{19}{3}, \frac{14}{3}, \frac{19}{3}\right)}$.

Answer 61E.

The equation of the surface is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{c}$$

Then $f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}$

$$f_x(x,y,z)=\frac{1}{2\sqrt{x}}$$

$$f_y(x,y,z)=\frac{1}{2\sqrt{y}}$$

$$f_z(x,y,z)=\frac{1}{2\sqrt{z}}$$

Let (x_0, y_0, z_0) be a point on the surface. Then the equation of the tangent plane at (x_0, y_0, z_0) is

$$f_x(x,y,z)(x-x_0)+f_y(x,y,z)(y-y_0)+f_z(x,y,z)(z-z_0)=0$$

$$\frac{(x-x_0)}{2\sqrt{x_0}}+\frac{(y-y_0)}{2\sqrt{y_0}}+\frac{(z-z_0)}{2\sqrt{z_0}}=0$$

Or $$\frac{x-x_0}{\sqrt{x_0}}+\frac{(y-y_0)}{\sqrt{y_0}}+\frac{z-z_0}{\sqrt{z_0}}=0$$

Or $$\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}$$

But $$\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}=\sqrt{c}$$

Then $$\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{c}$$

Then x – intercept is $\sqrt{x_0c}$ {putting $y = z = 0$

y – Intercept is $\sqrt{y_0c}$ {putting $x = z = 0$

z – Intercept is $\sqrt{z_0c}$ {putting $y = x = 0$

So the sum of the intercepts is

$$\sqrt{x_0c}+\sqrt{y_0c}+\sqrt{z_0c}=\sqrt{c}\left(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}\right)=c$$

Hence sum of the intercepts is constant

Answer 62E.

Consider the function

$$f(x,y)=3xe^{y}-x^{3}-e^{3y}$$

It is required to find the critical points of f and prove that the function has local maxima at these two critical points.

To find the critical points find the partial derivatives of f independently with respect to x and y and equate them to zero.

First find the partial derivatives.

$$f_x(x,y)=\frac{\partial}{\partial x}\left[3xe^{y}-x^{3}-e^{3y}\right]$$

$$=\frac{\partial}{\partial x}\left[3xe^{y}\right]+\frac{\partial}{\partial x}\left[-x^{3}\right]+\frac{\partial}{\partial x}\left[-e^{3y}\right]$$

Use sum rule

$$=3e^{y}\frac{\partial}{\partial x}[x]-\frac{\partial}{\partial x}\left[x^{3}\right]-\frac{\partial}{\partial x}\left[e^{3y}\right]$$

Use constant multiple rule

$$=3e^{y}(1)-3x^{2}-0$$

Use power rule and $\frac{\partial}{\partial x}(k)=0$

$$=3e^{y}-3x^{2} \quad \text{...... (1)}$$

$$f_y(x,y)=\frac{\partial}{\partial y}\left[3xe^{y}-x^{3}-e^{3y}\right]$$

$$=\frac{\partial}{\partial y}\left[3xe^{y}\right]+\frac{\partial}{\partial y}\left[-x^{3}\right]+\frac{\partial}{\partial y}\left[-e^{3y}\right]$$

Use sum rule

$$=3x\frac{\partial}{\partial y}\left[e^{y}\right]-\frac{\partial}{\partial y}\left[x^{3}\right]-\frac{\partial}{\partial y}\left[e^{3y}\right]$$

Use constant multiple rule

$$=3xe^{y}-0-3e^{3y}$$

Use power rule and $\frac{\partial}{\partial x}(k)=0$

$$=3xe^{y}-3e^{3y} \quad \text{...... (2)}$$

Set the derivatives equal to zero.

$f_x(x,y)=0$

$3e^y-3x^2=0$

$e^y=x^2$

Plug in $e^y=x^2$ in $f_y(x,y)=0$

$f_y(x,y)=0$

$3xe^y-3e^{3y}=0$

$3x(x^2)-3(x^2)^3=0$

$3x^3-3x^6=0$

$x^3(1-x^2)=0$

$x=0$ or $x=\pm 1$

So, the corresponding *y*-values are obtained by substitution in $e^y=x^2$.

$e^y=x^2$

$y=\ln x^2$

$y=\ln 0$

The values of *y* are not defined for $x=0, x=-1$ since $\ln 0, \ln(-1)$ is not defined.

So,

For $x=1$

$y=\ln 1$

$=0$

Therefore, the only possible critical point is $\boxed{(1,0)}$.

Classify the behaviour of the critical points of f.

Recall the second derivative test,

A function *f* has continuous partial derivatives on disk (a,b) and $f_x(a,b)=0, f_y(a,b)=0$.

Let

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

a. If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum

b. If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum

c. If $D<0$ then $f(a,b)$ is not a local minimum or local maximum.

To find $D(x,y)$, find $f_{xx}(x,y), f_{yy}(x,y)$ and $f_{xy}(x,y)$.

$$f_{xx} = \frac{\partial}{\partial x}\left(3e^y - 3x^2\right)$$

Differentiate (1) with respect to x

$$= \frac{\partial}{\partial x}\left(3e^y\right) + \frac{\partial}{\partial x}\left(-3x^2\right) \text{ Use sum rule}$$

$$= -6x$$

$$f_{yy} = \frac{\partial}{\partial y}\left(3xe^y - 3e^{3y}\right) \text{ Differentiate (2) with respect to } y$$

$$= \frac{\partial}{\partial y}\left(3xe^y\right) + \frac{\partial}{\partial y}\left(-3e^{3y}\right) \text{ Use sum rule}$$

$$= 3xe^y - 9e^{3y}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(3e^y - 3x^2\right)$$

$$= 3e^y$$

Substitute f_{xx}, f_{yy} and f_{xy} in D.

$$D = f_{xx}(x,y) f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2$$

$$= [-6x]\left[3xe^y - 9e^{3y}\right] - \left[3e^y\right]^2$$

$$= -18x^2e^y + 56xe^{3y} - 9e^{2y}$$

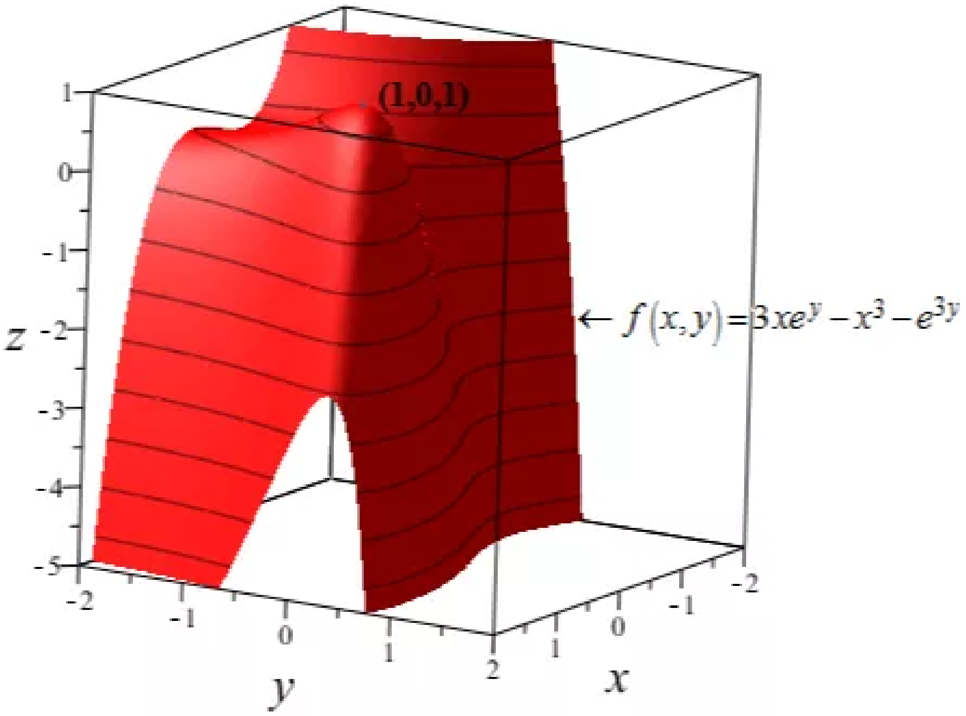
Find D, f_{xx} and behaviour for the three critical points and tabulate them as follows.

Critical point	Value of f	f_{xx}	D	Conclusion
$(1,0)$	$f(1,0)=1$	$f_{xx}(1,0) = -6 < 0$	$29 > 0$	Local maximum

Therefore, conclude that the local maxima exist at only one critical point and value of f is $\boxed{f(1,0)=1}$.

Thus, the highest point of the graph is $\boxed{(1,0,1)}$.

Sketch the graph of the function by choosing a viewing rectangle that displays the critical points exactly.



Answer 63E.

The given paraboloid is; $z=x^2+y^2$

Take $f(x,y,z)=z-x^2-y^2$

$f_x(x,y,z)=-2x$

$f_y(x,y,z)=-2y$

$f_z(x,y,z)=1$

Then $\vec{\nabla}f(x,y,z)=<-2x,-2y,1>$

And $\vec{\nabla}f(-1,1,2)=<2,-2,1>$

And the given ellipsoid is: $4x^2+y^2+z^2=9$

Take $g(x,y,z)=4x^2+y^2+z^2$

Then $g_x(x,y,z)=8x$

$g_y(x,y,z)=2y$

$g_z(x,y,z)=2z$

Then $\vec{\nabla} g(x,y,z) = <8x, 2y, 2z>$

And $\vec{\nabla} g(-1,1,2) = <-8, 2, 4>$

The tangent at the point of intersection of the two curves is perpendicular to both $\vec{\nabla} f$ and $\vec{\nabla} g$ at $(-1,1,2)$ and therefore the vector $\vec{v} = \vec{\nabla} f \times \vec{\nabla} g$ will be parallel to the tangent line

$$\vec{v} = \vec{\nabla} f \times \vec{\nabla} g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\hat{i} - 16\hat{j} - 12\hat{k}$$

Then the direction numbers of tangent lines are $<-10, -16, -12>$ and it passes through $(-1, 1, 2)$

Hence its equation is: $\dfrac{x+1}{-10} = \dfrac{y-1}{-16} = \dfrac{z-2}{-12}$

Or $x = -1 - 10t,\ y = 1 - 16t,\ z = 2 - 12t$

Answer 64E.

(A)

Let $f(x,y,z) = y + z$

And $g(x,y,z) = x^2 + y^2$

Then the gradient tangent line is perpendicular to both $\vec{\nabla} f(1,2,1)$ and $\vec{\nabla} g(1,2,1)$ and therefore it is parallel to the vector $\vec{\nabla} f(1,2,1) \times \vec{\nabla} g(1,2,1) = \vec{n}$ (say)

Now $\vec{\nabla} f(x,y,z) = <f_x, f_y, f_z>$
$= <0,1,1>$

And $\vec{\nabla} g(x,y,z) = <g_x, g_y, g_z>$
$= <2x, 2y, 0>$

Then $\vec{\nabla} f(1,2,1) = <0,1,1>$

And $\vec{\nabla} g(1,2,1) = <2,4,0>$

$$\text{Therefore } \vec{n} = \vec{\nabla} f \times \vec{\nabla} g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix}$$
$$= -4\hat{i} + 2\hat{j} - 2\hat{k}$$

Hence the parametric equations of line passing through $(1,2,1)$ and parallel to vector $<-4,2,-2>$ is $\boxed{x = 1 - 4t,\ y = 2 + 2t,\ z = 1 - 2t}$

Answer 65E.

(A)

The given surfaces are

$$F(x,y,z)=0\text{, and } G(x,y,z)=0$$

Then their partial derivatives are

$$F_x, F_y, F_z\text{, and } G_x, G_y, G_z$$

Let (x_0, y_0, z_0) be their point of intersection

Then the equation of tangent line to $F(x,y,z)=0$ at (x_0, y_0, z_0) is

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

The direction numbers of tangent live are

$$< F_x(x_0,y_0,z_0), F_y(x_0,y_0,z_0), F_z(x_0,y_0,z_0) >$$

Also the equation of tangent line to $G(x,y,z)=0$ at (x_0, y_0, z_0) is

$$\frac{x-x_0}{G_x(x_0,y_0,z_0)}=\frac{y-y_0}{G_y(x_0,y_0,z_0)}=\frac{z-z_0}{G_z(x_0,y_0,z_0)}$$

The direction numbers of tangent line to $G(x,y,z)=0$ is

$$< G_x(x_0,y_0,z_0), G_y v, G_z(x_0,y_0,z_0) >$$

If the two surfaces are orthogonal then their normal are perpendicular and thus tangents at the point of intersection are also perpendicular. Therefore

$$F_xG_x+F_yG_y+F_zG_z=0\text{, at } (x_0,y_0,z_0)$$

(It is given that $\nabla F \neq 0$ and $\nabla G \neq 0$ that is not all the respective partial derivatives are zero)

Conversely if $F_xG_x+F_yG_y+F_zG_z=0$ at (x_0,y_0,z_0)

That is the sum of the product of the direction numbers of the tangent lines is zero

That is the normals at (x_0,y_0,z_0) are perpendicular

That is the two surfaces are orthogonal

(B)

Take $\quad F(x,y,z)=x^2+y^2-z^2 \qquad$ --------------- (1)

Then $\quad F_x(x,y,z)=2x$

$$F_y(x,y,z)=2y,\quad F_z(x,y,z)=-2z$$

And take $G(x,y,z)=x^2+y^2+z^2-r^2$

Then $\quad G_x(x,y,z)=2x$

$$G_y(x,y,z)=2y$$

$$G_z(x,y,z)=2z$$

Consider $F_x G_x + F_y G_y + F_z G_z$

$$= (2x)(2x) + (2y)(2y) + (-2z)(-2z)$$
$$= 4x^2 + 4y^2 - 4z^2$$
$$= 4(x^2 + y^2 - z^2)$$
$$= 4(0) = 0 \qquad \{\text{Because of } (1)$$

This is true for every point. Hence we say that the given surfaces are orthogonal at every point of intersection.

Answer 66E.

(a) A function f is continuous at a point (x_0, y_0) if:

1. $f(x_0, y_0)$ exists
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

We check these conditions at the origin. First we evaluate the function at the origin:

$$f(0,0) = \sqrt[3]{0 \cdot 0}$$
$$= 0$$

Now we determine the limit at the origin:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \sqrt[3]{xy}$$

For the function to be continuous at the origin we need for this limit to exist and to equal 0, since $f(0,0) = 0$. We show this if for any $\varepsilon > 0$ we can find a $\delta > 0$ such that if $0 < |(x,y) - (0,0)| < \delta$ for any (x,y) in the domain of f we have $0 < |f(x,y) - 0| < \varepsilon$.

Given any $\varepsilon > 0$, we choose $\delta = \varepsilon^{3/2}$. Then suppose that :

$$0 < |(x,y) - (0,0)| < \delta$$
$$0 < \left|\sqrt{(x-0)^2 + (y-0)^2}\right| < \delta$$
$$0 < \sqrt{x^2 + y^2} < \delta$$
$$0 < \sqrt{x^2 + y^2} < \varepsilon^{3/2}$$

Now we note that $x < \sqrt{x^2 + y^2}$ and that $y < \sqrt{x^2 + y^2}$. Then we have

$$\begin{aligned}|f(x,y) - 0| &= \left|\sqrt[3]{xy}\right| \\ &< \left|\sqrt[3]{\left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+y^2}\right)}\right| \\ &< \left|\sqrt[3]{\left(\varepsilon^{3/2}\right)\left(\varepsilon^{3/2}\right)}\right| \\ &= \left|\sqrt[3]{\varepsilon^3}\right|\end{aligned}$$

$$= \varepsilon$$

Therefore, $0 < |f(x,y) - 0| < \varepsilon$. Hence f is continuous at the origin.

The partial derivative f_x exists at (x_0, y_0) if the following limit exists:

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

We evaluate this limit at the origin:

$$\begin{aligned}f_x(0,0) &= \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt[3]{h \cdot 0} - 0}{h} \\ &= \lim_{h \to 0}(0) \\ &= 0\end{aligned}$$

Therefore, f_x exists at the origin and is equal to 0.

Similarly, f_y exists at (x_0, y_0) if the following limit exists:

$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

We evaluate this limit at the origin:

$$\begin{aligned} f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[3]{0\cdot h} - 0}{h} \\ &= \lim_{h\to 0}(0) \\ &= 0 \end{aligned}$$

Therefore, f_y exists at the origin and is equal to 0.

The directional derivative of *f* at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle a, b\rangle$ exists if the following limit exists:

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

We evaluate this limit at the origin:

$$\begin{aligned} \lim_{h\to 0} \frac{f(0+ha, 0+hb) - f(0,0)}{h} &= \lim_{h\to 0} \frac{f(ha, hb) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt[3]{(ha)(hb)} - 0}{h} \\ &= \lim_{h\to 0} \frac{h^{2/3}(ab)^{1/3}}{h} \\ &= (ab)^{1/3} \lim_{h\to 0} \frac{1}{h^{1/3}} \end{aligned}$$

But this limit does not exist unless $a = 0$ or $b = 0$. Therefore, the directional derivatives in any direction other than in the directions $\mathbf{i} = \langle 1, 0\rangle$ or $\mathbf{j} = \langle 0, 1\rangle$ do not exist.

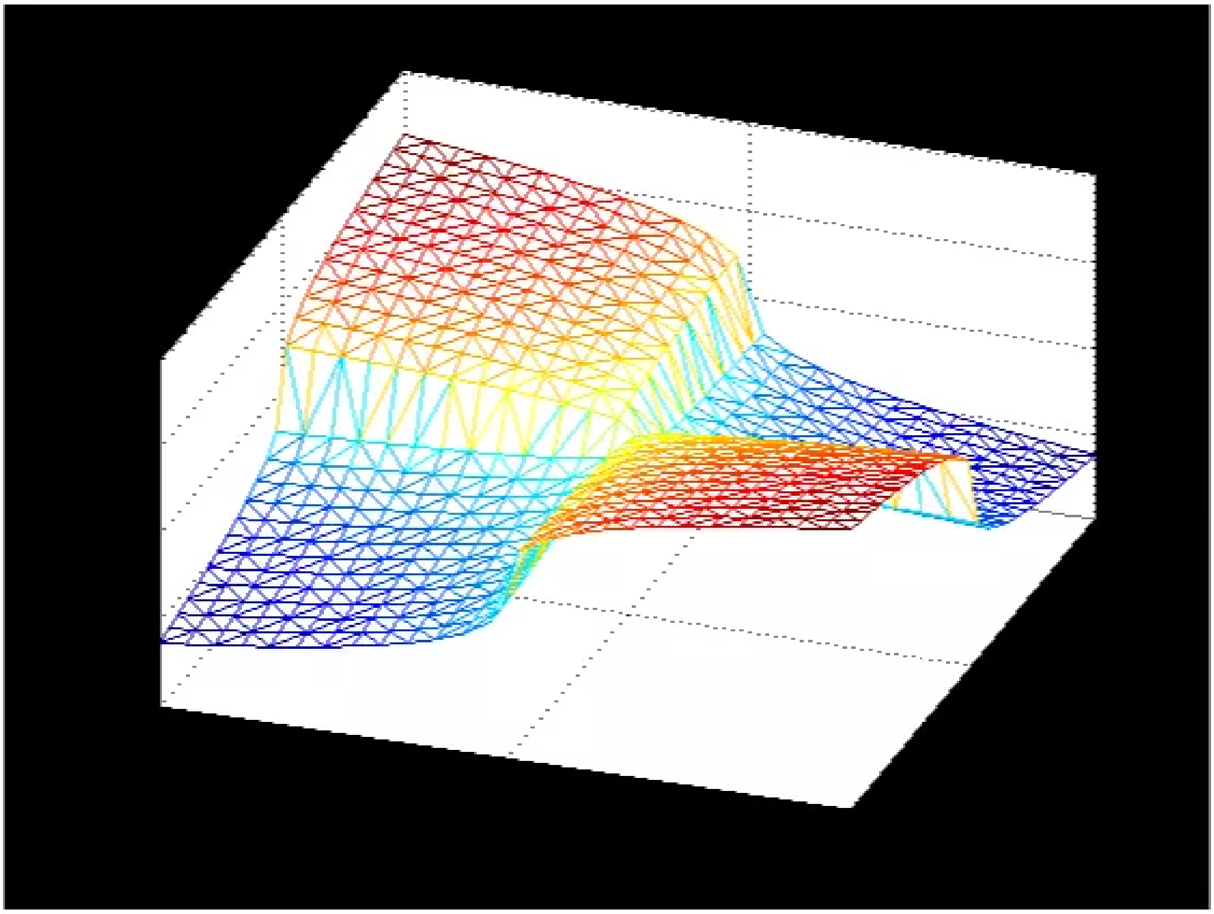
(b) We graph the function $z=\sqrt[3]{xy}$ near the origin using MATLAB. Using the following lines of code, we create a triangular grid on the domain $-0.5 \le x, y \le 0.5$:

```
[X Y] = meshgrid(-0.5:0.04:0.5);
>> gridval = delaunay(X,Y);
```

Now we evaluate the function $z=\sqrt[3]{xy}$ at each point in our grid, then we graph the function.

```
>> Z = nthroot((X.*Y),3);
>> trimesh(gridval,X,Y,Z)
```

Here is the resulting graph:



The graph shows that at the origin, the only trajectories which do not approach a sudden change in slope are the trajectories along the *x* and *y* axes. Hence, althought the function is continuous at the origin, the directional derivatives do not exist other than along the *x* and *y* axes.

Answer 67E.

The directional derivatives of f(x, y) at a given point in two non – parallel directions given by unit vector $\vec{u}$ and $\vec{v}$ are given, that is $D_u f$ and $D_v f$ are given

By the definition of directional derivatives

$$D_u f = \vec{\nabla} f . \vec{u}$$
$$= < f_x, f_y > . \vec{u}$$

If $\vec{u} = < a, b >$

Then $D_u f = < f_x, f_y > . < a, b > = A$ (say) {given}

Also $D_v f = \vec{\nabla} f . \vec{v}$
$$= < f_x, f_y > . \vec{v}$$

If $\vec{v} = < c, d >$

Then $D_v f = < f_x, f_y > . < c, d > = B$ (say) {given}

Hence we are given

$$a f_x + b f_y = A$$

And $c f_x + d f_y = B$

Therefore on solving these linear equations for f_x and f_y we can find f_x, f_y and hence $\vec{\nabla} f$

Answer 68E.

By definition, if $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

For the function $z = f(x, y)$, we note that $f(\mathbf{x}) - f(\mathbf{x}_0) = \Delta z$ and that:

$$\mathbf{x} \to \mathbf{x}_0$$
$$(x, y) \to (x_o, y_0)$$
$$(x - x_0, y - y_0) \to (0,0)$$
$$(\Delta x, \Delta y) \to (0,0)$$

Hence by substitution into the left hand side of the given equation:

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\frac{f(\mathbf{x})-f(\mathbf{x}_0)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z-\nabla f(\mathbf{x}_0)\cdot(x-x_0,y-y_0)}{|(x-x_0,y-y_0)|}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z-\langle f_x(\mathbf{x}_0),f_y(\mathbf{x}_0)\rangle\cdot\langle\Delta x,\Delta y\rangle}{|(\Delta x,\Delta y)|}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z-f_x(\mathbf{x}_0)\Delta x-f_y(\mathbf{x}_0)\Delta y}{|(\Delta x,\Delta y)|}$$

Now we substitute for Δz from the definition since z is differentiable:

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z-\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{|(\Delta x,\Delta y)|}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(f_x(\mathbf{x}_0)\Delta x+f_y(\mathbf{x}_0)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y\right)-f_x(\mathbf{x}_0)\Delta x-f_y(\mathbf{x}_0)\Delta y}{|(\Delta x,\Delta y)|}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\varepsilon_1\Delta x+\varepsilon_2\Delta y}{|(\Delta x,\Delta y)|}$$

Now we recognize that any quantity is no greater than its absolute value:

$$\begin{aligned}\varepsilon_1\Delta x+\varepsilon_2\Delta y&\le|\varepsilon_1\Delta x+\varepsilon_2\Delta y|\\&=|\langle\varepsilon_1,\varepsilon_2\rangle\cdot\langle\Delta x,\Delta y\rangle|\end{aligned}$$

We have the absolute value of a dot product. The Cauchy Schwartz Inequality tells us that for any two vectors ***a*** and ***b***, we have

$$|\mathbf{a}\cdot\mathbf{b}|\le|\mathbf{a}||\mathbf{b}|$$

Therefore,

$$|\langle\varepsilon_1,\varepsilon_2\rangle\cdot\langle\Delta x,\Delta y\rangle|\le|\langle\varepsilon_1,\varepsilon_2\rangle||\langle\Delta x,\Delta y\rangle|$$

Hence we can continue to evaluate our limit:

$$\begin{aligned}\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\varepsilon_1\Delta x+\varepsilon_2\Delta y}{\left|(\Delta x,\Delta y)\right|} &\le \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left|\varepsilon_1\Delta x+\varepsilon_2\Delta y\right|}{\left|(\Delta x,\Delta y)\right|} \\ &= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left|\langle\varepsilon_1,\varepsilon_2\rangle\cdot\langle\Delta x,\Delta y\rangle\right|}{\left|(\Delta x,\Delta y)\right|} \\ &\le \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left|\langle\varepsilon_1,\varepsilon_2\rangle\right|\left|\langle\Delta x,\Delta y\rangle\right|}{\left|(\Delta x,\Delta y)\right|} \\ &= \lim_{(\Delta x,\Delta y)\to(0,0)} \left|\langle\varepsilon_1,\varepsilon_2\rangle\right|\end{aligned}$$

Finally, by the definition, ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x,\Delta y)\to(0,0)$. Therefore,

$$\begin{aligned}\lim_{(\Delta x,\Delta y)\to(0,0)} \left|\langle\varepsilon_1,\varepsilon_2\rangle\right| &= \lim_{(\Delta x,\Delta y)\to(0,0)} \sqrt{(\varepsilon_1)^2+(\varepsilon_2)^2} \\ &= \sqrt{0^2+0^2} \\ &= 0\end{aligned}$$

Therefore, since this limit goes to 0, then:

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x})-f(\mathbf{x}_0)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{\left|\mathbf{x}-\mathbf{x}_0\right|}=0$$