



1062CH03

# PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

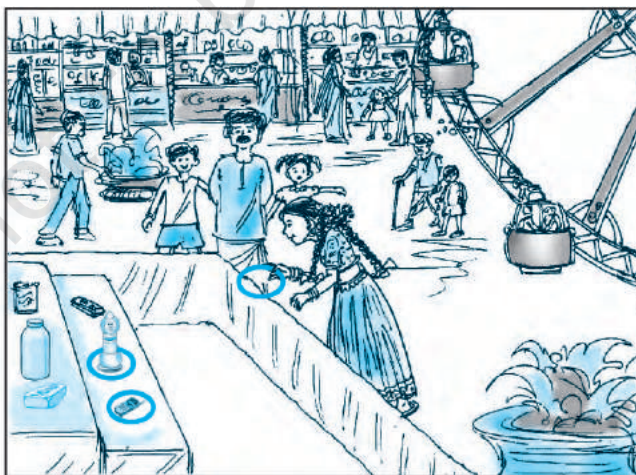
# 3

## 3.1 Introduction

You must have come across situations like the one given below :

Akhila went to a fair in her village. She wanted to enjoy rides on the Giant Wheel and play Hoopla (a game in which you throw a ring on the items kept in a stall, and if the ring covers any object completely, you get it). The number of times she played Hoopla is half the number of rides she had on the Giant Wheel. If each ride costs ₹ 3, and a game of Hoopla costs ₹ 4, how would you find out the number of rides she had and how many times she played Hoopla, provided she spent ₹ 20.

May be you will try it by considering different cases. If she has one ride, is it possible? Is it possible to have two rides? And so on. Or you may use the knowledge of Class IX, to represent such situations as linear equations in two variables.



Let us try this approach.

Denote the number of rides that Akhila had by  $x$ , and the number of times she played Hoopla by  $y$ . Now the situation can be represented by the two equations:

$$y = \frac{1}{2}x \quad (1)$$

$$3x + 4y = 20 \quad (2)$$

Can we find the solutions of this pair of equations? There are several ways of finding these, which we will study in this chapter.

### 3.2 Pair of Linear Equations in Two Variables

Recall, from Class IX, that the following are examples of linear equations in two variables:

$$2x + 3y = 5$$

$$x - 2y - 3 = 0$$

and

$$x - 0y = 2, \text{ i.e., } x = 2$$

You also know that an equation which can be put in the form  $ax + by + c = 0$ , where  $a$ ,  $b$  and  $c$  are real numbers, and  **$a$  and  $b$  are not both zero**, is called a linear equation in two variables  $x$  and  $y$ . (We often denote the condition  $a$  and  $b$  are not both zero by  $a^2 + b^2 \neq 0$ ). You have also studied that a solution of such an equation is a pair of values, one for  $x$  and the other for  $y$ , which makes the two sides of the equation equal.

For example, let us substitute  $x = 1$  and  $y = 1$  in the left hand side (LHS) of the equation  $2x + 3y = 5$ . Then

$$\text{LHS} = 2(1) + 3(1) = 2 + 3 = 5,$$

which is equal to the right hand side (RHS) of the equation.

Therefore,  $x = 1$  and  $y = 1$  is a solution of the equation  $2x + 3y = 5$ .

Now let us substitute  $x = 1$  and  $y = 7$  in the equation  $2x + 3y = 5$ . Then,

$$\text{LHS} = 2(1) + 3(7) = 2 + 21 = 23$$

which is not equal to the RHS.

Therefore,  $x = 1$  and  $y = 7$  is **not** a solution of the equation.

Geometrically, what does this mean? It means that the point  $(1, 1)$  lies on the line representing the equation  $2x + 3y = 5$ , and the point  $(1, 7)$  does not lie on it. So, **every solution of the equation is a point on the line representing it.**

In fact, this is true for any linear equation, that is, **each solution  $(x, y)$  of a linear equation in two variables,  $ax + by + c = 0$ , corresponds to a point on the line representing the equation, and vice versa.**

Now, consider Equations (1) and (2) given above. These equations, **taken together**, represent the information we have about Akhila at the fair.

These two linear equations are **in the same two variables  $x$  and  $y$** . Equations like these are called a *pair of linear equations in two variables*.

Let us see what such pairs look like algebraically.

The general form for a pair of linear equations in two variables  $x$  and  $y$  is

$$a_1x + b_1y + c_1 = 0$$

and

$$a_2x + b_2y + c_2 = 0,$$

where  $a_1, b_1, c_1, a_2, b_2, c_2$  are all real numbers and  $a_1^2 + b_1^2 \neq 0, a_2^2 + b_2^2 \neq 0$ .

Some examples of pair of linear equations in two variables are:

$$2x + 3y - 7 = 0 \text{ and } 9x - 2y + 8 = 0$$

$$5x = y \text{ and } -7x + 2y + 3 = 0$$

$$x + y = 7 \text{ and } 17 = y$$

Do you know, what do they look like geometrically?

Recall, that you have studied in Class IX that the geometrical (i.e., graphical) representation of a linear equation in two variables is a straight line. Can you now suggest what a pair of linear equations in two variables will look like, geometrically? There will be two straight lines, both to be considered together.

You have also studied in Class IX that given two lines in a plane, only one of the following three possibilities can happen:

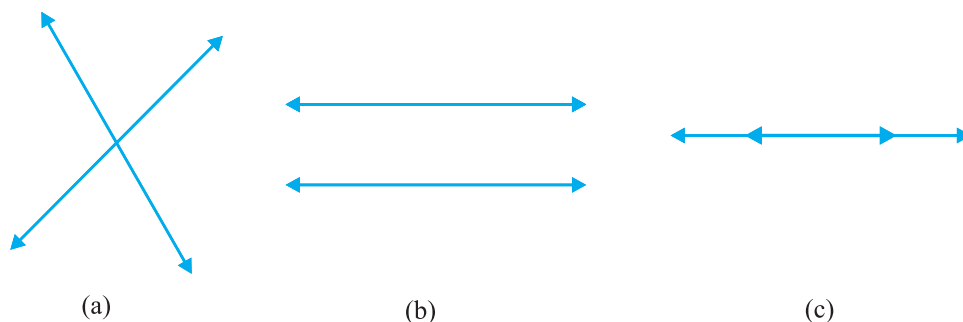
- (i) The two lines will intersect at one point.
- (ii) The two lines will not intersect, i.e., they are parallel.
- (iii) The two lines will be coincident.

We show all these possibilities in Fig. 3.1:

In Fig. 3.1 (a), they intersect.

In Fig. 3.1 (b), they are parallel.

In Fig. 3.1 (c), they are coincident.

**Fig. 3.1**

Both ways of representing a pair of linear equations go hand-in-hand—the algebraic and the geometric ways. Let us consider some examples.

**Example 1 :** Let us take the example given in Section 3.1. Akhila goes to a fair with ₹ 20 and wants to have rides on the Giant Wheel and play Hoopla. Represent this situation algebraically and graphically (geometrically).

**Solution :** The pair of equations formed is :

$$y = \frac{1}{2}x$$

i.e.,  $x - 2y = 0$  (1)

$$3x + 4y = 20$$
 (2)

Let us represent these equations graphically. For this, we need at least two solutions for each equation. We give these solutions in Table 3.1.

**Table 3.1**

$x$	0	2
$y = \frac{x}{2}$	0	1

(i)

$x$	0	$\frac{20}{3}$	4
$y = \frac{20 - 3x}{4}$	5	0	2

(ii)

Recall from Class IX that there are infinitely many solutions of each linear equation. So each of you can choose any two values, which may not be the ones we have chosen. Can you guess why we have chosen  $x = 0$  in the first equation and in the second equation? When one of the variables is zero, the equation reduces to a linear

equation in one variable, which can be solved easily. For instance, putting  $x = 0$  in Equation (2), we get  $4y = 20$ , i.e.,  $y = 5$ . Similarly, putting  $y = 0$  in Equation (2), we get  $3x = 20$ , i.e.,  $x = \frac{20}{3}$ . But as  $\frac{20}{3}$  is not an integer, it will not be easy to plot exactly on the graph paper. So, we choose  $y = 2$  which gives  $x = 4$ , an integral value.

Plot the points  $A(0, 0)$ ,  $B(2, 1)$  and  $P(0, 5)$ ,  $Q(4, 2)$ , corresponding to the solutions in Table 3.1. Now draw the lines  $AB$  and  $PQ$ , representing the equations  $x - 2y = 0$  and  $3x + 4y = 20$ , as shown in Fig. 3.2.

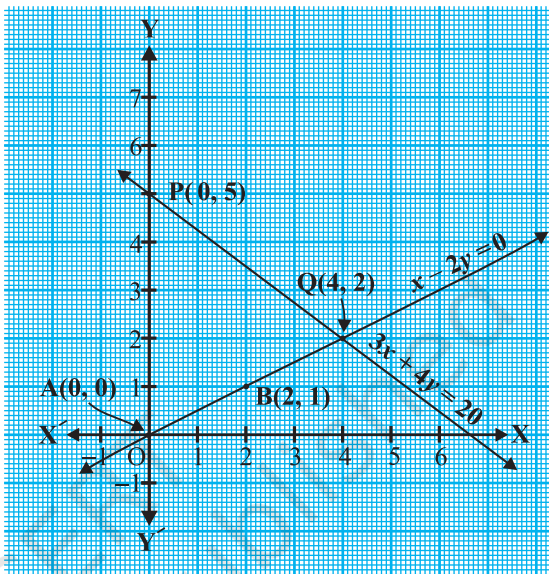


Fig. 3.2

In Fig. 3.2, observe that the two lines representing the two equations are intersecting at the point  $(4, 2)$ . We shall discuss what this means in the next section.

**Example 2 :** Romila went to a stationery shop and purchased 2 pencils and 3 erasers for ₹ 9. Her friend Sonali saw the new variety of pencils and erasers with Romila, and she also bought 4 pencils and 6 erasers of the same kind for ₹ 18. Represent this situation algebraically and graphically.

**Solution :** Let us denote the cost of 1 pencil by ₹  $x$  and one eraser by ₹  $y$ . Then the algebraic representation is given by the following equations:

$$2x + 3y = 9 \quad (1)$$

$$4x + 6y = 18 \quad (2)$$

To obtain the equivalent geometric representation, we find two points on the line representing each equation. That is, we find two solutions of each equation.

These solutions are given below in Table 3.2.

**Table 3.2**

$x$	0	4.5
$y = \frac{9-2x}{3}$	3	0

(i)

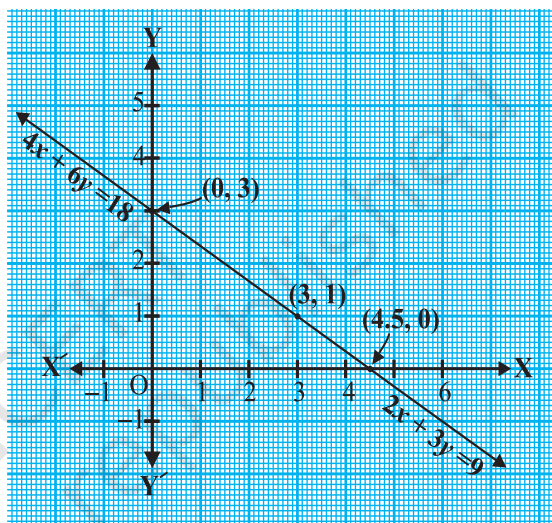
$x$	0	3
$y = \frac{18-4x}{6}$	3	1

(ii)

We plot these points in a graph paper and draw the lines. We find that both the lines coincide (see Fig. 3.3). This is so, because, both the equations are equivalent, i.e., one can be derived from the other.

**Example 3 :** Two rails are represented by the equations  $x + 2y - 4 = 0$  and  $2x + 4y - 12 = 0$ . Represent this situation geometrically.

**Solution :** Two solutions of each of the equations :



**Fig. 3.3**

$$x + 2y - 4 = 0 \quad (1)$$

$$2x + 4y - 12 = 0 \quad (2)$$

are given in Table 3.3

**Table 3.3**

$x$	0	4
$y = \frac{4-x}{2}$	2	0

(i)

$x$	0	6
$y = \frac{12-2x}{4}$	3	0

(ii)

To represent the equations graphically, we plot the points R(0, 2) and S(4, 0), to get the line RS and the points P(0, 3) and Q(6, 0) to get the line PQ.

We observe in Fig. 3.4, that the lines do not intersect anywhere, i.e., they are parallel.

So, we have seen several situations which can be represented by a pair of linear equations. We have seen their algebraic and geometric representations. In the next few sections, we will discuss how these representations can be used to look for solutions of the pair of linear equations.

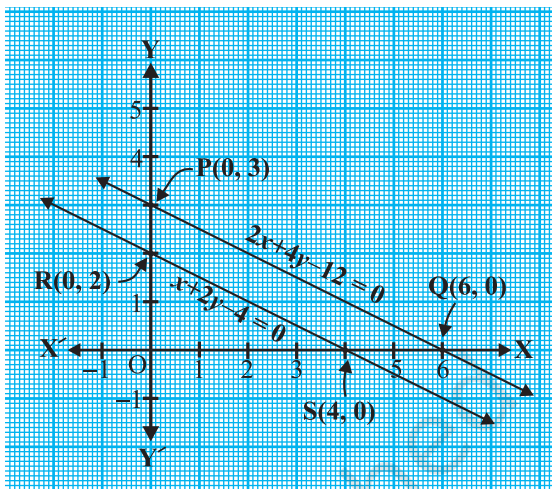


Fig. 3.4

### EXERCISE 3.1

1. Aftab tells his daughter, “Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be.” (Isn’t this interesting?) Represent this situation algebraically and graphically.
2. The coach of a cricket team buys 3 bats and 6 balls for ₹ 3900. Later, she buys another bat and 3 more balls of the same kind for ₹ 1300. Represent this situation algebraically and geometrically.
3. The cost of 2 kg of apples and 1 kg of grapes on a day was found to be ₹ 160. After a month, the cost of 4 kg of apples and 2 kg of grapes is ₹ 300. Represent the situation algebraically and geometrically.

### 3.3 Graphical Method of Solution of a Pair of Linear Equations

In the previous section, you have seen how we can graphically represent a pair of linear equations as two lines. You have also seen that the lines may intersect, or may be parallel, or may coincide. Can we solve them in each case? And if so, how? We shall try and answer these questions from the geometrical point of view in this section.

Let us look at the earlier examples one by one.

- In the situation of Example 1, find out how many rides on the Giant Wheel Akhila had, and how many times she played Hoopla.

In Fig. 3.2, you noted that the equations representing the situation are geometrically shown by two lines intersecting at the point (4, 2). Therefore, the



point (4, 2) lies on the lines represented by both the equations  $x - 2y = 0$  and  $3x + 4y = 20$ . And this is the only common point.

Let us verify algebraically that  $x = 4$ ,  $y = 2$  is a solution of the given pair of equations. Substituting the values of  $x$  and  $y$  in each equation, we get  $4 - 2 \times 2 = 0$  and  $3(4) + 4(2) = 20$ . So, we have verified that  $x = 4$ ,  $y = 2$  is a solution of both the equations. **Since (4, 2) is the only common point on both the lines, there is one and only one solution for this pair of linear equations in two variables.**

Thus, the number of rides Akhila had on Giant Wheel is 4 and the number of times she played Hoopla is 2.

- In the situation of Example 2, can you find the cost of each pencil and each eraser?

In Fig. 3.3, the situation is geometrically shown by a pair of coincident lines. The solutions of the equations are given by the common points.

Are there any common points on these lines? From the graph, we observe that every point on the line is a common solution to both the equations. So, the equations  $2x + 3y = 9$  and  $4x + 6y = 18$  have **infinitely many solutions**. This should not surprise us, because if we divide the equation  $4x + 6y = 18$  by 2, we get  $2x + 3y = 9$ , which is the same as Equation (1). That is, both the equations are equivalent. From the graph, we see that any point on the line gives us a possible cost of each pencil and eraser. For instance, each pencil and eraser can cost ₹ 3 and ₹ 1 respectively. Or, each pencil can cost ₹ 3.75 and eraser can cost ₹ 0.50, and so on.

- In the situation of Example 3, can the two rails cross each other?

In Fig. 3.4, the situation is represented geometrically by two parallel lines. Since the lines do not intersect at all, the rails do not cross. This also means that the equations have no common solution.

A pair of linear equations which has no solution, is called an *inconsistent* pair of linear equations. A pair of linear equations in two variables, which has a solution, is called a *consistent* pair of linear equations. A pair of linear equations which are equivalent has infinitely many distinct common solutions. Such a pair is called a *dependent* pair of linear equations in two variables. Note that a dependent pair of linear equations is always consistent.

We can now summarise the behaviour of lines representing a pair of linear equations in two variables and the existence of solutions as follows:



- (i) the lines may intersect in a single point. In this case, the pair of equations has a unique solution (consistent pair of equations).
- (ii) the lines may be parallel. In this case, the equations have no solution (inconsistent pair of equations).
- (iii) the lines may be coincident. In this case, the equations have infinitely many solutions [dependent (consistent) pair of equations].

Let us now go back to the pairs of linear equations formed in Examples 1, 2, and 3, and note down what kind of pair they are geometrically.

- (i)  $x - 2y = 0$  and  $3x + 4y - 20 = 0$  (The lines intersect)
- (ii)  $2x + 3y - 9 = 0$  and  $4x + 6y - 18 = 0$  (The lines coincide)
- (iii)  $x + 2y - 4 = 0$  and  $2x + 4y - 12 = 0$  (The lines are parallel)

Let us now write down, and compare, the values of  $\frac{a_1}{a_2}$ ,  $\frac{b_1}{b_2}$  and  $\frac{c_1}{c_2}$  in all the three examples. Here,  $a_1$ ,  $b_1$ ,  $c_1$  and  $a_2$ ,  $b_2$ ,  $c_2$  denote the coefficients of equations given in the general form in Section 3.2.

**Table 3.4**

Sl No.	Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratios	Graphical representation	Algebraic interpretation
1.	$x - 2y = 0$ $3x + 4y - 20 = 0$	$\frac{1}{3}$	$\frac{-2}{4}$	$\frac{0}{-20}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Exactly one solution (unique)
2.	$2x + 3y - 9 = 0$ $4x + 6y - 18 = 0$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{-9}{-18}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines	Infinitely many solutions
3.	$x + 2y - 4 = 0$ $2x + 4y - 12 = 0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-4}{-12}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution

From the table above, you can observe that if the lines represented by the equation

$$a_1x + b_1y + c_1 = 0$$

and  $a_2x + b_2y + c_2 = 0$

- are (i) intersecting, then  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ .
- (ii) coincident, then  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ .
- (iii) parallel, then  $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ .

In fact, the converse is also true for any pair of lines. You can verify them by considering some more examples by yourself.

Let us now consider some more examples to illustrate it.

**Example 4 :** Check graphically whether the pair of equations

$$x + 3y = 6 \quad (1)$$

and  $2x - 3y = 12 \quad (2)$

is consistent. If so, solve them graphically.

**Solution :** Let us draw the graphs of the Equations (1) and (2). For this, we find two solutions of each of the equations, which are given in Table 3.5

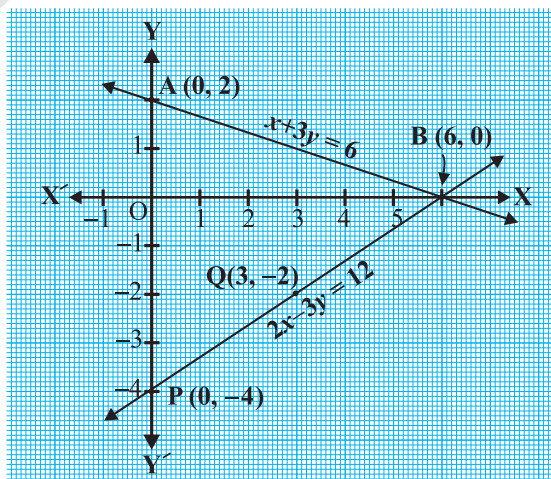
**Table 3.5**

$x$	0	6
$y = \frac{6-x}{3}$	2	0

$x$	0	3
$y = \frac{2x-12}{3}$	-4	-2

Plot the points A(0, 2), B(6, 0), P(0, -4) and Q(3, -2) on graph paper, and join the points to form the lines AB and PQ as shown in Fig. 3.5.

We observe that there is a point B(6, 0) common to both the lines AB and PQ. So, the solution of the pair of linear equations is  $x = 6$  and  $y = 0$ , i.e., the given pair of equations is consistent.



**Fig. 3.5**

**Example 5 :** Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$5x - 8y + 1 = 0 \quad (1)$$

$$3x - \frac{24}{5}y + \frac{3}{5} = 0 \quad (2)$$

**Solution :** Multiplying Equation (2) by  $\frac{5}{3}$ , we get

$$5x - 8y + 1 = 0$$

But, this is the same as Equation (1). Hence the lines represented by Equations (1) and (2) are coincident. Therefore, Equations (1) and (2) have infinitely many solutions.

Plot few points on the graph and verify it yourself.

**Example 6 :** Champa went to a ‘Sale’ to purchase some pants and skirts. When her friends asked her how many of each she had bought, she answered, “The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased”. Help her friends to find how many pants and skirts Champa bought.

**Solution :** Let us denote the number of pants by  $x$  and the number of skirts by  $y$ . Then the equations formed are :

$$y = 2x - 2 \quad (1)$$

and

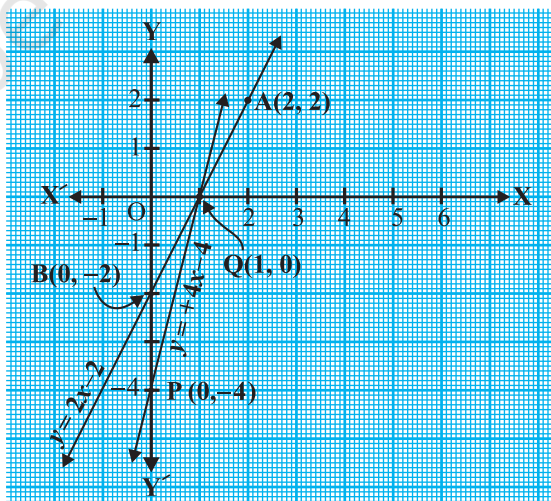
$$y = 4x - 4 \quad (2)$$

Let us draw the graphs of Equations (1) and (2) by finding two solutions for each of the equations. They are given in Table 3.6.

**Table 3.6**

$x$	2	0
$y = 2x - 2$	2	-2

$x$	0	1
$y = 4x - 4$	-4	0



**Fig. 3.6**

Plot the points and draw the lines passing through them to represent the equations, as shown in Fig. 3.6.

The two lines intersect at the point (1, 0). So,  $x = 1$ ,  $y = 0$  is the required solution of the pair of linear equations, i.e., the number of pants she purchased is 1 and she did not buy any skirt.

**Verify** the answer by checking whether it satisfies the conditions of the given problem.

### EXERCISE 3.2

- Form the pair of linear equations in the following problems, and find their solutions graphically.
  - 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.
  - 5 pencils and 7 pens together cost ₹ 50, whereas 7 pencils and 5 pens together cost ₹ 46. Find the cost of one pencil and that of one pen.
- On comparing the ratios  $\frac{a_1}{a_2}$ ,  $\frac{b_1}{b_2}$  and  $\frac{c_1}{c_2}$ , find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident:
  - $5x - 4y + 8 = 0$   
 $7x + 6y - 9 = 0$
  - $9x + 3y + 12 = 0$   
 $18x + 6y + 24 = 0$
  - $6x - 3y + 10 = 0$   
 $2x - y + 9 = 0$
- On comparing the ratios  $\frac{a_1}{a_2}$ ,  $\frac{b_1}{b_2}$  and  $\frac{c_1}{c_2}$ , find out whether the following pair of linear equations are consistent, or inconsistent.
  - $3x + 2y = 5$  ;  $2x - 3y = 7$
  - $2x - 3y = 8$  ;  $4x - 6y = 9$
  - $\frac{3}{2}x + \frac{5}{3}y = 7$  ;  $9x - 10y = 14$
  - $5x - 3y = 11$  ;  $-10x + 6y = -22$
  - $\frac{4}{3}x + 2y = 8$  ;  $2x + 3y = 12$
- Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution graphically:

- (i)  $x + y = 5$ ,  $2x + 2y = 10$   
 (ii)  $x - y = 8$ ,  $3x - 3y = 16$   
 (iii)  $2x + y - 6 = 0$ ,  $4x - 2y - 4 = 0$   
 (iv)  $2x - 2y - 2 = 0$ ,  $4x - 4y - 5 = 0$
5. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.
6. Given the linear equation  $2x + 3y - 8 = 0$ , write another linear equation in two variables such that the geometrical representation of the pair so formed is:  
 (i) intersecting lines (ii) parallel lines  
 (iii) coincident lines
7. Draw the graphs of the equations  $x - y + 1 = 0$  and  $3x + 2y - 12 = 0$ . Determine the coordinates of the vertices of the triangle formed by these lines and the  $x$ -axis, and shade the triangular region.

### 3.4 Algebraic Methods of Solving a Pair of Linear Equations

In the previous section, we discussed how to solve a pair of linear equations graphically. The graphical method is not convenient in cases when the point representing the solution of the linear equations has non-integral coordinates like  $(\sqrt{3}, 2\sqrt{7})$ ,

$(-1.75, 3.3)$ ,  $(\frac{4}{13}, \frac{1}{19})$ , etc. There is every possibility of making mistakes while reading such coordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall now discuss.

**3.4.1 Substitution Method :** We shall explain the method of substitution by taking some examples.

**Example 7 :** Solve the following pair of equations by substitution method:

$$7x - 15y = 2 \quad (1)$$

$$x + 2y = 3 \quad (2)$$

**Solution :**

**Step 1 :** We pick either of the equations and write one variable in terms of the other. Let us consider the Equation (2) :

$$x + 2y = 3$$

and write it as

$$x = 3 - 2y \quad (3)$$

**Step 2 :** Substitute the value of  $x$  in Equation (1). We get

$$7(3 - 2y) - 15y = 2$$

i.e.,  $21 - 14y - 15y = 2$

i.e.,  $-29y = -19$

Therefore,  $y = \frac{19}{29}$

**Step 3 :** Substituting this value of  $y$  in Equation (3), we get

$$x = 3 - 2\left(\frac{19}{29}\right) = \frac{49}{29}$$

Therefore, the solution is  $x = \frac{49}{29}$ ,  $y = \frac{19}{29}$ .

**Verification :** Substituting  $x = \frac{49}{29}$  and  $y = \frac{19}{29}$ , you can verify that both the Equations (1) and (2) are satisfied.

To understand the substitution method more clearly, let us consider it stepwise:

**Step 1 :** Find the value of one variable, say  $y$  in terms of the other variable, i.e.,  $x$  from either equation, whichever is convenient.

**Step 2 :** Substitute this value of  $y$  in the other equation, and reduce it to an equation in one variable, i.e., in terms of  $x$ , which can be solved. Sometimes, as in Examples 9 and 10 below, you can get statements with no variable. If this statement is true, you can conclude that the pair of linear equations has infinitely many solutions. If the statement is false, then the pair of linear equations is inconsistent.

**Step 3 :** Substitute the value of  $x$  (or  $y$ ) obtained in Step 2 in the equation used in Step 1 to obtain the value of the other variable.

**Remark :** We have **substituted** the value of one variable by expressing it in terms of the other variable to solve the pair of linear equations. That is why the method is known as the *substitution method*.

**Example 8 :** Solve Q.1 of Exercise 3.1 by the method of substitution.

**Solution :** Let  $s$  and  $t$  be the ages (in years) of Aftab and his daughter, respectively. Then, the pair of linear equations that represent the situation is

$$s - 7 = 7(t - 7), \text{ i.e., } s - 7t + 42 = 0 \quad (1)$$

and  $s + 3 = 3(t + 3), \text{ i.e., } s - 3t = 6 \quad (2)$

Using Equation (2), we get  $s = 3t + 6$ .

Putting this value of  $s$  in Equation (1), we get

$$(3t + 6) - 7t + 42 = 0,$$

i.e.,

$$4t = 48, \text{ which gives } t = 12.$$

Putting this value of  $t$  in Equation (2), we get

$$s = 3(12) + 6 = 42$$

So, Aftab and his daughter are 42 and 12 years old, respectively.

Verify this answer by checking if it satisfies the conditions of the given problems.

**Example 9 :** Let us consider Example 2 in Section 3.3, i.e., the cost of 2 pencils and 3 erasers is ₹ 9 and the cost of 4 pencils and 6 erasers is ₹ 18. Find the cost of each pencil and each eraser.

**Solution :** The pair of linear equations formed were:

$$2x + 3y = 9 \quad (1)$$

$$4x + 6y = 18 \quad (2)$$

We first express the value of  $x$  in terms of  $y$  from the equation  $2x + 3y = 9$ , to get

$$x = \frac{9 - 3y}{2} \quad (3)$$

Now we substitute this value of  $x$  in Equation (2), to get

$$\frac{4(9 - 3y)}{2} + 6y = 18$$

i.e.,

$$18 - 6y + 6y = 18$$

i.e.,

$$18 = 18$$

This statement is true for all values of  $y$ . However, we do not get a specific value of  $y$  as a solution. Therefore, we cannot obtain a specific value of  $x$ . This situation has arisen because both the given equations are the same. Therefore, Equations (1) and (2) have *infinitely many solutions*. Observe that we have obtained the same solution graphically also. (Refer to Fig. 3.3, Section 3.2.) We cannot find a unique cost of a pencil and an eraser, because there are many common solutions, to the given situation.

**Example 10 :** Let us consider the Example 3 of Section 3.2. Will the rails cross each other?



**Solution :** The pair of linear equations formed were:

$$x + 2y - 4 = 0 \quad (1)$$

$$2x + 4y - 12 = 0 \quad (2)$$

We express  $x$  in terms of  $y$  from Equation (1) to get

$$x = 4 - 2y$$

Now, we substitute this value of  $x$  in Equation (2) to get

$$2(4 - 2y) + 4y - 12 = 0$$

$$\text{i.e.,} \quad 8 - 12 = 0$$

$$\text{i.e.,} \quad -4 = 0$$

which is a false statement.

Therefore, the equations do not have a common solution. So, the two rails will not cross each other.

### EXERCISE 3.3

1. Solve the following pair of linear equations by the substitution method.

(i)  $x + y = 14$

(ii)  $s - t = 3$

$$x - y = 4$$

$$\frac{s}{3} + \frac{t}{2} = 6$$

(iii)  $3x - y = 3$

(iv)  $0.2x + 0.3y = 1.3$

$$9x - 3y = 9$$

$$0.4x + 0.5y = 2.3$$

(v)  $\sqrt{2}x + \sqrt{3}y = 0$

(vi)  $\frac{3x}{2} - \frac{5y}{3} = -2$

$$\sqrt{3}x - \sqrt{8}y = 0$$

$$\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$$

2. Solve  $2x + 3y = 11$  and  $2x - 4y = -24$  and hence find the value of ' $m$ ' for which  $y = mx + 3$ .

3. Form the pair of linear equations for the following problems and find their solution by substitution method.

(i) The difference between two numbers is 26 and one number is three times the other. Find them.

(ii) The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.

(iii) The coach of a cricket team buys 7 bats and 6 balls for ₹ 3800. Later, she buys 3 bats and 5 balls for ₹ 1750. Find the cost of each bat and each ball.

- (iv) The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹ 105 and for a journey of 15 km, the charge paid is ₹ 155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?
- (v) A fraction becomes  $\frac{9}{11}$ , if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes  $\frac{5}{6}$ . Find the fraction.
- (vi) Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages?

### 3.4.2 Elimination Method

Now let us consider another method of eliminating (i.e., removing) one variable. This is sometimes more convenient than the substitution method. Let us see how this method works.

**Example 11 :** The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly incomes.

**Solution :** Let us denote the incomes of the two person by ₹  $9x$  and ₹  $7x$  and their expenditures by ₹  $4y$  and ₹  $3y$  respectively. Then the equations formed in the situation is given by :

$$9x - 4y = 2000 \quad (1)$$

and

$$7x - 3y = 2000 \quad (2)$$

**Step 1 :** Multiply Equation (1) by 3 and Equation (2) by 4 to make the coefficients of  $y$  equal. Then we get the equations:

$$27x - 12y = 6000 \quad (3)$$

$$28x - 12y = 8000 \quad (4)$$

**Step 2 :** Subtract Equation (3) from Equation (4) to *eliminate*  $y$ , because the coefficients of  $y$  are the same. So, we get

$$(28x - 27x) - (12y - 12y) = 8000 - 6000$$

$$\text{i.e.,} \quad x = 2000$$

**Step 3 :** Substituting this value of  $x$  in (1), we get

$$9(2000) - 4y = 2000$$

$$\text{i.e.,} \quad y = 4000$$

So, the solution of the equations is  $x = 2000$ ,  $y = 4000$ . Therefore, the monthly incomes of the persons are ₹ 18,000 and ₹ 14,000, respectively.

**Verification :**  $18000 : 14000 = 9 : 7$ . Also, the ratio of their expenditures =  $18000 - 2000 : 14000 - 2000 = 16000 : 12000 = 4 : 3$

**Remarks :**

1. The method used in solving the example above is called the *elimination* method, because we eliminate one variable first, to get a linear equation in one variable. In the example above, we eliminated  $y$ . We could also have eliminated  $x$ . Try doing it that way.
2. You could also have used the substitution, or graphical method, to solve this problem. Try doing so, and see which method is more convenient.

Let us now note down these steps in the elimination method :

**Step 1 :** First multiply both the equations by some suitable non-zero constants to make the coefficients of one variable (either  $x$  or  $y$ ) numerically equal.

**Step 2 :** Then add or subtract one equation from the other so that one variable gets eliminated. If you get an equation in one variable, go to Step 3.

If in Step 2, we obtain a true statement involving no variable, then the original pair of equations has infinitely many solutions.

If in Step 2, we obtain a false statement involving no variable, then the original pair of equations has no solution, i.e., it is inconsistent.

**Step 3 :** Solve the equation in one variable ( $x$  or  $y$ ) so obtained to get its value.

**Step 4 :** Substitute this value of  $x$  (or  $y$ ) in either of the original equations to get the value of the other variable.

Now to illustrate it, we shall solve few more examples.

**Example 12 :** Use elimination method to find all possible solutions of the following pair of linear equations :

$$2x + 3y = 8 \quad (1)$$

$$4x + 6y = 7 \quad (2)$$

**Solution :**

**Step 1 :** Multiply Equation (1) by 2 and Equation (2) by 1 to make the coefficients of  $x$  equal. Then we get the equations as :

$$4x + 6y = 16 \quad (3)$$

$$4x + 6y = 7 \quad (4)$$

**Step 2 :** Subtracting Equation (4) from Equation (3),

$$(4x - 4x) + (6y - 6y) = 16 - 7$$

i.e.,  $0 = 9$ , which is a false statement.

Therefore, the pair of equations has no solution.

**Example 13 :** The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

**Solution :** Let the ten's and the unit's digits in the first number be  $x$  and  $y$ , respectively. So, the first number may be written as  $10x + y$  in the expanded form (for example,  $56 = 10(5) + 6$ ).

When the digits are reversed,  $x$  becomes the unit's digit and  $y$  becomes the ten's digit. This number, in the expanded notation is  $10y + x$  (for example, when 56 is reversed, we get  $65 = 10(6) + 5$ ).

According to the given condition.

$$(10x + y) + (10y + x) = 66$$

$$\text{i.e., } 11(x + y) = 66$$

$$\text{i.e., } x + y = 6 \quad (1)$$

We are also given that the digits differ by 2, therefore,

$$\text{either } x - y = 2 \quad (2)$$

$$\text{or } y - x = 2 \quad (3)$$

If  $x - y = 2$ , then solving (1) and (2) by elimination, we get  $x = 4$  and  $y = 2$ .

In this case, we get the number 42.

If  $y - x = 2$ , then solving (1) and (3) by elimination, we get  $x = 2$  and  $y = 4$ .

In this case, we get the number 24.

Thus, there are two such numbers 42 and 24.

**Verification :** Here  $42 + 24 = 66$  and  $4 - 2 = 2$ . Also  $24 + 42 = 66$  and  $4 - 2 = 2$ .

### EXERCISE 3.4

1. Solve the following pair of linear equations by the elimination method and the substitution method :

(i)  $x + y = 5$  and  $2x - 3y = 4$

(ii)  $3x + 4y = 10$  and  $2x - 2y = 2$

(iii)  $3x - 5y - 4 = 0$  and  $9x = 2y + 7$

(iv)  $\frac{x}{2} + \frac{2y}{3} = -1$  and  $x - \frac{y}{3} = 3$

2. Form the pair of linear equations in the following problems, and find their solutions (if they exist) by the elimination method :
- If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes  $\frac{1}{2}$  if we only add 1 to the denominator. What is the fraction?
  - Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?
  - The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.
  - Meena went to a bank to withdraw ₹ 2000. She asked the cashier to give her ₹ 50 and ₹ 100 notes only. Meena got 25 notes in all. Find how many notes of ₹ 50 and ₹ 100 she received.
  - A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹ 27 for a book kept for seven days, while Susy paid ₹ 21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.

### 3.4.3 Cross - Multiplication Method

So far, you have learnt how to solve a pair of linear equations in two variables by graphical, substitution and elimination methods. Here, we introduce one more algebraic method to solve a pair of linear equations which for many reasons is a very useful method of solving these equations. Before we proceed further, let us consider the following situation.

The cost of 5 oranges and 3 apples is ₹ 35 and the cost of 2 oranges and 4 apples is ₹ 28. Let us find the cost of an orange and an apple.

Let us denote the cost of an orange by ₹  $x$  and the cost of an apple by ₹  $y$ . Then, the equations formed are :

$$5x + 3y = 35, \text{ i.e., } 5x + 3y - 35 = 0 \quad (1)$$

$$2x + 4y = 28, \text{ i.e., } 2x + 4y - 28 = 0 \quad (2)$$

Let us use the elimination method to solve these equations.

Multiply Equation (1) by 4 and Equation (2) by 3. We get

$$(4)(5)x + (4)(3)y + (4)(-35) = 0 \quad (3)$$

$$(3)(2)x + (3)(4)y + (3)(-28) = 0 \quad (4)$$

Subtracting Equation (4) from Equation (3), we get

$$[(5)(4) - (3)(2)]x + [(4)(3) - (3)(4)]y + [4(-35) - (3)(-28)] = 0$$

Therefore,

$$x = \frac{-[(4)(-35) - (3)(-28)]}{(5)(4) - (3)(2)}$$

i.e.,

$$x = \frac{(3)(-28) - (4)(-35)}{(5)(4) - (2)(3)} \quad (5)$$

If Equations (1) and (2) are written as  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$ , then we have

$$a_1 = 5, b_1 = 3, c_1 = -35, a_2 = 2, b_2 = 4, c_2 = -28.$$

Then Equation (5) can be written as  $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ ,

Similarly, you can get  $y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

By simplifying Equation (5), we get

$$x = \frac{-84 + 140}{20 - 6} = 4$$

Similarly,

$$y = \frac{(-35)(2) - (5)(-28)}{20 - 6} = \frac{-70 + 140}{14} = 5$$

Therefore,  $x = 4, y = 5$  is the solution of the given pair of equations.

Then, the cost of an orange is ₹ 4 and that of an apple is ₹ 5.

**Verification :** Cost of 5 oranges + Cost of 3 apples = ₹ 20 + ₹ 15 = ₹ 35. Cost of 2 oranges + Cost of 4 apples = ₹ 8 + ₹ 20 = ₹ 28.

Let us now see how this method works for any pair of linear equations in two variables of the form

$$a_1x + b_1y + c_1 = 0 \quad (1)$$

and  $a_2x + b_2y + c_2 = 0 \quad (2)$

To obtain the values of  $x$  and  $y$  as shown above, we follow the following steps:

**Step 1 :** Multiply Equation (1) by  $b_2$  and Equation (2) by  $b_1$ , to get

$$b_2a_1x + b_2b_1y + b_2c_1 = 0 \quad (3)$$

$$b_1a_2x + b_1b_2y + b_1c_2 = 0 \quad (4)$$

**Step 2 :** Subtracting Equation (4) from (3), we get:

$$(b_2a_1 - b_1a_2)x + (b_2b_1 - b_1b_2)y + (b_2c_1 - b_1c_2) = 0$$

i.e.,  $(b_2a_1 - b_1a_2)x = b_1c_2 - b_2c_1$

So,  $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ , provided  $a_1b_2 - a_2b_1 \neq 0$  (5)

**Step 3 :** Substituting this value of  $x$  in (1) or (2), we get

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad (6)$$

Now, two cases arise :

**Case 1 :**  $a_1b_2 - a_2b_1 \neq 0$ . In this case  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ . Then the pair of linear equations has a unique solution.

**Case 2 :**  $a_1b_2 - a_2b_1 = 0$ . If we write  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$ , then  $a_1 = k a_2$ ,  $b_1 = k b_2$ .

Substituting the values of  $a_1$  and  $b_1$  in the Equation (1), we get

$$k(a_2x + b_2y) + c_1 = 0. \quad (7)$$

It can be observed that the Equations (7) and (2) can both be satisfied only if

$$c_1 = k c_2, \text{ i.e., } \frac{c_1}{c_2} = k.$$

If  $c_1 = k c_2$ , any solution of Equation (2) will satisfy the Equation (1), and vice versa. So, if  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$ , then there are infinitely many solutions to the pair of linear equations given by (1) and (2).

If  $c_1 \neq k c_2$ , then any solution of Equation (1) will not satisfy Equation (2) and vice versa. Therefore the pair has no solution.

We can summarise the discussion above for the pair of linear equations given by (1) and (2) as follows:

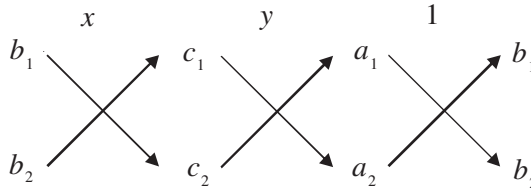
- (i) When  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ , we get a unique solution.
- (ii) When  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ , there are infinitely many solutions.
- (iii) When  $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ , there is no solution.



Note that you can write the solution given by Equations (5) and (6) in the following form :

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \quad (8)$$

In remembering the above result, the following diagram may be helpful to you :



The arrows between the two numbers indicate that they are to be multiplied and the second product is to be subtracted from the first.

For solving a pair of linear equations by this method, we will follow the following steps :

**Step 1 :** Write the given equations in the form (1) and (2).

**Step 2 :** Taking the help of the diagram above, write Equations as given in (8).

**Step 3 :** Find  $x$  and  $y$ , provided  $a_1b_2 - a_2b_1 \neq 0$

Step 2 above gives you an indication of why this method is called the **cross-multiplication method**.

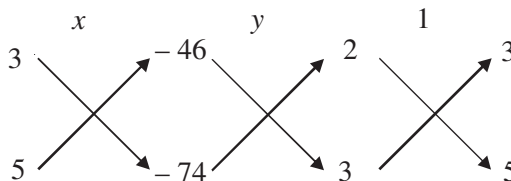
**Example 14 :** From a bus stand in Bangalore , if we buy 2 tickets to Malleswaram and 3 tickets to Yeshwanthpur, the total cost is ₹ 46; but if we buy 3 tickets to Malleswaram and 5 tickets to Yeshwanthpur the total cost is ₹ 74. Find the fares from the bus stand to Malleswaram, and to Yeshwanthpur.

**Solution :** Let ₹  $x$  be the fare from the bus stand in Bangalore to Malleswaram, and ₹  $y$  to Yeshwanthpur. From the given information, we have

$$2x + 3y = 46, \text{ i.e., } 2x + 3y - 46 = 0 \quad (1)$$

$$3x + 5y = 74, \text{ i.e., } 3x + 5y - 74 = 0 \quad (2)$$

To solve the equations by the cross-multiplication method, we draw the diagram as given below.



Then 
$$\frac{x}{(3)(-74) - (5)(-46)} = \frac{y}{(-46)(3) - (-74)(2)} = \frac{1}{(2)(5) - (3)(3)}$$

i.e., 
$$\frac{x}{-222 + 230} = \frac{y}{-138 + 148} = \frac{1}{10 - 9}$$

i.e., 
$$\frac{x}{8} = \frac{y}{10} = \frac{1}{1}$$

i.e., 
$$\frac{x}{8} = \frac{1}{1} \text{ and } \frac{y}{10} = \frac{1}{1}$$

i.e., 
$$x = 8 \text{ and } y = 10$$

Hence, the fare from the bus stand in Bangalore to Malleswaram is ₹ 8 and the fare to Yeshwanthpur is ₹ 10.

**Verification :** You can check from the problem that the solution we have got is correct.

**Example 15 :** For which values of  $p$  does the pair of equations given below has unique solution?

$$4x + py + 8 = 0$$

$$2x + 2y + 2 = 0$$

**Solution :** Here  $a_1 = 4$ ,  $a_2 = 2$ ,  $b_1 = p$ ,  $b_2 = 2$ .

Now for the given pair to have a unique solution :  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

i.e., 
$$\frac{4}{2} \neq \frac{p}{2}$$

i.e., 
$$p \neq 4$$

Therefore, for all values of  $p$ , except 4, the given pair of equations will have a unique solution.

**Example 16 :** For what values of  $k$  will the following pair of linear equations have infinitely many solutions?

$$kx + 3y - (k - 3) = 0$$

$$12x + ky - k = 0$$

**Solution :** Here,  $\frac{a_1}{a_2} = \frac{k}{12}$ ,  $\frac{b_1}{b_2} = \frac{3}{k}$ ,  $\frac{c_1}{c_2} = \frac{k-3}{k}$

For a pair of linear equations to have infinitely many solutions :  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

So, we need

$$\frac{k}{12} = \frac{3}{k} = \frac{k-3}{k}$$

or,

$$\frac{k}{12} = \frac{3}{k}$$

which gives  $k^2 = 36$ , i.e.,  $k = \pm 6$ .

Also,

$$\frac{3}{k} = \frac{k-3}{k}$$

gives  $3k = k^2 - 3k$ , i.e.,  $6k = k^2$ , which means  $k = 0$  or  $k = 6$ .

Therefore, the value of  $k$ , that satisfies both the conditions, is  $k = 6$ . For this value, the pair of linear equations has infinitely many solutions.

### EXERCISE 3.5

- Which of the following pairs of linear equations has unique solution, no solution, or infinitely many solutions. In case there is a unique solution, find it by using cross multiplication method.

(i)  $x - 3y - 3 = 0$

(ii)  $2x + y = 5$

$3x - 9y - 2 = 0$

$3x + 2y = 8$

(iii)  $3x - 5y = 20$

(iv)  $x - 3y - 7 = 0$

$6x - 10y = 40$

$3x - 3y - 15 = 0$

- For which values of  $a$  and  $b$  does the following pair of linear equations have an infinite number of solutions?

$2x + 3y = 7$

$(a - b)x + (a + b)y = 3a + b - 2$

- For which value of  $k$  will the following pair of linear equations have no solution?

$3x + y = 1$

$(2k - 1)x + (k - 1)y = 2k + 1$

- Solve the following pair of linear equations by the substitution and cross-multiplication methods :

$8x + 5y = 9$

$3x + 2y = 4$

- Form the pair of linear equations in the following problems and find their solutions (if they exist) by any algebraic method :

- (i) A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When a student A takes food for 20 days she has to pay ₹ 1000 as hostel charges whereas a student B, who takes food for 26 days, pays ₹ 1180 as hostel charges. Find the fixed charges and the cost of food per day.
- (ii) A fraction becomes  $\frac{1}{3}$  when 1 is subtracted from the numerator and it becomes  $\frac{1}{4}$  when 8 is added to its denominator. Find the fraction.
- (iii) Yash scored 40 marks in a test, getting 3 marks for each right answer and losing 1 mark for each wrong answer. Had 4 marks been awarded for each correct answer and 2 marks been deducted for each incorrect answer, then Yash would have scored 50 marks. How many questions were there in the test?
- (iv) Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?
- (v) The area of a rectangle gets reduced by 9 square units, if its length is reduced by 5 units and breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, the area increases by 67 square units. Find the dimensions of the rectangle.

### 3.5 Equations Reducible to a Pair of Linear Equations in Two Variables

In this section, we shall discuss the solution of such pairs of equations which are not linear but can be reduced to linear form by making some suitable substitutions. We now explain this process through some examples.

**Example 17 :** Solve the pair of equations:

$$\begin{aligned}\frac{2}{x} + \frac{3}{y} &= 13 \\ \frac{5}{x} - \frac{4}{y} &= -2\end{aligned}$$

**Solution :** Let us write the given pair of equations as

$$2\left(\frac{1}{x}\right) + 3\left(\frac{1}{y}\right) = 13 \quad (1)$$

$$5\left(\frac{1}{x}\right) - 4\left(\frac{1}{y}\right) = -2 \quad (2)$$

These equations are not in the form  $ax + by + c = 0$ . However, if we substitute

$\frac{1}{x} = p$  and  $\frac{1}{y} = q$  in Equations (1) and (2), we get

$$2p + 3q = 13 \quad (3)$$

$$5p - 4q = -2 \quad (4)$$

So, we have expressed the equations as a pair of linear equations. Now, you can use any method to solve these equations, and get  $p = 2$ ,  $q = 3$ .

You know that  $p = \frac{1}{x}$  and  $q = \frac{1}{y}$ .

Substitute the values of  $p$  and  $q$  to get

$$\frac{1}{x} = 2, \text{ i.e., } x = \frac{1}{2} \text{ and } \frac{1}{y} = 3, \text{ i.e., } y = \frac{1}{3}.$$

**Verification :** By substituting  $x = \frac{1}{2}$  and  $y = \frac{1}{3}$  in the given equations, we find that both the equations are satisfied.

**Example 18 :** Solve the following pair of equations by reducing them to a pair of linear equations :

$$\frac{5}{x-1} + \frac{1}{y-2} = 2$$

$$\frac{6}{x-1} - \frac{3}{y-2} = 1$$

**Solution :** Let us put  $\frac{1}{x-1} = p$  and  $\frac{1}{y-2} = q$ . Then the given equations

$$5\left(\frac{1}{x-1}\right) + \frac{1}{y-2} = 2 \quad (1)$$

$$6\left(\frac{1}{x-1}\right) - 3\left(\frac{1}{y-2}\right) = 1 \quad (2)$$

can be written as :  $5p + q = 2 \quad (3)$

$$6p - 3q = 1 \quad (4)$$

Equations (3) and (4) form a pair of linear equations in the general form. Now, you can use any method to solve these equations. We get  $p = \frac{1}{3}$  and  $q = \frac{1}{3}$ .

Now, substituting  $\frac{1}{x-1}$  for  $p$ , we have

$$\frac{1}{x-1} = \frac{1}{3},$$

i.e.,  $x - 1 = 3$ , i.e.,  $x = 4$ .

Similarly, substituting  $\frac{1}{y-2}$  for  $q$ , we get

$$\frac{1}{y-2} = \frac{1}{3}$$

i.e.,  $3 = y - 2$ , i.e.,  $y = 5$

Hence,  $x = 4$ ,  $y = 5$  is the required solution of the given pair of equations.

**Verification :** Substitute  $x = 4$  and  $y = 5$  in (1) and (2) to check whether they are satisfied.

**Example 19 :** A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours, it can go 40 km upstream and 55 km down-stream. Determine the speed of the stream and that of the boat in still water.

**Solution :** Let the speed of the boat in still water be  $x$  km/h and speed of the stream be  $y$  km/h. Then the speed of the boat downstream =  $(x + y)$  km/h,

and the speed of the boat upstream =  $(x - y)$  km/h

Also, 
$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

In the first case, when the boat goes 30 km upstream, let the time taken, in hour, be  $t_1$ . Then

$$t_1 = \frac{30}{x - y}$$



Let  $t_2$  be the time, in hours, taken by the boat to go 44 km downstream. Then  $t_2 = \frac{44}{x+y}$ . The total time taken,  $t_1 + t_2$ , is 10 hours. Therefore, we get the equation

$$\frac{30}{x-y} + \frac{44}{x+y} = 10 \quad (1)$$

In the second case, in 13 hours it can go 40 km upstream and 55 km downstream. We get the equation

$$\frac{40}{x-y} + \frac{55}{x+y} = 13 \quad (2)$$

Put  $\frac{1}{x-y} = u$  and  $\frac{1}{x+y} = v$  (3)

On substituting these values in Equations (1) and (2), we get the pair of linear equations:

$$30u + 44v = 10 \quad \text{or} \quad 30u + 44v - 10 = 0 \quad (4)$$

$$40u + 55v = 13 \quad \text{or} \quad 40u + 55v - 13 = 0 \quad (5)$$

Using Cross-multiplication method, we get

$$\frac{u}{44(-13) - 55(-10)} = \frac{v}{40(-10) - 30(-13)} = \frac{1}{30(55) - 44(40)}$$

i.e.,  $\frac{u}{-22} = \frac{v}{-10} = \frac{1}{-110}$

i.e.,  $u = \frac{1}{5}, \quad v = \frac{1}{11}$

Now put these values of  $u$  and  $v$  in Equations (3), we get

$$\frac{1}{x-y} = \frac{1}{5} \quad \text{and} \quad \frac{1}{x+y} = \frac{1}{11}$$

i.e.,  $x-y = 5$  and  $x+y = 11$  (6)

Adding these equations, we get

$$2x = 16$$

i.e.,  $x = 8$

Subtracting the equations in (6), we get

$$2y = 6$$

i.e.,  $y = 3$



Hence, the speed of the boat in still water is 8 km/h and the speed of the stream is 3 km/h.

**Verification :** Verify that the solution satisfies the conditions of the problem.

### EXERCISE 3.6

1. Solve the following pairs of equations by reducing them to a pair of linear equations:

$$(i) \quad \frac{1}{2x} + \frac{1}{3y} = 2$$

$$(ii) \quad \frac{2}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 2$$

$$\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6}$$

$$\frac{4}{\sqrt{x}} - \frac{9}{\sqrt{y}} = -1$$

$$(iii) \quad \frac{4}{x} + 3y = 14$$

$$(iv) \quad \frac{5}{x-1} + \frac{1}{y-2} = 2$$

$$\frac{3}{x} - 4y = 23$$

$$\frac{6}{x-1} - \frac{3}{y-2} = 1$$

$$(v) \quad \frac{7x-2y}{xy} = 5$$

$$(vi) \quad 6x+3y=6xy$$

$$\frac{8x+7y}{xy} = 15$$

$$2x+4y=5xy$$

$$(vii) \quad \frac{10}{x+y} + \frac{2}{x-y} = 4$$

$$(viii) \quad \frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}$$

$$\frac{15}{x+y} - \frac{5}{x-y} = -2$$

$$\frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = \frac{-1}{8}$$

2. Formulate the following problems as a pair of equations, and hence find their solutions:

- (i) Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current.
- (ii) 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work, and also that taken by 1 man alone.
- (iii) Roohi travels 300 km to her home partly by train and partly by bus. She takes 4 hours if she travels 60 km by train and the remaining by bus. If she travels 100 km by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus separately.

### EXERCISE 3.7 (Optional)\*

- The ages of two friends Ani and Biju differ by 3 years. Ani's father Dharam is twice as old as Ani and Biju is twice as old as his sister Cathy. The ages of Cathy and Dharam differ by 30 years. Find the ages of Ani and Biju.
- One says, "Give me a hundred, friend! I shall then become twice as rich as you". The other replies, "If you give me ten, I shall be six times as rich as you". Tell me what is the amount of their (respective) capital? [From the Bijaganita of Bhaskara II]  
[Hint :  $x + 100 = 2(y - 100)$ ,  $y + 10 = 6(x - 10)$ ].
- A train covered a certain distance at a uniform speed. If the train would have been 10 km/h faster, it would have taken 2 hours less than the scheduled time. And, if the train were slower by 10 km/h; it would have taken 3 hours more than the scheduled time. Find the distance covered by the train.
- The students of a class are made to stand in rows. If 3 students are extra in a row, there would be 1 row less. If 3 students are less in a row, there would be 2 rows more. Find the number of students in the class.
- In a  $\triangle ABC$ ,  $\angle C = 3\angle B = 2(\angle A + \angle B)$ . Find the three angles.
- Draw the graphs of the equations  $5x - y = 5$  and  $3x - y = 3$ . Determine the co-ordinates of the vertices of the triangle formed by these lines and the y axis.
- Solve the following pair of linear equations:
  - $$\begin{aligned} px + qy &= p - q \\ qx - py &= p + q \end{aligned}$$
  - $$\begin{aligned} ax + by &= c \\ bx + ay &= 1 + c \end{aligned}$$
  - $$\begin{aligned} \frac{x}{a} - \frac{y}{b} &= 0 \\ ax + by &= a^2 + b^2 \end{aligned}$$
  - $$\begin{aligned} (a - b)x + (a + b)y &= a^2 - 2ab - b^2 \\ (a + b)(x + y) &= a^2 + b^2 \end{aligned}$$
  - $$\begin{aligned} 152x - 378y &= -74 \\ -378x + 152y &= -604 \end{aligned}$$
- ABCD is a cyclic quadrilateral (see Fig. 3.7). Find the angles of the cyclic quadrilateral.

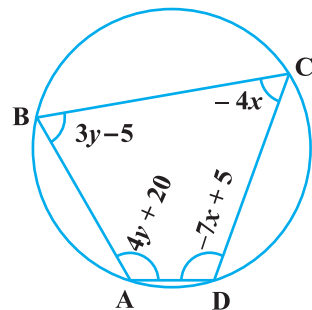


Fig. 3.7

\* These exercises are not from the examination point of view.

### 3.6 Summary

In this chapter, you have studied the following points:

1. Two linear equations in the same two variables are called a pair of linear equations in two variables. The most general form of a pair of linear equations is

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

where  $a_1, a_2, b_1, b_2, c_1, c_2$  are real numbers, such that  $a_1^2 + b_1^2 \neq 0, a_2^2 + b_2^2 \neq 0$ .

2. A pair of linear equations in two variables can be represented, and solved, by the:

- (i) graphical method
- (ii) algebraic method

3. Graphical Method :

The graph of a pair of linear equations in two variables is represented by two lines.

- (i) If the lines intersect at a point, then that point gives the unique solution of the two equations. In this case, the pair of equations is **consistent**.
  - (ii) If the lines coincide, then there are infinitely many solutions — each point on the line being a solution. In this case, the pair of equations is **dependent (consistent)**.
  - (iii) If the lines are parallel, then the pair of equations has no solution. In this case, the pair of equations is **inconsistent**.
4. Algebraic Methods : We have discussed the following methods for finding the solution(s) of a pair of linear equations :
    - (i) Substitution Method
    - (ii) Elimination Method
    - (iii) Cross-multiplication Method
  5. If a pair of linear equations is given by  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$ , then the following situations can arise :

- (i)  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$  : In this case, the pair of linear equations is consistent.

- (ii)  $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$  : In this case, the pair of linear equations is inconsistent.

- (iii)  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$  : In this case, the pair of linear equations is dependent and consistent.

6. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we alter them so that they are reduced to a pair of linear equations.