5. Definite Integral

- The definite integral $\int_a^b f(x)\,dx$ can be expressed as the sum of limits as

  $\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)h)]$, where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

- First fundamental theorem of integral calculus: Let f be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then, $A'(x) = f(x)\ \forall x \in [a, b]$
- Second fundamental theorem of integral calculus: Let f be a continuous function on the closed interval $[a, b]$ and let F be an anti-derivative of f. Then,

  $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

  Example 2: Find: $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

  Solution:

  $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x = F(x)$

  By second fundamental theorem, we have

  $$\therefore \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}x\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$$

  $$= \frac{\pi}{3} - \frac{\pi}{6}$$

  $$= \frac{\pi}{6}$$

- Definite integral: A definite integral is denoted by $\int_a^b f(x)\,dx$, where a is the lower limit and b is the upper limit of the integral. If $\int f(x)\,dx = F(x) + C$, then

  $$\int_a^b f(x)\,dx = F(b) - F(a)$$

- The definite integral $\int_a^b f(x)\,dx$ represents the area function A(x) since $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, $x \in [a, b]$, the x-axis, and the ordinates $x = a$ and $x = b$

- The steps for evaluating $\int_a^b f(x)\,dx$ by substitution method can be listed as:
- **Step l:** Considering the integral without limits, substitute $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form and the limits of integral are accordingly changed.

  Step 2: Integrate the new integrand with respect to the new variable, and then find the difference of the values at the obtained upper and lower limits.

Example:

Evaluate: $\int_1^2 \frac{3x^2}{1+x^3} dx$

Solution:

Put $1 + x^3 = t$

Then, $3x^2 dx = dt$

When $x = 1$, $t = 2$

$x = 2$, $t = 9$

$$\therefore \int_1^2 \frac{3x^2}{4+x^3} dx = \int_2^9 \frac{dt}{t} = [\log t]_2^9 = \log 9 - \log 2 = \log \frac{9}{2}$$

Evaluation of Definite Integrals through Integration by Parts

$$\int_a^b u.\, v\, dx = \left[u \int v\, dx\right]_a^b - \int_a^b \left[\frac{du}{dx} \int v\, dx\right] dx$$

Evaluation of Definite Integrals by Partial Fraction

In this method, we first simplify the rational function into two or more rational functions, which can be integrated more easily. We then integrate the expression using partial fraction as we do in case of indefinite integrals and put the limits to evaluate the definite integral.

- Some useful properties of definite integrals are as follows:
    - $\int_a^b f(x)dx = \int_a^b f(t)dt$
    - $\int_a^b f(x)dx = -\int_b^a f(x)dx$ · In particular, $\int_a^a f(x)dx = 0$
    - $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
    - $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
    - $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
    - $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
    - $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

- $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & \text{if } f \text{ is an even function i.e., if } f(-x) = f(x) \\ 0, \text{ if } f & \text{is an odd function i.e., if } f(-x) = -f(x) \end{cases}$$

Example 3: Evaluate: $\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\sin^6 x}{\sin^4 x - \cos^4 x} dx$

Solution:

Let $I = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\sin^6 x}{\sin^4 x - \cos^4 x} dx$ … (1)

Using the property of definite integrals, $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

$$I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{\sin^6 x}{\sin^4 x - \cos^4 x} dx$$

$$I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{\cos^6 x}{\cos^4 x - \sin^4 x} dx$$

Adding the above 2 equations we get,

$$2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{sin^6 x - cos^6 x}{sin^4 x - cos^4 x} dx$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{(sin^2 x)^3 - (cos^2 x)^3}{(sin^2 x - cos^2 x)(sin^2 x + cos^2 x)} dx$$

$$2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{(sin^2 x - cos^2 x)(sin^4 x + cos^4 x + \sin^2 x\, cos^2 x)}{(sin^2 x - cos^2 x)(sin^2 x + cos^2 x)} dx$$

$$2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \left(1 - sin^2 x\, cos^2 x\right) dx$$

$$2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \left(1 - \frac{sin^2 2x}{4}\right) dx$$

$$2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} 1 - \frac{\left(\frac{1 - \cos 4x}{2}\right)^2}{4}$$

$$2I = \frac{1}{16} \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \left[\frac{29}{2} + 2\cos 4x - \frac{\cos 8x}{2}\right] dx$$

$$I = \frac{1}{32}\left[\frac{29x}{2} + \frac{\sin 4x}{2} - \frac{\sin 8x}{16}\right]_{\frac{\pi}{2}}^{\frac{5\pi}{2}}$$

$$= \frac{1}{32}\left[29\pi + \frac{\sin 10\pi}{2} - \frac{\sin 20\pi}{16} - \frac{\sin 2\pi}{2} + \frac{\sin 4\pi}{16}\right]$$

$$= \frac{29\pi}{32}$$