7. Adjoint and Inverse of a Matrix

Exercise 7

1. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I.

$$\begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$$

Answer

Here, $A = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$

Now, we have to find adj A and for that we have to find co-factors:

a_{11} (co – factor of 2) = $(-1)^{1+1}(9) = (-1)^{2}(9) = 9$

a_{12} (co – factor of 3) = $(-1)^{1+2}(5) = (-1)^{3}(5) = -5$

a_{21} (co – factor of 5) = $(-1)^{2+1}(3) = (-1)^{3}(3) = -3$

a_{22} (co – factor of 9) = $(-1)^{2+2}(2) = (-1)^{4}(2) = 2$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 9 & -5 \\ -3 & 2 \end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix} 9 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$

Calculating A (adj A)

$A.(adj\ A) = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 9 + 3 \times (-5) & 2 \times (-3) + 3 \times 2 \\ 5 \times 9 + 9 \times (-5) & 5 \times (-3) + 9 \times 2 \end{bmatrix}$

$= \begin{bmatrix} 18 - 15 & -6 + 6 \\ 45 - 45 & -15 + 18 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= 3I

Calculating (adj A)A

$(adj\ A).A = \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$

$= \begin{bmatrix} 9 \times 2 + (-3) \times 5 & 9 \times 3 + (-3) \times 9 \\ -5 \times 2 + 2 \times 5 & -5 \times 3 + 2 \times 9 \end{bmatrix}$

$= \begin{bmatrix} 18 - 15 & 27 - 27 \\ -10 + 10 & -15 + 18 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= (3)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

= 3I

Calculating |A|.I

$|A|.I = \begin{vmatrix}2 & 3\\5 & 9\end{vmatrix} I$

If $A = \begin{matrix}a & b\\c & d\end{matrix}$, then determinant of A , is given by

$|A| = \begin{vmatrix}a & b\\c & d\end{vmatrix} = ad - bc$

= (2 × 9 – 3 × 5)I

= (18 – 15)I

= 3I

Thus, A(adj A) = (adj A)A = |A|I = 3I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix}9 & -3\\-5 & 2\end{bmatrix}$

2. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I. $\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$

Answer

Here, $A = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$

Now, we have to find adj A and for that we have to find co-factors:

a_{11} (co – factor of 3) = $(-1)^{1+1}(2) = (-1)^2(2) = 2$

a_{12} (co – factor of -5) = $(-1)^{1+2}(-1) = (-1)^3(-1) = 1$

a_{21} (co – factor of -1) = $(-1)^{2+1}(-5) = (-1)^3(-5) = 5$

a_{22} (co – factor of 2) = $(-1)^{2+2}(3) = (-1)^4(3) = 3$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}^T = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$

Calculating A (adj A)

$A.(adj\ A) = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$

$= \begin{bmatrix}3 \times 2 + (-5) \times 1 & 3 \times 5 + (-5) \times 3\\(-1) \times 2 + 2 \times 1 & (-1) \times 5 + 2 \times 3\end{bmatrix}$

$= \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= I

Calculating (adj A)A

$(adj\ A).A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 \times 3 + 5 \times (-1) & 2 \times (-5) + 5 \times 2 \\ 1 \times 3 + 3 \times (-1) & 1 \times (-5) + 3 \times 2 \end{bmatrix}$

$= \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

= I

Calculating |A|.I

$|A|.I = \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} I$

If $A = \begin{matrix} a & b \\ c & d \end{matrix}$, then determinant of A , is given by

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

= [3 × 2 – (-1) × (-5)]I

= [6 – (5)] I

= (1)I

= I

Thus, A(adj A) = (adj A)A = |A|I = I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

3. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I. $\begin{bmatrix} \text{Cos}\,\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Answer

Here, $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Now, we have to find adj A and for that we have to find co-factors:

a_{11} (co – factor of cos α) = $(-1)^{1+1}$(cos α)= $(-1)^2$(cos α) = cos α

a_{12} (co – factor of sin α) = $(-1)^{1+2}$(sin α) = $(-1)^3$(sin α) = -sin α

a_{21} (co – factor of sin α) = $(-1)^{2+1}$(sin α) = $(-1)^3$(sin α) = -sin α

a_{22} (co – factor of cos α) = $(-1)^{2+2}$(cos α)= $(-1)^4$(cos α) = cos α

$$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Now, adj A = Transpose of co-factor Matrix

$$\therefore adj\ A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Calculating A (adj A)

$$A.(adj\ A) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \times \cos\alpha + \sin\alpha \times (-\sin\alpha) & \cos\alpha \times (-\sin\alpha) + \sin\alpha \times \cos\alpha \\ \sin\alpha \times \cos\alpha + \cos\alpha \times (-\sin\alpha) & \sin\alpha \times (-\sin\alpha) + \cos\alpha \times \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & -\cos\alpha\sin\alpha + \cos\alpha\sin\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

= $(\cos^2$ α – $\sin^2$ α) I

Calculating (adj A)A

$$(adj\ A).A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \times \cos\alpha + (-\sin\alpha) \times \sin\alpha & \cos\alpha \times \sin\alpha + (-\sin\alpha) \times \cos\alpha \\ (-\sin\alpha) \times \cos\alpha + \cos\alpha \times \sin\alpha & (-\sin\alpha) \times \sin\alpha + \cos\alpha \times \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha - \cos\alpha\sin\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

= $(\cos^2$ α – $\sin^2$ α) I

Calculating |A|.I

$$|A|.I = \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} I$$

$$\text{If } A = \begin{matrix} a & b \\ c & d \end{matrix}, \text{then determinant of A , is given by}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

= [cos α × cos α – (sin α) × (sin α)]I

= $[\cos^2$ α – $\sin^2$ α] I

Thus, A(adj A) = (adj A)A = |A|I = I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

4. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Answer

Here, $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Now, we have to find adj A, and for that, we have to find co-factors:

$a_{11} = \begin{vmatrix} 1 & -2 \\ 0 & 3 \end{vmatrix} = 3 - (0) = 3$

$a_{12} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -(9 - (-2)) = -(9 + 2) = -11$

$a_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$a_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -(-3 - 0) = 3$

$a_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$

$a_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -\big(0 - (-1)\big) = -1$

$a_{31} = \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} = 2 - 2 = 0$

$a_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -(-2 - 6) = 8$

$a_{33} = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 1 - (-3) = 1 + 3 = 4$

$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 3 & -11 & -1 \\ 3 & 1 & -1 \\ 0 & 8 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$

Calculating A (adj A)

$A.(adj\ A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 + 11 - 2 & 3 - 1 - 2 & 0 - 8 + 8 \\ 9 - 11 + 2 & 9 + 1 + 2 & 0 + 8 - 8 \\ 3 - 0 - 3 & 3 + 0 - 3 & 0 + 0 + 12 \end{bmatrix}$

$= \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

$= 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= 12I

Calculating (adj A)A

$(adj\ A).A = \begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3+9+0 & -3+3+0 & 6-6+0 \\ -11+3+8 & 11+1+0 & -22-2+24 \\ -1-3+4 & 1-1+0 & -2+2+12 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= 12I

Calculating |A|.I

Expanding along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A|.I = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} I$$

= [1(3 – 0) – (-1){9 – (-2)} + 2(0 – 1)]I

= [3 + 1(11) + 2(-1)] I

= (3 + 11 – 2)I

= 12I

Thus, A(adj A) = (adj A)A = |A|I = 12I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$

5. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I.

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Answer

Here, $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

Now, we have to find adj A, and for that, we have to find co-factors:

$$a_{11} = \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} = 12 - (10) = 2$$

$$a_{12} = -\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} = -(-30 - (-25)) = -(-30 + 25) = 5$$

$$a_{13} = \begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} = 30 - 30 = 0$$

$$a_{21} = -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = -(-2 - (-2)) = 0$$

$$a_{22} = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 = 1$$

$$a_{23} = -\begin{vmatrix} 3 & -1 \\ 5 & -2 \end{vmatrix} = -\big(-6 - (-5)\big) = -(-6 + 5) = 1$$

$$a_{31} = \begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} = 5 - 6 = -1$$

$$a_{32} = -\begin{vmatrix} 3 & 1 \\ -15 & -5 \end{vmatrix} = -\big(-15 - (-15)\big) = -(-15 + 15) = 0$$

$$a_{33} = \begin{vmatrix} 3 & -1 \\ -15 & 6 \end{vmatrix} = 18 - 15 = 3$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Calculating A (adj A)

$$A.(adj\ A) = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 - 5 + 0 & 0 - 1 + 1 & -3 + 0 + 3 \\ -30 + 30 + 0 & 0 + 6 - 5 & 15 + 0 - 15 \\ 10 - 10 + 0 & 0 - 2 + 2 & -5 + 0 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= I

Calculating (adj A)A

$$(adj\ A).A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 + 0 - 5 & -2 + 0 + 2 & 2 + 0 - 2 \\ 15 - 15 + 0 & -5 + 6 + 0 & 5 - 5 + 0 \\ 0 - 15 + 15 & 0 + 6 - 6 & 0 - 5 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= I

Calculating |A|.I

Expanding along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A|.I = \begin{vmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix} I$$

= [3(12 – 10) – (-15){-2 – (-2)} + 5(5 – 6)]I

= [3(2) + 15(0) + 5(-1)] I

= (6 – 5)I

= I

Thus, A(adj A) = (adj A)A = |A|I = I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

6. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Answer

Here, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Now, we have to find adj A, and for that, we have to find co-factors:

$a_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2 - 3 = -1$

$a_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -(1 - 9) = 8$

$a_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5$

$a_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -(1 - 2) = 1$

$a_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = 0 - 6 = -6$

$a_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -(0 - 3) = 3$

$a_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$

$a_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(0 - 2) = 2$

$a_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 0 - 1 = -1$

$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

Calculating A (adj A)

$A.(adj\ A) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 0+8-10 & 0-6+6 & 0+2-2 \\ -1+16-15 & 1-12+9 & -1+4-3 \\ -3+8-5 & 3-6+3 & -3+2-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= -2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= -2I

Calculating (adj A)A

$$(adj\ A).A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1-3 & -1+2-1 & -2+3-1 \\ 0-6+6 & 8-12+2 & 16-18+2 \\ 0+3-3 & -5+6-1 & -10+9-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= -2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= -2I

Calculating |A|.I

Expanding along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A|.I = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} I$$

= [0(2 – 3) – (1){1 – 2} + 3(3 – 4)]I

= [0 – 1(-1) + 3(-1)] I

= (1 – 3)I

= -2I

Thus, A(adj A) = (adj A)A = |A|I = -2I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

7. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I. $\begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$

Answer

Here, $A = \begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} -1 & 4 \\ 8 & 2 \end{vmatrix} = -2 - 32 = -34$$

$$a_{12} = -\begin{vmatrix} 5 & 4 \\ 6 & 2 \end{vmatrix} = -(10 - 24) = -(-14) = 14$$

$$a_{13} = \begin{vmatrix} 5 & -1 \\ 6 & 8 \end{vmatrix} = 40 - (-6) = 40 + 6 = 46$$

$$a_{21} = -\begin{vmatrix} 7 & 3 \\ 8 & 2 \end{vmatrix} = -(14 - 24) = 10$$

$$a_{22} = \begin{vmatrix} 9 & 3 \\ 6 & 2 \end{vmatrix} = 18 - 18 = 0$$

$$a_{23} = -\begin{vmatrix} 9 & 7 \\ 6 & 8 \end{vmatrix} = -(72 - 42) = -30$$

$$a_{31} = \begin{vmatrix} 7 & 3 \\ -1 & 4 \end{vmatrix} = 28 - (-3) = 31$$

$$a_{32} = -\begin{vmatrix} 9 & 3 \\ 5 & 4 \end{vmatrix} = -(36 - 15) = -21$$

$$a_{33} = \begin{vmatrix} 9 & 7 \\ 5 & -1 \end{vmatrix} = -9 - 35 = -44$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -34 & 14 & 46 \\ 10 & 0 & -30 \\ 31 & -21 & -44 \end{bmatrix}^T = \begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix}$$

Calculating A (adj A)

$$A.(adj\ A) = \begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix} \begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix}$$

$$= \begin{bmatrix} -306 + 98 + 138 & 90 + 0 - 90 & 279 - 147 - 132 \\ -170 - 14 + 184 & 50 + 0 - 120 & 155 + 21 - 176 \\ -204 + 112 + 92 & 60 + 0 - 60 & 186 - 168 - 88 \end{bmatrix}$$

$$= \begin{bmatrix} -70 & 0 & 0 \\ 0 & -70 & 0 \\ 0 & 0 & -70 \end{bmatrix}$$

$$= -70 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

=-70 I

Calculating (adj A)A

$$(adj\ A).A = \begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix} \begin{bmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -306+50+186 & -238-10+248 & -102+40+62 \\ 126+0-126 & 98+0-168 & 42+0-42 \\ 414-150-264 & 322+30-352 & 138-120-88 \end{bmatrix}$$

$$= \begin{bmatrix} -70 & 0 & 0 \\ 0 & -70 & 0 \\ 0 & 0 & -70 \end{bmatrix}$$

$$= -70 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= -70 I

Calculating |A|.I

Expanding along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A|.I = \begin{vmatrix} 9 & 7 & 3 \\ 5 & -1 & 4 \\ 6 & 8 & 2 \end{vmatrix} I$$

= [9(-2 – 32) – (5){14 – 24} + 6(28 – (-3))]I

= [9(-34) – 5(-10) + 6(31)] I

= (-306 + 50 + 186)I

=-70 I

Thus, A(adj A) = (adj A)A = |A|I = -70 I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix}$

8. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I.

$$\begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}$$

Answer

Here, $A = \begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}$

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} 0 & 6 \\ 7 & 9 \end{vmatrix} = 0 - 42 = -42$$

$$a_{12} = -\begin{vmatrix} 1 & 6 \\ 2 & 9 \end{vmatrix} = -(9 - 12) = 3$$

$$a_{13} = \begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} = 7 - 0 = 7$$

$$a_{21} = -\begin{vmatrix} 5 & 3 \\ 7 & 9 \end{vmatrix} = -(45 - 21) = -24$$

$$a_{22} = \begin{vmatrix} 4 & 3 \\ 2 & 9 \end{vmatrix} = 36 - 6 = 30$$

$$a_{23} = -\begin{vmatrix} 4 & 5 \\ 2 & 7 \end{vmatrix} = -(28 - 10) = -18$$

$$a_{31} = \begin{vmatrix} 5 & 3 \\ 0 & 6 \end{vmatrix} = 30 - 0 = 30$$

$$a_{32} = -\begin{vmatrix} 4 & 3 \\ 1 & 6 \end{vmatrix} = -(24 - 3) = -(21) = -21$$

$$a_{33} = \begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix} = 0 - 5 = -5$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -42 & -3 & 7 \\ -24 & 30 & -18 \\ 30 & -21 & -5 \end{bmatrix}^T = \begin{bmatrix} -42 & -24 & 30 \\ -3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$$

Calculating A (adj A)

$$A.(adj\ A) = \begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix} \begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -168 + 15 + 21 & -96 + 150 - 54 & 120 - 105 - 15 \\ -42 + 0 + 42 & -24 + 0 - 108 & 30 + 0 - 30 \\ -84 + 21 + 63 & -48 + 210 - 162 & 60 - 147 - 45 \end{bmatrix}$$

$$= \begin{bmatrix} -132 & 0 & 0 \\ 0 & -132 & 0 \\ 0 & 0 & -132 \end{bmatrix}$$

$$= -132 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= -132I

Calculating (adj A)A

$$(adj\ A).A = \begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix} \begin{bmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -168 - 24 + 60 & -210 + 0 + 210 & -126 - 144 + 270 \\ 12 + 30 - 42 & 15 + 0 - 147 & 9 + 180 - 189 \\ 28 - 18 - 10 & 35 + 0 - 35 & 21 - 108 - 45 \end{bmatrix}$$

$$= \begin{bmatrix} -132 & 0 & 0 \\ 0 & -132 & 0 \\ 0 & 0 & -132 \end{bmatrix}$$

$$= -132 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= -132I

Calculating |A|.I

Expanding along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A|.I = \begin{vmatrix} 4 & 5 & 3 \\ 1 & 0 & 6 \\ 2 & 7 & 9 \end{vmatrix} I$$

= [4(0 – 42) – (1){45 – 21} + 2(30 – 0)]I

= [4(-42) – 1(24) + 2(30)] I

= (-168 – 24 + 60)I

= -132I

Thus, A(adj A) = (adj A)A = |A|I = -132I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$

9. Question

Find the adjoint of the given matrix and verify in each case that A. (adj A) = (adj A) =m |A|.I.

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer

Here, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, we have to find adj A, and for that, we have to find co-factors:

$$a_{11} = \begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha$$

$$a_{12} = -\begin{vmatrix} \sin\alpha & 0 \\ 0 & 1 \end{vmatrix} = -\sin\alpha$$

$$a_{13} = \begin{vmatrix} \sin\alpha & \cos\alpha \\ 0 & 0 \end{vmatrix} = 0$$

$$a_{21} = -\begin{vmatrix} -\sin\alpha & 0 \\ 0 & 1 \end{vmatrix} = \sin\alpha$$

$$a_{22} = \begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha$$

$$a_{23} = -\begin{vmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{vmatrix} = 0$$

$$a_{31} = \begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & 0 \end{vmatrix} = 0$$

$$a_{32} = -\begin{vmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{vmatrix} = 0$$

$$a_{33} = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = [\cos^2\alpha - \{-\sin^2\alpha\}] = [\cos^2\alpha + \sin^2\alpha] = 1$$

[$\because \cos^2$ α + $\sin^2$ α = 1]

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculating A (adj A)

$$A.(adj\ A) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\ sin\alpha - \sin\alpha\cos\alpha & 0 \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[$\because \cos^2$ α + $\sin^2$ α = 1]

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= I

Calculating (adj A)A

$$(adj\ A).A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & 0 \\ -\sin\alpha\cos\alpha + cos\ \alpha\ sin\ \alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[$\because \cos^2$ α + $\sin^2$ α = 1]

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= I

Calculating |A|.I

Expanding along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A|.I = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} I$$

= [0 – 0 + 1($\cos^2$ α – (-$\sin^2$ α))]I

= [$\cos^2$ α + $\sin^2$ α] I

= (1)I [∵$\cos^2$ α + $\sin^2$ α = 1]

= I

Thus, A(adj A) = (adj A)A = |A|I = I

⇒ A(adj A) = (adj A)A = |A|I

Hence Proved

Ans. $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. Question

If A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that adj A = A.

Answer

Here, $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

Now, we have to find adj A, and for that, we have to find co-factors:

$$a_{11} = \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 0 - 4 = -4$$

$$a_{12} = -\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(3-4) = -(-1) = 1$$

$$a_{13} = \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 4 - 0 = 4$$

$$a_{21} = -\begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -(-9+12) = -3$$

$$a_{22} = \begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = -12 + 12 = 0$$

$$a_{23} = -\begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = -(-16+12) = 4$$

$$a_{31} = \begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = -3 + 0 = -3$$

$$a_{32} = -\begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = -(-4+3) = 1$$

$$a_{33} = \begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = 0 + 3 = 3$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = A$$

Thus, adj A = A

Hence Proved

11. Question

If A = $\begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that adj A = 3A’.

Answer

We have, $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

To show: adj A = 3A’

Firstly, we find the Transpose of A i.e. A’

$$\text{Transpose of } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

So,

$A' = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ ...(i)

Now, we have to find adj A, and for that, we have to find co-factors:

$$a_{11} = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 1 - 4 = -3$$

$$a_{12} = -\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -(2 + 4) = -6$$

$$a_{13} = \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -4 - 2 = -6$$

$$a_{21} = -\begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} = -(-2 - 4) = 6$$

$$a_{22} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 + 4 = 3$$

$$a_{23} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2 + 4) = -6$$

$$a_{31} = \begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix} = 4 + 2 = 6$$

$$a_{32} = -\begin{vmatrix} -1 & -2 \\ 2 & -2 \end{vmatrix} = -(2 + 4) = -6$$

$$a_{33} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = -1 + 4 = 3$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}^T = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

Now, taking Adj A i.e.

$adj\ A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$

$= 3 \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

= 3A’ [from eq. (i)]

Hence Proved

12. Question

$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Answer

Here, $A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find the adj A and for that we have to find co-factors:

a_{11} (co – factor of 3) = $(-1)^{1+1}(2) = (-1)^2(2) = 2$

a_{12} (co – factor of -5) = $(-1)^{1+2}(-1) = (-1)^3(-1) = 1$

a_{21} (co – factor of -1) = $(-1)^{2+1}(-5) = (-1)^3(-5) = 5$

a_{22} (co – factor of 2) = $(-1)^{2+2}(3) = (-1)^4(3) = 3$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Calculating |A|

$|A| = \begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix}$

$\text{If A} = \begin{matrix} a & b \\ c & d \end{matrix}, \text{then determinant of A , is given by}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

= [3 × 2 – (-1) × (-5)]

= (6 – 5)

= 1

$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}}{1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Ans. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

13. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Answer

Here, $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find the adj A and for that we have to find co-factors:

a_{11} (co – factor of 4) = $(-1)^{1+1}(3) = (-1)^2(3) = 3$

a_{12} (co – factor of 1) = $(-1)^{1+2}(2) = (-1)^3(2) = -2$

a_{21} (co – factor of 2) = $(-1)^{2+1}(1) = (-1)^3(1) = -1$

a_{22} (co – factor of 3) = $(-1)^{2+2}(4) = (-1)^4(4) = 4$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$

Calculating |A|

$|A| = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$

$\text{If A} = \begin{matrix} a & b \\ c & d \end{matrix}, \text{then determinant of A, is given by}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

= [4 × 3 – 1 × 2]

= (12 – 2)

= 10

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}}{10} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{2}{10} & \frac{4}{10} \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Ans. $\begin{bmatrix} \frac{3}{10} & \frac{-1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

14. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$$

Answer

Here, $A = \begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Firstly, we find the adj A and for that we have to find co-factors:

a_{11} (co – factor of 2) = $(-1)^{1+1}(6) = (-1)^2(6) = 6$

a_{12} (co – factor of -3) = $(-1)^{1+2}(4) = (-1)^3(4) = -4$

a_{21} (co – factor of 4) = $(-1)^{2+1}(-3) = (-1)^3(-3) = 3$

a_{22} (co – factor of 6) = $(-1)^{2+2}(2) = (-1)^4(2) = 2$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$

Calculating |A|

$|A| = \begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix}$

$\text{If A} = \begin{matrix} a & b \\ c & d \end{matrix}, \text{then determinant of A , is given by}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

= [2 × 6 – (-3) × 4]

= (12 + 12)

= 24

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}}{24} = \frac{1}{24}\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{6}{24} & \frac{3}{24} \\ -\frac{4}{24} & \frac{2}{24} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ -\frac{1}{6} & \frac{1}{12} \end{bmatrix}$$

Ans. $\begin{bmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{-1}{6} & \frac{1}{12} \end{bmatrix}$

15. Question

Find the inverse of each of the matrices given below.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, when (ab – bc) $\neq$ 0

Answer

Here, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Firstly, we find the adj A and for that we have to find co-factors:

a_{11} (co – factor of a) = $(-1)^{1+1}(d) = (-1)^2(d) = d$

a_{12} (co – factor of b) = $(-1)^{1+2}(c) = (-1)^3(c) = -c$

a_{21} (co – factor of c) = $(-1)^{2+1}(b) = (-1)^3(b) = -b$

a_{22} (co – factor of d) = $(-1)^{2+2}(a) = (-1)^4(a) = a$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} \mathrm{d} & -c \\ -\mathrm{b} & a \end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix} \mathrm{d} & -c \\ -\mathrm{b} & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Calculating |A|

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\text{If A} = \begin{matrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{matrix}, \text{then determinant of A , is given by}$

$|\mathrm{A}| = \begin{vmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{vmatrix} = \mathrm{ad} - \mathrm{bc}$

= [a × d – c × b]

= ad – bc

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{(ad - bc)} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ans. $\frac{1}{(\mathrm{ad} - \mathrm{bc})}.\begin{bmatrix} \mathrm{d} & -\mathrm{b} \\ -\mathrm{c} & \mathrm{a} \end{bmatrix}$

16. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

Answer

We have, $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|\mathrm{A}| = (1)\begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} - (1)\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix}$$

= 1(1 – (-3)) – 1(-2 – 15) + 2(-2 – (-5))

= (1 + 3) – 1(-17) + 2(-2 + 5)

= 4 + 17 + 2(3)

= 21 + 6

= 27

Now, we have to find adj A, and for that, we have to find co-factors:

$$a_{11} = \begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} = 1 - (-3) = 1 + 3 = 4$$

$$a_{12} = -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -(-1 + 2) = -1$$

$$a_{13} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 - (-2) = 3 + 2 = 5$$

$$a_{21} = -\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = -(-2 - 15) = 17$$

$$a_{22} = \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} = -1 - 10 = -11$$

$$a_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -(3 - 4) = 1$$

$$a_{31} = \begin{vmatrix} 2 & 5 \\ -1 & -1 \end{vmatrix} = -2 - (-5) = -2 + 5 = 3$$

$$a_{32} = -\begin{vmatrix} 1 & 5 \\ 1 & -1 \end{vmatrix} = -(-1 - 5) = 6$$

$$a_{33} = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -1 - 2 = -3$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{bmatrix}^T = \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}}{27} = \frac{1}{27}\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$$

Ans. $\frac{1}{27}.\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$

17. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix}$$

Answer

We have, $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix}$

We have to find A-1 and $A^{-1} = \frac{adj\,A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A| = (2)\begin{vmatrix} 0 & -1 \\ 6 & 0 \end{vmatrix} - (3)\begin{vmatrix} -1 & 1 \\ 6 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}$$

= 2(0 – (-6)) – 3(0 – 6) + 2(1 – 0)

= 2(6) – 3(-6) + 2(1)

= 12 + 18 + 2

= 32

Now, we have to find adj A, and for that, we have to find co-factors:

$$a_{11} = \begin{vmatrix} 0 & -1 \\ 6 & 0 \end{vmatrix} = 0 - (-6) = 6$$

$$a_{12} = -\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -(0+2) = -2$$

$$a_{13} = \begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix} = 18 - 0 = 18$$

$$a_{21} = -\begin{vmatrix} -1 & 1 \\ 6 & 0 \end{vmatrix} = -(0-6) = 6$$

$$a_{22} = \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = 0 - 2 = -2$$

$$a_{23} = -\begin{vmatrix} 2 & -1 \\ 2 & 6 \end{vmatrix} = -\big(12 - (-2)\big) = -(12+2) = -14$$

$$a_{31} = \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 1 - 0 = 1$$

$$a_{32} = -\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -(-2-3) = 5$$

$$a_{33} = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} = 0 - (-3) = 3$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 6 & -2 & 18 \\ 6 & -2 & -14 \\ 1 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 18 & -14 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}}{32} = \frac{1}{32}\begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 18 & -14 & 3 \end{bmatrix}$$

Ans. $\frac{1}{32}.\begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 8 & -14 & 3 \end{bmatrix}$

18. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$$

Answer

We have, $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A| = (2)\begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix} - (2)\begin{vmatrix} -3 & 3 \\ -2 & 2 \end{vmatrix} + 3\begin{vmatrix} -3 & 3 \\ 2 & 3 \end{vmatrix}$$

= 2(4 – (-6)) – 2(-6 – (-6)) + 3(-9 – 6)

= 2(4 + 6) – 2(-6 + 6) + 3(-15)

= 2(10) – 2(0) – 45

= 20 – 45

= -25

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix} = 4 - (-6) = 4 + 6 = 10$$

$$a_{12} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -(4 - 9) = 5$$

$$a_{13} = \begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix} = -4 - 6 = -10$$

$$a_{21} = -\begin{vmatrix} -3 & 3 \\ -2 & 2 \end{vmatrix} = -\big(-6 - (-6)\big) = -(-6 + 6) = 0$$

$$a_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = -5$$

$$a_{23} = -\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} = -\big(-4 - (-9)\big) = -(-4 + 9) = -5$$

$$a_{31} = \begin{vmatrix} -3 & 3 \\ 2 & 3 \end{vmatrix} = -9 - 6 = -15$$

$$a_{32} = -\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = -(6 - 6) = 0$$

$$a_{33} = \begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} = 4 - (-6) = 4 + 6 = 10$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 10 & 5 & -10 \\ 0 & -5 & -5 \\ -15 & 0 & 10 \end{bmatrix}^T = \begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}}{(-25)} = -\frac{1}{25}\begin{bmatrix} 10 & 0 & -15 \\ 5 & -5 & 0 \\ -10 & -5 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10}{-25} & 0 & -\frac{15}{-25} \\ \frac{5}{-25} & -\frac{5}{-25} & 0 \\ -\frac{10}{-25} & -\frac{5}{-25} & \frac{10}{-25} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

Ans. $\frac{1}{5}.\begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$

19. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$$

Answer

We have, $A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A| = (0)\begin{vmatrix} 4 & 5 \\ -4 & -7 \end{vmatrix} - (3)\begin{vmatrix} 0 & -1 \\ -4 & -7 \end{vmatrix} + (-2)\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix}$$

= 0 – 3(0 – 4) – 2(0 – (-4))

= 12 – 2(4)

= 12 – 8

= 4

Now, we have to find adj A and for that we have to find co-factors:

$a_{11} = \begin{vmatrix} 4 & 5 \\ -4 & -7 \end{vmatrix} = -28 - (-20) = -28 + 20 = -8$

$a_{12} = -\begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} = -\left(-21 - (-10)\right) = -(-21 + 10) = 11$

$a_{13} = \begin{vmatrix} 3 & 4 \\ -2 & -4 \end{vmatrix} = -12 - (-8) = -12 + 8 = -4$

$a_{21} = -\begin{vmatrix} 0 & -1 \\ -4 & -7 \end{vmatrix} = -(0 - 4) = 4$

$a_{22} = \begin{vmatrix} 0 & -1 \\ -2 & -7 \end{vmatrix} = 0 - 2 = -2$

$a_{23} = -\begin{vmatrix} 0 & 0 \\ -2 & -4 \end{vmatrix} = -(0) = 0$

$a_{31} = \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 0 - (-4) = 4$

$a_{32} = -\begin{vmatrix} 0 & -1 \\ 3 & 5 \end{vmatrix} = -\left(0 - (-3)\right) = -3$

$a_{33} = \begin{vmatrix} 0 & 0 \\ 3 & 4 \end{vmatrix} = 0$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -8 & 11 & -4 \\ 4 & -2 & 0 \\ 4 & -3 & 0 \end{bmatrix}^T = \begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} -8 & -4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}}{4} = \frac{1}{4}\begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

Ans. $\frac{1}{4}.\begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$

20. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Answer

We have, $A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A| = (2)\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - (-3)\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix}$$

= 2(0 – 1) + 3(-2 – 4) – 1(-1 – 0)

= 2(-1) + 3(-6) – 1(-1)

= -2 – 18 + 1

= -19

$$A = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 0 - 1 = -1$$

$$a_{12} = -\begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} = -(-6 + 1) = 5$$

$$a_{13} = \begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix} = -3 - 0 = -3$$

$$a_{21} = -\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} = -(-2 - 4) = 6$$

$$a_{22} = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 4 + 4 = 8$$

$$a_{23} = -\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -(2 - 1) = -1$$

$$a_{31} = \begin{vmatrix} -1 & 4 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1$$

$$a_{32} = -\begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = -\big(2 - (-12)\big) = -(2 + 12) = -14$$

$$a_{33} = \begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} = 0 - 3 = -3$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -1 & 5 & -3 \\ 6 & 8 & -1 \\ -1 & -14 & -3 \end{bmatrix}^T = \begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & -1 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & 3 & -3 \end{bmatrix}}{(-19)} = -\frac{1}{19}\begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & -1 & -3 \end{bmatrix}$$

$$= \frac{1}{19}\begin{bmatrix} 1 & -6 & 1 \\ -5 & -8 & 14 \\ 3 & 1 & 3 \end{bmatrix}$$

Ans. $\frac{1}{19}.\begin{bmatrix} 1 & -6 & 1 \\ -5 & -8 & 14 \\ 3 & 1 & 3 \end{bmatrix}$

21. Question

Find the inverse of each of the matrices given below.

$$\begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$$

Answer

We have, $A = \begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|\mathrm{A}| = 8\begin{vmatrix} 0 & 6 \\ 1 & 6 \end{vmatrix} - (10)\begin{vmatrix} -4 & 1 \\ 1 & 6 \end{vmatrix} + 8\begin{vmatrix} -4 & 1 \\ 0 & 6 \end{vmatrix}$$

= 8(0 – 6) – 10(-24 – 1) + 8(-24 – 0)

= 8(-6) – 10(-25) + 8(-24)

= -48 + 250 – 192

= 250 – 240

= 10

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} 0 & 6 \\ 1 & 6 \end{vmatrix} = 0 - 6 = -6$$

$$a_{12} = -\begin{vmatrix} 10 & 6 \\ 8 & 6 \end{vmatrix} = -(60 - 48) = -12$$

$$a_{13} = \begin{vmatrix} 10 & 0 \\ 8 & 1 \end{vmatrix} = 10 - 0 = 10$$

$$a_{21} = -\begin{vmatrix} -4 & 1 \\ 1 & 6 \end{vmatrix} = -(-24 - 1) = 25$$

$$a_{22} = \begin{vmatrix} 8 & 1 \\ 8 & 6 \end{vmatrix} = 48 - 8 = 40$$

$$a_{23} = -\begin{vmatrix} 8 & -4 \\ 8 & 1 \end{vmatrix} = -\big(8 - (-32)\big) = -(8 + 32) = -40$$

$$a_{31} = \begin{vmatrix} -4 & 1 \\ 0 & 6 \end{vmatrix} = -24 - 0 = -24$$

$$a_{32} = -\begin{vmatrix} 8 & 1 \\ 10 & 6 \end{vmatrix} = -(48 - 10) = -38$$

$$a_{33} = \begin{vmatrix} 8 & -4 \\ 10 & 0 \end{vmatrix} = 0 - (-40) = 40$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} -6 & -12 & 10 \\ 25 & 40 & -40 \\ -24 & -38 & 40 \end{bmatrix}^T = \begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}}{10} = \frac{1}{10}\begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$$

Ans. $\frac{1}{10}.\begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$

22. Question

If A = $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19}A$.

Answer

Here, $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

To show: $A^{-1} = \frac{1}{19}A$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find the adj A and for that we have to find co-factors:

a_{11} (co – factor of 2) = $(-1)^{1+1}(-2) = (-1)^2(-2) = -2$

a_{12} (co – factor of 3) = $(-1)^{1+2}(5) = (-1)^3(5) = -5$

a_{21} (co – factor of 5) = $(-1)^{2+1}(3) = (-1)^3(3) = -3$

a_{22} (co – factor of -2) = $(-1)^{2+2}(2) = (-1)^4(2) = 2$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

Calculating |A|

$|A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}$

$\text{If A} = \begin{matrix} a & b \\ c & d \end{matrix}, \text{then determinant of A , is given by}$

$|\text{A}| = \begin{vmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{vmatrix} = \text{ad} - \text{bc}$

= [2 × (-2) – 3 × 5]

= (-4 – 15)

= -19

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}}{(-19)} = -\frac{1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$= \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

[Taking (-1) common from the matrix]

$A^{-1} = \frac{1}{19}A\left[\because A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}\right]$

Hence Proved

23. Question

If A = $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, show that $A^{-1} = A^2$.

Answer

We have, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

To show: $A^{-1} = A^2$

Firstly, we have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Calculating |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A| = (1)\begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} - (2)\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}$$

= 1(0) – 2(0) + 1(0 – (-1))

= 1(1)

= 1

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$a_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$a_{13} = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 0 - (-1) = 1$$

$$a_{21} = -\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = -0 = 0$$

$$a_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$a_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -\big(0 - (-1)\big) = -(1) = -1$$

$$a_{31} = \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 0 - (-1) = 1$$

$$a_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -(0 - 2) = 2$$

$a_{33} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1 - (-2) = -1 + 2 = 1$

$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ ...(i)

Calculating A^2

$A^2 = A.A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

= A^{-1} [from eq. (i)]

Thus, $A^2 = A^{-1}$

Hence Proved

24. Question

If A = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, prove that $A^{-1} = A^3$.

Answer

We have, $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

To show: $A^{-1} = A^3$

Firstly, we have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Calculating |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$|A| = (3)\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} - (2)\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} + 0\begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix}$

= 3(-3 – (-4)) – 2(-3 – (-4)) + 0

= 3(- 3 + 4) – 2(-3 + 4)

= 3(1) – 2(1)

= 3 – 2

= 1

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -3 - (-4) = -3 + 4 = 1$$

$$a_{12} = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -(2 - 0) = -2$$

$$a_{13} = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2 - 0 = -2$$

$$a_{21} = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -\left(-3 - (-4)\right) = -(-3 + 4) = -1$$

$$a_{22} = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$a_{23} = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = -(-3 - 0) = 3$$

$$a_{31} = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = -12 - (-12) = 0$$

$$a_{32} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -(12 - 8) = -4$$

$$a_{33} = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = \left(-9 - (-6)\right) = -9 + 6 = -3$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 1 & 1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}{1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \ldots(i)$$

Calculating A^3

$$A^2 = A.A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 6 + 0 & -9 + 9 - 4 & 12 - 12 + 4 \\ 6 - 6 + 0 & -6 + 9 - 4 & 8 - 12 + 4 \\ 0 - 2 + 0 & 0 + 3 - 1 & 0 - 4 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 8 + 0 & -9 + 12 - 4 & 12 - 16 + 4 \\ 0 - 2 + 0 & 0 + 3 + 0 & 0 - 4 + 0 \\ -6 + 4 + 0 & 6 - 6 + 3 & -8 + 8 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

= A^{-1} [from eq. (i)]

Thus, $A^3 = A^{-1}$

Hence Proved

25. Question

If A = $\frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ show that $A^{-1} = A'$.

Answer

We have, $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{8}{9} & \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{7}{9} \\ \frac{1}{9} & -\frac{8}{9} & \frac{4}{9} \end{bmatrix}$

To show: $A^{-1} = A'$

Firstly, we find the Transpose of A, i.e. A.'

$$\text{Transpose of } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

Here, $A = \begin{bmatrix} -\frac{8}{9} & \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{7}{9} \\ \frac{1}{9} & -\frac{8}{9} & \frac{4}{9} \end{bmatrix}$

So, $A' = \begin{bmatrix} -\frac{8}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} & -\frac{8}{9} \\ \frac{4}{9} & \frac{7}{9} & \frac{4}{9} \end{bmatrix}$ ...(i)

Now, we have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Calculating |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A| = \left(-\frac{8}{9}\right)\begin{vmatrix} \frac{4}{9} & \frac{7}{9} \\ -\frac{8}{9} & \frac{4}{9} \end{vmatrix} - \left(\frac{4}{9}\right)\begin{vmatrix} \frac{1}{9} & \frac{4}{9} \\ -\frac{8}{9} & \frac{4}{9} \end{vmatrix} + \left(\frac{1}{9}\right)\begin{vmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{7}{9} \end{vmatrix}$$

$$= -\frac{8}{9}\left[\frac{4}{9}\times\frac{4}{9} - \left(\frac{7}{9}\times\left(-\frac{8}{9}\right)\right)\right] - \frac{4}{9}\left[\frac{1}{9}\times\frac{4}{9} - \frac{4}{9}\times\left(-\frac{8}{9}\right)\right] + \frac{1}{9}\left[\frac{1}{9}\times\frac{7}{9} - \frac{4}{9}\times\frac{4}{9}\right]$$

$$= -\frac{8}{9}\left(\frac{16}{81} + \frac{56}{81}\right) - \frac{4}{9}\left(\frac{4}{81} + \frac{32}{81}\right) + \frac{1}{9}\left(\frac{7}{81} - \frac{16}{81}\right)$$

$$= -\frac{8}{9}\times\frac{72}{81} - \frac{4}{9}\times\frac{36}{81} + \frac{1}{9}\left(-\frac{9}{81}\right)$$

$$= -\frac{8\times 8}{81} - \frac{4\times 4}{81} - \frac{1}{81}$$

$= \frac{-64-1-16}{81}$

$= -\frac{81}{81}$

= -1

Now, we have to find adj A, and for that, we have to find co-factors:

$$a_{11} = \begin{vmatrix} \frac{4}{9} & \frac{7}{9} \\ -\frac{8}{9} & \frac{4}{9} \end{vmatrix} = \frac{4}{9} \times \frac{4}{9} - \left(\frac{7}{9} \times \left(-\frac{8}{9}\right)\right) = \frac{16}{81} + \frac{56}{81} = \frac{72}{81} = \frac{8}{9}$$

$$a_{12} = -\begin{vmatrix} \frac{4}{9} & \frac{7}{9} \\ \frac{1}{9} & \frac{4}{9} \end{vmatrix} = -\left[\frac{4}{9} \times \frac{4}{9} - \left(\frac{7}{9} \times \frac{1}{9}\right)\right] = -\left[\frac{16}{81} - \frac{7}{81}\right] = -\frac{9}{81} = -\frac{1}{9}$$

$$a_{13} = \begin{vmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{1}{9} & -\frac{8}{9} \end{vmatrix} = \left[\frac{4}{9} \times \frac{-8}{9} - \left(\frac{4}{9} \times \frac{1}{9}\right)\right] = \left[-\frac{32}{81} - \frac{4}{81}\right] = -\frac{36}{81} = -\frac{4}{9}$$

$$a_{21} = -\begin{vmatrix} \frac{1}{9} & \frac{4}{9} \\ -\frac{8}{9} & \frac{4}{9} \end{vmatrix} = -\left[\frac{1}{9} \times \frac{4}{9} - \left(-\frac{8}{9} \times \frac{4}{9}\right)\right] = -\left[\frac{4}{81} + \frac{32}{81}\right] = -\frac{36}{81} = -\frac{4}{9}$$

$$a_{22} = \begin{vmatrix} -\frac{8}{9} & \frac{4}{9} \\ \frac{1}{9} & \frac{4}{9} \end{vmatrix} = \left[\frac{-8}{9} \times \frac{4}{9} - \left(\frac{1}{9} \times \frac{4}{9}\right)\right] = \left[\frac{-32}{81} - \frac{4}{81}\right] = -\frac{36}{81} = -\frac{4}{9}$$

$$a_{23} = -\begin{vmatrix} -\frac{8}{9} & \frac{1}{9} \\ \frac{1}{9} & -\frac{8}{9} \end{vmatrix} = -\left[\frac{-8}{9} \times \frac{-8}{9} - \left(\frac{1}{9} \times \frac{1}{9}\right)\right] = -\left[\frac{64}{81} - \frac{1}{81}\right] = -\frac{63}{81} = -\frac{7}{9}$$

$$a_{31} = \begin{vmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{7}{9} \end{vmatrix} = \left[\frac{1}{9} \times \frac{7}{9} - \left(\frac{4}{9} \times \frac{4}{9}\right)\right] = \left[\frac{7}{81} - \frac{16}{81}\right] = -\frac{9}{81} = -\frac{1}{9}$$

$$a_{32} = -\begin{vmatrix} -\frac{8}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{7}{9} \end{vmatrix} = -\left[\frac{-8}{9} \times \frac{7}{9} - \left(\frac{4}{9} \times \frac{4}{9}\right)\right] = -\left[\frac{-56}{81} - \frac{16}{81}\right] = \frac{72}{81} = \frac{8}{9}$$

$$a_{33} = \begin{vmatrix} -\frac{8}{9} & \frac{1}{9} \\ \frac{4}{9} & \frac{4}{9} \end{vmatrix} = \left[\frac{-8}{9} \times \frac{4}{9} - \left(\frac{1}{9} \times \frac{4}{9}\right)\right] = \left[\frac{-32}{81} - \frac{4}{81}\right] = -\frac{36}{81} = -\frac{4}{9}$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} \frac{8}{9} & -\frac{1}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \\ -\frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \end{bmatrix}^T = \begin{bmatrix} \frac{8}{9} & -\frac{4}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{4}{9} & \frac{8}{9} \\ -\frac{4}{9} & -\frac{7}{9} & -\frac{4}{9} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} \frac{8}{9} & -\frac{4}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{4}{9} & \frac{8}{9} \\ -\frac{4}{9} & -\frac{7}{9} & -\frac{4}{9} \end{bmatrix}}{-1} = -\begin{bmatrix} \frac{8}{9} & -\frac{4}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{4}{9} & \frac{8}{9} \\ -\frac{4}{9} & -\frac{7}{9} & -\frac{4}{9} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{8}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{9} & -\frac{8}{9} \\ \frac{4}{9} & \frac{7}{9} & \frac{4}{9} \end{bmatrix}$$

= A’ [from eq. (i)]

Thus, A^{-1} = A’

Hence Proved

26. Question

Let D = diag [d_1, d_2, d_3], where none of d_1, d_2, d_3 is 0; prove that D^{-1} = diag [d_1^{-1}, d_2^{-1}, d_3^{-1}].

Answer

Given: D = diag [d_1, d_2, d_3]

It is also given that $d_1 \neq 0$, $d_2 \neq 0$, $d_3 \neq 0$

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

A diagonal matrix D = diag(d_1, d_2, ...d_n) is invertible iff all diagonal entries are non – zero, i.e. $d_i \neq 0$ for $1 \leq i \leq n$

If D is invertible then D^{-1} = diag(d_1^{-1}, ...d_n^{-1})

By the Inverting Diagonal Matrices Theorem, which states that

Here, it is given that $d_1 \neq 0$, $d_2 \neq 0$, $d_3 \neq 0$

∴ D is invertible

⇒ D^{-1} = diag [d_1^{-1}, d_2^{-1}, d_3^{-1}]

Hence Proved.

27. Question

If A = $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

Answer

Given: $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

To Verify: $(AB)^{-1} = B^{-1}A^{-1}$

Firstly, we find the $(AB)^{-1}$

Calculating AB

$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

$= \begin{bmatrix} 18+16 & 21+18 \\ 42+40 & 49+45 \end{bmatrix}$

$= \begin{bmatrix} 34 & 39 \\ 82 & 94 \end{bmatrix}$

We have to find $(AB)^{-1}$ and $(AB)^{-1} = \frac{adj\ (AB)}{|AB|}$

Firstly, we find the adj AB and for that we have to find co-factors:

a_{11} (co – factor of 34) = $(-1)^{1+1}(94) = (-1)^2(94) = 94$

a_{12} (co – factor of 39) = $(-1)^{1+2}(82) = (-1)^3(82) = -82$

a_{21} (co – factor of 82) = $(-1)^{2+1}(39) = (-1)^3(39) = -39$

a_{22} (co – factor of 94) = $(-1)^{2+2}(34) = (-1)^4(34) = 34$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 94 & -82 \\ -39 & 34 \end{bmatrix}$

Now, adj AB = Transpose of co-factor Matrix

$\therefore adj\ AB = \begin{bmatrix} 94 & -82 \\ -39 & 34 \end{bmatrix}^T = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

Calculating |AB|

$|AB| = \begin{vmatrix} 34 & 39 \\ 82 & 94 \end{vmatrix}$

$\text{If A} = \begin{matrix} a & b \\ c & d \end{matrix}, \text{then determinant of A, is given by}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

= [34 × 94 – (82) × (39)]

= (3196 – 3198)

= -2

$\therefore (AB)^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}}{-2} = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

Now, we have to find $B^{-1}A^{-1}$

Calculating B^{-1}

Here, $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

We have to find A^{-1} and $B^{-1} = \frac{adj\ B}{|B|}$

Firstly, we find the adj B and for that we have to find co-factors:

a_{11} (co – factor of 6) = $(-1)^{1+1}(9) = (-1)^2(9) = 9$

a_{12} (co – factor of 7) = $(-1)^{1+2}(8) = (-1)^3(8) = -8$

a_{21} (co – factor of 8) = $(-1)^{2+1}(7) = (-1)^3(7) = -7$

a_{22} (co – factor of 9) = $(-1)^{2+2}(6) = (-1)^4(6) = 6$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

Now, adj B = Transpose of co-factor Matrix

$\therefore adj\ B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}^T = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

Calculating |B|

$|B| = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix}$

$\text{If A} = \begin{matrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{matrix}, \text{then determinant of A , is given by}$

$|\text{A}| = \begin{vmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{vmatrix} = \text{ad} - \text{bc}$

= [6 × 9 – 7 × 8]

= (54 – 56)

= -2

$\therefore B^{-1} = \frac{adj\ B}{|B|} = \frac{\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}}{-2} = -\frac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$

Calculating A^{-1}

Here, $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find the adj A and for that we have to find co-factors:

a_{11} (co – factor of 3) = $(-1)^{1+1}(5) = (-1)^2(5) = 5$

a_{12} (co – factor of 2) = $(-1)^{1+2}(7) = (-1)^3(7) = -7$

a_{21} (co – factor of 7) = $(-1)^{2+1}(2) = (-1)^3(2) = -2$

a_{22} (co – factor of 5) = $(-1)^{2+2}(3) = (-1)^4(3) = 3$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

Now, adj A = Transpose of co-factor Matrix

$\therefore adj\ A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

Calculating |A|

$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix}$

$\text{If A} = \begin{matrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{matrix}, \text{then determinant of A , is given by}$

$|\text{A}| = \begin{vmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{vmatrix} = \text{ad} - \text{bc}$

= [3 × 5 – 2 × 7]

= (15 – 14)

= 1

$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}}{1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

Calculating $B^{-1}A^{-1}$

Here, $B^{-1} = -\frac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$ & $A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

So,

$B^{-1}A^{-1} = \left(-\frac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}\right)\left(\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\right)$

$= \frac{1}{-2}\begin{bmatrix} 45+49 & -18-21 \\ -40-42 & 16+18 \end{bmatrix}$

$= \frac{1}{-2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

So, we get

$(AB)^{-1} = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$ and $B^{-1}A^{-1} = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Hence verified

28. Question

If A = $\begin{bmatrix} 9 & -1 \\ 6 & -2 \end{bmatrix}$ and B = $\begin{bmatrix} -4 & 3 \\ 5 & -4 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

Answer

Given: $A = \begin{bmatrix} 9 & -1 \\ 6 & -2 \end{bmatrix}$ & $B = \begin{bmatrix} -4 & 3 \\ 5 & -4 \end{bmatrix}$

To Verify: $(AB)^{-1} = B^{-1}A^{-1}$

Firstly, we find the $(AB)^{-1}$

Calculating AB

$AB = \begin{bmatrix} 9 & -1 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 5 & -4 \end{bmatrix}$

$= \begin{bmatrix} -36-5 & 27+4 \\ -24-10 & 18+8 \end{bmatrix}$

$= \begin{bmatrix} -41 & 31 \\ -34 & 26 \end{bmatrix}$

We have to find $(AB)^{-1}$ and $(AB)^{-1} = \frac{adj\ (AB)}{|AB|}$

Firstly, we find the adj AB and for that we have to find co-factors:

a_{11} (co – factor of -41) = $(-1)^{1+1}(26) = (-1)^2(26) = 26$

a_{12} (co – factor of 31) = $(-1)^{1+2}(-34) = (-1)^3(-34) = 34$

a_{21} (co – factor of -34) = $(-1)^{2+1}(31) = (-1)^3(31) = -31$

a_{22} (co – factor of 26) = $(-1)^{2+2}(-41) = (-1)^4(-41) = -41$

$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 26 & 34 \\ -31 & -41 \end{bmatrix}$

Now, adj AB = Transpose of co-factor Matrix

$$\therefore adj\ AB = \begin{bmatrix} 26 & 34 \\ -31 & -41 \end{bmatrix}^T = \begin{bmatrix} 26 & -31 \\ 34 & -41 \end{bmatrix}$$

Calculating |AB|

$$|AB| = \begin{vmatrix} -41 & 31 \\ -34 & 26 \end{vmatrix}$$

If $A = \begin{matrix} a & b \\ c & d \end{matrix}$, then determinant of A, is given by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

= [-41 × 26 – (-34) × (31)]

= (-1066 + 1054)

= -12

$$\therefore (AB)^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 26 & -31 \\ 34 & -41 \end{bmatrix}}{-12} = -\frac{1}{12}\begin{bmatrix} 26 & -31 \\ 34 & -41 \end{bmatrix}$$

Now, we have to find $B^{-1}A^{-1}$

Calculating B^{-1}

Here, $B = \begin{bmatrix} -4 & 3 \\ 5 & -4 \end{bmatrix}$

We have to find A^{-1} and $B^{-1} = \frac{adj\ B}{|B|}$

Firstly, we find the adj B and for that we have to find co-factors:

a_{11} (co – factor of -4) = $(-1)^{1+1}(-4) = (-1)^2(-4) = -4$

a_{12} (co – factor of 3) = $(-1)^{1+2}(5) = (-1)^3(5) = -5$

a_{21} (co – factor of 5) = $(-1)^{2+1}(3) = (-1)^3(3) = -3$

a_{22} (co – factor of -4) = $(-1)^{2+2}(-4) = (-1)^4(-4) = -4$

$$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} -4 & -5 \\ -3 & -4 \end{bmatrix}$$

Now, adj B = Transpose of co-factor Matrix

$$\therefore adj\ B = \begin{bmatrix} -4 & -5 \\ -3 & -4 \end{bmatrix}^T = \begin{bmatrix} -4 & -3 \\ -5 & -4 \end{bmatrix}$$

Calculating |B|

$$|B| = \begin{vmatrix} -4 & 3 \\ 5 & -4 \end{vmatrix}$$

If $A = \begin{matrix} a & b \\ c & d \end{matrix}$, then determinant of A, is given by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

= [(-4) × (-4) – 3 × 5]

= (16 – 15)

= 1

$$\therefore B^{-1} = \frac{adj\ B}{|B|} = \frac{\begin{bmatrix} -4 & -3 \\ -5 & -4 \end{bmatrix}}{1} = \begin{bmatrix} -4 & -3 \\ -5 & -4 \end{bmatrix}$$

Calculating A^{-1}

Here, $A = \begin{bmatrix} 9 & -1 \\ 6 & -2 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find the adj A and for that we have to find co-factors:

a_{11} (co – factor of 9) = $(-1)^{1+1}(-2) = (-1)^2(-2) = -2$

a_{12} (co – factor of -1) = $(-1)^{1+2}(6) = (-1)^3(6) = -6$

a_{21} (co – factor of 6) = $(-1)^{2+1}(-1) = (-1)^3(-1) = 1$

a_{22} (co – factor of -2) = $(-1)^{2+2}(9) = (-1)^4(9) = 9$

$$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} -2 & -6 \\ 1 & 9 \end{bmatrix}$$

Now, adj A = Transpose of co-factor Matrix

$$\therefore adj\ A = \begin{bmatrix} -2 & -6 \\ 1 & 9 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 \\ -6 & 9 \end{bmatrix}$$

Calculating |A|

$$|A| = \begin{vmatrix} 9 & -1 \\ 6 & -2 \end{vmatrix}$$

If $A = \begin{matrix} a & b \\ c & d \end{matrix}$, then determinant of A , is given by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$= [9 \times (-2) - (-1) \times 6]$

$= (-18 + 6)$

$= -12$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} -2 & 1 \\ -6 & 9 \end{bmatrix}}{-12} = -\frac{1}{12}\begin{bmatrix} -2 & 1 \\ -6 & 9 \end{bmatrix}$$

Calculating $B^{-1}A^{-1}$

Here, $B^{-1} = \begin{bmatrix} -4 & -3 \\ -5 & -4 \end{bmatrix}$ & $A^{-1} = -\frac{1}{12}\begin{bmatrix} -2 & 1 \\ -6 & 9 \end{bmatrix}$

So,

$$B^{-1}A^{-1} = \left(\begin{bmatrix} -4 & -3 \\ -5 & -4 \end{bmatrix}\right)\left(-\frac{1}{12}\begin{bmatrix} -2 & 1 \\ -6 & 9 \end{bmatrix}\right)$$

$$= -\frac{1}{12}\begin{bmatrix} 8 + 18 & -4 - 27 \\ 10 + 24 & -5 - 36 \end{bmatrix}$$

$$= -\frac{1}{12}\begin{bmatrix} 26 & -31 \\ 34 & -41 \end{bmatrix}$$

So, we get

$(AB)^{-1} = -\frac{1}{12}\begin{bmatrix} 26 & -31 \\ 34 & -41 \end{bmatrix}$ and $B^{-1}A^{-1} = -\frac{1}{12}\begin{bmatrix} 26 & -31 \\ 34 & -41 \end{bmatrix}$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Hence verified

29. Question

Compute $(AB)^{-1}$ when $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ and B^{-1} -= $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

Answer

We have, $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

To find: $(AB)^{-1}$

We know that,

$(AB)^{-1} = B^{-1}A^{-1}$

and here, B^{-1} is given but we have to find A^{-1} and $A^{-1} = \frac{adj\,A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|A| = (1)\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} - (0)\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} + (3)\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$$

= 1(8 – 6) – 0 + 3(-3 – 4)

= 1(2) + 3(-7)

= 2 – 21

= -19

Now, we have to find adj A and for that we have to find co-factors:

$a_{11} = \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 8 - 6 = 2$

$a_{12} = -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -(0 + 9) = -9$

$a_{13} = \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = 0 - 6 = -6$

$a_{21} = -\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} = -(4 + 4) = -8$

$a_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$a_{23} = -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -(-2 - 3) = 5$

$a_{31} = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -3 - 4 = -7$

$a_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = -(-3-0) = 3$

$a_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 2 & -9 & -6 \\ -8 & -2 & 5 \\ -7 & 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -8 & -7 \\ -9 & -2 & 3 \\ -6 & 5 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} -1 & 6 & -1 \\ 5 & 8 & -14 \\ -3 & 3 & -3 \end{bmatrix}}{(-19)} = -\frac{1}{19}\begin{bmatrix} 2 & -8 & -7 \\ -9 & -2 & 3 \\ -6 & 5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} -2 & 8 & 7 \\ 9 & 2 & -3 \\ 6 & -5 & -2 \end{bmatrix}$

Now, we have

$B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ & $A^{-1} = \frac{1}{19}\begin{bmatrix} -2 & 8 & 7 \\ 9 & 2 & -3 \\ 6 & -5 & -2 \end{bmatrix}$

So,

$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}\left\{\frac{1}{19}\begin{bmatrix} -2 & 8 & 7 \\ 9 & 2 & -3 \\ 6 & -5 & -2 \end{bmatrix}\right\}$

$= \frac{1}{19}\begin{bmatrix} -2+18+0 & 8+4+0 & 7-6+0 \\ 0+27-6 & 0+6+5 & 0-9+2 \\ -2+0+12 & 8+0-10 & 7+0-4 \end{bmatrix}$

$= \frac{1}{19}\begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix}$

Ans. $\frac{1}{19}.\begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix}$

30. Question

Obtain the inverses of the matrices $\begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$. And, hence find the inverse of the matrix $\begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$.

Answer

Let $A = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$ & $C = \begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$

To find: A^{-1}, B^{-1} and C^{-1}

Calculating A^{-1}

We have, $A = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}$

We have to find A^{-1} and $A^{-1} = \frac{adj\ A}{|A|}$

Firstly, we find |A|

Expanding |A| along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|\mathrm{A}| = (1)\begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} - (0) + (0)$$

= 1(1 – 0)

= 1

Now, we have to find adj A and for that we have to find co-factors:

$$a_{11} = \begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$a_{12} = -\begin{vmatrix} 0 & p \\ 0 & 1 \end{vmatrix} = 0$$

$$a_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$a_{21} = -\begin{vmatrix} p & 0 \\ 0 & 1 \end{vmatrix} = -(p - 0) = -p$$

$$a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$a_{23} = -\begin{vmatrix} 1 & p \\ 0 & 0 \end{vmatrix} = -(0) = 0$$

$$a_{31} = \begin{vmatrix} p & 0 \\ 1 & p \end{vmatrix} = p^2 - 0 = p^2$$

$$a_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & p \end{vmatrix} = -(p - 0) = -p$$

$$a_{33} = \begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore adj\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ p^2 & -p & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj\ A}{|A|} = \frac{\begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix} \ldots\text{(i)}$$

Calculating B^{-1}

We have, $B = \begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$

We have to find B^{-1} and $B^{-1} = \frac{adj\ B}{|B|}$

Firstly, we find |A|

Expanding |B| along C_1, we get

$$|B| = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}(-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$|\mathrm{B}| = (1)\begin{vmatrix} 1 & 0 \\ q & 1 \end{vmatrix} - (q)\begin{vmatrix} 0 & 0 \\ q & 1 \end{vmatrix} + (0)$$

= 1(1 – 0) – q(0)

= 1

Now, we have to find adj B and for that we have to find co-factors:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$$

$$a_{11} = \begin{vmatrix} 1 & o \\ q & 1 \end{vmatrix} = 1 - 0 = 1$$

$$a_{12} = -\begin{vmatrix} q & 0 \\ 0 & 1 \end{vmatrix} = -q$$

$$a_{13} = \begin{vmatrix} q & 1 \\ 0 & q \end{vmatrix} = q^2$$

$$a_{21} = -\begin{vmatrix} 0 & 0 \\ q & 1 \end{vmatrix} = 0$$

$$a_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$a_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & q \end{vmatrix} = -(q - 0) = -q$$

$$a_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$a_{32} = -\begin{vmatrix} 1 & 0 \\ q & 0 \end{vmatrix} = 0$$

$$a_{33} = \begin{vmatrix} 1 & o \\ q & 1 \end{vmatrix} = 1$$

$$\therefore adj\ B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} 1 & -q & q^2 \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{adj\ B}{|B|} = \frac{\begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{bmatrix}}{1} = \begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{bmatrix} \ldots\text{(ii)}$$

Calculating C^{-1}

Here, $C = \begin{bmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}$$

⇒ C = AB $\left[\because A = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{bmatrix}\right]$

$\Rightarrow C^{-1} = (AB)^{-1}$

We know that,

$(AB)^{-1} = B^{-1}A^{-1}$

Substitute the values, we get

$$C^{-1} = (AB)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{bmatrix} \begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -p & p^2 \\ -q & pq+1 & -p^2q-p \\ q^2 & -q^2p-q & p^2q^2+pq+1 \end{bmatrix}$$

Ans. $\begin{bmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ -q & 1 & 1 \\ q^2 & -q & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -p & p^2 \\ -q & pq+1 & -qp^2-p \\ q^2 & -pq^2-q & p^2q^2+pq+1 \end{bmatrix}$.

31. Question

If A = $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$, verify that $A^2 - 4A - I = O$, and hence find A^{-1}.

Answer

Given: $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

To verify: $A^2 - 4A - I = 0$

Firstly, we find the A^2

$$A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 6+2 \\ 6+2 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$$

Taking LHS of the given equation .i.e.

$A^2 - 4A - I$

$$\Rightarrow \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} - 4\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} - \begin{bmatrix} 12 & 8 \\ 8 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} - \left\{\begin{bmatrix} 12 & 8 \\ 8 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} - \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

= 0

= RHS

∴ LHS = RHS

Hence verified

Now, we have to find A^{-1}

Finding A^{-1} using given equation

$A^2 - 4A - I = O$

Post multiplying by A^{-1} both sides, we get

$(A^2 - 4A - I)A^{-1} = OA^{-1}$

$\Rightarrow A^2.A^{-1} - 4A.A^{-1} - I.A^{-1} = O$ $[OA^{-1} = O]$

$\Rightarrow A.(AA^{-1}) - 4I - A^{-1} = O$ $[AA^{-1} = I]$

$\Rightarrow A(I) - 4I - A^{-1} = O$

$\Rightarrow A - 4I - A^{-1} = O$

$\Rightarrow A - 4I - O = A^{-1}$

$\Rightarrow A - 4I = A^{-1}$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3-4 & 2-0 \\ 2-0 & 1-4 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

Ans. $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

32. Question

Show that the matrix A = $\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the equation $x^2 + 4x - 42 = 0$ and hence find A^{-1}.

Answer

Given: $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$

To show: Matrix A satisfies the equation $x^2 + 4x - 42 = 0$

If Matrix A satisfies the given equation then

$A^2 + 4A - 42 = 0$

Firstly, we find the A^2

$$A^2 = A.A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 64+10 & -40+20 \\ -16+8 & 10+16 \end{bmatrix}$$

$$= \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix}$$

Taking LHS of the given equation .i.e.

$A^2 + 4A - 42$

$\Rightarrow \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix} + 4\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} - 42\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix} + \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix} - \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 74-32 & -20+20 \\ -8+8 & 26+16 \end{bmatrix} - \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix} - \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

= O

= RHS

∴ LHS = RHS

Hence matrix A satisfies the given equation $x^2 + 4x - 42 = 0$

Now, we have to find A^{-1}

Finding A^{-1} using given equation

$A^2 + 4A - 42 = O$

Post multiplying by A^{-1} both sides, we get

$(A^2 + 4A - 42)A^{-1} = OA^{-1}$

$\Rightarrow A^2.A^{-1} + 4A.A^{-1} - 42.A^{-1} = O$ [$OA^{-1} = O$]

$\Rightarrow A.(AA^{-1}) + 4I - 42A^{-1} = O$ [$AA^{-1} = I$]

$\Rightarrow A(I) + 4I - 42A^{-1} = O$

$\Rightarrow A + 4I - 42A^{-1} = O$

$\Rightarrow A + 4I - O = 42A^{-1}$

$\Rightarrow A^{-1} = \frac{1}{42}(A + 4I)$

$\Rightarrow A^{-1} = \frac{1}{42}\left\{\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$\Rightarrow A^{-1} = \frac{1}{42}\left\{\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\}$

$\Rightarrow A^{-1} = \frac{1}{42}\begin{Bmatrix} -8+4 & 5+0 \\ 2+0 & 4+4 \end{Bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{42}\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$

Ans. $\frac{1}{42}.\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$.

33. Question

If A = $\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$, show that $A^2 + 3A + 4I_2 = O$ and hence find A^{-1}.

Answer

Given: $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$

To verify: $A^2 + 3A + 4I = 0$

Firstly, we find the A^2

$$A^2 = A.A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 1+2 \\ -2-4 & -2+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -6 & 2 \end{bmatrix}$$

Taking LHS of the given equation .i.e.

$A^2 + 3A + 4I$

$$\Rightarrow \begin{bmatrix} -1 & 3 \\ -6 & 2 \end{bmatrix} + 3\begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & -3 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1+(-3) & 3+(-3) \\ -6+6 & 2+(-6) \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

= O

= RHS

∴ LHS = RHS

Hence verified

Now, we have to find A^{-1}

Finding A^{-1} using given equation

$A^2 + 3A + 4I = O$

Post multiplying by A^{-1} both sides, we get

$(A^2 + 3A + 4I)A^{-1} = OA^{-1}$

$\Rightarrow A^2.A^{-1} + 3A.A^{-1} + 4I.A^{-1} = O$ [$OA^{-1} = O$]

$\Rightarrow A.(AA^{-1}) + 3I + 4A^{-1} = O$ [$AA^{-1} = I$]

$\Rightarrow A(I) + 3I + 4A^{-1} = O$

$\Rightarrow A + 3I + 4A^{-1} = O$

$\Rightarrow 4A^{-1} = - A - 3I + O$

$\Rightarrow A^{-1} = \frac{1}{4}[-A - 3I]$

$\Rightarrow A^{-1} = \frac{1}{4}\left\{-\begin{bmatrix}-1 & -1\\ 2 & -2\end{bmatrix} - 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}$

$\Rightarrow A^{-1} = \frac{1}{4}\left\{\begin{bmatrix}1 & 1\\ -2 & 2\end{bmatrix} + \begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}\right\}$

$\Rightarrow A^{-1} = \frac{1}{4}\left\{\begin{bmatrix}1+(-3) & 1+0\\ -2+0 & 2+(-3)\end{bmatrix}\right\}$

$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix}-2 & 1\\ -2 & -1\end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix}-\frac{2}{4} & \frac{1}{4}\\ -\frac{2}{4} & -\frac{1}{4}\end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix}-\frac{1}{2} & \frac{1}{4}\\ -\frac{1}{2} & -\frac{1}{4}\end{bmatrix}$

Ans. $A^{-1} = \begin{bmatrix}-\frac{1}{2} & \frac{1}{4}\\ -\frac{1}{2} & -\frac{1}{4}\end{bmatrix}$

34. Question

If A = $\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$, find □ and □ such that A^2 + □ I = □ A. Hence, find A^{-1}. [CBSE 2005]

Answer

Given: $A = \begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$

To find: value of x and y

Given equation: $A^2 + xI = yA$

Firstly, we find the A^2

$A^2 = A.A = \begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$

$= \begin{bmatrix}9+7 & 3+5\\ 21+35 & 7+25\end{bmatrix}$

$= \begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix}$

Putting the values in given equation

$A^2 + xI = yA$

$\Rightarrow \begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix} + x\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = y\begin{bmatrix}3 & 1\\ 7 & 5\end{bmatrix}$

$\Rightarrow \begin{bmatrix}16 & 8\\ 56 & 32\end{bmatrix} + \begin{bmatrix}x & 0\\ 0 & x\end{bmatrix} = \begin{bmatrix}3y & y\\ 7y & 5y\end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16+x & 8 \\ 56 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

On Comparing, we get

16 + x = 3y …(i)

y = 8 …(ii)

56 = 7y …(iii)

32 + x = 5y …(iv)

Putting the value of y = 8 in eq. (i), we get

16 + x = 3(8)

⇒ 16 + x = 24

⇒ x = 8

Hence, the value of x = 8 and y = 8

So, the given equation become $A^2 + 8I = 8A$

Now, we have to find A^{-1}

Finding A^{-1} using given equation

$A^2 + 8I = 8A$

Post multiplying by A^{-1} both sides, we get

$(A^2 + 8I)A^{-1} = 8AA^{-1}$

⇒ $A^2.A^{-1} + 8I.A^{-1} = 8AA^{-1}$

⇒ $A.(AA^{-1}) + 8A^{-1} = 8I$ $[AA^{-1} = I]$

⇒ $A(I) + 8A^{-1} = 8I$

⇒ $A + 8A^{-1} = 8I$

⇒ $8A^{-1} = - A + 8I$

$$\Rightarrow A^{-1} = \frac{1}{8}[-A+8I]$$

$$\Rightarrow A^{-1} = \frac{1}{8}\left\{-\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{8}\left\{\begin{bmatrix} -3 & -1 \\ -7 & -5 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{8}\left\{\begin{bmatrix} -3+8 & -1+0 \\ -7+0 & -5+8 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$$

Ans. □ = 8, □ =8 and $A^{-1} = \frac{1}{8}.\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix}$.

35. Question

If A = $\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$. Find the value of λ so that $A^2 = \lambda A - 2I$. Hence, find A^{-1}.

[CBSE 2007]

Answer

Given: $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

To find: value of λ

Given equation: $A^2 = \lambda A - 2I$

Firstly, we find the A^2

$$A^2 = A.A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

Putting the values in given equation

$A^2 = \lambda A - 2I$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3\lambda & -2\lambda \\ 4\lambda & -2\lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3\lambda - 2 & -2\lambda - 0 \\ 4\lambda - 0 & -2\lambda - 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3\lambda - 2 & -2\lambda \\ 4\lambda & -2\lambda - 2 \end{bmatrix}$$

On Comparing, we get

$3\lambda - 2 = 1$ …(i)

$-2\lambda = -2$ …(ii)

$4\lambda = 4$ …(iii)

$-2\lambda - 2 = -4$ …(iv)

Solving eq. (iii), we get

$4\lambda = 4$

$\Rightarrow \lambda = 1$

Hence, the value of λ = 1

So, the given equation become $A^2 = A - 2I$

Now, we have to find A^{-1}

Finding A^{-1} using given equation

$A^2 = A - 2I$

Post multiplying by A^{-1} both sides, we get

$(A^2)A^{-1} = (A - 2I)\, A^{-1}$

$\Rightarrow A^2.A^{-1} = AA^{-1} - 2IA^{-1}$

$\Rightarrow A.(AA^{-1}) = I - 2A^{-1}$ $[AA^{-1} = I]$

$\Rightarrow A(I) = I - 2A^{-1}$

$\Rightarrow A + 2A^{-1} = I$

$\Rightarrow 2A^{-1} = - A + I$

$$\Rightarrow A^{-1} = \frac{1}{2}[-A + I]$$

$$\Rightarrow A^{-1} = \frac{1}{2}\left\{-\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\left\{\begin{bmatrix} -3 & 2 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\left\{\begin{bmatrix} -3+1 & 2+0 \\ -4+0 & 2+1 \end{bmatrix}\right\}$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$$

Ans. $\lambda = 1$, $A^{-1} = \frac{1}{2}.\begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$.

36. Question

Show that the $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation $A^3 - A^2 - 3A - I = O$, and hence find A^{-1}.

Answer

Given: $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

We have to show that matrix A satisfies the equation $A^3 - A^2 - 3A - I = O$

Firstly, we find the A^2

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-6 & 0+0-8 & -2+0-2 \\ -2+2+6 & 0+1+8 & 4-2+2 \\ 3-8+3 & 0-4+4 & -6+8+1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix}$$

Now, we have to calculate A^3

$$A^3 = A^2.A = \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+16-12 & 0+8-16 & 10-16-4 \\ 6-18+12 & 0-9+16 & -12+18+4 \\ -2+0+9 & 0+0+12 & 4+0+3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 & -10 \\ 0 & 7 & -10 \\ 7 & 12 & 7 \end{bmatrix}$$

Taking LHS of the given equation .i.e.

$A^3 - A^2 - 3A - I$

Putting the values, we get

$$\Rightarrow \begin{bmatrix} -1 & -8 & -10 \\ 0 & 7 & -10 \\ 7 & 12 & 7 \end{bmatrix} - \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1-(-5) & -8-(-8) & -10-(-4) \\ 0-6 & 7-9 & -10-4 \\ 7-(-2) & 12-0 & 7-3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -6 \\ -6 & -3 & 6 \\ 9 & 12 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1+5 & -8+8 & -10+4 \\ -6 & -2 & -14 \\ 7+2 & 12 & 4 \end{bmatrix} - \left\{\begin{bmatrix} 3 & 0 & -6 \\ -6 & -3 & 6 \\ 9 & 12 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$\Rightarrow \begin{bmatrix} 4 & 0 & -6 \\ -6 & -2 & -14 \\ 9 & 12 & 4 \end{bmatrix} - \left\{\begin{bmatrix} 3+1 & 0+0 & -6+0 \\ -6+0 & -3+1 & 6+0 \\ 9+0 & 12+0 & 3+1 \end{bmatrix}\right\}$$

$$\Rightarrow \begin{bmatrix} 4 & 0 & -6 \\ -6 & -2 & -14 \\ 9 & 12 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -6 \\ -6 & -2 & -14 \\ 9 & 12 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

= O

= RHS

∴ LHS = RHS

Hence, the given matrix A satisfies the equation $A^3 - A^2 - 3A - I$

Now, we have to find A^{-1}

Finding A^{-1} using given equation

$A^3 - A^2 - 3A - I$

Post multiplying by A^{-1} both sides, we get

$(A^3 - A^2 - 3A - I)A^{-1} = OA^{-1}$

$\Rightarrow A^3.A^{-1} - A^2.A^{-1} - 3A.A^{-1} - I.A^{-1} = O$ $[OA^{-1} = O]$

$\Rightarrow A^2.(AA^{-1}) - A.(AA^{-1}) - 3I - A^{-1} = O$

$\Rightarrow A^2(I) - A(I) - 3I - A^{-1} = O$ $[AA^{-1} = I]$

$\Rightarrow A^2 - A - 3I - A^{-1} = O$

$\Rightarrow O + A^{-1} = A^2 - A - 3I$

$\Rightarrow A^{-1} = A^2 - A - 3I$

$$\Rightarrow A^{-1} = [A^2 - A - 3I]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -5-1 & -8-0 & -4-(-2) \\ 6-(-2) & 9-(-1) & 4-2 \\ -2-3 & 0-4 & 3-1 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -6 & -8 & -4+2 \\ 6+2 & 9+1 & 2 \\ -5 & -4 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -6-3 & -8+0 & -2+0 \\ 8+0 & 10-3 & 2+0 \\ -5+0 & -4+0 & 2-3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

Ans. $A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$.

37. Question

Prove that: (i) adj I = I (ii) adj O = O (iii) I^{-1} = I.

Answer

(i) To Prove: adj I = I

We know that, I means the Identity matrix

Let I is a 2 × 2 matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we have to find adj I and for that we have to find co-factors:

a_{11} (co – factor of 1) = $(-1)^{1+1}(1) = (-1)^2(1) = 1$

a_{12} (co – factor of 0) = $(-1)^{1+2}(0) = (-1)^3(0) = 0$

a_{21} (co – factor of 0) = $(-1)^{2+1}(0) = (-1)^3(0) = 0$

a_{22} (co – factor of 1) = $(-1)^{2+2}(1) = (-1)^4(1) = 1$

$$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, adj I = Transpose of co-factor Matrix

$$\therefore adj\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, adj I = I

Hence Proved

(ii) To Prove: adj O = O

We know that, O means Zero matrix where all the elements of matrix are 0

Let O is a 2 × 2 matrix

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Calculating adj O

Now, we have to find adj O and for that we have to find co-factors:

a_{11} (co – factor of 0) = $(-1)^{1+1}(0) = 0$

a_{12} (co – factor of 0) = $(-1)^{1+2}(0) = 0$

a_{21} (co – factor of 0) = $(-1)^{2+1}(0) = 0$

a_{22} (co – factor of 0) = $(-1)^{2+2}(0) = 0$

$$\therefore \text{The co} - \text{factor matrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, adj O = Transpose of co-factor Matrix

$$\therefore adj\ O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Thus, adj O = O

Hence Proved

(iii) To Prove: $I^{-1} = I$

We know that,

$$I^{-1} = \frac{adj\ I}{|I|}$$

From the part(i), we get adj I

So, we have to find |I|

Calculating |I|

$$|I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

If $A = \begin{matrix} a & b \\ c & d \end{matrix}$, then determinant of A, is given by $$\mathbf{|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

= [1 × 1 – 0]

= 1

$$\therefore I^{-1} = \frac{adj\ I}{|I|} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, $I^{-1} = I$

Hence Proved