VECTORS [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. The scalar $\vec{A}\cdot(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$ equals

   a. 0
   b. $[\vec{A}\,\vec{B}\,\vec{C}]+[\vec{B}\,\vec{C}\,\vec{A}]$
   c. $[\vec{A}\,\vec{B}\,\vec{C}]$
   d. none of these

   (IIT-JEE 1981)

2. For non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$ holds if and only if

   a. $\vec{a}\cdot\vec{b}=0, \vec{b}\cdot\vec{c}=0$
   b. $\vec{b}.\vec{c}=0, \vec{c}\cdot\vec{a}=0$
   c. $\vec{c}\cdot\vec{a}=0, \vec{a}\cdot\vec{b}=0$
   d. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

   (IIT-JEE 1982)

3. The volume of the parallelepiped whose sides are given by $\overrightarrow{OA} = 2\hat{i} - 2\hat{j}$, $\overrightarrow{OB} = \hat{i} + \hat{j} - \hat{k}$ and $\overrightarrow{OC} = 3\hat{i} - \hat{k}$ is

a. 4/13 b. 4
c. 2/7 d. 2 **(IIT-JEE 1983)**

4. The points with position vectors $60\hat{i} + 3\hat{j}$, $40\hat{i} - 8\hat{j}$, $a\hat{i} - 52\hat{j}$ are collinear if

a. $a = -40$ b. $a = 40$
c. $a = 20$ d. none of these
(IIT-JEE 1983)

5. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{p}, \vec{q}$ and $\vec{r}$ be the vectors defined by the relations $\vec{p} = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{q} = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{r} = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$.
Then the value of the expression $(\vec{a} + \vec{b}) \cdot \vec{p} + (\vec{b} + \vec{c}) \cdot \vec{q} + (\vec{c} + \vec{a}) \cdot \vec{r}$ is

a. 0 b. 1
c. 2 d. 3 **(IIT-JEE 1988)**

6. Let a, b and c be distinct non-negative numbers. If vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar, then c is

a. the arithmetic mean of a and b
b. the geometric mean of a and b
c. the harmonic mean of a and b
d. equal to zero **(IIT-JEE 1993)**

7. Let α, β and γ be distinct and real numbers. The points with position vectors $\alpha i + \beta\hat{j} + \gamma\hat{k}$, $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$ and $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

a. are collinear b. form an equilateral triangle
c. form a scalene triangle d. form a right-angled triangle
(IIT-JEE 1994)

8. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{k} - \hat{i}$. If $\vec{d}$ is a unit vector such that $\vec{a} \cdot \vec{d} = 0 = [\vec{b}\ \vec{c}\ \vec{d}]$, then $\vec{d}$ equals

a. $\pm \dfrac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}}$ b. $\pm \dfrac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$

c. $\pm \dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ d. $\pm \hat{k}$ **(IIT-JEE 1995)**

9. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \dfrac{\vec{b} + \vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

a. $3\pi/4$ b. $\pi/4$
c. $\pi/2$ d. π **(IIT-JEE 1995)**

10. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$. If $|\vec{u}| = 3$, $|\vec{v}| = 4$ and $|\vec{w}| = 5$, then $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$ is

a. 47 b. -25 c. 0 d. 25
(IIT-JEE 1995)

11. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then $(\vec{a} + \vec{b} + \vec{c}) \cdot [(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})]$ equals

a. 0 b. $[\vec{a}\ \vec{b}\ \vec{c}]$

c. $2\,[\vec{a}\ \vec{b}\ \vec{c}]$ d. $-[\vec{a}\ \vec{b}\ \vec{c}]$
(IIT-JEE 1995)

12. $\vec{p}, \vec{q}$ and $\vec{r}$ are three mutually perpendicular vectors of the same magnitude. If vector $\vec{x}$ satisfies the equation $\vec{p} \times ((\vec{x} - \vec{q}) \times \vec{p}) + \vec{q} \times ((\vec{x} - \vec{r}) \times \vec{q}) + \vec{r} \times ((\vec{x} - \vec{p}) \times \vec{r}) = \vec{0}$, then $\vec{x}$ is given by

a. $\dfrac{1}{2}(\vec{p} + \vec{q} - 2\vec{r})$ b. $\dfrac{1}{2}(\vec{p} + \vec{q} + \vec{r})$

c. $\dfrac{1}{3}(\vec{p} + \vec{q} + \vec{r})$ d. $\dfrac{1}{3}(2\vec{p} + \vec{q} - \vec{r})$
(IIT-JEE 1997)

13. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent vectors and $|\vec{c}| = \sqrt{3}$, then

a. $\alpha = 1, \beta = -1$ b. $\alpha = 1, \beta = \pm 1$
c. $\alpha = -1, \beta = \pm 1$ d. $\alpha = \pm 1, \beta = 1$
(IIT-JEE 1998)

14. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $b = \hat{i} + \hat{j}$. If c is a vector such that $\vec{a} \cdot \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 30°, then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to

a. 2/3 b. 3/2
c. 2 d. 3 **(IIT-JEE 1999)**

15. Let $\vec{a} = \hat{i} - \hat{k}$, $\vec{b} = x\hat{i} + \hat{j} + (1 - x)\hat{k}$ and $\vec{c} = y\hat{i} + x\hat{j} + (1 + x - y)\hat{k}$. Then $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar for

a. some values of x b. some values of y
c. no values of x and y d. for all values of x and y
(IIT-JEE 2000)

16. Let $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and a unit vector $\vec{c}$ be coplanar. If $\vec{c}$ is perpendicular to $\vec{a}$, then $\vec{c}$ is

a. $\dfrac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ b. $\dfrac{1}{\sqrt{3}}(-\hat{i} - \hat{j} - \hat{k})$

c. $\dfrac{1}{\sqrt{5}}(\hat{i} - 2\hat{j})$ d. $\dfrac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

(IIT-JEE 2000)

17. If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ form the sides BC, CA and AB, respectively, of triangle ABC, then

a. $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=0$

b. $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

c. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}$

d. $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$ **(IIT-JEE 2000)**

18. Let vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$, respectively. Then the angle between P_1 and P_2 is

a. 0 b. $\pi/4$

c. $\pi/3$ d. $\pi/2$ **(IIT-JEE 2000)**

19. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit coplanar vectors, then the scalar triple product $[2\vec{a}-\vec{b}\ \ 2\vec{b}-\vec{c}\ \ 2\vec{c}-\vec{a}]$ is

a. 0 b. 1

c. $-\sqrt{3}$ d. $\sqrt{3}$ **(IIT-JEE 2000)**

20. If $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors, then $|\hat{a}-\hat{b}|^2+|\hat{b}-\hat{c}|^2+|\hat{c}-\hat{a}|^2$ does not exceed

a. 4 b. 9

c. 8 d. 6 **(IIT-JEE 2001)**

21. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other, then the angle between $\vec{a}$ and $\vec{b}$ is

a. 45° b. 60°

c. $\cos^{-1}(1/3)$ d. $\cos^{-1}(2/7)$

(IIT-JEE 2002)

22. Let $\vec{V}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{W}=\hat{i}+3\hat{k}$. If $\vec{U}$ is a unit vector, then the maximum value of the scalar triple product $[\vec{U}\ \vec{V}\ \vec{W}]$ is

a. -1 b. $\sqrt{10}+\sqrt{6}$

c. $\sqrt{59}$ d. $\sqrt{60}$ **(IIT-JEE 2002)**

23. The value of a so that the volume of parallelepiped formed by $\hat{i}+a\hat{j}+\hat{k}$, $\hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ is minimum is

a. -3 b. 3

c. $1/\sqrt{3}$ d. $\sqrt{3}$ **(IIT-JEE 2003)**

24. If $\vec{a}=(\hat{i}+\hat{j}+\hat{k})$, $\vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then $\vec{b}$ is

a. $\hat{i}-\hat{j}+\hat{k}$ b. $2\hat{j}-\hat{k}$

c. $\hat{i}$ d. $2\hat{i}$ **(IIT-JEE 2004)**

25. The unit vector which is orthogonal to the vector $5\hat{j}+2\hat{j}+6\hat{k}$ and is coplanar with vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ is

a. $\dfrac{2\hat{i}-6\hat{j}+\hat{k}}{\sqrt{41}}$ b. $\dfrac{2\hat{i}-3\hat{j}}{\sqrt{13}}$

c. $\dfrac{3\hat{i}-\hat{k}}{\sqrt{10}}$ d. $\dfrac{4\hat{i}+3\hat{j}-3\hat{k}}{\sqrt{34}}$

(IIT-JEE 2004)

26. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero, non-coplanar vectors and $\vec{b}_1=\vec{b}-\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$, $\vec{b}_2=\vec{b}+\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$,

$\vec{c}_1=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}+\dfrac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1$, $\vec{c}_2=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}-\dfrac{\vec{b}\cdot\vec{c}}{|\vec{b}_1|^2}\vec{b}_1$,

$\vec{c}_3=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}+\dfrac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1$, $\vec{c}_4=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}-\dfrac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b}_1$,

then the set of orthogonal vectors is

a. $(\vec{a}, \vec{b}_1, \vec{c}_3)$ b. $(\vec{a}, \vec{b}_1, \vec{c}_1)$

c. $(\vec{a}, \vec{b}_1, \vec{c}_2)$ d. $(\vec{a}, \vec{b}_2, \vec{c}_2)$

(IIT-JEE 2005)

27. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $1/\sqrt{3}$ is

a. $4\hat{i}-\hat{j}+4\hat{k}$ b. $3\hat{i}+\hat{j}-3\hat{k}$

c. $2\hat{i}+\hat{j}-2\hat{k}$ d. $4\hat{i}+\hat{j}-4\hat{k}$

(IIT-JEE 2006)

28. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar is

a. zero b. one

c. two d. three **(IIT-JEE 2007)**

29. Let $\vec{a}, \vec{b}, \vec{c}$ be units vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Which one of the following is correct?

a. $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\vec{0}$

b. $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq\vec{0}$

c. $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{a} \times \vec{c} = \vec{0}$

d. $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ are mutually perpendicular

(IIT-JEE 2007)

30. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t, the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\hat{a}\cot t + \hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$. then

a. $\hat{u} = \dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = (1 + \hat{a}\cdot\hat{b})^{1/2}$

b. $\hat{u} = \dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M = (1 + \hat{a}\cdot\hat{b})^{1/2}$

c. $\hat{u} = \dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = (1 + 2\hat{a}\cdot\hat{b})^{1/2}$

d. $\hat{u} = \dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M = (1 + 2\hat{a}\cdot\hat{b})^{1/2}$ **(IIT-JEE 2008)**

31. The edges of a parallelepiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = 1/2$. Then the volume of the parallelepiped is

a. $\dfrac{1}{\sqrt{2}}$ b. $\dfrac{1}{2\sqrt{2}}$ c. $\dfrac{\sqrt{3}}{2}$ d. $\dfrac{1}{\sqrt{3}}$

(IIT-JEE 2008)

32. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a} \times \vec{b})\cdot(\vec{c} \times \vec{d}) = 1$ and $\vec{a}\cdot\vec{c} = \dfrac{1}{2}$, then

a. $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar

b. $\vec{b}, \vec{c}$ and $\vec{d}$ are non-coplanar

c. $\vec{b}$ and $\vec{d}$ are non-parallel

d. $\vec{a}$ and $\vec{d}$ are parallel and $\vec{b}$ and $\vec{c}$ are parallel

(IIT-JEE 2009)

33. Two adjacent sides of a parallelogram $ABCD$ are given by $\overline{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$ and $\overline{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by

a. $\dfrac{8}{9}$ b. $\dfrac{\sqrt{17}}{9}$

c. $\dfrac{1}{9}$ d. $\dfrac{4\sqrt{5}}{9}$ **(IIT-JEE 2010)**

34. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}, 4\hat{i}, 3\hat{i} + 3\hat{j}$ and $-3\hat{j} + 2\hat{j}$, respectively. The quadrilateral $PQRS$ must be a

a. parallelogram, which is neither a rhombus nor a rectangle

b. square

c. rectangle, but not a square

d. rhombus, but not a square **(IIT-JEE 2010)**

35. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\dfrac{1}{\sqrt{3}}$, is given by

a. $\hat{i} - 3\hat{j} + 3\hat{k}$ b. $-3\hat{i} - 3\hat{j} + \hat{k}$

c. $3\hat{i} - \hat{j} + 3\hat{k}$ d. $\hat{i} + 3\hat{j} - 3\hat{k}$

(IIT-JEE 2011)

36. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a} + \vec{b}| = \sqrt{29}$ and $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$, then a possible value of $(\vec{a} + \vec{b})\cdot(-7\hat{i} + 2\hat{j} + 3\hat{k})$ is

a. 0 b. 3 c. 4 d. 8

(JEE Advanced 2012)

37. Let $\overline{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overline{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram $PQRS$, and $\overline{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overline{PT}, \overline{PQ}$ and $\overline{PS}$ is

a. 5 b. 20 c. 10 d. 30

(JEE Advanced 2013)

Multiple Correct Answers Type

1. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

a. 0 b. 1

c. $\dfrac{1}{4}\left(a_1^2 + a_2^2 + a_2^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$

d. $\dfrac{3}{4}\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)\left(c_1^2 + c_2^2 + c_3^2\right)$

(IIT-JEE 1986)

2. The number of vectors of unit length perpendicular to vectors $\vec{a} = (1, 1, 0)$ and $\vec{b} = (0, 1, 1)$ is

a. one b. two
c. three d. infinite **(IIT-JEE 1987)**

3. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$, whose projection on $\vec{a}$ is of magnitude $\sqrt{2/3}$, is

a. $2\hat{i} + 3\hat{j} - 3\hat{k}$ b. $2\hat{i} + 3\hat{j} + 3\hat{k}$
c. $-2\hat{i} - \hat{j} + 5\hat{k}$ d. $2\hat{i} + \hat{j} + 5\hat{k}$
(IIT-JEE 1993)

4. For three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ which of the following expressions is not equal to any of the remaining three?

a. $\vec{u} \cdot (\vec{v} \times \vec{w})$ b. $(\vec{v} \times \vec{w}) \cdot \vec{u}$
c. $\vec{v} \cdot (\vec{u} \times \vec{w})$ d. $(\vec{u} \times \vec{v}) \cdot \vec{w}$
(IIT-JEE 1998)

5. Which of the following expressions are meaningful?

a. $\vec{u} \cdot (\vec{v} \times \vec{w})$ b. $(\vec{u} \cdot \vec{v}) \cdot \vec{w}$
c. $(\vec{u} \cdot \vec{v})\, \vec{w}$ d. $\vec{u} \times (\vec{v} \cdot \vec{w})$
(IIT-JEE 1998)

6. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\, \vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$, then $|\vec{v}|$ is

a. $|\vec{u}|$ b. $|\vec{u}| + |\vec{u} \cdot \vec{a}|$
c. $|\vec{u}| + |\vec{u} \cdot \vec{b}|$ d. $|\vec{u}| + \vec{u} \cdot (\vec{a} + \vec{b})$
(IIT-JEE 1999)

7. Vector $\frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$ is

a. a unit vector
b. makes an angle $\pi/3$ with vector $(2\hat{i} - 4\hat{j} + 3\hat{k})$
c. parallel to vector $\left(-\hat{i} + \hat{j} - \frac{1}{2}\hat{k}\right)$
d. perpendicular to vector $3\hat{i} + 2\hat{j} - 2\hat{k}$
(IIT-JEE 1994)

8. Let $\vec{A}$ be a vector parallel to the line of intersection of planes P_1 and P_2. Plane P_1 is parallel to vectors $2\hat{j} + 3\hat{k}$ and $4\hat{j} - 3\hat{k}$ and P_2 is parallel to $\hat{j} - \hat{k}$ and $3\hat{i} + 3\hat{j}$. Then the angle between vector $\vec{A}$ and a given vector $2\hat{i} + \hat{j} - 2\hat{k}$ is

a. $\pi/2$ b. $\pi/4$ c. $\pi/6$ d. $3\pi/4$
(IIT-JEE 2006)

9. The vector(s) which is/are coplanar with vectors $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to vector $\hat{i} + \hat{j} + \hat{k}$, is/are

a. $\hat{j} - \hat{k}$ b. $-\hat{i} + \hat{j}$ c. $\hat{i} - \hat{j}$ d. $-\hat{j} + \hat{k}$
(IIT-JEE 2011)

10. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a non-zero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then

a. $\vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$ b. $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$
c. $\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$ d. $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{z} - \vec{y})$
(JEE Advanced 2014)

11. Let ΔPQR be a triangle. Let $\vec{a} = \overline{QR}$, $\vec{b} = \overline{RP}$ and $\vec{c} = \overline{PQ}$. If $|\vec{a}| = 12, |\vec{b}| = 4\sqrt{3}$ and $\vec{b} \cdot \vec{c} = 24$, then which of the following is (are) true?

a. $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$ b. $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$
c. $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$ d. $\vec{a} \cdot \vec{b} = -72$
(JEE Advanced 2015)

Matching Column Type

1. Match the statements given in Column I with the values of given in Column II.

Column I	Column II
(a) Root(s) of the equation $2\sin^2\theta + \sin^2 2\theta = 2$	(p) $\frac{\pi}{6}$
(b) Points of discontinuity of the function $f(x) = \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$, where $[y]$ denotes the largest integer less than or equal to y	(q) $\frac{\pi}{4}$
(c) Volume of the parallelepiped with its edges represented by the vectors $\hat{i} + \hat{j}, \hat{i} + 2\hat{j}$ and $\hat{i} + \hat{j} + \pi\hat{k}$	(r) $\frac{\pi}{3}$
(d) Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a} + \vec{b} + \sqrt{3}\,\vec{c} = 0$	(s) $\frac{\pi}{2}$
	(t) π

(IIT-JEE 2009)

2. Match the statements given in Column I with the values of given in Column II.

Column I	Column II
(a) A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If length $PQ = d$, then d^2 is	(p) –4
(b) The value of x satisfying $\tan^{-1}(x+3)-\tan^{-1}(x-3)=\sin^{-1}\left(\frac{3}{5}\right)$ are	(q) 0
(c) Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}\cdot\vec{b}=0, (\vec{b}-\vec{a})\cdot(\vec{b}+\vec{c})=0$ and $2\lvert\vec{b}+\vec{c}\rvert=\lvert\vec{b}-\vec{a}\rvert$. If $\vec{a}=\mu\vec{b}+4\vec{c}$, then the possible values of μ are	(r) 4
(d) Let f be the function on $[-\pi, \pi]$ given by $f(0)=9$ and $f(x)=\sin\left(\frac{9x}{2}\right)\Big/\sin\left(\frac{x}{2}\right)$ for $x\neq 0$. The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ is	(s) 5
	(t) 6

(IIT-JEE 2010)

3. Match the statements given in Column I with the values of given in Column II.

Column I	Column II
(a) Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	(p) 100
(b) Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	(q) 30
(c) Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	(r) 24
(d) Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	(s) 60

(JEE Advanced 2013)

4. Match the statements given in Column I with the values of given in Column II.

Column I	Column II
(a) If $\vec{a}=\hat{j}+\sqrt{3}\hat{k}, \vec{b}=-\hat{j}+\sqrt{3}\hat{k}$ and $\vec{c}=2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(b) If $\int_a^b (f(x)-3x)dx = a^2-b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(c) The value of $\frac{\pi^2}{\log_e 3}\int_{7/6}^{5/6} \sec(\pi x)\,dx$ is	(r) $\frac{\pi}{3}$
(d) the maximum value of $\left\lvert \text{Arg}\left(\frac{1}{1-z}\right)\right\rvert$ for $\lvert z\rvert = 1, z\neq 1$ is given by	(s) π
	(t) $\frac{\pi}{2}$

(JEE Advanced 2013)

5. Match the statements given in Column I with the values of given in Column II.

Column I	Column II
(p) Let $y(x)=\cos(3\cos^{-1}x), x\in[-1, 1], x\neq\pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals	(1) 1
(q) Let $A_1, A_2, \ldots, A_n$ $(n>2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a_k}$ be the position vector of the point A_k, $k = 1, 2, \ldots n$. If $\left\lvert\sum_{k=1}^{n-1}(\vec{a_k}\times\vec{a_{k+1}})\right\rvert = \left\lvert\sum_{k=1}^{n-1}(\vec{a_k}\cdot\vec{a_{k+1}})\right\rvert$, then the minimum value of n is	(2) 2

(r) If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line $x+y=8$, then the value of h is	(3) 8
(s) Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is	(4) 9

Codes:

	(p)	(q)	(r)	(s)
a.	(4)	(3)	(2)	(1)
b.	(2)	(4)	(3)	(1)
c.	(4)	(3)	(1)	(2)
d.	(2)	(4)	(1)	(3)

(JEE Advanced 2014)

6.

Column I	Column II
(a) In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha=2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(p) 1
(b) Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\ge 1\end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of α is (are)	(q) 2
(c) Let $\omega \ne 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$, then possible value(s) of n is (are)	(r) 3
(d) Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is an arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	(s) 4
	(t) 5

(JEE Advanced 2015)

7.

Column I	Column II
(a) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z respectively. If $2(a^2-b^2)=c^2$ and $\lambda=\frac{\sin(X-Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda)=0$ is (are)	(p) 1
(b) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1+\cos 2X-2\cos 2Y=2\sin X\sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	(q) 2
(c) In R^2, let $\sqrt{3}\hat{i}+\hat{j}, \hat{i}+\sqrt{3}\hat{j}$ and $\beta\hat{i}+(1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect of the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	(r) 3
(d) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x=0$, $x=2$, $y^2=4x$ and $y=\lvert\alpha x-1\rvert+\lvert\alpha x-2\rvert+\alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha)+\frac{8}{3}\sqrt{2}$, when $\alpha=0$ and $\alpha=1$, is (are)	(s) 5
	(t) 6

(JEE Advanced 2015)

Integer Answer Type

1. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $(2\vec{a}+\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})]$ is **(IIT-JEE 2010)**
2. Let $\vec{a}=-\hat{i}-\hat{k}, \vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}\cdot\vec{a}=0$, then the value of $\vec{r}\cdot\vec{b}$ is **(IIT-JEE 2011)**
3. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $\lvert\vec{a}-\vec{b}\rvert^2+\lvert\vec{b}-\vec{c}\rvert^2+\lvert\vec{c}-\vec{a}\rvert^2=9$, then $\lvert 2\vec{a}+5\vec{b}+5\vec{c}\rvert$ is. **(IIT-JEE 2012)**
4. Consider the set of eight vectors $V=\{a\hat{i}+b\hat{j}+c\hat{k};\ a, b, c \in \{-1, 1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **(JEE Advanced 2013)**

5. Let $\vec{a}, \vec{b}$, and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p, q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is **(JEE Advanced 2014)**

6. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r})$, $(\vec{p} - \vec{q} + \vec{r})$ and $(-\vec{p} - \vec{q} + \vec{r})$ are x, y and z, respectively, then the value of $2x + y + z$ is **(JEE Advanced 2015)**

Assertion-Reasoning Type

1. Let the vectors $\vec{PQ}, \vec{QR}, \vec{RS}, \vec{ST}, \vec{TU}$ and $\vec{UP}$ represent the sides of a regular hexagon.

Statement 1: $\vec{PQ} \times (\vec{RS} + \vec{ST}) \neq \vec{0}$

Statement 2: $\vec{PQ} \times \vec{RS} = \vec{0}$ and $\vec{PQ} \times \vec{ST} \neq \vec{0}$

 a. Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
 b. Statement 1 is true, statement 2 is true; statement 2 is NOT a correct explanation for statement 1.
 c. Statement 1 is true, statement 2 is false.
 d. Statement 1 is false, statement 2 is true.

(IIT-JEE 2007)

Fill in the Blanks Type

1. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be vectors of length, 3, 4 and 5, respectively. Let $\vec{A}$ be perpendicular to $\vec{B} + \vec{C}$, $\vec{B}$ to $\vec{C} + \vec{A}$ and $\vec{C}$ to $\vec{A} + \vec{B}$. Then the length of vector $\vec{A} + \vec{B} + \vec{C}$ is___________. **(IIT-JEE 1981)**
2. The unit vector perpendicular to the plane determined by $P(1, -1, 2)$, $Q(2, 0, -1)$ and $R(0, 2, 1)$ is___________. **(IIT-JEE 1983)**
3. The area of the triangle whose vertices are $A(1, -1, 2)$, $B(2, 1, -1)$, $C(3, -1, 2)$ is___________. **(IIT-JEE 1983)**
4. A, B, C and D are four points in a plane with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, such that $(\vec{a} - \vec{d}) \cdot (\vec{b} - \vec{c}) = (\vec{b} - \vec{d}) \cdot (\vec{c} - \vec{a}) = 0$. Then point D is the___________ of triangle ABC. **(IIT-JEE 1984)**
5. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\vec{A} = (1, a, a^2)$, $\vec{B} = (1, b, b^2)$, $\vec{C} = (1, c, c^2)$ are non-coplanar, then the product $abc =$___________. **(IIT-JEE 1985)**
6. If $\vec{A}, \vec{B}$ and $\vec{C}$ are three non-coplanar vectors, then $\frac{\vec{A} \cdot \vec{B} \times \vec{C}}{\vec{C} \times \vec{A} \cdot \vec{B}} + \frac{\vec{B} \cdot \vec{A} \times \vec{C}}{\vec{C} \cdot \vec{A} \times \vec{B}} =$___________. **(IIT-JEE 1985)**
7. If $\vec{A} = (1, 1, 1)$ and $\vec{C} = (0, 1, -1)$ are given vectors, then vector $\vec{B}$ satisfying the equations $\vec{A} \times \vec{B} = \vec{C}$ and $\vec{A} \cdot \vec{B} = 3$ is___________. **(IIT-JEE 1985)**
8. If the vectors $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ $(a, b, c \neq 1)$ are coplanar, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ $=$___________. **(IIT-JEE 1987)**
9. Let $\vec{b} = 4\hat{i} + 3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$, respectively, are given by___________. **(IIT-JEE 1987)**
10. The components of a vector $\vec{a}$ along and perpendicular to a non-zero vector $\vec{b}$ are___________ and _______, respectively. **(IIT-JEE 1988)**
11. A unit vector coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and perpendicular to $\hat{i} + \hat{j} + \hat{k}$ is___________. **(IIT-JEE 1992)**
12. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by vectors $\hat{i}$ and $\hat{i} + \hat{j}$ and the plane determined by vectors $\hat{i} - \hat{j}$ and $\hat{i} + \hat{k}$. The angle between $\vec{a}$ and vector $\hat{i} - 2\hat{j} + 2\hat{k}$ is___________. **(IIT-JEE 1996)**
13. If $\vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors and $\vec{a}$ is any vector, then $(\vec{a} \cdot \vec{b})\vec{b} + (\vec{a} \cdot \vec{c})\vec{c} + \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}(\vec{b} \times \vec{c}) =$___________. **(IIT-JEE 1996)**
14. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors having magnitudes 1, 1 and 2, respectively. If $\vec{a} \times (\vec{a} \times \vec{c}) + \vec{b} = \vec{0}$, then the acute angle between $\vec{a}$ and $\vec{c}$ is___________. **(IIT-JEE 1997)**

15. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = 10\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$, where O, A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If $p = kq$, then k = ___________. **(IIT-JEE 1997)**

True/False Type

1. Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be unit vectors such that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C} = 0$ and the angle between $\vec{B}$ and $\vec{C}$ be $\pi/3$. Then $\vec{A} = \pm 2(\vec{B}\times\vec{C})$. **(IIT-JEE 1981)**
2. If $\vec{X}\cdot\vec{A} = 0$, $\vec{X}\cdot\vec{B} = 0$ and $\vec{X}\cdot\vec{C} = 0$ for some non-zero vector $\vec{X}$, then $[\vec{A}\ \vec{B}\ \vec{C}] = 0$. **(IIT-JEE 1983)**
3. The points with position vectors $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$ and $\vec{a}+k\vec{b}$ are collinear for all real values of k. **(IIT-JEE 1984)**
4. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $(\vec{a}-\vec{b})\cdot(\vec{b}-\vec{c})\times(\vec{c}-\vec{a}) = 2\vec{a}\cdot\vec{b}\times\vec{c}$. **(IIT-JEE 1989)**

Subjective Type

1. From a point O inside a triangle ABC, perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB, respectively. Prove that the perpendiculars from A, B and C to the sides EF, FD and DE are concurrent. **(IIT-JEE 1978)**
2. A vector has components A_1, A_2 and A_3 in a right-handed rectangular Cartesian coordinate system $OXYZ$. The coordinate system is rotated about the z-axis through an angle $\pi/2$. Find the components of A in the new coordinate system in terms of A_1, A_2 and A_3. **(IIT-JEE 1983)**
3. If c is a given non-zero scalar, and $\vec{A}$ and $\vec{B}$ are given non-zero vectors such that $\vec{A}\perp\vec{B}$, then find vector $\vec{X}$ which satisfies the equations $\vec{A}\cdot\vec{X} = c$ and $\vec{A}\times\vec{X} = \vec{B}$. **(IIT-JEE 1983)**
4. The position vectors of the point A, B, C and D are $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$, respectively. If the points A, B, C and D lie on a plane, find the value of λ. **(IIT-JEE 1986)**
5. If A, B, C, D are any four points in space, prove that $|\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}| = 4$ (area of triangle ABC). **(IIT-JEE 1986)**
6. Let $OACB$ be a parallelogram with O at the origin and OC a diagonal. Let D be the midpoint of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio. **(IIT-JEE 1988)**
7. In a triangle OAB, E is the midpoint of BO and D is a point on AB such that $AD : DB = 2 : 1$. If OD and AE intersect at P, determine the ratio $OP : PD$ using the vector method. **(IIT-JEE 1989)**
8. If vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, show that
$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = \vec{0}.$$
(IIT-JEE 1989)
9. Let $\vec{A} = 2\hat{i}+\hat{k}$, $\vec{B} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{C} = 4\hat{i}-3\hat{j}+7\hat{k}$. Determine a vector $\vec{R}$ satisfying $\vec{R}\times\vec{B} = \vec{C}\times\vec{B}$ and $\vec{R}\cdot\vec{A} = 0$. **(IIT-JEE 1990)**
10. Determine the value of c so that for all real x, vectors $cx\hat{i}-6\hat{j}-3\hat{k}$ and $x\hat{i}+2\hat{j}+2cx\hat{k}$ make an obtuse angle with each other. **(IIT-JEE 1991)**
11. In a triangle ABC, D and E are points on BC and AC, respectively, such that $BD = 2DC$ and $AE = 3EC$. Let P be the point of intersection of AD and BE. Find BP/PE using the vector method. **(IIT-JEE 1993)**
12. If vectors $\vec{b}$, $\vec{c}$ and $\vec{d}$ are not coplanar, then prove that vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to $\vec{a}$. **(IIT-JEE 1994)**
13. The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $2\sqrt{2}/3$, find the position vectors of the point E for all its possible positions. **(IIT-JEE 1996)**
14. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-coplanar unit vectors, equally inclined to one another at an angle θ. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c} = p\vec{a}+q\vec{b}+r\vec{c}$, find scalars p, q and r in terms of θ. **(IIT-JEE 1997)**
15. If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are vectors such that $|\vec{B}| = |\vec{C}|$. Prove that $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})\cdot(\vec{B}+\vec{C}) = 0$. **(IIT-JEE 1997)**
16. Find all values of λ such that $x, y, z \neq (0, 0, 0)$ and $(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z = \lambda(x\hat{i}+y\hat{j}+z\hat{k})$, where, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the coordinate axes. **(IIT-JEE 1998)**

17. Prove, by vector method or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the midpoint of the parallel sides (you may assume that the trapezium is not a parallelogram.) **(IIT-JEE 1998)**

18. $A_1, A_2, \ldots, A_n$ are the vertices of a regular plane polygon with n sides and O as its centre. Show that $\sum_{i=1}^{n-1} (\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = ((1-n)(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$. **(IIT-JEE 1998)**

19. For any two vectors $\vec{u}$ and $\vec{v}$, prove that

a. $(\vec{u}.\vec{v})^2 + |\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2$ and

b. $(\vec{1} + |\vec{u}|^2)(\vec{1} + |\vec{v}|^2)$
$= (1 - \vec{u}.\vec{v})^2 + |\vec{u} + \vec{v} + (\vec{u} \times \vec{v})|^2$ **(IIT-JEE 1998)**

20. Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$, then prove that $|(\vec{u} \times \vec{v}) \cdot \vec{w}| \le 1/2$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$. **(IIT-JEE 1999)**

21. Show, by vector method, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices. **(IIT-JEE 2001)**

22. Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}$, $t \in [0, 1]$, where f_1, f_2, g_1, g_2 are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all and $\vec{A}(0) = 2\hat{i} + 3\hat{j}$, $\vec{A}(1) = 6\hat{i} + 2\hat{j}$, $\vec{B}(0) = 3\hat{i} + 2\hat{j}$ and $\vec{B}(1) = 2\hat{i} + 6\hat{j}$, then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t. **(IIT-JEE 2001)**

23. Find three-dimensional vectors $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ satisfying $\vec{v_1} \cdot \vec{v_1} = 4$, $\vec{v_1} \cdot \vec{v_2} = -2$, $\vec{v_1} \cdot \vec{v_3} = 6$, $\vec{v_2} \cdot \vec{v_2} = 2$, $\vec{v_2} \cdot \vec{v_3} = -5$, $\vec{v_3} \cdot \vec{v_3} = 29$. **(IIT-JEE 2001)**

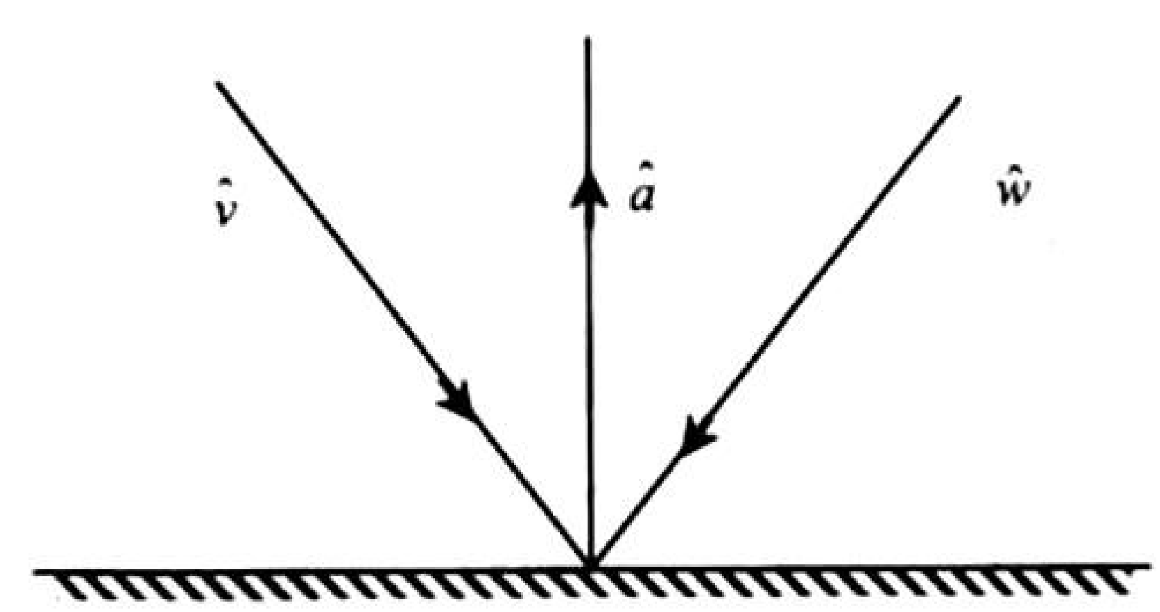
24. Let V be the volume of the parallelepiped formed by the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If a_r, b_r and c_r, where $r = 1, 2, 3$, are non-negative real numbers and $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$, show that $V \le L^3$. **(IIT-JEE 2002)**

25. $\vec{u}, \vec{v}$ and $\vec{w}$ are three non-coplanar unit vectors and α, β and γ are the angles between $\vec{u}$ and $\vec{v}$, $\vec{v}$ and $\vec{w}$, and $\vec{w}$ and $\vec{u}$, respectively, and $\vec{x}, \vec{y}$ and $\vec{z}$ are unit vectors along the bisectors of the angles α, β and γ, respectively. Prove that $[\vec{x} \times \vec{y} \;\; \vec{y} \times \vec{z} \;\; \vec{z} \times \vec{x}]$ $= \frac{1}{16}[\vec{u}\;\vec{v}\;\vec{w}]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$. **(IIT-JEE 2003)**

26. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are distinct vectors such that $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, prove that $(\vec{a} - \vec{d}) \cdot (\vec{b} - \vec{c}) \ne 0$, i.e., $\vec{a} \cdot \vec{b} + \vec{d} \cdot \vec{c} \ne \vec{d} \cdot \vec{b} + \vec{a} \cdot \vec{c}$. **(IIT-JEE 2004)**

27. P_1 and P_2 are planes passing through origin. L_1 and L_2 are two lines on P_1 and P_2, respectively, such that their intersection is the origin. Show that there exist points A, B and C, whose permutation A', B' and C', respectively, can be chosen such that **(i)** A is on L_1, B on P_1 but not on L_1 and C not on P_1; **(ii)** A' is on L_2, B' on P_2 but not on L_2 and C' not on P_2. **(IIT-JEE 2004)**

28. If the incident ray on a surface is along the unit vector $\hat{v}$, the reflected ray is along the unit vector $\hat{w}$ and the normal is along the unit vector $\hat{a}$ outwards, express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$. **(IIT-JEE 2005)**



Answer Key

JEE Advanced

Single Correct Answer Type

1. a.	**2.** d.	**3.** d.	**4.** a.	**5.** d.
6. b.	**7.** b.	**8.** a.	**9.** a.	**10.** b.
11. d.	**12.** b.	**13.** d.	**14.** b.	**15.** d.
16. a.	**17.** b.	**18.** a.	**19.** a.	**20.** b.
21. b.	**22.** c.	**23.** c.	**24.** c.	**25.** c.
26. c.	**27.** a.	**28.** c.	**29.** b.	**30.** a.
31. a.	**32.** c.	**33.** b.	**34.** a.	**35.** c.
36. c.	**37.** c.			

Multiple Correct Answers Type

1. c.	**2.** b.	**3.** a., c.	**4.** c.
5. a., c.	**6.** a., c.	**7.** a., c., d.	**8.** b., d.
9. a., d.	**10.** a., b., c.	**11.** a., c., d.	

Matching Column Type

1. (c) – (t); (d) – (r) **2.** (c) – (q), (s)
3. (a) – (r); (b) – (s); (c) – (p); (d) – (q)
4. (a) – (q) **5.** a.
6. (a) – (p), (q) **7.** (c) – (p, q)

Integer Answer Type

1. (5) **2.** (9) **3.** (3) **4.** (5)
5. (4) **6.** (9)

Assertion–Reasoning Type

1. c.

Fill in the Blanks Type

1. $5\sqrt{2}$ **2.** $\dfrac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$

3. $\sqrt{13}$ **4.** orthocenter **5.** -1

6. 0 **7.** $\dfrac{5}{3}\hat{i}+\dfrac{2}{3}\hat{j}+\dfrac{2}{3}\hat{k}$ **8.** 1

9. $2\hat{i}-\hat{j}$ **10.** $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ and $\vec{a}-\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

11. $\dfrac{\hat{j}-\hat{k}}{\sqrt{2}}$ or $\dfrac{-\hat{j}+\hat{k}}{\sqrt{2}}$ **12.** $\pi/4$ or $3\pi/4$

13. $\vec{a}$ **14.** $\pi/6$ **15.** 6

True/False Type

1. False **2.** True **3.** True **4.** False

Subjective Type

2. $A_2\hat{i}-A_1\hat{j}+A_3\hat{k}$ **3.** $\vec{X}=\dfrac{\vec{B}\times\vec{A}+c\vec{A}}{(\vec{A}\cdot\vec{A})}$

4. $-\dfrac{146}{17}$ **6.** 2 : 1 **9.** $-\hat{i}-8\hat{j}+2\hat{k}$

10. $-4/3 < c < 0$ **11.** 8:3.

13. $-\hat{i}+3\hat{j}+3\hat{k}$ or $3\hat{i}-\hat{j}-\hat{k}$

14. $p=\dfrac{1}{\sqrt{1+2\cos\theta}}, q=\dfrac{-2\cos\theta}{\sqrt{1+2\cos\theta}}, r=\dfrac{1}{\sqrt{1+2\cos\theta}}$

28. $\hat{w}=\hat{v}-2(\hat{a}\cdot\hat{v})\hat{a}$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. a. $\vec{A}\cdot(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$

$= \vec{A}\cdot[\vec{B}\times\vec{A}+\vec{B}\times\vec{B}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}+\vec{C}\times\vec{B}+\vec{C}\times\vec{C}]$

$= \vec{A}\cdot\vec{B}\times\vec{A}+\vec{A}\cdot\vec{B}\times\vec{C}+\vec{A}\cdot\vec{C}\times\vec{A}+\vec{A}\cdot\vec{C}\times\vec{B}$

(Using $\vec{a}\times\vec{a}=0$)

$= 0+[\vec{A}\,\vec{B}\,\vec{C}]+0+[\vec{A}\,\vec{C}\,\vec{B}]$

$= [\vec{A}\,\vec{B}\,\vec{C}]-[\vec{A}\,\vec{B}\,\vec{C}]$

$= 0$

2. d. $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

or $||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

or $|\vec{a}||\vec{b}||\vec{c}||\sin\theta\cos\alpha| = |\vec{a}||\vec{b}||\vec{c}|$

or $|\sin\theta||\cos\alpha| = 1$

$\Rightarrow$ $\theta=\pi/2$ and $\alpha=0$

i.e., $\vec{a}\perp\vec{b}$ and $\vec{c}\,||\,\hat{n}$ or $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$\Rightarrow$ $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$

3. d. Volume of parallelepiped = $[\vec{OA}\ \vec{OB}\ \vec{OC}]$

$$= \begin{vmatrix} 2 & -2 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 2(-1)+2(-1+3) = 2$$

4. a. Three points $A(\vec{a}), B(\vec{b}), C(\vec{c})$ are collinear if $\overline{AB}\,||\,\overline{AC}$

$\overrightarrow{AB} = -20\hat{i}-11\hat{j};\ \overrightarrow{AC} = (a-60)\hat{i}-55\hat{j}$

$\Rightarrow$ $\overrightarrow{AB}\,||\,\overrightarrow{AC} \Rightarrow \dfrac{a-60}{-20} = \dfrac{-55}{-11}$ or $a=-40$

5. d. Given that $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar. Therefore,

$[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$

Also, $\vec{p} = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{q} = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$,

$$\vec{r} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} \quad \text{(i)}$$

Now, $(\vec{a}+\vec{b})\cdot\vec{p} + (\vec{b}+\vec{c})\cdot\vec{q} + (\vec{c}+\vec{a})\cdot\vec{r}$

$$= (\vec{a}+\vec{b})\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} + (\vec{b}+\vec{c})\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} + (\vec{c}+\vec{a})\cdot\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{\vec{a}\cdot\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{\vec{b}\cdot\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{\vec{c}\cdot\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$[\because \vec{b}\cdot\vec{b}\times\vec{c} = \vec{c}\cdot\vec{c}\times\vec{a} = \vec{a}\cdot\vec{a}\times\vec{b} = 0]$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$= 1 + 1 + 1$
$= 3$

6. b. a, b and c are distinct negative numbers and vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

or $ac + c^2 - ab - ac = 0$

or $c^2 = ab$

Hence, a, c, b are in G.P.

So, c is the G.M. of a and b.

7. b. Let the given position vectors be of points A, B and C, respectively. Then

$$|\overrightarrow{AB}| = \sqrt{(\beta-\alpha)^2 + (\gamma-\beta)^2 + (\alpha-\gamma)^2}$$

$$|\overrightarrow{BC}| = \sqrt{(\gamma-\beta)^2 + (\alpha-\gamma)^2 + (\alpha-\beta)^2}$$

$$|\overrightarrow{CA}| = \sqrt{(\alpha-\gamma)^2 + (\beta-\alpha)^2 + (\gamma-\beta)^2}$$

$\because \quad |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$

Hence, ΔABC is an equilateral triangle.

8. a. Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$.

where $x^2 + y^2 + z^2 = 1$ (i)

($\vec{d}$ being a unit vector)

$\therefore \quad \vec{a}.\vec{d} = 0$

$\Rightarrow \quad x - y = 0$ or $x = y$ (ii)

$[\vec{b}\ \vec{c}\ \vec{d}] = 0$

$$\Rightarrow \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ x & y & z \end{vmatrix} = 0$$

or $x + y + z = 0$

or $2x + z = 0$ [using (ii)]

or $z = -2x$ (iii)

From (i), (ii) and (iii), we have

$x^2 + x^2 + 4x^2 = 1$

$x = \pm \dfrac{1}{\sqrt{6}}$

$$\therefore \quad \vec{d} = \pm\left(\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{2}{\sqrt{6}}\hat{k}\right)$$

$$= \pm\left(\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}\right)$$

9. a. Since $\vec{a}\times(\vec{b}\times\vec{c}) = \dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$

$$\therefore \quad (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c} = \frac{1}{\sqrt{2}}\vec{b} + \frac{1}{\sqrt{2}}\vec{c}$$

Since $\vec{b}$ and $\vec{c}$ are non-coplanar,

$$\vec{a}\cdot\vec{c} = \frac{1}{\sqrt{2}} \text{ and } \vec{a}\cdot\vec{b} = -\frac{1}{\sqrt{2}}$$

$\Rightarrow \quad \cos\theta = -\dfrac{1}{\sqrt{2}}$ (because $\vec{a}$ and $\vec{b}$ are unit vectors)

or $\theta = \dfrac{3\pi}{4}$

10. b. Since $\vec{u} + \vec{v} + \vec{w} = 0$, we have

$$|\vec{u} + \vec{v} + \vec{w}|^2 = 0$$

or $|\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}) = 0$

or $9 + 16 + 25 + 2(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}) = 0$

or $\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u} = -25$

11. d. $(\vec{a}+\vec{b}+\vec{c})\cdot[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})]$

$$= (\vec{a}+\vec{b}+\vec{c})\cdot[\vec{a}\times\vec{a} + \vec{a}\times\vec{c} + \vec{b}\times\vec{a} + \vec{b}\times\vec{c}]$$

$$= (\vec{a}+\vec{b}+\vec{c})\cdot[\vec{a}\times\vec{c} + \vec{b}\times\vec{a} + \vec{b}\times\vec{c}]$$

$$= \vec{a}\cdot\vec{b}\times\vec{c} + \vec{b}\cdot\vec{a}\times\vec{c} + \vec{c}\cdot\vec{b}\times\vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{c}]$$

$$= -[\vec{a}\ \vec{b}\ \vec{c}]$$

12. b. As $\vec{p}, \vec{q}$ and $\vec{r}$ are three mutually perpendicular vectors of same magnitude, so let us consider

$$\vec{p} = a\hat{i}, \vec{q} = a\hat{j}, \vec{r} = a\hat{k}$$

Also, let $\vec{x} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

Given that $\vec{x}$ satisfies the equation

$$\vec{p} \times [(\vec{x} - \vec{q}) \times \vec{p}] + \vec{q} \times [(\vec{x} - \vec{r}) \times \vec{q}] + \vec{r} \times [(\vec{x} - \vec{p}) \times \vec{r}] = 0 \quad \text{(i)}$$

Now, $\vec{p} \times [(\vec{x} - \vec{q}) \times \vec{p}] = \vec{p} \times [\vec{x} \times \vec{p} - \vec{q} \times \vec{p}]$

$$= \vec{p} \times (\vec{x} \times \vec{p}) - \vec{p} \times (\vec{q} \times \vec{p})$$

$$= (\vec{p} \cdot \vec{p})\vec{x} - (\vec{p} \cdot \vec{x})\vec{p} - (\vec{p} \cdot \vec{p})\vec{q} + (\vec{p} \cdot \vec{q})\vec{p}$$

$$= a^2\vec{x} - a^2 x_1 \hat{i} - a^3\hat{j} + 0$$

Similarly,

$$\vec{q} \times [(\vec{x} - \vec{r}) \times \vec{q}] = a^2\vec{x} - a^2 y_1\hat{j} - a^3\hat{k}$$

and $\vec{r} \times [(\vec{x} - \vec{p}) \times \vec{r}] = a^2\vec{x} - a^2 z_1\hat{k} - a^3\hat{i}$

Substituting these values in the equation, we get

$$3a^2\vec{x} - a^2(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) - a^2(a\hat{i} + a\hat{j} + a\hat{k}) = \vec{0}$$

or $3a^2\vec{x} - a^2\vec{x} - a^2(\vec{p} + \vec{q} + \vec{r}) = \vec{0}$

or $2a^2\vec{x} = (\vec{p} + \vec{q} + \vec{r})\, a^2$

or $\vec{x} = \frac{1}{2}(\vec{p} + \vec{q} + \vec{r})$

13. d. Given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent,

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0$$

or $1 - \beta = 0$

or $\beta = 1$

Also, given that $|\vec{c}| = \sqrt{3} \Rightarrow 1 + \alpha^2 + \beta^2 = 3$

Substituting the value of β, we get

$\alpha^2 = 1$ or $\alpha = \pm 1$

14. b. $|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}||\vec{c}| \sin 30°$

$$= \frac{1}{2}|\vec{a} \times \vec{b}||\vec{c}| \quad \text{(i)}$$

We have, $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$

$\Rightarrow \vec{a} \times \vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{9} = 3$

Also given $|\vec{c} - \vec{a}| = 2\sqrt{2}$

or $|\vec{c} - \vec{a}|^2 = 8$

or $|\vec{c}|^2 + |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{c} = 8$

Given $|\vec{a}| = 3$ and $\vec{a} \cdot \vec{c} = |\vec{c}|$, using these we get

$$|\vec{c}|^2 - 2|\vec{c}| + 1 = 0$$

or $(|\vec{c}| - 1)^2 = 0$

or $|\vec{c}| = 1$

Substituting values of $|\vec{a} \times \vec{b}|$ and $|\vec{c}|$ in (i), we get

$$|(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$$

15. d. $\vec{a} = \hat{i} - \hat{k}$

$$\vec{b} = x\hat{i} + \hat{j} + (1 - x)\hat{k}$$

$$\vec{c} = y\hat{i} + x\hat{j} + (1 + x - y)\hat{k}$$

$$\therefore \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1 - x \\ y & x & 1 + x - y \end{vmatrix}$$

$$= 1 + x - y - x^2 + y - x + x^2$$

$$= 1$$

16. a. As $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$, we take

$$\vec{c} = \alpha\vec{a} + \beta\vec{b}, \quad \text{(i)}$$

where α and β are scalars.

As $\vec{c}$ is perpendicular to $\vec{a}$, using (i), we get

$$0 = \alpha\,\vec{a} \cdot \vec{a} + \beta\,\vec{b} \cdot \vec{a}$$

or $0 = \alpha(6) + \beta(2 + 2 - 1) = 3(2\alpha + \beta)$

or $\beta = -2\alpha$

Thus, $\vec{c} = \alpha(\vec{a} - 2\vec{b}) = \alpha(-3\hat{j} + 3\hat{k}) = 3\alpha(-\hat{j} + \hat{k})$

$\therefore |\vec{c}|^2 = 18\alpha^2$

or $1 = 18\alpha^2$

or $\alpha = \pm \frac{1}{3\sqrt{2}}$

$\therefore \vec{c} = \pm \frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$

17. b. Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ (by triangle law). Therefore,

$$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$$

$\therefore \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\therefore \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

Similarly, by taking cross product with $\vec{b}$, we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

$\therefore \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

18. a. Given that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are vectors such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$ (i)

P_1 is the plane determined by vectors $\vec{a}$ and $\vec{b}$. Therefore, normal vector $\vec{n_1}$ to P_1 will be given by

$$\vec{n_1} = \vec{a} \times \vec{b}$$

Similarly, P_2 is the plane determined by vectors $\vec{c}$ and $\vec{d}$. Therefore, normal vector $\vec{n_2}$ to P_2 will be given by

$$\vec{n_2} = \vec{c} \times \vec{d}$$

Substituting the values of $\vec{n_1}$ and $\vec{n_2}$ in (i), we get

$$\vec{n_1} \times \vec{n_2} = \vec{0}$$

Hence, $\vec{n_1} \,||\, \vec{n_2}$

Hence, the planes will also be parallel to each other.

Thus, angle between the planes is 0.

19. a. Given $\vec{a}, \vec{b}$ and $\vec{c}$ are unit coplanar vectors so, $2\vec{a} - \vec{b}, 2\vec{b} - 2\vec{c}$ and $2\vec{c} - \vec{a}$ are also coplanar vectors, being a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

Thus, $[2\vec{a} - \vec{b}\ \ 2\vec{b} - \vec{c}\ \ 2\vec{c} - \vec{a}] = 0$

20. b. $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors.

Now, $x = |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2$

$$= 2(\hat{a}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{c}\cdot\hat{c}) - 2\hat{a}\cdot\hat{b} - 2\hat{b}\cdot\hat{c} - 2\hat{c}\cdot\hat{a}$$

$$= 6 - 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \quad \text{(i)}$$

Also, $|\hat{a} + \hat{b} + \hat{c}| \geq 0$

$\therefore \quad \hat{a}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{c}\cdot\hat{c} + 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq 0$

or $\quad 3 + 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq 0$

or $\quad 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq -3$

or $\quad -2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \leq 3$

or $\quad 6 - 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \leq 9 \quad$ (ii)

From (i) and (ii), $x \leq 9$

Therefore, x does not exceed 9.

21. b. Given that $\vec{a}$ and $\vec{b}$ are two unit vectors.

$\therefore \quad |\vec{a}| = 1$ and $|\vec{b}| = 1$

Also given that $(\vec{a} + 2\vec{b})\cdot(5\vec{a} - 4\vec{b}) = 0$

or $\quad 5|\vec{a}|^2 - 8|\vec{b}|^2 - 4\vec{a}\cdot\vec{b} + 10\vec{b}\cdot\vec{a} = 0$

or $\quad 5 - 8 + 6\vec{a}\cdot\vec{b} = 0$

or $\quad 6|\vec{a}||\vec{b}|\cos\theta = 3$

(Where θ is the angle between $\vec{a}$ and $\vec{b}$)

or $\quad \cos\theta = 1/2$

or $\quad \theta = 60°$

22. c. Given that $\vec{V} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{W} = \hat{i} + 3\hat{k}$ and $\vec{U}$ is a unit vector.

$\therefore \quad |\vec{U}| = 1$

Now, $[\vec{U}\,\vec{V}\,\vec{W}] = \vec{U}\cdot(\vec{V} \times \vec{W})$

$$= \vec{U}\cdot(2\hat{i} + \hat{j} - \hat{k}) \times (\hat{i} + 3\hat{k})$$

$$= \vec{U}\cdot(3\hat{i} - 7\hat{j} - \hat{k})$$

$$= \sqrt{3^2 + 7^2 + 1^2}\,\cos\theta$$

This is maximum when $\cos\theta = 1$

Therefore, maximum value of $[\vec{U}\,\vec{V}\,\vec{W}] = \sqrt{59}$

23. c. Volume of parallelepiped formed by $\vec{u} = \hat{i} + a\hat{j} + \hat{k}$, $\vec{v} = \hat{j} + a\hat{k}$, $\vec{w} = a\hat{i} + \hat{k}$ is

$$V = [\vec{u}\,\vec{v}\,\vec{w}] = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$$

$$= 1(1 - 0) - a(0 - a^2) + 1(0 - a)$$

$$= 1 + a^3 - a$$

For V to be minimum, $\dfrac{dV}{da} = 0$

$\Rightarrow \quad 3a^2 - 1 = 0$

or $\quad a = \pm\dfrac{1}{\sqrt{3}}$

But $a > 0$ or $a = \dfrac{1}{\sqrt{3}}$

24. c. $(\vec{a} \times \vec{b}) \times \vec{a} = (\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}$

$\therefore \quad (\hat{j} - \hat{k}) \times (\hat{i} + \hat{j} + \hat{k}) = (\sqrt{3})^2\vec{b} - (\hat{i} + \hat{j} + \hat{k})$

or $\quad 3\vec{b} = 3\hat{i}$ or $\vec{b} = \hat{i}$

25. c. Any vector coplanar to $\vec{a}$ and $\vec{b}$ can be written as

$$\vec{r} = \mu\vec{a} + \lambda\vec{b}$$

or $\quad \vec{r} = (\mu + 2\lambda)\,\hat{i} + (-\mu + \lambda)\,\hat{j} + (\mu + \lambda)\,\hat{k}$

Since $\vec{r}$ is orthogonal to $5\hat{j} + 2\hat{j} + 6\hat{k}$,

$$5(\mu + 2\lambda) + 2(-\mu + \lambda) + 6(\mu + \lambda) = 0$$

or $\quad 9\mu + 18\lambda = 0$

or $\quad \lambda = -\dfrac{1}{2}\mu$

$\therefore \quad \vec{r} = \lambda(3\hat{j} - \hat{k})$

Since $\hat{r}$ is a unit vector, $\hat{r} = \dfrac{3\hat{i} - \hat{k}}{\sqrt{10}}$.

26. c. We observe that

$$\vec{a}\cdot\vec{b_1} = \vec{a}\cdot\vec{b} - \left(\frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}\cdot\vec{a} = \vec{a}\cdot\vec{b} - \vec{a}\;\vec{b} = 0$$

$$\vec{a}\cdot\vec{c_2} = \vec{a}\cdot\left(\vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{c}\cdot\vec{b_1}}{|\vec{b_1}|^2}\vec{b_1}\right)$$

$$= \vec{a}\cdot\vec{c} - \frac{\vec{a}\cdot\vec{c}}{|\vec{a}|^2}|\vec{a}|^2 - \frac{\vec{c}\cdot\vec{b_1}}{|\vec{b_1}|^2}(\vec{a}\cdot\vec{b_1})$$

$$= \vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{c} - 0 \quad (\because \vec{a}\cdot\vec{b_1} = 0)$$

And $\vec{b_1}\cdot\vec{c_2} = \vec{b_1}\cdot\left(\vec{c} - \dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \dfrac{\vec{c}\cdot\vec{b_1}}{|\vec{b_1}|^2}\vec{b_1}\right)$

$$= \vec{b_1}\cdot\vec{c} - \frac{(\vec{c}\cdot\vec{a})(\vec{b_1}\cdot\vec{a})}{|\vec{a}|^2} - \frac{\vec{c}\cdot\vec{b_1}}{|\vec{b_1}|^2}\vec{b_1}\cdot\vec{b_1}$$

$$= \vec{b_1}\cdot\vec{c} - 0 - \vec{b_1}\cdot\vec{c} \quad (\text{Using } \vec{b_1}\cdot\vec{a} = 0)$$

$= 0$

27. a. A vector in the plane of $\vec{a}$ and $\vec{b}$ is

$$\vec{u} = \mu\vec{a} + \lambda\vec{b} = (\mu+\lambda)\hat{i} + (2\mu - \lambda)\hat{j} + (\mu+\lambda)\hat{k}$$

Projection of $\vec{u}$ on $\vec{c} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \dfrac{\vec{u}\cdot\vec{c}}{|\vec{c}|} = \dfrac{1}{\sqrt{3}}$

or $\quad \vec{u}\cdot\vec{c} = 1$

or $\quad |\mu + \lambda + 2\mu - \lambda - \mu - \lambda| = 1$

or $\quad |2\mu - \lambda| = 1$

or $\quad \lambda = 2\mu \pm 1$

$\Rightarrow \quad \vec{u} = 2\hat{i} + \hat{j} + 2\hat{k}$ or $4\hat{i} - \hat{j} + 4\hat{k}$

28. c. We know that three vectors are coplanar if their scalar triple product is zero. Thus,

$$\begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

or $$\begin{vmatrix} 2-\lambda^2 & 2-\lambda^2 & 2-\lambda^2 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

or $$(2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

or $$(2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(1+\lambda^2) & 0 \\ 0 & 0 & -(1+\lambda^2) \end{vmatrix} = 0$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

or $\quad (2 - \lambda^2)(1 + \lambda^2)^2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$

Hence, two real solutions.

29. b. Since $\vec{a}, \vec{b}, \vec{c}$ are unit vectors and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

Taking cross product with $\vec{a}$, we get

$$\vec{a}\times\vec{a} + \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = \vec{0}$$

$\Rightarrow \quad \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

Taking cross product with $\vec{b}$, we get

$$\vec{b}\times\vec{a} + \vec{b}\times\vec{b} + \vec{b}\times\vec{c} = \vec{0}$$

$\Rightarrow \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c}$

Thus, $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

Since, vectors form an equilateral triangle.

$$\vec{a}\times\vec{b} = \vec{b}\times\vec{c}\times\vec{a} \neq 0$$

30. a. $|\overrightarrow{OP}| = |\hat{a}\cos t + \hat{b}\sin t|$

$$= (\cos^2 t + \sin^2 t + 2\cos t \sin t\ \hat{a}\cdot\hat{b})^{1/2}$$

$$= (1 + 2\cos t \sin t\ \hat{a}\cdot\hat{b})^{1/2}$$

$$= (1 + \sin 2t\ \hat{a}\cdot\hat{b})^{1/2}$$

$\therefore \quad |\overrightarrow{OP}|_{max} = (1 + \hat{a}\cdot\hat{b})^{1/2}$ when $t = \pi/4$

$$\therefore \quad \hat{u} = \frac{\dfrac{\hat{a}}{\sqrt{2}} + \dfrac{\hat{b}}{\sqrt{2}}}{\dfrac{|\hat{a}+\hat{b}|}{\sqrt{2}}} = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$$

31. a. Volume of the parallelepiped is, $V = [\vec{a}\ \vec{b}\ \vec{c}]$

Now $[\vec{a}\ \vec{b}\ \vec{c}]^2 = [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}$$

$= 1/2$

$\therefore$ Volume of parallelepiped, $V = [\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{\sqrt{2}}$

32. c. $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\alpha\, \hat{n} = \sin\alpha\, \hat{n}_1, \alpha \in [0, \pi]$

$\vec{c} \times \vec{d} = \sin\beta\, \hat{n}_2, \beta \in [0, \pi]$

Now $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = 1$

$\Rightarrow \quad \sin\alpha \cdot \sin\beta\, (\hat{n}_1 \cdot \hat{n}_2) = 1,$

$\Rightarrow \quad \sin\alpha \sin\beta \cos\theta = 1$

where θ is the angle between $\vec{n}_1$ and $\vec{n}_2$

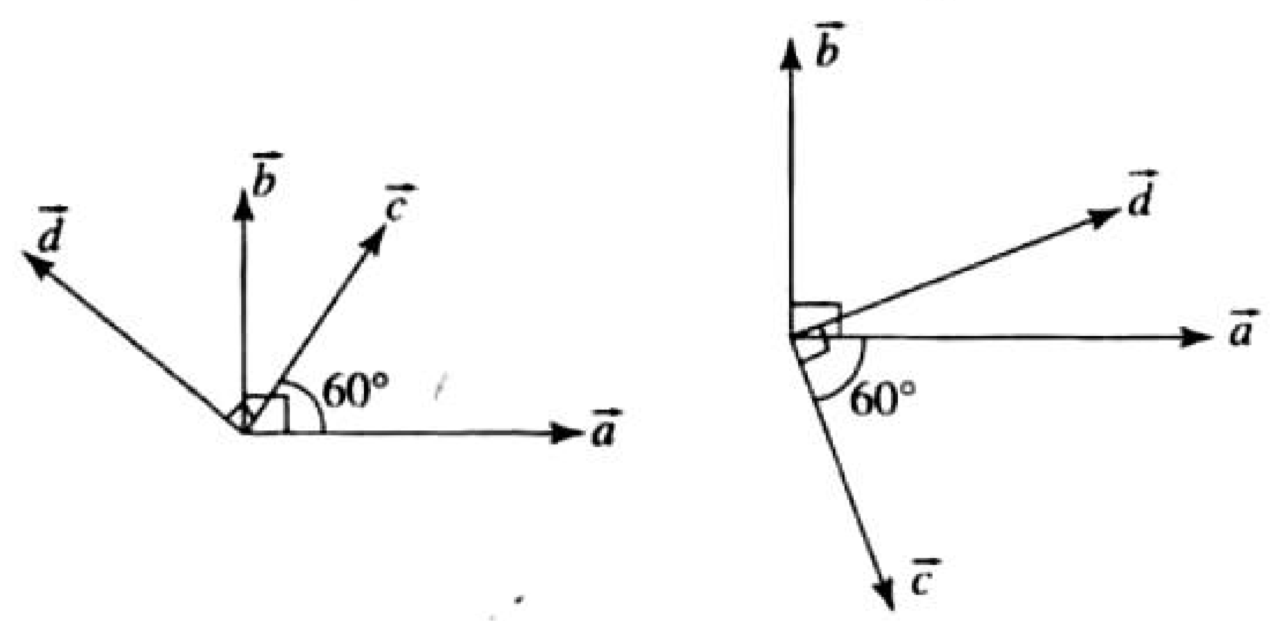
$\Rightarrow \quad \alpha = \pi/2, \beta = \pi/2$ and $\theta = 0$

Now, $\vec{a} \cdot \vec{c} = \frac{1}{2}$

$\Rightarrow \quad \cos\gamma = 1/2 \Rightarrow \gamma = \pi/3$

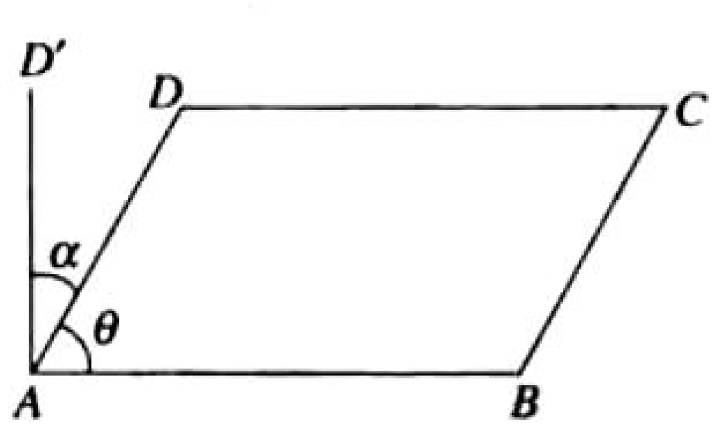
As $\vec{a} \times \vec{b} \parallel \vec{c} \times \vec{d}, \vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar.

There are two possibilities as shown in figure.



Thus $\vec{b}$ and $\vec{c}$ are non-parallel

33. b.

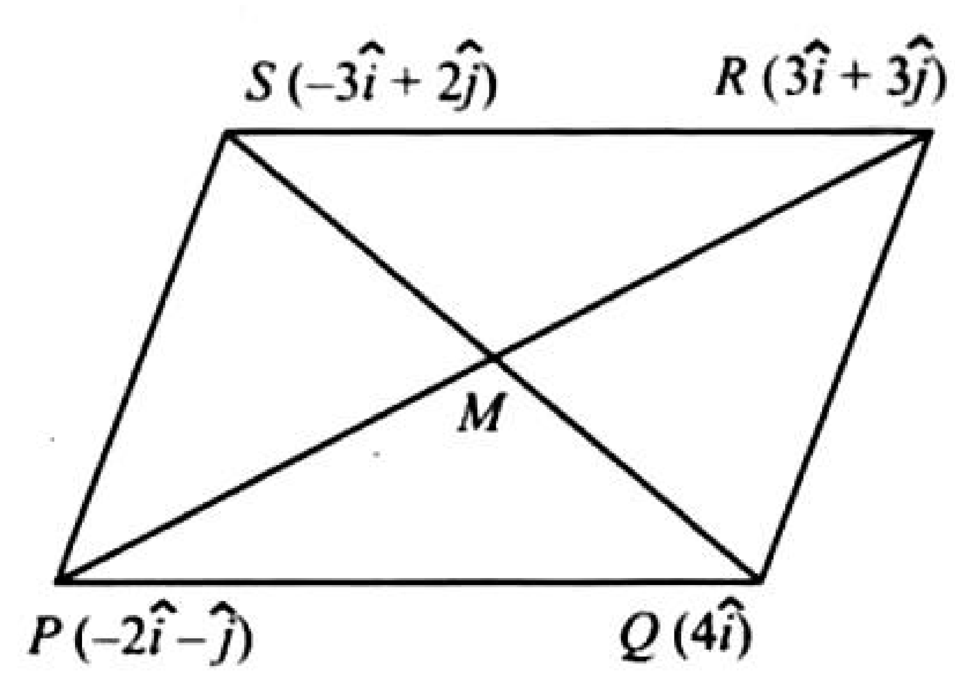


Angle between vectors $\vec{AB}$ and $\vec{AD}$ is given by

$$\cos\theta = \frac{\vec{AB} \cdot \vec{AD}}{|\vec{AB}| \cdot |\vec{AD}|} = \frac{-2 + 20 + 22}{\sqrt{4 + 100 + 121}\sqrt{1 + 4 + 4}} = \frac{8}{9}$$

$\Rightarrow \quad \cos\alpha = \cos(90° - \theta) = \sin\theta = \frac{\sqrt{17}}{9}$

34. a.



Evaluating midpoint of PR and QS which gives $M \equiv \left[\frac{\hat{i}}{2} + \hat{j}\right]$, same for both.

$\overline{PQ} = \overline{SR} = 6\hat{i} + \hat{j}$

$\overline{PS} = \overline{QR} = -\hat{i} + 3\hat{j}$

So, $\quad \overline{PQ} \cdot \overline{PS} \neq 0$

$\overline{PQ} \parallel \overline{SR}, \overline{PS} \parallel \overline{QR}$ and $|\overline{PQ}| = |\overline{SR}|, |\overline{PS}| = |\overline{QR}|$

Hence, $PQRS$ is a parallelogram but not rhombus or rectangle.

35. c. $\vec{v} = \lambda\vec{a} + \mu\vec{b}$

$= \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

Projection of $\vec{v}$ on $\vec{c}$

$\frac{\vec{v} \cdot \vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$

or $\quad \frac{[(\lambda + \mu)\hat{i} + (\lambda - \mu)\hat{j} + (\lambda + \mu)\hat{k}] \cdot (\hat{i} - \hat{j} - \hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

or $\quad \lambda + \mu - \lambda + \mu - \lambda - \mu = 1$

or $\quad \mu - \lambda = 1$

or $\quad \lambda = \mu - 1$

$\Rightarrow \quad \vec{v} = (\mu - 1)(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$= (2\mu - 1)\hat{i} - \hat{j} + (2\mu - 1)\hat{k}$

At $\mu = 2$, $\vec{v} = 3\hat{i} - \hat{j} + 3\hat{k}$

36. c. $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$

$(\vec{a} + \vec{b}) \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = \vec{0}$

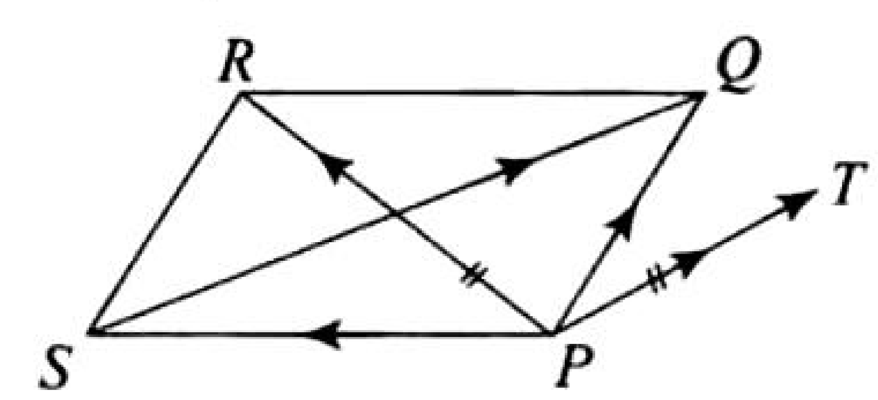
$\Rightarrow \quad \vec{a} + \vec{b} = \pm(2\hat{i} + 3\hat{j} + 4\hat{k}) \qquad$ [as $|\vec{a} + \vec{b}| = \sqrt{29}$]

$\Rightarrow \quad (\vec{a} + \vec{b}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k})$

$= \pm(-14 + 6 + 12)$

$= \pm 4$

37. c.



Area of base ($PQRS$)

$$= \frac{1}{2}|\overline{PR} \times \overline{SQ}| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & -4 \end{vmatrix}$$

$= \frac{1}{2}|-10\hat{i} + 10\hat{j} - 10\hat{k}|$

$= 5|\hat{i} - \hat{j} + \hat{k}| = 5\sqrt{3}$

Height = Projection of PT on $\hat{i} - \hat{j} + \hat{k}$

$= \left|\frac{1 - 2 + 3}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$

$\therefore \quad$ Volume $= (5\sqrt{3})\left(\frac{2}{\sqrt{3}}\right) = 10$ cu. unit

Multiple Correct Answers Type

1. **c.** We are given that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [\vec{a}\ \vec{b}\ \vec{c}]^2$

$$= (\vec{a} \times \vec{b} \cdot \vec{c})^2$$

$$= (|\vec{a} \times \vec{b}| \cdot 1 \cos 0°)^2$$

(since $\vec{c} \perp \vec{a}$ and $\vec{b}, \vec{c} \,||\, \vec{a} \times \vec{b}$)

$$= (|\vec{a} \times \vec{b}|)^2$$

$$= \left(|\vec{a}||\vec{b}| \cdot \sin\frac{\pi}{6}\right)^2$$

$$= \left(\frac{1}{2}\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\right)^2$$

$$= \frac{1}{4}\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

2. **b.** We know that if $\hat{n}$ is perpendicular to $\vec{a}$ as well as $\vec{b}$, then

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \text{ or } \frac{\vec{b} \times \vec{a}}{|\vec{b} \times \vec{a}|}$$

As $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ represent two vectors in opposite directions, we have two possible values of $\hat{n}$.

3. **a., c.** We have

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{b} = \hat{i} + 2\hat{j} - \hat{k},\ \vec{c} = \hat{i} + \hat{j} - 2\hat{k}$$

Any vector in the plane of $\vec{b}$ and $\vec{c}$ is

$$\vec{u} = \mu\vec{b} + \lambda\vec{c}$$

$$= \mu(\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k})$$

$$= (\mu + \lambda)\hat{i} + (2\mu + \lambda)\hat{j} - (\mu + 2\lambda)\hat{k}$$

Given that the magnitude of projection of $\vec{u}$ on $\vec{a}$ is $\sqrt{2/3}$. Thus,

$$\sqrt{\frac{2}{3}} = \left|\frac{\vec{u} \cdot \vec{a}}{|\vec{a}|}\right|$$

$$= \left|\frac{2(\mu + \lambda) - (2\mu + \lambda) - (\mu + 2\lambda)}{\sqrt{6}}\right|$$

or $|-\lambda - \mu| = 2$

$\Rightarrow$ $\lambda + \mu = 2$ or $\lambda + \mu = -2$

Therefore, the required vector is either

$$2\hat{i} + 3\hat{j} - 3\hat{k} \text{ or } -2\hat{i} - \hat{j} + 5\hat{k}.$$

4. **c.** $[\vec{u}\ \vec{v}\ \vec{w}] = [\vec{v}\ \vec{w}\ \vec{u}] = [\vec{w}\ \vec{u}\ \vec{v}]$

but $[\vec{v}\ \vec{u}\ \vec{w}] = -[\vec{u}\ \vec{v}\ \vec{w}]$

5. **a., c.** Dot product of two vectors gives a scalar quantity.

6. **a., c.** We have

$$\vec{v} = \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\,\hat{n} = \sin\theta\,\hat{n}$$

where $\vec{a}$ and $\vec{b}$ are unit vectors. Therefore,

$$|\vec{v}| = \sin\theta$$

Now, $\vec{u} = \vec{a} - (\vec{a} \cdot \vec{b})\vec{b}$

$$= \vec{a} - \vec{b}\cos\theta \text{ (where } \vec{a} \cdot \vec{b} = \cos\theta)$$

$\therefore$ $|\vec{u}|^2 = |\vec{a} - \vec{b}\cos\theta|^2$

$$= 1 + \cos^2\theta - 2\cos\theta \cdot \cos\theta$$
$$= 1 - \cos^2\theta = \sin^2\theta = |v|^2$$

$\Rightarrow$ $|\vec{u}| = |\vec{v}|$

Also, $\vec{u} \cdot \vec{b} = \vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{b})$

$$= \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{b} = 0$$

$\therefore$ $|\vec{u} \cdot \vec{b}| = 0$

$\therefore$ $|\vec{v}| = |\vec{u}| + |\vec{u} \cdot \vec{b}|$ is also correct.

7. **a., c., d.**

$$\vec{a} = \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$$

$$|\vec{a}|^2 = \frac{1}{9}(4 + 4 + 1) = 1 \text{ or } |\vec{a}| = 1$$

Let $\vec{b} = 2\hat{i} - 4\hat{j} + 3\hat{k}$. Then, angle between $\vec{a}$ and $\vec{b}$ is given by,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{\sqrt{29}} \Rightarrow \theta \neq \frac{\pi}{3}$$

Let $\vec{c} = -\hat{i} + \hat{j} - \frac{1}{2}\hat{k} = \frac{-3}{2}\hat{a} \Rightarrow \vec{c} \,||\, \vec{a}$

Let $\vec{d} = 3\hat{i} + 2\hat{j} - 2\hat{k}$. Then $\vec{a} \cdot \vec{d} = 0 \Rightarrow \vec{a} \perp \vec{d}$

8. **b., d.** Normal to plane P_1 is

$$\vec{n_1} = (2\hat{j} + 3\hat{k}) \times (4\hat{j} - 3\hat{k}) = -18\hat{i}$$

Normal to plane P_2 is

$$\vec{n_2} = (\hat{j} - \hat{k}) \times (3\hat{i} + 3\hat{j}) = 3\hat{i} - 3\hat{j} - 3\hat{k}$$

$\vec{A}$ is parallel to $\pm(\vec{n_1} \times \vec{n_2}) = \pm(-54\hat{j} + 54\hat{k})$

Now, the angle between $\vec{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is given by

$$\cos\theta = \pm\frac{(-54\hat{j}+54\hat{k})\cdot(2\hat{i}+\hat{j}-2\hat{k})}{54\sqrt{2}\cdot 3}$$

$$= \pm\frac{1}{\sqrt{2}}$$

$\therefore \quad \theta = \pi/4$ or $3\pi/4$

9. a., d. Any vector in the plane of $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$ is

$$\vec{r} = \lambda(\hat{i}+\hat{j}+2\hat{k}) + \mu(\hat{i}+2\hat{j}+\hat{k})$$

$$= (\lambda+\mu)\hat{i} + (\lambda+2\mu)\hat{j} + (2\lambda+\mu)\hat{k}$$

Also, $\vec{r}$ is perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$.

$\Rightarrow \quad \vec{r}\cdot\vec{c} = 0$

$\Rightarrow \quad \lambda+\mu = 0$

Possible vectors are $\hat{j}-\hat{k}$ or $-\hat{j}+\hat{k}$

10. a., b., c. According to the question

$$\vec{x}\cdot\vec{z} = \vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{z} = \sqrt{2}\cdot\sqrt{2}\cdot\cos\frac{\pi}{3} = 1$$

Given $\vec{a}$ is perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$

$\therefore \quad \vec{a} = \lambda_1(\vec{x}\times(\vec{y}\times\vec{z}))$

$\Rightarrow \quad \vec{a} = \lambda_1((\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z})$

$\Rightarrow \quad \vec{a} = \lambda_1(\vec{y}-\vec{z}) \qquad (1)$

Now $\vec{a}\cdot\vec{y} = \lambda_1(\vec{y}\cdot\vec{y} - \vec{y}\cdot\vec{z}) = \lambda_1(2-1)$

$\Rightarrow \quad \lambda_1 = \vec{a}\cdot\vec{y} \qquad (2)$

From (1) and (2), $\vec{a} = (\vec{a}\cdot\vec{y})(\vec{y}-\vec{z})$

Similarly, $\vec{b} = (\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$

Now, $\vec{a}\cdot\vec{b} = (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})[(\vec{y}-\vec{z})\cdot(\vec{z}-\vec{x})]$

$$= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})[1-1-2+1]$$

$$= -(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$$

11. a., c., d. $\vec{a}+\vec{b}+\vec{c} = 0$

$\Rightarrow \quad \vec{b}+\vec{c} = -\vec{a}$

$\Rightarrow \quad |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} = |\vec{a}|^2$

$\Rightarrow \quad 48 + |\vec{c}|^2 + 48 = 144$

$\Rightarrow \quad |\vec{c}|^2 = 48$

$\Rightarrow \quad |\vec{c}| = 4\sqrt{3}$

$\therefore \quad \frac{|\vec{c}|^2}{2} - |\vec{a}| = 24 - 12 = 12$

$$\frac{|\vec{c}|^2}{2} + |\vec{a}| = 36$$

Further,

$$\vec{a}+\vec{b} = -\vec{c}$$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{c}|^2$

$\Rightarrow \quad 144 + 48 + 2\vec{a}\cdot\vec{b} = 48$

$\Rightarrow \quad \vec{a}\cdot\vec{b} = -72$

$\because \quad \vec{a}+\vec{b}+\vec{c} = 0$

$\Rightarrow \quad \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = 0$

$\therefore \quad |\vec{a}\times\vec{b} + \vec{c}\times\vec{a}|$

$$= 2|\vec{a}\times\vec{b}|$$

$$= 2\sqrt{a^2b^2 - (\vec{a}\cdot\vec{b})^2}$$

$$= 2\sqrt{(144)(48) - (72)^2} = 48\sqrt{3}$$

Matching Column Type

1. (c) – (t); (d) – (r)

c. Volume $= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix} = \pi$

d. $\vec{a}+\vec{b}+\sqrt{3}\vec{c} = \vec{0}$

$\Rightarrow \quad \vec{a}+\vec{b} = -\sqrt{3}\vec{c}$

$\Rightarrow \quad |\vec{a}+\vec{b}| = \sqrt{3}$

$\Rightarrow \quad a^2+b^2+2\vec{a}\cdot\vec{b} = 3$

$\Rightarrow \quad 2 + 2\cos\alpha = 3$

$\Rightarrow \quad \alpha = \frac{\pi}{3}$

Note: Solutions of the remaining parts are given in their respective chapters.

2. (c) – (q), (s)

Since $\vec{a}\cdot\vec{b} = 0$

Let $\vec{b} = \lambda_1\hat{i}, \vec{a} = \lambda_2\hat{j}$

Now, $2|\vec{b}+\vec{c}| = |\vec{b}-\vec{a}|$ and $\vec{a} = \mu\vec{b} + 4\vec{c}$

$\Rightarrow \quad 2\left|\lambda_1\hat{i} + \frac{\lambda_2\hat{j} - \lambda_1\mu\hat{i}}{4}\right| = |\lambda_1\hat{i} - \lambda_2\hat{j}|$

$\Rightarrow \quad |\lambda_1(4-\mu)\hat{i} + \lambda_2\hat{j}| = 2|\lambda_1\hat{i} + \lambda_2\hat{j}|$

Squaring both sides, we get

$$\lambda_1^2(4-\mu)^2 + \lambda_2^2 = 4\lambda_1^2 + 4\lambda_2^2$$

$\Rightarrow \quad 3\lambda_2^2 = (12+\mu^2-8\mu)\lambda_1^2 \qquad (1)$

Also, $(\vec{b}-\vec{a})\cdot(\vec{b}+\vec{c}) = 0$

$\Rightarrow \quad (\lambda_1\hat{i} - \lambda_2\hat{j})\cdot\left(\lambda_1\hat{i} + \frac{\lambda_2\hat{j} - \lambda_1\mu\hat{i}}{4}\right) = 0$

$\Rightarrow \quad \frac{\lambda_1^2(4-\mu) - \lambda_2^2}{4} = 0$

$\Rightarrow \quad \lambda_2^2 = \lambda_1^2(4-\mu)$ (2)

From (1) and (2)

$12 + \mu^2 - 8\mu = 12 - 3\mu$

$\Rightarrow \quad \mu^2 - 5\mu = 0$

$\Rightarrow \quad \mu = 0, 5$

Note: Solutions of the remaining parts are given in their respective chapters.

3. (a) – (r); (b) – (s); (c) – (p); (d) – (q)

a. $[\vec{a}\ \vec{b}\ \vec{c}] = 2$

$\Rightarrow [2\vec{a}\times\vec{b}\quad 3\vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = 6[\vec{a}\ \vec{b}\ \vec{c}]^2$

$= 6 \times 4 = 24$

b. $[\vec{a}\ \vec{b}\ \vec{c}] = 5$

$\Rightarrow [3(\vec{a}+\vec{b})\ \vec{b}+\vec{c}\ 2(\vec{c}+\vec{a})]$

$= 6[(\vec{a}+\vec{b})\ \vec{b}+\vec{c}\ (\vec{c}+\vec{a})] = 12[\vec{a}\ \vec{b}\ \vec{c}] = 60$

c. Given $\frac{1}{2}|\vec{a}\times\vec{b}| = 20$

Now $\frac{1}{2}|(2\vec{a}+3\vec{b})\times(\vec{a}-\vec{b})|$

$= \frac{1}{2}|-2(\vec{a}\times\vec{b}) - 3(\vec{a}\times\vec{b})|$

$= \frac{5}{2}\times 40 = 100$

d. Given $|\vec{a}\times\vec{b}| = 30 \Rightarrow |(\vec{a}+\vec{b})\times\vec{a}| = |\vec{b}\times\vec{a}| = 30$

4. (a) – (q)

a. $\vec{a}\cdot\vec{b} = (\hat{j}+\sqrt{3}\hat{k})\cdot(-\hat{j}+\sqrt{3}\hat{k}) = -1+3 = 2$

$|\vec{a}| = 2, |\vec{b}| = 2$

$\therefore \cos\theta = \frac{2}{2\times 2} = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{3}$ but its value is $\frac{2\pi}{3}$ as its opposite to side of maximum length.

Note: Solutions of the remaining parts are given in their respective chapters.

5. a.

q. $(\vec{a_k}\times\vec{a}_{k+1}) = r^2 \sin\frac{2\pi}{n}$

$\vec{a_k}\cdot\vec{a}_{k+1} = r^2\cos\frac{2\pi}{n}$

Given $\left|\sum_{k=1}^{n-1}\vec{a_k}\times\vec{a}_{k+1}\right| = \left|\sum_{k=1}^{n-1}\vec{a_k}\cdot\vec{a}_{k+1}\right|$

$\Rightarrow \quad r^2(n-1)\sin\frac{2\pi}{n} = r^2(n-1)\cos\frac{2\pi}{n}$

$\Rightarrow \quad \tan\frac{2\pi}{n} = 1$

$\Rightarrow \quad \frac{2\pi}{n} = k\pi + \frac{\pi}{4}, k\in Z$

$\Rightarrow \quad n = \frac{8}{4k+1}$

$\Rightarrow \quad n = 8$ (when $k = 0$)

Note: Solutions of the remaining parts are given in their respective chapters.

6. (a) – (p), (q)

Projection of $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$.

So, $\left|(\alpha\hat{i}+\beta\hat{j})\cdot\left(\frac{\sqrt{3}\hat{i}+\hat{j}}{2}\right)\right| = \sqrt{3}$

$\Rightarrow \quad \sqrt{3}\alpha+\beta = \pm 2\sqrt{3}$

$\Rightarrow \quad \sqrt{3}\alpha + \left(\frac{\alpha-2}{\sqrt{3}}\right) = \pm 2\sqrt{3}$

$\Rightarrow \quad 3\alpha+\alpha-2 = \pm 6$

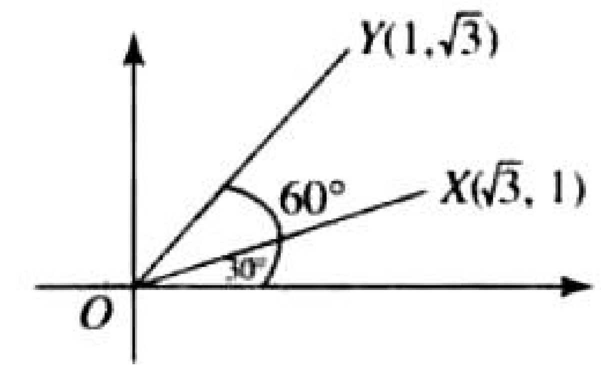
$\Rightarrow \quad 4\alpha = 8, -4$

$\Rightarrow \quad \alpha = 2, -1$

Note: Solutions of the remaining parts are given in their respective chapters.

7. (c) – (p), (q)

We have $\overline{OX} = \sqrt{3}\hat{i}+\hat{j}$ and $\overline{OY} = \hat{i}+\sqrt{3}\hat{j}$



Hence, equation of acute angle bisector of $\overline{OX}$ and $\overline{OY}$ is

$y = x$

or $\quad x - y = 0$

Now, distance of $\beta\hat{i} + (1-\beta)\hat{i} \equiv Z$ or $(\beta, 1-\beta)$ from $x-y=0$, is

$\Rightarrow \quad \left|\frac{\beta-(1-\beta)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$

$\Rightarrow \quad |2\beta-1| = 3$

$\Rightarrow \quad 2\beta - 1 = \pm 3$

$\Rightarrow \quad 2\beta = 4, -2$

$\Rightarrow \quad \beta = 2, -1$

$\Rightarrow \quad |\beta| = 2, 1$

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (5) $E = (2\vec{a}+\vec{b})\cdot[|\vec{a}|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a} - 2(\vec{a}\cdot\vec{b})\vec{b} + 2|\vec{b}|^2\vec{a}]$

$\vec{a}\cdot\vec{b} = \frac{2-2}{\sqrt{70}} = 0$

and $\quad |\vec{a}| = 1$ and $|b| = 1$

$\therefore \quad E = (2\vec{a}+\vec{b})\cdot[2|\vec{b}|^2\vec{a} + |\vec{a}|^2\vec{b}]$

$= 4|\vec{a}|^2|\vec{b}|^2 + 2|\vec{a}|^2(\vec{a}\cdot\vec{b}) + 2|\vec{b}|^2(\vec{b}\cdot\vec{a}) + |\vec{a}|^2|\vec{b}|^2$

$= 5|\vec{a}|^2|\vec{b}|^2 = 5$

2. (9) $\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$

Taking cross product with $\vec{a}$, we get

$\vec{a}\times(\vec{r}\times\vec{b}) = \vec{a}\times(\vec{c}\times\vec{b})$

or $\quad (\vec{a}\cdot\vec{b})\vec{r} - (\vec{a}\cdot\vec{r})\vec{b} = \vec{a}\times(\vec{c}\times\vec{b})$

or $\quad \vec{r} = -3\hat{i}+6\hat{j}+3\hat{k} \quad (\vec{a}\cdot\vec{b} = 1, \vec{a}\cdot\vec{r} = 0)$

$\Rightarrow \quad \vec{r}\cdot\vec{b} = 3+6 = 9$

3. (3) As $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2$

$= 3(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2)-|\vec{a}+\vec{b}+\vec{c}|^2$

$\Rightarrow \quad 3\times 3-|\vec{a}+\vec{b}+\vec{c}|=9$

or $\quad |\vec{a}+\vec{b}+\vec{c}|=0$

or $\quad \vec{a}+\vec{b}+\vec{c}=0$

or $\quad \vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow \quad |2\vec{a}+5(\vec{b}+\vec{c})|=|-3\vec{a}|=3|\vec{a}|=3.$

4. (5) Let (1, 1, 1), (–1, 1, 1), (1, –1, 1), (–1, –1, 1) be vectors $\vec{a},\vec{b},\vec{c},\vec{d}$, rest of the vectors are $-\vec{a},-\vec{b},-\vec{c},-\vec{d}$ and let us find the number of ways of selecting co–planar vectors.
Observe that out of any three coplanar vectors two will be collinear (anti parallel).
Number of ways of selecting the anti-parallel pair = 4
Number of ways of selecting the third vector = 6
Total = 24
Number of non-coplanar selections

$= {}^8C_3-24=32=2^5$

$\therefore \quad p=5$

5. (4) $\quad |\vec{a}|=|\vec{b}|=|\vec{c}|=1$

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=1/2$

Also, $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$

$\Rightarrow \quad \vec{a}\cdot(\vec{b}\times\vec{c})=p+q(\vec{a}\cdot\vec{b})+r(\vec{a}\cdot\vec{c})$

$\therefore \quad p+\frac{q}{2}+\frac{r}{2}=[\vec{a}\,\vec{b}\,\vec{c}] \quad (1)$

Similarly, taking dot product with vector $\vec{b}$, we get

$\frac{p}{2}+q+\frac{r}{2}=0 \quad (2)$

And, taking dot product with vector $\vec{c}$, we get

$\frac{p}{2}+\frac{q}{2}+r=[\vec{a}\,\vec{b}\,\vec{c}] \quad (3)$

Solving, (1), (2) and (3), we get

$p=r=-q$

$\Rightarrow \quad \frac{p^2+2q^2+r^2}{q^2}=4$

6. (9) According to question $\vec{s}=4\vec{p}+3\vec{q}+5\vec{r}$

and $\quad \vec{s}=x(-\vec{p}+\vec{q}+\vec{r})+y(\vec{p}-\vec{q}+\vec{r})+z(-\vec{p}-\vec{q}+\vec{r})$

$\therefore \quad -x+y-z=4 \quad (1)$

$x-y-z=3 \quad (2)$

$x+y+z=5 \quad (3)$

Adding (1) and (2), we get

$z=-\frac{7}{2}$

Adding (2) and (3), we get

$x=4$

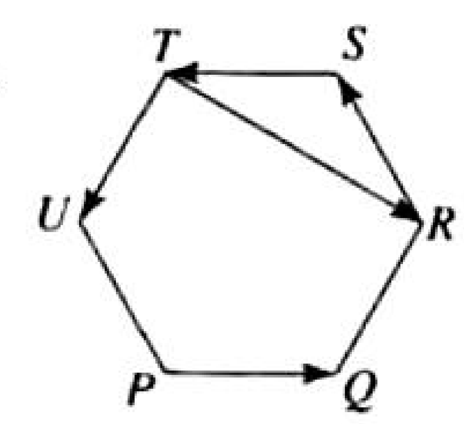
Adding (1) and (3), we get

$y=9/2$

$\therefore \quad 2x+y+z=2(4)+1=9$

Assertion-Reasoning Type

1. c.



$\vec{PQ}\times(\vec{RS}+\vec{ST})=0$

$\vec{PQ}\times\vec{RT}\neq 0 \quad (\because \vec{PQ}$ is not parallel to $\vec{TR})$

$\vec{PQ}\times\vec{RS}\neq\vec{0} \quad (\because \vec{PQ}$ is not parallel to $\vec{RS})$

$\vec{PQ}\times\vec{ST}=\vec{0} \quad (\because \vec{PQ}$ is parallel to $\vec{ST})$

$\vec{PQ}\neq\vec{TR}\ \because \vec{TR}$ is resultant of $\vec{SR}$ and $\vec{ST}$

Fill in the Blanks Type

1. Given that $|\vec{A}|=3;\ |\vec{B}|=4;\ |\vec{C}|=5$

$\vec{A}\perp(\vec{B}+\vec{C})\Rightarrow \vec{A}\cdot(\vec{B}+\vec{C})=0$

$\Rightarrow \quad \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}=0 \quad$ (i)

$\vec{B}\perp(\vec{C}+\vec{A})\Rightarrow \vec{B}\cdot(\vec{C}+\vec{A})=0$

$\Rightarrow \quad \vec{B}\cdot\vec{C}+\vec{B}\cdot\vec{A}=0 \quad$ (ii)

$\vec{C}\perp(\vec{A}+\vec{B})\Rightarrow \vec{C}\cdot(\vec{A}+\vec{B})=0$

$\Rightarrow \quad \vec{C}\cdot\vec{A}+\vec{C}\cdot\vec{B}=0 \quad$ (iii)

Adding (i), (ii) and (iii), we get

$2(\vec{A}\cdot\vec{B}+\vec{B}\cdot\vec{C}+\vec{C}\cdot\vec{A})=0 \quad$ (iv)

Now, $|\vec{A}+\vec{B}+\vec{C}|^2$

$=(\vec{A}+\vec{B}+\vec{C})\cdot(\vec{A}+\vec{B}+\vec{C})$

$=|\vec{A}|^2+|\vec{B}|^2+|\vec{C}|^2+2(\vec{A}\cdot\vec{B}+\vec{B}\cdot\vec{C}+\vec{C}\cdot\vec{A})$

$=9+16+25+0$

$=50$

$\therefore \quad |\vec{A}+\vec{B}+\vec{C}|=5\sqrt{2}$

2. Required unit vector

$$\hat{a}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{|\overrightarrow{PQ}\times\overrightarrow{PR}|}$$

$\overrightarrow{PQ}=\hat{i}+\hat{j}-3\hat{k};\ \overrightarrow{PR}=-\hat{i}+3\hat{j}-\hat{k}$

$\therefore \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\hat{i} + 4\hat{j} + 4\hat{k}$

$\therefore \quad |\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{64+16+16} = \sqrt{96} = 4\sqrt{6}$

$\therefore \quad \hat{n} = \dfrac{8\hat{i}+4\hat{j}+4\hat{k}}{4\sqrt{6}} = \dfrac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$

3. Area of $\Delta ABC = \dfrac{1}{2}\,|\overrightarrow{BA} \times \overrightarrow{BC}|$

$\overrightarrow{BA} = -\hat{i} - 2\hat{j} + 3\hat{k}$

$\overrightarrow{BC} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore \quad \text{Area} = \dfrac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 3 \\ 1 & -2 & 3 \end{vmatrix}$

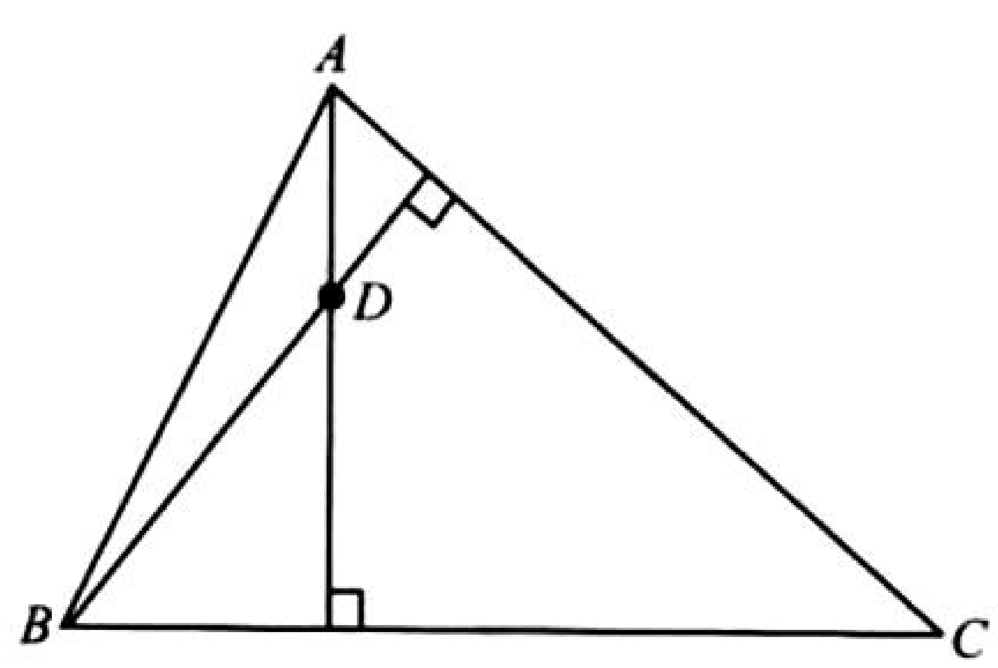
$= \dfrac{1}{2}\,|6\hat{j} + 4\hat{k}| = |3\hat{j} + 2\hat{k}|$

$= \sqrt{9+4} = \sqrt{13}$

4. Given that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are position vectors of points A, B, C and D, respectively, such that

$(\vec{a} - \vec{d})\cdot(\vec{b} - \vec{c}) = (\vec{b} - \vec{d})\cdot(\vec{c} - \vec{a}) = 0$

$\Rightarrow \quad \overrightarrow{DA}\cdot\overrightarrow{CB} = \overrightarrow{DB}\cdot\overrightarrow{AC} = 0$



$\Rightarrow \quad \overrightarrow{DA} \perp \overrightarrow{CB}$ and $\overrightarrow{DB} \perp \overrightarrow{AC}$

Clearly, D is the orthocentre of ΔABC.

5. Given that $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

Operating $C_2 \leftrightarrow C_3$ and then $C_1 \leftrightarrow C_2$ in first determinant

$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$(1 + abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

either $1 + abc = 0$ or $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

Also given that vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are non-coplanar.

$\therefore \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$

So we must have $1 + abc = 0$ or $abc = -1$

6. $\dfrac{\vec{A}\cdot\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\cdot\vec{B}} + \dfrac{\vec{B}\cdot\vec{A}\times\vec{C}}{\vec{C}\cdot\vec{A}\times\vec{B}} = \dfrac{[\vec{A}\,\vec{B}\,\vec{C}]}{[\vec{A}\,\vec{B}\,\vec{C}]} + \dfrac{-[\vec{A}\,\vec{B}\,\vec{C}]}{[\vec{A}\,\vec{B}\,\vec{C}]} = 0$

7. Given $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = \hat{j} - \hat{k}$

Let $\quad \vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$

Given that $\vec{A} \times \vec{B} = \vec{C} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$

or $\quad (z-y)\hat{i} + (x-z)\hat{j} + (y-x)\hat{k} = \hat{j} - \hat{k}$

$\Rightarrow \quad z - y = 0,\ x - z = 1$ and $y - x = -1$ (i)

Also, $\vec{A}\cdot\vec{B} = 3$

$\Rightarrow \quad x + y + z = 3$ (ii)

From (i) and (ii), we get

$y = 2/3,\ x = 5/3,\ z = 2/3$

$\therefore \quad \vec{B} = \dfrac{5}{3}\hat{i} + \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}$

8. Given that the vectors $\vec{u} = a\hat{i} + \hat{j} + \hat{k}$, $\vec{v} = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{w} = \hat{i} + \hat{j} + c\hat{k}$, where $a, b, c \neq 1$ are coplanar. Therefore,

$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

Operating $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$\begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix} = 0$

Expanding

$c(a-1)(b-1)+(1-b)(1-c)-(1-c)(a-1)=0$

$\therefore \quad \dfrac{c}{1-c}+\dfrac{1}{1-a}+\dfrac{1}{1-b}=0$

$\therefore \quad \dfrac{c}{1-c}+1+\dfrac{1}{1-a}+\dfrac{1}{1-b}=1$

$\therefore \quad \dfrac{1}{1-c}+\dfrac{1}{1-a}+\dfrac{1}{1-b}=1$

9. Let $\vec{c}=\alpha\hat{i}+\beta\hat{j}$

Given that $\vec{b}\perp\vec{c}$

$\therefore \quad \vec{b}\cdot\vec{c}=0.$

$\Rightarrow \quad (4\hat{i}+3\hat{j})\cdot(\alpha\hat{i}+\beta\hat{j})=0$

or $\quad 4\alpha+3\beta=0$

or $\quad \dfrac{\alpha}{3}=\dfrac{\beta}{-4}=\lambda$

or $\quad \alpha=3\lambda,\ \beta=-4\lambda$ (i)

Now let $\vec{a}=x\hat{i}+y\hat{j}$ be the required vectors.

Projection of $\vec{a}$ along $\vec{b}$ gives

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\frac{4x+3y}{\sqrt{4^2+3^2}}=1$$

or $\quad 4x+3y=5$ (ii)

Also projection of $\vec{a}$ along $\vec{c}$ gives

$$\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}=2$$

$\Rightarrow \quad \dfrac{\alpha x+\beta y}{\sqrt{\alpha^2+\beta^2}}=2$

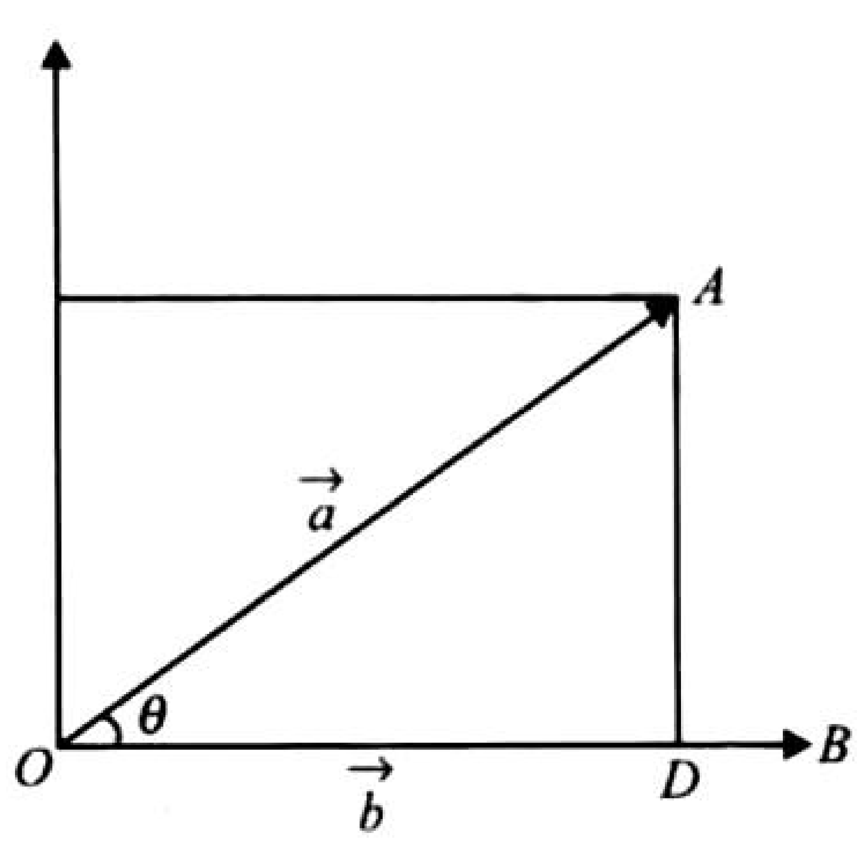
or $\quad 3\lambda x-4\lambda y=10\lambda$

or $\quad 3x-4y=10$ (iii)

Solving (ii) and (iii), we get $x=2,\ y=-1$

Therefore, the required vector is $2\hat{i}-\hat{j}$.

10.



Component of $\vec{a}$ along $\vec{b}$

$$\overrightarrow{OD}=OA\cos\theta\cdot\frac{\vec{b}}{|\vec{b}|}$$

$$=\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}=\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Component of $\vec{a}$ perpendicular to $\vec{b}$

$$\overrightarrow{DA}=\vec{a}-\overrightarrow{OD}$$

$$=\vec{a}-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

11. Let $x\hat{i}+y\hat{j}+z\hat{k}$ be a unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and also perpendicular to $\hat{i}+\hat{j}+\hat{k}$. Then

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}=0$$

or $\quad -3x+y+z=0$ (i)

and $\quad x+y+z=0$ (ii)

Solving (i) and (ii) by cross-product method, we get

$$\frac{x}{0}=\frac{y}{4}=\frac{z}{-4} \text{ or } \frac{x}{0}=\frac{y}{1}=\frac{z}{-1}=\lambda \text{ (say)}$$

$\Rightarrow \quad x=0,\ y=\lambda,\ z=-\lambda$

As $x\hat{i}+y\hat{j}+z\hat{k}$ is a unit vector, we have

$0+\lambda^2+\lambda^2=1$

or $\quad \lambda^2=\dfrac{1}{2}$ or $\lambda=\pm\dfrac{1}{\sqrt{2}}$

$\therefore \quad$ Required vector $=\dfrac{\hat{j}-\hat{k}}{\sqrt{2}}$ or $\dfrac{-\hat{j}+\hat{k}}{\sqrt{2}}$

12. A vector normal to the plane containing vectors $\hat{i}$ and $\hat{i}+\hat{j}$ is

$$\vec{p}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}=\hat{k}$$

A vector normal to the plane containing vectors $\hat{i}-\hat{j},\ \hat{i}+\hat{k}$ is

$$\vec{q}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix}=-\hat{i}-\hat{j}+\hat{k}.$$

Vector $\vec{a}$ is parallel to vector $\vec{p}\times\vec{q}$.

$$\vec{p}\times\vec{q}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{vmatrix}=\hat{i}-\hat{j}$$

Therefore, a vector in direction of $\vec{a}$ is $\hat{i}-\hat{j}$.

Now if θ is the angle between $\vec{a}$ and $\hat{i}-2\hat{j}+2\hat{k}$, then

$$\cos\theta = \pm\frac{1\cdot 1+(-1)\cdot(-2)}{\sqrt{1+1}\,\sqrt{1+4+4}} = \pm\frac{3}{\sqrt{2}\cdot 3}$$

$$\Rightarrow \quad \cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

13. Let $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be any three mutually perpendicular non-coplanar unit vectors and $\vec{a}$ be any vector, then

$$\vec{a} = (\vec{a}\cdot\vec{\alpha})\vec{\alpha} + (\vec{a}\cdot\vec{\beta})\vec{\beta} + (\vec{a}\cdot\vec{\gamma})\vec{\gamma}$$

Here $\vec{b}, \vec{c}$ are two mutually perpendicular vectors, therefore $\vec{b}, \vec{c}$ and $\dfrac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}$ are three mutually perpendicular non-coplanar unit vectors. Hence

$$\vec{a} = \left(\vec{a}\,.\,\vec{b}\right)\vec{b} + \left(\vec{a}\,.\,\vec{c}\right)\vec{c} + \left(\vec{a}\,.\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}\right)\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}$$

$$= \left(\vec{a}\,.\,\vec{b}\right)\vec{b} + \left(\vec{a}\,.\,\vec{c}\right)\vec{c} + \frac{\vec{a}\,.\left(\vec{b}\times\vec{c}\right)}{|\vec{b}\times\vec{c}|^2}\left(\vec{b}\times\vec{c}\right)$$

14. $\vec{a}\times(\vec{a}\times\vec{c}) + \vec{b} = \vec{0}$

or $\quad (\vec{a}\cdot\vec{c})\,\vec{a} - (\vec{a}\cdot\vec{a})\,\vec{c} + \vec{b} = \vec{0}$

or $\quad 2\cos\theta\cdot\vec{a} - \vec{c} + \vec{b} = \vec{0}$

(Using $|\vec{a}| = 1, |\vec{b}| = 1, |\vec{c}| = 2$)

or $\quad (2\cos\theta\,\vec{a} - \vec{c})^2 = (-\vec{b})^2$

or $\quad 4\cos^2\theta\cdot|\vec{a}|^2 + |\vec{c}|^2 - 2\cdot 2\cos\theta\cdot\vec{a}\cdot\vec{c} = |\vec{b}|^2$

or $\quad 4\cos^2\theta + 4 - 8\cos\theta\cdot\cos\theta = 1$

or $\quad 4\cos^2\theta - 8\cos^2\theta + 4 = 1$

or $\quad 4\cos^2\theta = 3$

or $\quad \cos\theta = \pm\sqrt{3}/2$

For θ to be acute, $\cos\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{6}$

15. q = Area of parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as adjacent sides

$= |\overrightarrow{OA}\times\overrightarrow{OC}|$

$= |\vec{a}\times\vec{b}|$

p = Area of quadrilateral $OABC$

$$= \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}| + \frac{1}{2}|\overrightarrow{OB}\times\overrightarrow{OC}|$$

$$= \frac{1}{2}[|\vec{a}\times(10\vec{a}+2\vec{b})| + |(10\vec{a}+2\vec{b})\times\vec{b}|]$$

$$= \frac{1}{2}|(12\,\vec{a}\times\vec{b})| = 6|\vec{a}\times\vec{b}|$$

$\Rightarrow \quad k = 6$

True/False Type

1. $\vec{A}, \vec{B}$ and $\vec{C}$ are three unit vectors such that

$$\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C} = 0 \qquad \text{(i)}$$

and the angle between $\vec{B}$ and $\vec{C}$ is $\pi/3$.

Now Eq. (i) shows that $\vec{A}$ is perpendicular to both $\vec{B}$ and $\vec{C}$. Thus,

$\vec{B}\times\vec{C} = \lambda\vec{A}$, where λ is any scalar.

or $\quad |\vec{B}\times\vec{C}| = |\lambda\vec{A}|$

or $\quad \sin\pi/3 = \pm\lambda$

(as $\pi/3$ is the angle between $\vec{B}$ and $\vec{C}$)

or $\quad \lambda = \pm\sqrt{3}/2$

$\Rightarrow \quad \vec{B}\times\vec{C} = \pm\dfrac{\sqrt{3}}{2}\vec{A}$

or $\quad \vec{A} = \pm\dfrac{2}{\sqrt{3}}(\vec{B}\times\vec{C})$

Therefore, the given statement is false.

2. $\vec{X}\cdot\vec{A} = 0 \Rightarrow$ either $\vec{A} = 0$ or $\vec{X}\perp\vec{A}$

$\vec{X}\cdot\vec{B} = 0 \Rightarrow$ either $\vec{B} = 0$ or $\vec{X}\perp\vec{B}$

$\vec{X}\cdot\vec{C} = 0 \Rightarrow$ either $\vec{C} = 0$ $\vec{X}\perp\vec{C}$

In any of the three cases,

$\vec{A}, \vec{B}, \vec{C} = 0$, $[\vec{A}\,\vec{B}\,\vec{C}] = 0$

Otherwise if $\vec{X}\perp\vec{A}, \vec{X}\perp\vec{B}$ and $\vec{X}\perp\vec{C}$, then $\vec{A}, \vec{B}$ and $\vec{C}$ are coplanar. Then

$[\vec{A}\,\vec{B}\,\vec{C}] = 0$

Therefore, the statement is true.

3. Let position vectors of points A, B and C be $\vec{a}+\vec{b}, \vec{a}-\vec{b}$ and $\vec{a}+k\vec{b}$, respectively.

Then $\overrightarrow{AB} = (\vec{a}-\vec{b}) - (\vec{a}+\vec{b}) = -2\vec{b}$

Similarly, $\overrightarrow{BC} = (\vec{a}+k\vec{b}) - (\vec{a}-\vec{b}) = (k+1)\vec{b}$

Clearly $\overrightarrow{AB} \parallel \overrightarrow{BC}$ $\forall k \in R$

Hence, A, B and C are collinear $\forall k \in R$

Therefore, the statement is true.

4. Clearly vectors $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ are coplanar

$\Rightarrow \quad [\vec{a}-\vec{b}\;\; \vec{b}-\vec{c}\;\; \vec{c}-\vec{a}] = 0$

Therefore, the given statement is false.

Subjective Type

1. Let the position vectors of points A, B, C, D, E and F be $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ and $\vec{f}$ w.r.t. O. Let perpendiculars from A to EF and from B to DF meet each other at H. Let position vectors of H be $\vec{r}$. We join CH. In order to prove the statement given in the question, it is sufficient to prove that CH is perpendicular to DE.

Now, as $OD \perp BC \Rightarrow \vec{d}\cdot(\vec{b}-\vec{c}) = 0$

$\Rightarrow \quad \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c}$ (i)

as $OE \perp AC \Rightarrow \vec{e}\cdot(\vec{c}-\vec{a}) = 0 \Rightarrow \vec{e}\cdot\vec{c} = \vec{e}\cdot\vec{a}$ (ii)

as $OF \perp AB \Rightarrow \vec{f}\cdot(\vec{a}-\vec{b}) = 0 \Rightarrow \vec{f}\cdot\vec{a} = \vec{f}\cdot\vec{b}$ (iii)

Also $AH \perp EF \Rightarrow (\vec{r}-\vec{a})\cdot(\vec{e}-\vec{f}) = 0$

$\Rightarrow \quad \vec{r}\cdot\vec{e} - \vec{r}\cdot\vec{f} - \vec{a}\cdot\vec{e} + \vec{a}\cdot\vec{f} = 0$ (iv)

and $BH \perp FD \Rightarrow (\vec{r}-\vec{b})\cdot(\vec{f}-\vec{d}) = 0$

$\Rightarrow \quad \vec{r}\cdot\vec{f} - \vec{r}\cdot\vec{d} - \vec{b}\cdot\vec{f} + \vec{b}\cdot\vec{d} = 0$ (v)

Adding (iv) and (v), we get

$$\vec{r}\cdot\vec{e} - \vec{a}\cdot\vec{e} + \vec{a}\cdot\vec{f} - \vec{r}\cdot\vec{d} - \vec{b}\cdot\vec{f} + \vec{b}\cdot\vec{d} = 0$$

or $\quad \vec{r}\cdot(\vec{e}-\vec{d}) - \vec{e}\cdot\vec{c} + \vec{d}\cdot\vec{c} = 0$ [Using (i), (ii) and (iii)]

or $\quad (\vec{r}-\vec{c})\cdot(\vec{e}-\vec{d}) = 0$

$\Rightarrow \quad \overrightarrow{CH}\cdot\overrightarrow{ED} = 0 \Rightarrow CH \perp ED$

2. Since vector $\vec{A}$ has components A_1, A_2 and A_3, in the coordinate system $OXYZ$,

$$\vec{A} = \hat{i}A_1 + \hat{j}A_2 + \hat{k}A_3$$

When given system is rotated through $\pi/2$, the new x-axis is along the old y-axis and the new y-axis is along the old negative x-axis; z remains same as before.
Hence, the components of A in the new system are $A_2, -A_1$ and A_3.

Therefore, $\vec{A}$ becomes $A_2\hat{i} - A_1\hat{j} + A_3\hat{k}$.

3. $\vec{A}\times\vec{X} = \vec{B}$

or $\quad (\vec{A}\times\vec{X})\times\vec{A} = \vec{B}\times\vec{A}$

or $\quad (\vec{A}\cdot\vec{A})\vec{X} - (\vec{X}\cdot\vec{A})\vec{A} = \vec{B}\times\vec{A}$

or $\quad (\vec{A}\cdot\vec{A})\vec{X} - c\vec{A} = \vec{B}\times\vec{A}$

or $\quad \vec{X} = \dfrac{\vec{B}\times\vec{A} + c\vec{A}}{(\vec{A}\cdot\vec{A})}$

4. Given that P.V.'s of points A, B, C and D are $3\hat{i}-2\hat{j}-\hat{k}, 2\hat{i}+3\hat{j}-4\hat{k}, -\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$, respectively.

Given that A, B, C and D lie in a plane if $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar. Therefore,

$$\begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & 1+\lambda \end{vmatrix} = 0$$

or $\quad -1(3+3\lambda-21) - 5(-4-4\lambda-3) - 3(-28-3) = 0$

or $\quad -3\lambda + 18 + 20\lambda + 35 + 93 = 0$

or $\quad 17\lambda = -146$

or $\quad \lambda = -\dfrac{146}{17}$

5. Let the position vectors of points A, B, C, D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, with respect to some origin.

$$|\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}|$$

$$= |(\vec{b}-\vec{a})\times(\vec{d}-\vec{c}) + (\vec{c}-\vec{b})\times(\vec{d}-\vec{a}) + (\vec{a}-\vec{c})\times(\vec{d}-\vec{b})|$$

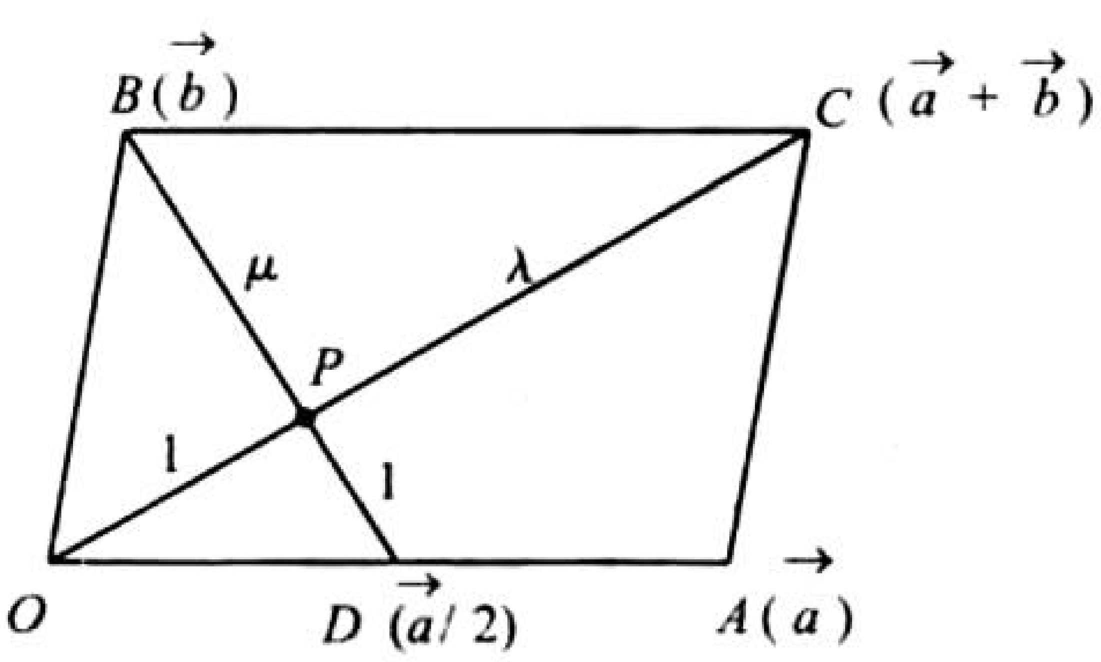
$$= 2|\vec{b}\times\vec{a} + \vec{c}\times\vec{b} + \vec{a}\times\vec{c}| \quad \text{(i)}$$

$$= 4\times\frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{b}-\vec{c})|$$

$$= 4 \times (\text{area of } \Delta ABC)$$

6. $OACB$ is a parallelogram with O as origin. Let with respect to O, position vectors of A and B be $\vec{a}$ and $\vec{b}$, respectively. Then P.V. of C is $\vec{a}+\vec{b}$.

Also D is the midpoint of OA; therefore, the position vector of D is $\vec{a}/2$.



CO and BD intersect each other at P.
Let P divide CO in the ratio $\lambda : 1$ and BD in the ratio $\mu : 1$. Then by section theorem, position vector of point P dividing CO in ratio $\lambda : 1$ is

$$\frac{\lambda\times 0 + 1\times(\vec{a}+\vec{b})}{\lambda+1} = \frac{\vec{a}+\vec{b}}{\lambda+1} \quad \text{(i)}$$

and position vector of point P dividing BD in the ratio $\mu : 1$ is

$$\frac{\mu\left(\dfrac{\vec{a}}{2}\right) + 1(\vec{b})}{\mu+1} = \frac{\mu\vec{a} + 2\vec{b}}{2(\mu+1)} \quad \text{(ii)}$$

As (i) and (ii) represent the position vector of the same point; hence,

$$\frac{\vec{a}+\vec{b}}{\lambda+1}=\frac{\mu\vec{a}+2\vec{b}}{2(\mu+1)}$$

Equating the coefficients of $\vec{a}$ and $\vec{b}$, we get

$$\frac{1}{\lambda+1}=\frac{\mu}{2(\mu+1)} \quad \text{(iii)}$$

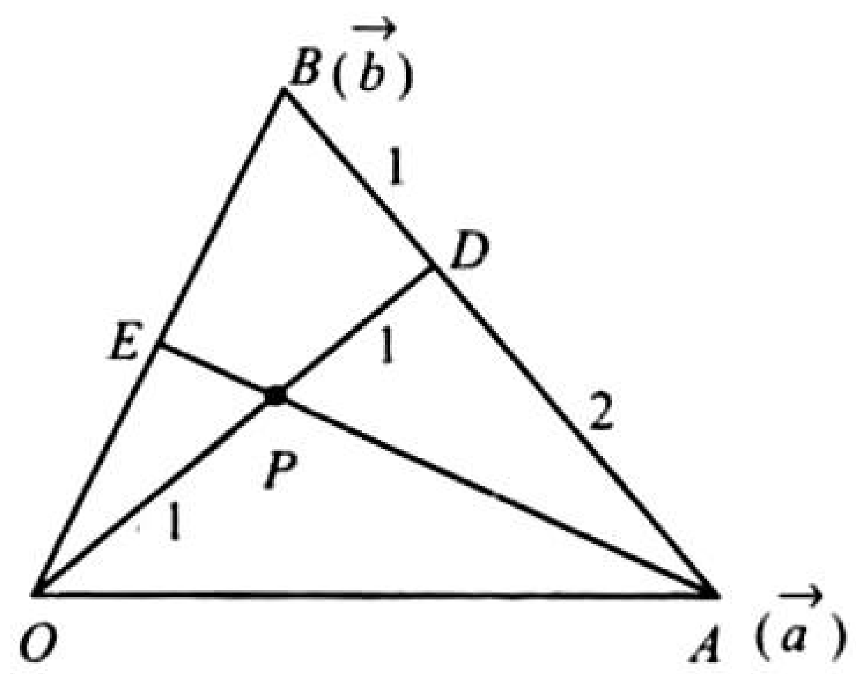
$$=\frac{1}{\mu+1} \quad \text{(iv)}$$

From (iv) we get $\lambda=\mu$, i.e., P divides CO and BD in the same ratio.

Putting $\lambda=\mu$ in Eq. (iii), we get $\mu=2$

Thus, the required ratio is 2 : 1.

7. With O as origin let $\vec{a}$ and $\vec{b}$ be the position vectors of A and B, respectively.



Then the position vector of E, the midpoint of OB, is $\vec{b}/2$.
Again since $AD : DB = 2 : 1$, the position vector of D is

$$\frac{1\cdot\vec{a}+2\vec{b}}{1+2}=\frac{\vec{a}+2\vec{b}}{3}$$

Let $\frac{OP}{PD}=\frac{1}{\lambda}$

$\Rightarrow$ P.V. of $P=\frac{\vec{a}+2\vec{b}}{3(\lambda+1)}$

Let $\frac{AP}{PE}=\frac{1}{\mu}$

$\Rightarrow$ P.V. of $P=\frac{\mu\vec{a}+\frac{\vec{b}}{2}}{\mu+1}$

Comparing P.V. of P, we get

$$\frac{1}{3(\lambda+1)}=\frac{\mu}{\mu+1} \text{ and } \frac{2}{3(\lambda+1)}=\frac{1}{2(\mu+1)}$$

Solving we get $\mu=\frac{1}{4} \Rightarrow \lambda=\frac{2}{3}$

$$\Rightarrow \frac{OP}{PD}=\frac{3}{2}$$

8. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three coplanar vectors. Therefore, there exist scalars x, y and z, not all zero, such that

$$x\vec{a}+y\vec{b}+z\vec{c}=\vec{0} \quad \text{(i)}$$

Taking dot product of $\vec{a}$ and (i), we get

$$x\,\vec{a}\cdot\vec{a}+y\,\vec{a}\cdot\vec{b}+z\,\vec{a}\cdot\vec{c}=0 \quad \text{(ii)}$$

Again taking dot product of $\vec{b}$ and (i), we get

$$x\,\vec{b}\cdot\vec{a}+y\,\vec{b}\cdot\vec{b}+z\,\vec{b}\cdot\vec{c}=0 \quad \text{(iii)}$$

Now Eqs. (i), (ii) and (iii) form a homogeneous system of equations, where x, y and z are not all zero,
Therefore the system must have a non-trivial solution, and for this, the determinant of coefficient matrix should be zero, i.e.,

$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix}=0$$

9. Given that $\vec{A}=2\hat{i}+\hat{k}$, $\vec{B}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{C}=4\hat{i}-3\hat{j}+7\hat{k}$ and to determine a vector $\vec{R}$ such that $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$ and $\vec{R}\cdot\vec{A}=0$. Let $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$

Then $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 7 \\ 1 & 1 & 1 \end{vmatrix}$$

or $\quad (y-z)\hat{i}-(x-z)\hat{j}+(x-y)\hat{k}=-10\hat{i}+3\hat{j}+7\hat{k}$

$\Rightarrow \quad y-z=-10, \quad$ (i)
$x-z=-3, \quad$ (ii)
$x-y=7 \quad$ (iii)

Also $\vec{R}\cdot\vec{A}=0$

$\Rightarrow \quad 2x+z=0 \quad$ (iv)

Substituting $y=x-7$ and $z=-2x$ from (iii) and (iv), respectively in (i), we get

$x-7+2x=-10$

$\Rightarrow \quad 3x=-3$

$\Rightarrow \quad x=-1, y=-8$ and $z=2$

10. We have, $\vec{a}=cx\hat{i}-6\hat{j}-3\hat{k}$

$\vec{b}=x\hat{i}+2\hat{j}+2cx\hat{k}$

Now we know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

As the angle between $\vec{a}$ and $\vec{b}$ is obtuse, $\cos\theta<0$

$\Rightarrow \quad \vec{a}\cdot\vec{b}<0$

$\Rightarrow \quad cx^2-12-6cx<0$

$\Rightarrow \quad c<0$ and $D<0$

$\Rightarrow \quad c<0$ and $36c^2+48c<0$

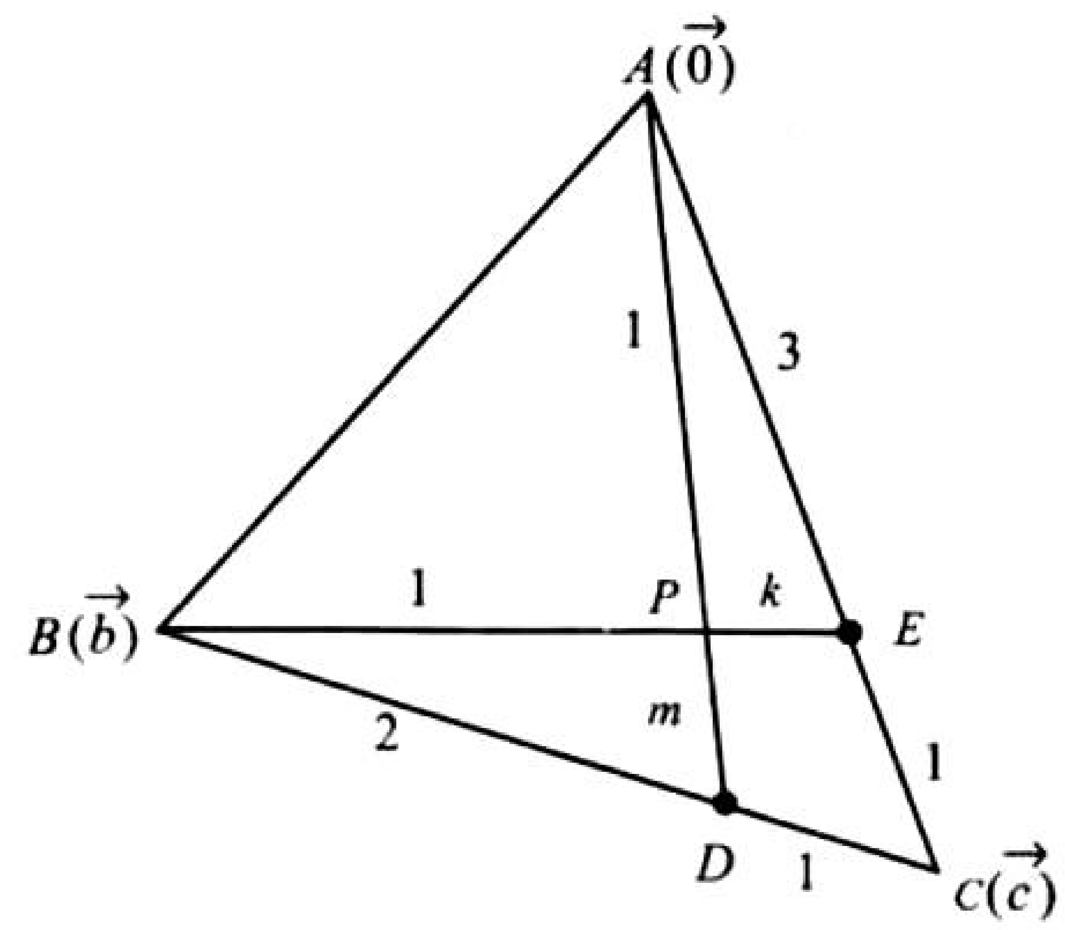
$\Rightarrow \quad c<0$ and $(3c+4)>0$

$\Rightarrow \quad c<0$ and $c>-4/3$

$\Rightarrow \quad -4/3<c<0$

11. Let the vertices of the triangle be $A(\vec{0}), B(\vec{b})$ and $C(\vec{c})$. Given that D divides BC in the ratio 2 : 1.

Therefore, position vector of D is $\dfrac{\vec{b}+2\vec{c}}{3}$.



E divides AC in the ratio 3 : 1.

Therefore, position vector of E is $\dfrac{\vec{0}+3\vec{c}}{4}=\dfrac{3\vec{c}}{4}$.

Let point of intersection P of AD and BE divide BE in the ratio 1 : k and AD in the ratio 1 : m. Then position vectors of P in these two cases are $\dfrac{k\vec{b}+1(3\vec{c}/4)}{k+1}$ and $\dfrac{m\vec{0}+m((\vec{b}+2\vec{c})/3)}{m+1}$, respectively.

Equating the position vectors of P in these two cases, we get

$$\frac{k\vec{b}}{k+1}+\frac{3\vec{c}}{4(k+1)}=\frac{m\vec{b}}{3(m+1)}+\frac{2m\vec{c}}{3(m+1)}$$

$$\Rightarrow \quad \frac{k}{k+1}=\frac{m}{3(m+1)} \text{ and } \frac{3}{4(k+1)}=\frac{2m}{3(m+1)}$$

Dividing, we have $\dfrac{4k}{3}=\dfrac{1}{2}$ or $k=\dfrac{3}{8}$

Required ratio is 8 : 3.

12. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

Here, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$$=-(\vec{c}\times\vec{d}\cdot\vec{b})\,\vec{a}+(\vec{c}\times\vec{d}\cdot\vec{a})\,\vec{b}$$

$$=[\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b}-[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} \quad \text{(i)}$$

$$(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})=-(\vec{d}\times\vec{b}\cdot\vec{c})\,\vec{a}+(\vec{d}\times\vec{b}\cdot\vec{a})\,\vec{c}$$

$$=[\vec{a}\ \vec{d}\ \vec{b}]\,\vec{c}-[\vec{c}\ \vec{d}\ \vec{b}]\,\vec{a} \quad \text{(ii)}$$

$(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

$$=(\vec{a}\times\vec{d}\cdot\vec{c})\,\vec{b}-(\vec{a}\times\vec{d}\cdot\vec{b})\,\vec{c}$$

$$=-[\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b}-[\vec{a}\ \vec{d}\ \vec{b}]\,\vec{c} \quad \text{(iii)}$$

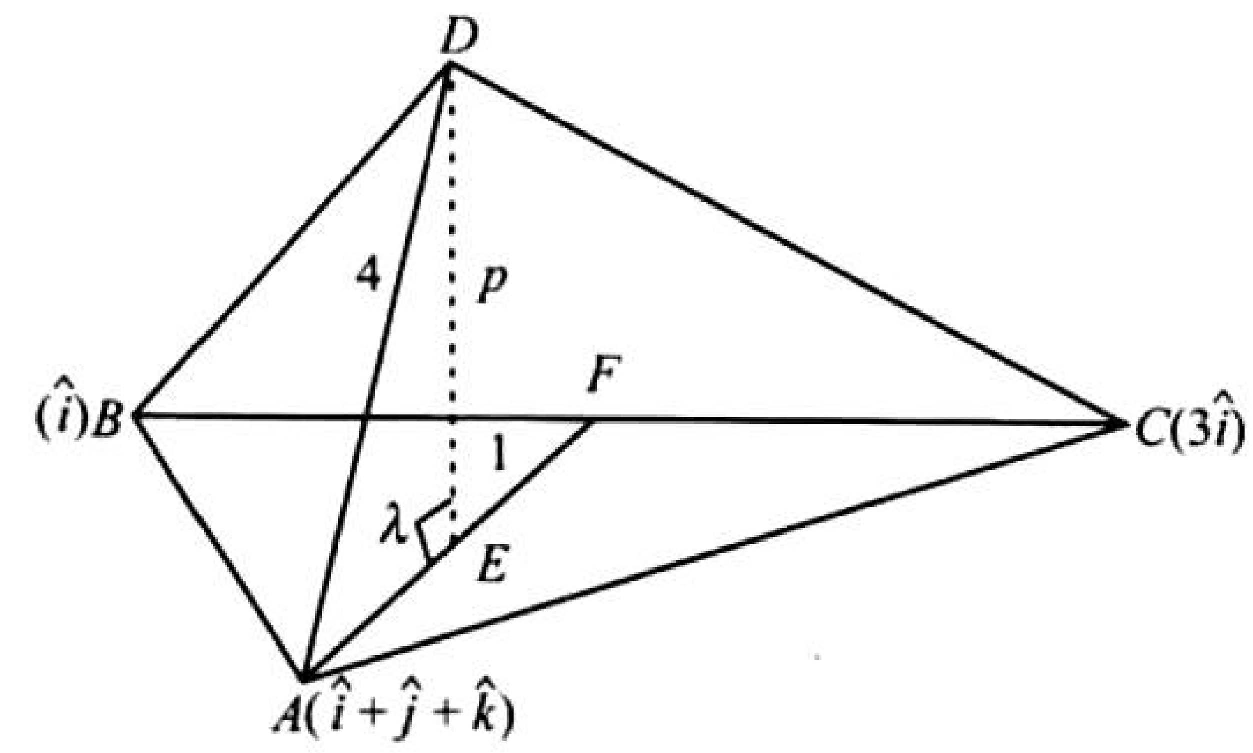
(**Note:** Here we have tried to write the given expression in such a way that we can get terms involving $\vec{a}$ and other similar terms which can get cancelled)

Adding (i), (ii) and (iii), we get

Given vector $=-2\,[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}=k\vec{a}$

Hence, given vector is parallel to $\vec{a}$.

13.



We are given $AD = 4$

Volume of tetrahedron $=\dfrac{2\sqrt{2}}{3}$

$\Rightarrow \quad \dfrac{1}{3}$ (Area of ΔABC) $p=\dfrac{2\sqrt{2}}{3}$

$\therefore \quad \dfrac{1}{2}|\overrightarrow{BA}\times\overrightarrow{BC}|\,p=2\sqrt{2}$

$\dfrac{1}{2}\,|(\hat{j}+\hat{k})\times 2\hat{i}|\,p=2\sqrt{2}$

or $\quad |\hat{j}-\hat{k}|\,p=2\sqrt{2}$

or $\quad \sqrt{2}\,p=2\sqrt{2}$ or $p=2$

We have to find the P.V. of point E. Let it divide median AF in the ratio λ : 1.

$$\text{P.V. of } E=\frac{\lambda\cdot 2\hat{i}+(\hat{i}+\hat{j}+\hat{k})}{\lambda+1} \quad \text{(i)}$$

$$\therefore \quad \overrightarrow{AE}=\text{P.V. or } E-\text{P.V. of } A=\frac{\lambda(\hat{i}-\hat{j}-\hat{k})}{\lambda+1}$$

$$\therefore \quad |\overrightarrow{AE}|^2=3\left(\frac{\lambda}{\lambda+1}\right)^2 \quad \text{(ii)}$$

In ΔAED,

Now, $4+3\left(\dfrac{\lambda}{\lambda+1}\right)^2=16$

$\therefore \quad \left(\dfrac{\lambda}{\lambda+1}\right)=\pm 2$

$\therefore \quad \lambda=-2$ or $-2/3$

Putting the value of λ in (i), we get the P.V. of possible positions of E as $-\hat{i}+3\hat{j}+3\hat{k}$ or $3\hat{i}-\hat{j}-\hat{k}$.

14. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors inclined at an angle θ with each other.

Also $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar. Therefore,

$$[\vec{a}\ \vec{b}\ \vec{c}]\neq 0.$$

Also given that $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$.

Taking dot product on both sides with $\vec{a}$, we get

$$p + q\cos\theta + r\cos\theta = [\vec{a}\ \vec{b}\ \vec{c}] \quad \text{(i)}$$

Similarly, taking dot product on both sides with $\vec{b}$ and $\vec{c}$, we get, respectively,

$$p\cos\theta + q + r\cos\theta = 0 \quad \text{(ii)}$$

and $$p\cos\theta + q\cos\theta + r = [\vec{a}\ \vec{b}\ \vec{c}] \quad \text{(iii)}$$

Adding (i), (ii) and (iii), we get

$$p + q + r = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta + 1} \quad \text{(iv)}$$

Multiplying (iv) by $\cos\theta$ and subtracting (i) from it, we get

$$p(\cos\theta - 1) = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta + 1} - [\vec{a}\ \vec{b}\ \vec{c}]$$

or $$p(\cos\theta - 1) = \frac{-[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta + 1}$$

or $$p = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{(1-\cos\theta)(1+2\cos\theta)}$$

Similarly, (iv) × cos θ – (ii) gives

$$q = \frac{-2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{(1+2\cos\theta)(1-\cos\theta)}$$

and (iv) × cos θ – (iii) gives

$$r(\cos\theta - 1) = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta + 1} - [\vec{a}\ \vec{b}\ \vec{c}]$$

or $$r = \frac{-[\vec{a}\ \vec{b}\ \vec{c}]}{(2\cos\theta + 1)(\cos\theta - 1)}$$

But we have to find p, q and r in terms of θ only.

So, let us find the value of $[\vec{a}\ \vec{b}\ \vec{c}]$.

We know that

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

On operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 1+2\cos\theta & \cos\theta & \cos\theta \\ 1+2\cos\theta & 1 & \cos\theta \\ 1+2\cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= (1 + 2\cos\theta)\begin{vmatrix} 1 & \cos\theta & \cos\theta \\ 1 & 1 & \cos\theta \\ 1 & \cos\theta & 1 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= (1 + 2\cos\theta)\begin{vmatrix} 0 & \cos\theta - 1 & 0 \\ 0 & 1 - \cos\theta & \cos\theta - 1 \\ 1 & \cos\theta & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$= (1 + 2\cos\theta)(1 - \cos\theta)^2$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = (1 - \cos\theta)\sqrt{1 + 2\cos\theta}$$

Thus, we get

$$p = \frac{1}{\sqrt{1+2\cos\theta}},\ q = \frac{-2\cos\theta}{\sqrt{1+2\cos\theta}},$$

$$r = \frac{1}{\sqrt{1+2\cos\theta}}$$

15. We have, $(\vec{A} + \vec{B}) \times (\vec{A} + \vec{C})$

$$= \vec{A}\times\vec{A} + \vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C}$$

$$= \vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C} \qquad (\because \vec{A}\times\vec{A} = \vec{0})$$

$$\therefore \quad [(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})$$

$$= [\vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C}]\times(\vec{B}\times\vec{C})$$

$$= (\vec{B}\times\vec{A})\times(\vec{B}\times\vec{C}) + (\vec{A}\times\vec{C})\times(\vec{B}\times\vec{C})$$

$$= \{(\vec{B}\times\vec{A})\cdot\vec{C}\}\vec{B} - \{(\vec{B}\times\vec{A})\cdot\vec{B}\}\vec{C}$$

$$+ \{(\vec{A}\times\vec{C})\cdot\vec{C}\}\vec{B} - \{(\vec{A}\times\vec{C})\cdot\vec{B}\}\vec{C}$$

$$= [\vec{B}\ \vec{A}\ \vec{C}]\vec{B} - [\vec{A}\ \vec{C}\ \vec{B}]\vec{C}$$

$$= [\vec{A}\ \vec{C}\ \vec{B}]\{\vec{B} - \vec{C}\}$$

Thus, L.H.S. of the given expression becomes

$$[\vec{A}\ \vec{C}\ \vec{B}](\vec{B} - \vec{C})\cdot(\vec{B} + \vec{C})$$

$$= [\vec{A}\ \vec{C}\ \vec{B}]\{(\vec{B} - \vec{C})\cdot(\vec{B} + \vec{C})\}$$

$$= [\vec{A}\ \vec{C}\ \vec{B}]\{|\vec{B}|^2 - |\vec{C}|^2\} = 0 \qquad (\because |B| = |C|)$$

16. $(\hat{i} + \hat{j} + 3\hat{k})x + (3\hat{i} - 3\hat{j} + \hat{k})y + (-4\hat{i} + 5\hat{j})z$

$$= \lambda(x\hat{i} + y\hat{j} + z\hat{k})$$

Comparing coefficient of $\hat{i}$, $x + 3y - 4z = \lambda x$

$$\Rightarrow \quad (1 - \lambda)x + 3y - 4z = 0 \quad \text{(i)}$$

Comparing coefficient of $\hat{j}$, $x - 3y + 5z = \lambda y$

$$\Rightarrow \quad x - (3 + \lambda)y + 5z = 0 \quad \text{(ii)}$$

Comparing coefficient of $\hat{k}$, $3x + y + 0z = \lambda z$

$$3x + y - \lambda z = 0 \quad \text{(iii)}$$

All the above three equations are satisfied for x, y and z not all zero if

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

or $(1-\lambda)[3\lambda+\lambda^2-5]-3[-\lambda-15]-4[1+9+3\lambda]=0$

or $\lambda^3+2\lambda^2+\lambda=0$

or $\lambda(\lambda+1)^2=0$

or $\lambda=0,-1$

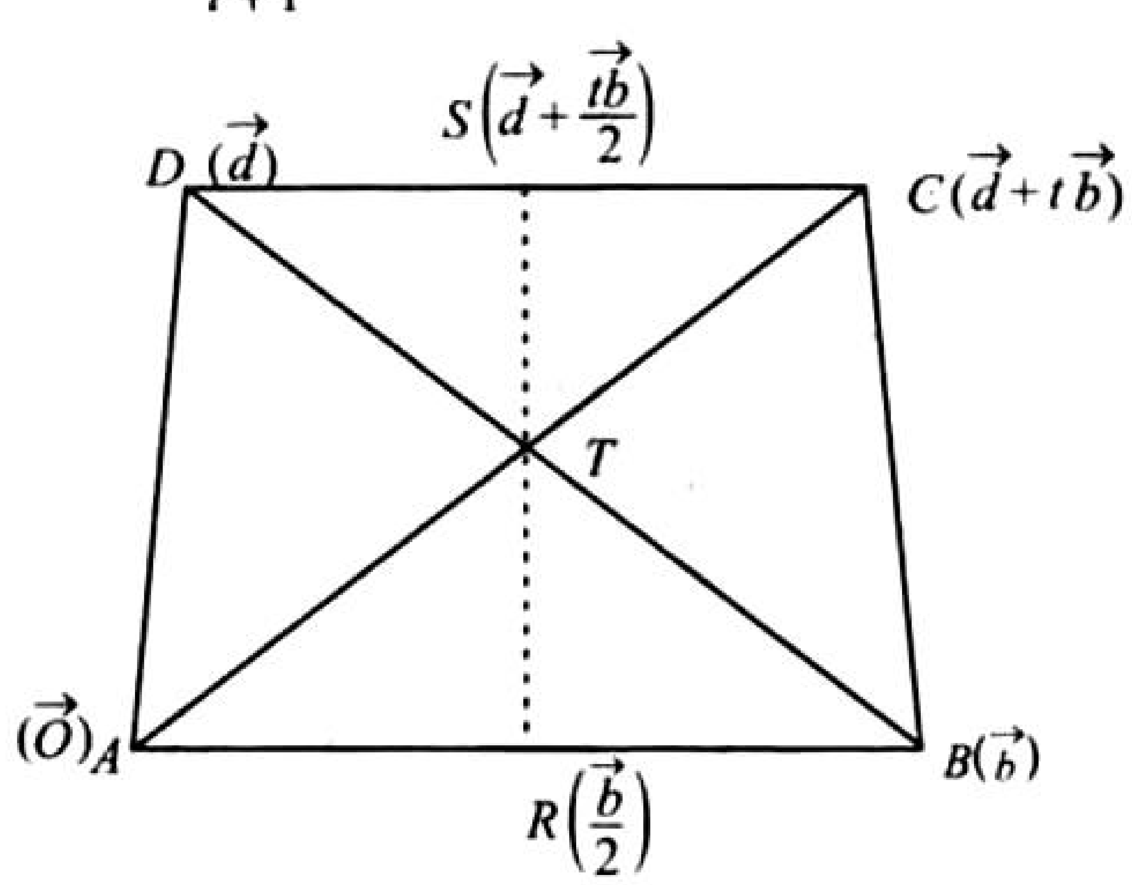
17. Let the P.V.s of the points A, B, C and D be $A(\vec{O})$, $B(\vec{b})$, $D(\vec{d})$ and $C(\vec{d}+t\vec{b})$.

For any point $\vec{r}$ on $\overrightarrow{AC}$ and $\overrightarrow{BD}$, $\vec{r}=\lambda(\vec{d}+t\vec{b})$ and $\vec{r}=(1-\mu)\vec{b}+\mu\vec{d}$, respectively.

For the point of intersection, say T, compare the coefficients.

$$\lambda=\mu,\ t\lambda=1-\mu=1-\lambda \text{ or } (t+1)\lambda=1$$

$$\therefore\quad \lambda=\frac{1}{t+1}=\mu$$



Therefore, $\vec{r}$ (position vector of T) $=\dfrac{\vec{d}+t\vec{b}}{t+1}$ (i)

Let R and S be the midpoints of the parallel sides AB and DC; then R is $\dfrac{\vec{b}}{2}$ and S is $\vec{d}+t\dfrac{\vec{b}}{2}$.

Let T divide SR in the ratio $m:1$.

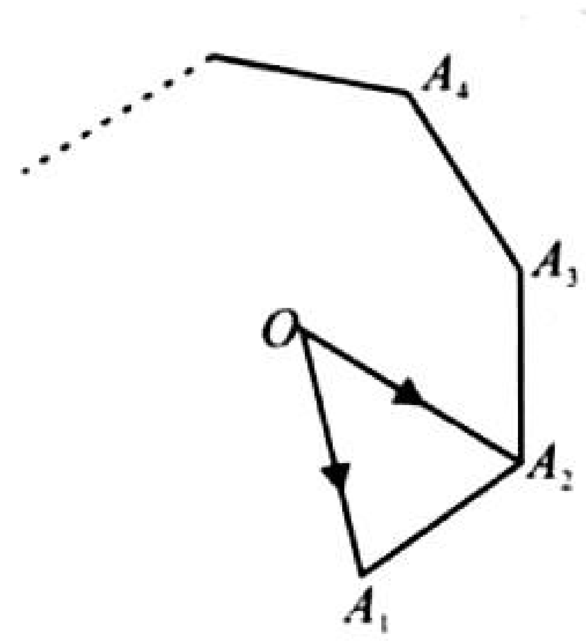
Position vector of T is $\dfrac{m\dfrac{\vec{b}}{2}+\vec{d}+t\dfrac{\vec{b}}{2}}{m+1}$, which is equivalent to $\dfrac{\vec{d}+t\vec{b}}{t+1}$.

Comparing coefficients of $\vec{b}$ and $\vec{d}$,

$$\frac{1}{m+1}=\frac{1}{t+1} \text{ and } \frac{m+t}{2(m+1)}=\frac{t}{t+1}.$$

From the first relation, $m=t$, which satisfies the second relation. Hence proved.

18. $\overrightarrow{OA}_1\ \overrightarrow{OA}_2,\ldots,\overrightarrow{OA}_n$. All vectors are of same magnitude, say a, and angle between any two consecutive vectors is the same, that is, $2\pi/n$. Let $\hat{p}$ be the unit vector parallel to the plane of the polygon.



Let $\overrightarrow{OA}_1\times\overrightarrow{OA}_2=a^2\sin\dfrac{2\pi}{n}\hat{p}$ (i)

Now, $\displaystyle\sum_{i=1}^{n-1}\overrightarrow{OA}_i\times\overrightarrow{OA}_{i+1}=\sum_{i=1}^{n-1}a^2\sin\frac{2\pi}{n}\hat{p}$

$=(n-1)\,a^2\sin\dfrac{2\pi}{n}\hat{p}$

$=(n-1)\,[-\overrightarrow{OA}_2\times\overrightarrow{OA}_1]$ [Using (i)]

$=(1-n)\,[\overrightarrow{OA_2}\times\overrightarrow{OA_1}]$

= R.H.S.

19. a. We have $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$

and $\vec{u}\times\vec{v}=|\vec{u}||\vec{v}|\sin\theta\,\hat{n}$

(Where θ is the angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$)

$\Rightarrow\quad (\vec{u}\cdot\vec{v})^2+|\vec{u}\times\vec{v}|^2$

$=|\vec{u}|^2|\vec{v}|^2(\cos^2\theta+\sin^2\theta)=|\vec{u}|^2|\vec{v}|^2$

b. $(1-\vec{u}\cdot\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

$=1-2\vec{u}\cdot\vec{v}+(\vec{u}\cdot\vec{v})^2+|\vec{u}|^2+|\vec{v}|^2+|\vec{u}\times\vec{v}|^2+2\vec{u}\cdot\vec{v}$

$(\because \vec{u}\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot(\vec{u}\times\vec{v})=0)$

$=1+|\vec{u}|^2+|\vec{v}|^2+(\vec{u}\cdot\vec{v})^2+|\vec{u}\times\vec{v}|^2$

$=1+|\vec{u}|^2+|\vec{v}|^2+|\vec{u}|^2|\vec{v}|^2$

$=(1+|\vec{u}|^2)(1+|\vec{v}|^2)$

20. $[\vec{u}\ \vec{v}\ \vec{w}]=(\vec{u}\times\vec{v})\cdot(\vec{v}-\vec{w}\times\vec{u})$

$=(\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{w})$

$=\begin{vmatrix}\vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{w}\\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{w}\end{vmatrix}$

Now, $\vec{u}\cdot\vec{u}=1$

$\vec{u}\cdot\vec{w}=\vec{u}\cdot(\vec{v}-\vec{w}\times\vec{u})=\vec{u}\cdot\vec{v}-[\vec{u}\ \vec{w}\ \vec{u}]=\vec{u}\cdot\vec{v}$

$\vec{v}\cdot\vec{w}=\vec{v}\cdot(\vec{v}-\vec{w}\times\vec{u})=1-[\vec{v}\ \vec{w}\ \vec{u}]=1-[\vec{u}\ \vec{v}\ \vec{w}]$

$\therefore\quad [\vec{u}\ \vec{v}\ \vec{w}]=\begin{vmatrix}1 & \cos\theta\\ \cos\theta & 1-[\vec{u}\ \vec{v}\ \vec{w}]\end{vmatrix}$

(θ is the angle between $\vec{u}$ and $\vec{v}$)

$=1-[\vec{u}\ \vec{v}\ \vec{w}]-\cos^2\theta$

$\therefore \quad [\vec{u}\ \vec{v}\ \vec{w}] = \frac{1}{2}\sin^2\theta \le \frac{1}{2}$

Equality holds when $\sin^2\theta = 1$, i.e., $\theta = \pi/2$, i.e., $\vec{u} \perp \vec{v}$.

21. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of A, B and C, respectively.
Let AD, BE and CF be the bisectors of $\angle A$, $\angle B$ and $\angle C$, respectively.
a, b and c are the lengths of sides BC, CA and AB, respectively.
Now AD divides BC in the ratio

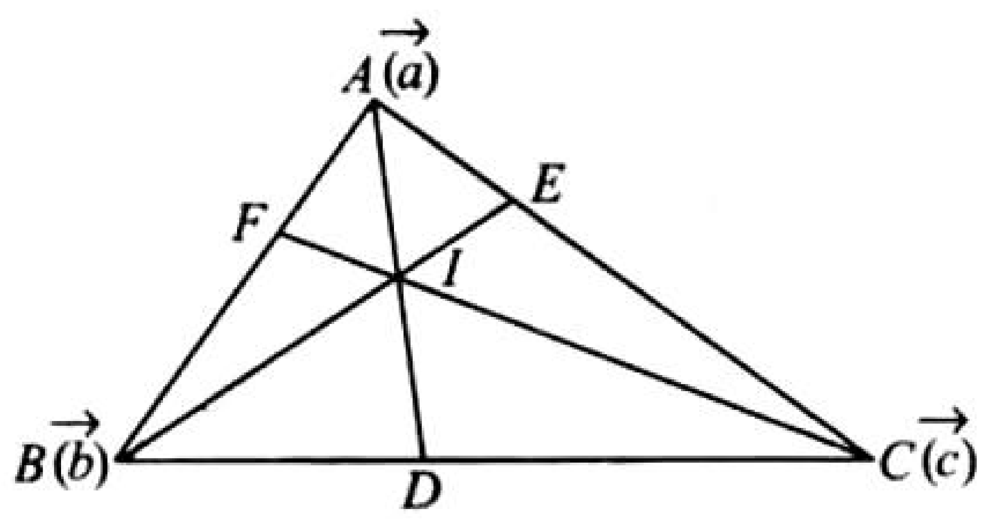
$$BD : DC = AB : AC = c : b.$$

Hence, the position vector of D is $\vec{d} = \frac{b\vec{b} + c\vec{c}}{b+c}$.

Let I be the point of intersection of BE and AD.
Then in ΔABC, BI is bisector of $\angle B$. Therefore,

$$DI : IA = BD : BA$$

But $\quad \frac{BD}{DC} = \frac{c}{b}$ or $\frac{BD}{BD+DC} = \frac{c}{c+b}$



or $\quad \frac{BD}{BC} = \frac{c}{c+b}$

or $\quad BD = \frac{ac}{b+c}$

$\therefore \quad DI : IA = \frac{ac}{b+c} : c = a : (b+c)$

$$\therefore \quad \text{P.V. of } I = \frac{a\vec{a} + \vec{d}(b+c)}{a+b+c}$$

$$= \frac{a\vec{a} + \left(\frac{b\vec{b} + c\vec{c}}{b+c}\right)(b+c)}{a+b+c}$$

$$= \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}$$

As P.V. of I is symmetrical in $\vec{a}, \vec{b}, \vec{c}$ and a, b, c, it must lie on CF as well.

22. $\vec{A}(t)$ is parallel to $\vec{B}(t)$ for some $t \in [0, 1]$ if and only if

$\frac{f_1(t)}{g_1(t)} = \frac{f_2(t)}{g_2(t)}$ for some $t \in [0, 1]$

or $\quad f_1(t)\cdot g_2(t) = f_2(t)g_1(t)$ for some $t \in [0, 1]$

Let $\quad h(t) = f_1(t)\cdot g_2(t) - f_2(t)\cdot g_1(t)$

$$h(0) = f_1(0)\cdot g_2(0) - f_2(0)\cdot g_1(0)$$

$$= 2\times 2 - 3\times 3 = -5 < 0$$

$$h(1) = f_1(1)\cdot g_2(1) - f_2(1)\cdot g_1(1)$$

$$= 6\times 6 - 2\times 2 = 32 > 0$$

Since h is a continuous function, and $h(0)\cdot h(1) < 0$, there are some $t \in [0, 1]$ for which $h(t) = 0$, i.e., $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for this t.

23. Given data are insufficient to uniquely determine the three vectors as there are only six equations involving nine variables (coefficients of vectors (v_1, v_2, v_3).
Therefore, we can obtain infinite number of sets of three vectors, $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, satisfying these conditions.
From the given data, we get

$$\vec{v}_1 \cdot \vec{v}_1 = 4 \Rightarrow |\vec{v}_1| = 2$$

$$\vec{v}_2 \cdot \vec{v}_2 = 2 \Rightarrow |\vec{v}_2| = \sqrt{2}$$

$$\vec{v}_3 \cdot \vec{v}_3 = 29 \Rightarrow |\vec{v}_3| = \sqrt{29}$$

Also $\vec{v}_1 \cdot \vec{v}_2 = -2$

$\Rightarrow \quad |v_1||v_2|\cos\theta = -2$

(where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$)

or $\quad \cos\theta = \frac{-1}{\sqrt{2}}$

or $\quad \theta = 135°$

Since any two vectors are always coplanar, let us suppose that $\vec{v}_1$ and $\vec{v}_2$ are in the x–y plane. Let $\vec{v}_1$ be along the positive direction of the x-axis. Then

$$\vec{v}_1 = 2\hat{i} \qquad (\because |\vec{v}_1| = 2)$$

As $\vec{v}_2$ makes an angle 135° with $\vec{v}_1$ and lies in the x–y plane, also $|\vec{v}_2| = \sqrt{2}$, we get

$$\vec{v}_2 = -\hat{i} \pm \hat{j}$$

Again let $\vec{v}_3 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

$$\vec{v}_3 \cdot \vec{v}_1 = 6 \Rightarrow 2\alpha = 6 \text{ or } \alpha = 3$$

and $\quad \vec{v}_3 \cdot \vec{v}_2 = -5 \Rightarrow -\alpha \pm \beta = -5$ or $\beta = \pm 2$

Also $\quad |\vec{v}_3| = \sqrt{29} \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 29$

$\Rightarrow \quad \gamma = \pm 4$

Hence $\vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$

24. Given that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_2\hat{k}$$

(where a_r, b_r, c_r ($r = 1, 2, 3$) are all non-negative real numbers)

Also, $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$

To prove $V \le L^3$, where V is the volume of the parallelepiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$$V = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad V = (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1) \qquad \text{(i)}$$

Now we know that A.M. $\ge$ G.M., therefore

$$\frac{(a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) + (a_3 + b_3 + c_3)}{3}$$
$$\ge [(a_1 + b_1 + c_1)(a_2 + b_2 + c_2)(a_3 + b_3 + c_3)]^{1/3}$$

$$\Rightarrow \quad L^3 \ge (a_1 + b_1 + c_1)(a_2 + b_2 + c_2)(a_3 + b_3 + c_3)$$
$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + 24 \text{ more such terms}$$
$$\ge a_1b_2c_3 + a_2b_3c_1 + a_3 b_1c_2$$
$$(\because a_r, b_r, c_r \ge 0, r = 1, 2, 3)$$
$$\ge (a_1 b_2c_3 + a_2 b_3c_1 + a_3 b_1 c_2) - (a_1b_3c_2 + a_2b_1c_2 + a_3b_2c_1) \qquad \text{(same reason)}$$
$$= V \qquad \text{[from (i)]}$$

Thus, $L^3 \ge V$

25. We know that $[\vec{x} \times \vec{y}\ \ \vec{y} \times \vec{z}\ \ \vec{z} \times \vec{x}] = [\vec{x}\ \vec{y}\ \vec{z}]^2$

Also a vector along the bisector of given two unit vectors $\vec{u}, \vec{v}$ is $\vec{u} + \vec{v}$.

A unit vector along the bisector is $\dfrac{\vec{u} + \vec{v}}{|\vec{u} + \vec{v}|}$.

$$|\vec{u} + \vec{v}|^2 = 1 + 1 + 2\vec{u}\cdot\vec{v} = 2 + 2\cos\alpha = 4\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \quad \vec{x} = \frac{\vec{u} + \vec{v}}{2\cos\frac{\alpha}{2}}$$

Similarly, $\vec{y} = \dfrac{\vec{v} + \vec{w}}{2\cos\beta/2}$ and $\vec{z} = \dfrac{\vec{u} + \vec{w}}{2\cos\gamma/2}$

$$\Rightarrow \quad [\vec{x}\ \vec{y}\ \vec{z}] = \frac{1}{8}[\vec{u} + \vec{v}\ \ \vec{v} + \vec{w}\ \ \vec{u} + \vec{w}] \times \sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$
$$= \frac{1}{8}2[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$
$$= \frac{1}{4}[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$\Rightarrow \quad [\vec{x} \times \vec{y}\ \ \vec{y} \times \vec{z}\ \ \vec{z} \times \vec{x}] = [\vec{x}\ \ \vec{y}\ \ \vec{z}]^2$$
$$= \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

26. Given that $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ (i)

and $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ (ii)

Subtracting (ii) from (i), we get

$$\vec{a} \times (\vec{c} - \vec{b}) = (\vec{b} - \vec{c}) \times \vec{d}$$

or $\vec{a} \times (\vec{c} - \vec{b}) = \vec{d} \times (\vec{c} - \vec{b})$

or $\vec{a} \times (\vec{c} - \vec{b}) - \vec{d} \times (\vec{c} - \vec{b}) = 0$

or $(\vec{a} - \vec{d}) \times (\vec{c} - \vec{b}) = 0$

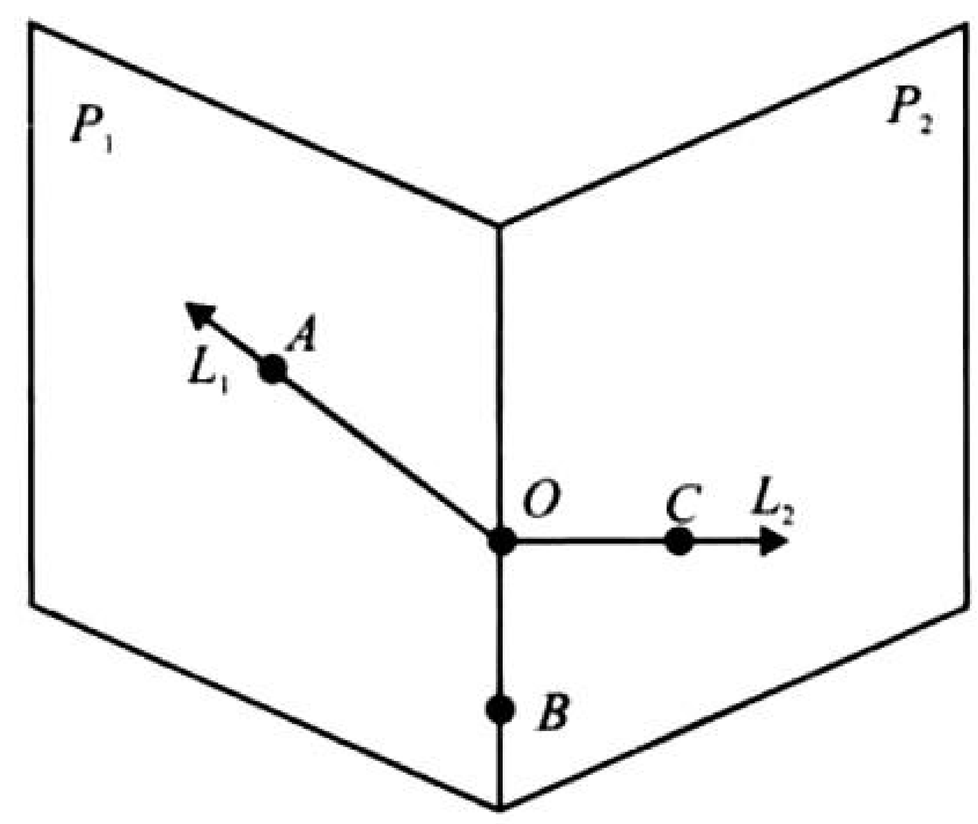
or $(\vec{a} - \vec{d}) \| (\vec{c} - \vec{b})$ $(\because \vec{a} - \vec{d} \ne 0, \vec{c} - \vec{b} \ne 0)$

Hence, the angle between $\vec{a} - \vec{d}$ and $\vec{c} - \vec{b}$ is either 0 or 180°.

$$\Rightarrow (\vec{a} - \vec{d}) \cdot (\vec{c} - \vec{b}) = |\vec{a} - \vec{d}||\vec{c} - \vec{b}|\cos 0 \ne 0$$

as $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ all are different.

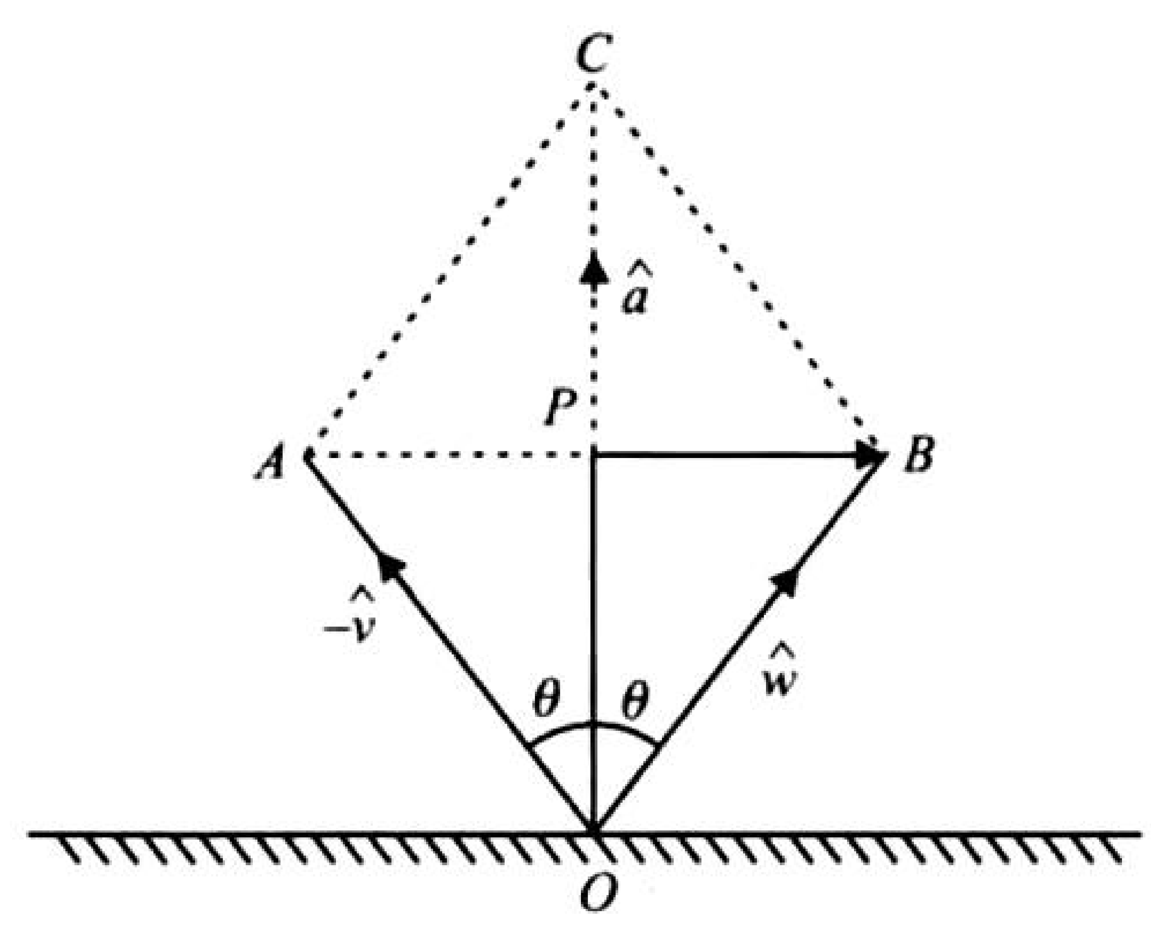
27. Figure shows the possible situation for planes P_1 and P_2 and the lines L_1 and L_2:



Now if we choose points A, B and C as A on L_1, B on the line of intersection of P_1 and P_2 but other than the origin and C on L_2 again other than the origin, then we can consider

A corresponds to one of A', B', C'.
B corresponds to one of the remaining of A', B' and C'.
C corresponds to third of A', B' and C', e.g.,
$A' \equiv C$; $B' \equiv B$; $C' \equiv A$
Hence, one permutation of $[A\ B\ C]$ is $[CBA]$. Hence proved.

28. Given that the incident ray is along $\hat{v}$, the reflected ray is along $\hat{w}$ and the normal is along $\hat{a}$, outwards. The given figure can be redrawn as shown in figure.



We know that the incident ray, the reflected ray, and the normal lie in a plane, and the angle of incidence is equal to the angle of reflection.

Therefore, $\hat{a}$ will be along the angle bisector of $\hat{w}$ and $-\hat{v}$, i.e.,

$$\hat{a} = \frac{\hat{w} + (-\hat{v})}{|\hat{w} - \hat{v}|} \qquad \text{(i)}$$

But $\hat{a}$ is a unit vector

where $|\hat{w} - \hat{v}| = OC = 2OP$

$$= 2\,|\hat{w}|\cos\theta = 2\cos\theta$$

Substituting this value in (i), we get

$$\hat{a} = \frac{\hat{w} - \hat{v}}{2\cos\theta}$$

or $\hat{w} = \hat{v} + (2\cos\theta)\,\hat{a}$

or $\hat{w} = \hat{v} - 2\,(\hat{a}\cdot\hat{v})\,\hat{a}$ $\qquad (\because\ \hat{a}\cdot\hat{v} = -\cos\theta)$