

Application of Derivatives

- For a quantity y varying with another quantity x , satisfying the rule $y = f(x)$, the rate of change of y with respect to x is given by $\frac{dy}{dx}$ or $f'(x)$

The rate of change of y with respect to x at the point $x = x_0$ is given by $\left. \frac{dy}{dx} \right|_{x=x_0}$ or $f'(x_0)$.

- If the variables x and y are expressed in form of $x = f(t)$ and $y = g(t)$, then the rate of change of y with respect to x is given by $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ provided $f'(t) \neq 0$

a. A function $f: (a, b) \rightarrow \mathbf{R}$ is said to be

- increasing on (a, b) , if $x_1 < x_2$ in (a, b)
- decreasing on (a, b) , if $x_1 < x_2$ in (a, b)

OR

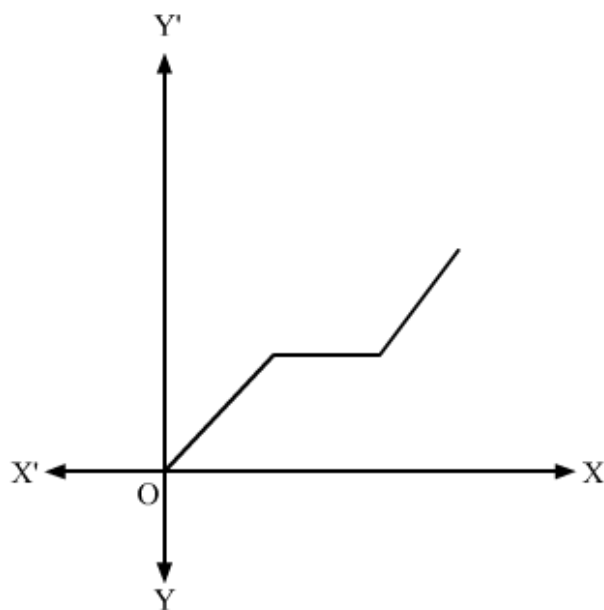
If a function f is continuous on $[a, b]$ and differentiable on (a, b) , then

- f is increasing in $[a, b]$, if $f'(x) > 0$ for each $x \in (a, b)$
- f is decreasing in $[a, b]$, if $f'(x) < 0$ for each $x \in (a, b)$
- f is constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$

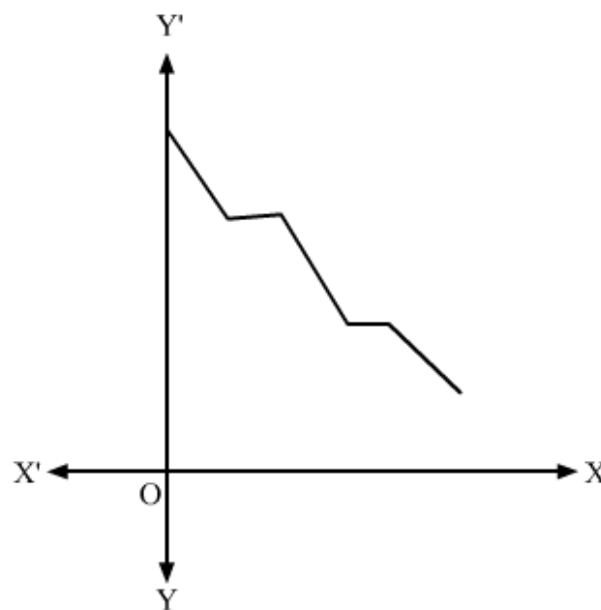
a. A function $f: (a, b) \rightarrow \mathbf{R}$ is said to be

- strictly increasing on (a, b) , if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) < f(x_2) \forall x_1, x_2 \in (a, b)$
- strictly decreasing on (a, b) , if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) > f(x_2) \forall x_1, x_2 \in (a, b)$

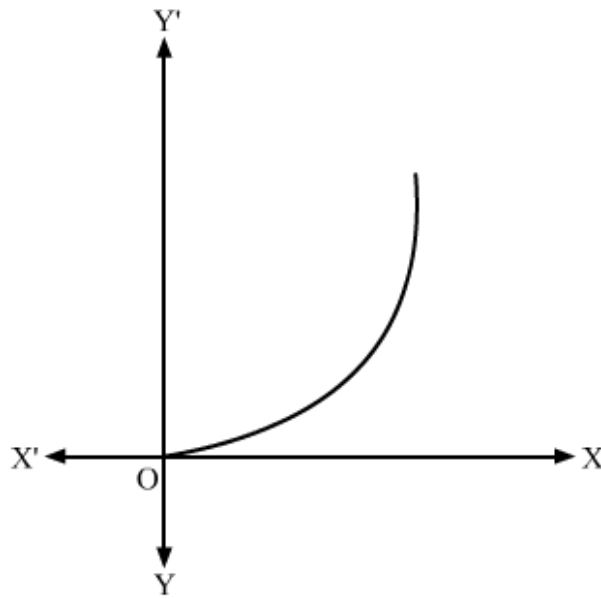
a. The graphs of various types of functions can be shown as follows:



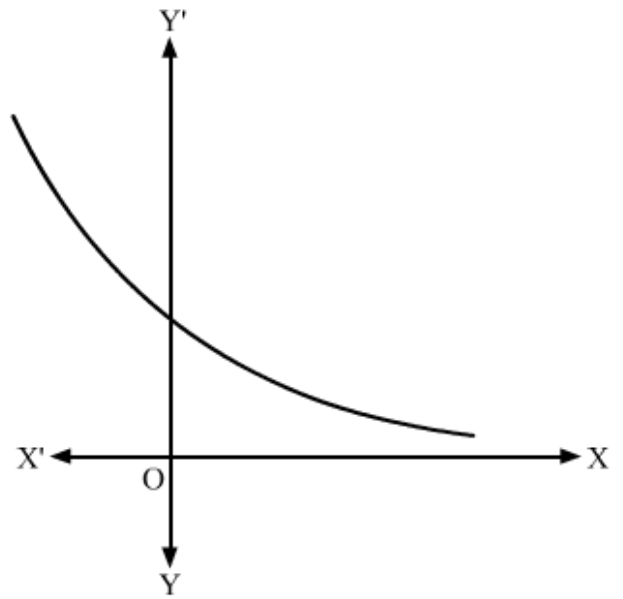
Increasing Function



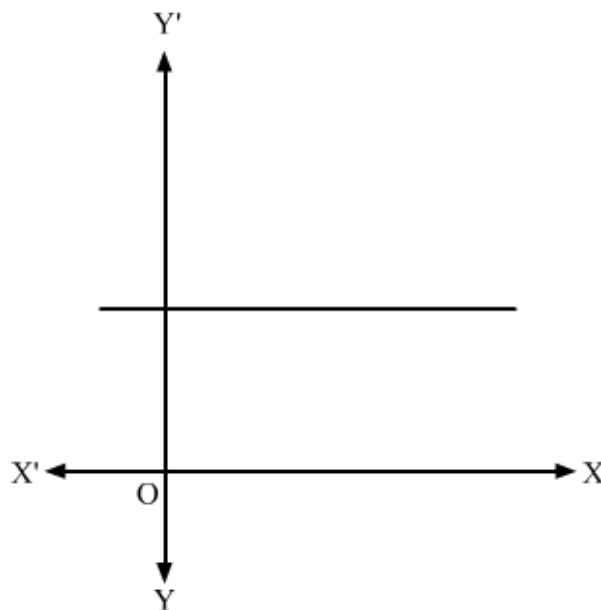
Decreasing Function



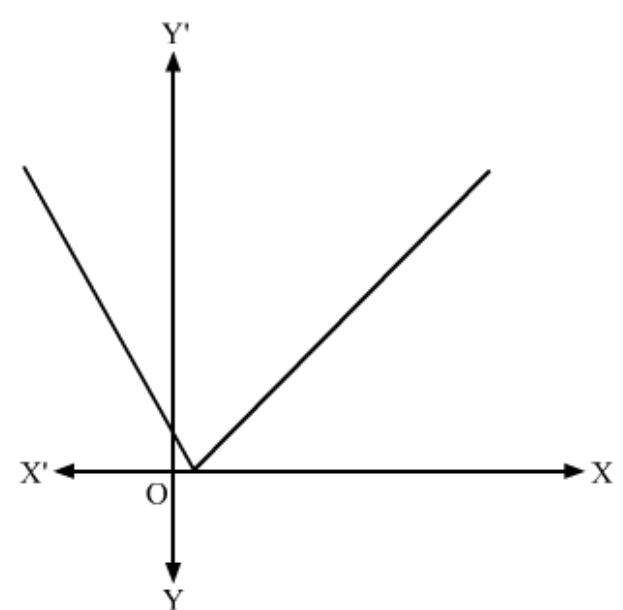
Strictly Increasing Function



Strictly Decreasing Function



Constant Function



Neither increasing or
creasing function

Example 1: Find the intervals in which the function f given by $f(x) = \sqrt{3} \sin x - \cos x, x \in [0, 2\pi]$ is strictly increasing or decreasing.

Solution:

$$f(x) = \sqrt{3} \sin x - \cos x$$

$$\therefore f'(x) = \sqrt{3} \cos x + \sin x$$

$$f'(x) = 0 \text{ gives } \tan x = -\sqrt{3}$$

$$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

The points $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$ divide the interval $[0, 2\pi]$ into three disjoint intervals,

$$\left[0, \frac{2\pi}{3}\right), \left(\frac{2\pi}{3}, \frac{5\pi}{3}\right), \left(\frac{5\pi}{3}, 2\pi\right].$$

$$\text{Now, } f'(x) > 0, \text{ if } x \in \left[0, \frac{2\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right]$$

f is strictly increasing in the intervals $\left[0, \frac{2\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right]$.

$$\text{Also, } f'(x) < 0, \text{ if } x \in \left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$$

f is strictly decreasing in the interval $\left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$.

- For the curve $y = f(x)$, the slope of tangent at the point (x_0, y_0) is given by $\left.\frac{dy}{dx}\right|_{(x_0, y_0)}$ or $f'(x_0)$.
- For the curve $y = f(x)$, the slope of normal at the point (x_0, y_0) is given by $\frac{-1}{\left.\frac{dy}{dx}\right|_{(x_0, y_0)}}$ or $\frac{-1}{f'(x_0)}$.
- The equation of tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by, $y - y_0 = f'(x_0) \times (x - x_0)$
- If $f'(x_0)$ does not exist, then the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is parallel to the y -axis and its equation is given by $x = x_0$.
- The equation of normal to the curve $y = f(x)$ at the point (x_0, y_0) is given by, $y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$
- If $f'(x_0)$ does not exist, then the normal to the curve $y = f(x)$ at the point (x_0, y_0) is parallel to the x -axis and its equation is given by $y = y_0$.
- If $f'(x_0) = 0$, then the respective equations of the tangent and normal to the curve $y = f(x)$ at the point (x_0, y_0) are $y = y_0$ and $x = x_0$.

- Let $y = f(x)$ and let Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x i.e., $\Delta y = f(x + \Delta x) - f(x)$

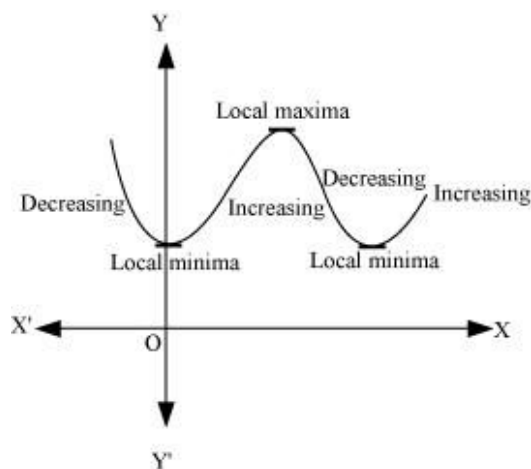
Then, $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$ is a good approximation of Δy , when $dx = \Delta x$ is relatively small and we denote it by $dy \approx \Delta y$.

- **Maxima and Minima:** Let a function f be defined on an interval I . Then, f is said to have
 - maximum value in I , if there exists $c \in I$ such that $f(c) > f(x)$, $\forall x \in I$ [In this case, c is called the point of maxima]

- minimum value in I , if there exists $c \in I$ such that $f(c) < f(x)$, $\forall x \in I$ [In this case, c is called the point of minima]
- an extreme value in I , if there exists $c \in I$ such that c is either point of maxima or point of minima [In this case, c is called an extreme point]

Note: Every continuous function on a closed interval has a maximum and a minimum value.

- **Local maxima and local minima:** Let f be a real-valued function and c be an interior point in the domain of f . Then, c is called a point of
 - local maxima, if there exists $h > 0$ such that $f(c) > f(x)$, $\forall x \in (c - h, c + h)$ [In this case, $f(c)$ is called the local maximum value of f]
 - local minima, if there exists $h > 0$ such that $f(c) < f(x)$, $\forall x \in (c - h, c + h)$ [In this case, $f(c)$ is called the local minimum value of f]



- A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f .
- **First derivative test:** Let f be a function defined on an open interval I . Let f be continuous at a critical point c in I . Then:
 - If $f'(x)$ changes sign from positive to negative as x increases through c , i.e. if $f'(x) > 0$ at every point sufficiently close to and to the left of c , and $f'(x) < 0$ at every point sufficiently close to and to the right of c , then c is a point of local maxima.
 - If $f'(x)$ changes sign from negative to positive as x increases through c , i.e. if $f'(x) < 0$ at every point sufficiently close to and to the left of c , and $f'(x) > 0$ at every point sufficiently close to and to the right of c , then c is a point of local minima.
 - If $f'(x)$ does not change sign as x increases through c , then c is neither a point of local maxima nor a point of local minima. Such a point c is called point of inflection.
- **Second derivative test:** Let f be a function defined on an open interval I and $c \in I$. Let f be twice differentiable at c and $f'(c) = 0$ Then:
 - If $f''(c) < 0$, then c is a point of local maxima. In this situation, $f(c)$ is local maximum value of f .
 - If $f''(c) > 0$, then c is a point of local minima. In this situation, $f(c)$ is local minimum value of f .
 - If $f''(c) = 0$, then the test fails. In this situation, we follow first derivative test and find whether c is a point of maxima or minima or a point of inflection.

Example 1: Find all the points of local maxima or local minima of the function f given by $f(x) = x^3 - 12x^2 + 36x - 4$.

Solution:

We have,

$$f(x) = x^3 - 12x^2 + 36x - 4$$

$$\therefore f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12)$$

$$\text{and } f''(x) = 3(2x - 8) = 6(x - 4)$$

$$\text{Now, } f'(x) = 0 \text{ gives } x^2 - 8x + 12 = 0$$

$$\Rightarrow (x - 2)(x - 6) = 0$$

$$\Rightarrow x = 2 \text{ or } x = 6$$

$$\text{However, } f''(2) = -12 \text{ and } f''(6) = 12$$

Therefore, the point of local maxima and local minima are at the points $x = 2$ and $x = 6$ respectively.

The local maximum value is $f(2) = 28$

The local minimum value is $f(6) = -4$