Quadrilaterals

Introduction

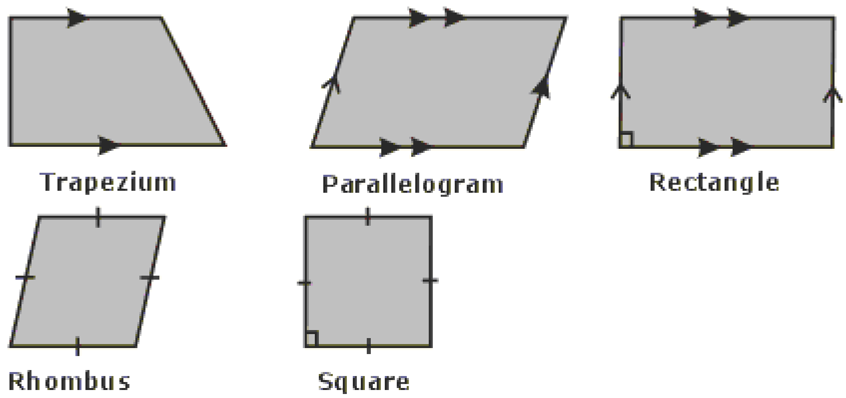
We are familiar with plane figures bounded by straight line segments as sides. They are known as **Polygons**.

Polygon is a word of Greek origin. It means figure with many angles implying many sides.

Squares, rectangles and other four-sided geometric figures formed by the union of four line segments are called **Quadrilaterals**.

Quadrilateral is a four-sided polygon.

Examples of Quadrilaterals



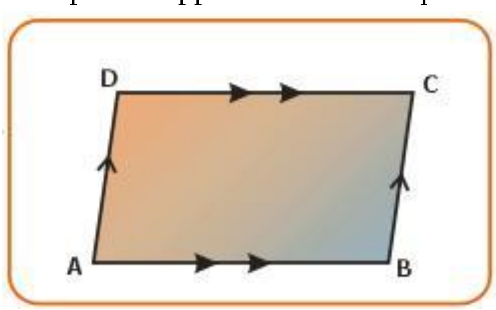
Parallelograms

Parallelogram is a quadrilateral whose opposite sides are parallel and equal.

A rectangle, a rhombus and a square are considered as parallelograms.

A trapezium is quadrilateral with exactly one pair of opposite sides being parallel. Hence, it is not a parallelogram.

Each pair of opposite sides are equal and parallel.



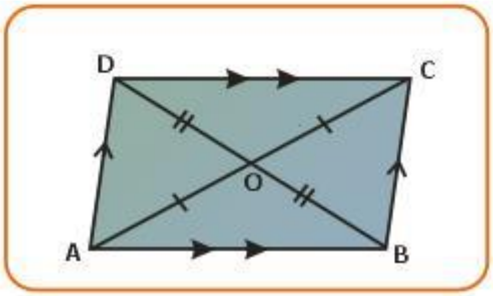
In the diagram,

Opposite sides: AB||DC and AD||BC

AB = DC and AD = BC

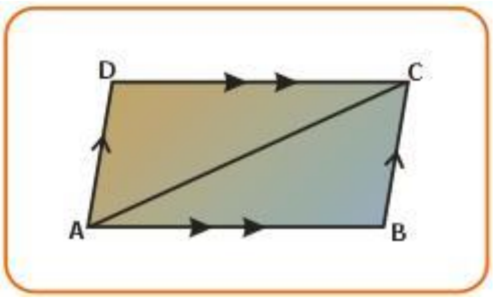
- Opposite angles are equal.
  In the diagram, $\hat{A} = \hat{C}$ and $\hat{B} = \hat{D}$

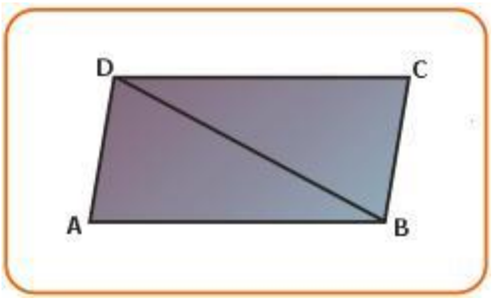
- Diagonals of a parallelogram bisect each other.



In the diagram, OD = OB and OA = OC

- Each diagonal divides the parallelogram into two congruent triangles.





In the diagrams, $\Delta ABC \cong \Delta CDA$
$\Delta ABD \cong \Delta CDB$

Opposite Sides of a quadrilateral:

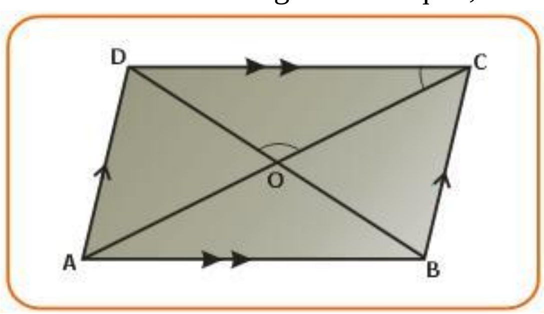
Two sides of a quadrilateral, which have no common point, are called opposite sides.
In the diagram, AB and DC is one pair of opposite sides.
AD and BC is the other pair of opposite sides.

Consecutive sides of a quadrilateral:
Two sides of a quadrilateral, which have a common end point, are called consecutive sides. In the diagram,
AB and BC is one pair of consecutive sides.
BC, CD; CD, DA; and DA, AB are the other three pairs of consecutive sides.

Opposite angles of a quadrilateral:
Two angles, which do not include a side in their intersection, are called the opposite angles of a quadrilateral.
In the diagram, $\hat{A}$ and $\hat{C}$ is one pair of opposite angles, $\hat{B}$ and $\hat{D}$ are another pair of opposite angles.

Consecutive angles of a quadrilateral:
Two angles of a quadrilateral, which include a side in their intersection, are called consecutive angles.
In the diagram, $\hat{A}$ and $\hat{B}$ is one pair of consecutive angles, $\hat{B}, \hat{C}$; $\hat{C}, \hat{D}$; and $\hat{D}, \hat{A}$ are the other three pairs of consecutive angles.

Theorem 1
Statement:
The diagonals of a parallelogram bisect each other.
If two sides of a triangle are unequal, the longer side has the greater angle opposite to it.



ABCD is a parallelogram in which diagonals AC and BD intersect each other at O.

To prove:
The diagonals AC and BD bisect each other i.e, AO=OC and BO=DO.

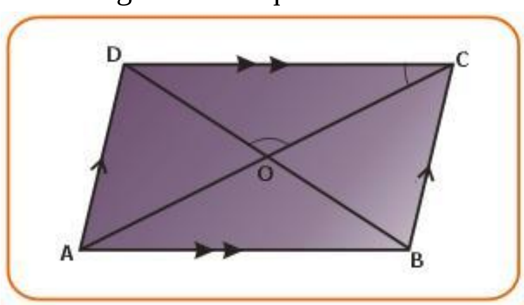
Proof:
AB||CD (by definition of parallelogram)
AC is a transversal.
$\therefore O\hat{A}B = O\hat{C}D$ ----(i)
(alternate angles are equal in a parallelogram)

Also AB=DC (opposite sides are equal in a parallelogram)
Now in Δ AOB and Δ COD,
AB=DC (opposite sides of parallelogram are equal)
$O\hat{A}B = O\hat{C}D$ (proved by (i))
$A\hat{O}B = C\hat{O}D$ (Vertically opposite angles are equal)
∴ Δ AOB ≅ Δ COD (AAS congruency condition)
∴ AO=OC and OB=OD (corresponding parts of congruent triangles are congruent)
i.e., the diagonals of a parallelogram bisect each other.

Sufficient Conditions for a Quadrilateral to be a Parallelogram
We can state the defining property of a parallelogram as follows:
"If a quadrilateral is a parallelogram, then its opposite sides are equal".
Converse
"If both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram".
The converse statement stated above is a necessary condition for a quadrilateral to be a parallelogram. Similarly, we may formulate the following two other conditions for a quadrilateral to be a parallelogram.

- "If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram".
- "If either pair of opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram".

Theorem 2
Statement:
If the diagonals of a quadrilateral bisect each other then the quadrilateral is a parallelogram.



Given:
ABCD is a quadrilateral in which diagonals AC and BD
intersect at O such that AO=OC and BO=OD.

To prove:
ABCD is a parallelogram.

Proof:

In triangles AOB and COD,

AO = CO (given)

BO = OD (given)

$A\hat{O}B = C\hat{O}D$ (Vertically opposite angles are equal)

∴ ΔAOB ≅ Δ COD (SAS congruency condition)

∴ $O\hat{A}B = O\hat{C}D$ (cpct)

Since these are alternate angles made by the transversal AC intersecting AB and CD.

∴ AB||CD

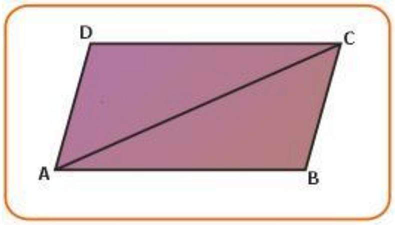
Similarly, AD||BC

Hence ABCD is a parallelogram.

Theorem 3

Statement:

A quadrilateral is a parallelogram if one pair of opposite sides are equal and parallel.



Given:

ABCD is a quadrilateral in which AB||CD and AB=CD.

To prove:

ABCD is a parallelogram.

Construction:

Join AC.

Proof:

In triangles ABC and ADC,

AB=CD (given)

$B\hat{A}C = A\hat{C}D$ (alternate angles are equal)

AC = AC (common side)

∴ Δ ABC ≅ ΔCDA (SAS congruency condition)

$B\hat{C}A = D\hat{A}C$ (corresponding parts of corresponding triangles)

Since these are alternate angles, AD||BC.

Thus in the quadrilateral ABCD, AB||CD and AD||BC

∴ ABCD is a parallelogram.

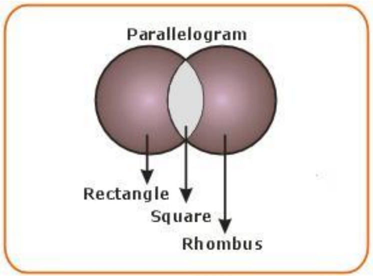
Special Parallelograms

Rectangles, rhombuses and squares belong to the set of parallelograms. Each of these may be defined as follows:

- A is an equilateral and equiangular parallelogram.

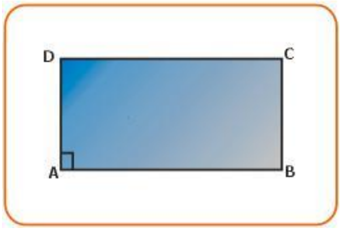
Thus a square is both a rectangle and a rhombus.

The relations among the special parallelograms can be pictorially represented in the figure given below:



Since every rectangle and every rhombus must be a parallelogram, they are shown as subsets of a parallelogram and since a square is both a rectangle and rhombus, it is represented by the overlapping shaded section.

Rectangle



A rectangle is a parallelogram with one of its angles as a right angle.

In the above diagram, let $\hat{A} = 90^0$

Since, $AD || BC$, $\hat{A} + \hat{B} = 180^0$

(sum of interior angles on the same side of transversal AB)

$\therefore \hat{B} = 90^0$

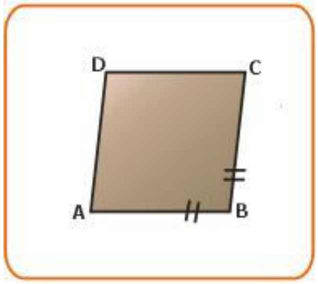
$AB || CD$ and $\hat{A} = 90^0$ (given)

$\therefore \hat{A} + \hat{D} = 180^0$

$\therefore \hat{D} = 90^0$

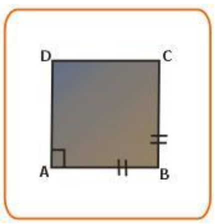
$\therefore \hat{C} = 90^0$

Corollary: Each of the four angles of a rectangle is a right angle.
Rhombus



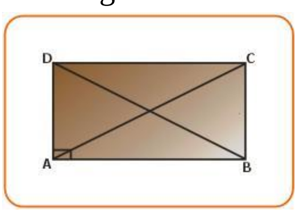
A rhombus is a parallelogram with a pair of its consecutive sides equal.
ABCD is a rhombus in which AB = BC.
Since a rhombus is a parallelogram, AB = DC and BC = AD.
Thus AB = BC = CD = AD.

Corollary: All the four sides of a rhombus are equal (congruent).
Square



A square is a rectangle with a pair of its consecutive sides equal.
Since square is a rectangle, each angle of a rectangle is a right angle and AB=DC, BC = CD.
Thus AB = BC = CD = AD.
Each of the four angles of a square is a right angle and each of the four sides is of the same length.

Theorem 4
Statement:
The diagonals of a rectangle are equal in length.



Given:
ABCD is a rectangle.
AC and BD are diagonals.

To prove:
AC=BD
Proof:

Let $A = 90^0$ (by definition of rectangle)

$\hat{A} + \hat{B} = 180^0$ (consecutive interior angles)

$\hat{A} = \hat{B} = 90^0$

Now in triangles, ABD and ABC,
AB=AB (common side)

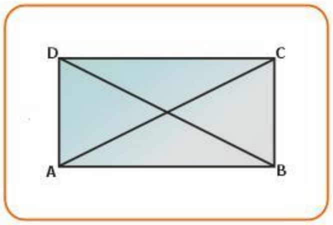
$\hat{A} = \hat{B} = 90^0$ (each angle is a right angle)

AD=BC (opposite sides of parallelogram)

$\therefore \Delta ABD \cong \Delta BAC$

$\therefore$ BD=AC (corresponding parts of corresponding triangles)
Hence the theorem is proved.

Converse of Theorem 4
Statement:
If two diagonals of a parallelogram are equal, it is a rectangle.



Given:
ABCD is a parallelogram in which AC=BD.

To prove:
Parallelogram ABCD is a rectangle.

Proof:
In triangles ABC and DBC,
AB=DC (opposite sides of parallelogram)
BC=BC (common side)
AC=BD (given)

$\therefore \Delta ABC \cong \Delta DCB$ (SSS congruency condition)

$\therefore A\hat{B}C = D\hat{C}B$ (Corresponding parts of corresponding triangles)

But these angles are consecutive interior angles on the same side of transversal BC and AB||DC.

$\therefore A\hat{B}C + D\hat{C}B = 180^0$

$\text{But } A\hat{B}C = D\hat{C}B$

$\therefore \; A\hat{B}C = D\hat{C}B = 90^0$

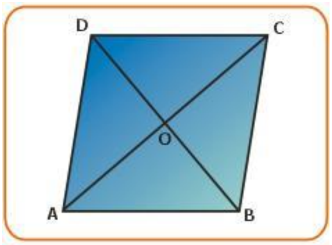
∴ By definition of rectangle, parallelogram ABCD is a rectangle.

Hence the theorem is proved.

Theorem 5

Statement:

The diagonals of a rhombus are perpendicular to each other.



Given:

ABCD is a rhombus. Diagonal AC and BD intersect at O.

To prove:

AC and BD bisect each other at right angles.

Proof:

A rhombus is a parallelogram such that

AB=DC=AD=BC ---(i)

Also the diagonals of a parallelogram bisect each other.

Hence BO=DO and AO=OC ---(ii)

Now compare triangles AOB and AOD,

AB=AD (from (i) above)

BO=DO (from (ii) above)

AO=AO (common side)

$\therefore \; \Delta AOB \cong \Delta AOD$ (SSS congruency condition)

$\therefore \; A\hat{O}B = A\hat{O}D$ (corresponding parts of corresponding parts)

BD is a straight line segment.

$\therefore \; A\hat{O}B + A\hat{O}D = 180^0$

$\text{But } A\hat{O}B = A\hat{O}D \qquad \text{(proved)}$

$A\hat{O}B = A\hat{O}D = \frac{180^0}{2} = 90^0$

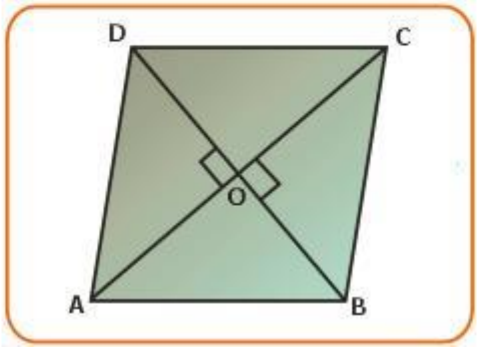
i.e., the diagonals bisect at right angles.

Hence the theorem is proved.

Converse of Theorem 5

Statement:

If the diagonals of a parallelogram are perpendicular then it is a rhombus.



Given:

ABCD is a parallelogram in which AC and BD are perpendicular to each other.

To prove:

ABCD is a rhombus.

Proof:

Let AC and BD intersect at right angles at O.

$A\hat{O}B = 90^0$

In triangles AOD and COD,

AO=OC (diagonals bisect each other)

OD=OD (common side)

$A\hat{O}D = C\hat{O}D = 90^0$ (given)

$\therefore \Delta AOD \cong \Delta COD$ (SAS congruency condition)

AD = DC

i.e., the adjacent sides are equal.

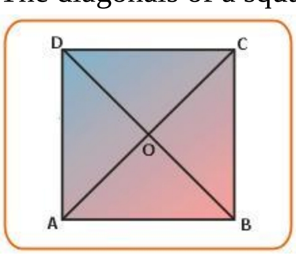
∴ By definition, ABCD is a rhombus.

Hence the theorem is proved.

Theorem 6

Statement:

The diagonals of a square are equal and perpendicular to each other.



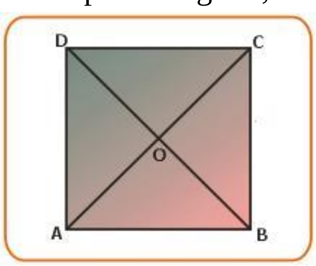
Given:
ABCD is a square.
AC and BD are diagonals intersecting at O.

To prove:
$AC = BD$ and $AC \perp BD$

Proof:
AB=AD (sides of a square are equal)
AB||DC (opposite sides of a square are parallel)
∴ ABCD is parallelogram with consecutive sides equal.

∴ ABCD is a rhombus. (by definition)
Since the diagonals of a rhombus are perpendicular to each other, AC □□BD.
∴ ABCD is a parallelogram.
$AB = AD$ and $\hat{A} = 90^0$
∴ ABCD is a rectangle with a pair of its consecutive sides equal.
Since the diagonals of a rectangle are equal, AC=BD.
∴ Diagonal AC=Diagonal BD and AC □□BD
Hence the theorem is proved.

Converse of Theorem 6
Statement:
If in a parallelogram, the diagonals are equal and perpendicular, then it is a square.



Given:
ABCD is a parallelogram.
AC=BD and AC ∠ BD

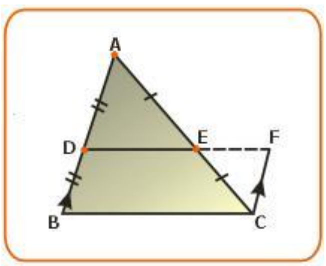
To prove:
ABCD is a square.

Proof:
Since the diagonals AC and BD are equal,

ABCD is a rectangle - - -(i)
(Diagonal property of rectangle)
Since the diagonals are perpendicular to each other.
ABCD is a rhombus.
∴ AB=AD - - -(ii)
ABCD is a rectangle. (from i)
With consecutive sides equal. (from ii)
∴ ABCD is a square. (by definition of a square)
Hence the theorem is proved.

The Mid-point Theorem
Parallel lines and Triangles
So far we have proved various theorems on parallelograms. Let us now apply these theorems to prove a few interesting and useful facts about a triangle.

Statement:
"The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and equal to half of it".



Given:
In Δ ABC, AD=DB and AE=EC

To prove:

i) $DE || BC$

ii) $DE = \frac{1}{2} BC$

Construction:
Analysis for construction shows **Analysis for construction:** that you have to draw CF||BD Think how you can complete to meet DE produced at F. a parallelogram with DB and BC as consecutive sides. You will find the need to draw CF||DB.

Proof:
In triangles, ADE and CEF,
AE=EC (given)

$A\hat{E}D = C\hat{E}F$ (vertically opposite angles)

$D\hat{A}E = E\hat{C}F$ (alternate angles, AD||CF by construction)

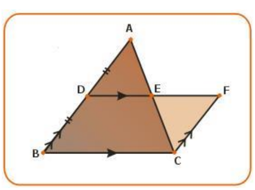
$\therefore \Delta ADE \cong \Delta CFE$ (ASA congruency condition)

∴ AD=CF and DE=EF (corresponding parts of corresponding triangles)
But AD=DB (given)
∴ DB=CF ----(i)
(AD is equal to both DB and CF)
In quadrilateral DBCF,
DB=CF and DB||CF
∴ DBCF is a parallelogram. (By definition of parallelogram)
∴ DF=BC
(opposite sides of a parallelogram are equal)
and DF||BC ----(ii)
But DE = EF (proved above)
And DF=DE+EF
=2 DE
and DF=BC (from (ii))
∴ BC=2 DE

or $DE = \frac{1}{2} BC$

Hence the theorem is proved.

Converse of Mid-point Theorem
Statement:
A straight line drawn through the mid-point of one side and parallel to another side of a triangle bisects the third side.

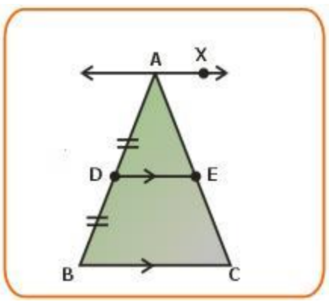


Given:
ΔABC in which D is the mid-point of AB and DE||BC.

To prove:
E is the mid-point of AC. i.e., to prove AE=EC.

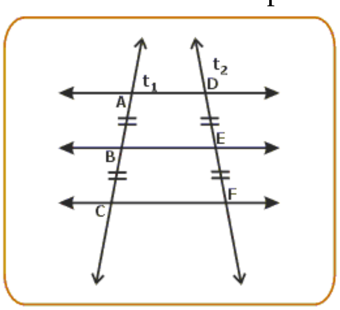
Construction:
Since DE||BC, you can complete a parallelogram with DB and BC as consecutive sides.
Hence draw EF||BD to meet DE produced at F.

Proof:
In quadrilateral DBCF,
DB||CF (by construction)
DF||BC (given)
∴ DBCF is a parallelogram.
∴ DB=CF ----(i) (opposite sides of a parallelogram)
But DB=AD ----(ii) (given)
From (i) and (ii), AD=CF
Now compare triangles, AED and CEF,
AD = CF
$A\hat{E}D = E\hat{C}F$ (vertically opposite angles)
$D\hat{A}E = E\hat{C}F$ (alternate angles, AD||CF)
$\therefore \Delta AED \cong \Delta CEF$ (ASA congruency condition)
∠AE = EC (CPCT)
That is E is the midpoint of AC.
Hence the theorem is proved.
Recall the above **theorem** and apply it to the diagram given.



In the diagram if D is the mid-point of AB and DE is drawn parallel to BC, then E Will be the midpoint of AC i.e., AE = EC.

Now if AX is drawn parallel to BC, then also AE =EC if AD = DB.

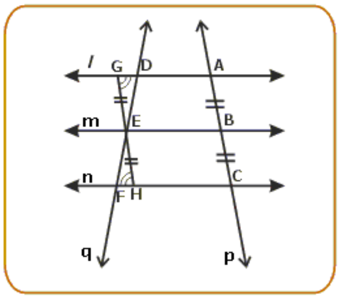


Now draw three parallel lines AB, CD, EF as shown in the figure.
Draw a transversal t_1 such that AB = BC.

Now draw another transversal t_2.
Measure DE and EF. You will find that DE = EF and AB = BC.
In the diagram, AD, BE and CF are three parallel lines.
AB and BC are equal intercepts made on t_1.
If any transversal is drawn, the intercepts made on it will also be equal.

The Intercept Theorem

Statement:
If there are three or more parallel lines and the intercepts made by them on one transversal are equal, the corresponding intercepts of any transversal are also equal.



Given:
l||m||n
P is a transversal intersecting l, m and n at A, B and C respectively such that AB=BC.
Q is another transversal drawn to cut l, m and n at D, E and F respectively. DE and EF are the intercepts made on q.

To prove:
DE = EF

Construction:
Draw a line through E parallel to p intersecting l in G, n in H respectively.

Proof:
AG||BE (given)
GE||AB (by construction)
∴ AGEB is a parallelogram
∴ GE=AB ----(i) (opposite sides of parallelogrm)

Similarly BEHC is a parallelogram.

∴ EH=BC ----(ii) (opposite sides of parallelogrm)
But AB=BC (given)

From (i) and (ii), GE=EH

Now compare triangles GED and EFH,

GE=EH (proved)

$G\hat{E}D = F\hat{E}H$ (vertically opposite angles)

$D\hat{G}E = F\hat{H}E$ (alternate angles, GD||FH)

$\therefore \Delta GED \cong \Delta HEF$ (ASA congruency condition)

$\therefore$ DE=EF (corresponding parts of corresponding triangles)

Hence the theorem is proved.