CHAPTER 08

Moment Distribution Method of Analysis

8.1 Introduction

Moment distribution method is a displacement method of analysis of kinematically indeterminate structure. It is based on stiffness approach. Moment distribution method is not an exact method but a method of successive approximation where any degree of accuracy can be obtained by repeated iteration. This method was suggested by Prof. Hardy cross in early 1930. It is also known as Hardy Cross method of successive approximation.

8.2 Stiffness

Stiffness for a member at a joint is the moment (force) required to produce unit rotation (displacement) at that joint. Stiffness at a joint depends upon end conditions and properties of cross-sections.

Consider a propped cantilever AB as shown in figure.

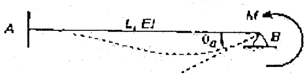


Fig. 8.1

For above propped cantilever, if anticlockwise moment is applied at end B, then rotation at B is given by

$$\theta_B = \frac{ML}{4EI}$$

$$\Rightarrow \quad M = \frac{4EI}{L}\theta_B \quad \text{(moment required to produce } \theta_B)$$

if θ_B = unity, then moment required to produce unit rotation is known as stiffness.

$$\therefore \quad K_{BA} = \frac{4EI}{L} \quad (\because \theta = 1)$$

Now consider a simply supported beam AB as shown in figure

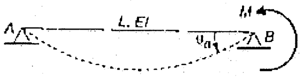


Fig. 8.2

For above beam, if anticlockwise moment M is applied at end B, the rotation at B is given by

$$\theta_B = \frac{ML}{3EI}$$

$$\Rightarrow \quad M = \frac{3EI}{L}\theta_B \qquad \text{(moment required to produce } \theta_B\text{)}$$

if θ_B = unity, then

$$K_{BA} = \frac{3EI}{L} \qquad [\because \theta_B = 1]$$

Therefore, it can be say that stiffness of a member when farther end is fixed is given by $\frac{4EI}{L}$ and stiffness of a member when farther end is hinged is given by $\frac{3EI}{L}$ and stiffness of a member when farther end is free is zero.

Consider a continuous beam shown below.

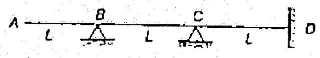


Fig. 8.3

EI is constant.

$$K_{BA} = 0 \qquad \text{(Farther end is free)}$$

$$K_{CB} = \frac{3EI}{L} \qquad \text{(Farther end } B \text{ is hinged)}$$

$$K_{CD} = \frac{4EI}{L} \qquad \text{(Farther end } D \text{ is fixed)}$$

Consider a rigid jointed from shown below.

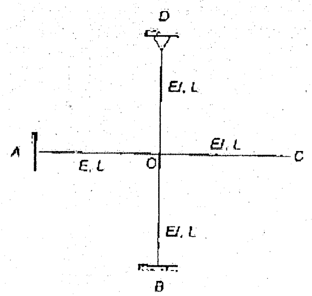


Fig. 8.4

$$K_{OA} = \frac{4EI}{L} \qquad \text{(Farther end } B \text{ is fixed)}$$

$$K_{OB} = \frac{4EI}{L} \qquad \text{(Farther end } B \text{ is fixed)}$$

$$K_{OC} = 0 \qquad \text{(Farther end } C \text{ is free)}$$

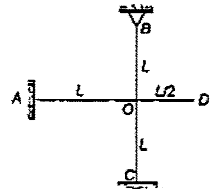
$$K_{OD} = \frac{3EI}{L} \qquad \text{(Farther end D is hinged)}$$

Example 8.1 A steel frame is shown in the given figure.

If joint O of the frame is rigid, the rotational stiffness of the frame at point O is given by

(a) $\frac{11EI}{L}$ (b) $\frac{10EI}{L}$

(c) $\frac{8EI}{L}$ (d) $\frac{6EI}{L}$



EI = constant

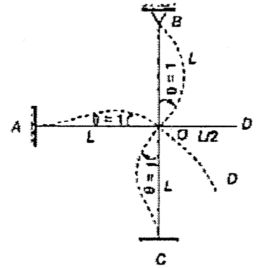
Ans. (a)

The rotational stiffness is the moment required for unit rotation at O.

$$K_0 = K_{OA} + K_{OB} + K_{OC} + K_{OD}$$

$$K_0 = \frac{4EI}{L} + \frac{3EI}{L} + \frac{4EI}{L} + 0$$

$$K_0 = \frac{11EI}{L}$$



8.3 Carry over Factors

It is defined as the ratio of the moment at the fixed far end to the moment at the rotating near end.

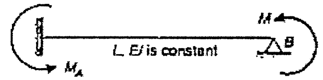


Fig. 8.5

Mathematically:

$$\text{Carry Over Factor} = \frac{\text{Carry over moment at Farther end}}{\text{Applied moment at near end}}$$

$$\text{C.O.F} = \frac{M_A}{M}$$

For above beam,

$$M(x \text{ from } B) = R_B x + M$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = R_B x + M \qquad ...(i)$$

integrating (i), we get

$$EI\frac{dy}{dx} = R_B\frac{x^2}{2} + Mx + C_1 \qquad ...(ii)$$

again integrating (*ii*), we get

$$EI\,y = R_B\frac{x^3}{6} + \frac{Mx^2}{2} + C_1x + C_2 \qquad ...(iii)$$

using end conditions,

at $x = 0$, $y = 0$

$\therefore$ $$0 = 0 + 0 + 0 + C_2$$

$\Rightarrow$ $$C_2 = 0$$

at $x = L$, $y = 0$

$\therefore$ $$0 = R_B\frac{L^3}{6} + \frac{ML^2}{2} + C_1L \qquad ...(iv)$$

at $x = L$, $\frac{dy}{dx} = 0$

$\therefore$ $$0 = \frac{R_BL^2}{2} + ML + C_1 \qquad ...(v)$$

From (*iv*) and (*v*),

$$R_B = \frac{-3}{2}\frac{M}{L}$$

On putting value of R_B into (*v*), we get

$$C_1 = \frac{-ML}{4}$$

On putting value of C_1 and R_B into (*ii*), we get

$$EI\frac{dy}{dx} = \frac{-3}{2}\frac{M}{L}\cdot\frac{x^2}{2} + Mx - \frac{ML}{4}$$

at $x = 0$, $\frac{dy}{dx} = \theta_B$

$$\theta_B = \frac{ML}{4EI}$$

if $\theta_B = 1$, them $M = K$ (Stiffness)

$\therefore$ $$K = \frac{4EI}{L}$$

$$\Sigma M_A = 0$$

$\Rightarrow$ $$M + R_{B\times L} + M_A = 0$$

$\Rightarrow$ $$M_A = \frac{M}{2}$$

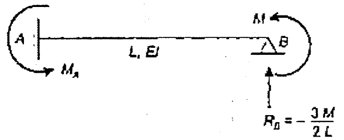


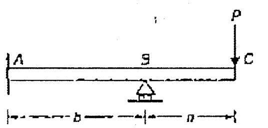
Fig. 8.6

Hence, carry over factor $= \frac{M_A}{M} = \frac{1}{2}$

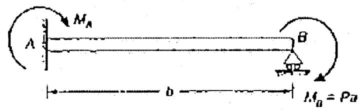
Example 8.2 The magnitude of the bending moment at the fixed support of the beam is equal to

(a) Pa (b) $\frac{Pa}{2}$

(c) Pb (d) $P(a + b)$

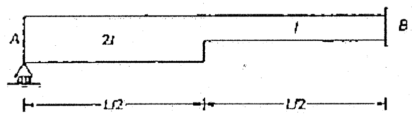


Ans. (b)

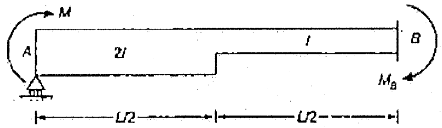


The moment at A will be, $M_A = \frac{M_B}{2} = \frac{Pa}{2}$

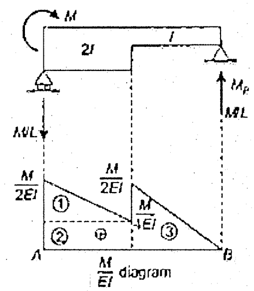
Example 8.3 What is the carry-over factor from A to B while using moment distribution for analysing beam as shown in the figure given below?



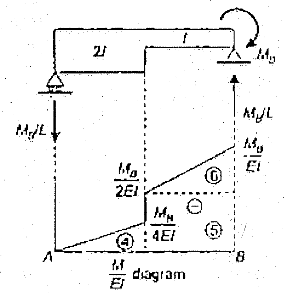
Solution:



Effect of M:



Effect of M_B:



Using area moment theorem,

Deflection at A = Moment of area of $\frac{M}{EI}$ diagram between A and B about A

$\Rightarrow$ $$\Delta_A = A_1\times\frac{1}{3}\times\frac{1}{2} + A_2\times\frac{L}{4} + A_3\times\frac{2}{3}L + A_4\times\frac{2}{3}\times\frac{L}{2} + A_5\times\frac{3L}{4} + A_6\times\frac{5}{6}L$$

$\Rightarrow$ $$0 = \left(\frac{1}{2}\times\frac{L}{2}\times\frac{M}{4EI}\times\frac{L}{2}\right) + \left(\frac{M}{4EI}\times\frac{L}{2}\times\frac{L}{4}\right) + \left(\frac{1}{2}\times\frac{M}{2EI}\times\frac{L}{2}\times\frac{2L}{3}\right)$$

$$-\left(\frac{1}{2}\times\frac{L}{2}\times\frac{M_B}{4EI}\times\frac{L}{3}\right)-\left(\frac{M_B}{2EI}\times\frac{L}{2}\times\frac{3L}{4}\right)-\left(\frac{1}{2}\times\frac{M_B}{2EI}\times\frac{L}{2}\times\frac{5}{6}L\right)$$

$$\Rightarrow \quad 0 = \frac{ML^2}{32EI}+\frac{ML^2}{32EI}+\frac{ML^2}{12}-\frac{M_BL^2}{48}-\frac{3M_BL^2}{16}-\frac{5M_BL^2}{48}$$

$$\Rightarrow \quad \frac{5}{16}M_B = \frac{7}{48}M$$

$$\Rightarrow \quad M_B = \frac{7}{15}M$$

$$\therefore \quad C_{or} = \frac{M_B}{M} = \frac{7}{15}$$

8.4 Distribution Theorem

If a moment which is applied to a structural joint to produce rotation without translation gets distributed to the connecting members at the joint in the proportion of their stiffness or relative stiffness.

$$M_{OA}:M_{OB}:M_{OC}:M_{OD}:M_{OE} = \frac{4E_1I_1}{L_1}:\frac{4E_2I_2}{L_2}:\frac{3E_3I_3}{L_3}:\frac{4E_4I_4}{L_4}:\frac{3E_5I_5}{L_5}$$

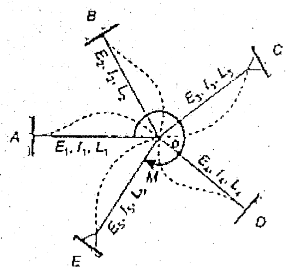


Fig. 8.7

Do You Know? $\frac{3}{4}\frac{I}{L}$ is called the relative stiffness of a member whose farther end is hinged, whereas $\frac{I}{L}$ is relative stiffness of a member whose farther end is fixed.

8.5 Distribution Factor

Distribution factor for a member at a joint is the ratio of the stiffness (or relative stiffness) of the member to the total stiffness (or total relative stiffness) of all the members meeting at that joint

$$DF = \frac{\text{Stiffness of the member}}{\text{Total stiffness of the joint}}$$

or

$$DF = \frac{\text{Relative stiffness of the member}}{\text{Total stiffness of the joint}}$$

Remember: Summation of Distribution factors for all the members at a joint is one.

Consider a rigid jointed Frame as shown in figure below; E = constant and joint O is rigid

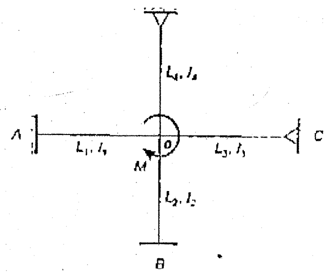


Fig. 8.8

Stiffness of OA, $K_1 = \frac{4EI_1}{L_1}$

Stiffness of OB, $K_2 = \frac{4EI_2}{L_2}$

Stiffness of OC, $K_3 = \frac{3EI_3}{L_3}$

Stiffness of OD, $K_4 = \frac{3EI_4}{L_4}$

Total stiffness of the joint, $\Sigma K = K_1 + K_2 + K_3 + K_4$

$$= \frac{4EI_1}{L_1}+\frac{4EI_2}{L_2}+\frac{3EI_3}{L_3}+\frac{3EI_4}{L_4}$$

DF for member OA at joint O = $\frac{\text{Stiffness of OA at O}}{\text{Total stiffness at joint}}$

$$(DF)_{OA} = \frac{K_1}{\Sigma K}$$

Similarly, DF of member OB at joint $O = \frac{K_2}{\Sigma K}$

DF of member OB at joint $O = \frac{K_3}{\Sigma K}$

DF of member OB at joint $O = \frac{K_4}{\Sigma K}$

Applied moment M is distributed in all members in the proportions of their stiffness. Let M_1, M_2, M_3 and M_4 are the moment distribution in the members OA, OB, OC, OD respectively at joint O.

$$\therefore \quad M_1 = \frac{K_1}{\Sigma K}M$$

$$M_2 = \frac{K_2}{\Sigma K}M$$

$$M_3 = \frac{K_3}{\Sigma K}M$$

$$M_4 = \frac{K_4}{\Sigma K}M$$

Note that, $M_1 + M_2 + M_3 + M_4 = M$

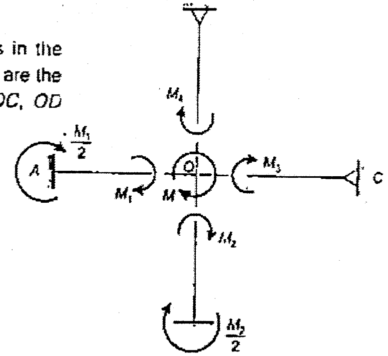
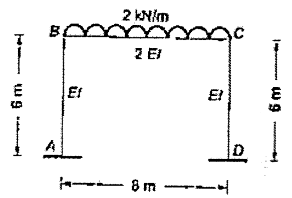


Fig. 8.9

Do you know? Distribution factors are the property of rigid joint, it is not the property of hinge joint. So distribution factor of a hinge joint is always zero.

Example 8.4 For the frame shown below, the distribution factors for members BC and BA at joint B are



(a) 0.4, 0.6
(b) 0.5, 0.5
(c) 0.6, 0.4
(d) 0.7, 0.3

[IES : 2002]

Ans. (c)

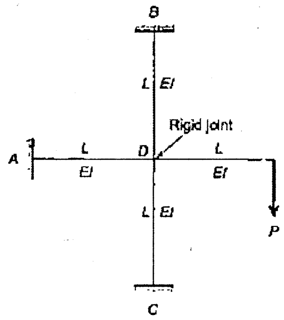
$$(DF)_{BC} = \frac{\text{Stiffness of BC at joint } B}{\text{Total stiffness at joint } B}$$

$$(DF)_{BC} = \frac{\frac{4E(2I)}{8}}{\frac{4E(2I)}{8} + \frac{4EI}{6}} = \frac{EI}{EI + \frac{2}{3}EI} = \frac{3}{5} = 0.6$$

we know summation of *DF* at joint is one

$$\therefore \quad (DF)_{BA} = 1 - 0.6 = 0.4$$

Example 8.5 The given figure shows a frame loaded with a single concentrated load *P*. The fixed-end moment developed at support *A* will be

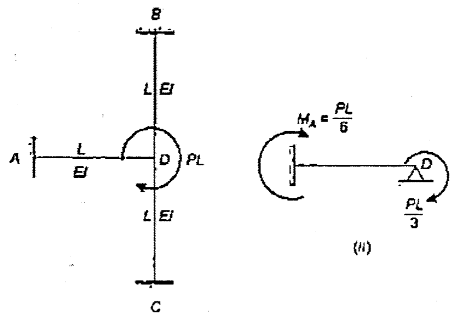


(a) $\frac{PL}{8}$ (b) $\frac{PL}{6}$

(c) $\frac{PL}{4}$ (d) $\frac{PL}{3}$

[IES : 1995]

Ans. (b)



(i) (ii)

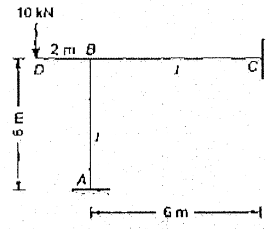
$$(DF)_{DA} = \frac{\frac{4EI}{L}}{\frac{4EI}{L} + \frac{4EI}{L} + \frac{4EI}{L}} = \frac{1}{3}$$

Distribute moment in *DA*, $M_{DA} = \frac{1}{3} \times PL$

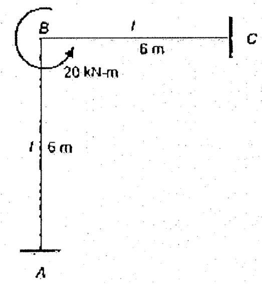
Since farther end is fixed, thus $\frac{1}{2} M_{DA}$ will be transferred at end *A*

$$\therefore \quad M_A = \frac{PL}{6}$$

Example 8.6 Find the value of θ_B for the beam shown in figure below.



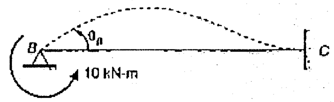
Solution:



DF for *BA* = *BC* = 0.5

Distributed moments in *BC* = *BA* = 0.5 × 20 = 10 kN-m

Consider member *BC* separately.



we know, $\theta_B = \frac{ML}{4EI}$ ↻

Here, M = 10 kN-m, L = 6 m

$\therefore$ $\theta_B = \frac{ML}{4EI}$ ↻

Here, M = 10 kN-m, L = 6 m

$\therefore$ $\theta_B = \frac{10 \times 6}{4EI} = \frac{15}{EI}$ anticlock wise

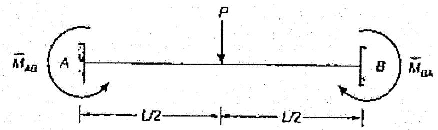
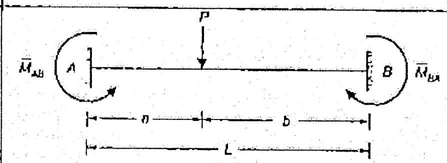
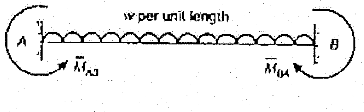
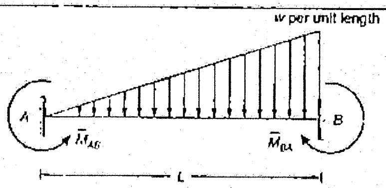
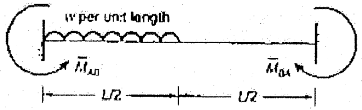
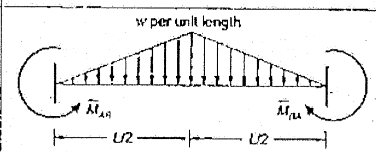
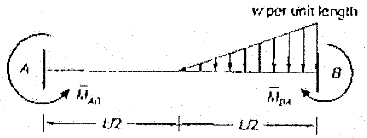
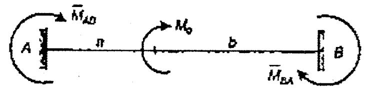
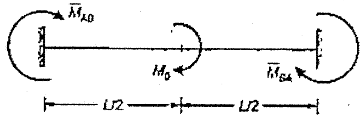
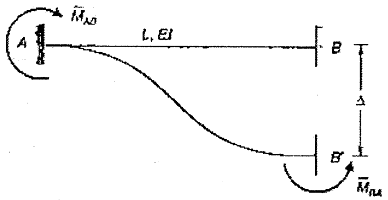
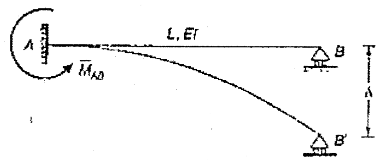
8.6 Fixed End Moments

When both the ends of a beam are fixed, the moments develop at the fixed ends due to applied load are known as fixed end moments.

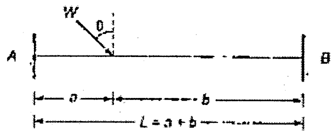
Sign Convention

- All clockwise moments are positive.
- All anticlockwise moments are negative.

8.6.1 Standard Results of Fixed End Moments

S.No.	Standard Cases of Loading	Fixed End Moments
1.		$\bar{M}_{AB} = -\frac{PL}{8}$ $\bar{M}_{BA} = +\frac{PL}{8}$
2.		$\bar{M}_{AB} = -\frac{Pab^2}{L^2}$ $\bar{M}_{BA} = +\frac{Pa^2b}{L^2}$
3.		$\bar{M}_{AB} = -\frac{wL^2}{12}$ $\bar{M}_{BA} = +\frac{wL^2}{12}$
4.		$\bar{M}_{AB} = -\frac{wL^2}{30}$ $\bar{M}_{BA} = +\frac{wL^2}{20}$
5.		$\bar{M}_{AB} = -\frac{11}{192}wL^2$ $\bar{M}_{BA} = -\frac{5}{192}wL^2$
6.		$\bar{M}_{AB} = -\frac{5}{96}wL^2$ $\bar{M}_{BA} = +\frac{5}{96}wL^2$
7.		$\bar{M}_{AB} = -\frac{7}{960}wL^2$ $\bar{M}_{BA} = +\frac{23}{960}wL^2$
8.	 The resultant sign depends on the ratio of $a : L$ and $b : L$ if $a > \frac{L}{3}$, then $\bar{M}_{AB}$ = +ve (clockwise) if $b > \frac{L}{3}$, then $\bar{M}_{BA}$ = +ve (clockwise) it means that if moments acts in middle third strip, then at both fixed ends, moment will be clockwise if applied moment is clockwise and vice-versa.	$\bar{M}_{AB} = \frac{M_0 b(3a-L)}{L^2}$ $\bar{M}_{BA} = \frac{M_0 a(3b-L)}{L^2}$
9.		$\bar{M}_{AB} = +\frac{M_0}{4}$ (in the direction applied moment)
10.	If support B settles by Δ.  **NOTE:** If settlement of support causes rotation of member in clockwise direction then fixed end moment developed will be in anticlockwise in direction or vice-versa.	$\bar{M}_{AB} = -\frac{6EI\Delta}{L^2}$ $\bar{M}_{BA} = -\frac{6EI\Delta}{L^2}$
11.		$\bar{M}_{AB} = -\frac{3EI\Delta}{L^2}$ $\bar{M}_{BA} = 0$

Example 8.7 For the fixed beam as shown in the figure below, what is the fixed end moment at A for the given loading?



(a) $\frac{Wab^2}{L^2}\cos^2\theta$ (b) $\frac{Wa^2b}{L^2}\cos\theta$

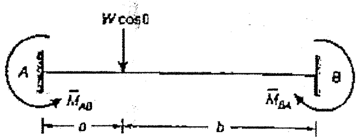
(c) $\frac{Wab^2}{L^2}\cos\theta$ (d) $\frac{Wa^2b}{L^2}\cos^2\theta$

[IES : 2006]

Ans. (c)

Vertical component of Load $= W\cos\theta$

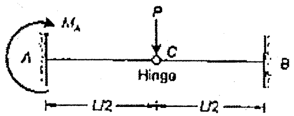
The fixed end moment at A,



$$\bar{M}_{AB} = \frac{Pab^2}{L^2}$$

$$\bar{M}_{AB} = \frac{W\cos\theta\, ab^2}{L^2} \Rightarrow \frac{Wab^2}{L^2}\cos\theta$$

Example 8.8 The fixed beam AB has a hinge C at mid span. A concentrated load P is applied at C. What is the fixed end moment M_A?



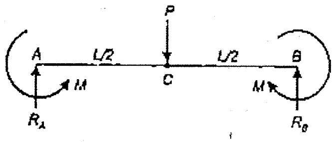
(a) PL (b) $\frac{PL}{2}$

(c) $\frac{PL}{4}$ (d) $\frac{PL}{8}$

[IES : 2008]

Ans. (c)

Since loading is symmetrical



$\therefore$ $M_A = M_B = M$

$\Sigma M_B = 0$

$\Rightarrow$ $R_A \times L - \frac{PL}{2} = 0$

$\Rightarrow$ $R_A = \frac{P}{2}$

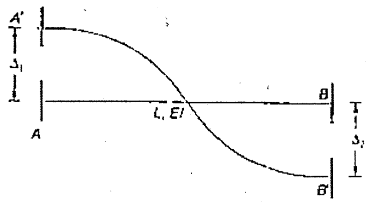
Also, $M_C = 0$ (From left)

$\Rightarrow$ $R_A \times \frac{L}{2} - M = 0$

$\Rightarrow$ $M = R_A \times \frac{L}{2} = \frac{P}{2} \times \frac{L}{2}$

$\Rightarrow$ $M = \frac{PL}{4}$

Example 8.9 For the beam shown in figure supports A and B settles Δ_1 upward and Δ_2 downward then fixed end moment developed at supports will be



(a) $\frac{6EI(\Delta_1 + \Delta_2)}{L^2}$ (b) $\frac{3EI(\Delta_1 + \Delta_2)}{L^2}$

(c) $\frac{6EI(\Delta_1 - \Delta_2)}{L^2}$ (d) $\frac{3EI(\Delta_1 - \Delta_2)}{L^2}$

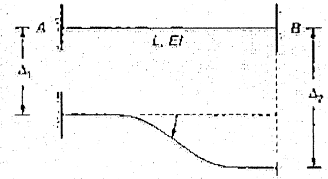
Ans. (a)

$$\bar{M}_{AB} = \bar{M}_{BA} = M = (\bar{M})_{\text{due to }\Delta_1} + (\bar{M})_{\text{due to }\Delta_2}$$

$\therefore$ $$M = \frac{6EI\Delta_1}{L^2} + \frac{6EI\Delta_2}{L^2}$$

$$M = \frac{6EI(\Delta_1 + \Delta_2)}{L^2}$$

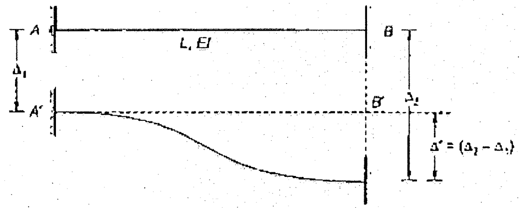
Example 8.10 If support A settles by Δ_1 and B settles by Δ_2 downward ($\Delta_2 > \Delta_1$) then fixed end moment will be



(a) $\frac{6EI(\Delta_1 + \Delta_2)}{L^2}$ (b) $-\frac{6EI(\Delta_2 - \Delta_1)}{L^2}$

(c) $\frac{3EI(\Delta_1 + \Delta_2)}{L^2}$ (d) $\frac{3EI(\Delta_2 - \Delta_1)}{L^2}$

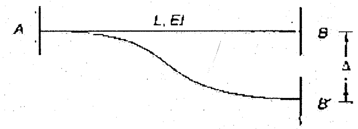
Ans. (b)



$$\bar{M}_{AB} = \bar{M}_{BA} = \frac{6EI\Delta'}{L^2}$$

$$= \frac{-6EI(\Delta_2 - \Delta_1)}{L^2}$$

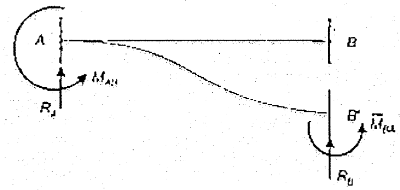
Example 8.11 Beam shown in figure fixed at both ends. If one of the support B settle down by Δ then shear force in beam AB will be



(a) Zero (b) $\frac{6EI\Delta}{L^3}$

(c) $\frac{12EI\Delta}{L^2}$ (d) $\frac{12EI\Delta}{L^3}$

Ans. (d)



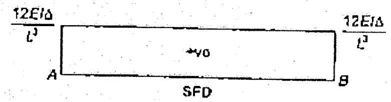
$\Sigma M_B = 0$

$\Rightarrow$ $R_A \times L - \frac{6EI\Delta}{L^2} - \frac{6EI\Delta}{L^2} = 0$

$\Rightarrow$ $R_A = \frac{12EI\Delta}{L^3}$

$\Rightarrow$ S_x (x from A) $= +R_A$

$$S_t = +\frac{12EI\Delta}{L^3} \quad \text{(constant)}$$



8.7 Bending Moment Diagram Using Stiffness Approach

Step-1. Consider a fixed beam AB carrying a point load P at mid span as shown in figure.

Find fixed end moment at ends A and B and then above beam can be considered as combined effect of following loading separately.

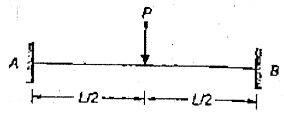
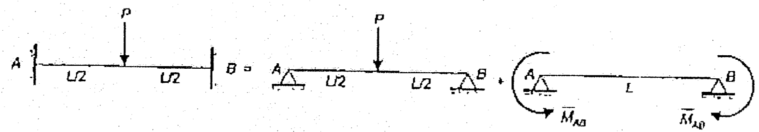


Fig. 8.10



(i) Simply supported beam

(ii) Simply supported beam subjected to end moment only

Fig. 8.11

Step-2. Draw BMD for simply supported Beam (Free BMD) and fixed beam subjected to fixed end moment (fixed BMD) separately

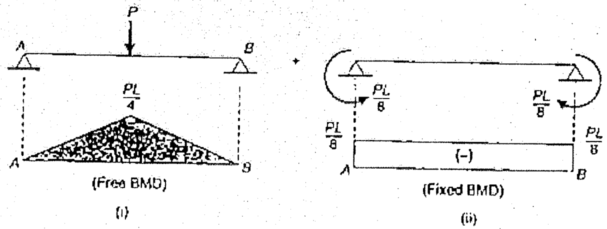


Fig. 8.12

For fixed BMD, Hogging end moments are taken Negative while plotting BMD and vice-versa.

Step-3. The resultant BMD is obtained by superimposing Free BMD and Fixed BMD in such a way that the common BMD area get canceled and resulting area represents final BMD for given loading

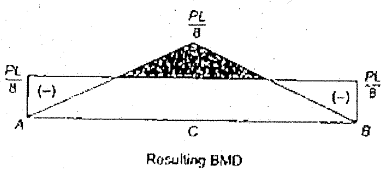
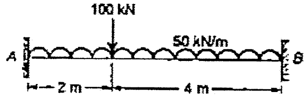


Fig. 8.13

Example 8.12 Draw bending moment diagram for following beam.



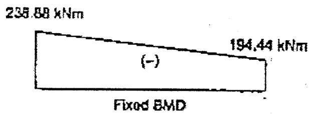
Solution:

(i) Fixed end moments:

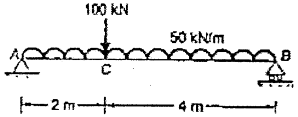
$$\bar{M}_{AB} = -\frac{WL^2}{12} - \frac{Pab^2}{L^2} = -\frac{50\times 6^2}{12} - \frac{100\times 2\times 4^2}{6^2} = -238.88 \text{ kN-m}$$

$$\bar{M}_{BA} = \frac{WL^2}{12} + \frac{Pa^2b}{L^2} = \frac{50\times 6^2}{12} + \frac{100\times 2^2\times 4}{6^2} = 194.44 \text{ kN-m}$$

(ii) Fixed end BMD:



(iii) Free BMD:



Let R_A and R_B the vertical reactions at simply supported ends A and B respectively.

$$\Sigma F_y = 0$$

$\Rightarrow \quad R_A + R_B = 100 + 50\times 6$

$\Rightarrow \quad R_A + R_B = 400 \quad \ldots(i)$

$$\Sigma M_B = 0$$

$\Rightarrow \quad R_A\times 6 - 100\times 4 - 50\times 6\times 3 = 0$

$\Rightarrow \quad R_A = 216.67 \text{ kN } (\uparrow)$

From equation (*i*), $\quad R_B = 183.33 \text{ kN } (\uparrow)$

Portion *AC*:

$$M_x(x \text{ from } A) = R_A\cdot x - W\frac{x^2}{2} \qquad [0 \le x \le 2]$$

$$M_x = 216.67x - 25x^2 \text{ (Parabolic)}$$

at $x = 0$, $M_A = 0$

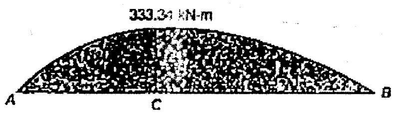
at $x = 2$, $M_C = 333.34$ kN-m

Portion *BC*:

$$M_x(x \text{ from } B) = R_B\cdot x - \frac{Wx^2}{2} \qquad [0 \le x \le 4]$$

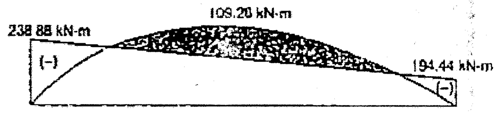
$$M_x = 183.33x - 25x^2 \quad \text{(Parabolic)}$$

at $x = 0$, $M_B = 0$
at $x = 4$, $M_C = 333.34$ kN-m



(Free BMD)

(*iv*) **Resulting BMD:** Resulting BMD can be obtained by superimposing fixed BMD and free BMD.



(Resulting BMD)

8.8 Procedure of Analysis by Moment Distribution Method

Step-1. Find fixed end moment for each member considering each end to be fixed.
Sign convention for fixed end moment:
Clockwise moment = +ve
Anticlockwise moment = –ve

Step-2. Find distribution factors for all members meeting at a Joint. Each joint is considered rigid.

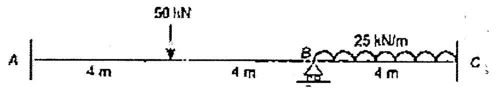
Step-3. Find unbalance moment at each joint. Distribute the balancing moment at each joint according to their distribution factor and transfer carry over moments to their farther ends if farther ends are fixed.

Step-4. Find final end moments at the ends when all joints are balanced.

Step-5. Draw BMD for given loading by stiffness approach as discussed in previous article.

Sign conventions:
+ve moment = sagging
–ve moment = hogging

Example 8.13 For the beam shown in figure analyse and draw Bending moment diagram using moment distribution method.

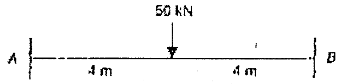


EI is constant.

Solution:

Fixed end moment:

Consider each span as a fixed beam and find corresponding fixed end moments.

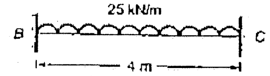


$$\bar{M}_{AB} = -\frac{PL}{8} = \frac{-50\times 8}{8} = -50 \text{ kN-m}$$

$$\bar{M}_{BA} = +\frac{PL}{8} = \frac{50\times 8}{8} = +50 \text{ kN-m}$$

$$\bar{M}_{BC} = -\frac{wL^2}{12} = \frac{-25\times 4^2}{12} = -33.33 \text{ kN-m}$$

$$\bar{M}_{CB} = +\frac{wL^2}{12} = \frac{+25\times 4^2}{12} = +33.33 \text{ kN-m}$$



Distribution Factors

Joint	Member	Stiffness	Total Stiffness	D.F.
B	BA	$\frac{4EI}{8}$	$\frac{12EI}{8}$	$\frac{1}{3}$
	BC	$\frac{4EI}{4}$		$\frac{2}{3}$

OR

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{8}$	$\frac{3I}{8}$	$\frac{1}{3}$
	BC	$\frac{I}{4}$		$\frac{2}{3}$

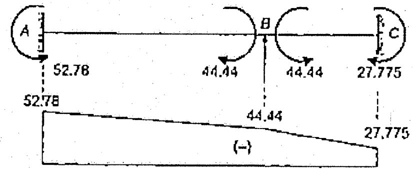
	A	B (1/3)	B (2/3)	C
F.E.M.	–50	+50	–33.33	+33.33
Balancing moment		–5.56	–11.11	
C.O.M.	–2.78			–5.555
Final end moments	–52.78	+44.44	–44.44	+27.775

At joint *B* resultant moment is 16.67 kN-m which is unbalance. Hence a balancing moment of –16.67 kN-m is applied at joint *B* in order to make net moment at joint zero. By the distribution theorem balancing moment get distributed in the proportions of their *DF*

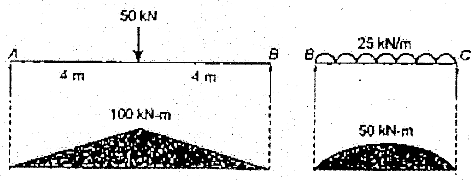
Since Farther ends A and B are fixed so a carry over moment transfer from joint to ends A and C.

$$\text{Carry over moment at } A = \frac{-5.56}{2} = -2.78 \text{ kN-m}$$

$$\text{Carry over moment at } C = \frac{-11.11}{2} = -5.555 \text{ kN-m}$$

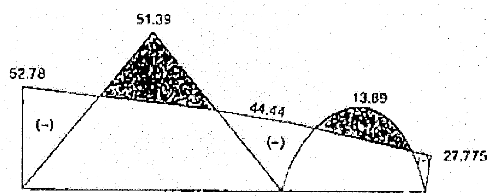


End moment diagram

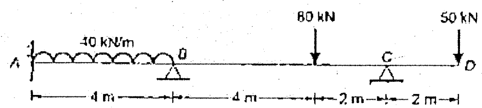


(Free BMD)

Resultant Bending Moment Diagram



Example 8.14 Analyse the continuous beam as shown in figure below by moment distribution method. Draw the Bending moment and elastic curve.



EI is constant.

Solution:

Fixed End Moments:

$$\bar{M}_{AB} = -\frac{wL^2}{12} = -\frac{40\times 4^2}{12} = -53.33 \text{ kN-m}$$

$$\bar{M}_{BA} = +\frac{wL^2}{12} = +\frac{40\times 4^2}{12} = +53.33 \text{ kN-m}$$

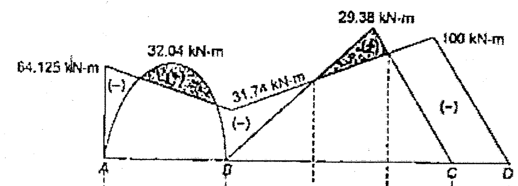
$$\bar{M}_{BC} = +\frac{Pab^2}{L^2} = -\frac{80\times 4\times 2^2}{6^2} = -35.56 \text{ kN-m}$$

$$\bar{M}_{CB} = +\frac{Pa^2b}{L^2} = +\frac{80\times 4^2\times 2}{6^2} = +71.11 \text{ kN-m}$$

$$\bar{M}_{CD} = -50\times 2 = -100 \text{ kN-m}$$

Joint	Member	Stiffness	Total Stiffness	D.F.
B	BA	$\frac{4EI}{4}$	1.5EI	0.67
	BC	$\frac{3EI}{6}$		0.33
C	CB	$\frac{4EI}{6}$	0.67EI	1
	CD	0		0

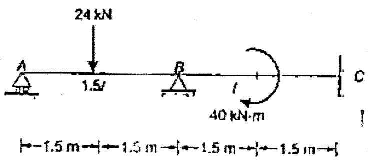
	A	B		C		D
		1	0	1	0	
F.E.M.	-53.33	+53.33	-35.56	+71.11	-100	0
Balancing moment		-11.90	-5.87	+28.89		
C.O.M.	-5.95		+14.45			
Balancing moment		-9.69	-4.76			
C.O.M.	-4.845					
Final End moment	-64.125	31.74	-31.74	100	-100	0



BMD


Elastic Curve

Example 8.15 Analyse the beam loaded as shown in figure below using moment distribution method. Portion *AB* has a second moment of area 1.5 *I* and *BC* has this value as *I*. Draw the BMD and SFD.



Solution:

Fixed end moments:

$$\bar{M}_{AB} = +\frac{P \times L}{8} = \frac{-24 \times 3}{8} = -9 \text{ kN-m}$$

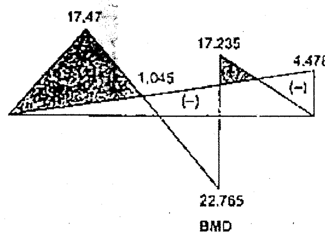
$$\bar{M}_{BA} = +\frac{PL}{8} = \frac{24 \times 3}{8} = +9 \text{ kN-m}$$

$$\bar{M}_{BC} = \frac{M_0}{4} = \frac{40}{4} = +10 \text{ kN-m}$$

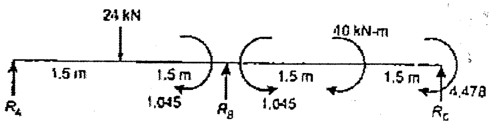
$$\bar{M}_{CB} = \frac{M_0}{4} = \frac{40}{4} = +10 \text{ kN-m}$$

Joint	Member	Stiffness	Total Stiffness	D. F.
B	BA	$\frac{3E(1.5I)}{3}$	2.83EI	0.53
	BC	$\frac{4EI}{3}$		0.47

	A	B (0.53)	B (0.47)	C
F.E.M.	−9	+9	+10	+10
Bal. Moment	+9	−10.07	−8.93	
C.O.M.		+4.5		−4.465
Bal. Moment		−2.385	−2.115	
C.O.M.				−1.057
Final end moments	0	+1.045	−1.045	+4.478



For SFD:



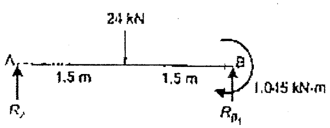
Consider free body diagram of member *AB*.

$$\Sigma M_B = 0$$

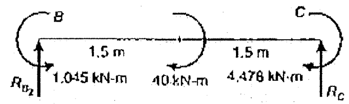
$\Rightarrow \quad R_A \times 3 + 1.045 - 24 \times 1.5 = 0$

$\Rightarrow \quad R_A = \dfrac{24 \times 1.5 + 1.045}{3}$

$= 12.34 \text{ kN} (\uparrow)$



Consider free body diagram of member *BC*.



$$\Sigma M_B = 0$$

$\Rightarrow \quad -R_C \times 3 + 40 + 4.478 - 1.045 = 0$

$\Rightarrow \quad R_C = \dfrac{-1.045 + 40 + 4.478}{3}$

$= +14.47 \text{ kN} (\uparrow)$

Take, $\quad \Sigma F_y = 0$

$\Rightarrow \quad R_A + R_B + R_C = 24 \text{ kN}$

$\Rightarrow \quad R_B = 24 - R_A - R_C$

$\Rightarrow \quad R_B = 24 - 12.34 - 14.47$

$\Rightarrow \quad R_B = -2.81 \text{ kN}(\downarrow)$

Reaction R_B can also be found as

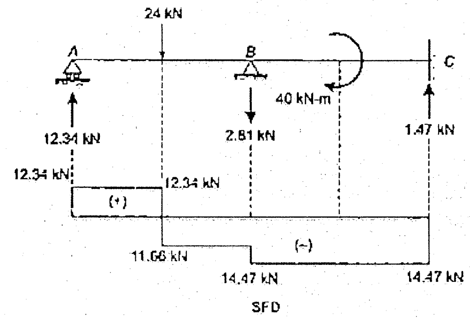
$$R_B = R_{B_1} + R_{B_2} \quad \text{...(i)}$$

Since, $\quad R_{B_1} = 24 - 12.34 = 11.66 \text{ kN} (\uparrow)$ and

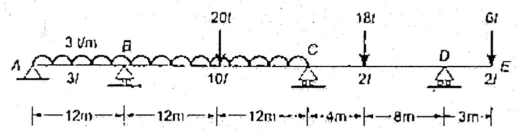
$R_{B_2} = -R_C = -14.47 \text{ kN} (\downarrow)$

$\therefore \quad R_B = 11.66 - 14.47 \text{ kN}$

$\Rightarrow \quad R_B = -2.81 \text{ kN}$



Example 8.16 Analyse the continuous beam shown in figure by moment distribution method. Also draw shear force and bending moment diagram.



Solution:

Fixed End Moments:

$$\bar{M}_{AB} = \frac{-3 \times 12^2}{12} = -36 \text{ t-m}$$

$$\bar{M}_{BA} = \frac{+3 \times 12^2}{12} = +36 \text{ t-m}$$

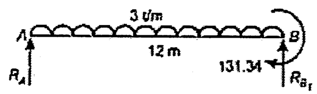
$$\bar{M}_{BC} = \frac{-3\times 24^2}{12} + \left(\frac{-20\times 24}{8}\right)$$
$$= -204 \text{ t-m}$$
$$\bar{M}_{CB} = +204 \text{ t-m}$$
$$\bar{M}_{CD} = \frac{-18\times 4\times 8^2}{12^2} = -32 \text{ t-m}$$
$$\bar{M}_{DC} = \frac{+18\times 8\times 4^2}{12^2} = +16 \text{ t-m}$$
$$\bar{M}_{DE} = -6\times 3 = -18 \text{ t-m}$$

Joint	Member	Stiffness	Total Stiffness	D. F.
B	BA	$3\times\frac{3I}{12} = 0.75I$	2.42I	0.31
	BC	$\frac{4\times 10I}{24} = 1.67I$		0.69
C	CB	$\frac{4\times 10I}{24} = 1.67I$	2.17I	0.77
	CD	$3\times\frac{2I}{12} = 0.5I$		0.23
D	DC	$\frac{4\times(2I)}{12} = 0.67I$	0.67I	1
	DE	0		0

	A	B (0.31)	B (0.69)	C (0.77)	C (0.23)	D (1)	D (0)	E
F.E.M	−36	+36	−204	+204	−32	+16	−18	0
Balancing moment	+36	+52.08	+115.92	−132.44	−39.56	+2		
C.O.M		+18	−66.22	+57.96	+1			
Balancing moment		+14.95	+33.27	−45.34	−13.57			
C.O.M			−22.67	+16.64				
Balancing moment		+7.03	+15.64	−12.81	−3.83			
C.O.M			−6.41	+7.82				
Balancing moment		+1.99	+4.42	−6.02	−1.82			
C.O.M			−3.01	+2.21				
Balancing moment		+0.93	+2.08	−1.70	−0.51			
C.O.M			−0.85	+1.04				
Balancing moment		+0.25	+0.60	−0.80	−0.24			
C.O.M			−0.40	+0.30				
Balancing moment		+0.12	+0.28	−0.24	−0.06			
C.O.M			−0.12	0.14				
Balancing moment		+0.04	+0.08	−0.11	−0.03			
Final End Moment		+131.34	−131.34	+90.61	−90.61	+18	−18	0

Support Reactions

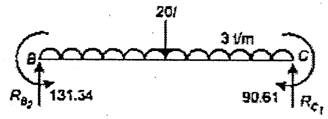
Span *AB*:



$$\Sigma M_B = 0$$
$$\Rightarrow \quad R_A\times 12 - \frac{3\times 12^2}{2} + 131.34 = 0$$
$$\Rightarrow \quad R_A = \frac{1}{12}\left[\frac{3\times 12^2}{2} - 131.34\right] = 7.055\ t$$

Also,
$$\Sigma M_A = 0$$
$$\Rightarrow \quad R_{B_1}\times 12 - 131.34 - 3\times 12\times 6 = 0$$
$$\Rightarrow \quad R_{B_1}\times 12 - 131.34 - 3\times 12\times 6 = 0$$
$$\Rightarrow \quad R_{B_1} = \frac{1}{12}[131.34 + 3\times 12\times 6] = 28.945\ t$$

Span *BC*:

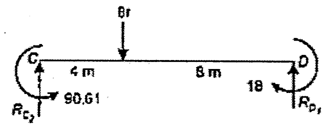


$$\Sigma M_c = 0$$
$$\Rightarrow R_{B_2}\times 24 - 131.34 - 3\times\frac{24^2}{2} - 20\times 12 + 90.61 = 0$$
$$\Rightarrow \quad R_{B_2} = 47.697\ t$$

So,
$$R_B = R_{B_1} + R_{B_2}$$
$$= 28.945 + 47.697 = 76.642\ t\ (\uparrow)$$

Also,
$$\Sigma M_B = 0$$
$$\Rightarrow R_{C_1}\times 24 - 90.61 - 20\times 12 - \frac{3\times 24^2}{2} + 131.24 = 0$$
$$\Rightarrow \quad R_{C_1} = \frac{1}{24}\left[90.61 + 240 + \frac{3\times 24^2}{2} - 131.34\right] = 44.303$$

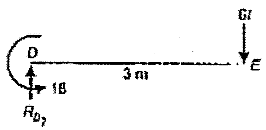
Span *CD*:



$$\Sigma M_D = 0$$
$$\Rightarrow \quad R_{C_2}\times 12 - 90.61 - 18\times 8 + 18 = 0$$
$$\Rightarrow \quad R_{C_2} = \frac{1}{12}[90.61 + 18\times 8 - 18] = 18.05$$
$$\therefore \quad R_C = R_{C_1} + R_{C_2} = 44.303 + 18.05 = 62.353\ t\ (\uparrow)$$

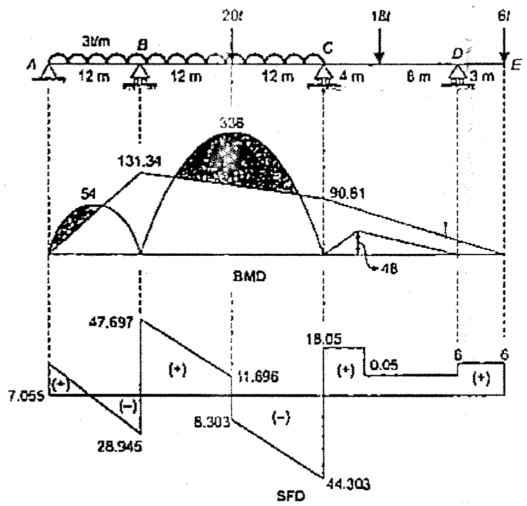
Also,
$$\Sigma M_C = 0$$
$$\Rightarrow \quad R_{D_1}\times 12 - 18 - 18\times 4 + 90.61 = 0$$
$$\Rightarrow \quad R_{D_1} = \frac{1}{12}[18 + 72 - 90.61] = -0.05t(\downarrow)$$

Span DE:

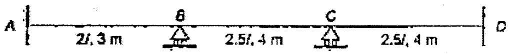


$$\therefore \quad R_{D_2} = 6\ t\ \uparrow$$

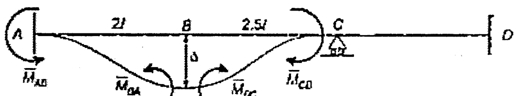
Now,
$$R_D = R_{D_1} + R_{D_2}$$
$$= -0.05 + 6 = 5.95\ t\ (\uparrow)$$



Example 8.17 Determine the support moment if support B settles by 5 mm. Take $EI = 45000$ kN-m²



Solution:



$$\bar{M}_{AB} = -\frac{6E(2I)\Delta}{L^2} = \frac{-6E \times 2I \times 5 \times 10^{-3}}{3^2} = -300 \text{ kN-m}$$

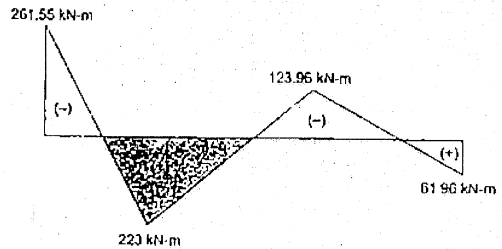
$$\bar{M}_{BA} = \bar{M}_{AB} = -300 \text{kN-m}$$

$$\bar{M}_{BC} = \frac{6E(2.5I)\Delta}{4^2} = \frac{6E \times 2.5I \times 5 \times 10^{-3}}{4^2} = 211 \text{ kN-m}$$

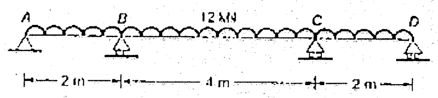
$$\bar{M}_{CB} = \bar{M}_{BC} = 211 \text{ kN-m}$$

Joint	Member	Stiffness	Total Stiffness	D.F.
B	BA	$\frac{2I}{3}$	1.29I	0.51
	BC	$\frac{2.5I}{4}$		0.49
C	CB	$\frac{2.5I}{4}$	1.25I	0.5
	CD	$\frac{2.5I}{4}$		0.5

	A	B (0.51)	B (0.49)	C (0.5)	C (0.5)	D
F.E.M	−300	−300	+211	+211	0	0
Balancing moment		+45.39	+43.61	−105.5	−105.5	
C.O.M	+22.70		−52.75	+21.80		−52.75
Balancing moment		+26.91	+25.84	−10.9	−10.9	
C.O.M	−13.45		−5.45	+12.92		−5.45
Balancing moment		+2.78	+2.67	−6.46	−6.46	
C.O.M	+1.39		−3.23	+1.335		−3.23
Balancing moment		+1.65	+1.58	−0.667	−0.667	
C.O.M	+0.825		−3.23	+1.335		−0.333
Balancing moment		+0.17	+0.163	−0.667	−0.395	
C.O.M	+0.085		−0.1973	+0.081		−0.197
Balancing moment		+0.099	+0.098	−0.395	−0.04	
Final End Moment	−261.55	−223	+223	+123.96	−123.96	−61.96



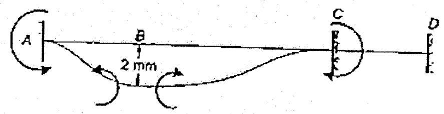
Example 8.18 Draw bending moment diagram for the continuous beam shown in figure if the support B and C sink by 2 mm and 5 mm respectively $I_{AB} = I_{CD} = 2 \times 10^7$ mm⁴, $I_{BC} = 4 \times 10^7$ mm⁴ and $E = 200$ kN/mm²



Solution:

Fixed end moments:

Effect of sinking of support *B*.



Span *AB*:

$$\bar{M}_{AB} = -\frac{wL_{AB}^2}{12} - \frac{6EI_{AB}\Delta_B}{L_{AB}^2}$$

$$= -\frac{12 \times 2^2}{12} - \frac{6 \times 200 \times 2 \times 10^7 \times 2}{2^2 \times 10^9}$$

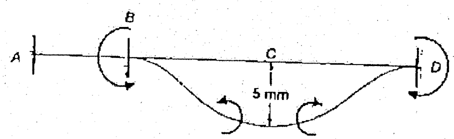
$$= -4 - 12 = -16 \text{ kN-m}$$

$$\bar{M}_{BA} = +\frac{wL_{AB}^2}{12} - \frac{6EI_{AB}\Delta_B}{L_{AB}^2}$$

$$= +4 - 12 = -8 \text{ kN-m}$$

Span *BC*:

Effect of sinking of support *C*.



$$\bar{M}_{BC} = -\frac{wL_{BC}^2}{12} - \frac{6EI_{BC}\Delta_B}{L_{BC}^2}$$

$$= -\frac{12 \times 4^2}{12} + \frac{6 \times 200 \times 4 \times 10^7 \times 2}{4^2 \times 10^9} - \frac{6 \times 200 \times 4 \times 10^7 \times 5}{4^2 \times 10^9}$$

$$= -16 + 6 - 15 = -25 \text{ kN-m}$$

$$\bar{M}_{CB} = +\frac{wL_{BC}^2}{12} + \frac{6EI_{BC} \times \Delta_C}{L_{BC}^2} - \frac{6EI_{BC} \times \Delta_C}{L_{BC}^2}$$

$$= +16 + 16 - 15 = +7 \text{ kN-m}$$

$$\bar{M}_{CB} = +\frac{wL_{BC}^2}{12} + \frac{6EI_{BC} \times \Delta_B}{L_{BC}^2} - \frac{6EI_{BC} \times \Delta_C}{L_{BC}^2}$$

$$= +16 + 16 - 15 = +7 \text{ kN-m}$$

Span *CD*:

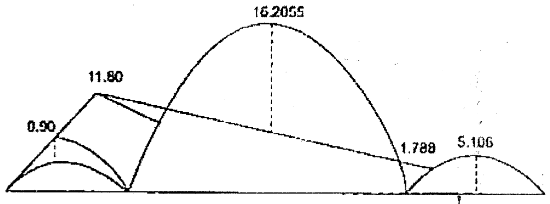
$$\bar{M}_{CD} = -\frac{wL_{CD}^2}{12} + \frac{6EI_{CD} \times \Delta_C}{L_{CD}^2}$$

$$= -\frac{12 \times 2^2}{12} + \frac{6 \times 200 \times 2 \times 10^7 \times 5}{2^2 \times 10^9}$$

$$= -4 + 30 = 34 \text{ kN-m}$$

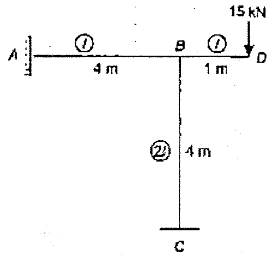
Joint	Member	Stiffness	Total Stiffness	D.F.
B	*BA*	$\frac{3EI}{2}$	3.5*EI*	$\frac{3}{7}$
	BC	$\frac{4E(2I)}{4}$		$\frac{4}{7}$
C	*CB*	$\frac{4E(2I)}{4}$	3.5*EI*	$\frac{4}{7}$
	CD	$\frac{3EI}{2}$		$\frac{3}{7}$

	A	*B* ($\frac{3}{7}$)	*B* ($\frac{4}{7}$)	*C* ($\frac{4}{7}$)	*C* ($\frac{3}{7}$)	
F.E.M	−16	−8	−25	+7	+26	+34
End Correction	+16	+8			−17	−34
Corrected F.E.M		0	−25	+7	+9	0
Balancing moment		10.72	14.28	−9.15	−6.85	
C.O.M			−4.575	+7.14		
Balancing moment		+1.961	+2.614	−4.08	−3.06	
C.O.M			−2.04	1.307		
Balancing moment		+0.875	+1.165	−0.746	−0.561	
C.O.M			−0.373	0.582		
Balancing moment		+0.16	+0.213	−0.332	−0.25	
C.O.M			−0.166	0.106		
Balancing moment		+0.072	+0.094	−0.060	−0.046	
C.O.M			−0.03	+0.047		
Balancing moment		+0.013	0.017	−0.026	−0.021	
Final End Moment	0	+13801	−13801	+1.788	−1.788	0



BMD

Example 8.19 Analyse the Frame shown in figure. Also draw BMD.



Solution:

Fixed end moments:

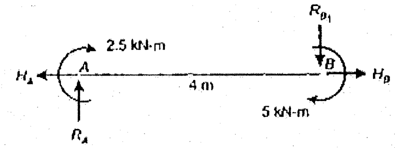
$$\bar{M}_{AB} = \bar{M}_{BA} = 0$$
$$\bar{M}_{BC} = \bar{M}_{CB} = 0$$
$$\bar{M}_{BD} = -15 \times 1 = -15$$
$$\bar{M}_{DB} = 0$$

Joint	Member	Stiffness	Total Stiffness	D.F.
B	BA	$\frac{I}{4}$	$\frac{3I}{4}$	$\frac{1}{3}$
	BC	$\frac{2I}{4}$		$\frac{2}{3}$
	BD	0		0

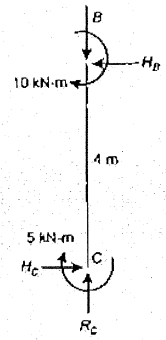
	A	B ($\frac{1}{3}$)	B ($\frac{2}{3}$)	C	B (0)	D
F.E.M	0	0	0	0	−15	0
Balancing Correction		+5	+10		0	
C.O.M	+2.5			+5		
Final End Moments	+2.5	+5	+10	+5	−15	0

Support Reactions:

Consider free body diagram of member *AB*,



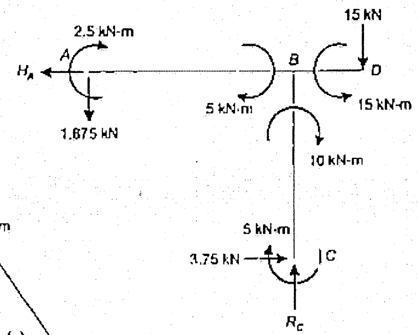
$$\Sigma F_x = 0$$
$$\Rightarrow \quad H_A = H_B \quad ...(i)$$
$$\Sigma F_y = 0$$
$$\Rightarrow \quad R_A = R_{B_1} \quad ...(ii)$$
$$\Sigma M_B = 0$$
$$\Rightarrow \quad R_A \times 4 + 2.5 + 5 = 0$$
$$R_A = -\frac{7.5}{4} = -1.875 \text{ kN}(\downarrow)$$



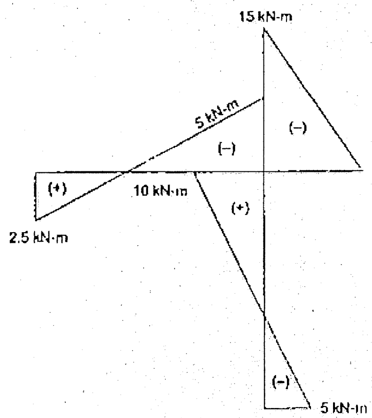
Consider free body diagram of member *BC*,

$$\Sigma M_B = 0$$
$$\Rightarrow \quad 5 + 10 - H_C \times 4 = 0$$
$$\Rightarrow \quad H_C = 3.75 \text{ kN} (\rightarrow)$$

Consider free body equilibrium of entire frame,

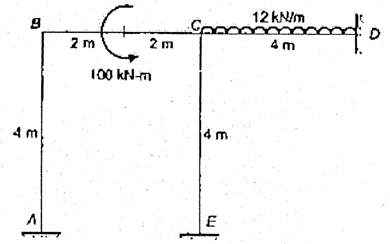


$$\Sigma F_x = 0$$
$$\Rightarrow \quad H_A = 3.75 \text{ kN} (\leftarrow)$$
$$\Sigma F_y = 0$$
$$\Rightarrow \quad R_C - 1.875 - 15 = 0$$
$$\Rightarrow \quad R_C = 16.875 \text{ kN} (\uparrow)$$



BMD

Example 8.20 Draw the bending moment diagram for the loaded frame shown in figure. *EI* is constant.



Solution:

Fixed end moments:

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{4}$	$\frac{2I}{4}$	$\frac{1}{2}$
	BC	$\frac{I}{4}$		$\frac{1}{2}$
C	CB	$\frac{I}{4}$	$\frac{3I}{4}$	$\frac{1}{3}$
	CD	$\frac{I}{4}$		$\frac{1}{3}$
	CE	$\frac{I}{4}$		$\frac{1}{3}$

$$\bar{M}_{AB} = \bar{M}_{BA} = 0$$

$$\bar{M}_{BC} = \bar{M}_{CB} = -\frac{M_0}{4} = -\frac{100}{4} = -25 \text{ kN/m}$$

$$\bar{M}_{CE} = \bar{M}_{EC} = 0$$

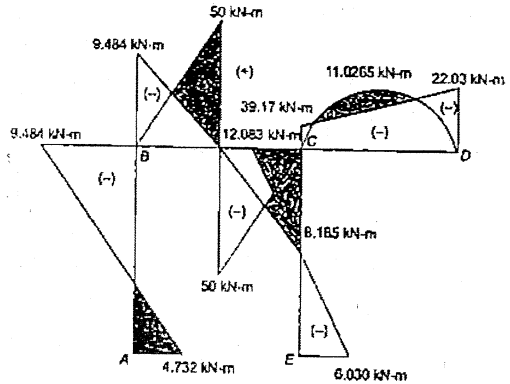
$$\bar{M}_{CD} = -\frac{12 \times 4^2}{12} = -16 \text{ kNm}$$

$$\bar{M}_{DC} = \frac{12 \times 4^2}{12} = 16 \text{ kNm}$$

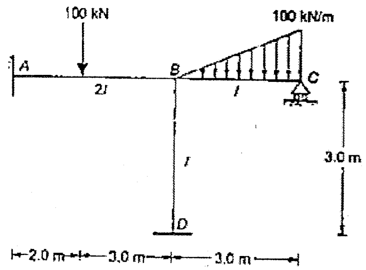
A (AB)	B (BA)	B (BC)	C (CB)	C (CD)	D (DC)	C (CE)	E (EC)
	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$		$\frac{1}{3}$	
0	0	−25	−25	−16	+16	0	0
	+12.5	+12.5	+13.66	+13.66		+13.66	
+6.25		+6.83	+6.25		+6.83		+6.83
	−3.415	−3.415	−2.083	−2.083		−2.083	
−1.707		−1.041	−1.707		−1.041		−1.041
	+0.520	+0.520	+0.569	+0.569		0.569	
+0.26		+0.284	+0.26		+0.284		+0.284
	−0.142	−0.142	−0.086	−0.086		−0.086	
−0.071		−0.043	−0.071		−0.043		−0.043
	+0.021	+0.021	+0.023	+0.023		+0.023	
+4.732	+9.484	−9.484	−8.185	−3.917	+22.03	+12.083	+6.030

Bending Moment Diagram:

Taking outer face as reference



Example 8.21 Analyse the frame shown in figure below by moment distribution method. Draw the BMD. The second moment of inertia are indicated in the figure.



[IES : 1998]

Solution:

(*i*) Distribution Factors:

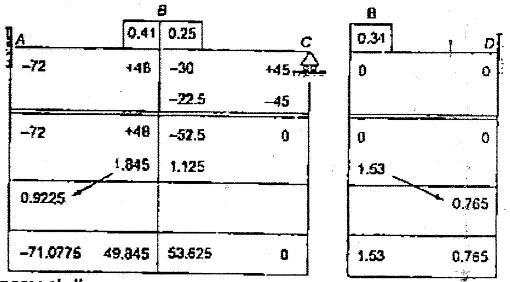
Joint	Members	Relative Stiffness	Total relative stiffness	D.F.
B	BA	$\frac{2I}{5} = \frac{24I}{60}$	$\frac{59I}{60}$	0.41
	BC	$\frac{3}{4} \times \frac{I}{3} = \frac{15I}{60}$		0.25
	BD	$\frac{I}{3} = \frac{20I}{60}$		0.34

(*ii*) Fixed End Moments:

$\bar{M}_{AB} = \frac{-100\times2\times3^2}{5^2} = +72$ kNm $\bar{M}_{BA} = \frac{100\times3\times2^2}{5^2} = +48$ kNm

$\bar{M}_{BC} = -\frac{wL^2}{30} = -\frac{100\times3^2}{30} = -30$ kNm $\bar{M}_{CB} = \frac{wL^2}{20} = \frac{100\times3^2}{20} = 45$ kNm

$\bar{M}_{BD} = \bar{M}_{DB} = 0$



(*iii*) Bending moment diagram:

Taking top face of the beam *ABC* as reference face and left face of the column *BD* as reference face.

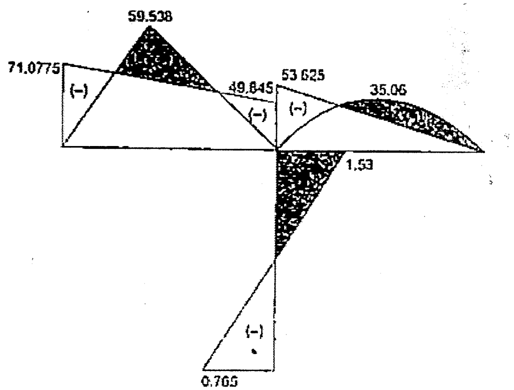
For simply supported span *AB*,

$$\text{Maximum moment} = \frac{100\times2\times3}{5} = 120 \text{ kN-m (Sagging)}$$

For simply supported span *BC*,

$$\text{Maximum moment} = \frac{wL^2}{9\sqrt{3}} = \frac{100\times3^2}{9\sqrt{3}} = 57.73 \text{ kNm (Sagging)}$$

M_{max} occur at a distance $\frac{L}{\sqrt{3}} = \sqrt{3}$ from *B*.



8.9 Sway Analysis

8.9.1 Frames without Sway

(*i*) When Resultant horizontal force is zero and columns are at same level.

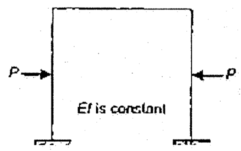


Fig. 8.14

(*ii*) When $\Sigma F_x = 0$ vertical loading is symmetrical, columns are at same level and stiffness of columns is constant.

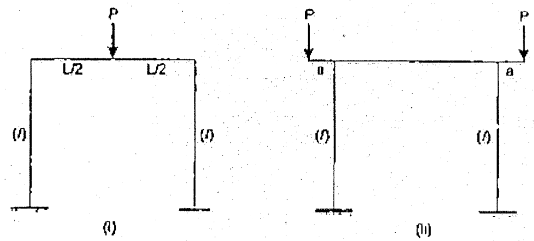


Fig. 8.15

(*iii*) When unyielding supports are presents at beam level, then the horizontal displacements are prevented.

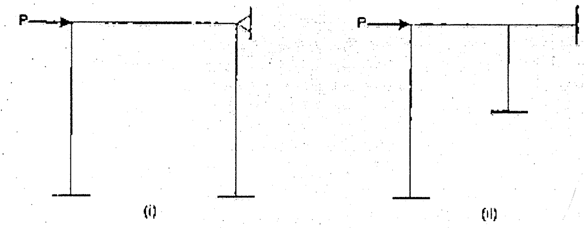


Fig. 8.16

8.9.2 Frames with Sway

(*i*) When resultant horizontal force is not zero then the unbalance force acts as sway force and sway will occurs in the direction of sway force.

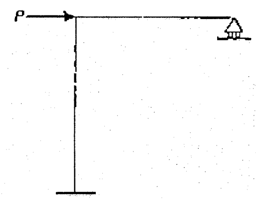


Fig. 8.17

(ii) When frame has symmetrical loading but column has different stiffness then frame will sway in that direction along which column has less stiffness.

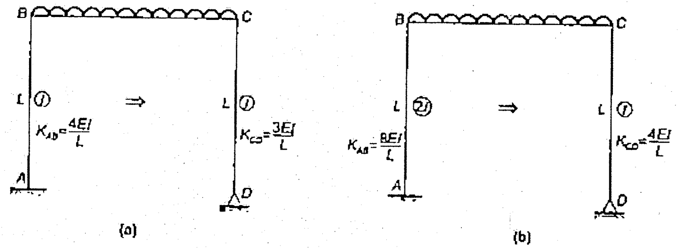


Fig. 8.18

(iii) When columns are at different levels, then sway will take place along that direction which column has less stiffness. *EI* is constant.

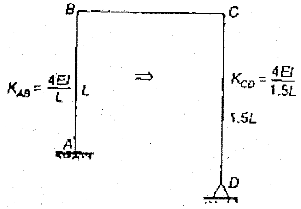


Fig. 8.19

(iv) When stiffness is symmetrical but loading is eccentric then sway will take place.

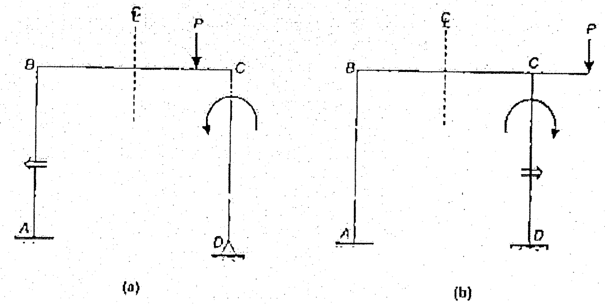


Fig. 8.20

8.9.3 Beam Sway (Joint Displacement)

(i) When properties of material (i.e *EI*) changed at a point

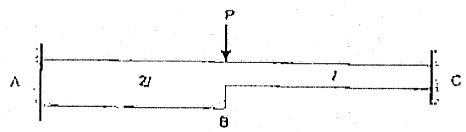


Fig. 8.21

(ii) When internal hinges are provided within span.

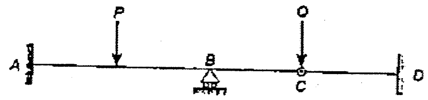
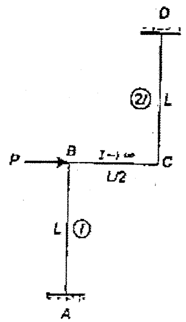


Fig. 8.22

Example 8.22 For the given frame as shown in figure shown below, the ratio of end moments at support *A* to end moments at support *D* is

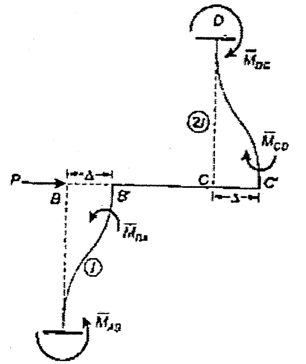


(a) 1.5
(b) 2
(c) 1
(d) 0.5

Ans. (d)

In above frame sway will take place in the direction of force *P*.

Let Δ be the sway displacement in the direction of sway.



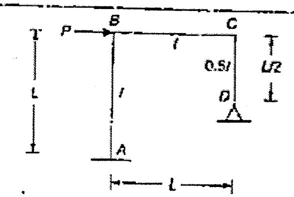
$$\bar{M}_{AB} = \bar{M}_{BA} = \frac{-6EI\Delta}{L^2}$$

$$\bar{M}_{CD} = \bar{M}_{DC} = \frac{+6E(2I)L}{L^2} = \frac{12EI\Delta}{L^2}$$

$$\therefore \quad \frac{\bar{M}_A}{\bar{M}_D} = \frac{+6EI\frac{\Delta}{L^2}}{12EI\frac{\Delta}{L^2}}$$

$$\frac{\bar{M}_A}{\bar{M}_D} = 0.5$$

Example 8.23 The given figure shows a portal frame with one end fixed and other hinged. The ratio of the fixed end moments M_{BA}/M_{CD} due to side sway will be

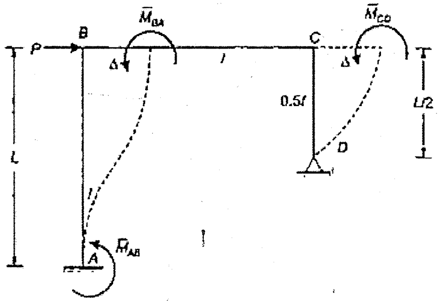


(a) 1.0
(b) 2.0
(c) 2.5
(d) 4.0

[IES : 1995]

Ans. (a)

Let Δ be the sway displacement in the direction of sway.

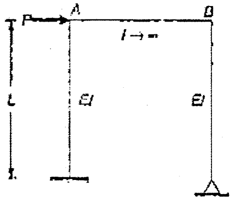


$$M_{BA} = \bar{M}_{BA} = -\frac{6EI\Delta}{L^2}$$

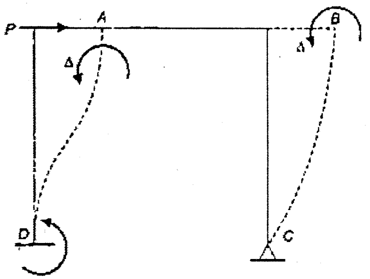
$$M_{CD} = \bar{M}_{CD} = \frac{-3E(0.5I)\Delta}{(0.5L)^2}$$

$$\therefore \quad \frac{M_{BA}}{M_{CD}} = \frac{-6EI\frac{\Delta}{L^2}}{-3\times 0.5EI\frac{\Delta}{(0.5L)^2}} = 1$$

Example 8.24 For the rigid frame shown in the figure below, find the force required for moving the girder AB through a horizontal displacement Δ.



Solution:

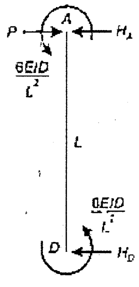


Consider free body diagram of member AD and BC separately:

(i) $\Sigma M_A = 0$

$$\Rightarrow \quad H_D\times L - \frac{6EI\Delta}{L^2} - \frac{6EI\Delta}{L^2} = 0$$

$$\Rightarrow \quad H_D = \frac{12EI\Delta}{L^3}$$

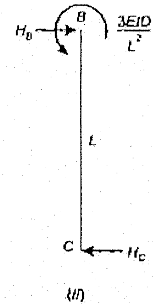


(i)

(ii) $\Sigma M_B = 0$

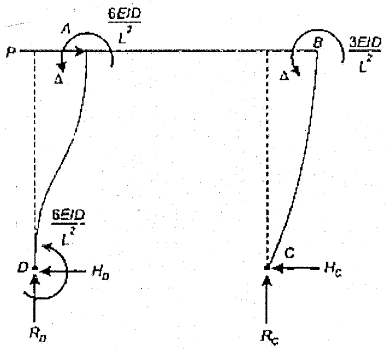
$$\Rightarrow \quad H_C\times L - \frac{3EI\Delta}{L^2} H_C\times L = 0$$

$$\Rightarrow \quad H_C = \frac{3EI\Delta}{L^3}$$



(ii)

Also consider free body diagram of whole frame.



$$\Sigma F_x = 0$$

$$\Rightarrow \quad P - H_D - H_C = 0$$

$$\Rightarrow \quad P = H_D + H_C$$

$$= \frac{12EI\Delta}{L^3} + \frac{3EI\Delta}{L^3} = \frac{15EI\Delta}{L^3}$$

8.9.4 Procedure of Sway Analysis

Case-1: When Sway (Δ) is known

Then the effect of given loading is considered separately and effect of sway (Δ) separately to compute fixed end moment. The net fixed end moments due to combined effect of sway and given loading are found and entered in table and distributed.

Case-2: When Sway (Δ) is not known but it is observe to occur

Then the effect of given loading is considered separately and effect of sway (Δ) separately to compute fixed end moment. The net fixed end moments due to combined effect of sway and given loading are found and entered in table and distributed.

Step-1: Non Sway analysis

Neglect the effect of sway (Δ). Calculate fixed end moments corresponding to given loading. Entered these fixed end moment into table and distribute according to their distribution factors. At last find the corresponding final end moments which are called Non Sway moments

Step-2:

Check ΣF_x for the portal frame.

If $\Sigma F_x = 0$ then there is no horizontal sway.

If $\Sigma F_x \neq 0$ then frame sway under the resultant sway force $S = \Sigma F_x$ to find ΣF_x, the transverse reactions (Horizontal reactions) can be found by using free body diagrams of vertical members subjected to external loading and non sway moments.

$$\Sigma F_x = H_A + H_D + P_2$$

If $\Sigma F_x = 0$, there is no sway

If $\Sigma F_x \neq 0$, then there is a sway and sway force will be $S = \Sigma F_x$

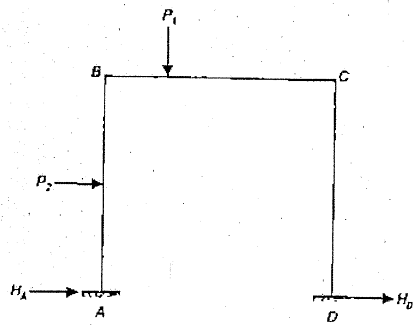


Fig. 8.23

Step-3:

To find effect of sway force. Remove all external loading and apply sway force S at the panel. Let Δ be the displacement caused by sway force.

Calculate fixed end moments due to sway

$$\bar{M}_{AB} = \bar{M}_{BA} = -\frac{6EI\Delta}{L^2}$$

$$\bar{M}_{BC} = \bar{M}_{CB} = 0$$

$$\bar{M}_{CD} = \bar{M}_{DC} = -\frac{6EI\Delta}{L^2}$$

where Δ is not known but it is proportional to sway force S.

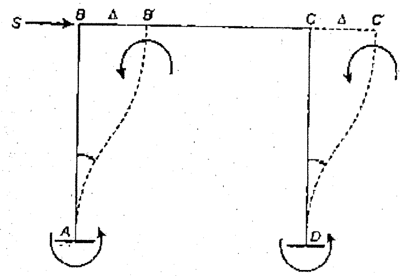


Fig. 8.24

Step-4:

Find the ratio of above sway fixing moments:

$$\bar{M}_{AB} : \bar{M}_{BA} : \bar{M}_{BC} : \bar{M}_{CB} : \bar{M}_{CD} : \bar{M}_{DC} : : -1 : -1 : 0 : 0 : -1 : -1$$

Take any arbitrary value of sway fixing moments in above ratios. For example:

$$\bar{M}_{AB} : \bar{M}_{BA} : \bar{M}_{BC} : \bar{M}_{CB} : \bar{M}_{CD} : \bar{M}_{DC} = -10 : -10 : 0 : 0 : -10 : -10$$

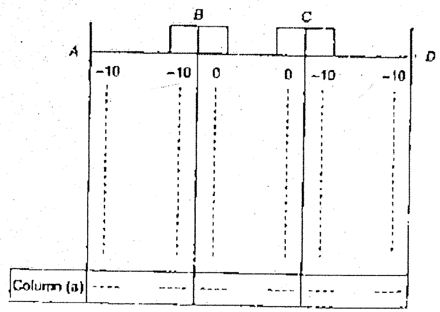
$$\bar{M}_{AB} : \bar{M}_{BA} : \bar{M}_{BC} : \bar{M}_{CB} : \bar{M}_{CD} : \bar{M}_{DC} = -8 : -8 : 0 : 0 : -8 : -8$$

Step-5:

The arbitrary values of sway fixing moments assumed above entered in table, which are balance and distributed according to the distribution factors. This procedure is called sway analysis.

At the end of table the moments obtain are not actual sway moment because the assumed fixing moments are arbitrary values.

The moments obtain at the end of table in column A are due to some sway force S'. Let H_A' and H_D' are the transverse reactions due to sway force such that $\Sigma F_x = 0$ i.e.



$$S' + H_A' + H_D' = 0$$

The transverse reactions can be found by free body diagrams of vertical members subjected to moments given in column (a).

Thus the value of S' can be found by

$$S' = -\left(H_A' + H_D'\right)$$

The moments given in column A are due to some sway force S'. Hence Actual sway moments will be

$$ASM = \frac{\text{Column (a)}}{S'} \times S$$

Step-6:

Final End moments = non sway moment + Actual sway moments

$$FEM = NSM + ASM$$

The final reactions can be found by the free body diagrams of members subjected to given loading and final end moments.

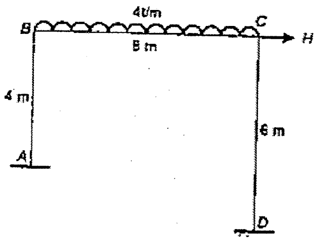
Finally, BMD can be draw by superimposing final end moments diagram and free BMD of each span.

Example 8.25 A rigid portal frame ABCD has two unequal vertical leg $AB = 4$ m and $CD = 6$ m with members BC horizontal and 8 m long. The bases A and D are fixed and all the members have same moment of Inertia I. The member BC carries a UDL of 4 t/m and a horizontal force H acts at C. Find the magnitude and sense of H such that the portal does not sway under the given loading. Draw the BM diagram for the portal under the same loading conditions. Use the moment distribution method.

Solution:

Non Sway Analysis:

Fixed End Moments:



$$\bar{M}_{AB} + \bar{M}_{BA} = 0$$

$$\bar{M}_{BC} = -\frac{wL^2}{12} = -\frac{4 \times 8^2}{12} = -21.33 \text{ t-m}$$

$$\bar{M}_{CB} = +\frac{wL^2}{12} = +21.33 \text{ t-m}$$

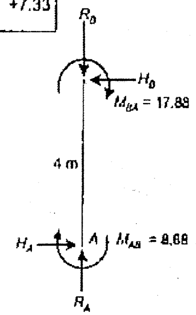
$$\bar{M}_{CD} = \bar{M}_{DC} = 0$$

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{4}$	$\frac{3I}{8}$	$\frac{2}{3}$
	BC	$\frac{I}{8}$		$\frac{1}{3}$
C	CB	$\frac{I}{8}$	$\frac{7I}{24}$	$\frac{3}{7}$
	CD	$\frac{I}{6}$		$\frac{4}{7}$

	A	B		C		D
		$\frac{2}{3}$	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{4}{7}$	
F.E.M	0	0	−21.33	+21.33	0	0
Bal. Moment		+14.22	+7.11	−9.14	−12.19	
C.O.M	+7.11		−4.57	+3.55		−6.09
Bal. Moment		+3.05	+1.52	−1.52	−2.08	
C.O.M	+1.2		−0.76	+0.76		−1.04
Bal. Moment		+0.51	+0.25	−0.33	−0.43	
C.O.M	+0.25		−0.16	+0.12		−0.22
Bal. Moment		+0.10	+0.06	−0.05	−0.07	
Final End Moments	+8.88	+17.88	−17.88	−14.72	−14.72	+7.33

Support Reactions:

Consider free body diagram of member *AB*,

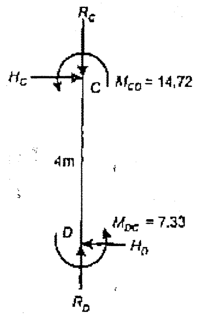


$$\Sigma M_B = 0$$

$\Rightarrow$ $$H_A \times 4 - 17.88 - 8.88 = 0$$

$\Rightarrow$ $$H_A = \frac{1}{4}[17.88 + 8.88] = 6.69 \text{ t } (\rightarrow)$$

Now consider free body diagram of member *CD*,



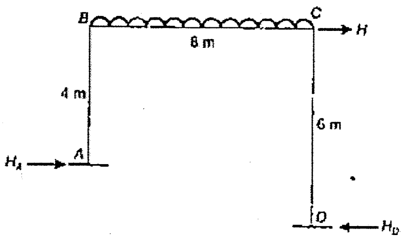
$$\Sigma M_C = 0$$

$\Rightarrow$ $$H_D \times 6 - 7.33 - 14.72 = 0$$

$\Rightarrow$ $$H_D = \frac{1}{6}[7.33 + 14.72]$$

$\Rightarrow$ $$H_D = 3.68 \text{ t } (\leftarrow)$$

Now consider equilibrium of entire frame



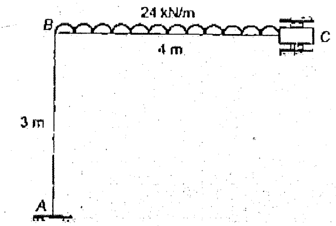
$$\Sigma F_x = 0$$

$\Rightarrow$ $$H_A + H - H_D = 0$$

$\Rightarrow$ $$H = H_D - H_A = 3.68 - 6.69 = -3.01 \text{t } (\leftarrow)$$

The force required is 3.01 t (←) for which portal does not sway under the given loading.

Example 8.26 For the frame shown in figure. Calculate end moments of the members. *EI* is constant.



Solution:

Distribution Factors

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{3} = \frac{4I}{12}$	$\frac{7I}{12}$	$\frac{4}{7}$
	BC	$\frac{I}{4} = \frac{3I}{12}$		$\frac{3}{7}$

1. Non Sway Analysis:

Fixed End Moments:

$$\bar{M}_{AB} = \bar{M}_{BA} = 0$$

$$\bar{M}_{BC} = -\frac{wL^2}{12} = -\frac{24 \times 4^2}{12} = -32 \text{ kNm}$$

$$\bar{M}_{CB} = +\frac{wL^2}{12} = +\frac{24 \times 4^2}{12} = +32 \text{ kNm}$$

	A	B		C
		$\frac{4}{7}$	$\frac{3}{7}$	
F.E.M	0	0	−32	+32
Bal. Moment		+18.285	+13.715	
C.O.M	+9.1425			+6.8575
N.S.M	+9.1425	+18.285	−18.285	+38.8575

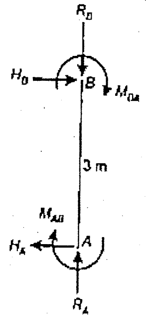
Consider free body diagram of member AB,

$\Sigma M_B = 0;$ $\quad H_A \times 3 + M_{AB} + M_{BA} = 0$

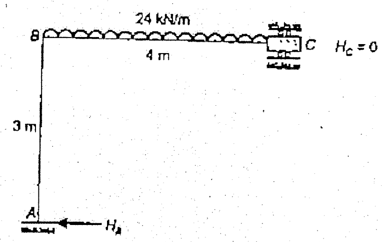
$$H_A = -\left[\frac{M_{AB} + M_{BA}}{3}\right]$$

$$= -\left(\frac{9.1425 + 18.285}{3}\right)$$

$$= -9.1425 \text{ kN or } 9.1425 \text{ kN } (\rightarrow)$$



Check ΣF_x for whole frame



$$\Sigma F_x = H_A(\leftarrow)$$

$\therefore$ $\quad$ 9.1425 kN or 9.1425 kN $(\rightarrow)$

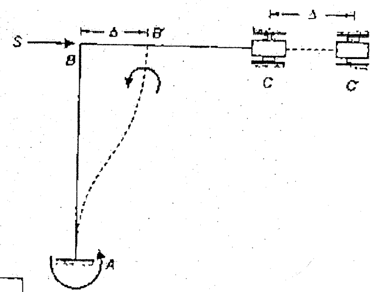
Since $\Sigma F_x \neq 0$, Hence Resultant horizontal force is unbalance.
Hence actual sway force,

$$S = \Sigma F_x = 9.1425 \text{kN} (\rightarrow)$$

Sway Analysis

Remove the given loading and apply sway force along as shown in figure.

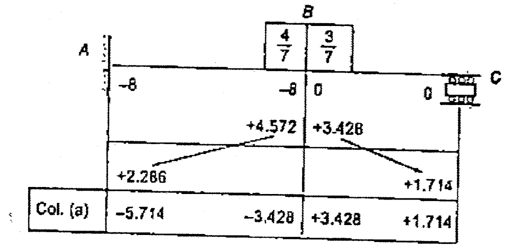
Let Δ be the transverse sway displacement as shown above. Due to sway force the fixing moment developed will be in anticlockwise direction.



$$\bar{M}_{AB} = \bar{M}_{BA} = -\frac{6EI\Delta}{L^2}$$

$$\bar{M}_{BC} = \bar{M}_{CB} = 0$$

$\bar{M}_{AB}$	:	$\bar{M}_{BA}$	:	$\bar{M}_{BC}$	:	$\bar{M}_{CB}$
$\frac{-6EI\Delta}{L^2}$	:	$\frac{-6EI\Delta}{L^2}$	:	0	:	0
−1	:	−1	:	0	:	0
−8	:	−8	:	0	:	0

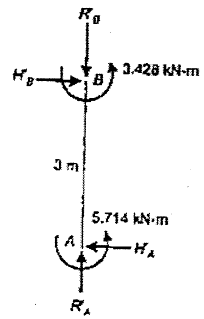


Let 'S'' be the sway force for which the moments given in Col. (a)

Consider free body diagram of member AB

$\Sigma M_B = 0;$ $\quad H_A' \times 3 - 5.714 - 3.428 = 0$

$$\therefore \quad H_A' = 3.0473 \text{ kN}(\leftarrow)$$



For equilibrium of whole frame

$$S' + H_A' + H_C' = 0$$

$$S' = -H_A' = -3.0473 \text{ kN or } 3.0473 \text{ kN}(\rightarrow)$$

Thus for a sway force 3.0473 kN $(\rightarrow)$ the sway moments are as per Col. (a). Hence for actual sway force of 9.1425 kN the corresponding actual sway moments will be

$$A.S.M = \frac{S}{S'} \times \text{col.}(a)$$

	M_{AB}	M_{BA}	M_{BC}	M_{CB}
Col. (a)	−5.714	−3.428	3.428	1.714
Actual Sway Moments $\frac{S}{S'} \times$ Col. (a)	−17.14	−10.28	10.28	5.15
Non Sway Moment	−9.14	18.28	−18.28	38.85
Final End Moments (A.S.M + N.S.M)	−8	8	−8	44

Actual Reactions

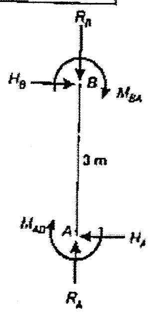
Consider equilibrium of member AB.

$\Sigma M_B = 0;$

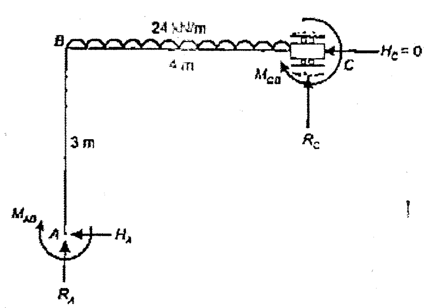
$$H_A \times 3 - M_{AB} + M_{BA} = 0$$

$$H_A = -\left(\frac{M_{AB} + M_{BA}}{3}\right)$$

$$= \left(-\frac{-8 + 8}{3}\right) = 0$$



Now consider, equilibrium of entire frame



$\Sigma F_y = 0;$ $\quad R_A + R_C = 24 \times 4 = 96$ ...(i)

$\Sigma M_C = 0;$

$$H_A \times 3 + R_A \times 4 + M_{AB} + M_{CB} - 14 \times 4 \times 2 = 0$$

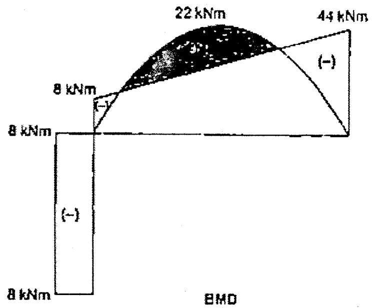
$$0 + 4R_A - 8 + 44 - 192 = 0$$

$$R_A = 39 \text{ kN } (\uparrow)$$

From equation (i), $\quad R_C = 96 - 39 = 57 \text{ kN } (\uparrow)$

Bending Moment Diagram:

Consider outer face as reference.



Example 6.27 Analyse the frame loaded as shown in figure below and draw the BMD. The frame is fixed at *A* and hinged at *D*. The relative second moment of the areas are also indicated in the figure.

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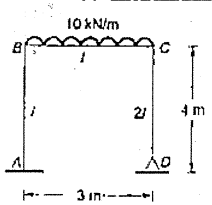


Solution:

Non-Sway Analysis:

Fixed End Moments

$$\bar{M}_{AB} = \bar{M}_{BA} = 0$$

$$\bar{M}_{BC} = \frac{-wL^2}{12} = -\frac{10 \times 3^2}{12} = -7.5 \text{ kN·m}$$

$$\bar{M}_{CB} = +\frac{wL^2}{12} = +7.5 \text{ kN·m}$$

$$\bar{M}_{CD} = \bar{M}_{DC} = 0$$

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{4}$	$\frac{7I}{12}$	$\frac{3}{7}$
	BC	$\frac{I}{3}$		$\frac{4}{7}$
C	CB	$\frac{I}{3}$	$\frac{17I}{24}$	$\frac{8}{17}$
	CD	$\frac{3}{4} \times \frac{2I}{4}$		$\frac{9}{17}$

	A	B: $\frac{3}{7}$	B: $\frac{4}{7}$	C: $\frac{8}{17}$	C: $\frac{9}{17}$	D
F.E.M	0	0	–7.5	+7.5	0	0
Bal. Moment		+3.21	+4.29	–3.53	–3.97	
C.O.M	+1.605		+1.765	+2.145		
Bal. Moment		+0.756	+1.009	–1.009	–1.136	
C.O.M	+0.378		–0.505	+0.505		
Bal. Moment		+0.216	+0.289	–0.238	–0.267	
C.O.M	+0.108		–0.119	+0.145		
Bal. Moment		+0.051	+0.068	–0.068	–0.077	
C.O.M	+0.026		–0.034	+0.034		
Bal. Moment		+0.015	+0.019	–0.016	–0.018	
Non Sway Moment	+2.117	+4.248	–4.248	+5.468	–5.468	0

Consider free body equilibrium of member AB and CD

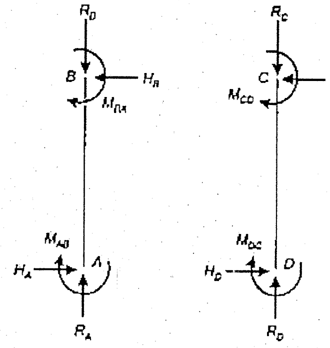
$\Sigma M_B = 0$ (for member AB)

$$M_{AB} + M_{BA} - H_A \times L_{AB} = 0$$

$$H_A = \frac{M_{AB} + M_{BA}}{L_{AB}}$$

$$H_A = \frac{2.117 + 4.248}{4}$$

$$H_A = 1.59 \text{ kN } (\rightarrow)$$



(a) (b)

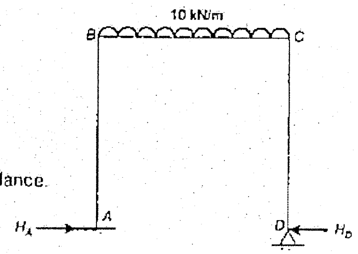
$\Sigma M_C = 0$ (for member CD)

$$M_{CD} + M_{DC} - H_D \times L_{CD} = 0$$

$$H_D = \frac{M_{CD} + M_{DC}}{L_{CD}}$$

$$H_D = \frac{-5.468 + 0}{4}$$

$$H_D = -1.37 \text{ kN } (\leftarrow)$$



Now check ΣF_x for whole frame

$$\Sigma F_x = H_A - H_D (\rightarrow)$$

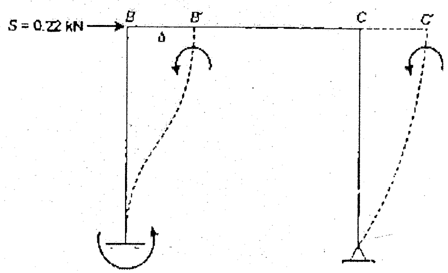
$$= 1.59 - 1.37$$

$$= 0.22 \text{ kN } (\rightarrow)$$

Since, $\Sigma F_x \neq 0$. Hence Resultant horizontal force is unbalance. Hence actual sway force.

$$S = \Sigma F_x = 0.22 \text{ kN } (\rightarrow)$$

Sway Analysis:



Remove the given loading and apply sway force alone as shown above. Let Δ be the transverse sway displacement. Due to sway force the fixing moment developed will be in anticlockwise directions.

$$\bar{M}_{AB} = \bar{M}_{BA} = \frac{-6EI\Delta}{L^2} = -\frac{6EI\Delta}{4^2} = -\frac{6EI\Delta}{16}$$

$$\bar{M}_{BC} = \bar{M}_{CB} = 0$$

$$\bar{M}_{CD} = \frac{-3E(2I)\Delta}{L^2} = \frac{-6EI\Delta}{4^2} = \frac{-6EI\Delta}{16}$$

$$\bar{M}_{DC} = 0$$

$\bar{M}_{AB}$	$\bar{M}_{BA}$	$\bar{M}_{BC}$	$\bar{M}_{CB}$	$\bar{M}_{CD}$	$\bar{M}_{DC}$
$\frac{-6EI\Delta}{16}$	$\frac{-6EI\Delta}{16}$	0	0	$\frac{-6EI\Delta}{16}$	0
−1	−1	0	0	−1	0
−10	−10	0	0	−10	0

	A	B		C		D
		$\frac{3}{7}$	$\frac{4}{7}$	$\frac{8}{17}$	$\frac{9}{17}$	
F.E.M	−10	−10	0	0	−10	0
Bal. Moment		+4.286	+5.714	+4.706	+5.294	
C.O.M	+2.143		+2.353	+2.857		
Bal. Moment		−1.008	−1.345	−1.344	−1.513	
C.O.M	−0.504		−0.672	−0.673		
Bal. Moment		+0.288	+0.384	+0.317	+0.356	
C.O.M	+0.144		+0.159	+0.192		
Bal. Moment		−0.068	−0.091	−0.091	−0.101	
C.O.M	−0.034		−0.045	−0.046		
Bal. Moment		+0.02	+0.026	+0.022		
Col. (a)	+8.251	−6.482	−6.482	+5.94	−5.94	0

$$H_A' = \frac{M_{AB} + M_{BA}}{L_{AB}} = \frac{-8.251 - 6.482}{4}$$

$$H_A' = 3.683 \text{ kN } (\leftarrow)$$

$$H_D' = \frac{M_{CD} + M_{DC}}{L_{CD}} = \frac{-5.94 + 0}{4}$$

$$H_D' = -1.485 \text{ kN}$$

$$H_D' = 1.485 \text{ kN } (\leftarrow)$$

Let S' be the sway force for which the moments given in Col. (a). For equilibrium of whole frame,

$$S' + H_A' + H_C' = 0$$

$$S' = -(3.683 + 1.485)\ (\leftarrow)$$

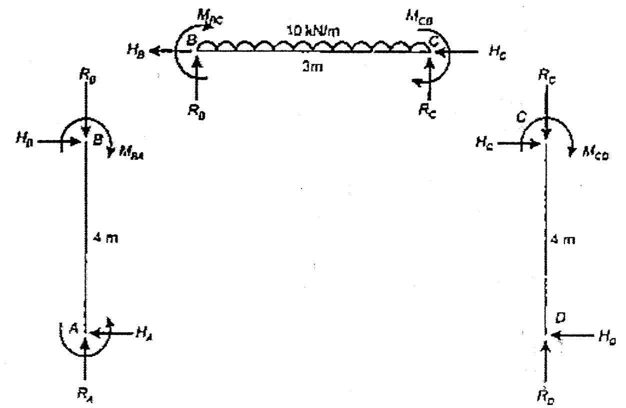
$$S' = 5.168\ \text{kN}\ (\rightarrow)$$

Thus for a sway force of 5.168 kN (→) the sway moments are as per Col. (a). Hence for actual sway force of 0.22 kN the corresponding actual sway moments will be

$$A.S.M = \frac{S}{S'} \times \text{col.}(a)$$

	M_{AB}	M_{BA}	M_{BC}	M_{CB}	M_{CD}	M_{CD}
Col. (a)	−8.251	−6.482	+6.482	+5.94	−5.94	0
Actual Sway Moments $\frac{S}{S'} \times$ **Col. (a)**	−0.35	−0.276	+0.276	+0.253	−0.253	0
Non Sway Moment	+2.177	−4.248	−4.248	+5.468	−5.468	0
Final End Moments (A.S.M + N.S.M)	+1.766	−3.972	−3.972	+5.721	−5.721	0

Actual Reactions:



$\Sigma M_B = 0$ (for member AB)

$$H_A \times 4 + M_{AB} + M_{BA} = 0$$

$$H_A = -\frac{(M_{AB} + M_{BA})}{4}$$

$$H_A = -\frac{(1.766 + 3.972)}{4} = -1.44\ \text{kN}(\leftarrow)$$

$$H_A = 1.44\ \text{kN}\ (\rightarrow)$$

$\Sigma_{MC} = 0$ (for member CD)

$$H_D \times 4 + M_{CD} = 0$$

$$H_D = -\frac{M_{CD}}{4} = \frac{-(-5.721)}{4} = 1.43\ \text{kN}\ (\leftarrow)$$

$\Sigma_{MC} = 0$ (for member BC)

$$R_B \times 3 + M_{BC} + M_{CB} - 10 \times 3 \times 1.5 = 0$$

$$3R_B = 45 - M_{BC} - M_{CB}$$

$$R_B = \frac{1}{3}\left[45 - (-3.972) - 5.721\right]$$

$$R_B = 14.42\ \text{kN}\ (\uparrow)$$

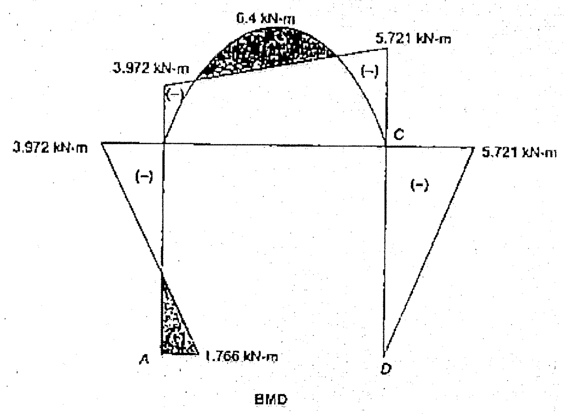
$\Sigma F_y = 0$ (for member BC) $\quad R_C = +10 \times 3 - 14.42 = 15.58\ (\uparrow)$

$\Sigma F_y = 0$ (for member AB) $\quad R_A = R_B = 14.42\ \text{kN}\ (\uparrow)$

$\Sigma F_y = 0$ (for member CD) $\quad R_D = R_C = 15.58\ \text{kN}\ (\uparrow)$

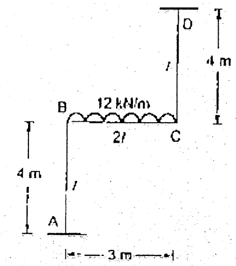
Bending Moment Diagram:

Now taking outer face of the portal frame as reference face.



BMD

Example 8.28 Analyse the rigid frame shown in figure below by moment distribution method.



[IES : 2001]

Solution:

Distribution Factors:

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{4} = \frac{3I}{12}$	$\frac{11I}{12}$	$\frac{3}{11}$
	BC	$\frac{2I}{3} = \frac{8I}{12}$		$\frac{8}{11}$
C	CB	$\frac{2I}{3} = \frac{8I}{12}$	$\frac{11I}{12}$	$\frac{8}{11}$
	CD	$\frac{I}{4} = \frac{3I}{12}$		$\frac{3}{11}$

(*i*) Non sway analysis:

Fixed End Moments,

$$\bar{M}_{AB} = \bar{M}_{BA} = 0$$

$$\bar{M}_{BC} = -\frac{12 \times 3^2}{12} = -9 \text{ kNm}$$

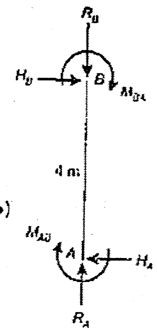
$$\bar{M}_{CB} = \frac{12 \times 3^2}{12} = 9 \text{ kNm}$$

$$\bar{M}_{CD} = \bar{M}_{DC} = 0$$

	AB	BA	BC	CB	CD	DC
D.F.		$\frac{3}{11}$	$\frac{8}{11}$	$\frac{8}{11}$	$\frac{3}{11}$	
F.E.M	0	0	−9	+9	0	0
Bal. Moments		+2.46	+6.54	−6.54	−2.46	
C.O.M	+1.23		−3.27	+3.27		−1.23
Bal. Moments		+0.9	+2.37	−2.37	−0.9	
C.O.M	+0.45		−1.185	+1.185		−0.45
Bal. Moments		+0.324	+0.861	−0.861	−0.324	
C.O.M	+0.162		−0.430	+0.430		−0.162
Bal. Moments		+0.118	+0.312	−0.312	−0.118	
C.O.M	+0.059		−0.156	+0.156		−0.059
Bal. Moments		+0.043	+0.113	−0.113	−0.043	
C.O.M	+0.021		−0.056	+0.056		−0.021
Bal. Moments		+0.016	+0.040	−0.040	−0.016	
N.S.M	+1.922	+3.860	−3.860	+3.860	−3.860	−1.922

Consider free body diagram of member *AB*.

$\Sigma M_B = 0$: $\quad H_A \times 4 + M_{AB} + M_{BA} = 0$



$$H_A = -\frac{1}{4}(M_{AB} + M_{BA})$$

$$= -\frac{1}{4}(1.922 + 3.86)$$

$$= 1.4457 \text{ kN } (\leftarrow) \text{ or } 1.4457 \text{ kN } (\rightarrow)$$

Similarly, for member *CD*,

$\Sigma M_C = 0$;

$$-H_D \times 4 + M_{DC} + M_{CD} = 0$$

$$H_D = \frac{M_{CD} + M_{DC}}{4} = \frac{(-3.861 - 1.922)}{4}$$

$$= -1.4457 \text{ kN } (\leftarrow) \text{ or } 1.4457 \text{ kN } (\rightarrow)$$

Check ΣF_x for entire frame

$$\Sigma F_x = 1.4457 + 1.4457 = 2.8914 \text{ kN } (\rightarrow)$$

Since $\Sigma F_x \neq 0$,

Hence resultant horizontal force is unbalance.

Hence, actual sway force will be

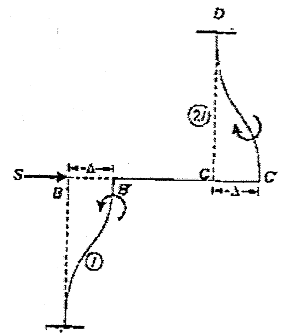
$$S = \Sigma F_x = 2.8914 \text{ kN } (\rightarrow)$$

(ii) Sway Analysis:

The rigid frame will now be analysed for a sway force of 6.725 kN acting from left to right.

The initial moments for *AB* will be negative while for *CD*, they are positive.

Let Δ be the transverse sway displacement as above due to sway forces.



$$\bar{M}_{AB} = \bar{M}_{BA} = -\frac{6EI\Delta}{L^2} = -\frac{6EI\Delta}{4^2} = -\frac{6EI\Delta}{16}$$

$$\bar{M}_{BC} = \bar{M}_{CB} = 0$$

$$\bar{M}_{CD} = \bar{M}_{DC} = +\frac{6EI\Delta}{L^2} = +\frac{6EI\Delta}{4^2} = +\frac{6EI\Delta}{16}$$

$\bar{M}_{AB}$	$\bar{M}_{BA}$	$\bar{M}_{BC}$	$\bar{M}_{CB}$	$\bar{M}_{CD}$	$\bar{M}_{DC}$
$\frac{-6EI\Delta}{16}$	$\frac{-6EI\Delta}{16}$	0	0	$\frac{6EI\Delta}{16}$	$\frac{6EI\Delta}{16}$
−10	−10	0	0	10	10

	A	B		C		D
DF		3/11	3/11	8/11	3/11	
F.E.M	−10	−10	0	0	+10	+10
B.M		+2.73	+7.27	−7.27	−2.73	
C.O.M	+1.36		−3.64	+3.64		−1.36
B.M		+0.99	+2.65	−2.65	−0.99	
C.O.M	+0.49		−1.33	+1.33		−0.49
B.M		+0.36	+0.97	−0.97	−0.36	
C.O.M	+0.18		−0.48	+0.48		−0.18
B.M		+0.13	+0.35	−0.35	−0.13	
C.O.M	+0.06		−0.18	+0.18		−0.06
B.M		+0.05	+0.13	−0.13	−0.05	
C.O.M	+0.02		−0.07	+0.07		−0.02
B.M		+0.02	+0.05	−0.05	−0.02	
col. (a)	−7.89	−5.72	+5.72	−5.72	+5.72	+7.89

Let S' be the sway force for which the moments are given in Col. (a).

Horizontal reaction at A,

$$H'_A = \frac{-7.89 - 5.72}{4} = -3.4025 \text{ kN } (\rightarrow) = 3.4025 \text{ kN } (\leftarrow)$$

Horizontal reaction at D,

$$H'_D = -\left(\frac{5.72 + 7.89}{4}\right) = -3.4025 \text{ kN } (\rightarrow) = 3.4025 \text{ kN } (\leftarrow)$$

∴ Sway force, $S = 3.4025 + 3.4025 = 6.805$ kN $(\rightarrow)$

Thus for a sway force of 6.805 kN, the sway moments are as per Col. (a). Hence for actual sway force of 2.8914 kN, the corresponding sway moments will be $\frac{S}{S'} \times$ Col (a).

	M_{AB}	M_{BA}	M_{BC}	M_{CB}	M_{CD}	M_{CD}
Col. (a)	−7.89	−5.72	+5.72	−5.72	+5.72	+7.89
Actual Sway Moments $\frac{S}{S'}$ **× Col. (a)**	−7.80	−5.65	+5.65	−5.65	+5.65	+7.80
Non Sway Moment	+4.47	+8.98	−8.98	+8.98	−8.98	−4.47
Final End Moments (A.S.M + N.S.M)	−3.33	+3.33	−3.33	+3.33	−3.33	+3.33

Actual Reaction:

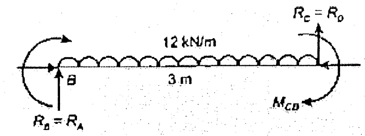
Actual horizontal reaction at A,

$$H_A = -\frac{(M_{AB} + M_{BA})}{4} = -\frac{(-1.43 + 1.43)}{4} = 0$$

Actual horizontal reaction at D,

$$H_D = \frac{M_{CD} + M_{DC}}{4} = \frac{(-1.43 + 1.43)}{4} = 0$$

Consider equilibrium of member BC.



$\Sigma F_y = 0;$ $R_A + R_D = 36$ ...(i)

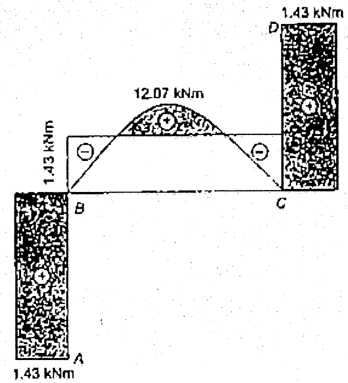
$\Sigma M_C = 0;$

$M_{BC} + M_{CD} + R_A \times 3 - 12 \times 3 \times 1.5 = 0$

$-1.43 + 1.43 + 3R_A - 54 = 0$

$R_A = 18$ kN (↑)

From eq. (i) we get, $R_D = 18$ kN (↑)



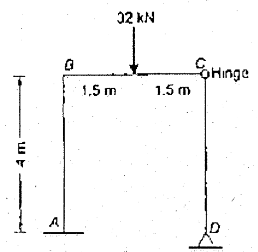
Bending moment Diagram:

Taking right hand side face of columns as reference face and top face of the beam as reference face.

Maximum simply supported moment in BC $= \frac{12 \times 3^2}{8}$

$= 13.5$ kN-m (Sagging)

Example 8.29 Analyse the portal frame shown in figure.



EI is constant.

Solution:

Distribution factors:

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{4}$	$\frac{2I}{4}$	$\frac{1}{2}$
	BC	$\frac{3}{4}\frac{I}{3}=\frac{I}{4}$		$\frac{1}{2}$
C	CB	0	0	0
	CD	0		0

(i) Non Sway Analysis:

$$\bar{M}_{AB} = \bar{M}_{BA} = 0$$

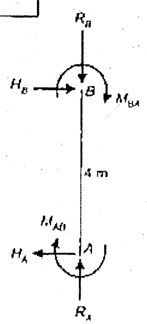
$$\bar{M}_{CD} = \bar{M}_{DC} = 0$$

$$\bar{M}_{BC} = -\frac{32\times 3}{8} = -12 \text{ kNm}$$

$$\bar{M}_{CB} = \frac{32\times 3}{8} = 12 \text{ kNm}$$

	A	B	B	C	C	
D.F.		$\frac{1}{2}$	$\frac{1}{2}$	0	0	
F.E.M	0	0	−12	+12	0	0
End correction			−6	−12		
Corrected F.E.M	0	0	−18	0	0	0
B.M		+9	+9			
C.O.M	+4.5					
N.S.M	+4.5	+9	−9	0	0	0

Consider free body diagram of member *AB*,



$\Sigma M_B = 0$;

$$H_A \times 4 + M_{AB} + M_{BA} = 0$$

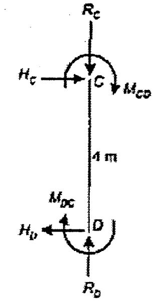
$$H_A = -\frac{1}{4}[M_{AB} + M_{BA}]$$

$$= -\frac{1}{4}(4.5+9)$$

$$= -3.375 \text{ kN}$$

or 3.375 kN (→)

Similarly, consider free body diagram of member *CD*.



$\Sigma M_C = 0$: $\quad H_A \times 4 + M_{CD} + M_{DC} = 0$

∴ $\quad H_D \times 4 + M_{CD} + M_{DC} = 0$

∴ $\quad H_D = 0$

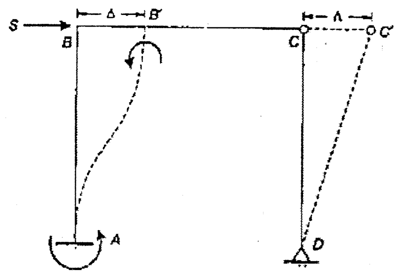
Check ΣF_x.

$$\Sigma F_x = -3.375 \text{ kN} (\leftarrow)$$

Since $\Sigma F_x \neq 0$, hence resultant horizontal force is unbalance.

∴ Actual sway force, $\quad \delta = 3.375$ kN (→)

(ii) Sway Analysis: Remove given loading and apply sway force along member BC as shown in figure.

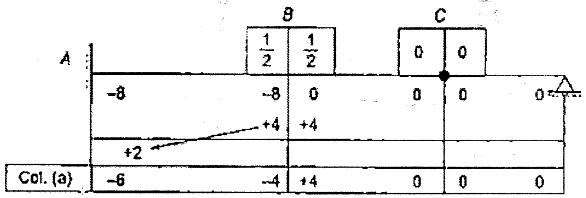


Let Δ be the transverse sway displacement as shown above. Due to sway force the fixing moment developed will be in anticlockwise direction.

$$\bar{M}_{AB} = \bar{M}_{BA} = -\frac{6EI\Delta}{4^2} = -\frac{6EI\Delta}{16}$$

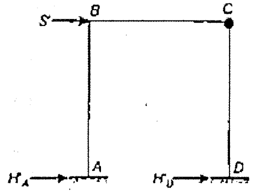
$$\bar{M}_{BC} = \bar{M}_{CB} = \bar{M}_{CD} = \bar{M}_{DC} = 0$$

$\bar{M}_{AB}$	$\bar{M}_{BA}$	$\bar{M}_{BC}$	$\bar{M}_{CB}$	$\bar{M}_{CD}$	$\bar{M}_{DC}$
$\frac{-6EI\Delta}{16}$	$\frac{-6EI\Delta}{16}$	0	0	0	0
−1	−1	0	0	0	0
−8	−8	0	0	0	0



Moment given in column (a) are due to some sway force S' such that,

$$S' + H'_A + H'_D = 0$$



Horizontal reaction at A, $H'_A = -\frac{1}{4}(M_{AB} + M_{BA})(\leftarrow)$

$$= -\frac{1}{4}(-6-4) = 2.5 \text{ kN}(\leftarrow)$$

Horizontal reaction at D, $H'_D = -\frac{1}{4}(M_{CD} + M_{DC}) = 0$

$\therefore$ $S' - 2.5 + 0 = 0$

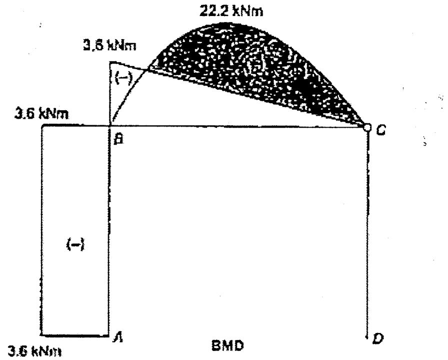
$$S' = 2.5 \text{ kN } (\rightarrow)$$

Thus for a sway force 2.5 kN ($\rightarrow$) the sway moments are as per Col. (a).

Hence for actual sway force of 3.375 kN the corresponding actual sway moment will be

$$\text{A.S.M} = \frac{S}{S'} \times \text{Col.(a)}$$

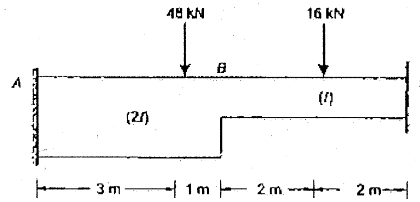
	M_{AB}	M_{BA}	M_{BC}	M_{CB}	M_{CD}	M_{CD}
Col. (a)	−6	−4	+4	0	0	0
Actual Sway Moments $\frac{S}{S'} \times$ Col. (a)	−8.1	−5.4	+5.4	0	0	0
Non Sway Moment	+4.5	+9	−9	0	0	0
Final End Moments (A.S.M + N.S.M)	−3.6	+3.6	−3.6	0	0	0



BMD

8.9.5 Sway Analysis of Non-prismatic Beams

Example 8.30 Analyse the fixed beam shown in figure by using moment distribution method.



Solution:

Distribution Factor:

Consider the imaginary support at B.

Joint	Member	Relative Stiffness	Total Stiffness	D.F.
B	BA	$\frac{2I}{4}$	$\frac{3I}{4}$	$\frac{2}{3}$
	BC	$\frac{I}{4}$		$\frac{1}{3}$

(I) Non sway analysis:

Fixed end moments.

$$\bar{M}_{AB} = -\frac{48 \times 3 \times 1^2}{4^2} = -9 \text{ kNm}$$

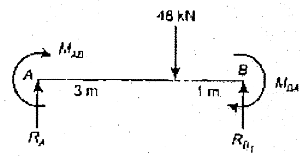
$$\bar{M}_{BA} = +\frac{48 \times 1 \times 3^2}{4^2} = +27 \text{ kNm}$$

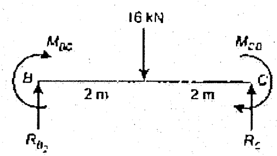
$$\bar{M}_{BC} = -\frac{16 \times 4}{8} = -8 \text{ kNm}$$

$$\bar{M}_{CB} = +\frac{16 \times 4}{8} = +8 \text{ kNm}$$

	A	B (2/3)	B (1/3)	C
F.E.M	−9	+27	−8	+8
B.M		−12.66	−6.34	
C.O.M	−6.33			−3.17
N.S.M	−15.33	+14.34	−14.34	+4.83

Consider free body equilibrium of span AB and BC.





$\Sigma M_B = 0$ (for span AB)

$$R_A \times 4 + M_{AB} + M_{BA} - 48 \times 1 = 0$$

$$R_A = \frac{48-(M_{AB}+M_{BA})}{4} = \frac{48-(-15.33+14.34)}{4}$$

$$R_A = 12.2475 \text{ kN} (\uparrow)$$

$\Sigma M_B = 0$ (for span BC)

$$-R_C \times 4 + M_{BC} + M_{CB} + 16 \times 2 = 0$$

$$R_C = \frac{32+(M_{BC}+M_{CB})}{4} = \frac{32+(-14.34+4.83)}{4}$$

$$R_C = 5.6225 \text{ kN} (\uparrow)$$

Check ΣF_y: (for whole beam)

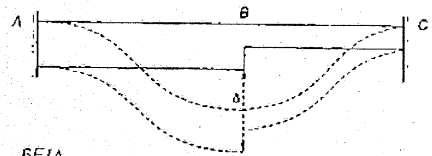
$$\Sigma F_y = 48 + 16 - R_A - R_C$$
$$= 48 + 16 - 12.2475 - 5.6225$$
$$= 46.13 \text{ kN} (\downarrow)$$

Since, $\Sigma F_y \neq 0$. Hence there will be a sway. So the sway force will be

$$S = 46.13 \text{ kN} (\downarrow)$$

(ii) Sway Analysis:

Remove all external loading and apply sway force at junction where I changes. Let Δ be the displacement due to sway force.



$$\bar{M}_{AB} = \bar{M}_{BA} = \frac{-6E(2I)\Delta}{L^2}$$

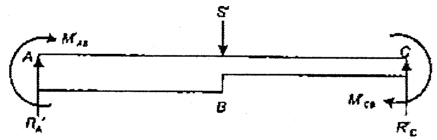
$$= \frac{-12EI\Delta}{4^2} = \frac{-12EI\Delta}{16}$$

$$\bar{M}_{BC} = \bar{M}_{CB} = +\frac{6EI\Delta}{L^2} = +\frac{6EI\Delta}{4^2} = +\frac{6EI\Delta}{16}$$

$\bar{M}_{AB}$	:	$\bar{M}_{BA}$	:	$\bar{M}_{BC}$	:	$\bar{M}_{CB}$
$\frac{-12EI\Delta}{16}$	:	$\frac{-12EI\Delta}{16}$	:	$\frac{6EI\Delta}{16}$	:	$\frac{6EI\Delta}{16}$
-12	:	-12	:	+6	:	+6

	A	B ($\frac{2}{3}$)	B ($\frac{1}{3}$)	C
F.E.M	-12	-12	+6	+6
Bal. Moment		+4	+2	
C.O.M	+2			+1
Col. (a)	-10	-8	+8	+7

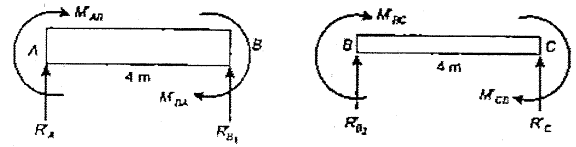
Moments given in column (a) are due to some sway force S'.



For equilibrium of whole beam

$$S' = R'_A + R'_C \qquad \text{...(i)}$$

R'_A and R'_C can be obtained by considering free body equilibrium of span AB and BC separately.



$\Sigma M_B = 0$ (for span AB)

$$R'_A \times 4 + M'_{AB} + M'_{BA} = 0$$

$$R'_A = -\frac{(M'_{AB}+M'_{BA})}{4} = -\frac{(-10-8)}{4} = 4.5 \text{ kN}(\uparrow)$$

$\Sigma M_B = 0$ (for span BC)

$$-R'_C \times 4 + M'_{BC} + M'_{CB} = 0$$

$$R'_C = \frac{(M'_{BC}+M'_{CB})}{4} = \frac{(8+7)}{4} = 3.75 \text{ kN}(\uparrow)$$

On substituting values of R'_A and R'_C in (i), we get

$$S' = 4.5 + 3.75 = 8.25 \text{ kN} (\uparrow)$$

Thus for a sway force of 8.25 kN (↓) the sway moments are as per Col. (a). Hence for actual sway force of 46.13 kN (↓) the corresponding actual sway moments will be

$$A.S.M = \frac{S}{S'} \times \text{col}(a)$$

	M_{AB}	M_{BA}	M_{BC}	M_{CB}
Col. (a)	-10	-8	+8	+7
Actual Sway Moments $\frac{S}{S'} \times$ Col. (a)	-55.91	-44.73	+44.73	39.14
Non Sway Moment	-15.33	+14.34	-14.34	4.83
Final End Moments (A.S.M + N.S.M)	-71.24	+30.39	+30.39	43.97

Final Reactions:

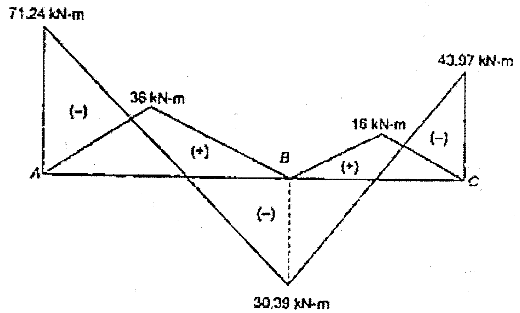
$\Sigma M_B = 0$ (for span AB)

$$R_A = \frac{48-(M_{AB}+M_{BA})}{4} = \frac{48-(-71.24-30.39)}{4} = 37.4075 \text{ kN} (\uparrow)$$

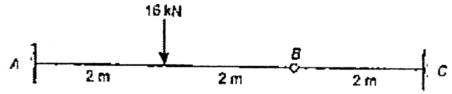
$\Sigma M_B = 0$ (for span BC)

$$R_C = \frac{32+(M_{BC}+M_{CB})}{4} = \frac{32+(30.39+43.97)}{4} = 26.59 \text{ kN} (\uparrow)$$

Bending Moment Diagram:



Example 8.31 **Draw Bending moment diagram for the beam shown in figure below.**



EI **is constant.**

Solution:

Non Sway Analysis:

Fixed end moments: For calculation of fixed end moments hinge join is considered as fixed joint.

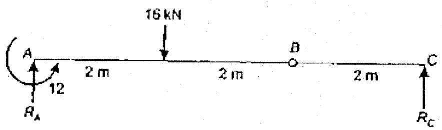
$$\bar{M}_{AB} = \frac{-PL}{8} = \frac{-16\times 4}{8} = -8 \text{ kN-m}$$

$$\bar{M}_{BA} = +\frac{PL}{8} = +8 \text{ kN-m}$$

$$\bar{M}_{BC} = \bar{M}_{CB} = 0 = 0$$

	A		B (internal hinge)	
F.E.M	−8	+8	0	
Internal hinge Correction		−8		
C.O.M	−4			
Col. (a)	−12	0	0	0

Consider free body diagram of beam,



$M_C = 0$, (from left)

$R_A \times 4 - 16 \times 2 - 12 = 0$

$$R_A = \frac{1}{4}[16\times 2+12] = 11 \text{ kN} (\uparrow)$$

Also, $M_C = 0$ (from right)

$R_C \times 2 = 0$

$R_C = 0$

Check ΣF_y,

$\Sigma F_y = 16 - R_A - R_C$

$\Sigma F_y = 16 - 11 - 0 = 5 \text{ kN} (\downarrow)$

Sway Analysis:

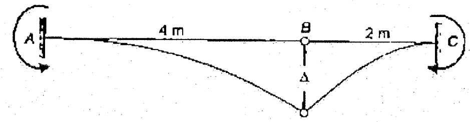
Remove loading and apply sway force at joint B. Let Δ be the displacement due to sway.

$$\bar{M}_{AB} = -\frac{3EI\Delta}{L^2} = \frac{-3EI\Delta}{4^2} = \frac{-3EI\Delta}{16}$$

$\bar{M}_{BA} = 0$

$M_{BC} = 0$

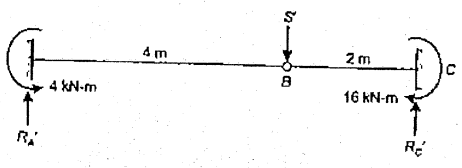
$$\bar{M}_{CB} = +\frac{3EI\Delta}{L^2} = \frac{+3EI\Delta}{4}$$



M_{AB}		M_{BA}		M_{BC}		M_{CB}
$\frac{-3EI\Delta}{16}$		0		0		$\frac{-3EI\Delta}{4}$
$-\frac{1}{16}$		0		0		$\frac{1}{4}$
−4		0		0		+16

	A		B	C
F.E.M	−4	0	0	+16
Col. (a)	−4	0	0	+16

Let S' be the some sway force for which the moments given in Col. (a) for equilibrium of whole beam



$\Sigma F_y = 0$:

$$R_A' + R_C' = S' \quad ...(i)$$

$$M_B = 0 \quad \text{(from left)}$$

$$R_A' \times 4 - 4 = 0$$

$$R_A' = 1 \text{ kN } (\uparrow)$$

$$M_B = 0 \quad \text{(from right)}$$

$$-R_C \times 2 + 16 = 0$$

$$R_C = 8 \text{ kN } (\uparrow)$$

On putting value of R_A' and R_C' in (i), we get

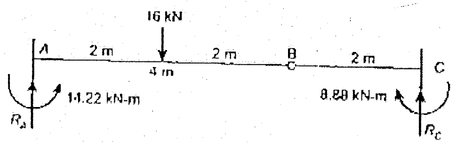
$$S' = 1 + 8 = 9 \text{ kN } (\downarrow)$$

Thus for a sway force of 9 kN (↓) the sway moments are as per Col. (a). Hence for actual sway force of 9 kN (↓) the corresponding actual sway moments will be.

$$A.S.M = \frac{S}{S'} \times \text{col.}(a)$$

	M_{AB}	M_{BA}	M_{BC}	M_{CB}
Col. (a)	−4	0	0	+16
Actual Sway Moments $\frac{S}{S'} \times$ Col. (a)	−2.22	0	0	+8.88
Non Sway Moment	−12	0	0	0
Final End Moments (A.S.M + N.S.M)	−14.22	0	0	+8.88

Actual Reactions:



$$M_B = 0 \quad \text{(from left)}$$

$$R_A \times 4 - 14.22 - 16 \times 2 = 0$$

$$R_A = \frac{1}{4}(14.22 + 16 \times 2)$$

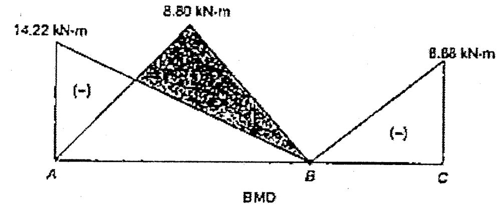
$$R_A = 11.55 \text{ kN } (\uparrow)$$

$$M_B = 0 \quad \text{(from right)}$$

$$-R_C \times 2 + 8.88 = 0$$

$$R_C = \frac{1}{2} \times 8.88 = 4.44 \text{ kN } (\uparrow)$$

Bending Moment Diagram:



8.9.6 Sway of Skew Frames

When there is sway in the frame as shown in figure, the movement of joint will be at 90° to the original position of member.

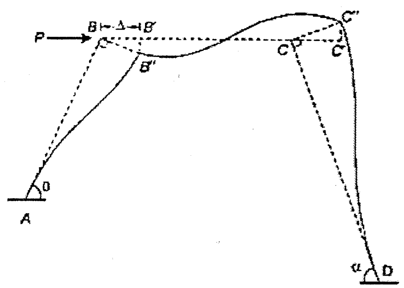


Fig. 8.25

Let Δ is lateral deflection due to Sway

$$\Delta = BB'$$

Actual displacement of joint B in the direction of sway = BB''

$$\Delta_{AB} = \frac{\Delta}{\sin\theta}$$

Actual displacement of joint C in the direction of sway = CC''

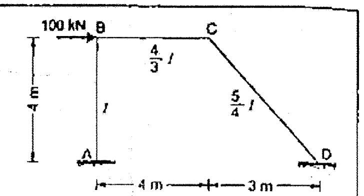
$$\Delta_{CD} = \Delta \cot \alpha$$

Actual displacement of member BC in the direction of sway

$$\Delta_{BC} = -(\Delta \cot\theta + \Delta \cot \alpha)$$

Example 8.32 Analyse the frame $ABCD$ shown in figure using moment distribution method. The properties of the sections geometry of the frame and the loading are given in figure. Draw the BM and axial force diagrams.

[IES : 2004]

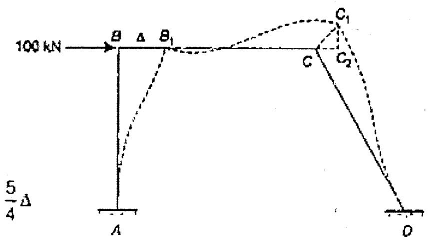


Solution:

There are no external loads acting on the portal from except the sway force of 100 kN from left to right. Hence only the sway analysis will be carried out.

Joint	Member	Relative Stiffness	Total Relative Stiffness	D.F.
B	BA	$\frac{I}{4}$	$\frac{7I}{12}$	$\frac{3}{7}$
	BC	$\frac{3I}{3}\times\frac{1}{4}=\frac{I}{3}$		$\frac{4}{7}$
C	CB	$\frac{4I}{3}\times\frac{1}{4}=\frac{I}{3}$	$\frac{7I}{12}$	$\frac{4}{7}$
	CD	$\frac{5I}{4}\times\frac{1}{4}=\frac{I}{4}$		$\frac{3}{7}$

Let the frame $ABCD$ deflect to the position, AB_1C_1D due to sway force



Let, $BB_1 = \Delta$

∴ $CC_2 = \Delta$

Now in ΔCC_1C_2

$$\cos\theta = \frac{CC_2}{CC_1}$$

⇒ $$CC_1 = \frac{\Delta}{(4/5)} = \frac{5}{4}\Delta$$

Again, $$\sin\theta = \frac{C_1C_2}{CC_1}$$

⇒ $$C_1C_2 = \frac{5}{4}\Delta\times\frac{3}{5}$$

$$C_1C_2 = \frac{3}{4}\Delta$$

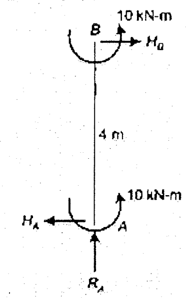
$$\text{Fixed end moment for } AB = \frac{-6EI\Delta BB_1}{L_{AB}^2} = \frac{-6EI\Delta}{4^2} = \frac{-6EI\Delta}{16}$$

$$\text{Fixed end moment for } BC = \frac{+6E\left(\frac{4}{3}I\right)\times\Delta C_1C_2}{L_{BC}^2} = \frac{6E\times\frac{4}{3}I\times\frac{3}{4}\Delta}{4^2} = \frac{6EI\Delta}{16}$$

$$\text{Fixed end moment for } CD = \frac{-6E\left(\frac{5}{4}I\right)\times\Delta_{CC_1}}{L_{CD}^2} = \frac{-6E\left(\frac{5}{4}I\right)\times\frac{5}{4}\Delta}{5^2} = -\frac{6EI\Delta}{16}$$

$\bar{M}_{AB}$	:	$\bar{M}_{BA}$	:	$\bar{M}_{BC}$	:	$\bar{M}_{CB}$	:	$\bar{M}_{CD}$	:	$\bar{M}_{DC}$
$\frac{-6EI\Delta}{16}$	:	$\frac{-6EI\Delta}{16}$	:	$\frac{6EI\Delta}{16}$	:	$\frac{6EI\Delta}{16}$	:	$\frac{-6EI\Delta}{16}$	:	$\frac{-6EI\Delta}{16}$
–1	:	–1	:	1	:	1	:	–1	:	–1
–10	:	–10	:	10	:	10	:	–10	:	–10

	A	B		C		D
D.F.		$\frac{3}{7}$	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	
F.E.M	–10	–10	+10	+10	–10	–10
Col. (a)	–10	–10	+10	+10	–10	–10



Horizontal Reaction at A,

$\Sigma M_B = 0$:

$$-10-10+H_A\times 4 = 0$$

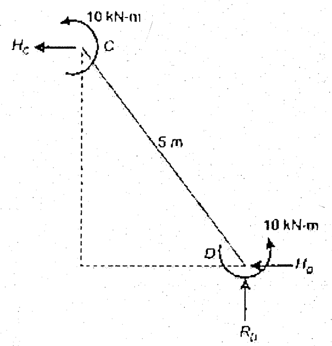
$$H_A = \frac{20}{4} = 5\text{ kN } (\leftarrow)$$

Horizontal Reaction at D,

$M_C = 0$:

$$-10-10+H_D\times 4 = 0$$

∴ $$H_D = \frac{20}{4} = 5\text{ kN } (\leftarrow)$$



Let S' be the sway force for which the moments given in Col. (a). For equilibrium

$$\Sigma f_x = 0$$

$$S' + H_A + H_D = 0$$

$$S' = -10\text{ kN } (\leftarrow)$$

$$S' = 10\text{ kN } (\rightarrow)$$

Thus for a sway force of 100 kN the sway moments are as per Col. (a). Hence for actual sway force of 100 kN the corresponding actual sway moments will be

$$A.S.M = \frac{S}{S'}\times\text{col (a)}$$

	M_{AB}	M_{BA}	M_{BC}	M_{CB}	M_{CD}	M_{DC}
Col. (a)	−10	−10	+10	+10	−10	−10
Actual Sway Moments $\frac{S}{S'} \times$ Col. (a)	−100	−100	+100	+100	−100	−100
Non Sway Moment	0	0	0	0	0	0
Final End Moments (A.S.M + N.S.M)	−100	−100	+100	+100	−100	−100

Vertical reaction at *D*,

$$R_D = \frac{M_{BC} + M_{CB}}{4} = \frac{100 + 100}{4}$$

$$= 50 \text{ kN} (\uparrow)$$

Vertical Reaction at *A*,

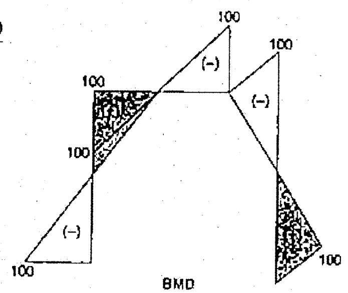
$$R_A = -R_D = -50 \text{ kN} (\uparrow)$$

$$= 50 \text{ kN} (\downarrow)$$

Horizontal Reaction, $H_A = \frac{M_{AB} + M_{BA}}{4} = \frac{-200}{4}$

$$= -50 \text{ kN} (\leftarrow)$$

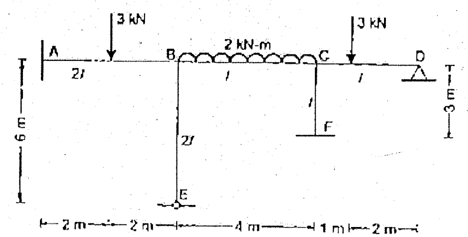
Horizontal Reaction, $H_D = 100 - 50 = 50 \text{ kN} (\leftarrow)$



BMD

Illustrative Examples

Example 8.33 Find the end moments of the frame shown below. Use Moment Distribution method. Also draw BMD.



Solution:

Fixed end moments:

$$\bar{M}_{AB} = \frac{-P \times L}{8} = \frac{-3 \times 4}{8} = -1.5 \text{ kN-m} \qquad \bar{M}_{BA} = \frac{+PL}{8} = +1.5 \text{ kN-m}$$

$$\bar{M}_{BC} = \frac{-wL^2}{12} = \frac{-2 \times 4^2}{12} = -2.667 \text{ kN-m} \qquad \bar{M}_{CB} = \frac{+wL^2}{12} = +2.667 \text{ kN-m}$$

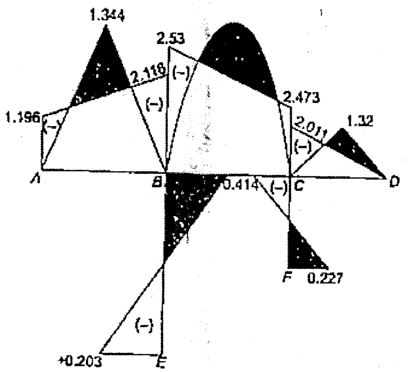
$$\bar{M}_{BE} = \bar{M}_{EB} = 0 \qquad \bar{M}_{CF} = \bar{M}_{FC} = 0$$

$$\bar{M}_{CD} = \frac{-Pab^2}{L^2} = \frac{-3 \times 1 \times 2^2}{3^2} = -1.333 \text{ kN-m} \qquad \bar{M}_{DC} = \frac{+Pa^2b}{L^2} = \frac{+3 \times 1^2 \times 2}{3^2} = +0.667 \text{ kN-m}$$

Joint	Member	Stiffness	Total Stiffness	D.F.
B	BA	$\frac{8EI}{4}$	$\frac{13EI}{3}$	$\frac{6}{13}$
	BC	$\frac{4EI}{4}$		$\frac{3}{13}$
	BE	$\frac{8EI}{4}$		$\frac{4}{13}$
C	CB	$\frac{4EI}{4}$	$\frac{10EI}{3}$	$\frac{3}{10}$
	CD	$\frac{3EI}{4}$		$\frac{3}{10}$
	CF	$\frac{4EI}{4}$		$\frac{4}{10}$

	A	B	B	C	C	D	B	E	C	F
D.F.		$\frac{6}{13}$	$\frac{3}{13}$	$\frac{3}{10}$	$\frac{3}{10}$		$\frac{4}{13}$		$\frac{4}{10}$	
	−1.5	+1.5	−2.667	+2 667	−1.333	+0.667	0	0	0	0
					−0.333	−0.667				
	−1.5	+1.5	−2.667	+2.667	−1.666	0	0	0	0	0
		+0.538	+0 269	−0.3	−0.3		+0.36		−0.4	
	+0.269		−0.15	+0.134				0.18		−0.2
		+0.069	+0.034	−0.040	−0.040		0.047		−0.054	
	0.0345		−0.02	+0.017				0.023		−0.027
		+0.009	+0.004	−0.005	−0.005		0.007		−0.007	
	−1.196	+2.116	−2.53	+2.473	−2.011	0	0.414	+3.3	−0.461	−0.227

M_{AB} = −1.196 kN-m
M_{BA} = +2.116 kN-m
M_{BC} = −2.53 kN-m
M_{BE} = +0.414 kN-m
M_{CB} = +2.473 kN-m
M_{CD} = −2.011 kN-m
M_{DC} = 0
M_{CF} = −0.461 kN-m
M_{FC} = −0.227 kN-m
M_{EB} = +0.203 kN-m



Summary

- The moment distribution is a displacement method. It is based on stiffness approach.
- For a beam if farther end is fixed then stiffness of member is $\frac{4EI}{L}$ and if farther end is hinged then stiffness of member is $\frac{3EI}{L}$.
- The distribution factor of a member is the ratio of stiffness of member to the total stiffness at joint.
- If moment is applied at a joint and farther end is fixed then a carry over moment half of applied moment get transferred to fixed end.
- Carry over factor is given by

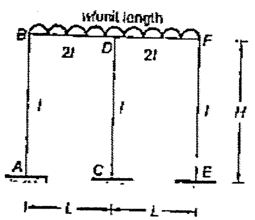
$$C.O.F = \frac{M_A}{M}$$

where, M_A = moment transferred at fixed end
M = moment applied at joint

- The summation of distribution factors at a joint is always one.

Objective Brain Teasers

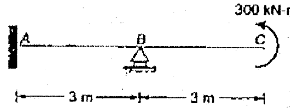
Q.1 In the frame shown in the figure below, the value of M_{CD} will be



(a) $\frac{wL^2}{12}$ (b) $\frac{wL^2}{6}$

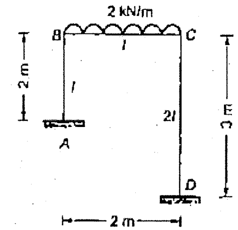
(c) $\frac{wH^2}{6}$ (d) zero

Q.2 A propped cantilever of uniform flexural rigidity is loaded as shown in the given figure. The bending moment at fixed end A is



(a) 150 kN-m hogging
(b) 300 kN-m sagging
(c) 150 kN-m sagging
(d) 300 kN-m hogging

Q.3 In the portal frame shown in the given figure, the ratio of sway moments in columns *AB* and *CD* will be equal to



(a) 1/3 (b) 2/3
(c) 9/8 (d) 13/8

Q.4 The portal frame shown in figure-I was analyzed, and the final column moments were found to be as shown in figure-II. The value of P is

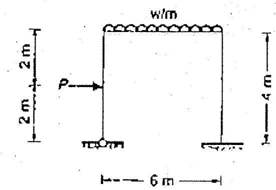


Figure-I

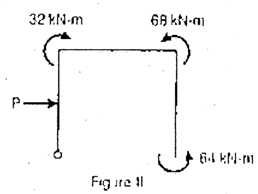
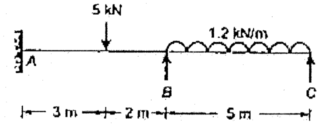


Figure II

(a) 23 kN (b) 41 kN
(c) 45 kN (d) 50 kN

Q.5 For the beam *AB* shown in the figure, the fixed end moments at ends *A* and *B* will be respectively

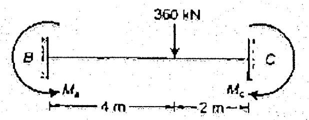


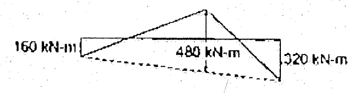
(a) − 3.6 kN-m and + 1.1 kN-m
(b) − 5.4 kN-m and + 4.6 kN-m
(c) − 2.4 kN-m and +3.6 kN-m
(d) − 3.6 kN-m and +6 kN-m

Q.6 If a point load acting at the mid-span of a fixed beam of uniform section produces fixed end moments of 60 kN-m, then the same load spread uniformly over the entire span will produce fixed end moments equal to

(a) 20 kN-m (b) 30 kN-m
(c) 40 kN-m (d) 45 kN-m

Q.7 The load diagram and bending moment of a beam are shown in the following figures:

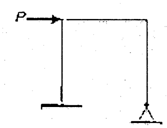


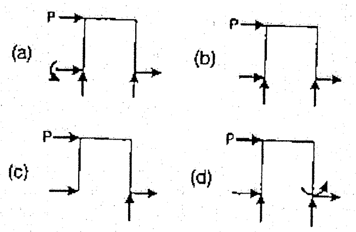


The shear force at B would be

(a) 93.33 kN (b) 120 kN
(c) 146.66 kN (d) 200 kN

Q.8 Which one among the following is the correct free body diagram for a portal frame shown in Figure given below?





Q.9 Match List-I (Method of analysis) with List-II (Unknowns being evaluated) and select the correct answer using the codes given below the lists:

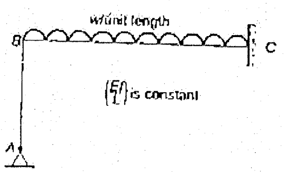
List-I
A. Flexibility Method
B. Stiffness Method
C. Kani's Method
D. Moment Distribution Method

List-II
1. Degrees of freedom
2. Redundant forces
3. Rotations by incremental iteration and unknown sways of plane frames
4. Displacement, rotations and sways of plane frames

Codes:

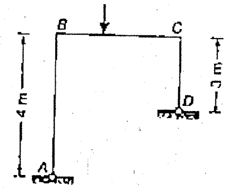
	A	B	C	D
(a)	2	1	4	3
(b)	3	4	1	2
(c)	2	4	1	3
(d)	3	1	4	2

Q.10 What is the ratio of magnitudes of moments in the member BC at the ends B and C in the figure given below?



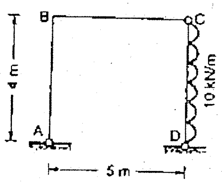
(a) 1 : 1 (b) 3 : 1
(c) 3 : 4 (d) 1 : 3

Q.11 For the frame as shown in the figure below, the final end moment M_{BC} has been calculated as −40 kN-m. What is the end moment M_{CD}?



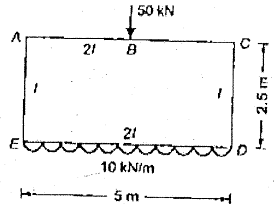
(a) +40 kN-m (b) −40 kN-m
(c) +30 kN-m (d) −30 kN-m

Q.12 What is the value of vertical reaction at A for the frame shown in figure below?



(a) 0 (b) 10 kN
(c) 16 kN (d) 20 kN

Q.13 The distribution factors for members AE and AC of the box section are



(a) 0.5 and 0.5 (b) 0.6 and 0.4
(c) 0.25 and 0.75 (d) 1 and zero

Q.14 The force required to produce a unit displacement (translation without rotation) at either one-third point of a fixed beam of span l and of uniform flexural rigidity EI is

(a) $\frac{729EI}{l^3}$ (b) $\frac{724EI}{l^3}$
(c) $\frac{724EI}{3l^3}$ (d) $\frac{729EI}{2l^3}$

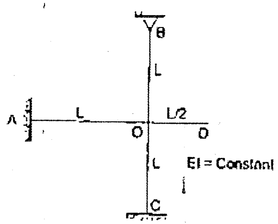
Q.15 The moment required to rotate the near end of a prismatic beam through unit angle without translation, when the far end is fixed, is

(a) $\frac{EI}{L}$ (b) $\frac{2EI}{L}$
(c) $\frac{3EI}{L}$ (d) $\frac{4EI}{L}$

Q.16 The moment-distribution method in structural analysis falls in the category of
(a) Displacement method
(b) Force method
(c) Flexibility method
(d) First order approximate method

Q.17 A fixed end beam of uniform cross-section is loaded uniformly throughout the span. What is the proportion of the bending moment at the centre to the end moment considering only elastic conditions?
(a) 1 : 1 (b) 1 : 2
(c) 1 : 4 (d) 2 : 3

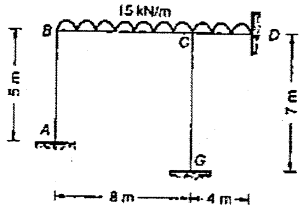
Q.18 A steel frame is shown in the given figure.



If joint O of the frame is rigid, the rotational stiffness of the frame at point O is given by

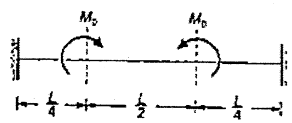
(a) $\frac{11EI}{L}$ (b) $\frac{10EI}{L}$
(c) $\frac{8EI}{L}$ (d) $\frac{6EI}{L}$

Q.19 For the frame shown in the figure, the distribution factors for members CB, CD and CG are respectively (Assume EI as constant)



(a) $\frac{14}{29}, \frac{8}{29}$ and $\frac{7}{29}$ (b) $\frac{7}{29}, \frac{14}{29}$ and $\frac{8}{29}$
(c) $\frac{7}{29}, \frac{8}{29}$ and $\frac{14}{29}$ (d) $\frac{14}{29}, \frac{7}{29}$ and $\frac{8}{29}$

Q.20 A fixed beam is subjected to moment M_0 as shown in the figure below. The fixed end moments will be



(a) zero (b) M_0
(c) $M_0/2$ (d) $2M_0$

Q.21 Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I
A. Strain energy method
B. Slope deflection
C. Moment distribution
D. Kani's method

List-II
1. Successive approximation
2. Flexibility method
3. Iteration process
4. Stiffness method

Codes:

	A	B	C	D
(a)	1	4	2	3
(b)	2	3	1	4
(c)	1	3	2	4
(d)	2	4	1	3

Q.22 Consider the following statements:
Hardy Cross method of moment distribution can be applied to analyze
1. continuous beams including non-prismatic structures
2. continuous beams with prismatic elements
3. structures with intermediate hinges
4. rigid frames

Which of these statements are correct?
(a) 1, 2, 3 and 4 (b) Only 1, 2 and 3
(c) Only 1, 2 and 4 (d) Only 3 and 4

Q.23 The displacement method is also referred to as which one of the following?
(a) Minimum strain energy method
(b) Maxwell-Mohr method
(c) Consistent deformation method
(d) Slope-deflection method

Q.24 If the free end of a cantilever of span L and flexure rigidity EI undergoes a unit displacement (without rotation), what is the bending moment induced at the fixed end?
(a) $\frac{3EI}{L^2}$ (b) $\frac{4EI}{L^2}$
(c) $\frac{5EI}{L^2}$ (d) $\frac{6EI}{L^2}$

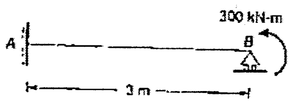
Answers

1. (d)	2. (a)	3. (c)	4. (d)	5. (c)
6. (c)	7. (a)	8. (a)	9. (a)	10. (d)
11. (d)	12. (c)	13. (a)	14. (d)	15. (d)
16. (a)	17. (b)	18. (a)	19. (b)	20. (c)
21. (d)	22. (a)	23. (d)	24. (d)	

Hints and Explanations:

1. (d)
The symmetry of the loading and frame does not allow any bending moment in column CD.

2. (a)

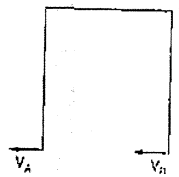


The moment is transferred to point B directly.
If M_{BA} = 300 kN-m

$M_{AB} = \frac{1}{2} \times 300 = 150$ kN-m as the carry over factor for beam is $\frac{1}{2}$. The direction of moment will be anticlockwise i.e. hogging.

3. (c)
Sway moment in column,
$$AB = \frac{6EI\Delta}{(2)^2} = \left(\frac{3EI\Delta}{2}\right)$$
Sway moment in column,
$$CD = \frac{6E(2I)\Delta}{(3)^2} = \frac{4EI\Delta}{3}$$

4. (d)



Shear force in left column,
$$V_A = \frac{P}{2} - \frac{32}{4} = \left(\frac{P}{2} - 8\right) \text{ kN [Towards left]}$$
Shear force in right column
$$= \frac{64+68}{4} = 33 \text{ kN [Towards left]}$$
Now, $\Sigma F_x = 0$ for frame
$$\frac{P}{2} - 8 + 33 = P$$
$\Rightarrow$ $P = 50$ kN

5. (c)
The fixed end moments can be given as
$$FEM_{AB} = \frac{-Wab^2}{L^2} = -\frac{5 \times 3 \times (2)^2}{(5)^2}$$
$$= -2.4 \text{ kN-m}$$
$$FEM_{BA} = \frac{Wba^2}{L^2} = \frac{5 \times 2 \times (3)^2}{(5)^2} = 3.6 \text{ kN-m}$$

6. (c)
Fixed end moment due to central point load,
$$M_1 = \frac{PL}{8} = 60 \text{ kN-m}$$
The fixed end moment due to uniformly distributed load is $\frac{PL}{12}$. Thus it is equal to 40 kN-m.

7. (a)
Shear force at B, taking sum of moment about C is zero.
$$V_B = \frac{360 \times 2}{6} - \left(\frac{320-160}{6}\right) = 93.33 \text{ kN}$$

8. (a)
There will be three reactions (2 forces and one moment) at fixed end and two reactions at hinged support. So (a) represents the free body diagram correctly.

9. (a)
Force or flexibility method uses redundant forces while stiffness or displacement method of analysis uses degrees of freedom.

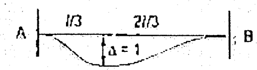
10. (d)
At joint B:
The stiffness of member BC:
$$K_{BC} = \frac{4EI}{L}$$
The stiffness of member AB:
$$K_{BA} = \frac{3EI}{L}$$
So distribution factors
$$D_{BC} = \frac{K_{BC}}{K_{BC} + K_{BA}} = \frac{4}{7}$$
$$D_{BA} = \frac{3}{7}$$
The fixed end moments
$$FEM_{BC} = -\frac{wL^2}{12}; FEM_{CB} = \frac{wL^2}{12}$$
Releasing moment at B, the member end moment for AB at end B.
$$M_{BA} = -\frac{3}{7}\left(\frac{wL^2}{12}\right)$$
$$M_{BC} = -\frac{4}{7}\left(\frac{wL^2}{12}\right) + \frac{wL^2}{12}$$
With carry over factor of $\frac{1}{2}$, the moment at end C of member BC is
$$M_{BC} = -\frac{2}{7}\left(\frac{wL^2}{12}\right) + \frac{wL^2}{24}$$
$$= -\frac{9}{7}\left(\frac{wL^2}{12}\right) = \frac{3wL^2}{28}$$
$$\therefore \quad \frac{M_{BC}}{M_{CB}} = \frac{1}{3}$$

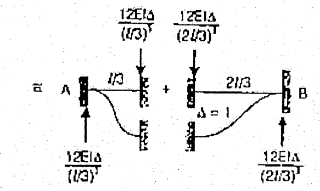
11. (d)
The shear force at end A should be equal and opposite to shear force at D. Let −40 kNm denotes the clockwise moment so moment at end B of column AB is 40 kNm anticlockwise and the shear force at A is 10 kN towards left.
Therefore shear force at D is 10 kN towards right.
Thus end moment at C is 30 kNm clockwise or −30 kNm.

12. (c)
Let the vertical reaction at A be V_A upwards. Taking moments about D, we get
$$\Sigma M_D = 0$$
$\Rightarrow$ $V_A \times 5 - 10 \times 4 \times 2 = 0$
$\Rightarrow$ $V_A = \frac{80}{5}$
$\Rightarrow$ $V_A = 16$ kN

13. (a)
Distribution factor of AE
$$= \frac{\text{Relative stiffness of } AE}{\text{Total relative stiffness}} = \frac{I/2.5}{\frac{I}{2.5} + \frac{2I}{5}} = 0.5$$
Distribution factor of $AC = 1 - 0.5 = 0.5$

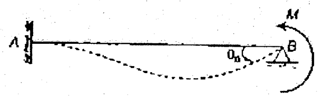
14. (d)





$$P = \frac{12EI\Delta}{(l/3)^3} + \frac{12EI\Delta}{(2l/3)^3}\Delta = \text{unity} = \frac{729EI}{2l^3}$$

15. (d)



$$\theta_B = \frac{ML}{4EI}$$

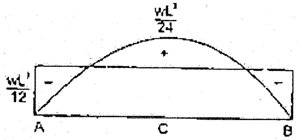
$$\Rightarrow \quad M = \frac{4EI}{L}\theta_B$$

Given, $\theta_B = 1$ unit

$$\therefore \quad M = \frac{4EI}{L}$$

17. (b)

Bending moment diagram of a fixed beam subjected to UDL throughout the span is given



$$\frac{M_C}{M_A} = \frac{wL^2/24}{wL^2/12} = \frac{12}{24} = \frac{1}{2}$$

18. (a)

Rotational stiffness is the ratio of moment and rotation at O. In other words rotational stiffness is the moment required for unit rotation at O.

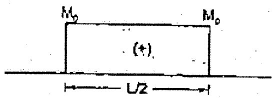
$$M = \frac{4EI}{L} + \frac{3EI}{L} + \frac{4EI}{L} = \frac{11EI}{L}$$

19. (b)

Member	Relative Stiffness	Distribution factor
CB	EI/8	7/29
CD	EI/4	14/29
CG	EI/7	8/29

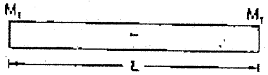
20. (c)

The free moment diagram of the beam is



Free moment diagram

Superimposing fixed end moment diagram and considering the fact that slopes at both end of the beam is zero.



Fixed moment diagram

Area under fixed moment diagram = Area under free moment diagram

$$M_1 L = M_0 \frac{L}{2}$$

$$\therefore \quad M_1 = \frac{M_0}{2}$$

So the fixed end moments will be $M_0/2$.

22. (a)

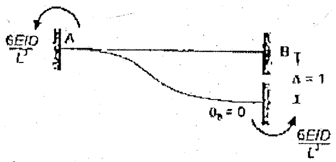
Hardy cross method or moment distribution method can be applied even when there is intermediate hinges.

23. (d)

Slope deflection method, moment distribution method, Kani's method, stiffness matrix method are all displacement methods.

24. (d)

The free end of the cantilever will undergo a displacement Δ without any rotation only when it is made fixed. This can be visualized by the given figure.

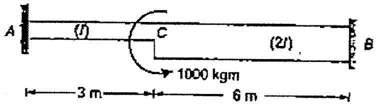


∴ Moment induced at

$$A = \frac{6EI\Delta}{L^2} = \frac{6EI}{L^2} \qquad [\because \Delta = 1]$$

Conventional Practice Questions

Q.1 Analyse the fixed beam shown in figure by moment distribution method.

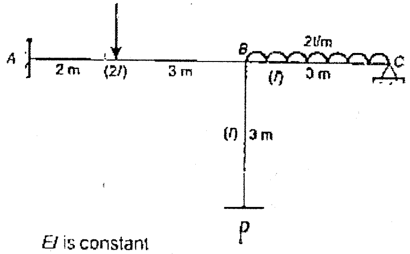


Ans. $M_{AB} = -108.11$, $M_{CA} = -378.38$, $M_{CB} = -621.62$, $M_{BC} = -351.35$

Q.2 A fixed base rectangular portal in which all the members have the same moment of inertia has a span of 6 m and height 4.5 m. Analyse the frame by moment distribution method when it is subjected to a horizontal sway load of 10t acting at the beam level from left to right and draw the bending moment diagram.

Ans. $M_{AB} = -13.26$, $M_{BA} = -9.25$, $M_{BC} = +9.25$, $M_{CB} = +9.25$, $M_{CD} = -9.25$, $M_{DC} = -13.26$

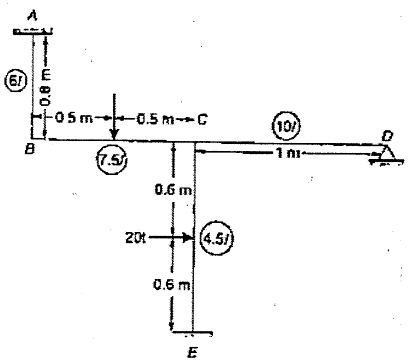
Q.3 A continuous beam ABC fixed at A and hinged at C is supported by a column BD which is rigidly connected to beam ABC at B. It is loaded as shown in figure. Analyse the from by moment distribution method.



EI is constant

Ans. $M_{AB} = -7.72$, $M_{BA} = +3.76$, $M_{BC} = -2.89$, $M_{BD} = -0.87$, $M_{BA} = 0$, $M_{BC} = -0.43$

Q.4 In the loaded structure shown in figure end A and E are fixed and end D is hinged and the joints B and C are rigid. EI values for various members are as shown in figure.



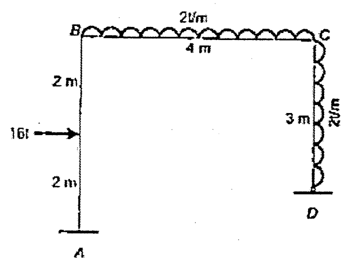
Ans. $M_{AB} = 1.71$ t-m, $M_{BA} = +3.46$ t-m, $M_{BC} = -3.465$ t-m, $M_{CD} = +2.83$ t-m, $M_{CE} = 1.06$ t-m, $M_{CD} = -3.89$ t-m, $M_{DC} = 0$, $M_{EC} = -3.965$ t-m, $R_A = 20.64$ t, $H_A = 6.47$ t, $H_E = 12.42$ t

Q.5 Analyse the portal Frame shown in figure, below by the moment distribution method.

(i) Compute the reaction components at the base.

(ii) Draw the bending moment diagram

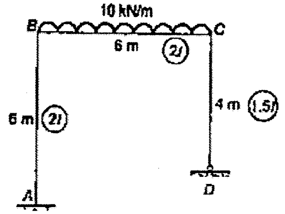
The support A to be built in and support D to be hinged.



$$I_{AB} : I_{BC} : I_{CD} = 2 : 1 : 1$$

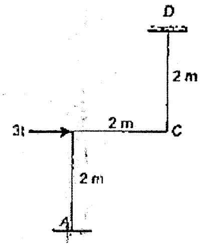
Ans. $H_A = 11.505$ t, $H_D = 1.51$ t, $R_D = 4.81$ t, $R_A = 1.51$ t

Q.6 For a rigid frame shown in figure find end moments and draw bending moment diagram.



Ans. $M_{AB} = 0$, $M_{BA} = 21.39$, $M_{BC} = -21.42$
$M_{CB} = 14.25$, $M_{CD} = -14.25$, $M_{DC} = 0$

Q.7 By using moment distribution method analyse the frame shown in figure and determine end moments. Take EI constant.



Ans. $M_{AB} = -2$ t-m, $M_{BA} = 1$ t-m, $M_{BC} = 1$ t-m
$M_{CB} = -12$ t-m, $M_{CD} = 3.46$ t-m
$M_{DC} = 1.01$ t-m, $R_A = 1.38$ t, $R_D = 3.38$ t
$H_A = 0.558$ t (←), $H_D = 0.558$ t (→)