Unit 6

Integrals

Teaching Points

• **Antiderivative or Primitives :** If $\frac{d}{dx}[f(x)] = \phi(x),$ the f(x) is called antideivative of f(x). As we know that $\frac{d}{dx}[f(x)+c] = \phi(x)$ so f(x) + c is also antiderivative of f(x), which depends on C (constant) as C may attain infinitely many values, therefore antderivative of a function is not unique.

Now f(x) + c is called the indefinite integral of f(x) w.r.t 'x' which is written as $\int \phi(x)dx = f(x) + C$ and C is known as constant of integration.

• **Integration :** The process of finding integral is called integration. Thus differentiation and integration are inverse process.

$\therefore$ $\frac{d}{ds}\left[\int f(x)dx\right]$ = .f(x) and $\int \frac{d}{dx}[f(x)]dx = f(x)$

• **Rules of Integration :**

1. $\int k.f(x)dx = k\int f(x)dx$ where k is any constant

2. $\int [\alpha.f_1(x) \pm \beta f_2(x)]\, dx = \alpha\int f_1(x)dx \pm \beta\int f_2(x)dx$ where a and b are constants.

Fundamental Integration Formulae

- $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
- $\int \frac{1}{x} dx = \log|x| + c$
- $\int e^x dx = e^x + c$
- $\int a^x dx = \frac{a^x}{\log_e^a} + c$
- $\int \sin x dx = -\cos x + c$

- $\int \cos x dx = \sin x + c$
- $\int \sec^2 x dx = \tan x + c$
- $\int \text{cosec}^2 x dx = -\cot x + c$
- $\int \sec x \tan x dx = \sec x + c$
- $\int \text{cosec}\, x \cot x dx = -\text{cosec}\, x + c$
- $\int \tan x dx = -\log|\cos x| + c = \log|\sec x| + c$
- $\int \sec x dx = \log|\sec x + \tan x| + c = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$
- $\int \text{cosec}\, x \cot x dx = \log|\text{cosec}\, x - \cot x| + c = \log \tan \frac{x}{2} + c$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c = -\cos^{-1} x + c$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c = -\cot^{-1} x + c$
- $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c = -\text{cosec}^{-1} x + c$
- $\int f(ax+b) dx = \frac{F(ax+b)}{a} + c, a \neq 0$

If $\int f(x) dx = F(x) + c$ i.e. if the integral of function of x is known, then if in place of x we have linear function of x, the integral is of same form but it is divided by coefficient of x.

● Following are some substitutions which are useful in evaluating integrals

Expression	**Substitutions**
$a^2 + x^2$ or $\sqrt{a^2 + x^2}$	$x = a\tan\theta / a\cot\theta$
$a^2 - x^2$ or $\sqrt{a^2 - x^2}$	$x = a\sin\theta / a\cos\theta$
$x^2 - a^2$ or $\sqrt{x^2 - a^2}$	$x = a\sec\theta / a\ \mathrm{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$a = a\cos 2\theta$

● **Special Integrals**

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2 - x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}dx = \log\left|x + \sqrt{x^2 + a^2}\right| + c$$

Use of Partial Fractions

● In some cases we find integral using partial fractions of the following types.

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$ (where Dr. can be factorised into linear factors)

$$\frac{px+q}{(ax+b)(cx+d)^2} = \frac{A}{(cx+b)} + \frac{B}{(x+d)^2} + \frac{C}{cx+d}$$ (where Dr. can be factorised into repeated linear factors)

$$\frac{px^2+qx+r}{(ax+b)(cx^2+dx+e)} = \frac{A}{ax+b} + \frac{bx+c}{cx^2+dx+e}$$

When $cx^2 + dx + e$ can not fractorised further.

When we have to evaluate the integral of the type.

$$\int e^x[f(x) + f'(x)]dx = e^x f(x) + c$$

● **Some special Integrals**

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + c$$

● **Definite Integral :** Let f(x) be a continous function defined on the closed integral [a, b] and F(x) be an antiderivative of f(x) then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

● Definite Integral as a limit of sum

$$\int_a^b f(x)dx = \underset{h \to 0}{Lt}\, h[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + \overline{n-1}h)]$$

where nh = b – a

● **Properties of definite Integral**

P(1) : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

P(2) : $\int_a^b f(x)dx = \int_b^a f(t)dt$

P(3) : $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

P(4) : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

P(5) : $\int_0^a f(x)dx = \int_a^a f(a-x)dx$

P(6) : $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

P(7) : $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x) \\ 0 \quad\quad\quad\quad \text{if } f(2a-x) = -f(x) \end{cases}$

P(8) : $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ if } f(-x) = f(x) \\ 0 \quad\quad\quad\quad \text{if } f(-x) = -f(x) \end{cases}$

Question for Practice

Evaluate the following Integrals

Very Short Answer Type Questions (1 Mark)

1. $\int \frac{\cos 6x + x}{3x^2 + \sin 6x} dx$ 2. $\int \sin^2 x dx$

3. $\int \frac{1}{x(2+3\log x)} dx$ 4. $\int x^3 \sin(x^4) dx$

5. $\int \frac{1}{\sqrt{x} + \sqrt{x-1}} dx$ 6. $\int e^x \left(\log x + \frac{1}{x} \right) dx$

7. $\int \frac{x^4+1}{x^2+1} dx$ 8. $\int \frac{1}{1+\sin x} dx$

9. $\int \frac{x}{(x+1)^2} dx$ 10. $\int_{-1}^{1} x^{99} \cos^4 x \, dx$

11. $\int_0^{3/2} [x] \, dx$ 12. $\int_0^1 \log\left(\frac{1-x}{1+x}\right) dx$

13. $\int_{-1}^{1} |x|\, dx$ 14. $\int_{0}^{2\pi} |\sin x|\, dx$

15. $\int \frac{1}{e^x + 1} dx$

Short Answer Type Questions (4 Mark)

16. $\int e^x \left(\log x + \frac{1}{x^2} \right) dx$ 17. $\int \sin^3 x \cos^5 x\, dx$

18. $\int \frac{\sin(x+a)}{\sin(x-a)} dx$ 19. $\int \frac{1}{\sin(x-a)\cos(x-b)} dx$

20. $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$ 21. $\int \frac{1}{\sqrt{\cos^3 x . \cos(x+a)}} dx$

22. $\int \sqrt{\sec x - 1}\, dx$ 23. $\int \frac{x}{x^4 + x^2 + 1} dx$

24. $\int \left(\sqrt{\tan x} + \sqrt{\cot x} \right) dx$ 25. $\int \frac{\sin x + \cos x}{\sqrt{\sin(2x)}} dx$

26. $\int \frac{1}{x\sqrt{(\log x)^2 + 3\log x - 4}} dx$ 27. $\int \frac{5x-2}{3x^2 + 2x + 1} dx$

28. $\int x\sqrt{1 + x - x^2}\, dx$ 29. $\int \frac{x^2}{x^2 + 5x + 3} dx$

30. $\int \frac{1}{x(x^9 - 1)} dx$ 31. $\int \frac{x^2 + x + 2}{(x+1)(x+2)} dx$

32. $\int \frac{1}{x^{1/2} + x^{1/3}} dx$ 33. $\int \frac{a^x . \tan^{-1}(a^x)}{a^{2x+1}} dx$

34. $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx$ 35. $\int \frac{1}{(\sin x)^{\frac{3}{4}} (\cos x)^{\frac{5}{4}}} dx$

36. $\int e^x \left[\frac{2+\sin 2x}{1+\cos 2x}\right] dx$ 37. $\int x^5 \sin(x^3) dx$

38. $\int_{-4}^{0} \{|x| + |x+3| + |x+6|\} dx$ 39. $\int_{0}^{\pi/2} \frac{x \sin 2x}{\sin^4 x + \cos^4 x} dx$

40. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$ 41. $\int_{0}^{3/2} |x \cos \pi x| dx$

42. $\int_{1}^{3} |x^2 - 2x| dx$ 43. $\int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

44. $\int_{0}^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$ 45. $\int_{-1}^{1} \log\left(\frac{1+\sin x}{1-\sin x}\right) dx$

Long Answer Type Questions (6 Mark)

46. $\int \sqrt{\tan x}\, dx$ 47. $\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta} d\theta$

48. $\int_{0}^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ 49. $\int_{0}^{\pi/2} \log(\cos x) dx$

50. $\int_{0}^{1} \frac{\log(1+x)}{1+x^2} dx$ 51. $\int \frac{dx}{\sin x(3+2\cos x)}$

52. $\int \frac{2\sin 2\theta - \cos\theta}{6-\cos^2\theta - 4\sin\theta} d\theta$ 53. Evaluate $\int_{2}^{5} (x^2 + 3x) dx$ as limit of sum.

54. Evaluate $\int_{1}^{2} (x^2 + x + 1) dx$ as limit of sum.

55. Evaluate $\int_{0}^{3} (3x^2 + e^{2x}) dx$ as limit of sum.

Answers

1. $\frac{1}{6}\log|3x^2+\sin 6x|+c$ 2. $\frac{x}{2}-\frac{\sin 2x}{2}+c$

3. $\frac{1}{3}\log|2+3\log x|+c$ 4. $-\frac{1}{4}\cos(x^4)+c$

5. $\frac{2}{3}\left[x^{\frac{3}{2}}-(x-1)^{\frac{3}{2}}\right]+c$ 6. $e^x\log x+c$

7. $x^3-x+2\tan^{-1}(x)+c$ 8. $\tan x-\tan x\sec x+c$

9. $\log|x+1|+\frac{1}{(x+1)}+c$ 10. 0

11. $\frac{1}{2}$ 12. 0

13. 1 14. 4

15. $-\log|1+e^{-x}|+c$ 16. $e^x\left[\log x-\frac{1}{x}\right]+c$

17. $-\frac{\cos^6 x}{6}+\frac{\cos^8 x}{8}+c$ 18. $x\cos 2a+\sin 2a\log|\sin(x-a)|+c$

19. $\frac{1}{\cos(a-b)}.\log\left|\frac{\sin(x-a)}{\cos(x-b)}\right|+c$

20. $\frac{1}{(a^2-b^2)}\log|a^2\sin^2 x+b^2\cos^2 x|+c$

21. $\frac{-2}{\sin a}\sqrt{\cos a-\sin a\tan x}+c$

22. $-\log\left|\left(\frac{2\cos x+1}{2}\right)+\sqrt{\cos^2 x+\cos x}\right|+c$

23. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$ 24. $\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$

25. $\sin^{-1}(\sin x-\cos x)+c$

26. $\log\left|\left(\frac{2\log x+3}{2}\right)+\sqrt{(\log x)^2+3\log x-4}\right|+c$

27. $\frac{5}{6}\log|3x^2+2x+1|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+c$

28. $\frac{-1}{3}(1+x-x^2)^{\frac{3}{2}}+\frac{1}{2}\left[\left(\frac{2x-1}{4}\right)\sqrt{1+x-x^2}+\frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right]+c$

29. $x-\frac{5}{2}\log|x^2+5x+3|+\frac{19}{2\sqrt{13}}\log\left|\frac{2x+5-\sqrt{13}}{2x+5+\sqrt{13}}\right|+c$

30. $\frac{1}{9}\log\left|\frac{x^9-1}{x^9}\right|+c$ 31. $x+2\log|x+1|-4\log|x+2|+c$

32. $6\left[\frac{x^{1/2}}{3}+\frac{x^{1/3}}{2}+x^{1/6}+\log|x^{1/6}-1|\right]+c$

33. $\frac{1}{2\log a}[\tan^{-1}(a^x)]^2+c$ 34. $x+\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+c$

35. $4(\tan x)^{\frac{1}{4}}+c$ 36. $e^x\tan x+c$

37. $\frac{1}{3}[-x^3\cos(x^3)+\sin(x^3)]+c$ 38. 29

39. $\frac{\pi^2}{8}$ 40. aπ

41. $\frac{5\pi-2}{2\pi^2}$ 42. 2

43. $\frac{\pi^2}{4}$ 44. $\frac{\pi}{2}$

45. 0

46. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+\frac{1}{2\sqrt{2}}\log\left|\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right|+c$

47. $\frac{-1}{3}\log|1+\tan x|+\frac{1}{6}\log|\tan^2 x-\tan x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+c$

48. $\pi\left(\frac{\pi}{2}-1\right)$ 49. $\frac{-\pi}{2}\log 2$

50. $\frac{\pi}{8}\log 2$ 51. $\frac{1}{10}\log|1-\cos x|-\frac{1}{2}\log|1+\cos x|+\frac{2}{5}\log|3+2\cos x|+c$

52. $2\log|\sin^2\theta-4\sin\theta+5|+7\tan^{-1}(\sin\theta-2)+c$

53. $\frac{141}{2}$ 54. $\frac{29}{6}$ 55. $\frac{(53+e^6)}{2}$

Hints

19. $\frac{1}{\cos(a-b)}\int\frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}dx$

$= \frac{1}{\cos(a-b)}\int\frac{\cos[(x-b)-(x-a)]}{\sin(x-a)\cos(x-b)}dx$

$= \frac{1}{\cos(a-b)}\int[\cot(x-a)+\tan(x-b)]dx$

22. $\int\sqrt{\frac{1-\cos x}{\cos x}}dx$

$= \int\frac{\sin x}{\sqrt{\cos x+\cos^2 x}}dx$ Put cos x = t

35. $\int\frac{1}{(\sin x)^{\frac{3}{4}}(\cos x)^{\frac{5}{4}}}dx$

Dividing num. and den. by $\cos^2 x$, we get

$= \int\frac{\sec^2 x dx}{(\tan x)^{\frac{3}{4}}}\quad\left[\because\ \frac{3}{4}+\frac{5}{4}=\frac{8}{4}=2\right]$

Now put tan x = t

39. I = $\int_0^{\pi/2} \frac{x \sin 2x}{\sin^4 x + \cos^4 x} dx$

$\therefore$ I = $\frac{\pi}{4} \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$ $\left[\text{Using prop. } \int_0^a f(x)dx = \int_0^a f(a-x)dx \right]$

I = $\frac{\pi}{4} \int_0^{\pi/2} \frac{2 \tan x \sec^2 x}{\tan^4 x + 1} dx$

Put $\tan^2 x = t$

40. I = $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$

I = $\int_{-a}^{a} \frac{a-x}{\sqrt{a^2 - x^2}} dx$

I = $\int_{-a}^{a} \frac{a\,dx}{\sqrt{a^2 - x^2}} - \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}} dx$

I = $2a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} - 0$ $\left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x) = f(x);\ = 0 \text{ if } f(-x) = -f(x) \right]$

I = aπ.

41. $\int_0^{3/2} |x \cos \pi x| \, dx = \int_0^{1/2} (x \cos \pi x) dx + \int_{1/2}^{3/2} (-x \cos \pi x) dx$

Use integral by parts to evaluate $\int (x \cos \pi x) dx.$

46. I = $\int \sqrt{\tan x} \, dx$ Put $\tan x = t^2$

I = $\int \frac{2t^2}{t^4 + 1} dt$

I = $\int \frac{t^2 + 1}{t^4 + 1} dt + \int \frac{t^2 - 1}{t^4 + 1} dt$

I = $\int \frac{1+1/t^2}{t^2+\frac{1}{t^2}}dt + \int \frac{1-1/t^2}{t^2+1/t^2}dt$

I = $\int \frac{1+1/t^2}{\left(t-\frac{1}{t}\right)^2+2}dt + \frac{1-1/t^2}{(t+1/t)^2-2}dt$

Put $t-\frac{1}{t}=a$ and $t+\frac{1}{t}=b$

I = $\int \frac{da}{a^2+(\sqrt{2})^2} + \int \frac{db}{b^2-(\sqrt{2})^2}$

50. I = $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Put x = tanθ

I = $\int_0^{\pi/4} \log(1+\tan\theta)d\theta$

I = $\int_0^{\pi/4} \log 2 d\theta - \mathrm{I}$ $\left[\text{Using property } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

I = $\frac{\pi}{8}\log 2.$

51. $\int \frac{dx}{\sin x(3+2\cos x)}$

= $\int \frac{\sin x dx}{(1-\cos^2 x)(3+2\cos x)}$ Put cos x = t

= $\int \frac{1}{(t-1)(t+1)(3+2t)}dt$

52. $\int_2^5 (x^2+3x)dx$

As $\int_a^b f(x)dx = \underset{h\to 0}{Lt}\, h\sum_{r=1}^{n} f(a+rh)$, where $h=\frac{b-a}{n}$

$\therefore \int_2^5 (x^2+3x)dx = \underset{h\to 0}{Lt}\, h\sum_{r=1}^{n}[10+7rh+r^2.h^2]$

$= \underset{h\to 0}{Lt}\, h\left[\sum_{r=1}^{n}10+7h\sum_{r=1}^{n}r+h^2\sum_{r=1}^{n}r^2\right]$

$= \underset{h\to 0}{Lt}\, h\left[10n+7h\frac{n(n+1)}{2}+h^2\frac{n(n+1)(2n+1)}{6}\right]$

$= \underset{h\to 0}{Lt}\, h\left[10nh+\frac{7(nh)(nh+h)}{2}+\frac{(nh)(nh+h)(2nh+h)}{6}\right]$

$= \frac{141}{2}$