Chapter - Integrals

Topic-1: Standard Integrals, Integration by Substitution, Itegration by Parts

1 MCQs with One Correct Answer

1. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{\frac{9}{2}}} dx$ equals (for some arbitrary constant K) **[2012]**

(a) $-\frac{1}{(\sec x+\tan x)^{\frac{11}{2}}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(b) $\frac{1}{(\sec x+\tan x)^{\frac{11}{2}}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(c) $-\frac{1}{(\sec x+\tan x)^{\frac{11}{2}}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(d) $\frac{1}{(\sec x+\tan x)^{\frac{11}{2}}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

2. Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant C, the value of J – I equals **[2008]**

(a) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+C$

(b) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+C$

(c) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+C$

(d) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+C$

3. The value of the integral $\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx$ is **[1995S]**

(a) $\sin x - 6\tan^{-1}(\sin x)+c$

(b) $\sin x - 2(\sin x)^{-1}+c$

(c) $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x)+c$

(d) $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x)+c$

4 Fill in the Blanks

4. If $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx = Ax + B\log(9e^{2x}-4) + C$, then $A =$ ...., $B =$ .... and $C =$ .... **[1990 - 2 Marks]**

6 MCQs with One or More than One Correct Answer

5. Let b be a nonzero real number. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f(0)=1$. If the derivative f' of f satisfies the equation

$$f'(x)=\frac{f(x)}{b^2+x^2}$$

for all $x\in\mathbb{R}$, then which of the following statements is/are TRUE? **[Adv. 2020]**

(a) If $b>0$, then f is an increasing function

(b) If $b<0$, then f is a decreasing function

(c) $f(x)f(-x)=1$ for all $x\in\mathbb{R}$

(d) $f(x)-f(-x)=0$ for all $x\in\mathbb{R}$

6. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be functions satisfying $f(x+y)=f(x)+f(y)+f(x)f(y)$ and $f(x)=xg(x)$ for all $x,y\in\mathbb{R}$. If $\lim_{x\to 0} g(x)=1$, then which of the following statements is/are TRUE? **[Adv. 2020]**

(a) f is differentiable at every $x\in\mathbb{R}$

(b) If $g(0)=1$, then g is differentiable at every $x\in\mathbb{R}$

(c) The derivative $f'(1)$ is equal to 1
(d) The derivative $f'(0)$ is equal to 1

9 Assertion and Reason/Statement Type Questions

7. Let $F(x)$ be an indefinite integral of $\sin^2 x$.
STATEMENT-1 : The function $F(x)$ satisfies $F(x+\pi) = F(x)$ for all real x. because
STATEMENT-2 : $\sin^2(x+\pi) = \sin^2 x$ for all real x. [2007 -3 marks]
(a) Statement-1 is True, statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True.

10 Subjective Problems

8. For any natural number m, evaluate
$$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx,\ x>0.$$
[2002 - 5 Marks]

9. Evaluate $\int \sin^{-1}\left(\dfrac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$. [2001 - 5 Marks]

10. Evaluate $\int \dfrac{(x+1)}{x(1+xe^x)^2}\,dx$. [1996 - 2 Marks]

11. Find the indefinite integral $\int \cos 2\theta \ln\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$ [1994 - 5 Marks]

12. Find the indefinite integral $\int\left(\dfrac{1}{\sqrt[3]{x}+\sqrt[4]{4}}+\dfrac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$ [1992 - 4 Marks]

13. Evaluate $\int (\sqrt{\tan x}+\sqrt{\cot x})dx$ [1989 - 3 Marks]

14. Evaluate : $\int\left[\dfrac{(\cos 2x)^{1/2}}{\sin x}\right]dx$ [1987 - 6 Marks]

15. Evaluate the following $\int\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ [1985 - 2½ Marks]

16. Evaluate the following $\int \dfrac{dx}{x^2(x^4+1)^{3/4}}$ [1984 - 2 Marks]

17. Evaluate $\int (e^{\log x}+\sin x)\cos x\,dx$. [1981 - 2 Marks]

18. Evaluate $\int \dfrac{x^2\,dx}{(a+bx)^2}$ [1979]

19. Evaluate $\int \dfrac{\sin x}{\sin x-\cos x}\,dx$ [1978]

Topic-2: Integration of the Forms: $\int e^x(f(x)+f'(x))dx$, $\int e^{kx}(kf(x)+f'(x))dx$, Integration by Partial Fractions, Integration of Different Expression of e^x

1 MCQs with One Correct Answer

1. $\displaystyle\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx=$ [2006 - 3M, –1]

(a) $\dfrac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (b) $\dfrac{\sqrt{2x^4-2x^2+1}}{x^3}+c$

(c) $\dfrac{\sqrt{2x^4-2x^2+1}}{x}+c$ (d) $\dfrac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

10 Subjective Problems

2. Integrate : $\displaystyle\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx.$ [1999 - 5 Marks]

3. Evaluate : $\displaystyle\int \frac{(x-1)e^x}{(x+1)^3}\,dx$ [1983 - 2 Marks]

Topic-3: Evaluation of Definite Integral by Substitution, Properties of Definite Integrals

1 MCQs with One Correct Answer

1. Let $f:(0,1)\to\mathbb{R}$ be the function defined as $f(x)=\sqrt{n}$ if $x\in\left[\dfrac{1}{n+1},\dfrac{1}{n}\right)$ where $n\in\mathbb{N}$. Let $g:(0,1)\to\mathbb{R}$ be a function such that $\displaystyle\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}\,dt<g(x)<2\sqrt{x}$ for all $x\in(0,1)$. Then $\lim\limits_{x\to 0} f(x)g(x)$ [Adv. 2023]

(a) does **NOT** exist (b) is equal to 1
(c) is equal to 2 (d) is equal to 3

2. For positive integer n, define

$$f(n) = n + \frac{16+5n-3n^2}{4n+3n^2} + \frac{32+n-3n^2}{8n+3n^2} + \frac{48-3n-3n^2}{12n+3n^2} + \ldots + \frac{25n-7n^2}{7n^2}.$$

Then, the value of $\lim_{n\to\infty} f(n)$ is equal to **[Adv. 2022]**

(a) $3+\frac{4}{3}\log_e 7$ (b) $4-\frac{3}{4}\log_e\left(\frac{7}{3}\right)$

(c) $4-\frac{4}{3}\log_e\left(\frac{7}{3}\right)$ (d) $3+\frac{3}{4}\log_e 7$

3. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+e^x}dx$ is equal to **[Adv. 2016]**

(a) $\frac{\pi^2}{4}-2$ (b) $\frac{\pi^2}{4}+2$ (c) $\pi^2-e^{\frac{\pi}{2}}$ (d) $\pi^2+e^{\frac{\pi}{2}}$

4. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\operatorname{cosec} x)^{17}\,dx$ is equal to **[Adv. 2014]**

(a) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u+e^{-u}\right)^{16} du$

(b) $\int_0^{\log(1+\sqrt{2})} \left(e^u+e^{-u}\right)^{17} du$

(c) $\int_0^{\log(1+\sqrt{2})} \left(e^u-e^{-u}\right)^{17} du$

(d) $\int_0^{\log(1+\sqrt{2})} 2\left(e^u-e^{-u}\right)^{16} du$

5. Let $f:[0,2]\to R$ be a function which is continuous on $[0,2]$ and is differentiable on $(0,2)$ with $f(0)=1$. Let $F(x)=\int_0^{x^2} f\left(\sqrt{t}\right)dt$ for $x\in[0,2]$. If $F'(x)=f'(x)$ for all $x\in(0,2)$, then $F(2)$ equals **[Adv. 2014]**

(a) e^2-1 (b) e^4-1
(c) $e-1$ (d) e^4

6. Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real number) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1} f(x)\,dx$ lies in the interval **[Adv. 2013]**

(a) $(2e-1, 2e)$ (b) $(e-1, 2e-1)$

(c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

7. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is **[2012]**

(a) 0 (b) $\frac{\pi^2}{2}-4$ (c) $\frac{\pi^2}{2}+4$ (d) $\frac{\pi^2}{2}$

8. The value of $\int_{\sqrt{\ell n 2}}^{\sqrt{\ell n 3}} \frac{x\sin x^2}{\sin x^2+\sin(\ell n 6-x^2)}dx$ is **[2011]**

(a) $\frac{1}{4}\ell n\frac{3}{2}$ (b) $\frac{1}{2}\ell n\frac{3}{2}$ (c) $\ell n\frac{3}{2}$ (d) $\frac{1}{6}\ell n\frac{3}{2}$

9. Let f be a real-valued function defined on the interval $(-1,1)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}\,dt$, for all $x\in(-1,1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to **[2010]**

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{e}$

10. Let f be a non-negative function defined on the interval $[0,1]$. If $\int_0^x\sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)\,dt,\quad 0\le x\le 1,$ and $f(0)=0$, then **[2009]**

(a) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(b) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(c) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

(d) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

11. $\int_{-2}^{0}\{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}\,dx$ is equal to **[2005S]**

(a) -4 (b) 0 (c) 4 (d) 6

12. If $\int_{\sin x}^{1} t^2 f(t)\,dt = 1-\sin x$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is **[2005S]**

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) 3 (d) $\sqrt{3}$

13. The value of the integral $\int_0^1\sqrt{\frac{1-x}{1+x}}dx$ is **[2004S]**

(a) $\frac{\pi}{2}+1$ (b) $\frac{\pi}{2}-1$ (c) -1 (d) 1

14. If $f(x)$ is differentiable and $\int_0^{t^2} xf(x)dx = \frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals **[2004S]**

(a) 2/5 (b) –5/2 (c) 1 (d) 5/2

15. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in **[2003S]**

(a) $(-2, 2)$ (b) no value of x
(c) $(0, \infty)$ (d) $(-\infty, 0)$

16. The integral $\int_{-1/2}^{1/2} \left([x] + \ell n\left(\frac{1+x}{1-x}\right)\right) dx$ equal to **[2002S]**

(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $2\ell n\left(\frac{1}{2}\right)$

17. Let $T > 0$ be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x + T) = f(x)$.

If $I = \int_0^T f(x)dx$ then the value of $\int_3^{3+3T} f(2x)dx$ is **[2002S]**

(a) $3/2I$ (b) $2I$ (c) $3I$ (d) $6I$

18. Let $f(x) = \int_1^x \sqrt{2 - t^2}\, dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are **[2002S]**

(a) ± 1 (b) $\pm\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2}$ (d) 0 and 1

19. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x f(t)dt$. If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals **[2001S]**

(a) 5/4 (b) 7 (c) 4 (d) 2

20. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x} dx$, $a > 0$, is **[2001S]**

(a) π (b) $a\pi$ (c) $\pi/2$ (d) 2π

21. The value of the integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is: **[2000S]**

(a) 3/2 (b) 5/2 (c) 3 (d) 5

22. If $f(x) = \begin{cases} e^{\cos x} \sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise,} \end{cases}$ then $\int_{-2}^{3} f(x)dx =$ **[2000S]**

(a) 0 (b) 1 (c) 2 (d) 3

23. Let $g(x) = \int_0^x f(t)dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$, for $t \in [0,1]$ and $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$. Then $g(2)$ satisfies the inequality **[2000S]**

(a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$ (b) $0 \le g(2) < 2$
(c) $\frac{3}{2} < g(2) \le \frac{5}{2}$ (d) $2 < g(2) < 4$

24. If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\, dx$ is **[1999 - 2 Marks]**

(a) $-\pi$ (b) 0 (c) $-\pi/2$ (d) $\pi/2$

25. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to **[1999 - 2 Marks]**

(a) 2 (b) –2 (c) 1/2 (d) –1/2

26. If $g(x) = \int_0^x \cos^4 t\, dt$, then $g(x+\pi)$ equals **[1997 - 2 Marks]**

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$
(c) $g(x)\, g(\pi)$ (d) $\frac{g(x)}{g(\pi)}$

27. The value of $\int_{\pi}^{2\pi} [2\, Sinx]\, dx$ where [.] represents the greatest integer function is **[1995S]**

(a) $\frac{-5\pi}{3}$ (b) $-\pi$ (c) $\frac{5\pi}{3}$ (d) -2π

28. If $f(x) = A\sin\left(\frac{\pi x}{2}\right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)dx = \frac{2A}{\pi}$, then constants A and B are **[1995S]**

(a) $\frac{\pi}{2}$ and $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ and $\frac{3}{\pi}$
(c) 0 and $\frac{-4}{\pi}$ (d) $\frac{4}{\pi}$ and 0

29. The value of $\int_0^{\pi/2} \frac{dx}{1 + \tan^3 x}$ is **[1993 - 1 Marks]**

(a) 0 (b) 1 (c) $\pi/2$ (d) $\pi/4$

30. Let $f: R \to R$ and $g: R \to R$ be continous functions. Then the value of the integral $\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)]\, [g(x) - g(-x)]\, dx$ is **[1990 - 2 Marks]**

(a) π (b) 1 (c) -1 (d) 0

31. For any integer n the integral — $\int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x dx$ has the value **[1985 - 2 Marks]**

(a) π (b) 1 (c) 0 (d) none of these

32. The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ is **[1983 - 1 Mark]**

(a) $\pi/4$ (b) $\pi/2$
(c) π (d) none of these

33. Let a, b, c be non-zero real numbers such that
$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$$
Then the quadratic equation $ax^2+bx+c=0$ has **[1981 - 2 Marks]**
(a) no root in (0, 2)
(b) at least one root in (0, 2)
(c) a double root in (0, 2)
(d) two imaginary roots

34. The value of the definite integral $\int_0^1 (1+e^{-x^2})\,dx$ is **[1981 - 2 Marks]**
(a) -1 (b) 2
(c) $1+e^{-1}$ (d) none of these

2 Integer Value Answer/ Non-Negative Integer

35. The greatest integer less than or equal to
$$\int_1^2 \log_2\left(x^3+1\right)dx + \int_1^{\log_2 9}\left(2^x-1\right)^{\frac{1}{3}}dx$$
is ______________. **[Adv. 2022]**

36. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If $I = \int_0^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$, then the value of $9I$ is ____. **[Adv. 2021]**

37. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.
If $F:[0,\pi]\to\mathbb{R}$ is defined by $F(x) = \int_0^x f(t)\,dt$, and if
$$\int_0^{\pi} (f'(x)+F(x))\cos x\,dx = 2,$$
then the value of $f(0)$ is ______ **[Adv. 2020]**

38. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$, then $27I^2$ equals ____. **[Adv. 2019]**

39. The value of the integral $\int_0^{\frac{1}{2}}\frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{\frac{1}{4}}}dx$ is _____. **[Adv. 2018]**

40. Let $f:\mathrm{R}\to\mathrm{R}$ be a differentiable function such that $f(0)=0$, $f\left(\frac{\pi}{2}\right)=3$ and $f'(0)=1$.
If $g(x) = \int_x^{\frac{\pi}{2}}[f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]\,dt$ for $x\in\left(0,\frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$ ______. **[Adv. 2018]**

41. The total number of distinct $x\in[0,1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x-1$ is **[Adv. 2016]**

42. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is **[Adv. 2015]**

43. Let $f: R\to R$ be a function defined by $f(x) = \begin{cases}[x], & x\le 2\\ 0, & x>2\end{cases}$ where $[x]$ is the greatest integer less than or equal to x, if $I = \int_{-1}^{2}\frac{xf(x^2)}{2+f(x+1)}dx$, then the value of $(4I-1)$ is **[Adv. 2015]**

44. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is **[Adv. 2014]**

45. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by
$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd,}\\ 1+[x]-x & \text{if } [x] \text{ is even}\end{cases}$$
Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx$ is **[2010]**

46. Let f: $\mathbf{R}\to\mathbf{R}$ be a continuous function which satisfies
$$f(x) = \int_0^x f(t)\,dt.$$
Then the value of $f(\ln 5)$ is **[2009]**

3 Numeric/ New Stem Based Questions

47. Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$, $i = 1, 2$, and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$ be functions such that $g_1(x) = 1$, $g_2(x) = |4x-\pi|$ and $f(x) = \sin^2 x$, for all $x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$

Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x).g_i(x)dx,\ i = 1,2$

The value of $\frac{16S_1}{\pi}$ is ____. **[Adv. 2021]**

48. The value of $\frac{48S_2}{\pi^2}$ is ____. **[Adv. 2021]**

49. The value of the integral
$$\int_0^{\pi/2}\frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$$ equals _____ **[Adv. 2019]**

4 Fill in the Blanks

50. Let $\frac{d}{dx}F(x)=\frac{e^{\sin x}}{x}, x>0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}dx=F(k)-F(1)$ then one of the possible values of k is .... **[1997 - 2 Marks]**

51. The value of $\int_1^{e^{37}} \frac{\pi\sin(\pi \ln x)}{x}\,dx$ is ________. **[1997 - 2 Marks]**

52. For $n>0$, $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx=$ ________. **[1996 - 1 Mark]**

53. If for nonzero x, $af(x)+bf\left(\frac{1}{x}\right)=\frac{1}{x}-5$ where $a\neq b$, then $\int_1^2 f(x)\,dx=$ ________. **[1996 - 2 Marks]**

54. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is ________. **[1994 - 2 Marks]**

55. The value of $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi}d\phi$ is ________. **[1993 - 2 Marks]**

56. The value of $\int_{-2}^{2} |1-x^2|\,dx$ is ________. **[1989 - 2 Marks]**

57. The integral $\int_0^{1.5}[x^2]dx$, where [ ] denotes the greatest integer function, equals ________. **[1988 - 2 Marks]**

58. $f(x)=\begin{vmatrix} \sec x & \cos x & \sec^2 x+\cot x\,\mathrm{cosec}\,x \\ \cos^2 x & \cos^2 x & \mathrm{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.

Then $\int_0^{\pi/2} f(x)\,dx=$ ________. **[1987 - 2 Marks]**

5 True / False

59. The value of the integral $\int_0^{2a}\left[\frac{f(x)}{\{f(x)+f(2a-x)\}}\right]dx$ is equal to a. **[1988 - 1 Mark]**

6 MCQs with One or More than One Correct Answer

60. Consider the equation **[Adv. 2022]**

$$\int_1^e \frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}dx=1,\quad a\in(-\infty,0)\cup(1,\infty).$$

Which of the following statements is/are TRUE?
(a) **No** a satisfies the above equation
(b) An integer a satisfies the above equation
(c) An irrational number a satisfies the above equation
(d) More than one a satisfy the above equation

61. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that

$$f(0)=1 \text{ and } \int_0^{\frac{\pi}{3}} f(t)dt=0$$

Then which of the following statements is (are) TRUE? **[Adv. 2021]**
(a) The equation $f(x)-3\cos 3x=0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$
(b) The equation $f(x)-3\sin 3x=-\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$
(c) $\lim_{x\to\infty}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$
(d) $\lim_{x\to\infty}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

62. Which of the following inequalities is/are TRUE? **[Adv. 2020]**
(a) $\int_0^1 x\cos x\,dx\geq\frac{3}{8}$ (b) $\int_0^1 x\sin x\,dx\geq\frac{3}{10}$
(c) $\int_0^1 x^2\cos x\,dx\geq\frac{1}{2}$ (d) $\int_0^1 x^2\sin x\,dx\geq\frac{2}{9}$

63. Let $f:R\to R$ be given by $f(x)=(x-1)(x-2)(x-5)$. Define

$$F(x)=\int_0^x f(t)dt, x>0.$$

Then which of the following options is/are correct? **[Adv. 2019]**
(a) F has a local maximum at $x=2$
(b) F has a local minimum at $x=1$
(c) F has two local maxima and one local minimum in $(0,\infty)$
(d) $F(x)<0$ for all $x\in(0,5)$

64. If $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$, then **[Adv. 2017]**
(a) $I>\log_e 99$ (b) $I<\log_e 99$
(c) $I<\frac{49}{50}$ (d) $I>\frac{49}{50}$

65. If $g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)dt$, then **[Adv. 2017]**
(a) $g'\left(\frac{\pi}{2}\right)=-2\pi$ (b) $g'\left(-\frac{\pi}{2}\right)=2\pi$
(c) $g'\left(\frac{\pi}{2}\right)=2\pi$ (d) $g'\left(-\frac{\pi}{2}\right)=-2\pi$

66. Let $f: R \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)? [Adv. 2017]

(a) $x^9 - f(x)$ (b) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

(c) $e^x - \int_0^x f(t)\sin t\,dt$ (d) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

67. Let $f'(x) = \dfrac{192x^3}{2+\sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are [Adv. 2015]

(a) $m = 13, M = 24$ (b) $m = \frac{1}{4}, M = \frac{1}{2}$

(c) $m = -11, M = 0$ (d) $m = 1, M = 12$

68. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are) [Adv. 2015]

(a) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ (b) $\int_0^{\pi/4} f(x)dx = 0$

(c) $\int_0^{\pi/4} xf(x)dx = \frac{1}{6}$ (d) $\int_0^{\pi/4} f(x)dx = 1$

69. The option(s) with the values of a and L that satisfy the following equation is(are) [Adv. 2015]

$$\frac{\int_0^{4\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt}{\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt} = L?$$

(a) $a = 2, L = \dfrac{e^{4\pi}-1}{e^{\pi}-1}$ (b) $a = 2, L = \dfrac{e^{4\pi}+1}{e^{\pi}+1}$

(c) $a = 4, L = \dfrac{e^{4\pi}-1}{e^{\pi}-1}$ (d) $a = 4, L = \dfrac{e^{4\pi}+1}{e^{\pi}+1}$

70. Let $f: (0,\infty) \to R$ be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$. Then [Adv. 2014]

(a) $f(x)$ is monotonically increasing on $[1,\infty)$

(b) $f(x)$ is monotonically decreasing on (0, 1)

(c) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0,\infty)$

(d) $f(2^x)$ is an odd function of x on R

71. Let $f: [a, b] \to [1,\infty)$ be a continuous function and let $g: R \to R$ be defined as [Adv. 2014]

$$g(x) = \begin{cases} 0, & \text{if } x < a, \\ \int_a^x f(t)dt, & \text{if } a \le x \le b, \text{ then} \\ \int_a^b f(t)dt, & \text{if } x > b. \end{cases}$$

(a) $g(x)$ is continuous but not differentiable at a

(b) $g(x)$ is differentiable on R

(c) $g(x)$ is continuous but not differentiable at b

(d) $g(x)$ is continuous and differentiable at either (a) or (b) but not both

72. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is (are) true? [2010]

(a) $f''(x)$ exists for all $x \in (0,\infty)$

(b) $f'(x)$ exists for all $x \in (0,\infty)$ and f' is continuous on $(0,\infty)$, but not differentiable on $(0,\infty)$

(c) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha,\infty)$

(d) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0,\infty)$

73. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are) [2010]

(a) $\frac{22}{7} - \pi$ (b) $\frac{2}{105}$

(c) 0 (d) $\frac{71}{15} - \frac{3\pi}{2}$

74. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$ $n = 0, 1, 2, \ldots$, then [2009]

(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(c) $\sum_{m=1}^{10} I_{2m} = 0$ (d) $I_n = I_{n+1}$

75. Let $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and

$g(x) = \int_0^x f(t)\,dt, x \in [0, 3]$ then $g(x)$ has [2006 - 5M, –1]

(a) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$

(b) local maxima at $x = 1$ and local minima at $x = 2$

(c) no local maxima

(d) no local minima

76. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3 (t-3)^5 \, dt$ has a local minimum at $x =$ [1999 - 3 Marks]

(a) 0 (b) 1 (c) 2 (d) 3

77. Let $f(x)$ be a non-constant twice differentiable function definied on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then, [2008]

(a) $f''(x)$ vanishes at least twice on [0, 1]

(b) $f'\left(\frac{1}{2}\right) = 0$

(c) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$

(d) $\int_{0}^{1/2} f(t)\, e^{\sin \pi t} \, dt = \int_{1/2}^{1} f(1-t)\, e^{\sin \pi t} \, dt$

78. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x. Then $\int_{-1}^{1} f(x)\,dx$ is [1998 - 2 Marks]

(a) 1 (b) 2

(c) 0 (d) 1/2

79. If $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$, then the value of $f(1)$ is [1998 - 2 Marks]

(a) 1/2 (b) 0 (c) 1 (d) −1/2

7 Match the Following

80.

List - I	List - II
P. The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0) = 0$ and $\int_0^1 f(x)\,dx = 1$, is	**1.** 8
Q. The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	**2.** 2
R. $\int_{-2}^{2} \frac{3x^2}{\left(1+e^x\right)} dx$ equals	**3.** 4
S. $\dfrac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_{0}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$	**4.** 0

[Adv. 2014]

	P	Q	R	S
(a)	3	2	4	1
(b)	2	3	4	1
(c)	3	2	1	4
(d)	2	3	1	4

8 Comprehension/Passage Based Questions

PASSAGE - 1

Let $f: \left[0, \frac{\pi}{2}\right] \to [0,1]$ be the function defined by $f(x) = \sin^2 x$ and let $g: \left[0, \frac{\pi}{2}\right] \to [0,\infty)$ be the function defined by $g(x) = \sqrt{\frac{\pi x}{2} - x^2}$.

81. The value of $2\int_0^{\frac{\pi}{2}} f(x)g(x)dx - \int_0^{\frac{\pi}{2}} g(x)dx$ is ______. [Adv. 2024]

82. The value of $\frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} f(x)g(x)dx$ is ______. [Adv. 2024]

PASSAGE - 2

Let $\psi_1: [0,\infty) \to \mathbb{R}$, $\psi_2: [0,\infty) \to \mathbb{R}$, $f: [0,\infty) \to \mathbb{R}$, and $g: [0,\infty) \to \mathbb{R}$, be functions such that $f(0) = f(0) = 0$,

$\psi_1(x) = e^{-x} + x, \; x \ge 0,$

$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \; x \ge 0,$

$f(x) = \int_{-x}^{x} (|t| - t^2) e^{-t^2} dt, \; x > 0$

and $g(x) = \int_0^{x^2} \sqrt{t}\, e^{-t} dt, \; x > 0$ [Adv. 2021]

83. Which of the following statements is **TRUE** ?

(a) $f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = \frac{1}{3}$

(b) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + ax$

(c) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(d) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

84. Which of the following statements is **TRUE** ?

(a) $\psi_1(x) \le 1$, for all $x > 0$

(b) $\psi_2(x) \le 0$, for all $x > 0$

(c) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(d) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

PASSAGE - 3

Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0, F(3) = -4$ and $F'(x) < 0$ for all $x \in \left(\frac{1}{2}, 3\right)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$. **[Adv. 2015]**

85. The correct statement(s) is(are)

(a) $f'(1) < 0$

(b) $f(2) < 0$

(c) $f'(x) \ne 0$ for any $x \in (1, 3)$

(d) $f'(x) = 0$ for some $x \in (1, 3)$

86. If $\int_1^3 x^2 F'(x)dx = -12$ and $\int_1^3 x^3 F''(x)dx = 40$, then the correct expression(s) is (are)

(a) $9f'(3) + f'(1) - 32 = 0$ (b) $\int_1^3 f(x)dx = 12$

(c) $9f'(3) - f'(1) + 32 = 0$ (d) $\int_1^3 f(x)dx = -12$

PASSAGE - 4

Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in IR$ and let

$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt$ for all $x \in (1, \infty)$. **[2012]**

87. Consider the statements:

P : There exists some $x \in \mathbb{R}$ such that, $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that, $2f(x) + 1 = 2x(1 + x)$

Then

(a) both P and Q are true

(b) P is true and Q is false

(c) P is false and Q is true

(d) both P and Q are false

88. Which of the following is true?

(a) g is increasing on $(1, \infty)$

(b) g is decreasing on $(1, \infty)$

(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$

(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

PASSAGE - 5

Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by

$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2.$ **[2008]**

89. Which of the following is true?

(a) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$

(b) $(2-a)^2 f''(1) - (2+a)^2 f''(-1) = 0$

(c) $f'(1)f'(-1) = (2-a)^2$

(d) $f'(1)f'(-1) = -(2+a)^2$

90. Which of the following is true?

(a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$

(b) $f(x)$ is increasing on $(-1, 1)$ and has a local minimum at $x = 1$

(c) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$

(d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$

91. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2} dt$. Which of the following is true? **[2008]**

(a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$

(d) $g'(x)$ does not change sign on $(-\infty, \infty)$

PASSAGE - 6

Let the definite integral be defined by the formula $\int_a^b f(x)dx = \frac{b-a}{2}(f(a) + f(b))$. For more accurate result for $c \in (a, b)$, we can use $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = F(c)$ so that for $c = \frac{a+b}{2}$, we get $\int_a^b f(x)dx = \frac{b-a}{4}(f(a) + f(b) + 2f(c))$.

92. $\int_0^{\pi/2} \sin x \, dx =$ **[2006 - 5M, –2]**

(a) $\frac{\pi}{8}\left(1+\sqrt{2}\right)$ (b) $\frac{\pi}{4}\left(1+\sqrt{2}\right)$

(c) $\frac{\pi}{8\sqrt{2}}$ (d) $\frac{\pi}{4\sqrt{2}}$

93. If $\lim_{x \to a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$, then $f(x)$ is of maximum degree

(a) 4 (b) 3 (c) 2 (d) 1

94. If $f''(x) < 0 \ \forall x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

(a) $\frac{f(b)-f(a)}{b-a}$ (b) $\frac{2(f(b)-f(a))}{b-a}$

(c) $\frac{2f(b)-f(a)}{2b-a}$ (d) 0

10 Subjective Problems

95. Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx$

[2004 - 4 Marks]

96. If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\, d\theta$, then find $\frac{dy}{dx}$ at $x = \pi$

[2004 - 2 Marks]

97. If f is an even function then prove that

$$\int_0^{\pi/2} f(\cos 2x)\cos x\, dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx$$

[2003 - 2 Marks]

98. Let $f(x), x \geq 0$, be a non–negative continuous function, and let $F(x) = \int_0^x f(t)\,dt, x \geq 0$. If for some $c > 0, f(x) \leq cF(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$.

[2001 - 5 Marks]

99. For $x>0$, let $f(x) = \int_1^x \frac{\ln t}{1+t}dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$. Here, $\ln t = \log_e t$.

[2000 - 5 Marks]

100. Integrate $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}dx$. **[1999 - 5 Marks]**

101. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = 2\int_0^1 \tan^{-1} x\, dx$.

Hence or otherwise, evaluate the integral

$\int_0^1 \tan^{-1}(1-x+x^2)\, dx$. **[1998 - 8 Marks]**

102. Let $a + b = 4$, where $a < 2$, and let $g(x)$ be a differentiable function.

If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\, dx + \int_0^b g(x)dx$ increases as $(b - a)$ increases. **[1997 - 5 Marks]**

103. Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$.

[1997 - 5 Marks]

104. Evaluate the definite integral :

$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$ **[1995 - 5 Marks]**

105. Evalute $\int_2^3 \frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$. **[1993 - 5 Marks]**

106. Determine a positive integer $n \leq 5$, such that

$\int_0^1 e^x(x-1)^n dx = 16 - 6e$ **[1992 - 4 Marks]**

107. Evaluate $\int_0^{\pi} \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$ **[1991 - 4 Marks]**

108. If 'f' is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2!$ **[1991 - 4 Marks]**

109. Prove that for any positive integer k,

$\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + \ldots\ldots\ldots\ldots + \cos(2k-1)x]$

Hence prove that $\int_0^{\pi/2} \sin 2kx \cot x\, dx = \frac{\pi}{2}$

[1990 - 4 Marks]

110. Show that $\int_0^{\pi/2} f(\sin 2x)\sin x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$ **[1990 - 4 Marks]**

111. If f and g are continuous function on $[0, a]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$,

then show that $\int_0^a f(x)g(x)dx = \int_0^a f(x)dx$

[1989 - 4 Marks]

112. Evaluate $\int_0^1 \log[\sqrt{1-x} + \sqrt{1+x}]dx$ **[1988 - 5 Marks]**

113. Investigate for maxima and minima the function

[1988 - 5 Marks]

$$f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]dt$$

114. Evaluate : $\int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\sin x}, 0<\alpha<\pi$ [1986 - 2½ Marks]

115. Evaluate the following : $\int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ [1985 - 2½ Marks]

116. Given a function $f(x)$ such that [1984 - 4 Marks]
(i) it is integrable over every interval on the real line and
(ii) $f(t+x)=f(x)$, for every x and a real t, then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a.

117. Evaluate the following $\int_0^{\frac{1}{2}} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ [1984 - 2 Marks]

118. Evaluate : $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx$ [1983 - 3 Marks]

119. Find the value of $\int_{-1}^{3/2} |x\sin\pi x|\,dx$ [1982 - 3 Marks]

120. Show that $\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$. [1982 - 2 Marks]

Topic-4: Summation of Series by Integration

1 MCQs with One Correct Answer

1. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\,ln(1+t)}{t^4+4}dt$ is [2010]

(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$

2. If $l(m,n)=\int_0^1 t^m(1+t)^n\,dt,$ then the expression for $l(m, n)$ in terms of $l(m+1, n-1)$ is [2003S]

(a) $\frac{2^n}{m+1}-\frac{n}{m+1}l(m+1,n-1)$

(b) $\frac{n}{m+1}l(m+1,n-1)$

(c) $\frac{2^n}{m+1}+\frac{n}{m+1}l(m+1,n-1)$

(d) $\frac{m}{n+1}l(m+1,n-1)$

2 Integer Value Answer/Non-Negative Integer

3. Let the function $f: [1,\infty)\to\mathbb{R}$ be defined by

$$f(t)=\begin{cases}(-1)^{n+1}2, & \text{if } t=2n-1, n\in\mathbb{N},\\ \frac{(2n+1-t)}{2}f(2n-1)+ & \text{if } 2n-1<t<2n+1,\\ \frac{(t-(2n-1))}{2}f(2n+1), & n\in\mathbb{N}.\end{cases}$$

Define $g(x)=\int_1^x f(t)dt, x\in(1,\infty)$. Let α denote the number of solutions of the equation $g(x)=0$ in the interval (1, 8] and $\beta=\lim_{x\to 1^+}\frac{g(x)}{x-1}$. Then the value of $\alpha+\beta$ is equal to _____. [Adv. 2024]

4. For $x\in\mathbb{R}$, let $\tan^{-1}(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the minimum value of the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\int_0^{x\tan^{-1}x}\frac{e^{(t-\cos t)}}{1+t^{2023}}dt$ is [Adv. 2023]

3 Numeric/ New Stem Based Questions

Question Stem

Let $f_1:(0,\infty)\to\mathbb{R}$ and $f_2:(0,\infty)\to\mathbb{R}$ be defined by

$$f_1(x)=\int_0^x\prod_{j=1}^{21}(t-j)^j\,dt,\ x>0$$

and $f_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450, x>0,$
where for any positive integer n and real numbrs $a_1, a_2, ..., a_n$, $\Pi_{i=1}^n a_i$ denotes the product of $a_1, a_2, ..., a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0,\infty)$

5. The value of $2m_1+3n_1+m_1n_1$ is _____. [Adv. 2021]
6. The value of $6m_2+4n_2+8m_2n_2$ is _____. [Adv. 2021]
7. For each positive integer n, let

$$y_n=\frac{1}{n}(n+1)(n+2)...(n+n)^{\frac{1}{n}}$$

For $x\in\mathbb{R}$, let $[x]$ be the greatest integer less than or equal to x. If $\lim_{n\to\infty}y_n=L$, then the value of $[L]$ is _____. [Adv. 2018]

8. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t\,|f(f(t))|\,dt$ for all $x \in [-1, 2]$. If $\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[Adv. 2015]**

9. For, $a \in R, |a| > 1$, let

$$\lim_{x \to \infty} \left(\frac{1 + \sqrt[3]{2} + \ldots.. + \sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \ldots. + \frac{1}{(an+n)^2} \right)} \right) = 54$$

Then the possible value(s) of a is/are **[Adv. 2019]**

(a) -9 (b) 7 (c) -6 (d) 8

6 MCQs with One or More than One Correct Answer

10. Let $f(x) = \lim_{n \to \infty} \left(\frac{n^n (x+n)\left(x + \frac{n}{2}\right) \ldots \left(x + \frac{n}{n}\right)}{n!\left(x^2 + n^2\right)\left(x^2 + \frac{n^2}{4}\right) \ldots \left(x^2 + \frac{n^2}{n^2}\right)} \right)^{\frac{x}{n}}$, for all $x > 0$. Then **[Adv. 2016]**

(a) $f\left(\frac{1}{2}\right) \geq f(1)$ (b) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$

(c) $f'(2) \leq 0$ (d) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

8 Comprehension/Passage Based Questions

Given that for each $a \in (0, 1)$, $\lim_{h \to 0^+} \int_{h}^{1-h} t^{-a}(1-t)^{a-1}\,dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. **[Adv. 2014]**

11. The value of $g\left(\frac{1}{2}\right)$ is

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

12. The value of $g'\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) π (c) $-\frac{\pi}{2}$ (d) 0

10 Subjective Problems

13. Let $I_m = \int_0^{\pi} \frac{1 - \cos mx}{1 - \cos x}\,dx$. Use mathematical induction to prove that $I_m = m\pi, m = 0, 1, 2, \ldots\ldots$ **[1995 - 5 Marks]**

14. Show that $\int_0^{n\pi + v} |\sin x|\,dx = 2n + 1 - \cos v$ where n is a positive integer and $0 \leq v < \pi$. **[1994 - 4 Marks]**

15. Show that : $\lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{6n} \right) = \log 6$ **[1981 - 2 Marks]**

Answer Key

Topic-1 : Standard Integrals, Integration by Substitution, Integration by Parts

1. (c) **2.** (c) **3.** (c) **4.** $\frac{-3}{2}, \frac{35}{36}, R$ **5.** (a, c) **6.** (a, b, d) **7.** (d)

Topic-2 : Integration of the Forms: $\int e^x(f(x) + f'(x))dx$, $\int e^{kx}(df(x) + f'(x))dx$, Integration by Partial Fractions, Integration of Different Expressions of e^x

1. (d)

Topic-3 : Evaluation of Definite Integral by Substitution, Properties of Definite Integrals

1. (c) **2.** (b) **3.** (a) **4.** (a) **5.** (b) **6.** (d) **7.** (b) **8.** (a) **9.** (b) **10.** (c)
11. (c) **12.** (c) **13.** (b) **14.** (a) **15.** (d) **16.** (a) **17.** (c) **18.** (a) **19.** (c) **20.** (c)
21. (b) **22.** (c) **23.** (b) **24.** (c) **25.** (a) **26.** (a) **27.** (a) **28.** (d) **29.** (d) **30.** (d)
31. (c) **32.** (a) **33.** (b) **34.** (d) **35.** (5) **36.** (182) **37.** (4) **38.** (4) **39.** (2) **40.** (2)
41. (1) **42.** (9) **43.** (0) **44.** (2) **45.** (4) **46.** (0) **47.** (2) **48.** (1.5) **49.** (0.5) **50.** (16)
51. (2) **52.** (π^2) **53.** $\frac{1}{a^2-b^2}\left[a(\log 2-5)+\frac{7b}{2}\right]$ **54.** $\frac{1}{2}$ **55.** $\pi(\sqrt{2}-1)$ **56.** (4) **57.** $2-\sqrt{2}$
58. $-\left(\frac{15\pi+32}{60}\right)$ **59.** (True) **60.** (c, d) **61.** (a,b,c) **62.** (a,b,d) **63.** (a, b, d) **64.** (b, d) **65.** (Bonus)
66. (a, b) **67.** (d) **68.** (a, b) **69.** (a, c) **70.** (a, c, d) **71.** (a, c) **72.** (b, c) **73.** (a) **74.** (a, b, c)
75. (a, b) **76.** (b, d) **77.** (a, b, c, d) **78.** (a) **79.** (a) **80.** (d) **81.** (0) **82.** (0.25) **83.** (c) **84.** (d)
85. (a, b, c) **86.** (c, d) **87.** (c) **88.** (b) **89.** (a) **90.** (a) **91.** (b) **92.** (a) **93.** (d) **94.** (b)

Topic-4 : Reduction Formulae for Definite Integration, Gamma & Beta Function, Walli's Formula, Summation of Series by Integration

1. (b) **2.** (a) **3.** (5) **4.** (0) **5.** (57) **6.** (6) **7.** (1) **8.** (7) **9.** (a, d) **10.** (b, c)
11. (a) **12.** (d)

Hints & Solutions

Topic-1: Standard Integrals, Integration by Substitution, Integration by Parts

1. (c) $I=\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$

Let $\sec x+\tan x=t \Rightarrow \sec x-\tan x=\frac{1}{t}$

$\Rightarrow \sec x=\frac{1}{2}\left(t+\frac{1}{t}\right)$ and $\sec x(\sec x+\tan x)\,dx=dt$

$\Rightarrow \sec x\,dx=\frac{dt}{t}$

$\therefore \quad I=\frac{1}{2}\int \frac{\left(t+\frac{1}{t}\right)dt}{t^{9/2}.t}=\frac{1}{2}\int\left(t^{-9/2}+t^{-13/2}\right)dt$

$=\frac{-1}{7}t^{-7/2}-\frac{1}{11}t^{-11/2}+K$

$=-\frac{1}{7t^{7/2}}-\frac{1}{11t^{11/2}}+K=-\frac{1}{t^{11/2}}\left(\frac{1}{11}+\frac{t^2}{7}\right)+K$

$=\frac{-1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

2. (c) Given $I=\int \frac{e^x}{e^{4x}+e^{2x}+1}dx,$

$J=\int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx=\int \frac{e^{3x}}{e^{4x}+e^{2x}+1}dx$

$\therefore \quad J-I=\int \frac{e^x(e^{2x}-1)}{e^{4x}+e^{2x}+1}dx$

Let $e^x=t \Rightarrow e^x dx=dt$

$\therefore \quad J-I=\int \frac{t^2-1}{t^4+t^2+1}dt=\int \frac{1-\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}dt$

Let $t+\frac{1}{t}=u \Rightarrow \left(1-\frac{1}{t^2}\right)dt=du$

$\therefore \quad J-I=\int \frac{du}{u^2-1}=\frac{1}{2}\log\left|\frac{u-1}{u+1}\right|+c$

$=\frac{1}{2}\log\left|\frac{\frac{t^2+1}{t}-1}{\frac{t^2+1}{t}+1}\right|+c=\frac{1}{2}\log\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right|+c$

3. (c) $I=\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx=\int \frac{(\cos^2 x+\cos^4 x)\cos x}{\sin^2 x(1+\sin^2 x)}dx$

$=\int \frac{[1-\sin^2 x+(1-\sin^2 x)^2]\cos x}{\sin^2 x(1+\sin^2 x)}dx$

$=\int \frac{(2-3\sin^2 x+\sin^4 x)\cos x}{\sin^2 x(1+\sin^2 x)}dx$

Put $\sin x=t \Rightarrow \cos x\,dx=dt$

$I=\int \frac{2-3t^2+t^4}{t^4+t^2}dt,\ I=\int\left(1+\frac{2}{t^2}-\frac{6}{t^2+1}\right)dt$

$=t-\frac{2}{t}-6\tan^{-1}t+c$

$=\sin x-2(\sin x)^{-1}-6\tan^{-1}(\sin x)+c$

4. $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx=Ax+B\ln(9e^{2x}-4)+C$

$\Rightarrow \frac{d}{dx}[Ax+B\ln(9e^{2x}-4)+C]=\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}$

$=A+B.\frac{18e^{2x}}{9e^{2x}-4}$

$\Rightarrow A+\frac{18Be^x}{9e^x-4e^{-x}}=\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}$

$\Rightarrow \frac{(9A+18B)e^x-4Ae^{-x}}{9e^x-4e^{-x}}=\frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}$

$\Rightarrow 9A+18B=4;\ -4A=6 \Rightarrow A=\frac{-3}{2};$

$B=\left(4+\frac{27}{2}\right)\frac{1}{18}=\frac{35}{36}$ and C can have any real value.

5. (a, c) Given that $f'(x)=\frac{f(x)}{b^2+x^2}$

$\int \frac{f'(x)}{f(x)}dx=\int \frac{dx}{x^2+b^2}$

$\Rightarrow \ln|f(x)|=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+c$

Now $f(0)=1$

$\therefore c=0$

$\therefore |f(x)|=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)} \Rightarrow f(x)=\pm e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\because f(0)=1 \therefore f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\because f'(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}} \times \frac{1}{b^2+x^2} > 0$ for all value of b.

$\therefore f(x)$ is increasing function for all values of b.

So, option (a) is true.

$$f(-x) = e^{-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$\therefore f(x).f(-x) = e^0 = 1$

So, option (c) is true.

6. (a, b, d) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot \lim_{x\to 0} g(x) = 0$

$$\because\ f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow\ f'(x) = \lim_{h\to 0}\frac{f(x)+f(h)+f(x)\cdot f(h)-f(x)}{h}$$

$$\Rightarrow\ f'(x) = \lim_{h\to 0}\frac{f(h)}{h}(1+f(x))$$

$$\left[\because \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{hg(\mathrm{h})}{h} = 0\right]$$

$\Rightarrow\ f'(x) = 1 + f(x)$

$$\Rightarrow\ \frac{f'(x)}{1+f(x)} = 1 \Rightarrow \int\frac{f'(x)}{1+f(x)}dx = \int dx$$

$\Rightarrow\ \ln(1+f(x)) = x + c$

$\Rightarrow\ 1 + f(x) = e^{x+c}\ (\because f(0) = 0 \Rightarrow c = 0)$

$\Rightarrow\ f(x) = e^x - 1 \Rightarrow f'(x) = e^x$

So, $f(x)$ is differentiable at every $x\in R$

So, option (a) is true.

$f'(0) = e^0 = 1$

So, option (d) is true.

$$g(x) = \frac{f(x)}{x} = \begin{cases}\frac{e^x-1}{x} & ; & x\neq 0\\ 1 & ; & x = 0\end{cases}$$

$\therefore$ $g(x)$ is differentiable at every $x\in R$

So, option (b) is true.

7. **(d)** $F(x) = \int\sin^2 x\,dx = \frac{1}{2}\int(1-\cos 2x)dx$

$$= \frac{1}{4}(2x - \sin 2x) + C$$

Now, $F(x+\pi) = \frac{1}{4}(2x+2\pi-\sin(2x+2\pi)) + C$

$$= \frac{1}{4}[2x+2\pi-\sin 2x] + C \neq F(x)$$

$\therefore$ Statement -1 is false.

Also $\sin^2(x+\pi) = \sin^2 x, \forall x\in R$

$\therefore$ Statement -2 is true.

8. $I = \int(x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}dx$

$$= \int(x^{3m}+x^{2m}+x^m)\left[\frac{2x^{3m}+3x^{2m}+6x^m}{x^m}\right]^{1/m}dx$$

$$= \int\left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(2x^{3m}+3x^{2m}+6x^m)^{1/m}dx$$

Put $2x^{3m}+3x^{2m}+6x^m = y$

$$\Rightarrow\ 6m\left(x^{3m-1}+x^{2m-1}+x^{m-1}\right)dx = dy$$

$$\Rightarrow\ \left(x^{3m-1}+x^{2m-1}+x^{m-1}\right)dx = \frac{1}{6m}dy$$

$$\therefore\ I = \frac{1}{6m}\int y^{1/m}dy = \frac{1}{6m}\left(\frac{y^{\frac{1}{m}+1}}{\frac{1}{m}+1}\right)+c = \frac{1}{6}\left(\frac{y^{\frac{m+1}{m}}}{m+1}\right)+c$$

$$= \frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1}+c$$

9. $I = \int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$= \int\sin^{-1}\left[\frac{x+1}{\sqrt{x^2+2x+\frac{13}{4}}}\right]dx$$

$$= \int\sin^{-1}\left[\frac{x+1}{\sqrt{(x+1)^2)+(3/2)^2}}\right]dx$$

Put $x + 1 = 3/2\tan\theta$, $dx = \frac{3}{2}\sec^2\theta d\theta$

$$\therefore\ I = \int\sin^{-1}\left[\frac{\left(\frac{3}{2}\tan\theta\right)}{\sqrt{\frac{9}{4}\tan^2\theta+\frac{9}{4}}}\right].\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{2}\int\sin^{-1}(\sin\theta).\sec^2\theta d\theta$$

$$= \frac{3}{2}\int\theta\sec^2\theta\,d\theta = \frac{3}{2}\left[\theta\tan\theta - \int\tan\theta\,d\theta\right]$$

$$= \frac{3}{2}[\theta\tan\theta - \log|\sec\theta|] + c$$

$$= \frac{3}{2}\left[\frac{2}{3}(x+1)\tan^{-1}\left[\frac{2}{3}(x+1)\right]\right.$$

$$\left.-\log\sqrt{1+\frac{4}{9}(x+1)^2]}\right]+c$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(9+4x^2+8x+4)$$

$$+\frac{3}{4}\log 9 + c$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13)+c$$

10. $I = \int \frac{(x+1)}{x(1+xe^x)^2}dx = \int \frac{e^x(x+1)}{xe^x(1+xe^x)^2}dx$

Put $1+xe^x = t \Rightarrow (xe^x + e^x)\,dx = dt$

$\therefore \quad I = \int \frac{dt}{(t-1)t^2} = \int\left(\frac{1}{1-t}+\frac{1}{t}+\frac{1}{t^2}\right)dt$

$= -\log|1-t| + \log|t| - \frac{1}{t} + c$

$= \log\left|\frac{t}{1-t}\right| - \frac{1}{t} + c = \log\left|\frac{1+xe^x}{-xe^x}\right| - \frac{1}{1+xe^x} + c$

$= \log\left(\frac{1+xe^x}{xe^x}\right) - \frac{1}{1+xe^x} + c$

11. Let $I = \int \cos 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$

Now we observe that

$\frac{d}{d\theta}\left\{\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}$

$= \frac{d}{d\theta}[\ln(\cos\theta+\sin\theta) - \ln(\cos\theta-\sin\theta)]$

$= \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} + \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

$= \frac{\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta} = \frac{2}{\cos 2\theta}$

On integrating I with respect to θ, by parts we get

$I = \frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \int \frac{\sin 2\theta}{2}\cdot\frac{2}{\cos 2\theta}d\theta$

$= \frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \int \tan 2\theta\, d\theta$

$= \frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \frac{1}{2}\ln\sec 2\theta + c$

12. $I = \int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}} + \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$

$= \int \frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx + \int \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx$

$I = I_1 + I_2$...(i)

where $I_1 = \int \frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx$ and $I_2 = \int \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx$

Let $x = y^{12}$ so that $dx = 12\,y^{11}\,dy$

$\therefore \quad I_1 = \int \frac{12y^{11}}{y^4+y^3}dy = 12\int \frac{y^8}{1+y}dy$

$= 12\int\left(y^7 - y^6 + y^5 - y^4 + y^3 - y^2 + y - 1 + \frac{1}{y+1}\right)dy$

$= 12\left[\frac{y^8}{8} - \frac{y^7}{7} + \frac{y^6}{6} - \frac{y^5}{5} + \frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2} - y + \log|y+1|\right] + c_1$

$= \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3} - 4x^{1/4} + 6x^{1/6} - 12x^{1/12} + 12\log|x^{1/12}+1| + c_1$ ...(ii)

Now, $I_2 = \int \frac{\ln(1+(x)^{1/6})}{(x)^{1/3}+(x)^{1/2}}dx$

Let $x = z^6$ so that $dx = 6z^5\,dz$

$\therefore \quad I_2 = \int \frac{\ln(1+z)}{z^2+z^3}.6z^5 dz = \int \frac{6z^3\ln(z+1)}{z+1}dz$

Put $z + 1 = t$

$\Rightarrow \; dz = dt$

$\therefore \quad I_2 = \int \frac{6(t-1)^3\ln t}{t}\,dt = 6\int (t^2 - 3t + 3 - \frac{1}{t})\ln t\, dt$

$= 6\left[\int (t^2 - 3t + 3)\ln t\, dt - \int \frac{1}{t}\ln t\; dt\right]$

$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t\right)\ln t - \int\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t\right).\frac{1}{t}dt - \frac{(\ln t)^2}{2}\right]$

$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t\right)\ln t - \int\left(\frac{t^2}{3} - \frac{3}{2}t + 3\right)dt - \frac{(\ln t)^2}{2}\right]$

$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t\right)\ln t - \left(\frac{t^3}{9} - \frac{3t^2}{4} + 3t\right) - \frac{(\ln t)^2}{2}\right] + c_2$

Now, $t = 1 + z = 1 + x^{1/6}$

$\therefore \; I_2 = 6\left[\left\{\frac{(1+x^{1/6})^3}{3} - \frac{3}{2}(1+x^{1/6})^2 + 3(1+x^{1/6})\right\}\ln(1+x^{1/6}) - \left\{\frac{(1+x^{1/6})^3}{9} - \frac{3}{4}(1+x^{1/6})^2 + 3(1+x^{1/6})\right\} - \frac{[\ln(1+x^{1/6})^2]}{2}\right] + c_2$ ...(iii)

From (i), (ii) and (iii), we get $I = I_1 + I_2$

$\Rightarrow \; I = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/2} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/13} - 4x^{1/4} + 6x^{1/6} - 12x^{1/12} + 12\log|x^{1/12}+1|$

$+ 6\left[\left\{\frac{(1+x^{1/6})^3}{3} - \frac{3}{2}(1+x^{1/6})^2 + 3(1+x^{1/6})\right\}\ln(1+x^{1/6})\right.$

$$-\left\{\frac{(1+x^{1/6})^3}{9}-\frac{3}{4}(1+x^{1/6})^2+3(1+x^{1/6})\right\}$$

$$\left.-\frac{\left[\ln(1+x^{1/6})^2\right]}{2}+c\right]$$

13. $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}dx$

$=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx\ =\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}$

Put $\sin x-\cos x=t\ \Rightarrow\ (\cos x+\sin x)\,dx=dt$

also $(\sin x-\cos x)^2=t^2\ \Rightarrow\ 1-\sin 2x=t^2$

$\Rightarrow\ \sin 2x=1-t^2\quad \therefore\ I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$

$=\sqrt{2}\sin^{-1}t+c\ =\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$

14. $I=\int\frac{\sqrt{\cos 2x}}{\sin x}dx$

$=\int\sqrt{\frac{\cos^2 x-\sin^2 x}{\sin^2 x}}\,dx=\int\sqrt{\cot^2 x-1}\,dx$

Let $\cot x=\sec\theta\Rightarrow -\operatorname{cosec}^2 x\,dx=\sec\theta\tan\theta\,d\theta$

$\therefore\ I=\int\sqrt{\sec^2\theta-1}.\frac{\sec\theta\tan\theta}{-(1+\sec^2\theta)}d\theta$

$=-\int\frac{\sec\theta.\tan^2\theta}{1+\sec^2\theta}d\theta=-\int\frac{\sin^2\theta}{\cos\theta+\cos^3\theta}d\theta$

$=-\int\frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta=-\int\frac{(1+\cos^2\theta)-2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}d\theta$

$=-\int\sec\theta d\theta+2\int\frac{\cos\theta}{1+\cos^2\theta}d\theta$

$=-\log|\sec\theta+\tan\theta|+2\int\frac{\cos\theta}{2-\sin^2\theta}d\theta$

$=-\log|\sec\theta+\tan\theta|+2.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right|+c$

Now $\cot x=\sec\theta\Rightarrow\tan x=\cos\theta\Rightarrow\sin\theta=\sqrt{1-\tan^2 x}$

$\therefore\ \mathrm{I}=-\log\left|\cot x+\sqrt{\cot^2 x-1}\right|+\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right|+c$

15. $I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Put $x=\cos^2\theta\ \Rightarrow\ dx=-2\cos\theta\sin\theta\,d\theta$

$\therefore\ \ I=-\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.2\sin\theta\cos\theta\,d\theta$

$=-\int\frac{\sin\theta/2}{\cos\theta/2}.2.2\sin(\theta/2)\cos(\theta/2)\cos\theta\,d\theta$

$=-2\int(1-\cos\theta)\cos\theta\,d\theta=-2\int(\cos\theta-\cos^2\theta)d\theta$

$=-2\int\left(\cos\theta-\frac{1+\cos 2\theta}{2}\right)d\theta$

$=-2\left[\sin\theta-\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c\right]$

$=-2\sqrt{1-x}+\left[\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}\right]+c$

$=-\ 2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+c$

16. $I=\int\frac{dx}{x^2\left(x^4+1\right)^{3/4}}=\int\frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $1+\frac{1}{x^4}=t\ \Rightarrow\ \frac{-4}{x^5}dx=dt\ \ \Rightarrow\ \frac{dx}{x^5}=-\frac{dt}{4}$

$\therefore\quad I=\int\frac{-dt}{4t^{3/4}}=\frac{-1}{4}\left(\frac{t^{-3/4+1}}{\frac{-3}{4}+1}\right)+c$

$=-t^{1/4}+c=-\left(1+\frac{1}{x^4}\right)^{1/4}+c$

17. $\int(e^{\log x}+\sin x)\cos x\,dx$

$=\int(x+\sin x)\cos x\,dx\ =\int x\cos x+\frac{1}{2}\int\sin 2x\,dx$

$=x\sin x-\int\sin x\,dx+\frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$

$=x\sin x+\cos x-\frac{1}{4}\cos 2x+c$

18. $I=\int\frac{x^2dx}{(a+bx)^2}$

Let $a+bx=t\ \Rightarrow\ x=\left(\frac{t-a}{b}\right)\ \Rightarrow\ dx=\frac{dt}{b}$

$\therefore\quad I=\frac{1}{b^3}\int\frac{(t-a)^2}{t^2}dt=\frac{1}{b^3}\int\frac{t^2-2at+a^2}{t^2}dt$

$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt\ =\frac{1}{b^3}\left[t-2a\log|t|-\frac{a^2}{t}\right]+c$

$=\frac{1}{b^3}\left[a+bx-2a\log|a+bx|-\frac{a^2}{a+bx}\right]+c$

19. $I\ =\int\frac{\sin x}{\sin x-\cos x}dx$

$=\frac{1}{2}\int\frac{\sin x+\cos x+\sin x-\cos x}{\sin x-\cos x}dx$

$=\frac{1}{2}\int\frac{\cos x+\sin x}{\sin x-\cos x}dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x-\cos x}dx$

$=\frac{1}{2}\log|\sin x-\cos x|+\frac{x}{2}+c$

Topic-2: Integration of the Forms: $\int e^x(f(x)+f'(x))dx$, $\int e^{kx}(df(x)+f'(x))dx$, Integration by Partial Fractions, Integration of Different Expressions of e^x

1. (d) $I=\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx=\int \frac{x^2-1}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$

$$=\frac{1}{4}\int \frac{\frac{4}{x^3}-\frac{4}{x^5}}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$$

Put $2-\frac{2}{x^2}+\frac{1}{x^4}=t \Rightarrow \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx=dt$

$$\therefore\ I=\frac{1}{4}\int \frac{dt}{\sqrt{t}}=\frac{2\sqrt{t}}{4}+c=\frac{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}{2}+c$$

$$=\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$$

2. $I=\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx.$

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

On comparing the coefficient of A, B, C, D and E from both sides and solving them, we get

$$A=-\frac{1}{2}, B=\frac{1}{2}, C=\frac{1}{2}, D=0 \text{ and } E=2.$$

$$\therefore\quad I=-\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{x+1}{x^2+1}dx+2\int\frac{dx}{(x^2+1)^2}$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+2I_1+c_1$$

where $I_1=\int\frac{dx}{(x^2+1)^2}$,

Now put $x=\tan\theta$

$$\therefore\ I_1=\int\frac{\sec^2\theta}{\sec^4\theta}d\theta=\int(\cos^2\theta)\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta$$

$$\Rightarrow\ I_1=\frac{1}{2}(\theta+\frac{1}{2}\sin 2\theta)=\frac{1}{2}\tan^{-1}x+\frac{1}{4}.\frac{2x}{1+x^2}$$

$$\therefore\quad I=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{3}{2}\tan^{-1}x$$

$+\frac{x}{1+x^2}+c$ where c is constant of integration.

3. $I=\int\frac{(x-1)e^x}{(x+1)^3}dx=\int\frac{(x+1-2)e^x}{(x+1)^3}dx$

$$=\int\left[\frac{1}{(x+1)^2}-\frac{2}{(x+1)^3}\right]e^x dx=\frac{e^x}{(x+1)^2}+c$$

$$[\because\ \int e^x(f(x)+f'(x))dx=e^x f(x)]$$

Topic-3: Evaluation of Definite Integral by Substitution, Properties of Definite Integrals

1. (c) Given that $\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt<g(x)<2\sqrt{x}$

$$\Rightarrow \int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt\,\sqrt{n}\le f(x)g(x)\le 2\sqrt{x}\,\sqrt{n}$$

Since,

$$\Rightarrow \int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt=\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}-\sin^{-1}x-x\sqrt{1-x^2}$$

$$\therefore \lim_{x\to 0}\frac{\left(\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}-\sin^{-1}x-x\sqrt{1-x^2}\right)}{\sqrt{x}}\le$$

$$f(x)g(x)\le\frac{2\sqrt{x}}{\sqrt{x}}$$

$$\Rightarrow 2\le\lim_{x\to 0}f(x)g(x)\le 2.$$

$$\Rightarrow \lim_{x\to 0}f(x)g(x)=2.$$

2. (b) Given that

$$f(n)=n+\frac{16+5n-3n^2}{4n+3n^2}+\frac{32+n-3n^2}{8n+3n^2}+...+\frac{25n-7n^2}{7n^2}$$

$$=\left(\frac{16+5n-3n^2}{4n+3n^2}+1\right)+\left(\frac{32+n-3n^2}{8n+3n^2}+1\right)+...+\left(\frac{25n-7n^2}{7n^2}+1\right)$$

$$\Rightarrow f(n)=\frac{9n+16}{4n+3n^2}+\frac{9n+32}{8n+3n^2}+......+\frac{25n}{7n^2}$$

$$=\sum_{r=1}^{n}\frac{9n+16r}{4rn+3n^2}=\frac{1}{n}\sum_{r=1}^{n}\frac{9+16\left(\frac{r}{n}\right)}{4\left(\frac{r}{n}\right)+3}$$

Let $\frac{1}{n}=dx$ and $\frac{r}{n}=x$ when $r=1$ and $n\to\infty$ then $x\to 0$

when $r=n$, then $x\to 1$

$$\therefore \lim_{n\to\infty}f(n)=\int_0^1\frac{9+16x}{4x+3}dx$$

$$=\int_0^1\frac{(16x+12)-3}{4x+3}dx$$

$$=\left(4x-\frac{3}{4}\ln|4x+3|\right)\Big|_0^1\ =4-\frac{3}{4}\ln\frac{7}{3}$$

3. (a) $I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$ ....(i)

$I=\int_{-\pi/2}^{\pi/2}\frac{e^x x^2\cos x}{1+e^x}dx$ ....(ii)

$\left[\because \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right]$

On adding (i) and (ii), we get

$2I=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx=2\int_0^{\pi/2}x^2\cos x\,dx$

$I=\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\frac{\pi}{2}}=\frac{\pi^2}{4}-2$

4. (a) Let $I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\operatorname{cosec}x)^{17}dx$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\operatorname{cosec}x+\cot x+\operatorname{cosec}x-\cot x)^{16}\,2\operatorname{cosec}x\,dx$

$I=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\operatorname{cosec}x+\cot x+\frac{1}{\operatorname{cosec}x+\cot x}\right)^{16}.\operatorname{cosec}x\,dx$

Let $\operatorname{cosec}x+\cot x=e^u$

$\Rightarrow(-\operatorname{cosec}x\cot x-\operatorname{cosec}^2x)\,dx=e^u\,du$

$\Rightarrow-\operatorname{cosec}x\,dx=du$

Also at $x=\frac{\pi}{4}$, $u=ln\left(\sqrt{2}+1\right)$

and at $x=\frac{\pi}{2}$, $u=ln\ 1=0$

$\therefore I=-2\int_{ln(\sqrt{2}+1)}^{0}\left(e^u+e^{-u}\right)^{16}du=2\int_0^{ln(\sqrt{2}+1)}\left(e^u+e^{-u}\right)^{16}du$

5. (b) $F(x)=\int_0^{x^2}f\left(\sqrt{t}\right)dt$ for $x\in[0,2]$

$\Rightarrow F'(x)=f(x).2x$

Now $F'(x)=f'(x)\,\forall x\in(0,2)$

$\Rightarrow f(x).2x=f'(x)\Rightarrow\frac{f'(x)}{f(x)}=2x$

On integrating w.r.t. x, we get

$ln\,f(x)=x^2+c\Rightarrow f(x)=e^{x^2+c}=e^{x^2}.\,e^c$

Since $f(0)=1\Rightarrow 1=e^c$, $\therefore f(x)=e^{x^2}$

Hence $F(x)=\int_0^{x^2}e^x dx=e^{x^2}-1$, $\therefore F(2)=e^4-1$

6. (d) We have $f'(x)-2f(x)<0$

$\Rightarrow e^{-2x}f'(x)-2e^{-2x}f(x)<0\Rightarrow\frac{d}{dx}(e^{-2x}f(x))<0$

$\Rightarrow e^{-2x}f(x)$ is strictly decreasing function on $\left[\frac{1}{2},1\right]$

$\therefore\ e^{-2x}f(x)<e^{-1}f\left(\frac{1}{2}\right)$ or $f(x)<e^{2x-1}$

Also given that $f(x)$ is positive function so $f(x)>0$

$\therefore\ 0<f(x)<e^{2x-1}$

$\Rightarrow\ 0<\int_{1/2}^1 f(x)dx<\int_{1/2}^1 e^{2x-1}dx$

$\Rightarrow\ 0<\int_{1/2}^1 f(x)dx<\left[\frac{e^{2x-1}}{2}\right]_{1/2}^1$

$\Rightarrow\ \int_{1/2}^1 f(x)dx\in\left(0,\frac{e-1}{2}\right)$

7. (b) $\int_{-\pi/2}^{\pi/2}\left[x^2+\ln\left(\frac{\pi+x}{\pi-x}\right)\right]\cos x\,dx$

$=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx+\int_{-\pi/2}^{\pi/2}\ln\left(\frac{\pi+x}{\pi-x}\right)\cos x\,dx$

$=2\int_0^{\pi/2}x^2\cos x\,dx+0$ [$\because x^2\cos x$ is an even function and $\ln\left(\frac{\pi+x}{\pi-x}\right)\cos x$ is an odd function]

$=2\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}=\frac{\pi^2}{2}-4$

8. (a) $I=\frac{1}{2}\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}}\frac{2x\sin x^2}{\sin x^2+\sin(\ell n6-x^2)}dx$

Let $x^2=t\Rightarrow 2x\,dx=dt$

Also, when $x=\sqrt{\ell n2}$, then, $t=\ell n2$

when $x=\sqrt{\ell n3}$, then, $t=\ell n3$

$\therefore I=\frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin t\,dt}{\sin t+\sin(\ell n6-t)}$ ...(i)

$\Rightarrow I=\frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin(\ell n6-t)}{\sin t+\sin(\ell n6-t)}dt$ ...(ii)

$\left[\because\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

On adding equations (i) and (ii), we get

$2I=\frac{1}{2}\int_{\ell n2}^{\ell n3}1\,dt=\frac{1}{2}(\ell n3-\ell n2)=\frac{1}{2}\ell n\frac{3}{2}\Rightarrow I=\frac{1}{4}\ell n\frac{3}{2}$

9. **(b)** $e^{-x}f(x)=2+\int_0^x\sqrt{1+t^4}\,dt\ \forall\ x\in(-1,1)$

On differentiating, we get

$-e^{-x}f(x)+e^{-x}f'(x)=0+\sqrt{1+x^4}$

$\Rightarrow -f(0)+f'(0)=1\Rightarrow f'(0)=1+f(0)=1+2=3$

Now $f^{-1}(f(x))=x$

$\Rightarrow \left[\left(f^{-1}\right)'(f(x))\right]f'(x)=1$

$\Rightarrow \left(f^{-1}\right)'(f(0))f'(0)=1\ \Rightarrow \left(f^{-1}\right)'(2)=\frac{1}{3}$

10. **(c)** Given that f is a non negative function defined on $[0, 1]$ and $\int_0^x\sqrt{1-(f'(t))^2}\ dt=\int_0^x f(t)\,dt,\ \ 0\le x\le 1$

Differentiating both sides with respect to x, we get

$\sqrt{1-[f'(x)]^2}=f(x)$

$\Rightarrow 1-[f'(x)]^2=[f(x)]^2\ \Rightarrow\ [f'(x)]^2=1-[f(x)]^2$

$\Rightarrow \frac{d}{dx}f(x)=\pm\sqrt{1-[f(x)]^2}\ \Rightarrow \pm\frac{d\,f(x)}{\sqrt{1-[f(x)]^2}}=dx$

Integrating both sides with respect to x, we get

$\pm\int\frac{d\,f(x)}{\sqrt{1-[f(x)]^2}}=\int dx\Rightarrow\ \pm\sin^{-1}f(x)=x+C$

$\because$ Given that $f(0)=0\ \Rightarrow\ C=0$

Hence $f(x)=\pm\sin x$

But as $f(x)$ is a non negative function on $[0, 1]$

$\therefore\ f(x)=\sin x.$

Now $\sin x<x,\forall\,x>0$

$\therefore\ \ f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}.$

11. **(c)** $I=\int_{-2}^{0}[x^3+3x^2+3x+3+(x+1)\cos(x+1)]dx$

$=\left[\frac{x^4}{4}+x^3+\frac{3x^2}{2}+3x+(x+1)\sin(x+1)+\cos(x+1)\right]_{-2}^{0}$

$=(\sin 1+\cos 1)-(4-8+6-6+\sin 1+\cos 1)\ =4$

12. **(c)** $\int_{\sin x}^{1}t^2f(t)\,dt=1-\sin x$

$\Rightarrow\ \frac{d}{dx}\int_{\sin x}^{1}t^2f(t)\,dt=\frac{d}{dx}(1-\sin x)$

$\Rightarrow\ -\sin^2 x\,f(\sin x).\cos x=-\cos x$

$\Rightarrow\ f(\sin x)=\frac{1}{\sin^2 x}\ \Rightarrow\ f(1/\sqrt{3})=\frac{1}{(1/\sqrt{3})^2}=3$

13. **(b)** $I=\int_0^1\sqrt{\frac{1-x}{1+x}}dx\ =\int_0^1\frac{1-x}{\sqrt{1-x^2}}dx$

$=\sin^{-1}x\Big|_0^1-\left(-\frac{1}{2}\right)\int_0^1\frac{2x}{\sqrt{1-x^2}}dx$

$=\frac{\pi}{2}+\frac{1}{2}\left[2\sqrt{1-x^2}\Big|_0^1\right]\ =\frac{\pi}{2}-1$

14. **(a)** $\int_0^{t^2}xf(x)dx=\frac{2}{5}t^5$ (Here, $t>0$)

Differentiating both sides w.r.t. t [Using Leibnitz theorem]

$\Rightarrow\ t^2f(t^2)\times 2t-0=\frac{2}{5}\times 5t^4\ \Rightarrow\ f(t^2)=t$

$\left[\because,\ \frac{d}{dx}\int_{\phi(x)}^{\psi(x)}f(t)dt=f[\psi(x)].\psi'(x)-f[\phi(x)]\,\phi'(x)\right]$

$\therefore\ \ f\left(\frac{4}{25}\right)=\frac{2}{5}$

15. **(d)** We have $f(x)=\int_{x^2}^{x^2+1}e^{-t^2}dt$

Then $f'(x)=e^{-(x^2+1)^2}.2x-e^{-x^4}.2x$

$\left[\because,\ \frac{d}{dx}\int_{\phi(x)}^{\psi(x)}f(t)dt=f[\psi(x)].\psi'(x)-f[\phi(x)]\,\phi'(x)\right]$

$\because (x^2+1)^2>x^4\ \therefore f'(x)>0,\forall x<0$

$\therefore$ $f(x)$ increases when $x<0$

16. **(a)** Let $I=\int_{-1/2}^{1/2}\left([x]+\ln\left(\frac{1+x}{1-x}\right)\right)dx$

$=\int_{-1/2}^{1/2}[x]dx+\int_{-1/2}^{1/2}\ln\left(\frac{1+x}{1-x}\right)dx$

$=\int_{-1/2}^{0}(-1)\,dx+0=-1/2$

$\left[\because \log\left(\frac{1+x}{1-x}\right)\text{is an odd function}\right]$

17. **(c)** Given that $T>0$ is a fixed real number. f is continuous $\forall x\in R$ such that $f(x+T)=f(x)$

$\Rightarrow f$ is a periodic function of period T

Also given $I=\int_0^T f(x)dx$

Then let $I_1=\int_3^{3+3T} f(2x)dx$

Put $2x=z \Rightarrow dx=\frac{dz}{2}$

also as $x\to 3, z\to 6$; as $x\to 3+3T, z\to 6+6T$

$$I_1=\frac{1}{2}\int_6^{6+6T} f(z)dz$$

$$=\frac{1}{2}\left[\int_6^T f(z)dz+\sum_{n=1}^{5}\int_{nT}^{(n+1)T} f(z)dz+\int_{6T}^{6T+6} f(z)dz\right]$$

Now, $\int_{nT}^{(n+1)T} f(z)dz=\int_0^T f(nT+u)du,$

where $z=nT+u$

$$=\int_0^T f(u)du=\text{I} \qquad [\because f(nT+u)=f(u)]$$

Similarly, we can show that

$$\int_{6T}^{6T+6} f(z)dz=\int_0^6 f(z)dz$$

$$\therefore I_1=\frac{1}{2}\left[\int_6^T f(z)dz+5I+\int_0^6 f(z)dz\right]$$

$$=\frac{1}{2}\left[\int_0^T f(z)dz+5I\right]=\frac{1}{2}(6I)=3\text{I}$$

18. **(a)** Here $f(x)=\int_1^x \sqrt{2-t^2}\,dt \Rightarrow f'(x)=\sqrt{2-x^2}$

$$\left[\because \frac{d}{dx}\int_{\psi(x)}^{\psi(x)} f(t)\,dt=f[\psi(x)]\psi'(x)-f[\phi(x)]\phi'(x)\right]$$

Hence, the given equation $x^2-f'(x)=0$ becomes

$$x^2-\sqrt{2-x^2}=0 \Rightarrow x^2=\sqrt{2-x^2} \Rightarrow x=\pm 1$$

19. **(c)** Given : $F(x)=\int_0^x f(t)dt$ and $F(x^2)=x^2(1+x)$

$\therefore$ $F'(x)=f(x)$ and $F'(x^2).2x=2x+3x^2$

$$\Rightarrow F'(x^2)=\left(\frac{2+3x}{2}\right) \Rightarrow f(x^2)=\frac{2+3x}{2}$$

$$[\because F'(x)=f(x)]$$

$$\therefore f(4)=\frac{2+3\times 2}{2}=\frac{8}{2}=4$$

20. **(c)** $I=\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$ .....(i)

Put $x=-y \Rightarrow dx=-dy$

$$I=-\int_{\pi}^{-\pi}\frac{\cos^2 y}{1+a^{-y}}dy=\int_{-\pi}^{\pi}\frac{a^y\cos^2 y}{1+a^y}dy$$

$$I=-\int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{1+a^x}dx\left[\because\int_a^b f(y)dy=\int_a^b f(x)dx\right] \text{ ...(ii)}$$

On adding (i) and (ii), we get

$$2I=\int_{-\pi}^{\pi}\frac{(1+a^x)\cos^2 x}{(1+a^x)}dx=\int_{-\pi}^{\pi}\cos^2 x dx$$

$$2I=2\int_0^{\pi}\cos^2 x dx \qquad \text{[Even function]}$$

$$I=2\int_0^{\pi/2}\cos^2 x dx \qquad \text{....(iii)}$$

$$\left[\because\int_0^{2a} f(x)dx=2\int_0^a f(x)dx \text{ if } f(2a-x)=f(x)\right]$$

$$=2\int_0^{\pi/2}\sin^2 x dx \qquad \text{....(iv)}$$

On adding (iii) and (iv), we get

$$2I=2\int_0^{\pi/2}(\cos^2 x+\sin^2 x)dx=2.\pi/2=\pi \quad \therefore I=\pi/2$$

21. **(b)** Let $I=\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$

We know that for $\frac{1}{e}<x<1, \log_e x<0$ and hence

$$\frac{\log_e x}{x}<0$$

Also for $1<x<e^2$, $\log_e x>0$ and hence $\frac{\log_e x}{x}>0$

$$\therefore \quad I=\int_{1/e}^{1}\left[-\frac{\log_e x}{x}\right]dx+\int_1^{e^2}\frac{\log_e x}{x}dx$$

$$=-\frac{1}{2}\left[(\log_e x)^2\right]_{1/e}^{1}+\frac{1}{2}\left[(\log_e x)^2\right]_1^{e^2}=\frac{1}{2}+2=\frac{5}{2}.$$

22. **(c)** If $f(x)=\begin{cases} e^{\cos x}\sin x & for\,|x|\le 2 \\ 2 & \text{otherwise}\end{cases}$

$$\Rightarrow f(x)=\begin{cases} e^{\cos x}\sin x & for -2\le x\le 2 \\ 2 & \text{otherwise}\end{cases}$$

$$\int_{-2}^{3} f(x)dx=\int_{-2}^{2} f(x)dx+\int_2^3 f(x)dx$$

$$= \int_{-2}^{2} e^{\cos x} \sin x dx + \int_{2}^{3} 2dx = 0 + 2[x]_2^3$$

[$\because$ $e^{\cos x}$ sin x is an odd function.]

$= 2\,[3 - 2] = 2$

23. **(b)** $g(x) = \int_0^x f(t)dt$

$$\Rightarrow \ g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt$$

Now, $\frac{1}{2} \le f(t) \le 1$ for $t \in [0,1]$

$$\Rightarrow \int_0^1 \frac{1}{2} dt \le \int_0^1 f(t)dt \le \int_0^1 1dt$$

$$\Rightarrow \ \frac{1}{2} \le \int_0^1 f(t)dt \le 1 \qquad \text{... (i)}$$

Again, $0 \le f(t) \le \frac{1}{2}$ for $t \in [1,2]$

$$\Rightarrow \ \int_1^2 0dt \le \int_1^2 f(t)dt \le \int_1^2 \frac{1}{2}dt$$

$$\Rightarrow \ 0 \le \int_1^2 f(t)dt \le \frac{1}{2} \qquad \text{... (ii)}$$

From (i) and (ii), we get

$$\frac{1}{2} \le \int_0^1 f(t)dt + \int_1^2 f(t)dt \le \frac{3}{2} \ \Rightarrow \frac{1}{2} \le g(2) \le \frac{3}{2}$$

Hence, $0 \le g(2) \le 2$ is the most appropriate solution.

24. **(c)**
$$[2\sin x] = \begin{cases} 1, & \text{if } \frac{\pi}{2} \le x < \frac{5\pi}{6} \\ 0, & \text{if } \frac{5\pi}{6} \le x < \pi \\ -1, & \text{if } \pi \le x < \frac{7\pi}{6} \\ -2, & \text{if } \frac{7\pi}{6} \le x \le \frac{3\pi}{2} \end{cases}$$

$$\therefore \quad I = \int_{\pi/2}^{5\pi/6} 1.dx + \int_{5\pi/6}^{\pi} 0.dx + \int_{\pi}^{7\pi/6} (-1)dx + \int_{7\pi/6}^{3\pi/2} (-2)dx$$

$$= \left[\frac{5\pi}{6} - \frac{\pi}{2}\right] + 0 - 1\left[\frac{7\pi}{6} - \pi\right] - 2\left[\frac{3\pi}{2} - \frac{7\pi}{6}\right]$$

$$= \frac{2\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6} = \frac{-3\pi}{6} = \frac{-\pi}{2}$$

25. **(a)** $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ ... (i)

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi - x)} \left[\because \int_a^b f(x)dx = \int_a^b (f(a+b-x)\,dx \right]$$

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1 - \cos x} \qquad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x} \right) dx$$

$$= \int_{\pi/4}^{3\pi/4} 2\operatorname{cosec}^2 x\, dx = -2\left[\cot x\right]_{\pi/4}^{3\pi/4}$$

$$= -2[\cot 3\pi/4 - \cot \pi/4] = -2(-1-1) = 4$$

$\therefore \quad I = 2$

26. **(a)** $g(x) = \int_0^x \cos^4 t\, dt$

$$\therefore\ g(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt = \int_0^{\pi} \cos^4 t\, dt + \int_{\pi}^{x+\pi} \cos^4 t\, dt$$

$$g(x+\pi) = g(\pi) + I, \text{ where } I = \int_{\pi}^{x+\pi} \cos^4 t\, dt$$

Put $t = \pi + y \Rightarrow dt = dy$

Also as $t \to \pi, y \to 0$

As $t \to x + \pi, y \to x$

$$\therefore \quad I = \int_0^x \cos^4(\pi + y)\, dy = \int_0^x \cos^4 y\, dy$$

$$= \int_0^x \cos^4 t\, dt = g(x)$$

Hence, $g(x+\pi) = g(\pi) + g(x)$

27. **(a)** Let $I = \int_{\pi}^{2\pi} [2\sin x]dx$

Now, $\pi \le x < 7\pi/6 \Rightarrow -1 \le 2\sin x < 0$

$\Rightarrow [2\sin x] = -1$

and $7\pi/6 \le x < 11\pi/6 \Rightarrow -2 \le 2\sin x < -1$

$\Rightarrow$ [2 sin x] = – 2

$\frac{11\pi}{6} \le x \le 2\pi$

$-1 < 2 \sin x < 0$

$[2\sin x] = -1$

$$\therefore \quad I = \int_{\pi}^{7\pi/6} (-1)\,dx + \int_{7\pi/6}^{11\pi/6} (-2)\,dx + \int_{11\pi/6}^{2\pi} (-1)\,dx$$

$$= \left(-\frac{7\pi}{6} + \pi\right) + 2\left(-\frac{11\pi}{6} + \frac{7\pi}{6}\right) + \left(-2\pi + \frac{11\pi}{6}\right)$$

$$= -\frac{\pi}{6} - \frac{8\pi}{6} - \frac{\pi}{6} = -\frac{10\pi}{6} = \frac{-5\pi}{3}$$

28. **(d)** $f(x) = A\sin(\pi x/2) + B$

$$\Rightarrow f'(x) = \frac{A\pi}{2}\cos\left(\frac{\pi x}{2}\right) \Rightarrow f'\left(\frac{1}{2}\right) = \frac{A\pi}{2}\cos\frac{\pi}{4} = \sqrt{2}$$

$$\Rightarrow A = 4/\pi \text{ and } \int_0^1 f(x)dx = \frac{2A}{\pi}$$

$$\Rightarrow \int_0^1 \left[A\sin\left(\frac{\pi x}{2}\right) + B\right]dx = \frac{2A}{\pi}$$

$$\Rightarrow \left|-\frac{2A}{\pi}\cos\left(\frac{\pi x}{2}\right) + Bx\right|_0^1 = \frac{2A}{\pi}$$

$$\therefore \quad B + \frac{2A}{\pi} = \frac{2A}{\pi} \Rightarrow B = 0$$

29. **(d)** Let $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^3 x} = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x + \cos^3 x}dx$ ... (i)

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^3 x + \sin^3 x}dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x + \sin^3 x}{\sin^3 x + \cos^3 x}dx = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2}; \ \therefore \ I = \frac{\pi}{4}$$

30. **(d)** $I = \int_{-\pi/2}^{\pi/2} \{f(x) + f(-x)\}\{g(x) - g(-x)\}dx$

Let $F(x) = (f(x) + f(-x))(g(x) - g(-x))$

then $F(-x) = (f(-x) + f(x))(g(-x) - g(x))$

$$= -[f(x) + f(-x)][g(x) - g(-x)] = -F(x)$$

Hence, $F(x)$ is an odd function, $\therefore \ I = 0$

31. **(c)** $I = \int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x dx, n \in Z$ ... (i)

$$= \int_0^{\pi} e^{\cos^2(\pi - x)}\cos^3[(2n+1)(\pi - x)]dx$$

$$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\therefore \quad I = \int_0^{\pi} e^{\cos^2 x}\cos^3[(2n+1)\pi - (2n+1)x]dx$$

$$I = \int_0^{\pi} (-e^{\cos^2 x}\cos^3(2n+1)x)\,dx \quad \text{...(ii)}$$

On adding (i) and (ii), we get

$2I = 0 \Rightarrow I = 0$

32. **(a)** $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}dx$ ...(i)

$$= \int_0^{\pi/2} \frac{\sqrt{\cot(\pi/2 - x)}}{\sqrt{\cot(\pi/2 - x)} + \sqrt{\tan(\pi/2 - x)}}dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} = dx \quad \text{... (ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cot x} + \sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}} = dx = \int_0^{\pi/2} 1.dx$$

$$= [x]_0^{\pi/2} = \pi/2, \quad \therefore$$

$I = \pi/4$

33. **(b)** $\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx$

$$= \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx$$

$$= \int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx + \int_1^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx$$

$$\Rightarrow \int_1^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = 0$$

We know that if $\int_{\alpha}^{\beta} f(x)dx = 0$, then $f(x)$ is + ve on some part of (α, β) and – ve on other part of (α, β).

But here $1 + \cos^8 x$ is always + ve,

$\therefore$ $ax^2 + bx + c$ is + ve on some part of [1 , 2] and – ve on other part [1, 2]

$\therefore$ $ax^2 + bx + c = 0$ has at least one root in (1, 2).

Hence, $ax^2 + bx + c = 0$ has at least one root in (0, 2).

34. **(d)**

$$\int_0^1 (1 + e^{-x^2})dx = \int_0^1 \left(1 + 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\infty\right)dx$$

$$= \left[2x - \frac{x^3}{3.1!} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \ldots\infty\right]_0^1$$

$$= \left[2 - \frac{1}{3.1!} + \frac{1}{5.2!} - \frac{1}{7.3!} + \ldots\infty\right]$$

35. **(5)** $\because$ $I=\int_1^2 \log_2(x^3+1)\,dx+\int_1^{\log_2 9}(2^x-1)^{\frac{1}{3}}dx$

Let $I_2=\int_1^{\log_2 9}(2^x-1)^{\frac{1}{3}}dx$

Put $2^x-1=t^3 \Rightarrow 2^x \ln 2dx = 3t^2dt$

$\Rightarrow \quad dx=\dfrac{3t^2}{\ln 2(t^3+1)}dt$

when $x=1$, $t=1$ when $x=\log_2 9$, $t=2$

$\therefore \quad I_2=\int_1^2 \dfrac{3t^3}{(t^3+1)\ln 2}dt$

So, $I=\int_1^2 \log_2(x^3+1)dx+\int_1^2 \dfrac{3t^3}{\ln 2(t^3+1)}dt$

$\Rightarrow \quad I=\int_1^2\left(\log_2\left(t^3+1\right)+t\,.\dfrac{3t^2}{(t^3+1)\ln 2}\right)dt$

$$\left[\because \int_a^b f(x)dx=\int_a^b f(t)dt\right]$$

$=\int_1^2 d(t.\log_2(t^3+1))$

$=t\,.\log_2(t^3+1)\Big|_1^2$

$=2\log_2 9-1\log_2 2$

$=2\log_2 9-1$

$\because \quad 8<9<2^{7/2} \Rightarrow 3<\log_2 9<\dfrac{7}{2}$

$\Rightarrow \quad 5<2\log_2 9-1<6$

So, $[I]=5$

36. **(182)** Let $y=\dfrac{10x}{x+1}$, $\qquad 0\le x\le 10$

$\Rightarrow \quad xy+y=10x$

$\Rightarrow \quad x=\dfrac{y}{10-y}$

$\Rightarrow \quad 0\le\dfrac{y}{10-y}\le 10$

$\Rightarrow \quad \dfrac{y}{y-10}\le 0$ and $\dfrac{y}{10-y}\le 10$

$\Rightarrow \quad \dfrac{11y-100}{y-10}\ge 0$

$\Rightarrow$ sign scheme: $+$ $-$ $+$ on 0, 10; $+$ $-$ $+$ on $\dfrac{100}{11}$, 10

$y\in[0,10)$ and $y\in\left(-\infty,\dfrac{100}{11}\right]U(10,\infty)$

$\Rightarrow \quad y\in\left[0,\dfrac{100}{11}\right]$

$\sqrt{y}\in\left[0,\dfrac{10}{\sqrt{11}}\right] \quad \Rightarrow \quad \left[\sqrt{y}\right]=\{0,1,2,3\}.$

Case I : $0\le\dfrac{10x}{x+1}<1$

$\therefore \quad x\in\left[0,\dfrac{1}{9}\right)$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=0$

Case II : $1\le\dfrac{10x}{x+1}<4$

$\therefore \quad x\in\left[\dfrac{1}{9},\dfrac{2}{3}\right)$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=1$

Case III : $4<\dfrac{10x}{x+1}<9$

$\therefore \quad x\in\left[\dfrac{2}{3},9\right)$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=2$

Case IV : $x\in[9,10)$

$\Rightarrow$ then $\left[\sqrt{\dfrac{10x}{x+1}}\right]=3$

$\therefore \quad I=\int_0^{1/9}0.dx+\int_{1/9}^{2/3}1.dx+\int_{2/3}^{9}2.dx+\int_9^{10}3.dx$

$=\left(\dfrac{2}{3}-\dfrac{1}{9}\right)+2\left(9-\dfrac{2}{3}\right)+3(10-9)$

$=\dfrac{5}{9}+\dfrac{50}{3}+3=\dfrac{182}{9}$

$\therefore \quad 9I=182.$

37. **(4.00)** $F(x)=\int_0^x f(t).dt$

$\Rightarrow F'(x)=f(x)$

$I=\int_0^\pi f'(x).\cos x dx+\int_0^\pi F(x)\cos x dx=2 \qquad \ldots(i)$

$I=\int_0^\pi f'(x).\cos x dx+\int_0^\pi F(x)\cos x dx=2$

Using by parts

$\Rightarrow I=(\cos x.f(x))_0^\pi+\int_0^\pi \sin x.f(x)dx+\int_0^\pi F(x)\cos x dx=2$

$\Rightarrow I=6-f(0)+\int_0^\pi \sin x.F'(x)dx+\int_0^\pi F(x)\cos x dx=2$

$$I = 6 - f(0) + \int_0^{\pi} F'(x)\sin x dx + \int_0^{\pi} F(x)\cos x dx = 2$$

$$\Rightarrow I = 6 - f(0) + [\sin x F(x)]_0^{\pi} - \int_0^{\pi} F(x)\cos x dx$$

$$+\int_0^{\pi} F(x)\cos x dx = 2$$

$$\Rightarrow I = 6 - f(0) + 0 = 2$$
$$\Rightarrow f(0) = 4$$

38. **(4)** $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{\left(1+e^{\sin x}\right)(2-\cos 2x)}$ ...(i)

$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{\left(1+e^{-\sin x}\right)(2-\cos 2x)}$$

$$\left[\text{using} \int_a^b f(x)\text{dx} = \int_a^b f(a+b-x)dx\right]$$

$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{e^{\sin x}}{\left(e^{\sin x}+1\right)(2-\cos 2x)} \quad \text{...(ii)}$$

Adding (i) and (ii):

$$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{1+e^{\sin x}}{\left(e^{\sin x}+1\right)(2-\cos 2x)}dx$$

$$= \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}dx$$

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{dx}{1+2\sin^2 x}$$

$$= \frac{2}{\pi}\int_0^{\pi/4} \frac{\sec^2 x}{\sec^2 x + 2\tan^2 x}dx = \frac{2}{\pi}\int_0^{\pi/4} \frac{\sec^2 x}{1+3\tan^2 x}dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

At $x = 0$, $t = 0$; At $x = \frac{\pi}{4}$, $t = 1$

$$\therefore I = \frac{2}{\pi}\int_0^1 \frac{1}{1+3t^2}dt = \frac{2}{\pi}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}t\right)\right]_0^1$$

$$= \frac{2}{\pi}\left[\frac{1}{\sqrt{3}} \times \frac{\pi}{3}\right] = \frac{2}{3\sqrt{3}}$$

$$\therefore 27 I^2 = 4$$

39. **(2)** Let $I = \int_0^{\frac{1}{2}} \frac{(1+\sqrt{3})\text{dx}}{\left[(1+\text{x})^2 (1-\text{x})^6\right]^{1/4}}$

$$= \int_0^{\frac{1}{2}} \frac{(1+\sqrt{3})\text{dx}}{(1+\text{x})^2 \left[\frac{(1-\text{x})^6}{(1+\text{x})^6}\right]^{1/4}}$$

Now put $\frac{1-\text{x}}{1+\text{x}} = \text{t} \Rightarrow \frac{-2\text{dx}}{(1+\text{x})^2} = \text{dt}$

$$\therefore \ I = \int_1^{1/3} \frac{(1+\sqrt{3})\,\text{dt}}{-2\text{t}^{6/4}} = \frac{-(1+\sqrt{3})}{2} \times \left|\frac{-2}{\sqrt{\text{t}}}\right|_1^{1/3}$$

$$= \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) = 2$$

40. **(2)** Given $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$, $f'(0) = 1$

$$g(x) = \int_x^{\frac{\pi}{2}} \left[f'(t)\,\text{cosec}\, t - \cot t\, \text{cosec}\, t\, f(t)\right]dt$$

$$g(x) = \int_x^{\frac{\pi}{2}} \frac{d}{dt}\left(f(t)\,\text{cosec}\, t\right)dt$$

$$= f\left(\frac{\pi}{2}\right)\text{cosec}\frac{\pi}{2} - f(x)\,\text{cosec}\, x$$

$$= 3 - f(x)\,\text{cosec}\, x = 3 - \frac{f(x)}{\sin x}$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left(3 - \frac{f(x)}{\sin x}\right) = 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}$$

$$= 3 - \lim_{x\to 0}\frac{f'(x)}{\cos x} = 3 - f'(0) = 3 - 1 = 2$$

41. **(1)** Let $f(x) = \int_0^x \frac{t^2}{1+t^4}dt - 2x + 1$

$$\Rightarrow f'(x) = \frac{x^2}{1+x^4} - 2 < 0, \forall x \in [0,1]$$

$\therefore$ f is decreasing on [0, 1]

Also f (0) = 1

and $f(1) = \int_0^1 \frac{t^2}{1+t^4}dt - 1$

For $0 \le t \le 1 \Rightarrow 0 \le \frac{t^2}{1+t^4} < \frac{1}{2}$

$$\therefore \int_0^1 \frac{t^2}{1+t^4}dt < \frac{1}{2}$$

$\Rightarrow f(1) < 0$

$\therefore$ f (x) crosses *x*-axis exactly once in [0, 1]

$\therefore$ f (x) = 0 has exactly one root in [0, 1]

42. **(9)** $\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Let $9x + 3\tan^{-1}x = t \Rightarrow \frac{12+9x^2}{1+x^2}dx = dt$

$$\therefore\quad \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\therefore\quad \log_e\left|1+e^{9+\frac{3\pi}{4}}-1\right| - \frac{3\pi}{4} = 9$$

43. **(0)** $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}dx$

$-1 < x < 2 \Rightarrow 0 < x^2 < 4$

Also $0 < x^2 < 1 \Rightarrow f(x^2) = [x^2] = 0$

$1 \le x^2 < 2 \Rightarrow f(x^2) = [x^2] = 1$

$2 \le x^2 < 3 \Rightarrow f(x^2) = 0$ (using definition of f)

$3 \le x^2 < 4 \Rightarrow f(x^2) = 0$ (using definition of f)

Also $1 \le x^2 < 2 \Rightarrow 1 \le x < \sqrt{2}$

$\Rightarrow\ 2 \le x+1 < \sqrt{2}+1$

$\Rightarrow\ f(x+1) = 0$

$$\therefore\quad I = \int_1^{\sqrt{2}} \frac{x\times 1}{2+0}dx = \left[\frac{x^2}{4}\right]_1^{\sqrt{2}} = \frac{1}{4}$$

$\Rightarrow\ 4I = 1$ or $4I - 1 = 0$

44. **(2)** $\int_0^1 4x^3\left[\frac{d^2}{dx^2}\left(1-x^2\right)^5\right]dx$

$$= 4x^3\left[\frac{d}{dx}\left(1-x^2\right)^5\right]\Bigg|_0^1 - \int_0^1\left[\frac{d}{dx}\left(1-x^2\right)^5\right].12x^2dx$$

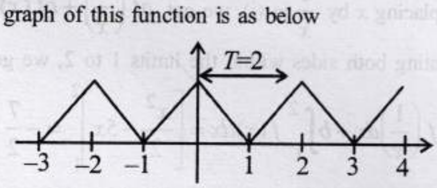
$$= -12x^2\left(1-x^2\right)^5\Big|_0^1 + \int_0^1\left(1-x^2\right)^5.24xdx$$

$$= -12\int_0^1\left(1-x^2\right)^5.(-2x)\,dx$$

$$= -12\left(\frac{\left(1-x^2\right)^6}{6}\right)_0^1 = 2$$

45. **(4)** Given function is $f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{ is even}\end{cases}$

The graph of this function is as below



Clearly $f(x)$ is periodic with period 2

Also $\cos \pi x$ is periodic with period 2

$\therefore\ f(x)\cos\pi x$ is periodic with period 2

$$\therefore\ I = \frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx$$

$$= \frac{\pi^2}{10}\times 10\int_0^2 f(x)\cos\pi x\,dx$$

$$= \pi^2\left[\int_0^1(1-x)\cos\pi x\,dx + \int_1^2(x-1)\cos\pi x\,dx\right]$$

$$= \pi^2\left[\left\{(1-x)\frac{\sin\pi x}{\pi}\Bigg|_0^1 + \int_0^1\frac{\sin\pi x}{\pi}dx\right\} + \left\{(x-1)\frac{\sin\pi x}{\pi}\Bigg|_1^2 - \int_1^2\frac{\sin\pi x}{\pi}dx\right\}\right]$$

$$= \pi^2\left[\left(-\frac{1}{\pi^2}\cos\pi x\right)\Bigg|_0^1 - \left(-\frac{1}{\pi^2}\cos\pi x\right)_1^2\right]$$

$$= \left[\left(-\cos\pi + \cos 0\right) - \left(-\cos 2\pi + \cos\pi\right)\right] = 4$$

46. **(0)** Given that $f(x) = \int_0^x f(t)dt$

Clearly $f(0) = 0$. Also $f'(x) = f(x)\ \Rightarrow\ \frac{f'(x)}{f(x)} = 1$

Integrating both sides with respect to x, we get

$$\int\frac{f'(x)}{f(x)}dx = \int 1\,dx$$

$\Rightarrow\ \ln f(x) = x + \ln C\quad \Rightarrow f(x) = Ce^x$

Now $f(0) = 0\ \Rightarrow\ Ce^0 = 0\ \Rightarrow\ C = 0$

$\therefore\quad f(x) = 0\ \forall\ x \Rightarrow f(\ln 5) = 0$

47. **(2.00)** $S_1 = \int_{\pi/8}^{3\pi/8} f(x)\,dx \equiv \int_{\pi/8}^{3\pi/8}\sin^2 xdx$ . . .(i)

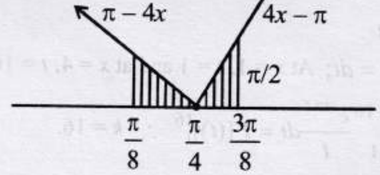
$$= \int_{\pi/8}^{3\pi/8}\sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)dx = \int_{\pi/8}^{3\pi/8}\cos^2 xdx \quad \text{. . . (ii)}$$

Adding (i) and (ii) we get

$$2S_1 = \int_{\pi/8}^{3\pi/8}(\sin^2 x + \cos^2 x)\,dx = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$$

$$\Rightarrow\quad \frac{16S_1}{\pi} = 2$$

48. **(1.50)**



$$S_2 = \int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx$$

$$= \int_{\pi/8}^{3\pi/8}\sin^2 x\,|4x-\pi|\,dx \quad \text{. . . (i)}$$

$$= \int_{\pi/8}^{3\pi/8}\sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)\left|4\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)-\pi\right|dx$$

$$= \int_{\pi/8}^{3\pi/8}\left(\cos^2 x\right)|\pi-4x|\,dx$$

$$= \int_{\pi/8}^{3\pi/8} \cos^2 x |4x - \pi| dx \quad \ldots (ii)$$

Adding equations (i) and (ii) we get

$$2S_2 = \int_{\pi/8}^{3\pi/8} |4x-\pi|(\sin^2 x + \cos^2 x)dx = \int_{\pi/8}^{3\pi/8} |4x-\pi| dx$$

$$= 2 \times \frac{1}{2} \times \frac{\pi}{8} \times \frac{\pi}{2} = \frac{\pi^2}{16}$$

$$\Rightarrow \quad \frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$$

49. **(0.50)** $I = \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^5} d\theta \quad \ldots(i)$

$$I = \int_0^{\pi/2} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\sin\theta} + \sqrt{\cos\theta}\right)^5} d\theta \quad \ldots(ii)$$

Adding two values of I, we get:

$$\frac{2}{3} I = \int_0^{\pi/2} \frac{1}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^4} d\theta$$

$$\frac{2}{3} I = \int_0^{\pi/2} \frac{\sec^2\theta}{\left(1 + \sqrt{\tan\theta}\right)^4} d\theta$$

(put $\tan\theta = t^2 \Rightarrow \sec^2\theta \, d\theta = 2t \, dt$)

$$\Rightarrow \quad \frac{2I}{3} = \int_0^{\infty} \frac{2t\,dt}{(1+t)^4}$$

$$\Rightarrow \quad I = 3\int_0^{\infty} \frac{t+1-1}{(t+1)^4} dt = 3\int_0^{\infty} \left(\frac{1}{(t+1)^3} - \frac{1}{(t+1)^4} \right) dt$$

$$= 3\left[-\frac{1}{2(t+1)^2} + \frac{1}{3(t+1)^3} \right]_0^{\infty} = 3 \times \frac{1}{6}$$

$$\Rightarrow \quad I = \frac{1}{2} = 0.50$$

50. $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1) = [F(x)]_1^k$

Put $x^2 = t$

$\Rightarrow \quad 2x dx = dt$; At $x = 1$, $t = 1$ and at $x = 4$, $t = 16$

$\therefore \quad I = \int_1^{16} \frac{e^{\sin t}}{t} dt = F[(t)]_1^{16} \quad \therefore k = 16.$

51. Let $I = \int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ and $\pi \ln x = t$

$\Rightarrow \quad \frac{\pi}{x} dx = dt$, Also as $x \to 1, t \to 0, x \to e^{37}, t \to 37\pi$

$$\therefore \quad I = \int_0^{37\pi} \sin t \, dt = [-\cos t]_0^{37\pi} = -\cos 37\pi + 1$$

$$= -(-1) + 1 = 2$$

52. Let $I = \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \quad \ldots(i)$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)} dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx \right]$$

$$I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \quad \ldots(ii)$$

On adding (i) and (ii), we get

$$2I = \int_0^{2\pi} \frac{2\pi \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \Rightarrow I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow \quad I = 2\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a (x)dx \text{ if } f(2a - x) = f(x) \right]$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \quad \ldots (iii)$$

[Using above property again]

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx \quad \ldots(iv)$$

$$\left[\because \int_0^a f(x)dx = \int_0^a (a-x)dx \right]$$

On adding (iii) and (iv), we get

$$2I = 4\pi \int_0^{\pi/2} 1.dx = 4\pi\left(\frac{\pi}{2} - 0\right) = 2\pi^2 \Rightarrow I = \pi^2$$

53. $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad \ldots(i)$

Integrating both sides within the limits 1 to 2, we get

$$a\int_1^2 f(x)dx + b\int_1^2 f\left(\frac{1}{x}\right)dx = [\log x - 5x]_1^2 = \log 2 - 5 \quad \ldots(ii)$$

On replacing x by $\frac{1}{x}$ in (i), we get $af\left(\frac{1}{x}\right) + bf(x) = x - 5$

Integrating both sides within the limits 1 to 2, we get

$$a\int_1^2 f\left(\frac{1}{x}\right)dx + b\int_1^2 f(x)dx = \left[\frac{x^2}{2} - 5x\right]_1^2 = -\frac{7}{2} \quad \ldots (iii)$$

Eliminate $\int_1^2 f\left(\frac{1}{x}\right)$ between (ii) and (iii) by multiplying (ii) by a and (iii) by b and subtracting

$$\therefore \quad (a^2 - b^2)\int_1^2 f(x)dx = a(\log 2 - 5) + b.\frac{7}{2}$$

$$\therefore \quad \int_1^2 f(x)dx = \frac{1}{(a^2 - b^2)}\left[a(\log 2 - 5) + \frac{7b}{2}\right]$$

54. $I=\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ ...(i)

$I=\int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$ ...(ii)

$\left[\text{Using} \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

On adding (i) and (ii), we get

$2I=\int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$

$\Rightarrow I=\frac{1}{2}\int_2^3 1dx=\frac{1}{2}(3-2)=\frac{1}{2}$

55. $I=\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi}d\phi$ ...(i)

$\Rightarrow I=\int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin(\pi-\phi)}d\phi$

$\left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

$\Rightarrow I=\int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin\phi}d\phi$ ... (ii)

On adding (i) and (ii), we get $2I=\int_{\pi/4}^{3\pi/4} \frac{\pi}{1+\sin\phi}d\phi$

$=\pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{1-\sin^2\phi}d\phi=\pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{\cos^2\phi}d\phi$

$=\pi\int_{\pi/4}^{3\pi/4} (\sec^2\phi-\sec\phi\tan\phi)\,d\phi$

$=\pi\left[\tan\phi-\sec\phi\right]_{\pi/4}^{3\pi/4}$

$=\pi\left[\tan 3\pi/4-\sec 3\pi/4-\tan\pi/4+\sec\pi/4\right]$

$=2\pi(\sqrt{2}-1) \Rightarrow I=\pi(\sqrt{2}-1)$

56. $I=\int_{-2}^{2}|1-x^2|dx=2\int_0^2|1-x^2|\,dx$

$\left[\because \int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx, \text{if } f \text{ is an even function}\right]$

$=2\int_0^1(1-x^2)dx+2\int_1^2(x^2-1)dx$

$=2\left[x-\frac{x^3}{3}\right]_0^1+2\left[\frac{x^3}{3}-x\right]_1^2 \quad =\frac{4}{3}+\frac{8}{3}=\frac{12}{3}=4$

57. $I=\int_0^{1.5}[x^2]dx$

We have $0<x<1.5 \Rightarrow 0<x^2<2.25$

$\therefore [x^2]=\begin{cases}0, & 0<x^2<1\\ 1, & 1\le x^2<2\\ 2, & 2\le x^2<(1.5)^2\end{cases}$

or $[x^2]=\begin{cases}0, & 0<x<1\\ 1, & 1\le x<\sqrt{2}\\ 2, & \sqrt{2}\le x<1.5\end{cases}$

$\therefore I=\int_0^{1.5}[x^2]dx=\int_0^1 0dx+\int_1^{\sqrt{2}}1dx+\int_{\sqrt{2}}^{1.5}2dx$

$=0+[x]_1^{\sqrt{2}}+[2x]_{\sqrt{2}}^{1.5} \quad =\sqrt{2}-1+3-2\sqrt{2}=2-\sqrt{2}$

58. $f(x)=\begin{vmatrix}\sec x & \cos x & \sec^2 x+\cot x\operatorname{cosec} x\\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x\\ 1 & \cos^2 x & \cos^2 x\end{vmatrix}$

$=\begin{vmatrix}0 & 0 & \sec^2 x+\cot x\operatorname{cosec} x-\cos x\\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x\\ 1 & \cos^2 x & \cos^2 x\end{vmatrix}$

$[R_1\to R_1-\sec x.R_3]$

On expanding along R_1, we get

$f(x)=(\sec^2 x+\cot x\operatorname{cosec} x-\cos x)(\cos^4 x-\cos^2 x)$

$=\left(\frac{1}{\cos^2 x}+\frac{\cos x}{\sin^2 x}-\cos x\right)\cos^2 x(\cos^2 x-1)$

$=-\sin^2 x-\cos^5 x$

$\therefore \int_0^{\pi/2} f(x)dx=-\int_0^{\pi/2}(\sin^2 x+\cos^5 x)dx$

$\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx=\frac{(n-1)(n-3)\ldots 2\,or\,1}{(n)(n-2)\ldots 2}$

Multiply the above by $\pi/2$ when n is even

$\therefore \int_0^{\pi/2} f(x)dx=-\left[\frac{1}{2}.\frac{\pi}{2}+\frac{4}{5}.\frac{2}{3}\right]=-\left[\frac{\pi}{4}+\frac{8}{15}\right]$

$=-\left(\frac{15\pi+32}{60}\right)$

59. **(True)** Let $I=\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx$ ... (i)

$=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f[2a-(2a-x)]}dx$

$[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\,]$

$I=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}$ ... (ii)

On adding (i) and (ii), we get

$2I=\int_0^{2a}\frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}dx=\int_0^{2a}1.dx$

$=[x]_0^{2a}=2a \Rightarrow I=a$

Hence, the given statement is true.

60. **(c, d)** We have, $\int_1^e \frac{(\log_e x)^{1/2}}{x(a-(\log_e x)^{3/2})^2}dx$

Let $I = \int_1^e f \frac{(\ln x)^{1/2}\, dx}{x(a-(\ln x)^{3/2})^2}$

Put $a - (\ln x)^{3/2} = t$

$\Rightarrow \; -\frac{3}{2}(\ln x)^{1/2} \cdot \frac{1}{x}dx = dt$

$\therefore \quad I = \int_a^{a-1} \frac{\left(-\frac{2}{3}\right)dt}{t^2} \qquad \left\{\because \begin{array}{l} at\ x=1, t=a \\ \text{and } at\ x=e, t=a-1 \end{array}\right\}$

$= \left(-\frac{2}{3}\right)\left[\frac{t^{1}}{-1}\right]_a^{a-1} = \frac{2}{3}\left(\frac{1}{a-1}-\frac{1}{a}\right)$

$\therefore \quad I = \left(\frac{2}{3}\right)\frac{1}{a(a-1)} = 1$

$\Rightarrow \quad 2 = 3a^2 - 3a \Rightarrow 3a^2 - 3a - 2 = 0$

$\Rightarrow \quad a = \frac{3 \pm \sqrt{9-4(3)(-2)}}{6}$

$a = \frac{3+\sqrt{33}}{6}, \frac{3-\sqrt{33}}{6}$

61. **(a,b,c)**

(a) Let $g(x) = \int_0^{\pi/3} (x)dx - 3\int_0^{\pi/3} \cos 3x\, dx = 0$

$g(x)$ is continuous and differentiable function and $g(0) = 0$, $g\left(\frac{\pi}{3}\right) = 0$

By Rolle's theorem

$g'(x) = f(x) - 3 \cos 3x = 0$ has least one solution in $\left(0, \frac{\pi}{3}\right)$

(b) Let $h(x) = f(x) - 3 \sin 3x + \frac{6}{\pi}$

Let $h(x) = \int_0^{\pi/3} f(x)dx - 3\int_0^{\pi/3} \sin 3x\, dx + \int_0^{\pi/3}\frac{6}{\pi}dx$

$= 0 - 2 + 2 = 0$

$h(x)$ is continuous and differentiable function and $h(0) = 0$ and $h\left(\frac{\pi}{3}\right) = 0$

By Rolle's theorem

$h'(x) = f(x) - 3 \sin 3x + \frac{6}{n} = 0$ has least one solution in $\left(0, \frac{\pi}{3}\right)$

(c) $\lim_{x\to 0} \frac{x\int_0^x f(t)dt}{1-e^{x^2}} = \lim_{x\to 0}\left(\frac{x^2}{1-e^{x^2}}\right)\frac{\int_0^x f(t)dt}{x}$

By L'Hospital's Rule

$= -1 \lim_{x\to 0}\frac{f(x)}{1} = -1 \qquad [\because f(0) = 1]$

(d) $\lim_{x\to 0} \frac{(\sin x)\int_0^x f(t)dt}{x^2} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\frac{\int_0^x f(t)dt}{x}$

By L'Hospital's Rule

$= 1 \lim_{x\to 0}\frac{f(x)}{1} = 1 \qquad [\because f(0) = 0]$

62. **(a,b,d)** (a) We know that,

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ and, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\because \cos x \geq 1 - \frac{x^2}{2}$

$\therefore \int_0^1 x\cos x \geq \int_0^1 x\left(1-\frac{x^2}{2}\right) = \frac{1}{2}-\frac{1}{8} \Rightarrow \int_0^1 x\cos x \geq \frac{3}{8}$

(b) $\because \sin x \geq x - \frac{x^3}{6}$

$\therefore \int_0^1 x\sin x \geq \int_0^1 x\left(x-\frac{x^3}{6}\right)dx$

$\Rightarrow \int_0^1 x\sin x \geq \frac{1}{3}-\frac{1}{30} \Rightarrow \int_0^1 x\sin x\, dx \geq \frac{3}{10}$

(c) $\because \cos x < 1 \Rightarrow x^2\cos x < x^2$

$\therefore \int_0^1 x^2\cos x\, dx < \int_0^1 x^2\, dx \Rightarrow \int_0^1 x^2\cos x\, dx < \frac{1}{3}$

(d) $\int_0^1 x^2\sin x\, dx \geq \int_0^1 x^2\left(x-\frac{x^3}{6}\right)dx$

$\Rightarrow \int_0^1 x^2\sin x\, dx \geq \frac{1}{4}-\frac{1}{36} \Rightarrow \int_0^1 x^2\sin x\, dx \geq \frac{2}{9}$

63. **(a, b, d)** $F(x) = (x-1)(x-2)(x-5)$

$F(x) = \int_0^x f(t)\, dt, x > 0$

$\Rightarrow \quad F'(x) = f(x) = (x-1)(x-2)(x-5)$

Put $F'(x) = 0 \Rightarrow x = 1, 2, 5$

$F''(x) = (x-2)(x-5) + (x-1)(x-5) + (x-1)(x-2)$

$F''(1) = +$ ve, $F''(2) = -$ve, $F''(5) = +$ve

$\therefore$ F(x) has local minima at x = 1 and x = 5 and local maxima at x = 2.

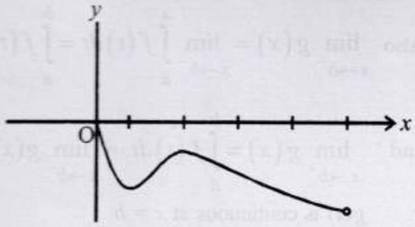
Also $F(x) = \int_0^x (t-1)(t-2)(t-5)dt$

$$= \int_0^x (t^3 - 8t^2 + 17t - 10)dt = \frac{x^4}{4} - \frac{8x^3}{3} + \frac{17x^2}{2} - 10x$$

$$= \frac{x}{12}(3x^3 - 32x^2 + 102x - 120)$$

$F(1), F(2), F(5) < 0$

Hence, approximate graph of $F(x)$ for $x \in (0, 5)$ is



$\therefore$ $F(x) < 0$, for all $x \in (0,5)$

Thus, options (a), (b) and (d) are correct but (c) is incorrect.

64. **(b, d)** $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{(k+1)}{x(x+1)} dx$; $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{(k+1)}{x(x+1)^2} dx$

$$\because \int_k^{k+1} \frac{1}{k+2} dx \le \int_k^{k+1} \frac{k+1}{x(x+1)} dx \le \int_k^{k+1} \frac{1}{k} dx$$

$$\Rightarrow I > \sum_{k=1}^{98} \frac{1}{k+2}$$

$$\Rightarrow I > \frac{1}{3} + ... + \frac{1}{100} > \frac{98}{100} \Rightarrow I > \frac{49}{50}$$

Also, $I > \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx = \sum_{k=1}^{98} [\log_e(k+1) - \log_e k]$

$I < \log_e 99$

65. **(Bonus)** $g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1}(t)\, dt$

$\Rightarrow g'(x) = \sin^{-1}(\sin 2x) \cdot 2\cos 2x - \sin^{-1}(\sin x) \cdot \cos x$

$$g'\left(\frac{\pi}{2}\right) = \sin^{-1}(\sin \pi) \cdot 2\cos \pi - \sin^{-1}\left(\sin\frac{\pi}{2}\right) \cos\frac{\pi}{2} = 0$$

$$g'\left(-\frac{\pi}{2}\right) = \sin^{-1}(\sin(-\pi)) \cdot 2\cos(-\pi) - \sin^{-1}\left(\sin\left(\frac{-\pi}{2}\right)\right) \cdot \cos\left(\frac{-\pi}{2}\right) = 0$$

66. **(a, b)** Let us check the given options one by one.

(a) Let $g(x) = x^9 - f(x)$

$\Rightarrow$ $g(0) = -f(0) < 0$ $\quad [\because f(x) \in (0, 1)]$

And $g(1) = 1 - f(1) > 0$

$\therefore$ $x^9 - f(x) = 0$ for some $x \in (0, 1)$

(b) Let $h(x) = x - \int_0^{\frac{\pi}{2} - x} f(t) \cos t\, dt$

$h(0) = -\int_0^{\frac{\pi}{2}} f(t) \cos t\, dt < 0$

and $h(1) = 1 - \int_0^{\frac{\pi}{2} - 1} f(t) \cos t\, dt > 0$

$\therefore$ $h(0) < 0$ and $h(1) > 0$

$\Rightarrow$ $h(x) = 0$ at some $x \in (0,1)$

$\therefore$ $h(x) = x - \int_0^{\frac{\pi}{2} - x} f(t) \cos t\, dt = 0$

at some $x \in (0, 1)$

(c) $e^x - \int_0^x f(t) \sin t\, dt$

$\because$ $x \in (0, 1)$ $\Rightarrow$ $e^x \in (1, e)$

and $0 < f(t) < 1$ and $0 < \sin t < 1$, $\forall$ $x \in (0, 1)$

$\therefore$ $0 < \int_0^x f(t) \sin t\, dt < 1$

$\therefore$ $e^x - \int_0^x f(t) \sin t\, dt \ne 0$ for any $x \in (0, 1)$

(d) $f(x) + \int_0^{\frac{\pi}{2}} f(t) \sin t\, dt$ is always positive $\forall$ $x \in (0, 1)$

67. **(d)** $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$

$$\Rightarrow \frac{192x^3}{3} \le f'(x) \le \frac{192x^3}{2} \Rightarrow 64x^3 \le f'(x) \le 96x^3$$

$$\Rightarrow \int_{1/2}^x 64x^3 dx \le \int_{1/2}^x f'(x)dx \le \int_{1/2}^x 96x^3 dx$$

$$\Rightarrow \frac{64x^4}{4} - \frac{64}{4} \times \frac{1}{16} \le \int_{1/2}^x f'(x)dx \le \frac{96x^4}{4} - \frac{96}{4 \times 16}$$

$$\Rightarrow 16x^4 - 1 \le \int_{1/2}^x f'(x) \le 24x^4 - \frac{3}{2}$$

$$\Rightarrow 16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}$$

$$\Rightarrow \int_{1/2}^1 (16x^4 - 1)dx \le \int_{1/2}^1 f(x)dx \le \int_{1/2}^1 \left(24x^4 - \frac{3}{2}\right) dx$$

$$\Rightarrow \left(\frac{16x^5}{5} - x\right)_{\frac{1}{2}}^1 \le \int_{1/2}^1 f(x)dx \le \left[\frac{24x^5}{5} - \frac{3}{2}x\right]_{\frac{1}{2}}^1$$

$$\Rightarrow 2.6 \le \int_{1/2}^1 f(x)dx \le 3.9$$

$\therefore$ Only (d) is the correct option.

68. **(a, b)** $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$
$= (7\tan^4 x - 3)(\tan^4 x + \tan^2 x)$
$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$

$$\int_0^{\pi/4} f(x)dx = \left[\tan^7 x - \tan^3 x\right]_0^{\pi/4} = 1 - 1 = 0$$

$$\therefore \int_0^{\pi/4} xf(x)dx = \left[x\left(\tan^7 x - \tan^3 x\right)\right]_0^{\pi/4} - \int_0^{\pi/4}\left(\tan^7 x - \tan^3 x\right)dx$$

$$= \int_0^{\pi/4} \tan^3 x(1 - \tan^2 x)\sec^2 x\,dx$$

$$= \left[\frac{\tan^4 x}{4} - \frac{\tan^6 x}{6}\right]_0^{\pi/4} = \frac{1}{12}$$

69. **(a, c)** Let $f(t) = e^t(\sin^6 at + \cos^4 at)$

$$\therefore f(k\pi + t) = e^{k\pi+t}(\sin^6 a(k\pi+t) + \cos^4 a(k\pi+t)) = e^{k\pi} f(t)$$

$$\therefore \frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}$$

$$= \frac{(1 + e^{\pi} + e^{2x} + e^{3\pi})\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt}$$

$$= 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

70. **(a, c, d)** $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$

$$\therefore f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{x}{x^2}e^{-\left(\frac{1}{x}+x\right)} = \frac{2}{x}e^{-\left(x+\frac{1}{x}\right)}$$

For $x \in [1, \infty)$, $f'(x) > 0$
$\therefore f$ is monotonically increasing on $[1, \infty)$ (a) is correct.
For $x \in (0, 1)$, $f'(x) > 0$
Hence, f is monotonically increasing on $(0, 1)$.
$\therefore$ (b) is not correct

$$f(x) + f\left(\frac{1}{x}\right) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t} + \int_{x}^{1/x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t} = 0$$

$\therefore$ (c) is correct.

Replacing x by 2^x in $f(x) + f\left(\frac{1}{x}\right) = 0$

We get $f(2^x) + f(2^{-x}) = 0$ or $f(2^x) = -f(2^{-x})$
Hence, $f(2^x)$ is an odd function.
$\therefore$ (d) is correct.

71. **(a, c)** Clearly $g(x)$ may or may not be continuous at $x = a$ or $x = b$.
But it is continuous at all value of x except $x = a$, b.
Let us check the continuity of $g(x)$ at $x = a$ and $x = b$

$$\lim_{x\to a^-} g(x) = 0$$

$$\lim_{x\to a^+} g(x) = \lim_{x\to a^+}\int_a^x f(t)dt = \int_a^a f(t)dt = 0$$

and $g(a) = \int_a^a f(t)dt = 0$

$\therefore$ $g(x)$ is continuous at $x = a$

Also $\lim_{x\to b^-} g(x) = \lim_{x\to b^-}\int_a^x f(t)dt = \int_a^b f(t)dt$

and $\lim_{x\to b^+} g(x) = \int_a^b f(t)dt = \lim_{x\to b^-} g(x) = g(b)$

$\therefore$ $g(x)$ is continuous at $x = b$
Therefore, $g(x)$ is continuous $\forall x \in R$

$$\text{Now } g'(x) = \begin{cases} 0, & x < a \\ f(x), & a \le x \le b \\ 0, & x > b \end{cases}$$

$g'(a^-) = 0$ and $g'(a^+) = f(a)$

$g'(b^-) = f(b)$ and $g'(b^+) = 0$

Since, $f(a), f(b) \in [1, \infty)$ $\therefore f(a), f(b) \neq 0$

$\therefore g'(a^-) \neq g'(a^+)$ and $g'(b^-) \neq g'(b^+)$

$\Rightarrow$ g is not differentiable at a and b.

72. **(b, c)** We have

$$f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt$$

$\Rightarrow f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$ which exists $\forall x \in (0, \infty)$

and $f'(x)$ has finite value $\forall x \in (0, \infty)$, *so* $f'(x)$ is continuous

Also $f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$

Which does not exist at the points where

$\sin x = -1$ like $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

$\therefore f'(x)$ is not differentiable.
$\therefore$ (a) is false but (b) is true

Now $\sqrt{1+\sin t} \ge 0 \Rightarrow \int_0^x \sqrt{1+\sin t}\,dt \ge 0, \forall x \in (0, \infty)$

And $\ln x > 0, \forall x \in (1, \infty) \Rightarrow f(x) > 0, \forall x \in (1, \infty)$

For $x \ge e^3$

$$f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\,dt \ge 3$$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x} \le \frac{1}{x} + \sqrt{2}, \forall x > 0$$

Now for $x \geq e^3$

$\Rightarrow \; 0 < f'(x) \leq \dfrac{1}{x} + \sqrt{2} < \dfrac{1}{e^3} + \sqrt{2} < 3 \; \forall x \in (e^3, \infty)$

$\Rightarrow \; |f'(x)| < |f(x)| \quad \therefore$ (c) is true.

Also $\lim\limits_{x \to \infty} f(x) = \infty$

$\therefore |f(x)| + |f'(x)|$ is not bounded.

$\therefore$ (d) is wrong.

73. **(a)** $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$

$$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1} x \right]_0^1 = \frac{22}{7} - \pi$$

74. **(a, b, c)** We have

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x} dx \quad \text{....(i)}$$

$$\Rightarrow \; I_n = \int_{-\pi}^{\pi} \frac{\sin n(-x)}{(1+\pi^{-x})\sin(-x)} dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \right]$$

$$\Rightarrow \; I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x} dx \quad \text{...(ii)}$$

Adding equations (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx = 2\int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

[since integrand is an even function]

$$\Rightarrow \; I_n = \int_0^{\pi} \frac{\sin nx}{\sin x} dx$$

$$\text{Now } I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx$$

$$= 2\int_0^{\pi} \cos(n+1)x \, dx = 2\left[\frac{\sin(n+1)x}{n+1} \right]_0^{\pi} = 0$$

$\therefore \quad I_{n+2} = I_n$

Also $I_1 = \displaystyle\int_0^{\pi} 1 dx = \pi$ and $I_0 = 0$

Hence $\displaystyle\sum_{m=1}^{10} I_{2m+1} = I_3 + I_5 + I_7 + ... + I_{21}$

$= 10\, I \; (\because \; I_{n+2} = I_n)$

$= 10\; \pi$

and $\displaystyle\sum_{m=1}^{10} I_{2m} = I_2 + I_4 + I_6 + ... + I_{20}$

$= 20 \times I_0 \; (\because \; I_{n+2} = I_n)$

$= 20 \times 0 = 0$

75. **(a, b)** Given $g(x) = \displaystyle\int_0^x f(t)dt$, $x \in [0, 3]$

$$\Rightarrow \; g'(x) = f(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$$

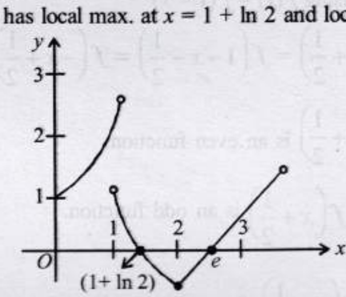
$\therefore \; g'(x) = 0 \Rightarrow e^{x-1} = 2$ or $x - e = 0$

$\Rightarrow \; x - 1) = \ln 2$ or $x = e \Rightarrow x = 1 + \ln 2$ or e

$$g''(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ -e^{x-1}, & 1 < x \leq 2 \\ 1, & 2 < x \leq 3 \end{cases}$$

$\therefore \quad g''(1 + \ln 2) = -2$ and $g''(e) = 1$

$\Rightarrow \; g(x)$ has local max. at $x = 1 + \ln 2$ and local min. at $x = e$.



Graph of $g'(x)$

Also from graph of $g'(x)$, it is clear that $g(x)$ has local max. at $x = 1$ and local min. at $x = 2$

76. **(b, d)** $f(x) = \displaystyle\int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5 \, dt$

$$\frac{dy}{dx} = f'(x) \Rightarrow x(e^x - 1)(x-1)(x-2)^3(x-3)^5 = 0$$

Critical points are 0, 1, 2, 3.

Consider change of sign of $\dfrac{dy}{dx}$ at $x = 3$.

When $x < 3$, then $\dfrac{dy}{dx} = -$ ve and when $x > 3$, $\dfrac{dy}{dx} = +$ve

Change of sign of $\dfrac{dy}{dx}$ is from $-$ve to $+$ve, hence minimum at $x = 3$. Again minimum and maximum occur alternately.

Hence, 2nd minimum occur at $x = 1$

77. **(a, b, c, d)**

$\therefore f(x)$ is a non constant twice differentiable function such that

$f(x) = f(1 - x) \;\; \Rightarrow f'(x) = -f'(1 - x)$ ...(i)

For $x=\frac{1}{2}$, we get $f'\left(\frac{1}{2}\right)=-f'\left(1-\frac{1}{2}\right)$

$\Rightarrow f'\left(\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)=0$ $\quad[\because f(x)=f(1-x)]$

$\Rightarrow f'\left(\frac{1}{2}\right)=0$

$\Rightarrow$ (b) is correct

For $x=\frac{1}{4}$, we get $f'\left(\frac{1}{4}\right)=-f'\left(\frac{3}{4}\right)$ $[\because f(x)=f(1-x)]$

but given that $f'\left(\frac{1}{4}\right)=0$

$\therefore f'\left(\frac{3}{4}\right)=f'\left(\frac{1}{4}\right)=0$

Hence, $f'(x)$ satisfies all conditions of Rolle's theorem for $x\in\left[\frac{1}{4},\frac{1}{2}\right]$ and $\left[\frac{1}{2},\frac{3}{4}\right]$. So there exists at least one point $a_1\in\left(\frac{1}{4},\frac{1}{2}\right)$ and at least one point $b_2\in\left(\frac{1}{2},\frac{3}{4}\right)$. Such that $f''(a_1)=0$ and $f''(b_2)=0$

$\therefore f''(x)$ varishes at least twice on [0, 1] $\Rightarrow$ (a) is correct.

Also using $f(x)=f(1-x)$

$\Rightarrow f\left(x+\frac{1}{2}\right)=f\left(1-x-\frac{1}{2}\right)=f\left(-x+\frac{1}{2}\right)$

$\Rightarrow f\left(x+\frac{1}{2}\right)$ is an even function.

$\Rightarrow \sin x.\ f\left(x+\frac{1}{2}\right)$ is an odd function.

$\Rightarrow \int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx=0$, $\therefore$ (c) is correct.

78. **(a)** $\int_{-1}^{1} f(x)dx=\int_{-1}^{1} x-[x]dx=\int_{-1}^{1} xdx-\int_{-1}^{1}[x]dx$

$$=0-\int_{-1}^{1}[x]dx \quad \text{...(i)}$$

$[\because x$ is an odd function$]$

Now $[x]=\begin{cases}-1, & \text{if } -1\le x<0\\ 0, & \text{if } 0\le x<1\\ 1, & \text{if } x=1\end{cases}$

$\therefore \int_{-1}^{1}[x]dx=\int_{-1}^{0}(-1)dx+\int_{0}^{1}0dx=[-x]_{-1}^{0}+0=-1$

Hence from (i) $\int_{-1}^{1} f(x)dx=1$

79. **(a)** $\int_0^x f(t)dt=x+\int_x^1 t\,f(t)dt$

Differentiating both sides w.r.t. x,

$f(x).1-f(0).0=1+1.f(1).0-xf(x).1$

$\Rightarrow (x+1)f(x)=1, \quad \Rightarrow f(x)=\frac{1}{x+1}$

$\therefore \quad f(1)=\frac{1}{2}$

80. **(d)** $P(2)$ Let $f(x)=ax^2+bx+c$

where $a, b, c\ge 0$ and a, b, c are integers.

$\because f(0)=0\Rightarrow c=0$

$\therefore f(x)=ax^2+bx$

Also $\int_0^1 f(x)dx=1$

$\Rightarrow \left[\frac{ax^3}{3}+\frac{bx^2}{2}\right]_0^1=1 \Rightarrow \frac{a}{3}+\frac{b}{2}=1\Rightarrow 2a+3b=6$

$\because$ a and b are integers

$a=0$ and $b=2$

or $a=3$ and $b=0$

$\therefore$ There are only 2 solutions.

$Q(3)$ $f(x)=\sin x^2+\cos x^2$

$f(x)$ is max. $\sqrt{2}$ at $x^2=\frac{\pi}{4}$ or $\frac{9\pi}{4}$

$\Rightarrow x=\pm\frac{\sqrt{\pi}}{2}$ or $\pm\frac{3\sqrt{\pi}}{2}\in\left[-\sqrt{13},\sqrt{13}\right]$

$\therefore$ There are four points.

$R(1)$ $I=\int_{-2}^{2}\frac{3x^2}{1+e^x}dx=\int_{-2}^{2}\frac{3x^2}{1+e^{-x}}dx$

$$\left[\text{Using}\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

$=\int_{-2}^{2}\frac{3x^2e^x}{1+e^x}dx$; $2I=\int_{-2}^{2}\frac{3x^2\left(1+e^x\right)}{1+e^x}dx=\int_{-2}^{2}3x^2dx$

$2I=\left(x^3\right)_{-2}^{2}=8-(-8)=16\Rightarrow I=8$

$S(4)$ $\dfrac{\int_{-1/2}^{1/2}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx}{\int_0^{1/2}\cos 2x\log\left(\frac{1+x}{1-x}\right)dx}=0$

$\because$ Numerator = 0, function being odd.

Hence option (d) is correct sequence.

81. (0) $f(x)=\sin^2 x, g(x)=\sqrt{\frac{\pi}{2}x-x^2}$

Here $f\left(\frac{\pi}{2}-x\right)=\cos^2 x, g\left(\frac{\pi}{2}-x\right)=g(x)$

Let $I_1 = 2\int_0^{\frac{\pi}{2}} f(x)g(x) = 2\int_0^{\frac{\pi}{2}} \sin^2 x.g(x)dx$ ...(i)

as $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$\Rightarrow I_1 = 2\int_0^{\frac{\pi}{2}} \cos^2 xg(x)dx$ ...(ii)

Euations (i) + (ii)

$\Rightarrow 2I_1 = 2\int_0^{\frac{\pi}{2}} g(x)dx \quad \Rightarrow I_1=\int_0^{\frac{\pi}{2}} g(x)dx$

$\Rightarrow 2\int_0^{\frac{\pi}{2}} f(x)g(x) - \int_0^{\frac{\pi}{2}} g(x)dx = 0$

82. (0.25) According to question 16

$2\int_0^{\frac{\pi}{2}} f(x)g(x)dx = \int_0^{\frac{\pi}{2}} g(x)dx = I_1$ (let)

Now, $I_1 = \int_0^{\frac{\pi}{2}} g(x)dx = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi}{2}x-x^2}\,dx$

$I_1 = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{\pi}{4}\right)^2-\left(\frac{\pi}{4}-x\right)^2}\,dx$

Put $\frac{\pi}{4}-x=t \Rightarrow dx=-dt$

$I_1 = -\int_{\frac{-\pi}{4}}^{\frac{-\pi}{4}} \sqrt{\left(\frac{\pi}{4}\right)^2-t^2}dt = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sqrt{\left(\frac{\pi}{4}\right)^2-t^2}dt$

$= 2\int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{\pi}{4}\right)^2-t^2}dt$

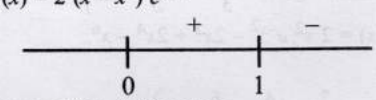
$=2\left[\frac{t}{2}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2}+\frac{\pi^2}{32}\sin^{-1}\left(\frac{4t}{\pi}\right)\right]_0^{\frac{\pi}{4}}=\frac{\pi^3}{32}$

Let $I=\frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} f(x)g(x)dx = \frac{8}{\pi^3}\times I_1=\frac{1}{4}=0.25$

83. (c) $\because \quad f(x)=\int_{-x}^{x}(|t|-t^2)e^{-t^2}dt,\ x>0$

$\therefore \quad f(x)=2\int_0^x (t-t^2)e^{-t^2}dt$

$f'(x)=2(x-x^2)e^{-x^2}$ . . . (i)



Hence option (d) is wrong.

$g'(x)=xe^{-x^2}2x$ . . . (ii)

Add (i) and (ii),

$f'(x)+g'(x)=2xe^{-x^2}$

$f(x)+g(x)=-e^{-x^2}+c$

$\because f(0)=g(0)=0$

$f(x)+g(x)=-e^{-x^2}+1$

$f(\sqrt{\ell n 3})+g(\sqrt{\ell n 3})=1-\frac{1}{3}=\frac{2}{3}$

So, option (a) is wrong.

$H(x)=\psi_1(x)-1-\alpha x=e^{-x}+-1-\alpha x$, $x\ge 1$ & $\alpha\in(1,x)$

$H(1)=e^{-1}+1-1-\alpha<0$

$H'(x)=-e^{-x}+1-\alpha<0 \quad \Rightarrow H(x)$ is decreasing

So option (b) is wrong.

(c) $\psi_2(x)=2(\psi_1(\beta)-1)$

Applying L.M.V.T to $\psi_2(x)$ in [0, x]

$\psi_2'(\beta)=\frac{\psi_2(x)-\psi_2(0)}{x}$

$2\beta-2+2e^{-\beta}=\frac{\psi_2(x)-0}{x}$

$\Rightarrow \quad \psi_2(x)=2x(\psi_1(\beta-1))$ has one solution

So, option (c) is correct.

84. (d)

(a) $\psi_1(x)=e^{-x}+x, x\ge 0$

$\psi_1'(x)=1-e^{-x}>0 \Rightarrow \quad \psi_1(x)$ is increasing

$\psi_1(x)\ge\psi_1(0)\forall x\ge 0 \Rightarrow \psi_1(x)\ge 1$

Option (a) is incorrect.

(b) $\psi_2(x)=x^2-2x+2-2e^{-x} \quad x\ge 0$

$\psi_2'(x)=2x-2+2e^{-x}=2\psi_1(x)-2\ge 0, \quad \forall x\ge 0$

$\Rightarrow \quad \psi_2(x)$ is increasing

$\Rightarrow \quad \psi_2(x)\ge\psi_2(0) \quad \Rightarrow \quad \psi_2(x)\ge 0$

So, option (b) is incorrect.

(c) $f(x) = 2\int_0^x (t - t^2)e^{-t^2}\,dt$ & $x \in \left(0, \frac{1}{2}\right)$

$= \int_0^x 2te^{-t^2}\,dt - \int_0^x 2t^2 e^{-t^2}\,dt$

$= -e^{-x^2}\Big|_0^x - \int_0^x 2t^2 e^{-t^2}\,dt$

Let $H(x) = f(x) - 1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5,\ x \in \left(0, \frac{1}{2}\right)$

$H(0) = 0$
$H'(x) = 2(x - x^2)\,e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$
$= -2x^2e^{-x^2} + 2x^2 - 2x^4$
$= 2x^2\,(1 - x^2 - e^{-x^2})$
$\because\ e^{-x} \geq 1 - x\ \forall\, x \geq 0$
$\Rightarrow\ H'(x) \leq 0\ \Rightarrow\ H(x)$ is decreasing

$\Rightarrow\ 1 - 1\,(x) < 0\ \forall x \in \left(0, \frac{1}{2}\right)$

Let $P(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7\ x \in \left(0, \frac{1}{2}\right)$

$P'(x) = 2\,x^2\,e^{-x^2} - 2x^2 + 2x^4 - x^6$

$$= 2x^2\left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{3} + \ldots\right) - 2x^2 + 2x^4 - x^6$$

$$= -\frac{x^8}{3} + \frac{x^{10}}{12}\ldots\ldots$$

$\Rightarrow\ P'(x) \leq 0\ \Rightarrow\ P(x)$ is decreasing
$\Rightarrow\ P(x) \leq 0$
So, option (d) is correct.

85. **(a, b, c)** $f(x) = xF(x) \Rightarrow f'(x) = F(x) + xF'(x)$
$\therefore\ f'(1) = F(1) + F'(1) = F'(1) < 0$

$$\left(\because F'(x) < 0, x \in \left(\frac{1}{2}, 3\right)\right)$$

$f(2) = 2F(2) < 0$, $(\because\ F'(x) < 0 \Rightarrow F$ is decreasing on $\left(\frac{1}{2}, 3\right)$

and $F'(1) = 0$, $F(3) = -4)$
$f'(x) = F(x) + x\,F'(x)$
For the same reason given above and $F'(x) < 0$ given.
$F(x) < 0\ \forall\ x \in (1, 3)$
$\therefore\ f'(x) \neq 0,\ x \in (1, 3)$

86. **(c, d)** $\int_1^3 x^2 F'(x)\,dx = -12$

$\Rightarrow\ \left[x^2 F(x)\right]_1^3 - \int_1^3 2x\,F(x)\,dx = -12$

$\Rightarrow\ 9\,F(3) - F(1)\ - 2\int_1^3 xF(x)\,dx = -12$

$\Rightarrow\ \int_1^3 xF(x)\,dx = -12 \Rightarrow \int_1^3 f(x)\,dx = -12$ ...(i)

Also $\int_1^3 x^3 F''(x)\,dx = 40$

$\Rightarrow\ \left[x^3 F'(x)\right]_1^3 - 3\int_1^3 x^2 F'(x)\,dx = 40$

$\Rightarrow\ \left[x^2\left(f'(x) - F(x)\right)\right]_1^3 - 3 \times (-12) = 40$

$$\begin{cases} \text{Using } xF'(x) = f'(x) - F(x) \\ \text{and} \int_1^3 x^2 F'(x)\,dx = -12 \end{cases}$$

$\Rightarrow\ 9(f'(3) - F(3)) - (f'(1) - F(1)) = 4$
$\Rightarrow\ 9f'(3) - 9 \times (-4) - f'(1) + 0 = 4\ \Rightarrow 9f'(3) - f'(1) + 32 = 0$

87. **(c)** For the statement P, $f(x) + 2x = 2\,(1 + x^2)$
$\Rightarrow\ (1 - x)^2 \sin^2 x + x^2 + 2x = 2(1 + x^2)$
$\Rightarrow\ (1 - x)^2 \sin^2 x = x^2 - 2x + 1 + 1$
$\Rightarrow\ (1 - x)^2 \sin^2 x = (1 - x)^2 + 1$
$\Rightarrow\ (1 - x)^2 \cos^2 x = -1$, which is not possible for any real value of x.
Hence P is not true.
Let $H(x) = 2f(x) + 1 - 2x\,(1 + x)$
$H(0) = 2f(0) + 1 - 0 = 1$
and $H(1) = 2f(1) + 1 - 4 = -3$
Hence, $H(x)$ has a solution in $(0, 1)$
Therefore, Q is true.

88. **(b)** $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\,dt,\ x \in (1, \infty)$

$\therefore\quad g'(x) = \left[\frac{2(x-1)}{x+1} - \ln x\right] f(x)$

Here $f(x) > 0$, $\forall\ x \in (1, \infty)$

Let $h(x) = \frac{2(x-1)}{x+1} - \ln x$

$\therefore\quad h'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)^2}{(x+1)^2 x} < 0, x \in (1, \infty)$

$\Rightarrow\ h(x)$ is decreasing function.
Hence, for $x > 1$, $h(x) < h(1) \Rightarrow h(x) < 0\ \forall\ x > 1$
$\Rightarrow\ g'(x) < 0\ \forall\ x \in (1, \infty)$
Therefore, $g(x)$ is decreasing on $(1, \infty)$.

89. **(a)** We have $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}$; $0 < a < 2$

$\Rightarrow\ f'(x) = \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2}$

$\Rightarrow\ (x^2 + ax + 1)^2 f'(x) = 2a(x^2 - 1)$

$\Rightarrow\ (x^2 + ax + 1)^2 f''(x) + 2(x^2 + ax + 1)$

$(2x + a) f'(x) = 4ax$ ...(i)

Putting $x = -1$ in equation (i), we get

$(2 - a)^2 f''(-1) = -4a$ ...(ii)

Putting $x = 1$ in equation (1), we get

$(2 + a)^2 f''(1) = 4a$ ...(iii)

Adding equations (ii) and (iii), we get

$(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

90. **(a)** We have $f'(x) = \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2}$

$f'(x) = 0 \Rightarrow x = -1,\ 1$ are the critical points.

$$\xleftarrow{\quad + \quad} -1 \xrightarrow{\quad - \quad} 1 \xrightarrow{\quad + \quad}$$

$\therefore f(x)$ is decreasing on $(-1, 1)$

Also using equation (1), $f''(-1) = \dfrac{-4a}{(2-a)^2} < 0$

and $f''(1) = \dfrac{4a}{(2+a)^2} > 0$

$\therefore x = -1$ is a point of local maximum and $x = 1$ is a point of local minimum.

91. **(b)** $g(x) = \int_0^{e^x} \dfrac{f'(t)}{1+t^2} dt \Rightarrow g'(x) = \dfrac{f'(e^x)}{1+e^{2x}} \cdot e^x$

$$= \frac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})} = \frac{2ae^x}{1+e^{2x}} \cdot \frac{e^{2x}-1}{(e^{2x}+ae^x+1)^2}$$

Now $g'(x) > 0$ for $e^{2x} - 1 > 0 \Rightarrow x > 0$

and $g'(x) < 0$ for $e^{2x} - 1 < 0 \Rightarrow x < 0$

$\therefore g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

92. **(a)** $\displaystyle\int_0^{\pi/2} \sin x\, dx = \frac{\left(\frac{\pi}{2}-0\right)}{4}\left(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}\right)$

$$= \frac{\pi}{8}(1+\sqrt{2})$$

93. **(d)** $\displaystyle\lim_{x\to a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$

$$\lim_{h\to 0} \frac{\int_a^{a+h} f(x)dx - \frac{h}{2}(f(a+h)+f(a))}{h^3} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h) - \frac{1}{2}[f(a)+f(a+h)] - \frac{h}{2}(f'(a+h))}{3h^2} = 0$$

[Using L'Hopital rule]

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$$

[Using L'Hopital rule]

$$\Rightarrow \lim_{h\to 0} -\frac{f''(a+h)}{12} = 0 \Rightarrow f''(x) = 0, \forall a \in R$$

$\Rightarrow f(x)$ must be of max. degree 1.

94. **(b)** $f''(x) < 0, \forall x \in (a, b)$, for $c \in (a, b)$

$$F(c) = \frac{c-a}{2}(f(a)+f(c)) + \frac{b-c}{2}(f(b)+f(c))$$

$$= \frac{b-a}{2}f(c) + \frac{c-a}{2}f(a) + \frac{b-c}{2}f(b)$$

$$\Rightarrow F'(c) = \frac{b-a}{2}f'(c) + \frac{1}{2}f(a) - \frac{1}{2}f(b)$$

$$= \frac{1}{2}[(b-a)f'(a) - f(b)]$$

$$F''(c) = \frac{1}{2}(b-a) \Rightarrow f''(c) < 0$$

$[\because f''(x) < 0, \forall x \in (a,b)$ and $b > a]$

$\therefore$ $F(c)$ is max. at the point $(c, f(c))$ where $F'(c) = 0$

$$\Rightarrow f'(c) = \frac{1}{2}\left(\frac{f(b)-f(a)}{b-a}\right).$$

95. Let $I = \displaystyle\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx$

$$= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx$$

The second integral becomes zero integrand being an odd function of x.

$$= 2\pi \int_0^{\pi/3} \frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$

{using the prop. of even function and also $|x| = x$ for $0 \le x \le \pi/3$}

Let $x + \pi/3 = y \Rightarrow dx = dy$

also as $x \to 0, y \to \pi/3$ as $x \to \pi/3, y \to 2\pi/3$

$\therefore$ The given integral becomes

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2-\cos y} = 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{3\tan^2 y/2 + 1} dy$$

$$= \frac{2\pi}{3} \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{\tan^2 y/2 + (1/\sqrt{3})^2} dy$$

$$= \frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}(\sqrt{3}\tan y/2)\right]_{\pi/3}^{2\pi/3} = \frac{4\pi}{\sqrt{3}}\left[\tan^{-1} 3 - \pi/4\right]$$

96. We have,

$$y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$$

Since, $\cos x$ is independent of θ

$$\therefore y(x) = \cos x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$$

$$\Rightarrow \frac{dy}{dx} = -\sin x \int_{\pi^2/16}^{x^2} \left[\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}\right] d\theta + \cos x \frac{d}{dx}\left[\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta\right]$$

$$= -\sin x \int_{\frac{\pi^2}{16}}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x\left[\frac{\cos x}{1+\sin^2 x}.2x - 0\right] \text{ (By Leibnitz theorem)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi} = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} .d\theta + \frac{(\cos^2\pi).2\pi}{1+\sin^2\pi}$$

$= 2\pi$

97. Let $I = \int_0^{\pi/2} f(\cos 2x)\cos x\, dx$ ...(i)

$$\Rightarrow I = \int_0^{\pi/2} f\left(\cos 2\left(\frac{\pi}{2}-x\right)\right)\cdot\cos\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow I = \int_0^{\pi/2} f(\cos 2x)\sin x\, dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\cos 2x)(\sin x + \cos x)dx$$

$$= \sqrt{2}\int_0^{\pi/2} f(\cos 2x)\left[\cos(x-\pi/4)\right]dx$$

Put $-x + \frac{\pi}{4} = t \Rightarrow -dx = dt$

$$\therefore \quad 2I = -\sqrt{2}\int_{\pi/4}^{-\pi/4} f\left[\cos\left(\frac{\pi}{2}-2t\right)\right]\cos t\, dt$$

$$\Rightarrow \quad 2I = \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cos t\, dt$$

$$\because \quad \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

$$\because \quad I = \sqrt{2}\int_0^{\pi/4} f(\sin 2t)\cos t\, dt$$

98. Given : $F(x) = \int_0^x f(t)dt$

$\therefore \quad F'(x) = f(x).1 - f(0).0$

[using Leibnitz theorem]

$\Rightarrow \quad F'(x) = f(x)$ ... (i), $\forall x \geq 0$

Also $F(0) = \int_0^0 f(t)dt = 0$

Now $f(x) \leq cF(x), \forall x \geq 0$ (Given)

$\therefore \quad f(0) \leq cF(0) = 0, \quad \therefore f(0) \leq 0$ ....(ii)

Now given that $f(x)$ is non- negative continuous function on $[0, \infty)$

$\therefore \quad f(x) \geq 0, \quad \therefore f(0) \geq 0$ ....(iii)

$\therefore$ From (ii) and (iii) $f(0) = 0$

Also given that $f(x) \leq cF(x) \forall x \geq 0$

$\Rightarrow \quad f(x) - cF(x) \leq 0$

$\Rightarrow \quad F'(x) - cF(x) \leq 0, \forall x \geq 0$ [using (i)]

$e^{-cx}F'(x) - ce^{-cx}F(x) \leq 0$

[By multiplying both sides by e^{-cx} (I.F.) and keeping in mind that $e^{-cx} > 0, \forall x$]

$\Rightarrow \quad \frac{d}{dx}[e^{-cx}F(x)] \leq 0$

$\Rightarrow \quad g(x) = e^{-cx}F(x)$ is a decreasing function on $[0, \infty)$ i.e., $g(x) \leq g(0)$ for all $x \geq 0$

But $g(0) = F(0) = 0, \therefore g(x) \leq 0, \forall x \geq 0$

$\Rightarrow \quad e^{-cx}F(x) \leq 0, \forall x \geq 0 \Rightarrow F(x) \leq 0, \forall x \geq 0$

$\therefore \quad f(x) \leq cF(x) \leq 0, \forall x \geq 0$

$\Rightarrow \quad f(x) \leq 0, \forall x \geq 0$

But given that $f(x) \geq 0 \Rightarrow f(x) = 0, \forall x \geq 0.$

99. $f(x) = \int_1^x \frac{\ln t}{1+t}dt$ for $x > 0$ (given)

Now $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1+t}dt$: Put $t = \frac{1}{u}$, so that

$dt = -\frac{1}{u^2}du$

Therefore $f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln(1/u)}{1+\frac{1}{u}}.\frac{(-1)}{u^2}du$

$$= \int_1^x \frac{\ln u}{u(u+1)}du = \int_1^x \frac{\ln t}{t(t+1)}dt$$

Now, $f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{1+t}dt + \int_1^x \frac{\ln t}{t(1+t)}dt$

$$= \int_1^x \frac{(1+t)\ln t}{t(1+t)}dt = \frac{1}{2}(\ln t)^2]_1^x = \frac{1}{2}(\ln x)^2$$

Put $x = e$, hence $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}(\ln e)^2 = \frac{1}{2}$

Hence Proved.

100. $I = \int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}dx$ ...(i)

$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^\pi \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}} \Rightarrow I = \int_0^\pi \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} \quad ...(ii)$$

Adding (i) and (ii), $2I = \int_0^\pi dx = \pi \Rightarrow I = \pi/2$

101. $\therefore I=\int_0^1 ydx=\int_0^1 \tan^{-1}xdx-\int_0^1 \tan^{-1}(x-1)dx$

$=\int_0^1 \tan^{-1}xdx-\int_0^1 \tan^{-1}\{(1-x)-1\}$

Since, $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$=\int_0^1 \tan^{-1}x\,dx-\int_0^1 (-\tan^{-1}x)\,dx=2\int_0^1 \tan^{-1}xdx$ (Proved)

$=2\left[x\tan^{-1}xdx-\frac{1}{2}\log(1+x^2)\right]_0^1$

$=\frac{\pi}{2}-\log 2$ ...... (i)

Now, $\int_0^1 \tan^{-1}(1-x+x^2)dx$

$=\int_0^1 \cot^{-1}\frac{1}{1-x+x^2}dx=\int_0^1\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{1-x+x^2}\right)dx$

$=\left[\frac{\pi}{2}x\right]_0^1-I=\frac{\pi}{2}-\left(\frac{\pi}{2}-\log 2\right)=\log 2$ by (i)

102. Let $b-a=t$, where $a+b=4$

$\Rightarrow\ a=\frac{4-t}{2}$ and $b=\frac{t+4}{2}$

Since $a<2$ and $b>2 \Rightarrow t>0$

Now $\int_0^a g(x)dx+\int_0^b g(x)dx$

$=\int_0^{\frac{4-t}{2}} g(x)dx+\int_0^{\frac{4+t}{2}} g(x)dx=\phi(t)$ [say]

$\Rightarrow\ \phi'(t)=g\left(\frac{4-t}{2}\right)\left(-\frac{1}{2}\right)+g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right)$

$\left[\because \frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)dt\right]=f[v(x)]\cdot v'(x)-f[u(x)]\cdot u'(x)\right]$

$=\frac{1}{2}\left[g\left(\frac{4+t}{2}\right)-g\left(\frac{4-t}{2}\right)\right]$

Since $g(x)$ is an increasing function (given)

$\therefore$ for $x_1>x_2 \Rightarrow g(x_1)>g(x_2)$

Here we have $\left(\frac{4+t}{2}\right)>\left(\frac{4-t}{2}\right)$

$\Rightarrow\ g\left(\frac{4+t}{2}\right)>g\left(\frac{4-t}{2}\right)$

$\Rightarrow\ \phi'(t)=\frac{1}{2}\left[g\frac{(4+t)}{2}-g\frac{(4-t)}{2}\right]>0 \Rightarrow \phi'(t)>0$

Hence $\phi(t)$ increase as t increases.

$\Rightarrow\ \int_0^a g(x)dx+\int_0^b g(x)dx$ increases as $(b-a)$ increases.

103. $\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}dx=I$ (say)

or $I=\int_{-\pi}^{\pi}\frac{2x}{1+\cos^2 x}dx+\int_{-\pi}^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$

$I=0+2\int_0^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx\quad\left[\because \frac{2x}{1+\cos^2 x}\text{ is an odd function}\right]$

or $I=4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ .....(i)

or $I=4\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\{\cos(\pi-x)\}^2}dx$

$=4\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

or $I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx-\text{I}$ [from (i)]

or $2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$

Putting $\cos x=t$, $-\sin x\,dx=dt$

When $x\to 0, t\to 1$ and when $x\to\pi, t\to -1$

$\Rightarrow I=2\pi\int_1^{-1}\frac{-dt}{1+r^2}=2\pi\int_{-1}^{1}\frac{dt}{1+t^2}$

$\because \int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$

$I=4\pi\int_0^1\frac{dt}{1+t^2}$

$\Rightarrow I=4\pi\left(\tan^{-1}t\right)_0^1=4\pi\{\tan^{-1}(1)-\tan^{-1}(0)\}$

$\Rightarrow I=4\pi\left\{\frac{\pi}{4}-0\right\}=\pi^2$

104. Let $I=\int_{-1\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

We know that $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$

Also $\sin^{-1}y+\cos^{-1}y=\frac{\pi}{2}$

$\therefore$ We get $\frac{\pi}{2}-\cos^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$

$\Rightarrow \cos^{-1}\left(\frac{2x}{1+x^2}\right)=\frac{\pi}{2}-2\tan^{-1}x$

$\therefore I=\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\left[\frac{\pi}{2}-2\tan^{-1}x\right]dx$

$=\frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4}{1-x^4}dx-2\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{x^4\tan^{-1}x}{1-x^4}dx$

$$= 2.\frac{\pi}{2}\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx - 2\times 0$$

$$\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f \text{ is even}$$
$$= 0 \text{ if } f \text{ is odd}$$

$$\therefore I = \pi\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx$$

$$\therefore I = -\pi\int_0^{1/\sqrt{3}} \frac{(1-x^4)-1}{1-x^4}dx$$

$$= -\pi\int_0^{1/\sqrt{3}} 1-\frac{1}{1-x^4}dx = -\pi\int_0^{1/\sqrt{3}}\left[1-\frac{1}{2}\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)\right]dx$$

$$= -\pi\left[x-\frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+x}{1+x}\right|+\tan^{-1}x\right)\right]_0^{1/\sqrt{3}}$$

$$= -\pi\left[\frac{1}{\sqrt{3}}-\frac{1}{2}\left(\frac{1}{2}\log\left|\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\right|-\tan^{-1}\frac{1}{\sqrt{3}}\right)-0\right]$$

$$= \frac{\pi}{12}[\pi+3\log(2+\sqrt{3})-4\sqrt{3}]$$

105. $I = \int_2^3 \frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx$

$$= \int_2^3 \frac{2x^3(x^2-1)+(x^2+1)^2}{(x^2+1)^2(x^2-1)}dx$$

$$= \int_2^3 \frac{2x^3}{(x^2+1)^2}+\int_2^3 \frac{1}{x^2-1}dx$$

$$= \int_2^3 \frac{x^2.2x}{(x^2+1)^2}+\left[\frac{1}{2}\log\frac{x-1}{x+1}\right]_2^3$$

Put $x^2+1=t,\ 2x\,dx = dt$

when $x\to 2, t\to 5$, $x\to 3, t\to 10$

$$\therefore I = \int_5^{10}\frac{t-1}{t^2}dt+\frac{1}{2}\left(\log\frac{2}{4}-\log\frac{1}{3}\right)$$

$$= \int_5^{10}\left(\frac{1}{t}-\frac{1}{t^2}\right)dt+\frac{1}{2}\log\frac{3}{2} = \left(\log|t|+\frac{1}{t}\right)_5^{10}+\frac{1}{2}\log\frac{3}{2}$$

$$= \log 10-\log 5+\frac{1}{10}-\frac{1}{5}+\frac{1}{2}\log\frac{3}{2}$$

$$= \frac{1}{2}\left[2\log 2+\log\frac{3}{2}\right]-\frac{1}{10}$$

$$= \frac{1}{2}\log 6-\frac{1}{10}$$

106. It is given that $\int_0^1 e^x(x-1)^n\,dx = 16-6e$

where $n\in N$ and $n\le 5$

To find the value of n.

Let $I_n = \int_0^1 e^x(x-1)^n\,dx$

$$= [(x-1)^n e^x]_0^1 - \int_0^1 n(x-1)^{n-1}e^x\,dx$$

$$= -(-1)^n - \int_0^1 n(x-1)^{n-1}e^x\,dx$$

$$I_n = (-1)^{n+1} - n\,I_{n-1} \quad \text{...(i)}$$

Also, $I_1 = \int_0^1 e^x(x-1)\,dx$

$$= [e^x(x-1)]_0^1 - \int_0^1 e^x dx = -(-1)-(e^x)_0^1$$

$$= -(e-1) = 2-e$$

Using eq. (i), $I_2 = (-1)^3 - 2I_1 = -1-2(2-e) = 2e-5$

Similarly, $I_3 = (-1)^4 - 3I_2 = 1-3(2e-5) = 16-6e$

$\therefore n = 3$

107. Let $I = \int_0^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$

Consider, $2x-\pi = y \Rightarrow dx = \frac{dy}{2}$. Also, $x = \left(\frac{\pi}{2}+\frac{y}{2}\right)$

When $x\to 0, y\to -\pi$ when $x\to\pi, y\to\pi$

$\therefore$ We get

$$I = \int_{-\pi}^{\pi} \frac{\left(\frac{\pi+y}{2}\right)\sin(\pi+y)\sin\left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{y}{2}\right)\right]}{y}.\frac{dy}{2}$$

$$= \frac{1}{4}\int_{-\pi}^{\pi}\left(\frac{\pi}{y}+1\right)(-\sin y)\sin\left(\frac{-\pi}{2}\sin\frac{y}{2}\right)dy$$

$$= \frac{\pi}{4}\int_{-\pi}^{\pi}\frac{\sin y\sin(\pi/2\sin y/2)}{y}dy$$

$$+\frac{1}{4}\int_{-\pi}^{\pi}\sin y\sin\left(\frac{\pi}{2}\sin\frac{y}{2}\right)dy$$

$$\because \int_{-a}^{a} f(x)\,dx = 0 \text{ if f is odd function}$$

$$= 2\int_0^a f(x)\,dx \text{ if } f \text{ is an even function]}$$

$$\therefore I = 0+\frac{2}{4}\int_0^{\pi}\sin y\sin(\pi/2\sin y/2)dy$$

$$\therefore I = \frac{1}{2}\int_0^{\pi} 2\sin y/2\cos y/2\sin(\pi/2\sin y/2)dy$$

Let $\sin y/2 = u \Rightarrow \frac{1}{2}\cos y/2\,dy = du$

$\Rightarrow \cos y/2\,dy = 2du$

Also as $y\to 0, u\to 0$ and as $y\to\pi, u\to 1$

$\therefore I = \int_0^1 2u \sin\left(\frac{\pi u}{2}\right) du$

$= \left[2u \frac{-\cos\frac{\pi u}{2}}{\pi/2}\right]_0^1 + \int_0^1 2.\frac{2}{\pi}\cos\left(\frac{\pi u}{2}\right) du$

$= 0 + \frac{4}{\pi}\left.\frac{\sin\left(\frac{\pi u}{2}\right)}{\pi/2}\right|_0^1 = \frac{8}{\pi^2}$

108. We are given that f is a continuous function and

$\int_0^x f(t)dt \to \infty \, as \, |x| \mapsto \infty$

To show that every line $y = mx$ intersects the curve

$y^2 + \int_0^x f(t)dt = 2.$ ...(i)

Let $y = mx$ intersects the given curve, then put $y = mx$ in the equation (i) of the curve

$m^2x^2 + \int_0^x f(t)dt = 2$ ...(ii)

Let $F(x) = m^2x^2 + \int_0^x f(t)dt - 2$

Then $F(x)$ is a continuous function as $f(x)$ is given to be continuous.

Also $F(x) \to \infty \, as \, |x| \mapsto \infty$

But $F(0) = -2$

Thus $F(0) = -$ve and $F(b) = +$ve where b is some value of x, and $F(x)$ is continuous.

Therefore $F(x) = 0$ for some value of $x \in (0, b)$ or eq. (i) is solvable for x.

Hence $y = mx$ intersects the given curve.

109. $\because$ $2 \sin x [\cos x + \cos 3x + \cos 5x \ldots + \cos (2k-1)x]$

$= 2 \sin x \cos x + 2 \sin x \cos 3x + 2\sin x \cos 5x + \ldots + 2 \sin x \cos (2k-1)x$

$= \sin 2x + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + \ldots + \{\sin 2kx - \sin (2k-2)x\}$

$= \sin 2kx$

$\therefore 2 [\cos x + \cos 3x + \cos 5x + \ldots + \cos (2k-1)x]$

$= \frac{\sin 2kx}{\sin x}$ ...(i)

Now, $\sin 2kx.\cot x = \frac{\sin 2kx}{\sin x}.\cos x$

$= 2 \cos x [\cos x + \cos 3x + \cos 5x + \ldots + \cos (2k-1)x]$ [from eqn (i)]

$= [2 \cos^2 x + 2 \cos x \cos 3x + 2 \cos x \cos 5x + \ldots + 2 \cos x \cos (2k-1)x]$

$= (1 + \cos 2x) + (\cos 4x + \cos 2x) + (\cos 6x + \cos 4x) + \ldots + \{\cos 2kx + \cos (2k-2)x\}$

$= 1 + 2 [\cos 2x + \cos 4x + \cos 6x + \ldots + \cos (2k-2)x] + \cos 2kx$

$\therefore \int_0^{\pi/2} (\sin 2kx).\cot x \, dx$

$= \int_0^{\pi/2} 1.dx + 2\int_0^{\pi/2} (\cos 2x + \cos 4x \ldots \cos(2k-2)x) dx + \int_0^{\pi/2} \cos(2k)x \, dx$

$= \frac{\pi}{2} + 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} + \left[\frac{\sin(2k)x}{2k}\right]_0^{\pi/2} = \frac{\pi}{2} + 2 \times 0 = \frac{\pi}{2}$

110. We have, $I = \int_0^{\pi/2} f(\sin 2x) \sin x dx$ ...(i)

$I = \int_0^{\pi/2} f(\sin 2x) \cos x dx$ ...(ii)

$[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} f(\sin 2x)(\cos x + \sin x)dx$

$\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ when $f(2a-x) = f(x)]$

$\therefore \quad 2I = 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$

$\Rightarrow I = \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)dx$

$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin(\pi/4 + x)dx$

$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\therefore \quad I = \sqrt{2}\int_0^{\pi/4} f\left[\sin\left(2\left(\frac{\pi}{4} - x\right)\right)\right]\sin(\pi/4 + \pi/4 - x)dx$

$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x dx$

Hence Proved.

111. Let $I=\int_0^a f(x)g(x)dx=\int_0^a f(a-x)g(a-x)dx$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$\therefore \quad I=\int_0^a f(x)(2-g(x))dx$

It is given that $f(a-x)=f(x)$ and $g(a-x)+g(x)=2$

$\therefore \quad I=2\int_0^a f(x)dx-\int_0^a f(x)g(x)dx$;

$\Rightarrow I=2\int_0^a f(x)dx-I \Rightarrow I=\int_0^a f(x)dx$

Hence the result.

112. Let $I=\int_0^1 1.\log[\sqrt{1-x}+\sqrt{1+x}]dx$

Intergrating by parts, we get

$I=[x\log(\sqrt{1-x}+\sqrt{1+x})]_0^1$

$$-\int_0^1 x.\frac{1}{\sqrt{1-x}+\sqrt{1+x}}.\left[\frac{-1}{2\sqrt{1-x}}+\frac{1}{2\sqrt{1+x}}\right]dx$$

$$=\log\sqrt{2}-\int_0^1 x\frac{(\sqrt{1+x}-\sqrt{1-x})}{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}-\sqrt{1-x})}.\frac{(\sqrt{1-x}-\sqrt{1+x})}{2\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\log 2+\frac{1}{2}\int_0^1\frac{1+x+1-x-2\sqrt{1-x^2}}{2\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}\log 2+\frac{1}{2}\int_0^1\frac{1}{\sqrt{1-x^2}}dx-\frac{1}{2}\int_0^1 1dx$$

$$=\frac{1}{2}\left[\log 2+\left(\sin^{-1}x\right)_0^1-(x)_0^1\right]=\frac{1}{2}[\log 2+\pi/2-1]$$

113. $f(x)=\int_1^x[2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]dt$

$\because \quad \frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)}g(t)dt\right]=g[\psi(x)]\psi'(x)-g[\phi(x)]\phi'(x)$

$\therefore \quad f'(x)=2(x-1)(x-2)^3+3(x-1)^2(x-2)^2$
$=(x-1)(x-2)^2(2x-4+3x-3)$
$=(x-1)(x-2)^2(5x-7)$

For extreme values, put $f'(x)=0 \Rightarrow x=1, 2, 7/5$

Now, $f''(x)=(x-2)^2(5x-7)+2(x-1)(x-2)(5x-7)+5(x-1)(x-2)^2$

$\because \quad f''(1)=1(-2)=-2<0, \therefore f$ is max. at $x=1$

$\because \quad f''(2)=0$

$\therefore \quad f$ is neither maximum nor minimum at $x=2$.

$$f''\left(\frac{7}{5}\right)=5\left(\frac{7}{5}-1\right)\left(\frac{7}{5}-2\right)^2=5\times\frac{2}{5}\times\frac{9}{25}=\frac{18}{25}>0$$

$\therefore \quad f(x)$ is minimum at $x=7/5$.

114. Let $I=\int_0^\pi\frac{x\,dx}{1+\cos\alpha\sin x}$ ...(i)

$$I=\int_0^\pi\frac{(\pi-x)dx}{1+\cos\alpha(\sin(\pi-x))}$$

[Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$]

$\therefore \quad I=\int_0^\pi\frac{(\pi-x)dx}{1+\cos\alpha\sin x}$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_0^\pi\frac{x+\pi-x}{1+\cos\alpha\sin x}dx=\int_0^\pi\frac{\pi}{1+\cos\alpha\sin x}dx$$

$$\therefore \quad I=\frac{\pi}{2}\int_0^\pi\frac{1}{1+\cos\alpha\sin x}dx$$

$$=\pi\int_0^{\pi/2}\frac{1}{1+\cos\alpha.\frac{2\tan x/2}{1+\tan^2 x/2}}dx$$

$$=\pi\int_0^{\pi/2}\frac{\sec^2 x/2}{1+\tan^2 x/2+2\cos\alpha\tan x/2}dx$$

Put $\tan x/2=t$, $\frac{1}{2}\sec^2\frac{x}{2}dx=dt \Rightarrow \sec^2 x/2dx=2dt$

Also when $x\to 0, t\to 0$ as $x\to\pi/2, t\to 1$

$$\therefore \quad I=\pi\int_0^1\frac{2dt}{t^2+(2\cos\alpha)t+1}$$

$$=2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+1-\cos^2\alpha}$$

$$=2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$$

$$=2\pi.\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^1$$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right)-\tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\cot\alpha/2)-\tan^{-1}(\cot\alpha)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}(\tan^{-1}(\pi/2-\alpha/2)) - \tan^{-1}(\tan(\pi/2-\alpha))\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2} + \alpha\right] = \frac{\pi\alpha}{\sin\alpha}$$

115. Let $I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x + \sin^4 x}dx$ ... (i)

$$I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin(\pi/2-x)\cos(\pi/2-x)}{\cos^4(\pi/2-x)+\sin^4(\pi/2-x)}dx$$

Since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Then, $I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin x\cos x}{\sin^4 x + \cos^4 x}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$$

$$\Rightarrow I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^4 x\left(\frac{\sin^4 x}{\cos^4 x}+1\right)}dx = \frac{\pi}{4}\int_0^{\pi/2}\frac{\sec^2 x\tan x}{\tan^4 x+1}dx$$

$$= \frac{\pi}{2\times 4}\int_0^{\pi/2}\frac{2\tan x\sec^2 x\,dx}{1+(\tan^2 x)^2}$$

Put $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\,dx = dt$

Also as $x \to 0, t \to 0$; as $x \to \pi/2, t \to \infty$

$$\therefore\ I = \frac{\pi}{8}\int_0^\infty \frac{dt}{1+t^2} = \frac{\pi}{8}[\tan^{-1} t]_0^\infty = \frac{\pi}{8}[\pi/2 - 0] = \pi^2/16$$

116. Let $\int f(x)dx = F(x)+c$

Then $F'(x) = f(x)$ ... (i)

Now $I = \int_a^{a+t} f(x)dx = F(a+t) - F(a)$

$$\therefore\ \frac{dI}{da} = F'(a+t) - F(a) = f(a+t) - f(a)$$

[Using eq. (i)]

$= f(a) - f(a) = 0$

[Using given condition]

This shows that I is independent of a.

117. Let $I = \int_0^{1/2} \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

Also when $x = 0, \theta = 0$

and when $x = 1/2, \theta = \pi/6$

Thus, $I = \int_0^{\pi/6} \frac{\sin\theta\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$

$$= \int_0^{\pi/6}\frac{\theta\sin\theta}{\cos\theta}\cos\theta\, d\theta \Rightarrow I = \int_0^{\pi/6}\theta\sin\theta\, d\theta$$

Intergrating the above by parts, we get

$$I = [\theta(-\cos\theta)]_0^{\pi/6} + \int_0^{\pi/6} 1.\cos\theta d\theta$$

$$= [-\theta\cos\theta + \sin\theta]_0^{\pi/6} = \frac{-\pi}{6}.\frac{\sqrt{3}}{2}+\frac{1}{2} = \frac{6-\pi\sqrt{3}}{12}$$

118. $I = \int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Let $\sin x - cox x = t \Rightarrow$ as $x \to 0, t \to -1$ as $x \to \pi/4, t \to 0$

$\Rightarrow (\cos x + \sin x)dx = dt$

Also, $t^2 = 1 - \sin 2x \Rightarrow \sin 2x = 1 - t^2$

$$I = \int_{-1}^0 \frac{dt}{9+16(1-t^2)} = \frac{1}{16}\int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right] = \frac{2\log 3}{40} = \frac{1}{20}\log 3$$

119. $\int_{-1}^{3/2} |x\sin\pi x|\, dx$

For $-1 \le x < 0 \Rightarrow -\pi < \pi x < 0 \Rightarrow \sin\pi x < 0$

$\Rightarrow x\sin\pi x > 0$

For $1 < x < 3/2$

$\Rightarrow \pi < \pi x < 3\pi/2 \Rightarrow \sin\pi x < 0 \Rightarrow x\sin\pi x < 0$

$$\therefore\ \int_{-1}^{3/2}|x\sin\pi x|\,dx = \int_{-1}^{1} x\sin\pi x\,dx + \int_1^{3/2}(-x\sin\pi x)ax$$

$$= 2\int_0^1 x\sin\pi x\,dx - \int_1^{3/2} x\sin\pi x\,dx$$

$$= 2\left[\frac{-x\cos\pi x}{\pi}+\frac{\sin\pi x}{\pi^2}\right]_0^1 - \left[\frac{-x\cos\pi x}{\pi}+\frac{\sin\pi x}{\pi^2}\right]_1^{3/2}$$

$$= 2\left[\frac{1}{\pi}\right] - \left[-\frac{1}{\pi^2}-\frac{1}{\pi}\right] = \frac{3}{\pi}+\frac{1}{\pi^2}$$

120. Let $I=\int_0^{\pi} xf(\sin x)dx$ ... (i)

$\Rightarrow I=\int_0^{\pi}(\pi-x)f(\sin x)dx$ ...(ii)

Adding (i) and (ii), we get,

$$2I=\int_0^{\pi}\pi f(\sin x)dx$$

$$I=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

Hence Proved.

Topic-4: Summation of Series by Integration

1. **(b)** $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\,ln(1+t)}{t^4+4}dt \quad \left[\frac{0}{0}\text{form}\right]$

Using L' Hospital's rule, we get

$$\lim_{x\to 0}\frac{\frac{x ln(1+x)}{x^4+4}}{3x^2}=\lim_{x\to\infty}\frac{ln\,(1+x)}{x}\cdot\frac{1}{3(x^4+4)}=1.\frac{1}{12}=\frac{1}{12}$$

2. **(a)** We have $I_{m,n}=\int_0^1 t^m(1+t)^n dt$

$$I_{m,n}=\left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1-\frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1}dt$$

[Using intergrating by parts]

$$I_{m,n}=\frac{2^n}{m+1}-\frac{n}{m+1}I_{m+1,n-1}$$

3. **(5)** $f(t)=\left(\frac{(2n+1)-t}{2}\right)(-1)^{n+1}2+\left(\frac{t-(2n-1)}{2}\right)$

$(-1)^{n+2}2,\ t\in(2n-1,2n+1)$

$\Rightarrow f(t)=2(-1)^{n+1}(2n-t),\ t\in(2n-1,2n+1)$

We have, $g(x)=\int_1^x f(t)dt, x\in(1,8]$

$$=\begin{cases}\int_1^x 2(2-t)dt, 1<x\le 3, n=1\\ \int_1^3 2(2-t)dt+\int_3^x(2t-8)dt, 3<x\le 5, n=2\\ \int_1^3 2(2-t)dt+\int_3^5(2t-8)dt+\int_5^x 2(6-t)dt,\\ \quad 5<x\le 7, n=3\\ \int_1^3 2(2-t)dt+\int_3^5(2t-8)dt+\int_5^7 2(6-t)dt+\\ \quad\int_7^x(2t-16)dt, x\in(7,8], n=4\end{cases}$$

$$=\begin{cases}-x^2+4x-3, 1<x\le 3,\\ x^2-8x+15, 3<x\le 5\\ -x^2+12x-35, 5<x\le 7\\ x^2-16x+63, 7<x\le 8\end{cases}$$

$$=\begin{cases}-(x-1)(x-3), 1<x\le 3\\ (x-3)(x-5), 3<x\le 5\\ -(x-5)(x-7), 5<x\le 7\\ (x-7)(x-9), 7<x\le 8\end{cases}$$

$\Rightarrow g(x)=0\Rightarrow x=3,5,7\Rightarrow\alpha=3$

$$\beta=\lim_{x\to 1^+}\left(\frac{g(x)}{x-1}\right)=\lim_{x\to 1^+}-\frac{(x-1)(x-3)}{x-1}=2$$

$\Rightarrow\alpha+\beta=5$

4. **(0)** Given, $f(x)=\int_0^{x\tan^{-1}x}\frac{e^{t-\cos t}}{1+t^{2023}}dt$

$$f'(x)=\frac{e^{x\tan^{-1}x-\cos(x\tan^{-1}x)}}{1+(x\tan^{-1}x)^{2023}}\cdot\left(\frac{x}{1+x^2}+\tan^{-1}x\right)$$

For $x<0$, $\tan^{-1}x\in\left(-\frac{\pi}{2},0\right)$

For $x\ge 0$, $\tan^{-1}x\in\left[0,\frac{\pi}{2}\right]$

$\Rightarrow x\tan^{-1}x\ge 0,\ \forall\ x\in R$

And $\frac{x}{1+x^2}+\tan^{-1}x=\begin{cases}>0 & \text{For } x>0\\ <0 & \text{For } x<0\\ 0 & \text{Fox } x=0\end{cases}$

i.e $\overset{-}{\quad}\bullet\overset{+}{\quad}$ f

point of minima

So, f(x) is minimum at x = 0.

Here minimum value is $f(0) = \int_0^0 = 0$.

5. **(57.00)** $f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j\, dt$

$$f_1'(x) = \prod_{j=1}^{21} (x-j)^j = (x-1)(x-2)^2(x-3)^3 \ldots (x-21)^{21}$$

Checking the sign scheme of $f_1'(x)$ at $x = 1, 2, 3, \ldots 21$

We get

$f_1(x)$ has local minima at x = 1, 5, 9, 13, 17, 21 and local maxima at 3, 7, 11, 15, 19.

$\Rightarrow \quad m_1 = 6, n_1 = 5$

So, $2m_1 + 3n_1 + m_1n_1$

$= 2 \times 6 + 3 \times 5 + 6 \times 5 = 57$

6. **(6.00)** $f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$

$\Rightarrow \quad f_2'(x) = 2 \times 49 \times 50(x-1)^{49} - 50 \times 12 \times 49(x-1)^{48}$

$= 50 \times 49 \times 2(x-1)^{48}(x-1-6)$

$= 50 \times 49 \times 2(x-1)^{48}(x-7)$

$f_2(x)$ has local minimum at

$x = 7$ and no local maxima

$\Rightarrow \quad m_2 = 1, n_2 = 0$

$6m_2 + 4n_2 + 8m_2n_2$

$= 6 \times 1 + 4 \times 0 + 8 \times 1 = 6$

7. **(1)** $y_n = \left(\frac{n+1}{n} \cdot \frac{n+2}{n} \cdot \frac{n+3}{n} \ldots \frac{n+n}{n}\right)^{1/n}$

$$\Rightarrow \log y_n = \frac{1}{n}\sum_{r=0}^{n} \log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \left(\lim_{n\to\infty} \log y_n\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n} \log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \log(1+x)dx = [x\log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}dx$$

$= \log 2 - [x - \log|1+x|]_0^1$

$= \log 2 - 1 + \log 2 = 2\log 2 - 1$

$= \log 4 - \log e = \log\left(\frac{4}{e}\right)$

$\therefore \quad L = \frac{4}{e} \Rightarrow [L] = \left[\frac{4}{e}\right] = 1$

8. **(7)** $\lim_{x\to 1} \frac{F(x)}{G(x)} = \frac{1}{14} \Rightarrow \lim_{x\to 1} \frac{\int_{-1}^{x} f(t)dt}{\int_{-1}^{x} t|f(f(t))|dt}$

$\because \quad \int_{-1}^{1} f(t)dt = 0$ and $\int_{-1}^{1} t|f(f(t))|dt = 0$

$f(t)$ being odd function

$\therefore$ Using L Hospital's rule, we get

$$\lim_{x\to 1} \frac{f(x)}{x|f(f(x))|} = \frac{1}{14}$$

$$\Rightarrow \frac{f(1)}{|f(f(1))|} = \frac{1}{14} \Rightarrow \frac{1/2}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

$$\Rightarrow \left|f\left(\frac{1}{2}\right)\right| = 7 \Rightarrow f\left(\frac{1}{2}\right) = 7$$

9. **(a, d)**

$$\lim_{n\to\infty} \frac{\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \ldots + \frac{1}{(an+n)^2}\right)} = 54$$

$$\Rightarrow \frac{\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{1}{\left(a+\frac{r}{n}\right)^2}} = 54$$

$$\Rightarrow \frac{\int_0^1 x^{1/3}dx}{\int_0^1 \frac{1}{(a+x)^2}dx} = 54 \Rightarrow \frac{\frac{3}{4}}{\frac{1}{a} - \frac{1}{a+1}} = 54$$

$\Rightarrow a^2 + a - 72 = 0 \Rightarrow (a+9)(a-8) = 0 \qquad \Rightarrow a = 8$ or -9

$\therefore$ options (a) and (d) are correct.

10. **(b, c)**

$$f(x) = \lim_{n\to\infty}\left[\frac{n^n(x+n)\left(x+\frac{n}{2}\right)\ldots\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)\ldots\left(x^2+\frac{n^2}{n^2}\right)}\right]^{x/n}$$

$$= \lim_{n\to\infty}\left[\frac{\left(\frac{x}{n}+1\right)\left(\frac{x}{n}+\frac{1}{2}\right)\ldots\left(\frac{x}{n}+\frac{1}{n}\right)}{\left(1.\frac{x^2}{n^2}+1\right)\left(2.\frac{x^2}{n^2}+\frac{1}{2}\right)\ldots\left(n.\frac{x^2}{n^2}+\frac{1}{n}\right)}\right]^{x/n}$$

$$\Rightarrow \ln f(x) = \lim_{n\to\infty}\frac{x}{n}\left[\sum_{r=1}^{n}\ln\left(\frac{x}{n}+\frac{1}{r}\right)-\sum_{r=1}^{n}\ln\left(\frac{rx^2}{n^2}+\frac{1}{r}\right)\right]$$

$$= \lim_{n\to\infty}\frac{x}{n}\left[\sum_{r=1}^{n}\ln\left(1+\frac{rx}{n}\right)-\ln\left(1+\frac{r^2x^2}{n^2}\right)\right]$$

$$= x\int_0^1 \ln(1+xy)\,dy - x\int_0^1 \ln\left(1+x^2y^2\right)dy$$

Let $xy = t \Rightarrow x\,dy = dt$

$$\therefore \quad \ln f(x) = \int_0^x \ln(1+t)\,dt - \int_0^x \ln\left(1+t^2\right)dt$$

$$\ln f(x) = \int_0^x \ln\left(\frac{1+t}{1+t^2}\right)dt \Rightarrow \frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right)$$

$$\Rightarrow \quad \frac{f'(2)}{f(2)} = \ln\left(\frac{3}{5}\right) < 0$$

$\Rightarrow f'(2) < 0 \quad \therefore$ (c) is correct

and $\dfrac{f'(3)}{f(3)} = \ln\left(\dfrac{2}{5}\right) < \dfrac{f'(2)}{f(2)}$

$\therefore$ (d) is not correct

$\because f'(x) = f(x)\ \ln\left(\dfrac{1+x}{1+x^2}\right) > 0, \ \forall x \in (0, 1)$

$\Rightarrow f'(x) > 0 \forall x \in (0,1)$

$\therefore$ f is an increasing function.

$\therefore \quad \dfrac{1}{2} < 1 \Rightarrow f\left(\dfrac{1}{2}\right) \le f(1)$

$\therefore$ (a) is not correct

and $\dfrac{1}{3} < \dfrac{2}{3} \Rightarrow f\left(\dfrac{1}{3}\right) \le f\left(\dfrac{2}{3}\right)$

$\therefore$ (b) is correct

11. **(a)** $g(a) = \lim\limits_{h\to 0^+}\displaystyle\int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$

$$\therefore\ g\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h} t^{-1/2}(1-t)^{-1/2}\,dt$$

$$= \lim_{h\to 0^+}\int_h^{1-h}\frac{1}{\sqrt{t(1-t)}}\,dt = \lim_{h\to 0^+}\int_h^{1-h}\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(t-\frac{1}{2}\right)^2}}\,dt$$

$$= \lim_{h\to 0^+}\left[\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_h^{1-h} = \lim_{h\to 0^+}\sin^{-1}(2t-1)\Big]_h^{1-h}$$

$$= \lim_{h\to 0^+}\left[\sin^{-1}(1-2h)-\sin^{-1}(2h-1)\right]$$

$$= \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) = \pi$$

12. **(d)** $g(a) = \lim\limits_{h\to 0^+}\displaystyle\int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$

$$g(a) = \lim_{h\to 0^+}\int_h^{1-h}(1-t)^{-a}\,t^{a-1}\,dt$$

$$\left(\text{using} \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right)$$

Also $g(1-a) = \lim\limits_{h\to 0^+}\displaystyle\int_h^{1-h} t^{a-1}(1-t)^{-a}\,dt$

Thus $g(a) = g(1-a)$

$\Rightarrow g'(a) = -g'(1-a) \Rightarrow g'(a) + g'(1-a) = 0$

Putting $a = \dfrac{1}{2}$ we get $g'\left(\dfrac{1}{2}\right) + g'\left(\dfrac{1}{2}\right) = 0$

or $g'\left(\dfrac{1}{2}\right) = 0$

13. Given $\quad I_m = \displaystyle\int_0^{\pi}\frac{1-\cos mx}{1-\cos x}\,dx$

To prove: $\quad I_m = m\pi,\ m = 0, 1, 2, \ldots\ldots$

For m = 0

$$I_0 = \int_0^{\pi}\frac{1-\cos 0}{1-\cos x}\,dx = \int_0^{\pi}\frac{1-1}{1-\cos x}\,dx = 0$$

$\because$ Result is true for m = 0

For m = 1,

$$I_1 = \int_0^{\pi}\frac{1-\cos x}{1-\cos x}\,dx = \int_0^{\pi} 1.dx$$

$$(x)_0^{\pi} = \pi - 0 = \pi$$

$\therefore$ Result is true for $m = 1$

Let the result be true for $m \le k$ i.e. $I_k = k\pi$ ....(i)

Consider $I_{k+1} = \displaystyle\int_0^{\pi}\frac{1-\cos(k+1)x}{1-\cos x}\,dx$

Now, $1-\cos(k+1)x$

$= 1 - \cos kx \cos x + \sin kx \sin x$

$= 1 + \cos kx \cos x + \sin kx \sin x - 2\cos kx \cos x$

$= 1 + \cos(k-1)x - 2\cos kx \cos x$

$= 2 - (1 - \cos(k-1)x) - 2\cos kx \cos x$

$= 2 - 2\cos kx + 2\cos kx - 2\cos kx \cos x$

$\qquad - [1 - \cos(k-1)x]$

$= 2(1 - \cos kx) + 2\cos kx(1 - \cos x) - (1 - \cos(k-1)x)$

$$\therefore\ I_{k+1}=\int_0^{\pi}\frac{2(1-\cos kx)+2\cos kx(1-\cos x)-(1-\cos(k-1)x)}{1-\cos x}dx$$

$$=2\int_0^{\pi}\frac{1-\cos kx}{1-\cos x}dx+2\int_0^{\pi}\cos kx dx-\int_0^{\pi}\frac{1-\cos(k-1)x}{1-\cos x}dx$$

$$=2I_k+2\left(\frac{\sin kx}{k}\right)_0^{\pi}-I_{k-1}$$

$$=2(k\pi)+2(0)-(k-1)\pi \qquad \text{[Using (i)]}$$

$$=(k+1)\pi$$

Thus result is true for $m=k+1$ as well. Therefore by the principle of mathematical induction, given statement is true for all $m=0, 1, 2,$ ....

14. To prove that $\int_0^{n\pi+v}|\sin x|\,dx=2n+1-\cos v$

Let $I=\int_0^{n\pi+v}|\sin x|\,dx$

$$=\int_0^{v}|\sin x|\,dx+\int_v^{n\pi+v}|\sin x|\,dx$$

Now we know that $|\sin x|$ is a periodic function of period π, So using the property.

$$=\int_a^{a+nT}f(x)dx=n\int_0^{T}f(x)dx$$

where $n\in I$ and $f(x)$ is a periodic function of period T

We get, $I=\int_0^{v}\sin x\,dx+n\int_0^{\pi}\sin x\,dx$

$[\therefore\ |\sin x|=\sin x \text{ for } 0\le x\le v]$

$$=(-\cos x)_0^v+n(-\cos x)_0^{\pi}=-\cos v+1+n(1+1)$$

$$=2n+1-\cos v=\text{R.H.S.}$$

15. We know that in integration as a limit sum

$$\int_0^1 f(x)dx=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}f(r/n)$$

Similarly the given series can be written as

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{6n}\right)=\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+\frac{r}{n}}=\int_0^5\frac{1}{1+x}dx=[\log|1+x|]_0^5=\log 6$$