POLYNOMIALS

2
CHAPTER

CONTENTS

➢ INTRODUCTION

Algebra is that branch of mathematics which treats the relation of numbers.

➢ CONSTANTS AND VARIABLES

In algebra, two types of symbols are used : constants and variables (literals).

◈ Constant :

It is a symbol whose value always remains the same, whatever the situation be.

For example: $5, -9, \frac{3}{8}, \pi, \frac{7}{15}$, etc.

◈ Variable :

It is a symbol whose value changes according to the situation.

For example : x, y, z, ax, a + x, 5y, – 7x, etc.

➢ ALGEBRAIC EXPRESSION

(a) An algebraic expression is a collection of terms separated by plus (+) or minus (–) sign. For example : 3x + 5y, 7y – 2x, 2x – ay + az, etc.

(b) The various parts of an algebraic expression that are separated by '+' or '–' sign are called terms.

For example :

Algebraic expression	No. of terms	Terms
(i) $-32x$	1	$-32x$
(ii) $2x + 3y$	2	2x and 3y
(iii) $ax - 5y + cz$	3	ax, –5y and cz
(iv) $\frac{3}{x} + \frac{y}{7} - \frac{xy}{8} + 9$	4	$\frac{3}{x}, \frac{y}{7}, -\frac{xy}{8}$ and 9 and so on.

Types of Algebraic Expressions :

(i) Monomial : An algebraic expression having only one term is called a monomial. For ex. 8y, –7xy, $4x^2$, abx, etc. 'mono' means 'one'.

(ii) Binomial : An algebraic expression having two terms is called a binomial. For ex. 8x + 3y, 8x + 3, 8 + 3y, a + bz, 9 – 4y, $2x^2 - 4z$, $6y^2 - 5y$, etc. 'bi' means 'two'

(iii) Trinomial : An algebraic expression having three terms is called a trinomial. For ex.

$ax - 5y + 8z$, $3x^2 + 4x + 7$, $9y^2 - 3y + 2x$, etc. 'tri means 'three'.

(iv) Multimonial : An algebraic expression having two or more terms is called a multimonial.

➢ FACTORS AND COEFFICIENTS

◈ Factor :

Each combination of the constants and variables, which form a term, is called a factor.

For examples :

(i) 7, x and 7x are factors of 7x, in which 7 is constant (numerical) factor and x is variable (literal) factor.

(ii) In $-5x^2y$, the numerical factor is –5 and literal factors are : x, y, xy, x^2 and x^2y.

◈ Coefficient :

Any factor of a term is called the coefficient of the remaining term.

For example :

(i) **In 7x** ; 7 is coefficient of x

(ii) **In $-5x^2y$;** 5 is coefficient of $-x^2y$; –5 is coefficient of x^2y,

Ex. 1 Write the coefficient of :

(i) x^2 in $3x^3 - 5x^2 + 7$

(ii) xy in 8xyz

(iii) –y in $2y^2 - 6y + 2$

(iv) x^0 in 3x + 7

Sol. (i) –5 (ii) 8z (iii) 6

(iv) Since $x^0 = 1$, Therefore

$3x + 7 = 3x + 7x^0$

coefficient of x^0 is 7.

➢ DEFINITION OF POLYNOMIAL

A polynomial is an algebraic expression in which each variable involved has power (exponent) a whole number.

For example :

In $3x^2 - y^5 + 8z$, the power of variable x in the term $3x^2$ is 2, the power of variable in the term $-y^5$ is 5 and the power of variable z in the term 8z is 1 ($z = z^1$). Therefore, the algebraic expression
$3x^2 - y^5 + 8z$ is a polynomial.

◈ Polynomial in one variable :

The algebraic expression like 8x, 7x + 3, 4y – 3, $8x^2 + 5$, $6 - z^2$, $x^2 - 5x + 6$, $3y^2 + 8y + 9$, etc. each of which involves only one variable (literal) are called polynomials in one variable.

◈ Polynomials in two or more variables :

An algebraic expression, whose term or involves/involve two or more variables (literals) such that the exponent of each variable is a whole number, is called a polynomial in two or more variables.

For examples :

(a) $3x^2 - 6xy + 8y^2$ is a polynomial in two variables x and y.

(b) $x + xy^3 - 8x^2yz - 15$ is a polynomial in three variables x, y and z.

➢ DEGREE OF A POLYNOMIAL

The greatest power (exponent) of the terms of a polynomial is called degree of the polynomial.

For example :

(a) In polynomial $5x^2 - 8x^7 + 3x$:

(i) The power of term $5x^2 = 2$

(ii) The power of term $-8x^7 = 7$

(iii) The power of 3x = 1

Since, the greatest power is 7, therefore degree of the polynomial $5x^2 - 8x^7 + 3x$ is 7

(b) The degree of polynomial :

(i) $4y^3 - 3y + 8$ is 3

(ii) 7p + 2 is 1($p = p^1$)

(iii) $2m - 7m^8 + m^{13}$ is 13 and so on.

❖ EXAMPLES ❖

Ex.2 Find which of the following algebraic expression is a polynomial.

(i) $3x^2 - 5x$ (ii) $x + \frac{1}{x}$

(iii) $\sqrt{y} - 8$ (iv) $z^5 - \sqrt[3]{z} + 8$

Sol. (i) $3x^2 - 5x = 3x^2 - 5x^1$

It is a polynomial.

(ii) $x + \frac{1}{x} = x^1 + x^{-1}$

It is not a polynomial.

(iii) $\sqrt{y} - 8 = y^{1/2} - 8$

Since, the power of the first term ($\sqrt{y}$) is $\frac{1}{2}$, which is not a whole number.

(iv) $z^5 - \sqrt[3]{z} + 8 = z^5 - z^{1/3} + 8$

Since, the exponent of the second term is 1/3, which in not a whole number. Therefore, the given expression is not a polynomial.

Ex.3 Find the degree of the polynomial :

(i) $5x - 6x^3 + 8x^7 + 6x^2$

(ii) $2y^{12} + 3y^{10} - y^{15} + y + 3$

(iii) x

(iv) 8

Sol. (i) Since the term with highest exponent (power) is $8x^7$ and its power is 7.

$\therefore$ The degree of given polynomial is 7.

(ii) The highest power of the variable is 15

$\Rightarrow$ degree = 15.

(iii) $x = x^1 \Rightarrow$ degree is 1.

(iv) $8 = 8x^0 \Rightarrow$ degree = 0

➤ TYPES OF POLYNOMIALS

(i) Constant Polynomial :

It is a polynomial with degree 0 (zero).

Ex. 20, –8, –1, 1, 5, 7, π, etc.

(ii) Linear Polynomial :

It is a polynomial with degree 1 (one).

Ex. $-8x, 3x, x, x + \sqrt{2}, \sqrt{3}x - 2, 5y - \frac{3}{2}, \frac{5}{7}z + 1$ etc.

(iii) Quadratic Polynomial :

It is a polynomial with degree 2 (two)

Ex. $6x^2, y^2, \frac{2}{5}z^2, x^2 - 3x, x^2 - 3, 8 - 3y^2, 5x^2 + 3x + 2, 7 - 2z + z^2$, etc.

(iv) Cubic Polynomial :

It is a polynomial with degree 3 (three).

Ex. $15y^3, x^3, 8z^3, x^3 - 5x^2, 3y^2 + y^3, \sqrt{2} + 3z - 2z^2 + 6z^3, 7y - 2 - 12y^3$, etc.

➤ VALUES OF A POLYNOMIAL

For a polynomial $f(x) = 3x^2 - 4x + 2$.

To find its value at x = 3;

replace x by 3 everywhere.

So, the value of $f(x) = 3x^2 - 4x + 2$ at x = 3 is

$f(3) = 3 \times 3^2 - 4 \times 3 + 2$

$= 27 - 12 + 2 = 17$.

Similarly, the value of polynomial

$f(x) = 3x^2 - 4x + 2$,

(i) at $x = -2$ is $f(-2) = 3(-2)^2 - 4(-2) + 2$

$= 12 + 8 + 2 = 22$

(ii) at $x = 0$ is $f(0) = 3(0)^2 - 4(0) + 2$

$= 0 - 0 + 2 = 2$

(iii) at $x = \frac{1}{2}$ is $f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2$

$= \frac{3}{4} - 2 + 2 = \frac{3}{4}$

Ex.4 Find the value of the polynomial $5x - 4x^2 + 3$ at:

(i) x = 0 (ii) x = –1

Sol. Let $p(x) = 5x - 4x^2 + 3$.

(i) At x = 0, $p(0) = 5 \times 0 - 4 \times (0)^2 + 3$

$= 0 - 0 + 3 = 3$

(ii) At x = –1, $p(-1) = 5(-1) - 4(-1)^2 + 3$

$= -5 - 4 + 3 = -6$

➤ ZEROES OF A POLYNOMIAL

If for x = a, the value of the polynomial p(x) is 0 i.e., p(a) = 0; then x = a is a zero of the polynomial p(x).

For example :

(i) For polynomial $p(x) = x - 2$; $p(2) = 2 - 2 = 0$

$\therefore$ x = 2 or simply 2 is a zero of the polynomial $p(x) = x - 2$.

(ii) For the polynomial $g(u) = u^2 - 5u + 6$;

$g(3) = (3)^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

$\therefore$ 3 is a zero of the polynomial $g(u)$

$= u^2 - 5u + 6$.

Also, $g(2) = (2)^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

$\therefore$ 2 is also a zero of the polynomial

$g(u) = u^2 - 5u + 6$

(a) Every linear polynomial has one and only one zero.

(b) A given polynomial may have more than one zeroes.

(c) If the degree of a polynomial is n; the largest number of zeroes it can have is also n.

For example :

If the degree of a polynomial is 5, the polynomial can have at the most 5 zeroes; if the degree of a polynomial is 8; largest number of zeroes it can have is 8.

(d) A zero of a polynomial need not be 0.

For example : If $f(x) = x^2 - 4$,

then $f(2) = (2)^2 - 4 = 4 - 4 = 0$

Here, zero of the polynomial $f(x) = x^2 - 4$ is 2 which itself is not 0.

(e) 0 may be a zero of a polynomial.

For example : If $f(x) = x^2 - x$,

then $f(0) = 0^2 - 0 = 0$

Here 0 is the zero of polynomial $f(x) = x^2 - x$.

❖ EXAMPLES ❖

Ex.5 Verify whether the indicated numbers are zeroes of the polynomial corresponding to them in the following cases :

(i) $p(x) = 3x + 1, x = -\frac{1}{3}$

(ii) $p(x) = (x + 1)(x - 2), x = -1, 2$

(iii) $p(x) = x^2, x = 0$

(iv) $p(x) = \lambda x + m, x = -\frac{m}{\lambda}$

(v) $p(x) = 2x + 1, x = \frac{1}{2}$

Sol. (i) $p(x) = 3x + 1$

$\Rightarrow p\left(-\frac{1}{3}\right) = 3 \times -\frac{1}{3} + 1 = -1 + 1 = 0$

$\therefore x = -\frac{1}{3}$ is a zero of $p(x) = 3x + 1$.

(ii) $p(x) = (x + 1)(x - 2)$

$\Rightarrow p(-1) = (-1 + 1)(-1 - 2) = 0 \times -3 = 0$

and, $p(2) = (2 + 1)(2 - 2) = 3 \times 0 = 0$

$\therefore x = -1$ and $x = 2$ are zeroes of the given polynomial.

(iii) $p(x) = x^2 \quad \Rightarrow \quad p(0) = 0^2 = 0$

$\therefore x = 0$ is a zero of the given polynomial

(iv) $p(x) = \lambda x + m \quad \Rightarrow \quad p\left(-\frac{m}{\lambda}\right) = \lambda\left(-\frac{m}{\lambda}\right) + m$

$= -m + m = 0$

$\therefore x = -\frac{m}{\lambda}$ is a zero of the given polynomial.

(v) $p(x) = 2x + 1 \quad \Rightarrow \quad p\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + 1$

$= 1 + 1 = 2 \neq 0$

$\therefore x = \frac{1}{2}$ is not a zero of the given polynomial.

Ex.6 Find the zero of the polynomial in each of the following cases :

(i) $p(x) = x + 5$ (ii) $p(x) = 2x + 5$

(iii) $p(x) = 3x - 2$

Sol. To find the zero of a polynomial p(x) means to solve the polynomial equation $p(x) = 0$.

(i) For the zero of polynomial $p(x) = x + 5$

$p(x) = 0 \quad \Rightarrow \quad x + 5 = 0 \quad \Rightarrow \quad x = -5$

$\therefore x = -5$ is a zero of the polynomial $p(x) = x + 5$.

(ii) $p(x) = 0 \quad \Rightarrow \quad 2x + 5 = 0$

$\Rightarrow 2x = -5$ and $x = \frac{-5}{2}$

$\therefore x = -\frac{-5}{2}$ is a zero of $p(x) = 2x + 5$.

(iii) $p(x) = 0 \Rightarrow \ 3x - 2 = 0$

$\Rightarrow \ 3x = 2$ and $x = \frac{2}{3}$.

$\therefore \ \ x = \frac{2}{3}$ is zero of $p(x) = 3x - 2$

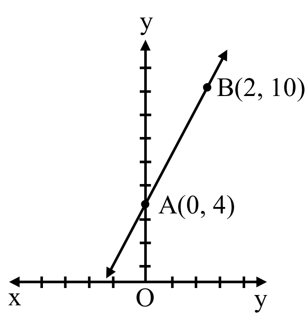
GEOMETRIC MEANING OF THE ZEROES OF A POLYNOMIAL

Let us consider linear polynomial $ax + b$. The graph of $y = ax + b$ is a straight line.

For example : The graph of $y = 3x + 4$ is a straight line passing

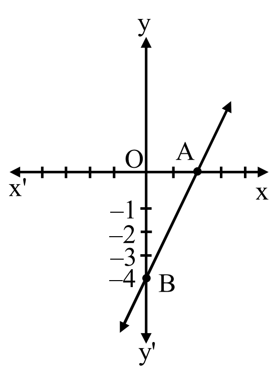
through (0, 4) and (2, 10).

x	0	2
$y = 3x + 4$	4	10
Points	A	B



(i) Let us consider the graph of $y = 2x - 4$ intersects the x-axis at $x = 2$. The zero $2x - 4$ is 2. Thus, the zero of the polynomial $2x - 4$ is the x-coordinate of the point where the graph $y = 2x - 4$ intersects the x-axis.

x	2	0
$y = 2x - 4$	0	-4
Points	A	B



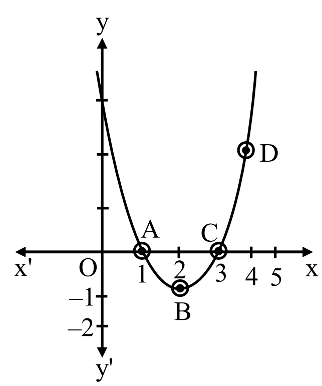
(ii) A general equation of a linear polynomial is $ax + b$. The graph of $y = ax + b$ is a straight line which intersects the x-axis at $\left(\frac{-b}{a}, 0\right)$.

Zero of the polynomial $ax + b$ is the x-coordinate of the point of intersection of the graph with x-axis.

(iii) Let us consider the quadratic polynomial $x^2 - 4x + 3$. The graph of $x^2 - 4x + 3$ intersects the x-axis at the point (1, 0) and (3, 0). Zeroes of the polynomial $x^2 - 4x + 3$ are the x-coordinates of the points of intersection of the graph with x-axis.

x	1	2	3	4	5
$y = x^2 - 4x + 3$	0	-1	0	3	8
Points	A	B	C	D	E

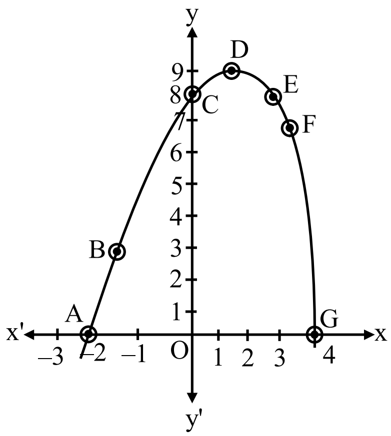
The shape of the graph of the quadratic polynomials is $\cup$ and the curve is known as parabola.



(iv) Now let us consider one more polynomial $-x^2 + 2x + 8$. Graph of this polynomial intersects the x-axis at the points (4, 0) (–2, 0).

Zeroes of the polynomial $-x^2 + 2x + 8$ are the x-coordinates of the points at which the graph intersects the x-axis. The shape of the graph of the given quadratic polynomial is $\cap$ and the curve is known as parabola.

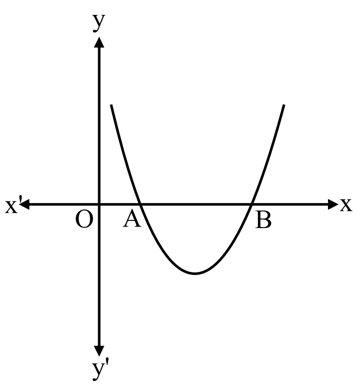
x	-2	-1	0	1	2	3	4
y	0	5	8	9	8	7	0
Points	A	B	C	D	E	F	G



The zeroes of a quadratic polynomial $ax^2 + bx + c$, $a \neq 0$ are the x-coordinates of the points where the graph of $y = ax^2 + bx + c$ intersects the x-axis. There are three types of the graph of $y = ax^2 + bx + c$.

Case I :

Graph of $y = ax^2 + bx + c$ intersects the x-axis at two distinct points A and B. The zeroes of the quadratic polynomial ax2 + bx + c are the x-coordinates of the points A and B.



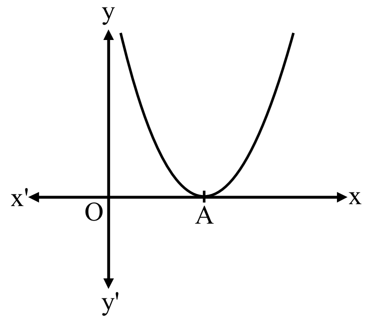
Condition : $b^2 - 4ac > 0$ and $a > 0$

For example : Quadratic polynomial $x^2 - 7x + 12$

Graph of $y = x^2 - 7x + 12$ will cut x-axis at the two distinct points (3, 0) and (4, 0). Zeroes of the polynomial are 3 and 4.

Case II :

Here the graph intersects the x-axis at exactly one point i.e., at two coincident points. These two coincident points A and B coincide and becomes one point A. Zero of the quadratic polynomial is the x-coordinate of point A.



Condition : $b^2 - 4ac = 0$ and $a > 0$

For example : $y = (x - 1)^2$

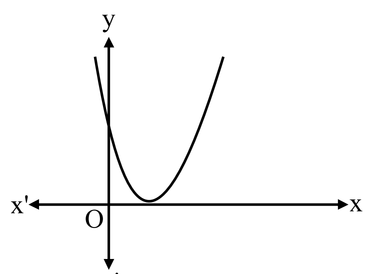
The graph of $y = (x - 1)^2$ will cut x-axis at one point (1, 0). Zero of the polynomial of the point of intersection with x-axis.

Case III :

Here the graph of the quadratic equation will not cut the x-axis. Either the graph will be completely above the x-axis or below the x-axis So the quadratic polynomial $ax^2 + bx + c$ has no zero in this case.

For example : $y = x^2 - 2x + 4$

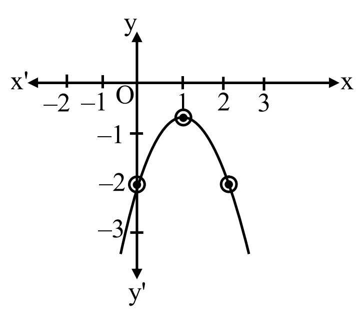
Graph of $y = x^2 - 2x + 4$ will not intersect the x-axis and the graph will be above the x-axis. The polynomial $x^2 - 2x + 4$ has no zero.



Let $y = -x^2 + 2x - 2$

Graph of $y = - x^2 + 2x - 2$ will not intersect the x-axis and the graph will be below the x-axis.

The polynomial $-x^2 + 2x - 2$ has no zero.



In Brief : It means that a polynomial of degree two has at most two zeroes.

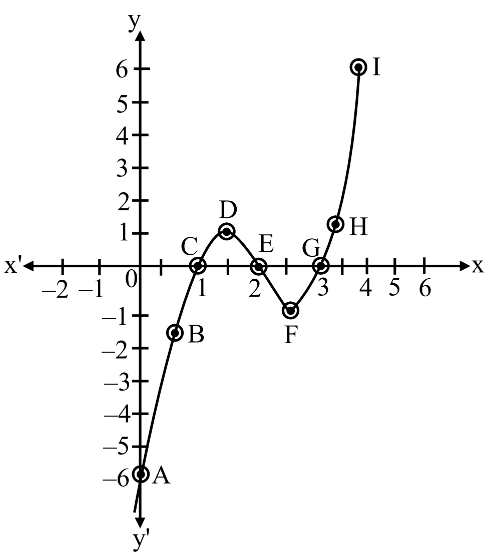
Cubic polynomial : Let us find out geometrically how many zeroes a cubic has.

Let consider cubic polynomial $x^3 - 6x^2 + 11x - 6$.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$y = x^3 - 6x^2 + 11x - 6$	-6	-1.875	0	0.375	0	-0.375	0	1.875	6
Points	A	B	C	D	E	F	G	H	I

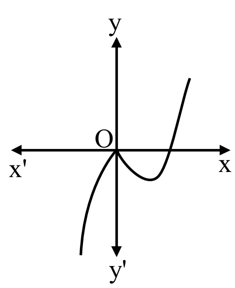
Case 1 :

The graph of the cubic equation intersects the x-axis at three points (1, 0), (2, 0) and (3, 0). Zeroes of the given polynomial are the x-coordinates of the points of intersection with the x-axis.



Case 2 :

The cubic equation $x^3 - x^2$ intersects the x-axis at the point (0, 0) and (1, 0). Zero of a polynomial $x^3 - x^2$ are the x-coordinates of the point where the graph cuts the x-axis.

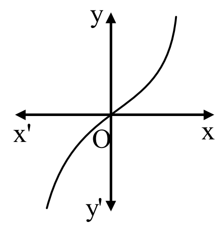


Zeroes of the cubic polynomial are 0 and 1.

Case 3 :

$y = x^3$

Cubic polynomial has only one zero.

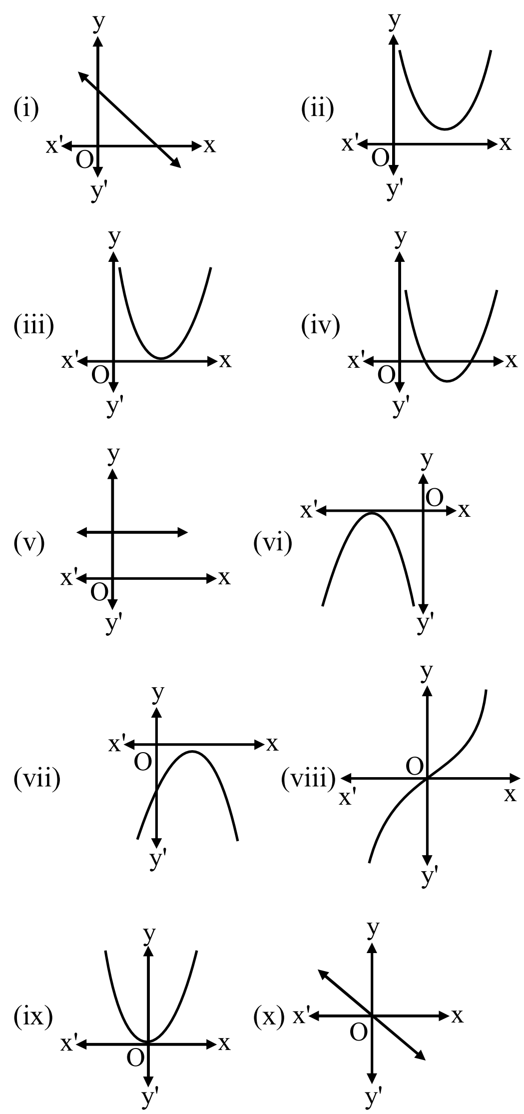


In brief : A cubic equation can have 1 or 2 or 3 zeroes or any polynomial of degree three can have at most three zeroes.

Remarks : In general, polynomial of degree n, the graph of y = p(x) passes x-axis at most at n points. Therefore, a polynomial p(x) of degree n has at most n zeroes.

❖ EXAMPLES ❖

Ex.7 Which of the following correspond to the graph to a linear or a quadratic polynomial and find the number of zeroes of polynomial.



Sol. (i) The graph is a straight line so the graph is of a linear polynomial. The number of zeroes is one as the graph intersects the x-axis at one point only.

(ii) The graph is a parabola. So, this is the graph of quadratic polynomial. The number of zeroes is zero as the graph does not intersect the x-axis.

(iii) Here the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).

(iv) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is two as the graph intersects the x-axis at two points.

(v) The polynomial is linear as the graph is straight line. The number of zeroes is zero as the graph does not intersect the x-axis.

(vi) The polynomial is quadratic as the graph is a parabola. The number of zeroes is 1 as the graph intersects the x-axis at one point (two coincident points) only.

(vii) The polynomial is quadratic as the graph is a parabola. The number of zeroes is zero, as the graph does not intersect the x-axis.

(viii) Polynomial is neither linear nor quadratic as the graph is neither a straight line nor a parabola is one as the graph intersects the x-axis at one point only.

(ix) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).

(x) The polynomial is linear as the graph is a straight line. The number of zeroes is one as the graph intersects the x-axis at only one point.

➢ WORKING RULE TO DIVIDE A POLYNOMIAL BY ANOTHER POLYNOMIAL

Step 1: First arrange the term of dividend and the divisor in the decreasing order of their degrees.

Step 2 : To obtain the first term of quotient divide the highest degree term of the dividend by the highest degree term of the divisor.

Step 3: To obtain the second term of the quotient, divide the highest degree term of the new dividend obtained as remainder by the highest degree term of the divisor.

Step 4 : Continue this process till the degree of remainder is less than the degree of divisor.

◈ Division Algorithm for Polynomial

If p(x) and g(x) are any two polynomials with

g(x) ≠ 0, then we can find polynomials q(x) and r(x) such that

$$p(x) = q(x) \times g(x) + r(x)$$

where r(x) = 0 or degree of r(x) < degree of g(x).

The result is called Division Algorithm for polynomials.

Dividend = Quotient × Divisor + Remainder

❖ EXAMPLES ❖

Ex.8 Divide $3x^3 + 16x^2 + 21x + 20$ by $x + 4$.

Sol.

$$\begin{array}{r|l}
 & 3x^2 + 4x + 5 \\
x+4 & 3x^3 + 16x^2 + 21x + 20 \\
 & 3x^3 + 12x^2 \\
 & - \quad - \\
\hline
 & \quad 4x^2 + 21x + 20 \\
 & \quad 4x^2 + 16x \\
 & \quad - \quad - \\
\hline
 & \qquad 5x + 20 \\
 & \qquad 5x + 20 \\
 & \qquad - \quad - \\
\hline
 & \qquad 0
\end{array}$$

First term of q(x) = $\frac{3x^3}{x} = 3x^2$

Second term of q(x) = $\frac{4x^2}{x} = 4x$

Third term of q(x) = $\frac{5x}{x} = 5$

Quotient = $3x^2 + 4x + 5$

Remainder = 0

Ex.9 Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below :

$p(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$

Sol. We have,

$p(x) = x^3 - 3x^2 + 5x - 3$ and $g(x) = x^2 - 2$

$$\begin{array}{r|l}
 & x - 3 \\
x^2-2 & x^3 - 3x^2 + 5x - 3 \\
 & x^3 \qquad - 2x \\
 & - \qquad + \\
\hline
 & - 3x^2 + 7x - 3 \\
 & - 3x^2 \qquad + 6 \\
 & + \qquad - \\
\hline
 & \qquad 7x - 9
\end{array}$$

First term of quotient is $\frac{x^3}{x^2} = x$

Second term of quotient is $\frac{-3x^2}{x^2} = -3$

We stop here since

degree of $(7x - 9)$ < degree of $(x^2 - 2)$

So, quotient = $x - 3$, remainder = $7x - 9$

Therefore,

Quotient × Divisor + Remainder

$= (x - 3)(x^2 - 2) + 7x - 9$

$= x^3 - 2x - 3x^2 + 6 + 7x - 9$

$= x^3 - 3x^2 + 5x - 3$ = Dividend

Therefore, the division algorithm is verified.

Ex.10 Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below

$p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

Sol. We have,

$p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

$$
\begin{array}{r|l}
 & x^2 + x - 3 \\
x^2 - x + 1 & x^4 - 3x^2 + 4x + 5 \\
 & x^4 - x^3 + x^2 \\
 & - \quad + \quad - \\ \hline
 & x^3 - 4x^2 + 4x + 5 \\
 & x^3 - x^2 + x \\
 & - \quad + \quad - \\ \hline
 & -3x^2 + 3x + 5 \\
 & -3x^2 + 3x - 3 \\
 & + \quad - \quad + \\ \hline
 & 8
\end{array}
$$

We stop here since

degree of (8) < degree of $(x^2 - x + 1)$.

So, quotient = $x^2 + x - 3$, remainder = 8

Therefore,

Quotient × Divisor + Remainder

$= (x^2 + x - 3)(x^2 - x + 1) + 8$

$= x^4 - x^3 + x^2 + x^3 - x^2 + x - 3x^2 + 3x - 3 + 8$

$= x^4 - 3x^2 + 4x + 5$ = Dividend

Therefore the Division Algorithm is verified.

Ex.11 Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm.

$t^2 - 3;\ 2t^4 + 3t^3 - 2t^2 - 9t - 12.$

Sol. We divide $2t^4 + 3t^3 - 2t^2 - 9t - 12$by $t^2 - 3$

$$
\begin{array}{r|l}
 & 2t^2 + 3t + 4 \\
t^2 - 3 & 2t^4 + 3t^3 - 2t^2 - 9t - 12 \\
 & 2t^4 \quad - 6t^2 \\
 & - \quad + \\ \hline
 & 3t^3 + 4t^2 + 9t - 12 \\
 & 3t^3 \quad - 9t \\
 & - \quad + \\ \hline
 & 4t^2 \quad - 12 \\
 & 4t^2 \quad - 12 \\
 & - \quad + \\ \hline
 & 0
\end{array}
$$

Here, remainder is 0, so $t^2 - 3$ is a factor of $2t^4 + 3t^3 - 2t^2 - 9t - 12$.

$2t^4 + 3t^3 - 2t^2 - 9t - 12$

$= (2t^2 + 3t + 4)(t^2 - 3)$

Ex.12 Check whether first polynomial is a factor of the second polynomial by applying the division algorithm.

$x^2 + 3x + 1,\ 3x^4 + 5x^3 - 7x^2 + 2x + 2$

Sol. We divide $3x^4+5x^3-7x^2+2x+2$ by $x^2 + 3x + 1$

$$
\begin{array}{r|l}
 & 3x^2 + 4x + 2 \\
x^2 - 3x + 1 & 3x^4 + 5x^3 - 7x^2 + 2x + 2 \\
 & 3x^4 + 9x^3 + 3x^2 \\
 & - \quad - \quad - \\ \hline
 & -4x^3 - 10x^2 + 2x + 2 \\
 & -4x^3 - 12x^2 - 4x \\
 & + \quad + \quad + \\ \hline
 & 2x^2 + 6x + 2 \\
 & 2x^2 + 6x^2 + 2 \\
 & - \quad - \quad - \\ \hline
 & 0
\end{array}
$$

Since, here remainder is zero.

Hence, $x^2 + 3x + 1$ is a factor of

$3x^4 + 5x^3 - 7x^2 + 2x + 2$.

Checking

$3x^4 + 5x^3 - 7x^2 + 2x + 2$

$= (3x^2 - 4x + 2)(x^2 + 3x + 1) + 0$

$= 3x^4 + 5x^3 - 7x^2 + 2x + 2$ = Dividend

➤ REMAINDER THEOREM AND FACTOR THEOREM

Remainder Theorem : Let p(x) be any polynomial of degree greater than or equal to one and let a be any real number. If p(x) is divided by the linear polynomial $x - a$, then the remainder is p(a).

Proof : Let p(x) be any polynomial with degree greater than or equal to 1. Suppose that when p(x) is divided by x – a, the quotient is q(x) and the remainder is r(x), i.e., p(x) = (x – a) q(x) + r(x)

Since the degree of x – a is 1 and the degree of r(x) is less than the degree of x – a, the degree of r(x) = 0. This means that r(x) is a constant, say r.

So, for every value of x, r(x) = r.

Therefore, p(x) = (x – a) q(x) + r

In particular, if x = a, this equation gives us

p(a) = (a – a) q(a) + r

= r,

which proves the theorem.

Ex.13 Find the remainder when $x^4 + x^3 – 2x^2 + x + 1$ is divided by x – 1.

Sol. Here, $p(x) = x^4 + x^3 – 2x^2 + x + 1$, and the zero of x – 1 is 1.

So, $p(1) = (1)^4 + (1)^3 – 2(1)^2 + 1 + 1$

= 2

So, by the Remainder Theorem, 2 is the remainder when $x^4 + x^3 – 2x^2 + x + 1$ is divided by x – 1. **Ans**

So, we can say

(i) Remainder obtained on dividing polynomial p(x) by x – a is equal to p(a) .

(ii) If a polynomial p(x) is divided by (x + a) the remainder is the value of p(x) at x = –a.

(iii) If a polynomial p(x) is divided by (ax –b), the remainder is the value of p(x) at $x = b/a$

(iv) If a polynomial p(x) is divided by (b–ax), the remainder is equal to the value of p(x) at $x = b/a$.

(v) (x–a) is a factor of polynomial p(x) if p(a) = 0

(vi) (x + a) is a factor of polynomial p(x) if p(–a) = 0

(vii) (ax – b) is a factor of polynomial p(x) if $p\left(\frac{b}{a}\right) = 0$.

(viii) (x – a)(x – b) is a factor of polynomial p(x), if p(a) = 0 and p(b) = 0.

❖ EXAMPLES ❖

Ex.14 Find the remainder when $4x^3 – 3x^2 + 2x – 4$ is divided by

(a) x – 1 (b) x + 2 (c) $x + \frac{1}{2}$

Sol. Let $p(x) = 4x^3 – 3x^2 + 2x – 4$

(a) When p(x) is divided by (x–1), then by remainder theorem, the required remainder will be p(1)

$p(1) = 4 (1)^3 – 3(1)^2 + 2(1) – 4$

$= 4 \times 1 – 3 \times 1 + 2 \times 1 – 4$

$= 4 – 3 + 2 – 4 = – 1$

(b) When p(x) is divided by (x + 2), then by remainder theorem, the required remainder will be p (–2).

$p(–2) = 4 (–2)^3 – 3 (–2)^2 + 2(–2) – 4$

$= 4 \times (–8) – 3 \times 4 – 4 – 4$

$= – 32 – 12 – 8 = – 52$

(c) When p(x) is divided by, $\left(x + \frac{1}{2}\right)$ then by remainder theorem, the required remainder will be

$$p\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) - 4$$

$$= 4 \times \left(-\frac{1}{8}\right) - 3 \times \frac{1}{4} - 2 \times \frac{1}{2} - 4$$

$$= -\frac{1}{2} - \frac{3}{4} - 1 - 4 = \frac{1}{2} - \frac{3}{4} - 5$$

$$= \frac{-2-3-20}{4} = \frac{-25}{4}$$

Ex.15 Determine the remainder when the polynomial $p(x) = x^4 – 3x^2 + 2x + 1$ is divided by x – 1.

Sol. By remainder theorem, the required remainder is equal to p(1).

Now, $p(x) = x^4 – 3x^2 + 2x + 1$

$\Rightarrow p(1) = (1)^4 – 3 \times 1^2 + 2 \times 1 + 1$

$= 1 – 3 + 2 + 1 = 1$

Hence required remainder = p(1) = 1

Ex.16 Find the remainder when the polynomial $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$ is divided by $x + 2$

Sol. We have, $x + 2 = x - (-2)$. So, by remainder theorem, when $f(x)$ is divided by $(x-(-2))$ the remainder is equal to $f(-2)$.

Now, $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$

$\Rightarrow f(-2) = 2(-2)^4 - 6(-2)^3 + 2(-2)^2 - (-2) + 2$

$\Rightarrow f(-2) = 2\times16 - 6\times -8 + 2\times4 + 2 + 2$

$\Rightarrow f(-2) = 32 + 48 + 8 + 2 + 2 = 92$

Hence, required remainder = 92

Ex.17 Find the remainder when

$p(x) = 4x^3 - 12x^2 + 14x - 3$ is divided by $g(x) = x - \frac{1}{2}$

Sol. By remainder theorem, we know that p(x) when divided by $g(x) = \left(x - \frac{1}{2}\right)$ gives a remainder equal to $p\left(\frac{1}{2}\right)$.

Now, $p(x) = 4x^3 - 12x^2 + 14x - 3$

$\Rightarrow p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^3 + 14\left(\frac{1}{2}\right) - 3$

$\Rightarrow p\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{12}{4} + \frac{14}{2} - 3$

$\Rightarrow p\left(\frac{1}{2}\right) = \frac{1}{2} - 3 + 7 - 3$

$\Rightarrow p\left(\frac{1}{2}\right) = \frac{3}{2}$

Hence, required remainder = $p\left(\frac{1}{2}\right) = \frac{3}{2}$

Ex.18 If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x-3)$, find the value of a.

Sol. Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + a$ be the given polynomials. The remainders when p(x) and q(x) are divided by (x–3) are p(3) and q(3) respectively.

By the given condition, we have

$p(3) = q(3)$

$\Rightarrow a\times3^3 + 4\times3^2 + 3\times3 - 4 = 3^3 - 4\times3 + a$

$\Rightarrow 27a + 36 + 9 - 4 = 27 - 12 + a$

$\Rightarrow 26a + 26 = 0$

$\Rightarrow 26a = -26$

$\Rightarrow a = -1$

Ex.19 Let R_1 and R_2 are the remainders when the polynomials $x^3 + 2x^2 - 5ax - 7$ and $x^3 + ax^2 - 12x + 6$ are divided by $x + 1$ and $x - 2$ respectively. If $2R_1 + R_2 = 6$, find the value of a.

Sol. Let $p(x) = x^3 + 2x^2 - 5ax - 7$ and

$q(x) = x^3 + ax^2 - 12x + 6$ be the given polynomials.

Now, R_1 = Remainder when p(x) is divided by $x + 1$

$\Rightarrow R_1 = p(-1)$

$\Rightarrow R_1 = (-1)^3 + 2(-1)^2 - 5a\times -1 - 7$

$[\because p(x) = x^3 + 2x^2 - 5ax - 7]$

$\Rightarrow R_1 = -1 + 2 + 5a - 7$

$\Rightarrow R_1 = 5a - 6$

And, R_2 = Remainder when q(x) is divided by $x - 2$

$\Rightarrow R_1 = q(2)$

$\Rightarrow R_2 = (2)^3 + a\times2^2 - 12\times2 + 6$

$[\because q(x) = x^3 + ax^2 - 12x - 6]$

$\Rightarrow R_2 = 8 + 4a - 24 + 6$

$\Rightarrow R_2 = 4a - 10$

Substituting the values of R_1 and R_2 in $2R_1 + R_2 = 6$, we get

$2(5a - 6) + (4a - 10) = 6$

$\Rightarrow 10a - 12 + 4a - 10 = 6$

$\Rightarrow 14a - 22 = 6$

$\Rightarrow 14a = 28$

$\Rightarrow a = 2$

Factor Theorem :

If p(x) is a polynomial of degree $n \geq 1$ and a is any real number, then (i) $x - a$ is a factor of p(x), if $p(a) = 0$, and (ii) $p(a) = 0$, if $x - a$ is a factor of p(x).

Proof : By the Remainder Theorem,

$p(x) = (x - a)\, q(x) + p(a)$.

(i) If $p(a) = 0$, then $p(x) = (x - a)\, q(x)$, which shows that $x - a$ is a factor of $p(x)$.

(ii) Since $x - a$ is a factor of $p(x)$, $p(x) = (x - a)\, g(x)$ for same polynomial $g(x)$. In this case, $p(a) = (a - a)\, g(a) = 0$.

Ex.20 Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.

Sol. The zero of $x + 2$ is -2. Let $p(x) = x^3 + 3x^2 + 5x + 6$ and $s(x) = 2x + 4$

Then, $p(-2) = (-2)^3 + 3(-2)^2 + 5(-2) + 6$

$= -8 + 12 - 10 + 6$

$= 0$

So, by the Factor Theorem, $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$.

Again, $s(-2) = 2(-2) + 4 = 0$

So, $x + 2$ is a factor of $2x + 4$. **Ans.**

To use factor theorem

Step 1 : $(x + a)$ is factor of a polynomial $p(x)$ if $p(-a) = 0$.

Step 2 : $(ax - b)$ is a factor of a polynomial $p(x)$ if $p(b/a) = 0$

Step 3 : $ax + b$ is a factor of a polynomial $p(x)$ if $p(-b/a) = 0$.

Step 4 : $(x - a)\,(x - b)$ is a factor of a polynomial $p(x)$ if $p(a) = 0$ and $p(b) = 0$.

❖ EXAMPLES ❖

Ex.21 Use the factor theorem to determine whether $x - 1$ is a factor of

(a) $x^3 + 8x^2 - 7x - 2$

(b) $2\sqrt{2}\,x^3 + 5\sqrt{2}\,x^2 - 7\sqrt{2}$

(c) $8x^4 + 12x^3 - 18x + 14$

Sol.(a) Let $p(x) = x^3 + 8x^2 - 7x - 2$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$

$p(1) = (1)^3 + 8(1)^2 - 7(1) - 2$

$= 1 + 8 - 7 - 2$

$= 9 - 9 = 0$

Hence $(x - 1)$ is a factor of $p(x)$.

(b) Let $p(x) = 2\sqrt{2}\,x^3 + 5\sqrt{2}\,x^2 - 7\sqrt{2}$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$.

$p(1) = 2\sqrt{2}\,(1)^3 + 5\sqrt{2}\,(1)^2 - 7\sqrt{2}$

$= 2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2}$

$= 7\sqrt{2} - 7\sqrt{2} = 0$

Hence $(x-1)$ is a factor of $p(x)$

(c) Let $p(x) = 8x^4 + 12x^3 - 18x + 14$

By using factor theorem, $(x-1)$ is a factor of $p(x)$ only when $p(1) = 0$

$p(1) = 8(1)^4 + 12(1)^3 - 18(1) + 14$

$= 8 + 12 - 18 + 14$

$= 34 - 18 = 16 \neq 0$.

Hence $(x-1)$ is not a factor of $p(x)$.

Ex.22 Factorize each of the following expression, given that $x^3 + 13x^2 + 32x + 20$. $(x+2)$ is a factor.

Sol. Let $p(x) = x^3 + 13x^2 + 32x + 20$

$= (x+2)$ is a factor of $p(x)$

$p(x) = (x+2)\,(x^2 + 11x + 10)$

$= (x + 2)\,(x^2 + 10x + x + 10)$

$= (x+2)\,(x + 10)\,(x + 1)$

Ex.23 Factorize $x^3 - 23x^2 + 142x - 120$

Sol. Let $p(x) = x^3 - 23x^2 + 142x - 120$

Constant term, $p(x)$ is -120

$\pm1, \pm2, \pm3, \pm4, \pm5, \pm10, \pm12$ ....±120

$P(1) = 1 - 23 + 142 - 120 = 0$

$\Rightarrow$ $x - 1$ is a factor of $p(x)$. We find the other factor by dividing $p(x)$ by $(x - 1)$

$p(x) = (x - 1)\,(x^2 - 22x + 120)$

$= (x - 1)\,(x^2 - 10x - 12x + 120)$

$= (x - 1)\,[x(x - 10) - 12(x - 10))$

$= (x - 1)\,(x - 10)\,(x - 12)$

Ex.24 Show that (x – 3) is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$

Sol. Let $p(x) = x^3 - 3x^2 + 4x - 12$ be the given polynomial. By factor theorem, (x – a) is a factor of a polynomial p(x) iff p(a) = 0. Therefore, in order to prove that x – 3 is a factor of p(x), it is sufficient to show that p(3) = 0. Now,

$p(x) = x^3 - 3x^2 + 4x - 12$

$\Rightarrow p(3) = 3^3 - 3 \times 3^2 + 4 \times 3 - 12$

$= 27 - 27 + 12 - 12 = 0$

Hence, (x – 3) is a factor of

$p(x) = x^3 - 3x^2 + 4x - 12$.

Ex.25 Show that (x – 1) is a factor of $x^{10} - 1$ and also of $x^{11} - 1$.

Sol. Let $f(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$.

In order to prove that (x – 1) is a factor of both f(x) and g(x), it is sufficient to show that f(1) = 0 and g(1) = 0.

Now, $f(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$

$\Rightarrow f(1) = 1^{10} - 1 = 0$ and $g(1) = 1^{11} - 1 = 0$

$\Rightarrow$ (x – 1) is a factor of both f(x) and g(x)

Ex.26 Show that x + 1 and 2x – 3 are factors of $2x^3 - 9x^2 + x + 12$.

Sol. Let $p(x) = 2x^3 - 9x^2 + x + 12$ be the given polynomial. In order to prove that x + 1 and 2x – 3 are factors of p(x), it is sufficient to show that p(–1) and p(3/2) both are equal to zero.

Now, $p(x) = 2x^3 - 9x^2 + x + 12$

$\Rightarrow p(-1) = 2 \times (-1)^3 - 9 \times (-1)^2 + (-1) + 12$

and, $p\left(\frac{3}{2}\right) = 2 \times \left(\frac{3}{2}\right)^3 - 9 \times \left(\frac{3}{2}\right)^3 + \frac{3}{2} + 12$

$\Rightarrow p(-1) = -2 - 9 - 1 + 12$ and

$p\left(\frac{3}{2}\right) = \frac{54}{8} - \frac{81}{4} + \frac{3}{2} + 12$

$\Rightarrow p(-1) = -12 + 12$ and

$p\left(\frac{3}{2}\right) = \frac{54 - 162 + 12 + 96}{8}$

$\Rightarrow p(-1) = 0$ and $p\left(\frac{3}{2}\right) = 0$

Hence, (x + 1) and (3x – 2) are factors of the given polynomial.

Ex.27 Find the value of k, if x + 3 is a factor of $3x^2 + kx + 6$.

Sol. Let $p(x) = 3x^2 + kx + 6$ be the given polynomial. Then, (x + 3) is a factor of p(x)

$\Rightarrow p(-3) = 0$

$\Rightarrow 3(-3)^2 + k \times (-3) + 6 = 0$

$\Rightarrow 27 - 3k + 6 = 0$

$\Rightarrow 33 - 3k = 0 \Rightarrow k = 11$

Hence, x + 3 is a factor of $3x^2 + kx + 6$ if k = 11.

Ex.28 If $ax^3 + bx^2 + x - 6$ has x + 2 as a factor and leaves a remainder 4 when divided by (x – 2), find the values of a and b.

Sol. Let $p(x) = ax^3 + bx^2 + x - 6$ be the given polynomial. Then, (x + 2) is a factor of p(x)

$\Rightarrow p(-2) = 0 \quad [x + 2 = 0 \Rightarrow x = -2]$

$\Rightarrow a(-2)^3 + b(-2)^2 + (-2) - 6 = 0$

$\Rightarrow -8a + 4b - 2 - 6 = 0 \Rightarrow -8a + 4b = 8$

$\Rightarrow -2a + b = 2$ ....(i)

It is given that p(x) leaves the remainder 4 when it is divided by (x – 2). Therefore,

$p(2) = 4 \quad [x - 2 = 0 \Rightarrow x = 2]$

$\Rightarrow a(2)^3 + b(2)^2 + 2 - 6 = 4$

$\Rightarrow 8a + 4b - 4 = 4 \Rightarrow 8a + 4b = 8$

$\Rightarrow 2a + b = 2$ ....(ii)

Adding (i) and (ii), we get

$2b = 4 \Rightarrow b = 2$

Putting b = 2 in (i), we get

$-2a + 2 = 2 \Rightarrow -2a = 0 \Rightarrow a = 0$.

Hence, a = 0 and b = 2.

Ex.29 If both x – 2 and $x - \frac{1}{2}$ are factors of $px^2 + 5x + r$, show that p = r.

Sol. Let $f(x) = px^2 + 5x + r$ be the given polynomial. Since x – 2 and $x - \frac{1}{2}$ are factors of f(x). Therefore,

$f(2) = 0$ and $f\left(\frac{1}{2}\right) = 0$

$$\left[\Theta\ x - 2 = 0 \Rightarrow x = 2 \text{ and } x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}\right]$$

$\Rightarrow\ p \times 2^2 + 5 \times 2 + r = 0$ and

$$p\left(\frac{1}{2}\right)^2 + 5 \times \frac{1}{2} + r = 0$$

$\Rightarrow\ 4p + 10 + r = 0$ and $\frac{p}{4} + \frac{5}{2} + r = 0$

$\Rightarrow\ 4p + r = -10$ and $\frac{p + 4r + 10}{4} = 0$

$\Rightarrow\ 4p + r = -10$ and $p + 4r + 10 = 0$

$\Rightarrow\ 4p + r = -10$ and $p + 4r = -10$

$\Rightarrow\ 4p + r = p + 4r$

[Θ RHS of the two equations are equal]

$\Rightarrow\ 3p = 3r \qquad \Rightarrow\ p = r$

Ex.30 If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$.

Sol. Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ be the given polynomial. Then, $(x^2 - 1)$ is a factor of p(x)

$\Rightarrow\ (x - 1)(x + 1)$ is a factor of p (x)

$\Rightarrow\ (x - 1)$ and $(x + 1)$ are factors of p(x)

$\Rightarrow\ p(1) = 0$ and $p(-1) = 0$

$[\ x - 1 = 0 \Rightarrow x = 1$ and $x + 1 = 0 \Rightarrow x = -1]$

$\Rightarrow a + b + c + d + e = 0$ and $a - b + c - d + e = 0$

Adding and subtracting these two equations, we get $2(a + c + e) = 0$ and $2(b + d) = 0$

$\Rightarrow\ a + c + e = 0$ and $b + d = 0$

$\Rightarrow\ a + c + e = b + d = 0$

Ex.31 Using factor theorem, show that a – b, b – c and c – a are the factors of

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Sol. By factor theorem, a – b will be a factor of the given expression if it vanishes by substituting a = b in it.

Substituting a = b in the given expression, we have

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$

$= b(b^2 - c^2) + b(c^2 - b^2) + c(b^2 - b^2)$

$= b^3 - bc^2 + bc^2 - b^3 + c(b^2 - b^2) = 0$

$\therefore$ (a – b) is a factor of

$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Similarly, we can show that (b – c) and (c – a) are also factors of the given expression.

Hence, (a – b), (b – c) and (c – a) are also factors of the given expression.

➢ APPLICATION OF REMAINDER THEOREM IN THE FACTORIZATION OF POLYNOMIALS

(i) Obtain the polynomial p(x)

(ii) Obtain the constant term in p(x) and find its all possible factors. For example, in the polynomial $x^4 + x^3 - 7x^2 - x + 6$ the constant term is 6 and its factors are ± 1, ± 2, ± 3, ± 6.

(iii) Take one of the factors, say a and replace x by it in the given polynomial. If the polynomial reduces to zero, then (x – a) is a factor of polynomial.

(iv) Obtain the factors equal in no. to the degree of polynomial. Let these are (x–a), (x–b), (x–c.).....

(v) Write p(x) = k (x–a) (x–b) (x–c) ..... where k is constant.

(vi) Substitute any value of x other than a,b,c ...... and find the value of k.

❖ EXAMPLES ❖

Ex.32 Factorize $x^2 + 4 + 9z^2 + 4x - 6xz - 12z$

Sol. The presence of the three squares viz.x^2, $(2)^2$, and $(3z)^2$ gives a clue that identity (vii) could be used. So we write.

$A = x^2 + (2)^2 + (3z)^2 + 4x - 6xz - 12z$

We note that the last two of the product terms are negative and that both of these contain z. Hence we write A as

$A = x^2 + (2)^2 + (-3z)^2 + 2.2x - 2.x.(-3z) +$

$2.2(-3z) = (x + 2 - 3z)^2$

$= (x + 2 - 3z)(x + 2 - 3z)$

Ex.33 Using factor theorem, factorize the polynomial $x^3 - 6x^2 + 11x - 6$.

Sol. Let $f(x) = x^3 - 6x^2 + 11x - 6$

The constant term in f(x) is equal to – 6 and factors of – 6 are ±1, ± 2, ± 3, ± 6.

Putting x = 1 in f(x), we have

$f(1) = 1^3 - 6 \times 1^2 + 11 \times 1 - 6$

$= 1 - 6 + 11 - 6 = 0$

$\therefore$ (x – 1) is a factor of f(x)

Similarly, x – 2 and x – 3 are factors of f(x).

Since f(x) is a polynomial of degree 3. So, it can not have more than three linear factors.

Let f(x) = k (x–1) (x– 2) (x – 3). Then,

$x^3 - 6x^2 + 11x - 6 = k(x-1)(x-2)(x-3)$

Putting x = 0 on both sides, we get

$-6 = k(0-1)(0-2)(0-3)$

$\Rightarrow -6 = -6k \Rightarrow k = 1$

Putting k = 1 in f(x) = k (x– 1) (x– 2) (x–3), we get

f(x) = (x–1) (x– 2) (x – 3)

Hence, $x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$

Ex.34 Using factor theorem, factorize the polynomial $x^4 + x^3 - 7x^2 - x + 6$.

Sol. Let $f(x) = x^4 + x^3 - 7x^2 - x + 6$

the factors of constant term in f(x) are ±1, ±2, ±3 and ± 6

Now,

f (1) = 1+ 1 – 7 – 1 + 6 = 8 – 8 = 0

$\Rightarrow$ (x – 1) is a factor of f(x)

f (–1) = 1– 1 – 7 + 1 + 6 = 8 – 8 = 0

$\Rightarrow$ x +1 is a factor of f(x)

$f(2) = 2^4 + 2^3 - 7 \times 2^2 - 2 + 6$

= 16 + 8 – 28 – 2 + 6 = 0

$\Rightarrow$ x–2 is a factor of f(x)

$f(-2) = (-2)^4 + (-2)^3 - 7(-2)^2 - (-2) + 6$

$= 16 - 8 - 28 + 2 + 6 = -12 \neq 0$

$\Rightarrow$ x + 2 is not a factor of f(x)

$f(-3) = (-3)^4 + (-3)^3 - 7(-3)^2 - (-3) + 6$

= 81 – 27 – 63 + 3 + 6 = 90 – 90 = 0

$\Rightarrow$ x + 3 is a factor of f (x)

Since f(x) is a polynomial of degree 4. So, it cannot have more than 4 linear factors

Thus, the factors of f (x) are (x–1), (x+1), (x–2) and (x+3).

Let f(x) = k (x–1) (x+1) (x–2) (x + 3)

$\Rightarrow x^4 + x^3 - 7x^2 - x + 6$

= k (x–1) (x +1) (x – 2) (x + 3)

Putting x = 0 on both sides, we get

$6 = k(-1)(1)(-2)(3) \Rightarrow 6 = 6k \Rightarrow k = 1$

Substituting k = 1 in (i), we get

$x^4 + x^3 - 7x^2 - x + 6 = (x-1)(x+1)(x-2)(x+3)$

Ex.35 Factorize, $2x^4 + x^3 - 14x^2 - 19x - 6$

Sol. Let $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ be the given polynomial. The factors of the constant term – 6 are ±1, ±2, ±3 and ±6, we have,

$f(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6$

= 2 – 1 – 14 + 19 – 6 = 21 – 21 = 0

and,

$f(-2) = 2(-2)^4 + (-2)^3 - 14(-2)^2 - 19(-2) - 6$

= 32 – 8 – 56 + 38 – 6 = 0

So, x + 1 and x + 2 are factors of f(x).

$\Rightarrow$ (x + 1) (x + 2) is also a factor of f(x)

$\Rightarrow x^2 + 3x + 2$ is a factor of f(x)

Now, we divide

$f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ by

$x^2 + 3x + 2$ to get the other factors.

Quotient: $2x^2 - 5x - 3$

Divisor $x^2 + 3x + 2$) $2x^4 + x^3 - 14x^2 - 19x - 6$

$2x^4 + 6x^3 + 4x^2$

–   –   –

$-5x^3 - 18x^2 - 19x - 6$

$-5x^3 - 15x^2 - 10x$

+   +   +

$-3x^2 - 9x - 6$

$-3x^2 - 9x - 6$

+   +   +

0

$\therefore 2x^4 + x^3 - 14x^2 - 19x - 6$

$= (x^2 + 3x + 2)(2x^2 - 5x - 3)$

$= (x + 1)(x + 2)(2x^2 - 5x - 3)$

Now $2x^2 - 5x - 3 = 2x^2 - 6x + x - 3$

= 2x (x – 3) + 1(x – 3)

= (x – 3) (2x + 1)

Hence, $2x^4 + x^3 - 14x^2 - 19x - 6$

= (x + 1) (x + 2) (x – 3) (2x+1)

Ex.36 Factorize, $9z^3 - 27z^2 - 100z + 300$, if it is given that $(3z+10)$ is a factor of it.

Sol. Let us divide $9z^3 - 27z^2 - 100z + 300$ by $3z + 10$ to get the other factors

$$\begin{array}{r|l} & 3z^2 - 19z + 30 \\ \hline 3z + 10 & 9z^3 - 27z^2 - 100z + 300 \\ & 9z^3 + 30z^2 \\ & - \quad - \\ \hline & -57z^2 - 100z + 300 \\ & -57z^2 - 190z \\ & + \quad + \\ \hline & 90z + 300 \\ & 90z + 300 \\ & - \quad - \\ \hline & 0 \end{array}$$

$\therefore$ $9z^3 - 27z^2 - 100z + 300$

$= (3z + 10)(3z^2 - 19z + 30)$

$= (3z + 10)(3z^2 - 10z - 9z + 30)$

$= (3z + 10)\{(3z^2 - 10z) - (9z - 30)\}$

$= (3z + 10)\{z(3z - 10) - 3(3z - 10)\}$

$= (3z + 10)(3z - 10)(z - 3)$

Hence, $9z^3 - 27z^2 - 100z + 300$

$= (3z + 10)(3z - 10)(z - 3)$

Ex.37 Simplify :

$$\frac{4x-2}{x^2-x-2} + \frac{3}{2x^2-7x+6} - \frac{8x+3}{2x^2-x-3}$$

Sol. $$\frac{2(2x-1)}{(x-2)(x+1)} + \frac{3}{(2x-3)(x-2)} - \frac{8x+3}{(2x-3)(x+1)}$$

The L.C.M. of the factors in the denominator is $(x - 2)(x + 1)(2x - 3)$

The given expression can be reduced to

$$\frac{2(2x-1)(2x-3)+3(x+1)-(8x+3)(x-2)}{(x-2)(x+1)(2x-3)}$$

$$= \frac{2(4x^2-8x+3)+3(x+1)-(8x^2-13x-6)}{(x-2)(x+1)(2x-3)}$$

$$= \frac{15}{(x-2)(x+1)(2x-3)}$$

Ex.38 Establish the identity

$$\frac{6x^2+11x-8}{3x-2} = (2x+5) + \frac{2}{3x-2}$$

Sol. $3x - 2$) $6x^2 + 11x - 8$ ($2x + 5$

$$\begin{array}{l} 6x^2 - 4x \\ \hline \quad 15x - 8 \\ \quad 15x - 10 \\ \hline \quad\quad 2 \end{array}$$

$$\therefore \frac{6x^2+11x-8}{3x-2} = (2x+5) + \frac{2}{3x-2}$$

➢ ALGEBRAIC IDENTITIES

$(a + b)^2 = a^2 + 2ab + b^2 = (-a - b)^2$
$(a - b)^2 = a^2 - 2ab + b^2$
$(a - b)(a + b) = a^2 - b^2$
$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$
$(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$
$(-a + b + c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$
$(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$
$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
$a^3 + b^3 = (a + b)^3 - 3ab(a + b)$
$= (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)^3 + 3ab(a - b)$
$= (a - b)(a^2 + ab + b^2)$
$a^3 + b^3 + c^3 - 3abc$
$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
if $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$

❖ EXAMPLES ❖

Ex.39 Expand each of the following :

(i) $(3x - 4y)^2$ (ii) $\left(\frac{x}{2} + \frac{y}{3}\right)^2$

Sol. (i) We have,

$(3x - 4y)^2 = (3x)^2 - 2 \times 3x \times 4y + (4y)^2$

$= 9x^2 - 24xy + 16y^2$

(ii) We have,

$$\left(\frac{x}{2} + \frac{y}{3}\right)^2 = \left(\frac{x}{2}\right)^2 + 2 \times \frac{x}{2} \times \frac{y}{3} + \left(\frac{y}{3}\right)^2$$

$$= \frac{x^2}{4} + \frac{1}{3}xy + \frac{y^2}{9}$$

Ex.40 Find the products :

(i) $(2x + 3y)(2x - 3y)$

(ii) $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

Sol.(i) We have,

$(2x + 3y)(2x - 3y)$

$= (2x)^2 - (3y)^2$ [Using: $(a+b)(a-b) = a^2 - b^2$]

$= (2x)^2 - (3y)^2 = 4x^2 - 9y^2$

(ii) We have,

$\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

$= \left(x^2 - \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2}\right)\left(x^4 + \frac{1}{x^4}\right)$

$= \left\{(x^2)^2 - \left(\frac{1}{x^2}\right)^2\right\}\left(x^4 + \frac{1}{x^4}\right)$

$= \left(x^4 - \frac{1}{x^4}\right)\left(x^4 + \frac{1}{x^4}\right) = (x^4)^2 - \left(\frac{1}{x^4}\right)^2$

$= x^8 - \frac{1}{x^8}$

Ex.41 Evaluate each of the following by using identities :

(i) 103×97 (ii) 103×103

(iii) $(97)^2$ (iv) $185 \times 185 - 115 \times 115$

Sol. (i) We have,

$103 \times 97 = (100 + 3)(100 - 3)$

$= (100)^2 - (3)^2 = 10000 - 9 = 9991$

(ii) We have,

$103 \times 103 = (103)^2$

$= (100 + 3)^2 = (100)^2 + 2 \times 100 \times 3 + (3)^2$

$= 10000 + 600 + 9 = 10609$

(iii) We have,

$(97)^2 = (100 - 3)^2$

$= (100)^2 - 2 \times 100 \times 3 + (3)^2$

$= 10000 - 600 + 9 = 9409$

(iv) We have,

$185 \times 185 - 115 \times 115$

$= (185)^2 - (115)^2 = (185 + 115)(185 - 115)$

$= 300 \times 70 = 21000$

Ex.42 If $x + \frac{1}{x} = 6$, find : $x^4 + \frac{1}{x^4}$

Sol. We have,

$x^2 + \frac{1}{x^2} = 34 \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = (34)^2$

$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2 \times x^2 \times \frac{1}{x^2} = 1156$

$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 1156 \Rightarrow x^4 + \frac{1}{x^4} = 1156 - 2$

$\Rightarrow x^4 + \frac{1}{x^4} = 1154$

Ex.43 If $x^2 + \frac{1}{x^2} = 27$, find the value of the $x - \frac{1}{x}$

Sol. We have,

$\left(x - \frac{1}{x}\right)^2 = x^2 - 2 \times x \times \frac{1}{x} + \frac{1}{x^2}$

$\Rightarrow \left(x - \frac{1}{x}\right)^2 = x^2 - 2 + \frac{1}{x^2}$

$\Rightarrow \left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$

$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 27 - 2$

$\left[\therefore x^2 + \frac{1}{x^2} = 27 \text{ (given)}\right]$

$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 25$

$\Rightarrow \left(x - \frac{1}{x}\right)^2 = (\pm 5)^2 \Rightarrow x - \frac{1}{x} = \pm 5$

Ex.44 If $x + y = 12$ and $xy = 32$, find the value of $x^2 + y^2$.

Sol. We have,

$(x + y)^2 = x^2 + y^2 + 2xy$

$\Rightarrow 144 = x^2 + y^2 + 2 \times 32$

[Putting $x + y = 12$ and $xy = 32$]

$\Rightarrow 144 = x^2 + y^2 + 64$

$\Rightarrow 144 - 64 = x^2 + y^2$

$\Rightarrow x^2 + y^2 = 80$

Ex. 45 Prove that :

$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$

$= [(a - b)^2 + (b - c)^2 + (c - a)^2]$

Sol. We have,

L.H.S. $= 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$

$= (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2)$

$+ (c^2 - 2ca + a^2)$ [Re-arranging the terms]

$= (a - b)^2 + (b - c)^2 + (c - a)^2 =$ R.H.S.

Hence, $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$

$= [(a - b)^2 + (b - c)^2 + (c - a)^2]$

Ex.46 If $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Sol. We have,

If $a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 2 \times 0$

[Multiplying both sides by 2]

$\Rightarrow (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2)$

$+ (c^2 - 2ac + a^2) = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow a - b = 0, b - c = 0, c - a = 0$

[$\therefore$ Sum of positive quantities is zero if and only if each quantity is zero]

$\Rightarrow a = b, b = c$ and $c = a$

$\Rightarrow a = b = c$

❖ EXAMPLES ❖

Ex.47 Write the following in expanded form :

(i) $(9x + 2y + z)^2$ (ii) $(3x + 2y - z)^2$

(iii) $(x - 2y - 3z)^2$ (iv) $(-x + 2y + z)^2$

Sol. Using the identity

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

(i) We have,

$(9x + 2y + z)^2$

$= (9x)^2 + (2y)^2 + z^2 + 2 \times 9x \times 2y$

$+ 2 \times 2y \times z + 2 \times 9x \times z$

$= 81x^2 + 4y^2 + z^2 + 36xy + 4yz + 18xz$

(ii) We have,

$(3x + 2y - z)^2$

$= [3x + 2y + (-z)]^2$

$= (3x)^2 + (2y)^2 + (-z)^2 + 2 \times 3x \times 2y$

$+ 2 \times 2y \times (-z) + 2 \times 3x \times (-z)$

$= 9x^2 + 4y^2 + z^2 + 12xy - 4yz - 6xz$

(iii) We have,

$(x - 2y - 3z)^2$

$= [x + (-2y) + (-3z)]^2$

$= x^2 + (-2y)^2 + (-3z)^2 + 2 \times x \times (-2y)$

$+ 2 \times (-2y) \times (-3z) + 2 \times (-3z) \times x$

$= x^2 + 4y^2 + 9z^2 - 4xy + 12yz - 6zx$

(iv) We have,

$(-x + 2y + z)^2$

$= [(-x) + 2y + z]^2$

$= (-x)^2 + (2y)^2 + z^2 + 2 \times (-x) \times (2y)$

$+ 2 \times 2y \times z + 2 \times (-x) \times z$

$= x^2 + 4y^2 + z^2 - 4xy + 4yz - 2zx$

Ex.48 If $a^2 + b^2 + c^2 = 20$ and $a + b + c = 0$, find $ab + bc + ca$.

Sol. We have,

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow 0^2 = 20 + 2(ab + bc + ca)$

$\Rightarrow -20 = 2(ab + bc + ca)$

$\Rightarrow -\frac{20}{2} = \left\{\frac{2(ab+bc+ca)}{2}\right\}$

$\Rightarrow ab + bc + ca = -10$

Ex.49 If $a + b + c = 9$ and $ab + bc + ca = 40$, find $a^2 + b^2 + c^2$.

Sol. We know that

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow 9^2 = a^2 + b^2 + c^2 + 2 \times 40$

$\Rightarrow 81 = a^2 + b^2 + c^2 + 80$

$\Rightarrow a^2 + b^2 + c^2 = 1$

Ex.50 If $a^2 + b^2 + c^2 = 250$ and $ab + bc + ca = 3$, find $a + b + c$.

Sol. We know that

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow (a + b + c)^2 = 250 + 2 \times 3$

$\Rightarrow (a + b + c)^2 = 256$

$\Rightarrow (a + b + c)^2 = (\pm 16)^2$

[Taking square root of both sides]

$\Rightarrow a + b + c = \pm 16$

❖ EXAMPLES ❖

Ex.51 Write each of the following in expanded form:

(i) $(2x + 3y)^3$ (ii) $(3x - 2y)^3$

Sol.(i) Replacing a by 2x and b by 3y in the identity

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$, we have

$(2x + 3y)^3 = (2x)^3 + (3y)^3 + 3 \times 2x \times 3y \times (2x + 3y)$

$= 8x^3 + 27^3 + 18xy \times 2x + 18xy \times 3y$

$= 8x^3 + 27y^3 + 36x^2y + 54xy^2$

(ii) Replacing a by 3x and b by 2y in the identity $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$, we have

$(3x - 2y)^3 = (3x)^3 - (2y)^3 - 3 \times 3x \times 2y \times (3x - 2y)$

$= 27x^3 - 8y^3 - 18xy \times (3x - 2y)$

$= 27x^3 - 8y^3 - 54x^2y + 36xy^2$

Ex.52 If $x + y = 12$ and $xy = 27$, find the value of $x^3 + y^3$.

Sol. We know that

$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

Putting $x + y = 12$ and $xy = 27$ in the above identity, we get

$12^3 = x^3 + y^3 + 3 \times 27 \times 12$

$\Rightarrow 1728 = x^3 + y^3 + 972$

$\Rightarrow x^3 + y^3 = 1728 - 972$

$\Rightarrow x^3 + y^3 = 756$

Ex.53 If $x - y = 4$ and $xy = 21$, find the value of $x^3 - y^3$.

Sol. We know that

$(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

Putting $x - y = 4$ and $xy = 21$, we get

$4^3 = x^3 - y^3 - 3 \times 21 \times 4$

$\Rightarrow 64 = x^3 - y^3 - 252 \Rightarrow 64 + 252 = x^3 - y^3$

$\Rightarrow x^3 - y^3 = 316$

Ex.54 If $x + \frac{1}{x} = 7$, find the value of $x^3 + \frac{1}{x^3}$.

Sol. We have,

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x}\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

Putting $x + \frac{1}{x} = 7$, we get

$$7^3 = x^3 + \frac{1}{x^3} + 3 \times 7$$

$$\Rightarrow 343 = x^3 + \frac{1}{x^3} + 21$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 343 - 21 \Rightarrow x^3 + \frac{1}{x^3} = 322$$

Ex.55 If $a + b = 10$ and $a^2 + b^2 = 58$, find the value of $a^3 + b^3$.

Sol. We know that

$(a + b)^2 = a^2 + b^2 + 2ab$

Putting $a + b = 10$ and $a^2 + b^2 = 58$, we get

$10^2 = 58 + 2ab \Rightarrow 100 = 58 + 2ab$

$\Rightarrow 100 - 58 = 2ab \Rightarrow 42 = 2ab$

$\Rightarrow ab = 21$ Thus, we have

$a + b = 10$ and $ab = 21$ Now,

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

$\Rightarrow 10^3 = a^3 + b^3 + 3 \times 21 \times 10$

[Putting $a + b = 10$ and $ab = 21$]

$\Rightarrow 1000 = a^3 + b^3 + 630$

$\Rightarrow 1000 - 630 = a^3 + b^3$

$\Rightarrow a^3 + b^3 = 370$

Ex.56 If $x^2 + \frac{1}{x^2} = 7$, find the value of $x^3 + \frac{1}{x^3}$.

Sol. We have,

$$\left(x+\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2 = 7 + 2 \left[\text{Putting } x^2 + \frac{1}{x^2} = 7\right]$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2 = 9 \quad \Rightarrow \left(x+\frac{1}{x}\right)^2 = 3^2$$

$$\Rightarrow x + \frac{1}{x} = 3$$

[Taking square root of both sides]

$$\Rightarrow \left(x+\frac{1}{x}\right)^3 = 3^3 \qquad \text{[Cubing both sides]}$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right) = 27$$

$$\Rightarrow \left(x^3+\frac{1}{x^3}\right) + 3 \times 3 = 27$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 27 - 9 \quad \Rightarrow x^3 + \frac{1}{x^3} = 18$$

Ex.57 If $x^4 + \frac{1}{x^4} = 47$. Find the value of $x^3 + \frac{1}{x^3}$.

Sol. We know that

$$\left(x^2+\frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2 \times x^2 + \frac{1}{x^2}$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2 = \left(x^4+\frac{1}{x^4}\right) + 2$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2 = 47 + 2 \left[\text{Putting } x^4 + \frac{1}{x^4} = 47\right]$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2 = 7^2 \Rightarrow x^2 + \frac{1}{x^2} = 7$$

[Taking square root of both sides]

Now, $\left(x+\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2 = 7 + 2 \quad \left[\text{using} : x^2 + \frac{1}{x^2} = 7\right]$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2 = 3^2 \quad \Rightarrow x + \frac{1}{x} = 3$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^3 = 3^3 \qquad \text{[Cubing both sides]}$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right) = 27$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3 \times 3 = 27 \left[\text{Putting } x + \frac{1}{x} = 3\right]$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 27 - 9 \Rightarrow x^3 + \frac{1}{x^3} = 18$$

Ex.58 If a + b = 10 and ab = 21, find the value of $a^3 + b^3$.

Sol. We know that

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$= (a + b)(a^2 + 2ab + b^2 - 2ab - ab)$

[Adding and subtracting 2ab in the second bracket]

$= (a + b)[(a + b)^2 - 3ab]$

$= 10 \times (10^2 - 3 \times 21)$

$= 10 \times (100 - 63) = 10 \times 37 = 370.$

Ex.59 If a – b = 4 and ab = 45, find the value of $a^3 - b^3$.

Sol. We have, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$= (a - b)(a^2 - 2ab + b^2 + 2ab + ab)$

$= (a - b)\{(a - b)^2 + 3ab\}$

$= 4 \times (4^2 + 3 \times 45) = 4 \times (16 + 135)$

$= 4 \times 151 = 604.$

❖ EXAMPLES ❖

Ex.60 If a + b + c = 0, then prove that

$a^3 + b^3 + c^3 = 3abc$

Sol. We know that

$a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

putting a + b + c = 0 on R.H.S., we get

$a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow a^3 + b^3 + c^3 = 3abc$

Ex.61 Find the following product :

$(x + y + 2z)(x^2 + y^2 + 4z^2 - xy - 2yz - 2zx)$

Sol. We have,

$(x + y + 2z)(x^2 + y^2 + 4z^2 - xy - 2yz - 2zx)$

$= (x + y + 2z)(x^2 + y^2 + (2z)^2 - x \times y - y \times 2z - 2z \times x)$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$,

where $a = x$, $b = y$, $c = 2z$

$= a^3 + b^3 + c^3 - 3abc$

$= x^3 + y^3 + (2z)^3 - 3 \times x \times y \times 2z$

$= x^3 + y^3 + 8z^3 - 6xyz$

Ex.62 If $a + b + c = 6$ and $ab + bc + ca = 11$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Sol. We know that

$a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$\Rightarrow a^3 + b^3 + c^3 - 3abc =$

$(a + b + c)\{(a^2 + b^2 + c^2) - (ab + bc + ca)\}$ ...(i)

Clearly, we require the values of $a + b + c$, $a^2 + b^2 + c^2$ and $ab + bc + ca$ to obtain the value of $a^3 + b^3 + c^3 - 3abc$. We are given the values of $a + b + c$ and $ab + bc + ca$. So, let us first obtain the value of $a^2 + b^2 + c^2$.

We know that

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$\Rightarrow 6^2 = a^2 + b^2 + c^2 + 2 \times 11$

[Putting the values of $a + b + c$ and $ab + bc + ca$]

$\Rightarrow 36 = a^2 + b^2 + c^2 + 22$

$\Rightarrow a^2 + b^2 + c^2 = 36 - 22$

$\Rightarrow a^2 + b^2 + c^2 = 14$

Now, putting $a + b + c = 6$, $ab + bc + ca = 1$ and $a^2 + b^2 + c^2 = 14$ in (i), we get

$a^3 + b^3 + c^3 - 3abc = 6 \times (14 - 11)$

$= 6 \times 3 = 18$.

Ex.63 If $x + y + z = 1$, $xy + yz + zx = -1$ and $xyz = -1$, find the value of $x^3 + y^3 + z^3$.

Sol. We know that :

$x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - 3xy - 3yz - 3zx)$

[Adding and subtracting $2xy + 2yz + 2zx$]

$\Rightarrow x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)\{(x + y + z)^2 - 3(xy + yz + zx)\}$

$\Rightarrow x^3 + y^3 + z^3 - 3 \times -1 = 1 \times \{(1)^2 - 3 \times -1\}$

[Putting the values of $x + y + z$, $xy + yz +$

$\Rightarrow x^3 + y^3 + z^3 + 3 = 4$

$\Rightarrow x^3 + y^3 + z^3 = 4 - 3$

$\Rightarrow x^3 + y^3 + z^3 = 1$

➤ TYPES OF FACTORIZATION

◈ Type I : Factorization by taking out the common factors.

Ex.64 Factorize the following expression :

$2x^2 y + 6xy^2 + 10x^2y^2$

Sol. $2x^2 y + 6xy^2 + 10x^2y^2 = 2xy(x + 3y + 5xy)$

◈ Type II : Factorization by grouping the terms.

Ex.65 Factorize the following expression :

$a^2 - b + ab - a$

Sol. $a^2 - b + ab - a$

$= a^2 + ab - b - a = (a^2 + ab) - (b + a)$

$= a(a + b) - (a + b) = (a + b)(a - 1)$

◈ Type III : Factorization by making a perfect square.

Ex.66 Factorize of the following expression :

$9x^2 + 12xy + 4y^2$

Sol. $9x^2 + 12xy + 4y^2$

$= (3x)^2 + 2 \times (3x) \times (2y) + (2y)^2$

$= (3x + 2y)^2$

Ex.67 Factorize of the following expression :

$$\frac{x^2}{y^2} + 2 + \frac{y^2}{x^{2'}}, x \neq 0, y \neq 0$$

Sol. $\frac{x^2}{y^2} + 2 + \frac{y^2}{x^{2'}}$

$$= \left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right).\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = \left(\frac{x}{y} + \frac{y}{x}\right)^2$$

Ex.68 Factorize of the following expression

$$\left(5x - \frac{1}{x}\right)^2 + 4\left(5x - \frac{1}{x}\right) + 4, x \neq 0$$

Sol. $= \left(5x - \frac{1}{x}\right)^2 + 4\left(5x - \frac{1}{x}\right) + 4$

$$= \left(5x - \frac{1}{x}\right)^2 + 2 \times \left(5x - \frac{1}{x}\right) \times 2 + 2^2$$

$$= \left(5x - \frac{1}{x} + 2\right)^2$$

◈ **Type IV : Factorizing by difference of two squares.**

Ex.69 Factorize

(a) $2x^2y + 6xy^2 + 10x^2y^2$

(b) $2x^4 + 2x^3y + 3xy^2 + 3y^3$

Sol. (a) $2x^2y + 6xy^2 + 10x^2y^2$

$= (2xy)(x + 3y + 5xy)$

(b) $2x^4 + 2x^3y + 3xy^2 + 3y^3$

$= (2x^4 + 2x^3y) + (3xy^2 + 3y^3)$

$= (2x^3 + 3y^2)(x + y)$

Ex.70 Factorize $4x^2 + 12xy + 9y^2$

Sol. Note that $4x^2 = (2x)^2 = a^2$ say, and $9y^2 = (3y)^2 = b^2$ say, where $a = 2x$ and $b = 3y$. This suggests the use of identity (i) may be used and the given expression is equal to
$(a + b)^2$. Hence

$4x^2 + 12xy + 9y^2$

$= (2x)^2 + 2(x)(3y) + (3y)^2$

$= (2x + 3y)^2$

$= (2x + 3y)(2x + 3y)$

If the expression A can be reduced to an expression, three of whose terms are the squares of some expression, then the identity (vii) may be useful.

Ex.71 Factorize each of the following expressions :

(i) $9x^2 - 4y^2$

(ii) $x^3 - x$

Sol. (i) $9x^2 - 4y^2 = (3x)^2 - (2y)^2$

$= (3x + 2y)(3x - 2y)$

(ii) $x^3 - x = x(x^2 - 1)$

$= x(x-1)(x+1)$

Ex.72 Factorize each of the following expressions :

(i) $36x^2 - 12x + 1 - 25y^2$

(ii) $a^2 - \frac{9}{a^2}, a \neq 0$

Sol. (i) $36x^2 - 12x + 1 - 25y^2$

$= (6x)^2 - 2 \times 6x \times 1 + 1^2 - (5y)^2$

$= (6x-1)^2 - (5y)^2$

$= \{(6x - 1) - 5y\}\{(6x - 1) + 5y\}$

$= (6x - 1 - 5y)(6x - 1 + 5y)$

$= (6x - 5y - 1)(6x + 5y - 1)$

(ii) $a^2 - \frac{9}{a^2} = (a)^2 - \left(\frac{3}{a}\right)^2$

$= \left(a - \frac{3}{a}\right)\left(a + \frac{3}{a}\right)$

Ex.73 Factorize the following algebraic expression :

$x^4 - 81y^4$

Sol. $x^4 - 81y^4 = [(x)^2]^2 - (9y^2)^2$

$= (x^2 - 9y^2)(x^2 + 9y^2)$

$= \{x^2 - (3y)^2\}(x^2 + 9y^2)$

$= (x - 3y)(x + 3y)(x^2 + 9y^2)$

Ex.74 Factorize the following expression:

$x(x+z) - y(y+z)$

Sol. $x(x+z) - y(y+z) = (x^2 - y^2) + (xz - yz)$

$= (x-y)(x+y) + z(x-y)$

$= (x-y)\{(x+y) + z\}$

$= (x-y)(x + y + z)$

Ex.75 Factorize the following expression :

$x^4 + x^2 + 1$

Sol. $x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2$

$= (x^2 + 1)^2 - x^2 = (x^2 + 1 - x)(x^2 + 1 + x)$

$= (x^2 - x + 1)(x^2 + x + 1)$

❖ **Type V : Factorizing the sum and difference of cubes of two quantities.**

(i) $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$

(iI) $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$

Ex.76 Factorize the following expression :

$a^3 + 27$

Sol. $a^3 + 27 = a^3 + 3^3 = (a + 3)(a^2 - 3a + 9)$

Ex.77 Simplify : $(x + y)^3 - (x - y)^3 - 6y(x^2 - y^2)$

Sol. Let $x + y = a$ and $x - y = b$.

Then, $ab = (x+y)(x-y) = x^2 - y^2$ and $a - b = (x+y) - (x-y) = 2y$

$\therefore \quad (x+y)^3 - (x-y)^3 - 6y(x^2 - y^2)$

$= a^3 - b^3 - 3ab(a - b) = (a - b)^3$

$= \{(x+y) - (x-y)\}^3 = (2y)^3 = 8y^3$

➤ FACTORIZATION OF THE QUADRATIC POLYNOMIAL BY SPLITTING THE MIDDLE TEAM

❖ **Type I : Factorization of Quadratic polynomials of the form $x^2 + bx + c$.**

(i) In order to factorize $x^2 + bx + c$ we have to find numbers p and q such that $p + q = b$ and $pq = c$.

(ii) After finding p and q, we split the middle term in the quadratic as $px + qx$ and get desired factors by grouping the terms.

❖ EXAMPLES ❖

Ex.78 Factorize each of the following expressions :

(i) $x^2 + 6x + 8$ (ii) $x^2 + 4x - 21$

Sol. (i) In order to factorize $x^2 + 6x + 8$, we find two numbers p and q such that $p + q = 6$ and $pq = 8$.

Clearly, $2 + 4 = 6$ and $2 \times 4 = 8$.

We know split the middle term 6x in the given quadratic as 2x + 4x, so that

$x^2 + 6x + 8 = x^2 + 2x + 4x + 8$

$= (x^2 + 2x) + (4x + 8)$

$= x(x + 2) + 4(x + 2)$

$= (x + 2)(x + 4)$

(ii) In order to factorize $x^2 + 4x - 21$, we have to find two numbers p and q such that

$p + q = 4$ and $pq = -21$

Clearly, $7 + (-3) = 4$ and $7 \times -3 = -21$

We now split the middle term 4x of $x^2 + 4x - 21$ as $7x - 3x$, so that

$x^2 + 4x - 21 = x^2 + 7x - 3x - 21$

$= (x^2 + 7x) - (3x + 21)$

$= x(x + 7) - 3(x + 7) = (x + 7)(x - 3)$

Ex.79 Factorize each of the following quadratic polynomials: $x^2 - 21x + 108$

Sol. In order to factorize $x^2 - 21x + 108$, we have to find two numbers such that their sum is -21 and the product 108.

Clearly, $-21 = -12 - 9$ and $-12 \times -9 = 108$

$\therefore \quad x^2 - 21x + 108 = x^2 - 12x - 9x + 108$

$= (x^2 - 12x) - (9x - 108)$

$= x(x - 12) - 9(x - 12) = (x - 12)(x - 9)$

Ex.80 Factorize the following by splitting the middle term : $x^2 + 3\sqrt{3}x + 6$

Sol. In order to factorize $x^2 + 3\sqrt{3}x + 6$, we have to find two numbers p and q such that $p + q = 3\sqrt{3}$ and $pq = 6$

Clearly, $2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$ and $2\sqrt{3} \times \sqrt{3} = 6$

So, we write the middle term $3\sqrt{3}x$ as $2\sqrt{3}x + \sqrt{3}x$, so that

$x^2 + 3\sqrt{3}x + 6$

$= x^2 + 2\sqrt{3}x + \sqrt{3}x + 6$

$= (x^2 + 2\sqrt{3}x) + (\sqrt{3}x + 6)$

$= (x^2 + 2\sqrt{3}x) + (\sqrt{3}x + 2\sqrt{3} \times \sqrt{3})$

$= x(x + 2\sqrt{3}) + \sqrt{3}(x + 2\sqrt{3})$

$= (x + 2\sqrt{3})(x + \sqrt{3})$

◈ **Type II : Factorization of polynomials reducible to the form $x^2 + bx + c$.**

Ex.81 Factorize : $(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$.

Sol. Let $a^2 - 2a = x$. Then,

$(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$

$= x^2 - 23x + 120$

Now, $x^2 - 23x + 120 = x^2 - 15x - 8x + 120$

$= (x^2 - 15x) - (8x - 120)$

$= x(x - 15) - 8(x - 15)$

$= (x - 15)(x - 8)$

Replacing x by $a^2 - 2a$ on both sides, we get

$(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$

$= (a^2 - 2a - 15)(a^2 - 2a - 8)$

$= (a^2 - 5a + 3a - 15)(a^2 - 4a + 2a - 8)$

$= \{(a(a - 5) + 3(a - 5)\}\ \{a(a - 4) + 2(a - 4)\}$

$= \{(a - 5)(a + 3)\}\ \{(a - 4)(a + 2)\}$

$= (a - 5)(a + 3)(a - 4)(a + 2)$

Ex.82 Factorize the following by splitting the middle term :

$x^4 - 5x^2 + 4$

Sol. Let $x^2 = y$. Then, $x^4 - 5x^2 + 4$

$= y^2 - 5y + 4$

Now, $y^2 - 5y + 4$

$= y^2 - 4y - y + 4$

$= (y^2 - 4y) - (y - 4)$

$= y(y - 4) - (y - 4)$

$= (y - 4)(y - 1)$

Replacing y by x^2 on both sides, we get

$x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1)$

$= (x^2 - 2^2)(x^2 - 1^2) = (x - 2)(x + 2)(x - 1)(x + 1)$

Ex.83 Factorize : $(x^2 - 4x)(x^2 - 4x - 1) - 20$

Sol. The given expression is

$(x^2 - 4x)(x^2 - 4x - 1) - 20$

$= (x^2 - 4x)^2 - (x^2 - 4x) - 20$

Let $x^2 - 4x = y$. Then,

$(x^2 - 4x)^2 - (x^2 - 4x) - 20 = y^2 - y - 20$

Now, $y^2 - y - 20$

$= y^2 - 5y + 4y - 20$

$= (y^2 - 5y) + (4y - 20)$

$= y(y - 5) + 4(y - 5)$

$= (y - 5)(y + 4)$

Thus, $y^2 - y - 20 = (y - 5)(y + 4)$

Replacing y by $x^2 - 4x$ on both sides, we get

$(x^2 - 4x)^2 - (x^2 - 4x) - 20$

$= (x^2 - 4x - 5)(x^2 - 4x + 4)$

$= (x^2 - 5x + x - 5)(x^2 - 2 \times x \times 2 + 2^2)$

$= \{x(x - 5) + (x - 5)\}(x - 2)^2$

$= (x - 5)(x + 1)(x - 2)^2$

◈ **Type III : Factorization of Expressions which are not quadratic but can factorized by splitting the middle term.**

Ex.84 If $x^2 + px + q = (x + a)(x + b)$, then factorize $x^2 + pxy + qy^2$.

Sol. We have,

$x^2 + px + q = (x + a)(x + b)$

$\Rightarrow\ x^2 + px + q = x^2 + x(a + b) + ab$

On equating the coefficients of like powers of x, we get

$p = a + b$ and $q = ab$

$\therefore\ x^2 + pxy + qy^2 = x^2 + (a + b)xy + aby^2$

$= (x^2 + axy) + (bxy + aby^2)$

$= x(x + ay) + by(x + ay)$

$= (x + ay)(x + by)$

Ex.85 Factorize the following expression

$x^2y^2 - xy - 72$

Sol. In order to factorize $x^2y^2 - xy - 72$, we have to find two numbers p and q such that $p + q = -1$ and $pq = -72$

clearly, $-9 + 8 = -1$ and $-9 \times 8 = -72$.

So, we write the middle term $-xy$ of

$x^2y^2 - xy - 72$ as $-9xy + 8xy$, so that

$x^2y^2 - xy - 72 = x^2y^2 - 9xy + 8xy - 72$

$= (x^2y^2 - 9xy) + (8xy - 72)$

$= xy(xy - 9) + 8(xy - 9)$

$= (xy - 9)(xy + 8)$

> **FACTORIZATION OF POLYNOMIALS OF THE FORM $ax^2 + bx + c, a \neq 0, 1$**

◈ Type I : Factorization of quadratic polynomials of the form $ax^2 + bx + c, a \neq 0, 1$

(i) In order to factorize $ax^2 + bx + c$. We find numbers l and m such that l + m = b and lm = ac

(ii) After finding l and m, we split the middle term bx as lx + mx and get the desired factors by grouping the terms.

❖ EXAMPLES ❖

Ex.86 Factorize the following expression :

$6x^2 - 5x - 6$

Sol. The given expression is of the form $ax^2 + bx + c$, where, a = 6, b = – 5 and c = –6.

In order to factorize the given expression, we have to find two numbers l and m such that

l + m = b = i.e, l + m = – 5

and lm = ac i.e. lm = 6 × – 6 = – 36

i.e., we have to find two factors of – 36 such that their sum is – 5. Clearly,

– 9 + 4 = – 5 and – 9 × 4 = – 36

∴ l = – 9 and m = 4

Now, we split the middle term – 5x of $x^2 - 5x - 6$ as – 9 x + 4x, so that

$6x^2 - 5x - 6 = 6x^2 - 9x + 4x - 6$

$= (6x^2 - 9x) + (4x - 6)$

$= 3x(2x - 3) + 2(2x - 3) = (2x - 3)(3x + 2)$

Ex.87 Factorize each of the following expressions :

(i) $\sqrt{3}x^2 + 11x + 6\sqrt{3}$

(ii) $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$

(iii) $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$

Sol. (i) The given quadratic expression is of the form $ax^2 + bx + c$, where $a = \sqrt{3}$, b = 11 and

$c = 6\sqrt{3}$.

In order to factorize it, we have to find two numbers l and m such that

$l + m = b = 11$ and $lm = ac = \sqrt{3} \times 6\sqrt{3} = 18$

Clearly, 9 + 2 = 11 and 9 × 2 = 18

∴ l = 9 and m = 2

Now, $\sqrt{3}x^2 + 11x + 6\sqrt{3}$

$= \sqrt{3}x^2 + 9x + 2x + 6\sqrt{3}$

$= (\sqrt{3}x^2 + 9x) + (2x + 6\sqrt{3})$

$= (\sqrt{3}x^2 + 3\sqrt{3} \times \sqrt{3}x) + (2x + 6\sqrt{3})$

$= \sqrt{3}x(x + 3\sqrt{3}) + 2(x + 3\sqrt{3})$

$= (\sqrt{3}x + 2)(x + 3\sqrt{3})$.

Hence, $\sqrt{3}x^2 + 11x + 6\sqrt{3}$

$= (\sqrt{3}x + 2)(x + 3\sqrt{3})$

(ii) Here, $a = 4\sqrt{3}$, b = 5 and $c = -2\sqrt{3}$

In order to factorize $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$, we have to find two numbers l and m such that

l + m = b = 5 and lm = ac

$= 4\sqrt{3} \times -2\sqrt{3} = -24$

Clearly, 8 + (– 3) = 5 and 8 × – 3 = – 24

∴ l = 8 and m = –3

Now, $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$

$= 4\sqrt{3}x^2 + 8x - 3x - 2\sqrt{3}$

$= (4\sqrt{3}x^2 + 8x) - (3x + 2\sqrt{3})$

$= 4x(\sqrt{3}x + 2) - \sqrt{3}(\sqrt{3}x + 2)$

$= (\sqrt{3}x + 2)(4x - \sqrt{3})$

(iii) The given quadratic polynomial is $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$.

Clearly, it is of the form $ax^2 + bx + c$, where $a = 7\sqrt{2}$, b = – 10 and $c = -4\sqrt{2}$.

In order to factorize $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$, we have to find two numbers l and m such that l + m = b = – 10 and

$l m = ac = 7\sqrt{2} \times -4\sqrt{2} = -56$

Clearly, – 14+4 = –10 and – 14 × 4 = – 56

∴ l = – 14 and m = 4

Now, we split the middle term – 10 x of $7\sqrt{2}\,x^2 - 10x - 4\sqrt{2}$ as – 14 x + 4x so that

$7\sqrt{2}\,x^2 - 10x - 4\sqrt{2}$

$= 7\sqrt{2}\,x^2 - 14x + 4x - 4\sqrt{2}$

$= (7\sqrt{2}\,x^2 - 14x) + (4x - 4\sqrt{2})$

$= (7\sqrt{2}\,x^2 - 7\sqrt{2} \times \sqrt{2}\,x) + (4x - 4\sqrt{2})$

$= 7\sqrt{2}\,x(x - \sqrt{2}) + 4(x - \sqrt{2})$

$= (x - \sqrt{2})(7\sqrt{2}\,x + 4)$

Ex.88 Factorize the following by splitting the middle term :

$$\frac{1}{3}x^2 - 2x - 9$$

Sol. In order to factorize $\frac{1}{3}x^2 - 2x - 9$, we have to find to number l and m such that

$l + m = -2$ and $l\,m = \frac{1}{3} \times -9 = -3$

Clearly, $-3 + 1 = -2$ and $-3 \times 1 = -3$

So, we write the middle term – 2x as

– 3x + x, so that

$\frac{1}{3}x^2 - 2x - 9 = \frac{1}{3}x^2 - 3x + x - 9$

$= (\frac{1}{3}x^2 - 3x) + (x - 9) = (\frac{1}{3}x^2 - \frac{9}{3}x) + (x - 9)$

$= (x-9)\left(\frac{1}{3}x + 1\right)$

❖ Type II : Factorization of trinomial expressions which are not quadratic but can be factorized by splitting the middle term.

Ex.89 Factorize the following trinomial by splitting the middle term :

$$8a^3 - 2a^2b - 15ab^2$$

Sol. Here $a^3 \times ab^2 = (a^2b)^2$ i.e., the product of the variables in first and last term is same as the square of the variables in the middle term. So, in order to factorize the given trinomial, we split the middle term

$-2a^2b$ as $-12a^2b + 10a^2b$, so that

$8a^3 - 2a^2b - 15ab^2$

$= 8a^3 - 12a^2b + 10a^2b - 15ab^2$

$= 4a^2(2a - 3b) + 5ab(2a - 3b)$

$= (2a - 3b)(4a^2 + 5ab)$

$= (2a - 3b)\,a(4a + 5b)$

$= a(2a - 3b)(4a + 5b)$

❖ Type III : Factorization of trinomial expressions reducible to quadratic expressions.

Ex. 90 Factorize each of the following expressions by splitting the middle term :

(i) $9(x - 2y)^2 - 4(x - 2y) - 13$

(ii) $2(x + y)^2 - 9(x + y) - 5$

(iii) $8(a + 1)^2 + 2(a + 1)(b + 2) - 15(b + 2)^2$

Sol. (i) The given expression is

$$9(x - 2y)^2 - 4(x - 2y) - 13.$$

Putting $x - 2y = a$, we get

$9(x - 2y)^2 - 4(x - 2y) - 13 = 9a^2 - 4a - 13$

Now, $9a^2 - 4a - 13 = 9a^2 - 13a + 9a - 13$

$= (9a^2 - 13a) + (9a - 13)$

$= a(9a - 13) + (9a - 13)$

$= (a + 1)(9a - 13)$

Replacing a by x – 2y on both sides, we get

$9(x - 2y)^2 - 4(x - 2y) - 13$

$= (x - 2y + 1)\{9(x - 2y) - 13\}$

$= (x - 2y + 1)(9x - 18y - 13)$

(ii) The given expression is

$2(x + y)^2 - 9(x + y) - 5$

Replacing x + y by a in the given expression, we have

$2(x + y)^2 - 9(x + y) - 5 = 2a^2 - 9a - 5$

Now, $2a^2 - 9a - 5 = 2a^2 - 10a + a - 5$

$= (2a^2 - 10a) + (a - 5)$

$= 2a(a - 5) + (a - 5) = (a - 5)(2a + 1)$

Replacing a by x + y on both sides, we get

$2(x + y)^2 - 9(x + y) - 5$

$= (x + y - 5)\{2(x + y) + 1\}$

$= (x + y - 5)(2x + 2y + 1).$

(iii) The given trinomial is

$8(a+1)^2 + 2(a+1)(b+2) - 15(b+2)^2$

Putting a + 1 = x and b + 2 = y, we have

$8(a+1)^2 + 2(a+1)(b+2) - 15(b+2)^2$

$= 8x^2 + 2xy - 15y^2$

$= 8x^2 + 12xy - 10xy - 15y^2$

$= 4x(2x + 3y) - 5y(2x + 3y)$

$= (2x + 3y)(4x - 5y)$

Replacing x by a + 1 and y by b + 2, we get

$8(a+1)^2 + 2(a+1)(b+2) - 15(b+2)^2$

$= \{2(a+1) + 3(b+2)\}\{4(a+1) - 5(b+2)\}$

$= (2a + 3b + 8)(4a - 5b - 6)$

➢ FACTORIZATION OF ALGEBRAIC EXPRESSIONS OF THE FORM $a^3 + b^3 + c^3$, WHEN $a + b + c = 0$

❖ EXAMPLES ❖

Ex.91 Factorize :

$(x-y)^3 + (y-z)^3 + (z-x)^3$

Sol. Let x – y = a, y– z = b and z – x = c, then,

$a + b + c = x - y + y - z + z - x = 0$

$\therefore\ a^3 + b^3 + c^3 = 3abc$

$\Rightarrow (x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x)$

Ex.92 Factorize :

$(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3$

Sol. We have,

let $x = a^2-b^2$, $y = b^2-c^2$ and $z = c^2-a^2$. Then,

$x + y + z = a^2-b^2 + b^2-c^2 + c^2-a^2 = 0$

$\therefore\ x^3 + y^3 + z^3 = 3xyz$

$\Rightarrow\ (a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3$

$= 3(a^2-b^2)(b^2-c^2)(c^2-a^2)$

$= 3(a+b)(a-b)(b+c)(b-c)(c+a)(c-a)$

$= 3(a+b)(b+c)(c+a)(a-b)(b-c)(c-a)$

Ex.93 Simplify : $\dfrac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3}$

Sol. We have,

$(a^2-b^2) + (b^2-c^2) + (c^2-a^2) = 0$

$\therefore\ (a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3$

$= 3(a^2-b^2)(b^2-c^2)(c^2-a^2)$

$= 3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)$

Similarly,

$(a-b) + (b-c) + (c-a) = 0$

$\Rightarrow (a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a)$

$\therefore\ \dfrac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3}$

$= \dfrac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$

$= (a+b)(b+c)(c+a)$

Ex.94 Find the value of $x^3 - 8y^3 - 36xy - 216$, when $x = 2y + 6$.

Sol. We have, $x^3 - 8y^3 - 36xy - 216$

$= x^3 + (-2y)^3 + (-6)^3 - 3(x)(-2y)(-6)$

$= (x - 2y - 6)(x^2 + 4y^2 + 36 + 2xy - 12y + 6x)$

$= 0 \times (x^2 + 4y^2 + 36 + 2xy - 12y + 6x)$

$[\because x = 2y + 6 \Rightarrow x - 2y - 6 = 0] = 0$

➢ FACTORIZATION OF $x^3 \pm y^3$

In order to factorize the algebraic expression expressible as the sum or difference of two cubes, we sue the following identities.

(i) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

(ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

❖ EXAMPLES ❖

Ex.95 Factorize $27x^3 + 64y^3$

Sol. $27x^3 + 64y^3 = (3x)^3 + (4y)^3$

$= (3x + 4y)\{(3x)^2 - (3x)(4y) + (4y)^2\}$,

$= (3x + 4y)(9x^2 - 12xy + 16y^2)$

Ex.96 Factorize $a^3 + 3a^2b + 3ab^2 + b^3 - 8$

Sol. $a^3 + 3a^2b + 3ab^2 + b^3 - 8 = (a+b)^3 - 2^3$

$= \{(a+b) - 2\}\{(a+b)^2 + (a+b).2 + 2^2\}$

$= (a + b - 2)(a^2 + 2ab + b^2 + 2a + 2b + 4)$

Ex.97 Factorize : $a^3 - 0.216$

Sol. We have, $a^3 - 0.216 = a^3 - (0.6)^3$

$= (a - 0.6)[a^2 + 0.6a + (0.6)^2]$

$= (a - 0.6)(a^2 + 0.6a + 0.36)$

Ex.98 Factorize :

(i) $(x+1)^3-(x-1)^3$ (ii) $8(x+y)^3-27(x-y)^3$

Sol. (i) $(x+1)^3-(x-1)^3$

$= \{(x+1)-(x-1)\}\{(x+1)^2+(x+1)(x-1)+(x-1)^2\}$

$= \{(x+1-x+1)\}\{(x^2+2x+1)+(x^2-1)+(x^2-2x+1)\}$

$= 2(x^2+2x+1+x^2-1+x^2-2x+1)$

$= 2(3x^2+1)$

(ii) We have, $8(x+y)^3-27(x-y)^3$

$= \{2(x+y)\}^3-\{3(x-y)\}^3$

$= \{2(x+y)-3(x-y)[\{2(x+y)\}^2-2(x+y)\times 3(x-y) + \{3(x-y)\}^2]$

$= (2x+2y-3x+3y)\ \{2(x^2+2xy+y^2) - 6(x^2-y^2)+3(x^2-2xy+y^2)\}$

$= (-x+5y)(2x^2+4xy+2y^2-6x^2+6y^2 + 3x^2-6xy+3y^2)$

$= (-x+5y)(-x^2-2xy+11y^2)$

Ex.99 Factorize : (i) x^6-y^6 (ii) $x^{12}-y^{12}$

Sol. (i) we have, x^6-y^6

$= (x^2)^3-(y^2)^3 = (x^2-y^2)\{(x^2)^2+x^2\times y^2+(y^2)^2\}$

$= (x^2-y^2)(x^4+x^2y^2+y^4)$

$= (x-y)(x+y)\{(x^4+2x^2y^2+y^4)-x^2y^2\}$

$= (x-y)(x+y)\{(x^2+y^2)^2-(xy)^2\}$

$= (x+y)(x-y)\{(x^2+y^2-xy)(x^2+y^2+xy)\}$

$= (x+y)(x-y)(x^2-xy+y^2)(x^2+xy+y^2)$

(ii) $x^{12}-y^{12} = (x^4)^3-(y^4)^3$

$= (x^4-y^4)\{(x^4)^2+x^4\times y^4+(y^4)^2\}$

$= (x^2)^2-(y^2)^2)\}\ (x^8+x^4y^4+y^8)$

$= (x^2-y^2)(x^2+y^2)(x^8+2x^4y^4+y^8-x^4y^4)$

$= (x-y)(x+y)(x^2+y^2)\ \{(x^4+y^4)^2-(x^2y^2)^2\}$

$= (x-y)(x+y)(x^2+y^2)\ \{(x^4+y^4-x^2y^2) (x^4+y^4+x^2y^2\}$

$= (x-y)(x+y)(x^2+y^2)(x^4+y^4-x^2y^2) \{(x^4+y^4+2x^2y^2)-x^2y^2)\}$

$= (x-y)(x+y)(x^2+y^2)(x^4+y^4-x^2y^2) \{(x^2+y^2)^2-(xy)^2\}$

$= (x-y)(x+y)(x^2+y^2)(x^4+y^4-x^2y^2) (x^2+y^2-xy)(x^2+y^2+xy)$

$= (x-y)(x+y)(x^2+y^2)(x^4+y^4-x^2y^2) (x^2-xy+y^2)(x^2+xy+y^2)$

Ex.100 Prove that :

$$\frac{0.87\times0.87\times0.87+0.13\times0.13\times0.13}{0.87\times0.87-0.87\times0.13+0.13\times0.13}=1$$

Sol. We have

$$\frac{0.87\times0.87\times0.87+0.13\times0.13\times0.13}{0.87\times0.87-0.87\times0.13+0.13\times0.13}$$

$$=\frac{(0.87)^3+(0.13)^3}{(0.87)^2-0.87\times0.13+(0.13)^2}$$

$$=\frac{a^3+b^3}{a^2-ab+b^2} \quad \text{where } a = 0.87 \text{ and } b = 0.13$$

$$=\frac{(a+b)(a^2-ab+b^2)}{(a^2-ab+b^2)}$$

$= a+b = (0.87+0.13) = 1$

➢ FACTORIZATION OF $x^3+y^3+z^3-3xyz$

(i) In order to factorize the algebraic expressions of the form $x^3+y^3+z^3-3xyz$

We use the following identity :

$x^3+y^3+z^3-3xyz$

$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

(ii) If $x+y+z=0$, then $x^3+y^3+z^3=3xyz$

❖ EXAMPLES ❖

Ex.101 Factorize : $8x^3+27y^3+z^3-18xyz$

Sol. We have,

$8x^3+27y^3+z^3-18xyz$

$= (2x)^3+(3y)^3+z^3-3\times 2x\times 3y+z$

$= (2x+3y+z)\ \{(2x)^2+(3y)^2+z^2-2x\times 3y -3y\times z-z\times 2x\}$

$= (2x+3y+z)\ \{(4x^2+9y^2+z^2-6xy-3yz-2zx\}$

Ex.102 Factorize :

$(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a)$

Sol. We have,

$(a+b)^3 + (b+c)^3 + (c+a)^3 - 3(a+b)(b+c)(c+a)$

$= \{(a+b) + (b+c) + (c+a)\}\ \{(a+b)^2 + (b+c)^2 + (c+a)^2 - (a+b)(b+c) - (b+c)(c+a) - (c+a)(a+b)\}$

$= (2a+2b+2c)\ \{(a^2 + 2ab + b^2)$

$+ (b^2 + 2bc + c^2)$

$+ (c^2 + 2ca + a^2) - (ab + ac + b^2 + bc)$

$- (bc + ba + c^2 + ca) - (ca + cb + a^2 + ab)\}$

$= 2(a+b+c)\ (2a^2 + 2b^2 + 2c^2 + 2ab + 2bc$

$+ 2ca - ab - ac - b^2 - bc - bc - ba - c^2 - ca$

$- ca - cb - a^2 - ab)$

$= 2(a+b+c)\ (a^2 + b^2 + c^2 - ab - bc - ca)$

Ex.103 Resolve $a^3 - b^3 + 1 + 3ab$ into factors

Sol. $a^3 - b^3 + 1 + 3ab$

$= a^3 + (-b)^3 + 1^3 - 3(a)(-b)(1)$

$= (a-b+1)\ (a^2 + b^2 + 1 + ab - a + b)$

$= (a-b+1)\ (a^2 + b^2 + ab - a + b + 1)$

Ex.104 Factorize : $2\sqrt{2}\ a^3 + 8b^3 - 27c^3 + 18\sqrt{2}\ abc$

Sol. $2\sqrt{2}\ a^3 + 8b^3 - 27c^3 + 18\sqrt{2}\ abc$

$= (\sqrt{2}\ a)^3 + (2b)^3 - (3c)^3 - 3(\sqrt{2}\ a)(2b)(-3c)$

$= (\sqrt{2}\ a + 2b - 3c)\ (2a^2 + 4b^2 + 9c^2 - 2\sqrt{2}\ ab$

$+ 6bc + 3\sqrt{2}\ ac)$

Ex.105 Prove that :

$a^3 + b^3 + c^3 - 3abc$

$= \frac{1}{2}\ (a+b+c)\ \{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

Sol. We have,

$a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)\ (a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)\ (2a^2 + 2b^2 + 2c^2 - 2ab$

$- 2bc - 2ca)$

$= \frac{1}{2}(a + b + c)\ \{(a^2 - 2ab + b^2) +$

$(b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)\}$

$= \frac{1}{2}(a + b + c)\ \{(a - b)^2 + (b-c)^2 + (c-a)^2\}$