CHAPTER 9

Sequence and Series

Progression literally means advance, onward movement in successive stages. In mathematics, progression implies an ordered sequence of numbers. It consists of a set of numbers arranged in a definite order. The numbers in the sequence are termed as terms of the sequence or series. The terms follow a well defined rule.

For example, 3, 7, 11, 15, 19, ...

Each term of this series exceeds its preceeding term by 4. The dots (....) indicate that the series goes on. On the basis of the above rule we can find other terms in the series. Thus, the term after 19 will be (19 + 4) = 23 and so on.

The generalised form of representing a series is

$$x_1, x_2, x_3, \ldots\ldots, x_n, \ldots..$$

Where x s denote different terms of the series.

x_1 : first term, x_2 : second term

x_n : nth term, n : positive integer, and so on.

nth term in the above series is represented by the following general expression:

$x_n = 4n - 1$ where $n = 1, 2, 3, \ldots.$

For $n = 1, x_1 = (4 \times 1) - 1 = 3$

$n = 2, x_2 = (4 \times 2) - 1 = 7$

$n = 3, x_3 = (4 \times 3) - 1 = 11$ and so on.

Similarly, in the series of positive odd numbers 1, 3, 5, 7, 9, ... the nth terms is given by

$x_n = 2n - 1$ where $n = 1, 2, 3, \ldots..$

The above series of odd numbers can be alternatively also given by

$x_n = 2n + 1$

Where $n = 0, 1, 2, 3 \ldots\ldots$

But in this case nth term of the series is given by putting the value of $(n - 1)$.

Using $x_n = 2n - 1, x_3 = 2(3 - 1) + 1 = 5$

We illustrate below another example.

13, 16, 19, 22, ...

Here, $x_n = 3n + 10$

Where $n = 1, 2, 3, \ldots..$

Ex. 1. Find the nth term of the series 11, 18, 25, 32, ...

Sol. $18 - 11 = 7$

$25 - 18 = 7$

$32 - 25 = 7$

The difference between consecutive terms is always 7.

Therefore, we can write the nth term as

$x_n = 7n + z$ where z is an unknown number.

We are given,

for $n = 1, x_1 = 11$

$\therefore\ 11 = (7 \times 1) + z$

or, $z = 11 - 7 = 4$

$\therefore x_n = 7n + 4$ **Ans.**

To check :

$x_2 = (7 \times 2) + 4 = 18$

$x_3 = (7 \times 3) + 4 = 25$

It satisfies other terms too of the series.

Ex. 2. Find the nth term of the series 9, 27, 81, 243, ...

Sol. We can see that the difference between consecutive terms are not equal. Next we try the ratio between consecutive terms.

$$\frac{27}{9} = 3, \frac{81}{27} = 3, \frac{243}{81} = 3$$

$\therefore$ We can write the series as

$n = 1 : x_1 = 9 = 9 \times 1 = 9 \times 3^0 = 9 \times 3^{(1-1)} = 9 \times 3^{(n-1)}$

$n = 2 : x_2 = 27 = 9 \times 3 = 9 \times 3^1 = 9 \times 3^{(2-1)} = 9 \times 3^{(n-1)}$

$n = 3 : x_3 = 81 = 9 \times 3 \times 3 = 9 \times 3^2 = 9 \times 3^{(3-1)} = 9 \times 3^{(n-1)}$

$n = 4 : x_4 = 243 = 9 \times 3 \times 3 \times 3 = 9 \times 3^3 = 9 \times 3^{(4-1)} = 9 \times 3^{(n-1)}$

$\therefore$ We can write the nth term of the series as

$x_n = 9 \times 3^{(n-1)}$

In this case we can further simplify it as

$x_n = 3^2 \times 3^{(n-1)}$

or, $x_n = 3^{(n-1)}$ **Ans.**

Alternatively : Starting from the fact that the common ratio between consecutive terms is equal to 3, we can write nth term as

$x_n = 9 \times 3^{(n-z)}$

But for $n = 1, x_1 = 9$

$\therefore\ 9 = 9 \times 3^{(1-z)}$

or, $1 = 3^{(1-z)}$ or, $3^\circ = 3^{(1-z)}$

or, $1 - z = 0$ or, $z = 1$

$\therefore x_n = 9 \times 3^{(n-1)} = 3^2 \times 3^{(n-1)} = 3^{(n+1)}$ **Ans.**

The series can be an infinite series or a finite series. An infinite series is a never ending series and number of terms in it is unlimited. A finite series is one which contains finite number of terms. In a finite series the last term is always identified.

A series of positive odd integers is an example of infinite series which is represented below :

1, 3, 5, 7, 9, 11, 13, ...

But the series given below is a finite series.

3, 5, 7, ... 13.

Here 3 is the first term and 13 is the last term. It can be further shown that this series contains total six terms, 3, 5, 7, 9, 11, 13.

Now we will discuss specific types of progression, namely, Arithmetic Progression, Geometric Progression and Harmonic Progression.

Arithmetic Progression (A.P.)

A sequence of numbers is said to be in Arithmetic Progression if the difference between any two consecutive numbers is always the same. In other words, each term in the sequence is obtained by adding (or subtracting) a fixed number to the preceding term. The difference between two consecutive terms of the series is termed as its *common difference (d)*.

Ex. 3. Show that the given series is in A.P. and find the common difference : 4, 12, 20, 28, 36, ...

Sol. $12 - 4 = 8;$ $20 - 12 = 8$

$28 - 20 = 8;$ $36 - 28 = 8$

Since the difference between any two consecutive terms of the series is the same i.e., 8, the series is in A.P.

The common difference for the series is 8, **Ans.**

Ex. 4. Show that the given series is in A.P.

252, 243, 234, 225, 216, ...

Sol. $252 - 243 = 9;$ $243 - 234 = 9$

$234 - 225 = 9;$ $225 - 216 = 9$

Since the difference between any two consecutive terms is the same i.e., 9, the series is in A.P.

Ex. 5. Show that the given series are in A.P. and find their common difference.

(*i*) – 4, – 7, – 10, – 13, – 16, ...

(*ii*) – 115, – 111, – 107, – 103, – 99, ...

Sol. (*i*) $-7-(-4) = -7+4 = -3$

$-10-(-7) = -10+7 = -3$

$-13-(-10) = -13+10 = -3$

$-16-(-13) = -16+13 = -3$

Since the difference between consecutive terms is the same, the series is in A.P. The common difference is – 3.

(*ii*) $-111-(115) = -111+115 = 4$

$-107-(-111) = -107+111 = 4$

$-103-(-107) = -103+107 = 4$

$-99-(-103) = -99+103 = 4$

Since the difference between consecutive terms is the same, the series is in A.P. The common difference is 4.

Ex. 6. Show that the following series are in A.P. and find their common difference.

(*i*) $a, 3a, 5a, 7a, 9a,$ ......

(*ii*) $(a + 2b), (a + 5b), (a + 8b), (a + 11b), (a + 14b)$,.......

(*iii*) $(a + 2b), (2a + 4b), (3a + 6b), (4a + 8b), (5a + 10b)$, ...

(*iv*) $(4a + 2b), (6a + b), (8a), ((10a - b), (12a - 2b)$, ...

Sol. A series is in A.P. if the difference between consecutive terms of the series is the same.

(*i*) $3a - a = 2a;$ $5a - 3a = 2a$

$7a - 5a = 2a;$ $9a - 7a = 2a$

The common difference is $2a$.

(*ii*) $(a + 5b) - (a + 2b) = 3b$

$(a + 8b) - (a + 5b) = 3b$

$(a + 11b) - (a + 8b) = 3b$

$(a + 14b) - (a + 11b) = 3b$

The common difference is $3b$.

(*iii*) $(2a + 4b) - (a + 2b) = a + 2b$

$(3a + 6b) - (2a + 4b) = a + 2b$

$(4a + 8b) - (3a + 6b) = a + 2b$

$(5a + 10b) - (4a + 8b) = a + 2b$

The common difference is $(a + 2b)$.

(*iv*) $(6a + b) - (4a + 2b) = 2a - b$

$(8a) - (6a + b) = 2a - b$

$(10a - b) - (8a) = 2a - b$

$(12a - 2b) - (10a - b) = 2a - b$

The common difference is $(2a - b)$.

In general, an arithmetic progression can be represented as : $a, (a + d), (a + 2d), (a + 3d)$ ....

Where a : first term

d : common difference

The nth term can be given by

$x_n = a + (n - 1)\,d$

Arithmetic Mean : Arithmetic mean of two quantities is half their sum.

When three quantities are in A.P., the middle quantity is the arithmetic mean of the other two quantities.

For example,

In (6, 9, 12), 9 is the arithmetic mean of 6 and 12.

$$\frac{6+12}{2} = 9$$

In $(x, x + d, x + 2d)$, $x + d$ is the arithmetic mean of x and $x + 2d$.

$$\frac{x+(x+2d)}{2} = \frac{2x+2d}{2} = x+d.$$

Let (x, y, z) be in A.P.

Then $y - x = z - y$

or, $2y = x + z$ or, $y = \dfrac{x+z}{2}$

Thus, the middle quantity i.e., y is the arithmetic mean of the other two quantities, x and z.

We will apply the concept of this Arithmetic mean to Arithmetic progression.

$a, a + d, a + 2d, a + 3d, a + 4d, a + 5d,$ ....

Leaving the first term which has only one neighbour, all other terms have two neighbours and are the arithmetic means of their two immediate neighbours.

$a + d$: $\frac{a+(a+2d)}{2} = \frac{2a+2d}{2} = a+d$

$a + 2d$: $\frac{(a+d)+(a+3d)}{2} = \frac{2a+4d}{2} = a+2d$

$a + 3d$: $\frac{(a+2d)+(a+4d)}{2} = \frac{2a+6d}{2} = a+3d$

$a + 4d$: $\frac{(a+3d)+(a+5d)}{2}$

$= \frac{2a+8d}{2} = a+4d$ and so on.

Any three consecutive terms in an A.P. can be written as : $(x - d), x, (x + d)$

$\frac{(x-d)+(x+d)}{2} = \frac{2x}{2} = x.$

The middle term is the arithmetic mean of the other two quantities.

We can extend the concept of Arithmetic mean to more than two quantities.

The Arithmetic mean (also termed as Average) of n quantities is equal to the sum of the n quantities divided by n.

$$\text{Arithmetic Mean} = \bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}$$

Ex. 7. Find the arithmetic mean of 12 and 20 and write them in A.P.

Sol. Arithmetic Mean= $\frac{12+20}{2} = \frac{32}{2} = 16$

The three numbers can be written in A.P. as 12, 16, 20.

Ex. 8. For each of the following pairs of numbers find the arithmetic mean, write them in A.P. and calculate the common difference.

(i) 17, 29

(ii) 33, 19

(iii) $x + 5, 3x + 25$

(iv) $7x - 12, x - 32$

Sol.

(i) Arithmetic Mean = $\frac{17+29}{2} = \frac{46}{2} = 23$

A.P. = (17, 23, 29)

$23 - 17 = 6$

$29 - 23 = 6$

∴ Common difference = 6

(ii) Arithmetic Mean = $\frac{33+19}{2} = \frac{52}{2} = 26$

A.P. = (33, 26, 19)

$26 - 33 = -7$

$19 - 26 = -7$

∴ Common difference = − 7

(iii) Arithmetic Mean

$= \frac{(x+5)+(3x+25)}{2} = \frac{4x+30}{2} = 2x+15$

A.P. = {(x + 5, (2x + 15), (3x + 15)}

$(2x + 15) - (x + 5) = x + 10$

$(3x + 25) - (2x + 15) = x + 10$

∴ Common difference= $x + 10$

(iv) Arithmetic **Mean**

$= \frac{(7x-12)+(x-32)}{2} = \frac{8x-44}{2} = 4x-22$

A.P.= {(7x − 12), (4x − 22), (x − 32)}

$(4x - 22) - (7x - 12) = (4x - 7x) - 22 + 12 = -3x - 10$

$= -(3x + 10)$

$(x - 32) - (4x - 22) = (x - 4x) - 32 + 22 = -3x - 10$

$= -(3x + 10)$

∴ Common difference= − (3x + 10)

Ex. 9. Find the arithmetic mean (average) for the following :

(i) 3, 9, 10, 12, 15, 17

(ii) 6, 4, 3, 14, 12, 11, 20

(iii) $a, 3a, 9a, 20a, 2a$

(iv) $(x + 3d), (2x - 2d), (5x + 4d), (4x + 3d)$

(v) $(20a + 15d), (12a + 10d), (7a + 5d)$

Sol. (i) There are 6 terms in this case.

∴ $n = 6$

Arithmetic Mean

$= \frac{3+9+10+12+15+17}{6} = \frac{66}{6} = 11$ **Ans.**

(ii) There are 7 terms in this case

∴ $n = 7$

Arithmetic Mean = $\frac{6+4+3+14+12+11+20}{7}$

$= \frac{70}{7} = 10$ **Ans.**

(iii) There are 5 terms in this case.

∴ $n = 5$

Arithmetic Mean

$= \frac{1+3a+9a+20a+2a}{5} = \frac{35a}{5} = 7a$ **Ans.**

(iv) There are 4 terms here.

∴ $n = 4$

Arithmetic Mean

$= \frac{(x+3d)+(2x-2d)+(5x+4d)+(4x+3d)}{4}$

$= \frac{(x+2x+5x+4x)+(3d-2d+4d+3d)}{4}$

$= \frac{12x+8d}{4} = 3x+2d$ **Ans.**

(*v*) It contains 3 terms.

$\therefore n = 3$

Arithmetic Mean

$$= \frac{(20a+15d)+(12a+10d)+(7a+5d)}{3}$$

$$= \frac{(20a+12a+7a)+(15d+10d+5d)}{3}$$

$$= \frac{39a+30d}{3} = 13a+10d$$ **Ans.**

The terms between any two given terms of a progression are called the means between these two terms.

For example,

5, 8, 11, 14, 17, 20, 23, 26.

In the above A.P. there are 6 arithmetic means between 5 and 26 namely 8, 11, 14, 17, 20 and 23. Similarly, there are 4 arithmetic means between 8 and 23 namely, 11, 14, 17 and 20 and 3 arithmetic means between 11 and 23 namely, 14, 17 and 20.

Ex. 10. Insert 5 arithmetic means between 8 and 32.

Sol. Since 8 is the first term and there has to be five more terms between 8 and 32, 32 must be the 7th term.

*n*th term, $x_n = a + (n-1)d$

where a : first term

d : common difference

n : corresponds to the *n*th term

$\therefore$ for $n = 7$

$x_7 = 8 + (7-1)d$

or, $32 = 8 + 6d$

or, $6d = 24$

or, $d = 4$

So, the five terms to be inserted between 8 and 32 given by $8 + d$, $8 + 2d$, $8 + 3d$, $8 + 4d$ and $8 + 5d$ are 12, 16, 20, 24, and 28 respectively.

Ex. 11. Insert n arithmetic means between x and y.

Sol. With n arithmetic means and the two given quantities x and y, the total no. of terms will be $(n + 2)$. Further, all the terms will be in A.P.

So, x is the first term

y is the $(n + 2)$th term

Let d be the common difference.

Then, $y = x + (n+2-1)d$

or, $y = x + (n+1)d$

or, $d = \frac{y-x}{n+1}$

$\therefore$ The required arithmetic means to be inserted are,

$$x+\frac{y-x}{n+1},\ x+\frac{2(y-x)}{n+1},\ x+\frac{3(y-x)}{n+1}, \ldots,\ x+\frac{n(y-x)}{n+1}$$

An A.P. is completely specified if any of the following are known :

(*i*) one term with its place and common difference

(*ii*) any two terms with their places in the A.P.

Ex. 12. In the A.P. {6, 11, 16, 21, 26, ...} find the common difference and the 12th term.

Sol. $11 - 6 = 5$, $16 - 11 = 5$

$21 - 16 = 5$, $26 - 21 = 5$

$\therefore$ The common difference is 5.

$x_n = a + (n-1)d$

For the 12th term, $n = 12$

$x_{12} = 6 + (12-1) \times 5 = 6 + 55 = 61$ **Ans.**

Ex. 13. The first term of an A.P. is 9 and the common difference is 6. Write the A.P.

Sol. A.P.= a, $a + d$, $a + 2d$, $a + 3d$, ...

$a = 9$ and $d = 6$

$\therefore$ A.P.= 9, 15, 21, 26, ... **Ans.**

Ex. 14. The first term and 6th term in an A.P. are 9 and 54 respectively. Write the A.P.

Sol. *n*th term in an A.P. is given by

$x_n = a + (n-1)d$

$a = 9$,

$x_6 = 54$, $n = 6$

$x_6 = 54 = 9 + (6-1)d$

or, $45 = 5d$

or, $d = 9$

A.P.= a, $a + d$, $a + 2d$, $a + 3d$, ...

$\therefore$ A.P.= 9, 18, 27, 36, 45, 54,..... **Ans.**

Ex. 15. The 4th and 7th terms in an A.P are 25 and 37 respectively. Write the A.P.

Sol. *n*th term in an A.P. is given by

$x_n = a + (n-1)d$

a : first term

d : common difference.

For $n = 4$, $x_4 = a + 3d$

or, $25 = a + 3d$ ...(*i*)

For $n = 7$, $x_7 = a + 6d$

or, $37 = a + 6d$ ...(*ii*)

Subtracting equation (*i*) from equation (*ii*)

$$\begin{array}{r} a + 6d = 37 \\ a + 3d = 25 \\ - \quad - \\ \hline 3d = 12 \\ d = 4 \end{array}$$

Putting $d = 4$ in (*i*) :

$a + 3d = 25$

or, $a = 25 - 3d$

or, $a = 25 - (3 \times 4)$

or, $a = 13$

$\therefore$ A.P. = 13, 17, 21, 25, 29, 33, 37.....**Ans.**

Ex. 16. A finite series in A.P. consists of 8 terms. If its first and last terms are 7 and 84 respectively, find the series.

Sol. First term, $a = 7$

Since the series has 8 terms, the last term is 8th term.

$\therefore x_8 = 84$

The series is in A.P. Therefore, we can write its nth term as

$x_n = a + (n-1)d$

For $n = 8$: $84 = 7 + (8-1)d$

or, $77 = 7d$

or, $d = 11$

$\therefore$ The series is {7, 18, 29, 40, 51, 62, 73, 84} **Ans.**

Note : In an arithmetic progression of finite number of terms the sum of any two terms equidistant from the beginning and the end is equal to the sum of the first and the last terms :

Let an A.P. consisting of n terms has

First term $= a$

Common difference $= d$

Last term $= l = a + (n-1)d$

A.P.= $a, a+d, a+2d, \ldots\ldots, l-2d, l-d, l$

rth term from the beginning

$= a + (r-1)d$

rth term from the end $= l - (r-1)d$

Sum $= a + (r-1)d + l - (r-1)d$.

$= a + l$ = first term + last term

It is also evident from the following :

$(a + d) + (l - d) = a + l$

$(a + 2d) + (l - 2d) = a + l$

Ex. 17. Find the sum of the series in A.P. containing n terms. Its first term is a, the last term l and the common difference d.

Sol. Let the sum of the series be S.

$S = a + (a + d) + (a + 2d) + \ldots. + (l - 2d) + (l - d) + l$ ...(*i*)

We can write it in reverse order as

$S = l + (l - d) + (l - 2d) + \ldots\ldots. + (a+2d) + (a + d) + a$ ...(*ii*)

Adding (*i*) and (*ii*) we get

$2S = (a + l) + (a + l) + (a + l) + \ldots\ldots. + (a + l) + (a + l) + (a + l)$ (n terms)

or, $2S = n(a + l)$

[Since there are n terms and equals $(a + l)$]

or, $S = \frac{1}{2}n(a + l)$

Also, $l = a + (n-1)d$

$\therefore S = \frac{1}{2}n[a + a + (n-1)d] = \frac{1}{2}n[2a + (n-1)d]$

$$\boxed{S = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]}$$

Ex. 18. Find the sum of first 15 terms of an A.P. whose first term is 10 and the common difference is 12.

Sol. $n = 10$, $a = 10$, $d = 12$

$S = \frac{1}{2}n[2a + (n-1)d] = \frac{1}{2} \times 15\,[(2 \times 10) + (15-1) \times 12]$

$= \frac{15}{2}[20+168] = \frac{15 \times 188}{2} = 1410$. **Ans.**

Ex. 19. An A.P. has 11 terms starting with 14 and the common difference is 6. Find the last term and the sum of A.P.

Sol. $n = 11$, $a = 14$, $d = 6$

Last term, $l = a + (n-1)d = 14 + (11-1) \times 6$

$= 14 + 60 = 74$

Sum of the A.P., $S = \frac{1}{2}n(a + l)$

$= \frac{1}{2} \times 11\,(14 + 74) = \frac{1}{2} \times 11 \times 88 = 484$. **Ans.**

Ex. 20. The first and the last terms of an A.P. with 20 terms are 8 and 198 respectively. What is sum of A.P.?

Sol. $n = 20$, $a = 8$, $l = 198$

Sum of the A.P., $S = \frac{1}{2}n(a + l) = \frac{1}{2} \times 20\,(8 + 198)$

$= 2060$. **Ans.**

Ex. 21. The first and the last terms of an A.P. are 12 and 180 and the sum of the A.P. is 2400. Find the no. of terms and the common difference.

Sol. $a = 12$, $l = 180$, $S = 2400$

$S = \frac{1}{2}n(a + l)$

or, $n = \frac{2S}{a+l}$ or, $n = \frac{2 \times 2400}{12+180}$

or, $n = \frac{4800}{192}$ or, $n = 25$

$l = a + (n-1)d$

or, $d = \frac{l-a}{n-1}$ or, $d = \frac{180-12}{25-1}$

or, $d = \frac{168}{24}$ or, $d = 7$.

Geometric Progression (G.P.)

A sequence of numbers (or terms) is said to be in geometric progression if the ratio of its two consecutive numbers (or terms) is constant. This constant ratio is called the common ratio. In other words, each number (or term) in the sequence is obtained by multiplying (or dividing) its preceding number (or term) by the common ratio.

Ex. 1. Show that {4, 12, 36, 108, 324, ...}is in G.P. Find the common ratio and write the next term.

Sol. $\frac{12}{4} = 3,\ \frac{36}{12} = 3,\ \frac{108}{36} = 3,\ \frac{324}{108} = 3$

We can see that the ratio between any two consecutive terms of the sequence is always the same. Hence, the given sequence is in G.P.

The common ratio is 3.

Next term = 324 × 3 = 972

Ex. 2. Show that the following series is in G.P. and find its common ratio and the next term.

12288, 3072, 768, 192, 48, ...

Sol. A series is in G.P. if the ratio of its any two consecutive terms is always the same.

$$\frac{3072}{12288} = \frac{1}{4};\ \frac{768}{3072} = \frac{1}{4}$$

The common ratio is $\frac{1}{4}$. The common ratio is 5.

Next term = $48 \times \frac{1}{4} = 12$

Ex. 3. Write the 6th term of the G.P. (3, 18, 108, 648, ...). Find the place in the G.P. where 839808 occurs.

Sol. $a = 3$

$r = \frac{18}{3} = 6 = \frac{108}{18} = \frac{648}{108}$

6th term : $n = 6$

Let the 6th term be denoted by x_6.

$x_n = ar^{n-1}$

$\therefore x_6 = ar^5 = 3 \times (6)^5 = 23328$

Now,

$839808 = 3 \times (6)^{n-1}$

$279936 = 6^{n-1}$

$(6)^7 = 6^{n-1}$

$n - 1 = 7$

$\therefore n = 8$

Geometric Mean : The square root of the product of two quantities is equal to their geometric mean.

When three quantities are in G.P., the middle quantity is the geometric mean of the other two quantities.

For example,

In {2, 14, 98}, 14 is the geometric mean of 2 and 98.

$\sqrt{2 \times 98} = \sqrt{196} = 14$

In {a, ar, ar^2} and {a, – ar, ar^2}, ± ar is the geometric mean of a and ar^2.

$\sqrt{a \times ar^2} = \sqrt{a^2r^2} = \sqrt{(ar)^2} = \pm ar.$

Let {x, y, z} be in G.P.

Then, $\frac{y}{x} = \frac{z}{y}$

or, $y^2 = xz$

or, $y = \pm\sqrt{xz}$

Thus, the middle quantity i.e., y is the geometric mean to the other two quantities x and z.

We will apply the concept of this geometric mean to geometric progression.

$a, ar, ar^2, ar^3, ar^4, ar^5, ...$

Leaving the first term which has only one neighbour, all other terms have two neighbours and are the geometirc means of their two immediate neighbours.

$\sqrt{a \times ar^2} = \sqrt{a^2r^2} = \pm ar = ar$ and $-ar$

$\sqrt{ar \times ar^3} = \sqrt{a^2r^4} = \pm ar^2 = ar^2$ and $-ar^2$

$\sqrt{ar^2 \times ar^4} = \sqrt{a^2r^6} = \pm ar^3 = ar^3$ and $-ar^3$

$\sqrt{ar^3 \times ar^5} = \sqrt{a^2r^8} = \pm ar^4 = ar^4$ and $-ar^4$ and so on. The – ve terms form complementary part as shown earlier.

Any three consecutive terms in a G.P. can be written as

$$\left\{\frac{a}{r}, a, ar\right\}, \left\{\frac{a}{r}, -a, ar\right\}$$

$$\sqrt{\frac{a}{r} \times ar} = \sqrt{a^2} = \pm a$$

Thus, the middle term is the geometric mean of the other two quantities.

We can extend the concept of Geometric Mean to more than two quantities.

The geometric mean of n quantities is equal to the nth root of the product of n quantities

G. M.,

$$\bar{\bar{x}} = \sqrt[n]{x_1 \times x_2 \times x_3 \times ......\times x_n} = (x_1 \times x_2 \times x_3 \times .... \times x_n)^{1/n}$$

Ex. 4. For the following pairs find the geometric mean, the common ratio and write G.P.

9, 144

Sol. G.M.= $\pm\sqrt{9 \times 144} = \pm 36$

G.P. = {9, 36, 144}

$\frac{36}{9} = 4,\ \frac{144}{36} = 4$

and, {9, –36, 144}

$\frac{-36}{9} = -4,\ \frac{144}{-36} = -4$

Common ratio = 4, and – 4 respectively.

Ex. 5. Find the geometric mean of the following : 2, 20, 25, 40, 80

Sol. There are 5 terms.

$\therefore \quad n = 5$

$$\text{G.M.} = \sqrt[5]{2\times 20\times 25\times 40\times 80}$$

$$= (3200000)^{1/5} = \left[(20)^5\right]^{1/5} = 20$$

Ex. 6. Instert 3 geometric means between 5 and 1280.

(1) 20, 80, 320 (2) 20, 60, 240
(3) 15, 60, 240 (4) 25, 50, 100

Sol. (1) Since 5 is the first term and there has to be 3 terms between 5 and 1280, 1280 must be the 5th term.

nth term $x_n = ar^{n-1}$

$n = 5,\ a = 5,\ x_5 = 1280$

$\therefore\ 1280 = 5r^{5-1}$

or, $256 = r^4$

or, $(4)^4 = r^4$

$\therefore\ r = 4$

The 3 terms to be inserted between 5 and 1280 given by $5r$, $5r^2$ and $5r^3$ are 20, 80 and 320 respectively.

Ex. 7. Insert n geometric means between x and y.

Sol. Since x is the first term and there has to be n terms between x and y, y must be $(n + 2)$th term.

General formula for nth term is

$x_n = ar^{n-1}$

For $(n + 2)$th term, $y = xr^{n+2-1}$

$$\frac{y}{x} = r^{n+1}$$

$$r = \left(\frac{y}{x}\right)^{\frac{1}{n+1}}$$

$\therefore$ n geometric means to be inserted between x and y are

$$x\left(\frac{y}{x}\right)^{\frac{1}{n+1}},\ x\left(\frac{y}{x}\right)^{\frac{2}{n+1}},\ x\left(\frac{y}{x}\right)^{\frac{3}{n+1}},\ \ldots\ldots\ldots\ldots\ldots,\ x\left(\frac{y}{x}\right)^{\frac{n}{n+1}}$$

Ex. 8. In the G.P. {2, 10, 50, 250, ...} find the common ratio and the 8th term respectively.

(1) 5, 156250 (2) 6, 156250
(3) 5, 155062 (4) 6, 155062

Sol. (1) $\frac{10}{2} = 5,\ \frac{50}{10} = 5,\ \frac{250}{50} = 5$

The common ratio is 5.

$x_n = ar^{n-1}$

For the 8th term, $n = 8$

$x_8 = 2(5)^{8-1} = 2(5)^7 = 156250.$

Ex. 9. The first term of a G.P. is 3 and the common ratio is 3. Write the G.P.

(1) 3, 9, 21, ... (2) 3, 9, 27, ...
(3) 3, 6, 9,.... (4) None of these

Sol. (2) G.P.= $a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots$

$a = 3,\quad r = 3$

G.P. = 3, 9, 27, 81, 243, ...

Ex. 10. The 2nd and the 4th terms in a G.P. are 12 and 432 respectively. Write the G.P.

(1) 2, 12, 72, ... (2) 2, −12, 72,....
(3) 3, −12, 48, ... (4) 3, 12, 48, ...

Sol. (3) nth term in a G.P. is given by

$x_n = ar^{n-1}$

Second term : $n = 2,\ x_2 = 12$

Fourth term : $n = 4,\ x_4 = 432$

$x_2 = ar$

$ar = 12$ ...(i)

$x_4 = ar^3$

$ar^3 = 432$ ...(ii)

Dividing equation (ii) by (i) :

$$\frac{ar^3}{ar} = \frac{432}{12}$$

$r^2 = 36$

$r = +6$ and -6

Substitute $r = +6$ in equation (i)

$6a = 12$

$a = 2$

Substitute $r = -6$ in equation (i)

$-6a = 12$

$a = -2$

For $r = +6$, G.P. = 2, 12, 72, 432, 2592, ...

For $r = -6$, G.P. = −2, 12, −72, 432, −2592, ...

Ex. 11. A finite G.P. consists of 6 terms. If its first and last terms are 5 and 84035, respectively, find the G.P.

(1) 5, 15, 45, ... (2) 5, 10, 20, ...
(3) 5, 35, 245, ... (4) None of these

Sol. (3) First term, $a = 5$

The last term is the 6th term

$n = 6,\ x_6 = 84035$

nth term of a G.P. is given by

$x_6 = ar^5$

or, $84035 = 5r^5$

or, $16807 = r^5$

or, $(7)^5 = r^5 \Rightarrow r = 7$

$\therefore$ G.P. = {5, 35, 245, 1715, 12005, 84035}

Ex. 12. Find the sum of finite n terms in G.P.

Sol. Let a : first term

r : common ratio

S_n : Sum of n terms of the G.P.

$S_n = a + ar + ar^2 + ar^3 + \ldots ar^{n-2} + ar^{n-1}$. ...(i)

Multiplying both sides by r,

$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n$ ...(ii)

Subtracting equation (ii) from (i) :

$S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

For $r \neq 1$, $S_n = a\dfrac{1-r^n}{1-r}$

When $r = 1$, $S_n = a + a + a + a + \ldots\ldots\ldots\ldots$ n terms.

$\therefore\ S_n = na$

We use, $S_n = a\frac{1-r^n}{1-r}$ if $-1 < r < 1$

$S_n = a\frac{r^n - 1}{r-1}$ if $r > 1$ or $r < -1$

$S_n = na$ if $r = 1$

Ex. 13. Find the sum of first 7 terms of a G.P. whose first term is 13 and the common ratio is 2.

(1) 851 (2) 651
(3) 1651 (4) 1681

Sol. (3) $n = 7$, $a = 13$, $r = 2$

$S_n = a\frac{r^n - 1}{r-1}$ $(\because\ r = 2 > 1)$

$S = 13 \times \frac{2^7 - 1}{2-1} = 13 \times 127 = 1651$

Ex. 14. Find the sum of first 8 terms of a G.P. whose first term is 13122 and the common ratio is $\frac{1}{3}$.

(1) 19680 (2) 19860
(3) 19806 (4) 19880

Sol. (1) $n = 8$, $a = 13122$, $r = \frac{1}{3}$

$S = a\frac{1-r^n}{1-r}$ $(\because\ r < 1)$

$S = 13122 \times \frac{1-\left(\frac{1}{3}\right)^8}{1-\frac{1}{3}} = 13122 \times \frac{1-\frac{1}{6561}}{\frac{2}{3}}$

$= 13122 \times \frac{3}{2} \times \frac{6560}{6561} = 3 \times 6560 = 19680$

Ex. 15. The first and the last terms of a finite G.P. are 6 and 24576 and its sum is 32766. Find the common ratio of the G.P.

(1) 3 (2) 4
(3) 5 (4) 6

Sol. (2) Let there be n terms in the G.P.

First term, $a = 6$

Last term, $l = 24576$

Sum, $S = 32766$

$l = ar^{n-1}$

$r^{n-1} = \frac{l}{a} = \frac{24576}{6}$

$r^{n-1} = 4096$ or, $r^n = 4096\,r$

$S_n = a\frac{r^n - 1}{r-1}$

or, $32766 = 6 \times \frac{r^n - 1}{r-1}$

or, $\frac{r^n - 1}{r-1} = \frac{32766}{6} = 5461$

Putting $r^n = 4096\,r$

$\therefore\ \frac{4096\,r - 1}{r-1} = 5461$

or, $4096\,r - 1 = 5461\,r - 5461$

or, $5461\,r - 4096\,r = 5461 - 1$

or, $1365\,r = 5460$

$\therefore\ r = 4$

Ex. 16. If each term of an A.P. is multiplied by any number, show that the new sequence is also in A.P.

Sol. Let $\{a, a + d, a + 2d, a + 3d, \ldots\ldots\}$ be the A.P.

On multiplying each term by any number (say x), we get the new sequence as

$\{ax, (ax + xd), (ax + 2xd), (ax + 3xd), \ldots\ldots\}$

We can see that this sequence too is in A.P. with first term as ax and the common difference as xd.

Ex. 17. Find the next two terms of the A.P. {15, 17, 19, 21, ...}

(1) 23, 25 (2) 24, 26
(3) 25, 27 (4) 24, 27

Sol. (1) $a = 15$

$d = 17 - 15 = 2$

Next two terms are 23 and 25.

Ex. 18. How many even numbers are there between 15 and 225?

(1) 102 (2) 103
(3) 104 (4) 105

Sol. (4) Between 15 and 225 the first even number is 16, second is 18 and the last one is 224. The sequence of even numbers form an A.P. with common difference 2.

$a = 16$, $d = 2$, $l = a + (n-1)d = 224$

$a + (n-1)d = 224$

$16 + (n-1) \times 2 = 224$

$2(n-1) = 208$

$n - 1 = 104$

$n = 105$

Ex. 19. How many multiples of 9 are there between 34 and 235 ?

(1) 23 (2) 24
(3) 25 (4) 26

Sol. (1) The A.P. of multiples of 9 can be written as

$\{9 \times 1, 9 \times 2, 9 \times 3, 9 \times 4, \ldots\ldots\} = \{9, 18, 27, 36, \ldots\ldots\ldots\}$

Here, $d = 9$

Between 34 and 235, the first multiple of 9 is 36 and the last is 234.

$a = 36$, $l = 234$, $d = 9$

$l = a + (n-1)d$

$234 = 36 + (n-1) \times 9$

$9(n-1) = 198$

$n - 1 = 22$

$n = 23$

Ex. 20. Find the missing terms in the A.P.

{13, _, _, _, 57, ...}

(1) 24, 35, 46 (2) 25, 36, 47

(3) 26, 37, 48 (4) 23, 34, 45

Sol. (1) $a = 13$

5th term, $x_5 = 57$

$x_5 = a + 4d$

$57 = 13 + 4d$

$$d = \frac{57-13}{4} = 11$$

$x_2 = 13 + 11 = 24$

$x_3 = 24 + 11 = 35$

$x_4 = 35 + 11 = 46$

Ex. 21. Find the sum of all multiples of 5 between 12 and 186.

(1) 3400 (2) 3500

(3) 3550 (4) 3600

Sol. (2) The sequence of multiples of 5 is an A.P. which has common difference 5.

Between 12 and 186,

the first term, $a = 15$

the last term, $l = 185$

Common difference, $d = 5$

$l = a + (n-1)d$

$$n = \frac{l-a}{d} + 1 = \frac{(185-15)}{5} + 1 = 35$$

$$S = \frac{n}{2}(a+l) = \frac{35}{2}(15+185) = 3500$$

Ex. 22. How many terms of the A.P. {9, 14, 19, ...} will give the sum 581?

(1) 11 (2) 12

(3) 13 (4) 14

Sol. (4) $a = 9$, $d = 14 - 9 = 5$, $S = 581$

$$S = \frac{n}{2}[2a + (n-1)d]$$

$$581 = \frac{n}{2}[(2\times9) + (n-1)\times5]$$

$2 \times 581 = n[18 + 5n - 5]$

$1162 = n[13 + 5n]$

$5n^2 + 13n - 1162 = 0$

$5n^2 - 70n + 83n - 1162 = 0$

$5n(n-14) + 83(n-14) = 0$

$(n-14)(5n+83) = 0$

$$\therefore\ n = 14, \text{ or } -\frac{83}{5}$$

Since the number of terms (n) can't be a negative fraction, $n = 14$

Ex. 23. Find the 10th term of the following G.P.

$5, 5\sqrt{2}, 10, 10\sqrt{2}, 20$

(1) 80 (2) $60\sqrt{2}$

(3) $80\sqrt{2}$ (4) 60

Sol. (3) $a = 5$

$$r = \frac{5\sqrt{2}}{5} = \sqrt{2} = (2)^{\frac{1}{2}}$$

$$x_{10} = 5\times\left[(2)^{\frac{1}{2}}\right]^9$$

$$= 5\times(2)^{\frac{9}{2}} = 5\times(2)^{4+\frac{1}{2}} = 5\times(2)^4\times(2)^{\frac{1}{2}}$$

$= 80\sqrt{2}$

Ex. 24. Find the missing terms in the following G.P.

96, —, —, —, 486, ...

(1) 144, 216, 324 (2) 142, 214, 322

(3) 140, 212, 320 (4) 120, 128, 136

Sol. (2) 486 is the 5th term in the G.P.

$a = 96$, $x_5 = 486$

$x_5 = ar^4$

$486 = 96r^4$

$$\text{or, } \frac{486}{96} = r^4 \Rightarrow \frac{81}{16} = r^4$$

$$\text{or, } \left(\frac{3}{2}\right)^4 = r^4 \Rightarrow r = \frac{3}{2}$$

$$\text{Second term} = 96\times\frac{3}{2} = 144$$

$$\text{Third term} = 144\times\frac{3}{2} = 216$$

$$\text{Fourth term} = 216\times\frac{3}{2} = 324$$

The three missing terms are 144, 216 and 324

Ex. 25. The sum of three numbers in G.P. is 26 and their product is 216. Find the numbers.

(1) 2, 6, 12 (2) 2, 6, 15

(3) 2, 6, 18 (4) 3, 6, 12

Sol. (3) Let the three numbers in G.P. be $\frac{a}{r}$, a, ar

$$\text{Sum} = \frac{a}{r} + a + ar = \frac{a + ar + ar^2}{r} = a\left(\frac{1+r+r^2}{r}\right)$$

$$\text{Product} = \frac{a}{r}\times a\times ar = a^3$$

$a^3 = 216 \Rightarrow a = 6$

$$a\left(\frac{1+r+r^2}{r}\right) = 26$$

or, $\frac{1+r+r^2}{r} = \frac{26}{a} = \frac{26}{6} = \frac{13}{3}$

or, $3r^2 + 3r + 3 = 13r$

or, $3r^2 - 10r + 3 = 0$

or, $3r^2 - 9r - r + 3 = 0$

or, $3r(r-3) - (r-3) = 0$

$(r-3)(3r-1) = 0$

$\therefore\ r = 3$ or $\frac{1}{3}$

G.P. = {2, 6, 18} or {18, 6, 2}

∴ The three numbers are 2, 6 and 18

Ex. 26. The product of three numbers in G.P. is 1728 and the sum of their products taken in pairs is 756. Find the numbers.

(1) 3, 12, 48 (2) 2, 8, 32
(3) 1, 4, 16 (4) None of these

Sol. (1) Let the three numbers in G.P. be $\frac{a}{r}, a, ar$

Product $= \frac{a}{r} \times a \times ar = a^3 = 1728$

or, $a = 12$

Sum of their products taken in pairs,

$$\left(\frac{a}{r}\times a\right) + (a \times ar) + \left(\frac{a}{r}\times ar\right) = \frac{a^2}{r} + a^2 r + a^2$$

$$= a^2\left(\frac{1}{r} + r + 1\right) = a^2\left(\frac{1+r^2+r}{r}\right)$$

According to the question

$$a^2\left(\frac{r^2+r+1}{r}\right) = 756$$

$$\frac{r^2+r+1}{r} = \frac{756}{(12)^2} = \frac{756}{144} = \frac{21}{4}$$

$\therefore\ 4r^2 + 4r + 4 = 21r$

or, $4r^2 - 17r + 4 = 0$

or, $4r^2 - 16r - r + 4 = 0$

or, $4r(r-4) - (r-4) = 0$

or, $(r-4)(4r-1) = 0$

or, $r = 4$ or $\frac{1}{4}$

G.P. = {3, 12, 48} or {48, 12, 3}

So, the three numbers are 3, 12 and 48

Ex. 27. The sum of first five terms of a G.P. is 31 and the sum of its first ten terms is 1023. Find the G.P.

(1) 1, 3, 6, 12, ... (2) 1, 2, 4, 8, ...
(3) 2, 4, 8, ... (4) 2, 4, 6, 8, ...

Sol. (2) $S_n = a\frac{r^n - 1}{r-1}$

$$S_5 = a\frac{r^5-1}{r-1} = 31$$

$$S_{10} = a\frac{r^{10}-1}{r-1} = 1023$$

$$\frac{S_{10}}{S_5} = \frac{r^{10}-1}{r^5-1} = \frac{(r^5+1)(r^5-1)}{r^5-1} = r^5+1$$

$$\therefore\ r^5 + 1 = \frac{S_{10}}{S_5} = \frac{1023}{31} = 33$$

$r^5 = 32$

$r = 2$

$$S_5 = 31 = a\frac{r^5-1}{r-1} = a\frac{32-1}{2-1} = 31a$$

$\therefore\ a = 1$

G. P. is {1, 2, 4, 8, ...}

Harmonic Progression (H.P.)

A sequecnce of numbers is said to be in H.P. if reciprocals of its terms form an arithmetic progression (A.P.). Conversely, if terms of a sequence are in A.P., their reciprocals form a H.P.

If H.P. = $\{x_1, x_2, x_3, x_4, ...\}$,

then $\left\{\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \frac{1}{x_4}, ...\right\}$ are in A.P.

Ex. 28. Show that $\left\{\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}\right\}$ is an H.P.

Sol. Reciprocals of the terms of the given sequence form the new sequence given by {3, 7, 11, 15, ...}

$7 - 3 = 4$

$11 - 7 = 4$

$15 - 11 = 4$

Since the difference between any two consecutive terms of the new sequence is always same, they form an A.P. Hence, the given sequence is an H.P.

Ex. 29. Show that $\left\{\frac{2}{11}, \frac{1}{7}, \frac{2}{17}, \frac{1}{10}, \frac{2}{23}, \frac{1}{13}, ...\right\}$ is an H.P.

Sol. The new sequence formed by reciprocals of the terms of the given sequence is

$$\left\{\frac{11}{2}, 7, \frac{17}{2}, 10, \frac{23}{2}, 13, ...\right\}$$

$$7 - \frac{11}{2} = \frac{3}{2},\ \frac{17}{2} - 7 = \frac{3}{2},\ 10 - \frac{17}{2} = \frac{3}{2},$$

$$\frac{23}{2} - 10 = \frac{3}{2},\ 13 - \frac{23}{2} = \frac{3}{2}$$

Since the difference between any two consecutive terms of the new sequence is always same, they form an A.P. Hence the given sequence is an H.P.

Ex. 30. Write the H.P. for $\left\{4, \frac{14}{3}, \frac{16}{3}, 6, \frac{20}{3}, ...\right\}$.

Sol. $\frac{14}{3} - 4 = \frac{2}{3},\ \frac{16}{3} - \frac{14}{3} = \frac{2}{3},\ 6 - \frac{16}{3} = \frac{2}{3},\ \frac{20}{3} - 6 = \frac{2}{3}.$

Since the difference between any two consecutive terms of the given sequence is always same, it is an A.P. Therefore, the reciprocals of the terms of the given sequence will be an H.P.

$$\text{H.P.} = \left\{\frac{1}{4}, \frac{3}{14}, \frac{3}{16}, \frac{1}{6}, \frac{3}{20}, \ldots\ldots\right\}$$

Ex. 31. Three quantities a, b, c, are in H.P. if and only if

$$\frac{a-b}{b-c} =$$

(1) $\frac{2a}{c}$ (2) $\frac{c}{2a}$

(3) $\frac{c}{a}$ (4) $\frac{a}{c}$

Sol. (4) a, b, c, are in H.P.

Then, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Common difference, $\frac{1}{b}-\frac{1}{a} = \frac{1}{c}-\frac{1}{b}$

$$\frac{a-b}{ab} = \frac{b-c}{bc}$$

$$\frac{a-b}{b-c} = \frac{ab}{bc} = \frac{a}{c}$$

$\therefore$ $a : c = (a-b) : (b-c)$

Harmonic Mean : The Harmonic Mean (H) of two given quantities a and b is given by $H = \frac{2ab}{a+b}$

When H is the Harmonic mean of a and b, $\{a, H, b\}$ form an H.P.

Then $\left\{\frac{1}{a}, \frac{1}{H}, \frac{1}{b}\right\}$ is an A.P.

Common difference, $\frac{1}{H}-\frac{1}{a} = \frac{1}{b}-\frac{1}{H}$

$$\frac{2}{H} = \frac{1}{a}+\frac{1}{b} = \frac{b+a}{ab}$$

$$\frac{2}{H} = \frac{a+b}{ab} \qquad \therefore H = \frac{2ab}{a+b}$$

Ex. 32. The first and 16th terms of an H.P. are $\frac{3}{2}$ and $\frac{1}{2}$ respectively. Find its 6th term.

(1) $\frac{10}{9}$ (2) $\frac{9}{10}$

(3) $\frac{5}{9}$ (4) $\frac{9}{5}$

Sol. (2) The first and 16th terms of the corresponding A.P. will be $\frac{2}{3}$ and 2 respectively

nth term of A.P., $x_n = a + (n-1)d$

$$\therefore d = \frac{x_n - a}{n-1}$$

For $n = 16$, $d = \frac{2-\frac{2}{3}}{16-1} = \frac{4}{3\times 15} = \frac{4}{45}$

6th term of the A.P. $n = 6$, $x_6 = a + 5d$

$$= \frac{2}{3}+5\times\frac{4}{45} = \frac{2}{3}+\frac{4}{9} = \frac{6+4}{9} = \frac{10}{9}$$

$\therefore$ 6th term of the H.P. is $\frac{9}{10}$

If x and y are two unequal positive distinct quantities, A, G, M are their arithmetic mean, geometric mean and harmonic mean respectively, then

(*i*) $AH = G^2$

(*ii*) $A > G > H$

But when $x = y$,

$A = G = H$

$\therefore$ In general, $A \geq G \geq H$.

Ex. 33. Find the Harmonic mean of 4 and 12 and show that the three numbers are in H.P.

Sol. $a = 4$, $b = 12$

$$H = \frac{2ab}{a+b} = \frac{2\times4\times12}{4+12} = \frac{96}{16}$$

$\therefore$ $H = 6$

$\{4, 6, 12\}$

New sequence formed by reciprocals $= \left\{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right\}$

$$\frac{1}{6}-\frac{1}{4} = \frac{2-3}{12} = -\frac{1}{12} = \frac{1}{12}-\frac{1}{6} = \frac{1-2}{12} = -\frac{1}{12}$$

Since the difference between any two consecutive terms is the same, the new sequence is A.P.

Hence, $\{4,6,12\}$ is an H.P.

Ex. 34. The arithmetic mean and the geometric mean of two numbers are 10 and 6 respectively. Find the harmonic mean and write the A.P., G.P. and H.P.

Sol. Let the two numbers be a and b.

$$H = \frac{G^2}{A} = \frac{(6)^2}{10} = \frac{36}{10} = \frac{18}{5}$$

$$A = \frac{a+b}{2} = 10$$

$\therefore$ $a + b = 20$ ...(*i*)

$G = \sqrt{ab} = 6$

$ab = 36$

$(a-b)^2 = (a+b)^2 - 4ab$

$(a-b)^2 = (20)^2 - (4 \times 36)$

$(a-b)^2 = 400 - 144$

$(a-b)^2 = 256$

$(a-b) = 16$ ...(*ii*)

Adding equations (*i*) and (*ii*) we get

$2a = 36$ or, $a = 18$

$b = 20 - a$ or, $b = 2$

A.P. = $\{18, 10, 2\}$

G.P. = $\{18, 6, 2\}$

H.P. $=\left\{18, \frac{18}{5}, 2\right\}$

Finite Series

Ex. 35. Find the sum of first n terms of the following series $\frac{1}{3.4}, \frac{1}{4.5}, \frac{1}{5.6}, \frac{1}{6.7}, \frac{1}{7.8}$..........

(1) $\frac{n}{n+3}$ (2) $\frac{n}{2(n+3)}$

(3) $\frac{n}{3(n+3)}$ (4) $\frac{n}{4(n+3)}$

Sol. (3) The terms of the given series can be represented by

$x_m = \frac{1}{(m+2)(m+3)} = \frac{1}{m+2} - \frac{1}{m+3}$

where m= 1, 2, 3, ...

$x_1 = \frac{1}{3.4} = \frac{1}{3} - \frac{1}{4}$

Similarly, $x_2 = \frac{1}{4} - \frac{1}{5}$

$x_3 = \frac{1}{5} - \frac{1}{6}$

$x_4 = \frac{1}{6} - \frac{1}{7}$

|

$x_n = \frac{1}{n+2} - \frac{1}{n+3}$

Adding the above terms we get

$S_n = \sum_{m=1}^{n} x_m$

$= \frac{1}{3} + \left(-\frac{1}{4} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{5}\right) + \left(-\frac{1}{6} + \frac{1}{6}\right) + ... - \frac{1}{n+3}$

$S_n = \frac{1}{3} - \frac{1}{n+3}$

$S_n = \frac{n+3-3}{3(n+3)}$

$S_n = \frac{n}{3(n+3)}$

Ex. 36. Find the sum to n terms of the series {5, 55, 555, ...}

(1) $\frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$ (2) $\frac{4}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

(3) $\frac{2}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$ (4) None of these

Sol. (1) $S_n = 5 + 55 + 555 + 5555 +$ ....to n terms
$= 5(1 + 11 + 111 + 1111 +$ .... to n terms)

$= \frac{5}{9}$ (9 + 99 + 999 + 9999 + ... to n terms)

$= \frac{5}{9}$ [(10 − 1) + (10^2 − 1) + (10^3 − 1) + (10^4 − 1) .... n terms]

$= \frac{5}{9}$ [(10 + 10^2 + 10^3 + 10^4 + ... + 10^n) − ($n \times 1$)]

The first part (10 + 10^2 + 10^3 + 10^4 + ... + 10^n) forms a G.P. whose first term, a = 10 and common ratio, r = 10

Sum of n terms of a G.P. $= a\frac{r^n - 1}{r-1}$

$\therefore S_n = \frac{5}{9}\left[\left(10 \times \frac{10^n - 1}{10-1}\right) - n\right]$

$\therefore S_n = \frac{5}{9}\left[\frac{10}{9}\left(10^n - 1\right) - n\right]$

Ex. 37. Find the sum of squares of first n odd natural numbers.

(1) $\frac{n(4n^2+1)}{3}$ (2) $\frac{n(4n^2-1)}{3}$

(3) $\frac{n(4n-1)}{3}$ (4) None of these

Sol. (2) The series of square odd natural numbers is {1^2, 3^2, 5^2, 7^2, ...}

The terms of this series is given by $(2m - 1)^2$ where m= 1, 2, 3, 4, ...

$(2m - 1)^2 = 4m^2 - 4m + 1$

$S_n = \sum_{m=1}^{n}(2m-1)^2 = 4\sum_{m=1}^{n}m^2 - 4\sum_{m=1}^{n}m + \sum_{m=1}^{n}1$

$= 4 \times \frac{n(n+1)(2n+1)}{6} - 4 \times \frac{n(n+1)}{2} + n$

$= \frac{2n\left(2n^2 + 3n + 1\right)}{3} - 2n(n+1) + n$

$= \frac{n\left(4n^2 + 6n + 2 - 6n - 6 + 3\right)}{3} = \frac{n\left(4n^2 - 1\right)}{3}$

Ex. 38. Find the sum of squares of first n even natural numbers.

(1) $\frac{2}{3}n(n-1)(2n+1)$ (2) $\frac{2}{3}n(n+1)(2n+1)$

(3) $\frac{2}{3}(n-1)(2n+1)$ (4) None of the. e

Sol. (2) The series of square even natural numbers is {2^2, 4^2, 6^2, 8^2, ...}

The terms of the series is given by $(2m)^2 = 4m^2$ where m= 1, 2, 3, 4, ...

$$S_n = 4\sum_{m=1}^{n} m^2$$

$$= 4\times\frac{n(n+1)(2n+1)}{6} = \frac{2}{3}n(n+1)(2n+1)$$

Ex. 39. Find the sum of first n terms of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ...$

(1) $\frac{1}{2}(-1)^{n+1}n(n+1)$ (2) $\frac{1}{2}n(n+1)$

(3) $\frac{1}{2}n(n-1)$ (4) None of these

Sol. (1) The given series can be written in two parts as

$(1^2 + 3^2 + 5^2 + ...) - (2^2 + 4^2 + 6^2 + ...)$

Let $S_1 = (1^2 + 3^2 + 5^2 + ...)$

and $S_2 = (2^2 + 4^2 + 6^2 + ...)$

$\therefore S_n = S_1 - S_2$

Case I : n is even.

Both S_1 and S_2 will have $\frac{n}{2}$ terms.

Refer Ex. (74) : Sum of squares of first n odd numbers

$$= \frac{n(4n^2-1)}{3}$$

Refer Ex.(75) : Sum of first n even numbers

$$= \frac{2n(n+1)(2n+1)}{3}$$

Putting $\frac{n}{2}$ in place of n we get

$$S_1 = \frac{\frac{n}{2}\left[4\left(\frac{n}{2}\right)^2 - 1\right]}{3} = \frac{n(n^2-1)}{6} = \frac{n(n+1)(n-1)}{6}$$

$$S_2 = \frac{2\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)\left(2\times\frac{n}{2}+1\right)}{3}$$

$$= \frac{n\left(\frac{n+2}{2}\right)(n+1)}{3} = \frac{n(n+1)(n+2)}{6}$$

$$\therefore S_n = \frac{n(n+1)(n-1)}{6} - \frac{n(n+1)(n+2)}{6}$$

$$S_n = \frac{n(n+1)[(n-1)-(n+2)]}{6}$$

$$S_n = \frac{n(n+1)(-3)}{6}$$

$$S_n = \frac{(-1)n(n+1)}{2} \quad ...(i)$$

Case II : n is odd.

S_1 contains $\frac{n+1}{2}$ terms and S_2 contains $\frac{n-1}{2}$ terms.

Putting $n = \frac{n+1}{2}$ in S_1 we get

$$S_1 = \frac{\left(\frac{n+1}{2}\right)\left[4\left(\frac{n+1}{2}\right)^2 - 1\right]}{3}$$

$$S_1 = \frac{(n+1)\left[(n+1)^2 - 1\right]}{6}$$

$$S_1 = \frac{(n+1)(n^2+2n)}{6}$$

$$S_1 = \frac{n(n+1)(n+2)}{6}$$

Putting $n = \frac{n-1}{2}$ in S_2 we get

$$S_2 = \frac{2\left(\frac{n-1}{2}\right)\left[\left(\frac{n-1}{2}\right)+1\right]\left[(n-1)+1\right]}{3}$$

$$S_2 = \frac{(n-1)\left(\frac{n+1}{2}\right)(n)}{3}$$

$$S_2 = \frac{n(n+1)(n-1)}{6}$$

$$\therefore S_n = \frac{n(n+1)(n+2)}{6} - \frac{n(n+1)(n-1)}{6}$$

$$S_n = \frac{n(n+1)[(n+2)-(n-1)]}{6}$$

$$S_n = \frac{n(n+1)(3)}{6}$$

$$S_n = \frac{n(n+1)}{2} \quad ...(ii)$$

Combining equations (i) and (ii) we can write

$$S_n = \frac{1}{2}(-1)^{n+1}n(n+1)$$

Ex. 40. Find the sum of the integers which are perfect squares and lie between 10 and 15630.

(1) 658861 (2) 658816

(3) 568861 (4) None of these

Sol. (1) The first perfect square lying between 10 and 15630 is $4^2 (= 16)$ and the last one is 125^2 (= 15625).

$\therefore S = 4^2 + 5^2 + 6^2 + ... + 125^2$.

$S = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + ...+ 125^2) - (1^2 + 2^2 + 3^2)$

Sum of the squares of first n integers

$$= \frac{n(n+1)(2n+1)}{6}$$

Putting $n = 125$ and $n = 3$ we get

$$S=\frac{125(125+1)(2\times125+1)}{6}-\frac{3(3+1)(2\times3+1)}{6}$$

$$S=\frac{(125\times126\times251)-(3\times4\times7)}{6}$$

$$S=\frac{3953166}{6}$$

$S = 658861$

Ex. 41. Find the sum of first n terms of the series $1, 2x, 3x^2, 4x^3, ...$

(1) $\frac{1-x^n}{(1-x)^2}$ (2) $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$

(3) $\frac{1+x^n}{(1+x)^2}-\frac{nx^n}{1-x}$ (4) None of these

Sol. (2) $S_n = 1 + 2x + 3x^2 + 4x^3 + ... + nx^{n-1}$

We can see that 1, 2, 3, 4, ...n are in A.P. with common difference and first term both as 1, and $x, x^2, x^3, ... x^{n-1}$ are in G.P. with common ratio and first term both as x. Hence, the given series is an Arithmetico–Geometric series.

$S_n = 1 + 2x + 3x^2 + 4x^3 + ... + nx^{n-1}$

Multiplying both sides by x we can write.

$xS_n = x + 2x^2 + 3x^2 + ... + (n-1)x^{n-1} + nx^n$

$\therefore S_n - xS_n = [1 + x + x^2 + x^3 + ... + x^{n-1}] - nx^n$

The terms within the bracket form a G.P. whose first term is 1, the common ratio is x and number of terms is n.

$$(1-x)S_n=\left[\frac{1-x^n}{1-x}\right]-nx^n$$

$$\therefore S_n=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$$

SOLVED QUESTIONS OF PREVIOUS YEARS' EXAMS.

Ex. 42. If $1+10+10^2+.........$ upto n terms $=\frac{10^n-1}{9}$, then the sum of the series

$4 + 44 + 444 + ...$ upto n term is

(1) $\frac{4}{9}(10^n-1)-\frac{4n}{9}$

(2) $\frac{4}{81}(10^n-1)-\frac{4n}{9}$

(3) $\frac{40}{81}(10^n-1)-\frac{4n}{9}$

(4) $\frac{40}{9}(10^n-1)-\frac{4n}{9}$

(SSC CPO Sub-Inspector Exam, 16.12.2007)

Sol. (3) Expression $= 4 + 44 + 444 + ...$ to n terms
$= 4(1 + 11 + 111 + ...$to n terms)

$= \frac{4}{9}(9 + 99 + 999 + ...$to n terms)

$= \frac{4}{9}[(10-1) + (100-1) + (1000-1) + ...$ to n terms]

$= \frac{4}{9}[(10 + 10^2 + 10^3 + ...$ to n terms$) - n]$

$= \frac{4}{9}[10(1 + 10 + 10^2 + ...$ to n terms$) - n]$

$= \frac{40}{9}.\frac{(10^n-1)}{9}-\frac{4}{9}n$

$[\because 1 + 10 + 10^2 + ...$ to n terms $= \frac{10^n-1}{9}]$

$= \frac{40}{81}(10^n-1)-\frac{4}{9}n$

Ex. 43. The sum of three consecutive numbers in G.P. is 21 and the sum of their squares is 189. The product of the numbers is

(1) 72 (2) 216
(3) 108 (4) 144

(RRB Kolkata Apprentice Supervisors' Exam, 14.10.2001)

Sol. (2) Let the three numbers in G.P. be a, ar, ar^2.

Given $a + ar + ar^2 = 21 \Rightarrow a(1 + r + r^2) = 21$ ... (i)

Also, $a^2 + a^2r^2 + a^2r^4 = 189 \Rightarrow a^2(1 + r^2 + r^4) = 189$ ... (ii)

So, $\frac{a^2(1+r^2+r^4)}{[a(1+r+r^2)]^2}=\frac{189}{(21)^2}=\frac{189}{441}=\frac{3}{7}$

or, $\frac{1+2r^2+r^4-r^2}{(1+r+r^2)^2}=\frac{3}{7}$ or, $\frac{(1+r^2)^2-r^2}{(1+r+r^2)^2}=\frac{3}{7}$

or, $\frac{(1+r^2+r)(1+r^2-r)}{(1+r+r^2)}=\frac{3}{7}$ or, $\frac{1+r^2-r}{1+r+r^2}=\frac{3}{7}$

or, $7(1 + r^2 - r) = 3(1 + r + r^2)$

or, $4r^2 - 10r + 4 = 0$

or, $2r^2 - 5r + 2 = 0$

or, $2r^2 - 4r - r + 2 = 0$

or, $2r(r-2) - (r-2) = 0$

or, $(r-2)(2r-1) = 0$

or, $r = 2, \frac{1}{2}$

From equation (i)

$a=\frac{21}{1+r+r^2}$

$\therefore$ For $r = 2$, $a = \frac{21}{1+2+4}$ and for $r = \frac{1}{2}, a = 12$

Hence, the three numbers are 3, 6, 12 or, 12, 6, 3

Their product = $3 \times 6 \times 12 = 216$

Ex. 44. If the 10th term of the sequence a, $a - b$, $a - 2b$, $a - 3b$, ... is 20 and the 20th term is 10, then the xth term of the series is

(1) $10 - x$ (2) $20 - x$
(3) $29 - x$ (4) $30 - x$

(SSC CPO Sub-Inspector Exam, 03.09.2006)

Sol. (4) a, $a - b$, $a - 2b$ ..... is an AP with first term = a and common difference = $-b$

Now,

$t_{10} = a + (10-1) \times (-b)$

$\Rightarrow 20 = a - 9b$ .... (i)

$t_{20} = a + (20-1)(-b)$

$\Rightarrow 10 = a - 19b$ ... (ii)

From equation (i) – (ii),

$20 - 10 = a - 9b - a + 19b$

$\Rightarrow 10b = 10 \Rightarrow b = 1$

From equation (i),

$20 = a - 9 \Rightarrow a = 29$

$\therefore t_x = 29 + (x-1) \times -1$

$= 29 - x + 1 = 30 - x$

Ex. 45. Given that $1^2 + 2^2 + 3^2 + ... + 20^2 = 2870$, the value of $(2^2 + 4^2 + 6^2 + ... + 40^2)$ is :

(1) 11480 (2) 5740
(3) 28700 (4) 2870

(SSC Graduate Level Prelim Exam., 13.11.2005)

Sol. (1) $2^2 + 4^2 + 6^2 + ... + 40^2$

$= 2^2 (1^2 + 2^2 + 3^2 + ... + 20^2)$

$= 4 \times 2870 = 11480$

Ex. 46. The mean of cubes of the first n natural numbers is

(1) n^2 (2) $\frac{n(n+1)(n+2)}{8}$

(3) $n^2 + n + 1$ (4) $\frac{n(n+1)^2}{4}$

(5) None of these

(RRB Bhubaneshwar (Technical) Exam, 03.06.2001)

Sol. (4) Sum of cubes of the first n natural numbers

$= \frac{[n(n+1)]^2}{4}$

Required average $= \frac{[n(n+1)]^2}{4n} = \frac{n^2(n+1)^2}{4n} = \frac{n(n+1)^2}{4}$

Ex. 47. Which of the following numbers belong to the series ?

4, 11, 18, 25, 32, 39, ...

(1) 2099 (2) 2096
(3) 2098 (4) 2097

(RRB Mahendrughat, Patna ASM Exam, 27.05.2001)

Sol. (4) $11 - 4 = 7$

$18 - 11 = 7$

$25 - 18 = 7$

The given series is an A.P. whose first term, $a = 4$ and common difference, $d = 7$

nth term, $x_n = a + (n - 1)d$.

or, $n = \frac{x_n - a}{d} + 1 = \frac{x_n - 4}{7} + 1$

'n' must be a whole number. This condition is satisfied only by $x_n = 2097$ out of the four given answer choices for others it comes out to be a fraction

$n = \frac{2097-4}{7} + 1 = \frac{2093}{7} + 1 = 299 + 1 = 300$

Ex. 48. A man starts going for morning walk every day. The distance walked by him on the first day was 2 kms. Everyday he walks half of the distance walked on the previous day. What can be the maximum total distance walked by him in his life time ?

(1) 4 kms. (2) 120 kms.
(3) 18 kms. (4) Data inadequate
(5) None of these

(SBI Bank P.O. Exam, 11.02.2001)

Sol. (1) The distance walked on the first day = 2 kms.

The distance walked on subsequent days is half the distance walked on the previous day.

$\therefore$ Total distance walked $= 2 + 1 + \frac{1}{2} + \frac{1}{4} + ...$

This is a geometric series whose first term, a = 2 and common ratio, $r = \frac{1}{2}$

Maximum total distance walked by the person in his life-time means the number of terms in the series would be infinite.

Hence, the series would be an infinite geometric series.

Sum of an infinite geometric series is given by

$S = \frac{a}{1-r}$ or, $S = \frac{2}{1 - \frac{1}{2}}$

$S = \frac{2}{\frac{1}{2}}$ or, S = 4 kms.

Ex. 49. The number of terms in the sequence 4, 11, 18, ..., 186 is

(1) 17 (2) 25
(3) 26 (4) 27

(Railway ASM Exam, January 2001)

Sol. (4) $11 - 4 = 7$, $18 - 11 = 7$

The given sequence is an A.P. whose first term, $a = 4$ common difference, $d = 7$

n^{th} term, $x_n = a + (n-1)d$

or, $n = \frac{x_n - a}{d} + 1 = \frac{186-4}{7} + 1$

$n = 27$

Ex. 50. The sum $(5^3 + 6^3 + ... + 10^3)$ is equal to

(1) 2295 (2) 2425
(3) 2495 (4) 2925

(SSC Graduate Level Prelim. Exam 27.02.200?)

Sol. (4) ? = 125 + 216 + 343 + 512 + 729 + 1000 = 2925

Alternatively,

Let $S = 5^3 + 6^3 + ... + 10^3$

$= (1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + ... + 10^3) - (1^3 + 2^3 + 3^3 + 4^3)$

We know that the sum of the cubes of first n natural numbers $= \left[\frac{n(n+1)}{2}\right]^2$

$\therefore$ Required answer= $\left[\frac{10(10+1)}{2}\right]^2 - \left[\frac{4(4+1)}{2}\right]^2$

$= \left[\frac{10\times 11}{2}\right]^2 - \left[\frac{4\times 5}{2}\right]^2 = (55)^2 - (10)^2$

$= (55 + 10)(55 - 10) = 65 \times 45 = 2925$ **Ans.**

Ex. 51. The sum 9 + 16 + 25 + 36 + ... + 100 is equal to

(1) 350 (2) 380

(3) 400 (4) 420

(SSC Graduate Level Prelim. Exam., 27.02.2000)

Sol. (2) 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 380

Alternatively,

9 + 16 + 25 + 36 + ... + 100

$= 3^2 + 4^2 + 5^2 + 6^2 + ... + 10^2$

$= (1^2 + 2^2 + 3^2 + ... + 10^2) - (1^2 + 2^2)$

We know that the sum of the squares of first n natural numbers $= \frac{n(n+1)(2n+1)}{6}$

Here,

Required answer

$= \frac{10(10+1)(2\times 10+1)}{6} - \frac{2(2+1)(2\times 2+1)}{6}$

$= \frac{10\times 11\times 21}{6} - \frac{2\times 3\times 5}{6} = 385 - 5 = 380$ **Ans.**

Ex. 52. If the 4th term of an arithmetic progression is 14 and the 12th term is 70, then the first term is

(1) – 10 (2) – 7

(3) + 7 (4) + 10

(SSC Graduate Level Prelim. Exam., 27.02.2000)

Sol. (1) $a_4 = a + (4 - 1) \times d$

$14 = a + 3d \Rightarrow a = 14 - 3d$ ....(*i*)

Also, $70 = a + 11d$ ....(*ii*)

After putting the value of a from equation (*i*) in equation (*ii*)

$14 - 3d + 11d = 70$

$8d = 70 - 14$

$\therefore d = 8$

$\therefore a = 14 - 24 = -10$

Ex. 53. The middle term of an antithmetic series 3, 7, 11 .... 147, is

(1) 71 (2) 75

(3) 79 (4) 83

(RRB Kolkata Supervisor (P. Way) Exam., 20.02.2000)

Sol. (2) 3, 7, 11 ...., 147

It is an arithmetic series whose

first term, $a = 3$

last term, $x_n = 147$

common difference, $d = 4$

$x_n = a + (n - 1)\, d$

$147 = 3 + (n - 1) \times 4$

$n - 1 = \frac{147 - 3}{4}$

$n - 1 = 36, \quad n = 37$

The given series consists of 37 terms. Therefore, its middle term will be

$\frac{37+1}{2}$ = 19th term

$x_{19} = 3 + (19 - 1)\, 4$

$= 3 + 18 \times 4 = 75$

$\therefore$ The middle term of the given arithmetic series is 75.

Ex. 54. The nth term of an arithmetic series is p and the sum of first n terms is S. The first term is

(1) $\frac{2S}{n} - p$ (2) $\frac{2S}{n} + p$

(3) $\frac{2S}{p} - n$ (4) $\frac{2S}{n}$

(RRB Kolkata Supervisor (P. Way) Exam., 20.02.2000)

Sol. (1) Sum (S) of first n terms of an arithmetic series is given by $S = \frac{n}{2}(a + p)$

where a : the first term and p : the nth term. $\frac{2S}{n} = a + p$

$a = \frac{2S}{n} - p$

Ex. 55. If $S_n = \left\{\frac{n(n+1)}{2}\right\}^2$ when S_n is the sum of first n terms of the series, then its rth term is

(1) r (2) r^2

(3) r^3 (4) $\frac{r^2(r+1)^2}{4}$

(RRB Kolkata Supervisor (P. Way) Exam., 20.02.2000)

Sol. (3) The sum of the first n terms of series is given to be

$S_n = \left(\frac{n(n+1)}{2}\right)^2$

It represents summation of series of cubes of first n natural numbers (Refer Ex. 53)

$\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

$\therefore$ rth term of the series is r^3.

IMPORTANT POINTS : AT A GLANCE

Constant : A symbol having a fixed numerical value is called a constant.

Examples : 8, – 7, $\frac{5}{9}$, π etc. are all constants.

Variables : A symbol which may be assigned different numerical vlaues is known as a variable.

Example : We know that area of circle is given by the formula $A = \pi r^2$ where r is the radius of the circle. Here, π is constant while A and r are variables.

Algebraic Expressions : A combination of constants and variables, connected by some or all of the operations +, –, × and ÷, is known as an algebraic expression.

Terms of An Algebraic Expression

The several parts of an algebraic expression separated by + or – operations are called the terms of the expression.

Examples : (i) $5 + 9x - 7x^2y + \frac{3}{7}xy$

(ii) $x^3 + 3x^2y + 3xy^2 + y^3 + 7$

Polynomials : An algebraic expression in which the variables involved have only non-negative integral powers is called a polynomial.

General Form : $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is a polynomial in variable x, where $a_0, a_1, a_2, a_3 ... a_n$ are real numbers and n is non-negative integer.

Examples : (i) $6x^3 - 4x^2 + 7x - 3$ is a polynomial in one variable x.

(ii) $9y^5 + 6y^4 + 7y^3 + 10y^2 - 8y + \frac{2}{7}$ is a polynomial in one variable y.

(iii) $3 + 2x^2 - 6x^2y + 5xy^2$ is a polynomial in two variables x and y.

(v) $5 + 8x^{5/2} + 7x^3$ is an expression but not a polynomial since it contains a term containing $x^{5/2}$ where $\frac{5}{2}$ is not a non-negative integer.

Coefficients : In the polynomial $6x^3 - 5x^2 + 5x - 7$ we say that coefficients of x^3, x^2 and x are 6, –5 and 5 respectively and we also say that –7 is the constant term in it.

Degree of A Polynomial in One Variable : In case of a polynomial in one variable, the highest power of the variable is called the degree of the polynomial.

Examples :

(i) $3x + 5$ is a polynomial in x of degree 1.

(ii) $4y^2 - \frac{7}{2}y + 5$ is a polynomial in y of degree 2.

Degree of A Polynomial in Two or More Variables :

In case of polynomials in more than one variable, the sum of the powers of the variables in each term is taken up and the highest sum so obtained is called the degree of polynomial.

Examples :

(i) $7x^2 - 5x^2y^2 + 3xy + 7y + 9$ is a polynomial in x and y of degree 4.

(ii) $4x^3y^3 - 5xy^2 + 2x^4 - 7$ is a polynomial in x and y of degree 6.

Polynomials of Various Degrees

(1) Linear Polynomial : A polynomial of degree 1 is called a linear polynomial.

Examples :

(i) $2x + 7$ is a linear polynomial in x.

(ii) $2x + y + 7$ is a linear polynomial in x and y.

(2) Quadratic Polynomial : A polynomial of degree 2 is called a quadratic polynomial.

Examples :

(i) $x^2 + 2x + 3$ is a quadratic polynomial in x.

(ii) $xx + yz + zx$ is a quadratic polynomial in x, y and z.

(3) Cubic Polynomial : A polynomial of degree 3 is called a cubic polynomial.

Examples :

(i) $5x^3 - 3x^2 + 7x + 7$ is a cubic polynomial in x and y.

(ii) $4x^2y + 7xy^2 + 7$ is a cubic polynomial in x and y.

(4) Biquadratic Polynomial : A polynomial of degree 4 is called a biquadratic polynomial.

Examples :

(i) $x^4 - 7x^3 + 15x^2 + 7x - 9$ is a biquadratic polynomial in x.

(ii) $x^2y^2 + xy^3 + y^4 - 8xy + 2y^2 + 9$ is a biquadratic polynomial in x and y.

Number of Terms in A Polynomial

(i) **Monomial :** A polynomial containing one nonzero term is called a monomial.

Example : 5, $3x$, $\frac{7xy}{4}$ are all monomials.

(ii) **Binomial :** A polynomial containing two nonzero terms is called a binomial.

Examples : $(7 + 5x)$, $(x - 7y)$, $(5x^2y + 3yz)$ are all binomials.

(iii) **Trinomial :** A polynomial containing three nonzero terms is called a trinomial.

Examples :

$(8 + 5x + x^2)$, $3x - 5xy + 7y^2$ are all trinomials.

Constant Polynomial

A polynomial containing one term only, consisting of a constant is called a constant polynomial.

Examples :

$7, -5, \frac{7}{9}$ etc. are all constant polynomials.

Clearly, the degree of a non-zero constant polynomial is zero.

Zero Polynomial

A polynomial consisting of one term, namely zero only, is called a zero polynomial. The degree of a zero polynomial is not defined.

Zeros of A Polynomial

Let $p(x)$ be a polynomial. If $p(a) = 0$, then we say that a is a zero of the polynomial $p(x)$. Finding the zeros of a polynomial $p(x)$ means solving the equation $p(x) = 0$.

Example : If $f(t) = 3t^2 - 10t + 6$, find $f(0)$.

Solution : $f(t) = 3t^2 - 10t + 6$

$\Rightarrow f(0) = 3 \times 0^2 - 10 \times 0 + 6 = 6$

Example : If $p(x) = 2x^2 - 5x + 4$, find $p(2)$

Solution : $p(x) = 2x^2 - 5x + 4$

$\Rightarrow p(2) = (2 \times 2^2 - 5 \times 2 + 4) = 8 - 10 + 4 = 2$

Example : Find a zero of polynomial $p(x) = x - 7$

Solution : $p(x) = x - 7$

Now, $p(x) = 0 \Rightarrow x - 7 = 0 \Rightarrow x = 7$

$\therefore$ 7 is a zero of polynomial $p(x)$

Addition and Difference of Two Polynomials

Addition of two polynomials is determined by arranging terms of same degrees with signs and adding the co-efficients. The operation of subtraction is similar to the operation of addition. Only difference is that the signs of the polynomial to be subtracted are changed and then operation of addition is performed.

Example : If $p(x)$

$= x^4 - 5x^3 + 3x + 9$ and $q(x) = 2x^4 - 3x^3 + 5x - 4$

Then, $p(x) + q(x)$

$= (x^4 - 5x^3 + 3x + 9) + (2x^4 - 3x^3 + 5x - 4)$

$= (x^4 + 2x^4) + (-5x^3 - 3x^3) + (3x + 5x) + (9 - 4)$

$= 3x^4 - 8x^3 + 8x + 5$

For the sake of convenience, the above operation can be written in the following form :

$p(x) = x^4 - 5x^3 + 3x + 9$

$q(x) = 2x^4 - 3x^3 + 5x - 4$

$\therefore p(x) + q(x) = 3x^4 - 8x^3 + 8x + 5$

and, $p(x) - q(x)$

$= (x^4 - 5x^3 + 3x + 9) - (2x^4 - 3x^3 + 5x - 4)$

$= (x^4 - 5x^3 + 3x + 9) + (-2x^4 + 3x^3 - 5x + 4)$

$= (x^4 - 2x^4) + (-5x^3 + 3x^3) + (3x - 5x) + (9 + 4)$

$= -x^4 - 2x^3 - 2x + 13$

For the sake of convenience,

$\Rightarrow \quad p(x) = x^4 - 5x^3 + 3x + 9$

$\quad\; q(x) = 2x^4 - 3x^3 + 5x - 4$

$\quad\;\; - \quad\; - \quad\;\; + \quad\; - \quad\; +$

$\therefore p(x) - q(x) = -x^4 - 2x^3 - 2x + 13$

Multiplication of Two Polynomials

To determine the product of two polynomials, the distributive law of multiplication is used first and then grouping is made of terms of same degrees for addition and subtraction.

$x^3 - 6x^2 + x + 1$

$x^2 - 3x + 2$

$x^5 - 6x^4 + x^3 + x^2$

$\quad - 3x^4 + 18x^3 - 3x^2 - 3x$

$\quad\quad + 2x^3 - 12x^2 + 2x + 2$

$x^5 - 9x^4 + 21x^3 - 14x^2 - x + 2$

Remember : $(-x) \times (-x) = +x^2$

$(+x) \times (-x) = -x^2$

$(x) \times (x) = x^2$ etc.

Or, $(-) \times (-) = +$

$(-) \times (+) = -$

$(+) \times (-) = -$

$(+) \times (+) = +$

Division of Polynomial by Another Polynomial

Let $p(x)$ and $q(x)$ be two polynomials and $q(x) \neq 0$. If we find two polynomials $g(x)$ and $r(x)$ such that

$p(x) = g(x)\, q(x) + r(x)$

i.e. Dividend = Divisor × Quotient + Remainder

Where degree of $r(x)$ < degree of $g(x)$, then we say that on dividing $p(x)$ by $q(x)$, the quotient is $g(x)$ and remainder is $r(x)$. If remainder $r(x)$ = zero, we say that $q(x)$ is a factor of $p(x)$.

Let's take a few examples to illustrate the method of division of a polynomial by a polynomial of lesser degree.

Example : Divide $p(x) = x^3 + 3x^2 - 12x + 4$ by $g(x) = x - 2$.

Solution :

$$\begin{array}{r|l} & x^2 + 5x - 2 \\ \hline x - 2 & x^3 + 3x^2 - 12x + 4 \\ & x^3 - 2x^2 \\ & - \quad + \\ \hline & \quad 5x^2 - 12x + 4 \\ & \quad 5x^2 - 10x \\ & \quad - \quad\; + \\ \hline & \quad\quad -2x + 4 \\ & \quad\quad -2x + 4 \\ & \quad\quad + \quad - \\ \hline & \quad\quad\quad 0 \end{array}$$

Note : It is to be noted that the degree of q(x) is less than that of $p(x)$ and polynomial of higher degree is always divided by a polynomial of lower degree. The operation of division ends when the remainder is either zero or the degree of remainder is less than that of divisor.

In the above example, the quotient is $x^2 + 5x - 2$ and remainder is zero. As the remainder is zero, $(x - 2)$ is a factor of $x^3 + 3x^2 - 12x + 4$.

Example : Divide $p(x) = x^3 - 14x^2 + 37x - 60$ by $g(x) = x - 2$.

Solution :

$$\begin{array}{r|l} & x^2 - 12x + 13 \\ \hline x - 2 & x^3 - 14x^2 + 37x - 60 \\ & x^3 - 2x^2 \\ & - \quad + \\ \hline & -12x^2 + 37x - 60 \\ & -12x^2 + 24x \\ & + \quad - \\ \hline & 13x - 60 \\ & 13x - 26 \\ & - \quad + \\ \hline & -34 \end{array}$$

Here, quotient $= x^2 - 12x + 13$ and remainder $= -34$

Since remainder $\neq 0$, then $(x - 2)$ is not a factor of $x^3 - 14x^2 + 37x - 60$.

Remainder Theorem

Let $f(x)$ be a polynomial of degree $n \geq 1$, and let a be any real number. When $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$.

Proof : Suppose that when $f(x)$ is divided by $(x - a)$, the quotient is $g(x)$ and the remainder is $r(x)$.

Then, degree $r(x) <$ degree $(x - a)$

$\Rightarrow$ degree $r(x) < 1$

$\Rightarrow$ degree $r(x) = 0$ [$\because$ degree of $(x - a) = 1$]

$\Rightarrow r(x)$ is constant, equal to r (say).

Thus, when $f(x)$ is divided by $(x - a)$, then the quotient is $g(x)$ and the remainder is r.

$\therefore f(x) = (x - a) \cdot g(x) + r$ ... (i)

Putting $x = a$ in (i), we get $r = f(a)$.

Thus, when $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$.

Remarks

(i) If a polynomial $p(x)$ is divided by $(x + a)$, the remainder is the value of $p(x)$ at $x = -a$ i.e. $p(-a)$

[$\because x + a = 0 \Rightarrow x = -a$]

(ii) If a polynomial $p(x)$ is divided by $(ax - b)$, the remainder is the value of $p(x)$ at $x = \frac{b}{a}$ i.e. $p\left(\frac{b}{a}\right)$.

[$\because ax - b = 0 \Rightarrow x = \frac{b}{a}$]

(iii) If a polynomial $p(x)$ is divided by $(ax + b)$, then remainder is the value of $p(x)$ at $x = -\frac{b}{a}$ i.e. $p\left(-\frac{b}{a}\right)$

[$\because ax + b = 0 \Rightarrow x = -\frac{b}{a}$]

(iv) If a polynomial $p(x)$ is divided by $b - ax$, the remainder is the value of $p(x)$ at $x = \frac{b}{a}$ i.e. $p\left(\frac{b}{a}\right)$

[$\because b - ax = 0 \Rightarrow x = \frac{b}{a}$]

Example : Let $p(x) = x^4 - 3x^2 + 2x + 5$. Find remainder when $p(x)$ is divided by $(x - 1)$.

Solution :

$$\begin{array}{r|l} & x^3 + x^2 - 2x \\ \hline x - 1 & x^4 + 0x^3 - 3x^2 + 2x + 5 \\ & x^4 - x^3 \\ & - \quad + \\ \hline & x^3 - 3x^2 + 2x + 5 \\ & x^3 - x^2 \\ & - \quad + \\ \hline & -2x^2 + 2x + 5 \\ & -2x^2 + 2x \\ & + \quad - \\ \hline & 5 \end{array}$$

Here, remainder $= 5$

Find the value of p(1) from the above example.

$p(1) = 1 - 3 \times 1 + 2 \times 1 + 5 = 5$

Thus, remainder obtained on dividing $p(x)$ by $(x - 1)$ is same as $p(1)$.

Factor Theorem

Let $p(x)$ be a polynomial of degree greater than or equal to 1 and a be a real number such that $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$.

Conversely, if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$

Proof : First, let $p(x)$ be a polynomial of degree greater than or equal to one and a be a real number such that $p(a) = 0$. then we have to show that $(x - a)$ is a factor of $p(x)$.

Let $q(x)$ be the quotient, when $p(x)$ is divided by $(x - a)$.

By remainder theorem,

$p(x)$ when divided by $(x - a)$ gives remainder equal to $p(a)$.

$\therefore p(x) = (x - a)\, q(x) + p(a)$

$\Rightarrow p(x) = (x - a)\, q(x)$ [$\because p(a) = 0$]

$\Rightarrow (x - a)$ is a factor of $p(x)$

Conversely, Let $(x - a)$ is a factor of $p(x)$. Then, we have to prove that $p(a) = 0$.

Now $(x - a)$ is a factor of $p(x)$

$\Rightarrow p(x)$, when divided by $(x - a)$ gives remainder zero. But by Remainder theorem,

$p(x)$ when divided by $(x - a)$ gives the remainder equal to $p(a)$.

$\therefore p(a) = 0$

Remarks

(i) $(x + a)$ is a factor of a polynomial iff (if and only if) $p(-a) = 0$

(ii) $(ax - b)$ is a factor of a polynomial iff $p\left(\frac{b}{a}\right) = 0$

(iii) $(ax + b)$ is a factor of a polynomial $p(x)$ iff $p\left(-\frac{b}{a}\right) = 0$

(iv) $(x - a)(x - b)$ are factors of a polynomial $p(x)$ iff $p(a) = 0$ and $p(b) = 0$

FACTORISATION

To express a sgiven polynomial as the product of polynomials, each of degree less than that of the given polynomial such that no such a factor has a factor of lower degree, is called factorisation.

Examples : (i) $x^2 - 16 = (x - 4)(x + 4)$

(ii) $x^2 - 3x + 2 = (x - 2)(x - 1)$

Formulae for Factorisation

(i) $(x + y)^2 = x^2 + y^2 + 2xy$

(ii) $(x - y)^2 = x^2 + y^2 - 2xy$

(iii) $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

(iv) $(x + y)^3 = x^3 + y^3 + 3xy(x - y)$

(v) $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

(vi) $x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)$

Thus, for factorisation, we have

(i) $x^2 - y^2 = (x - y)(x + y)$

(ii) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

(iii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

METHODS OF FACTORISATION

Method 1 : When each term of an expression has a common factor, we divide each term by this factor and take it out as a multiple as shown below :

Example : $36a^3b - 60a^2bc = 12a^2b(3a - 5c)$

Example : $x(x - y)^3 + 3x^2y(x - y)$

$= x(x - y)[(x - y)^2 + 3xy]$

$= x(x - y)[x^2 + y^2 - 2xy + 3xy]$

$= x(x - y)(x^2 + y^2 + xy)$

Method 2 : Sometimes in a given expression it is not possible to take out a common factor directly. However the terms of the expression are grouped in such a manner that we may have a common factor. This can now be factorised as discussed above.

Example : Factorise : $6ab - b^2 + 12ac - 2bc$

Sol. : $6ab - b^2 + 12ac - 2bc = (6ab + 12ac) - (b^2 + 2bc)$

$= 6a(b + 2c) - b(b + 2c)$

$= (b + 2c)(6a - b)$

Example : Factorise : $x^2 + 18x + 81$

Sol. $x^2 + 18x + 81 = x^2 + 18x + 9^2$

$= x^2 + 2 \times 9 \times x + 9^2$

$= (x + 9)^2$

Example : Factorise : $64x^2 - 16x + 1$

Sol. : $64x^2 - 16x + 1 = (8x)^2 - 16x + 1^2$

$= (8x)^2 - 2.(8x).1 + 1^2 = (8x - 1)^2$

Example : Factorise : $81 - 64x^2$

Sol. $81 - 64x^2 = 9^2 - (8x)^2$

$= (9 + 8x)(9 - 8x)$

Mothod 3 : Factorisation of Quadratic Trinomials

Case I : Polynomial of the form $x^2 + bx + c$. We find integers p and q such that $p + q = b$ and $pq = c$. Then,

$x^2 + bx + c = x^2 + (p + q)x + pq$

$= x^2 + px + qx + pq$

$= x(x + p) + q(x + p)$

$= (x + q)(x + p)$

Case II : Polynomial of the form $ax^2 + bx + c$. In this case, we find integers p and q such that $p + q = b$ and $pq = ac$.

Then, $ax^2 + bx + c = ax^2 + (p + q)x + \frac{pq}{a}$

$= a^2x^2 + apx + aqx + pq$

$= ax(ax + p) + q(ax + p)$

$= (ax + p)(ax + q)$

Example : Factorise : $x^2 + 9x + 18$

Sol. : We try to split 9 into two parts whose sum is 9 and product is 18.

Clearly, $6 + 3 = 9$ and $6 \times 3 = 18$

$\therefore x^2 + 9x + 18 = x^2 + 6x + 3x + 18$

$= x(x + 6) + 3(x + 6)$

$= (x + 3)(x + 6)$

Example : Factorise : $x^2 + 5x - 24$

Sol. : We try to split 5 into two parts whose sum is 5 and product is -24.

Clearly, $8 + (-3) = 5$ and $8 \times (-3) = -24$

$\therefore x^2 + 5x - 24 = x^2 - 3x + 8x - 24$

$(x^2 - 3x) + (8x - 24)$

$= x(x - 3) + 8(x - 3)$

$= (x - 3)(x + 8)$

Example : Factorise : $x^2 - 4x - 21$

Sol. : We try to split -4 into two parts whose sum is -4 and product is -21.

Clearly, $(-7) + 3 = -4$ and $(-7) \times 3 = -21$

$\therefore x^2 - 4x - 21 = x^2 - 7x + 3x - 21$

$= x(x - 7) + 3(x - 7)$

$= (x - 7)(x + 3)$

Method 4 : Factorisation of forms $x^3 - y^3$ and $x^3 + y^3$:

Remember these formulae :

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Example : Factorise : $x^3 - 27y^3$

Sol. : $x^3 - 27y^3 = (x)^3 - (3y)^3$

$= (x - 3y)\{(x)^2 + x \times 3y + (3y)^2\}$

$= (x - 3y)(x^2 + 3xy + 9y^2)$

Example : $8x^3 + 27$

Sol. : $8x^3 + 27 = (2x)^3 + (3)^3$

$= (2x + 3)\{(2x)^2 - 2x \times 3 + (3)^2\}$

$= (2x + 3)(4x^2 - 6x + 9)$

Method 5 : Factorisation of $x^3 + y^3 + z^3 - 3xyz$

Theorem : prove that

$x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Proof : $x^3 + y^3 + z^3 - 3xyz = (x^3 + y^3) + z^3 - 3xyz$

$= [(x + y)^3 - 3xy(x + y)] + z^3 - 3xyz$

$= u^3 - 3xyu + z^3 - 3xyz$, जहां $(x + y) = u$

$= (u^3 + z^3) - 3xy(u + z)$

$= (u + z)(u^2 - uz + z^2) - 3xy(u + z)$

$= (u + z)(u^2 + z^2 - uz - 3xy)$

$= (x + y + z)[(x + y)^2 + z^2 - (x + y)z - 3xy]$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Note : $x^2 + y^2 + z^2 - xy - yz - zx$

$= \frac{1}{2}(2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx)$

$= \frac{1}{2}\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right]$

Theorem : If $x + y + z = 0$ then prove that

$x^3 + y^3 + z^3 = 3xyz$

Sol : First Method :

$\because x + y + z = 0$

$\therefore x + y = -z$

Cubing both sides

$(x + y)^3 = (-z)^3$

$\Rightarrow x^3 + y^3 + 3xy(x + y) = -z^3$

$\Rightarrow x^3 + y^3 + 3xy(-z) = -z^3$

[Putting the value of $(x + y)$]

$\Rightarrow x^3 + y^3 - 3xyz = -z^3$

$\therefore x^3 + y^3 + z^3 = 3xyz$

Example : Factorise : $a^3 - 8b^3 + 64c^3 + 24abc$

Sol. : $a^3 - 8b^3 + 64c^3 + 24abc$

$= a^3 + (-2b)^3 + (4c)^3 - 3a \times (-2b)(4c)$

$= x^3 + y^3 + z^3 - 3xyz$

Where, $a = x$, $-2b = y$ and $4c = z$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$= [a + (-2b) + 4c][a^2 + (-2b)^2 + (4c)^2 - a(-2b) - (-2b)(4c) - a(4c)]$

$= (a - 2b + 4c)(a^2 + 4b^2 + 16c^2 + 2ab + 8bc - 4ac)$

G.C.D & L.C.M OF POLYNOMIALS

Greatest Common Divisor/Highest Common Factor (GCD/HCF) : The GCD of two polynomials $p(x)$ and $q(x)$ is that common divisor which has the highet degree among all common divisors and the coefficient of the highest degree term be positive.

Example : What is the HCF of $(x + 4)^2 (x - 3)^3$ and $(x - 1)(x + 4)(x - 3)^2$?

Sol. $p(x) = (x + 4)^2 (x - 3)^2$

$q(x) = (x - 1)(x + 4)(x - 3)^2$

We see that $(x + 4)(x - 3)^2$ is such a polynomial that is a common divisor and whose degree is highest among all common divisors.

$\therefore$ HCF $= (x + 4)(x - 3)^2$

Lowest Common Multiple (LCM) : The LCM of two polynomials $p(x)$ and $q(x)$ is a polynomial of lowest degree of which $p(x)$ and $q(x)$ both are multiples.

Example : Find the LCM of $(x - 1)(x + 2)^2$ and $(x - 1)^3 (x + 2)$.

Sol. $p(x) = (x - 1)(x + 2)^2$

$q(x) = (x - 1)^3 (x + 2)$

We make a polynomial by taking each factor of $p(x)$ and $q(x)$. If a factor is common in both, then we take that factor which has highest degree in $p(x)$ and $q(x)$.

$\therefore$ LCM $= (x - 1)^3 (x + 2)^2$

Relation Between Two Polynomials and Their HCF And LCM

$p(x) = 8(x^3 - 3x + 2)$ and

$q(x) = 14(x^2 + x - 2)$

Now, $p(x) = 8(x^2 - 3x + 2)$

$= 2 \times 2 \times 2 \times (x - 2)(x - 1)$

$q(x) = 14(x^2 + x - 2)$

$= 2 \times 7 \times (x + 2)(x - 1)$

HCF $= 2(x - 1)$

LCM $= 2 \times 2 \times 2 \times 7 \times (x - 1)(x - 2)(x + 2)$

$= 56(x - 1)(x - 2)(x + 2)$

$\therefore$ HCF $\times$ LCM $= 112(x - 1)^2 (x - 2)(x + 2)$

Also, $p(x) \times q(x)$

$= 112(x - 1)^2 (x - 2)(x + 2)$

$\therefore$ HCF $\times$ LCM $= p(x) \times q(x)$

$\therefore$ HCF of polynomials $\times$ LCM of same polynomials = Product of same polynomials.

Example : Find the HCF of following pair of polynomials :

$p(x) = (x^2 - 9)(x - 3)$

$q(x) = x^2 + 6x + 9$

Sol. $p(x) = (x^2 - 9)(x - 3)$

$= (x + 3)(x - 3)(x - 3)$

$= (x + 3)(x - 3)^2$

$q(x) = x^2 + 6x + 9$

$= x^2 + 3x + 3x + 9$

$= x(x + 3) + 3(x + 3) = (x + 3)(x + 3)$

$= (x + 3)^2$

HCF of $p(x)$ and $q(x) = x + 3$

Example : Find the LCM of $p(x) = (x + 3)(x - 2)^2$ and $q(x) = (x - 2)(x - 6)$.

Sol. $p(x) = (x + 3)(x - 2)^2$

$q(x) = (x - 2)(x - 6)$

HCF of $p(x)$ and $q(x) = (x - 2)$

LCM of $p(x)$ and $q(x) = \dfrac{p(x) \times q(x)}{\text{HCF}}$

$= \dfrac{(x+3)(x-2)^2 \times (x-2)(x-6)}{(x-2)}$

$= (x + 3)(x - 2)^2 (x - 6)$

ALGEBRAIC IDENTITIES

An algebraic identity is an algebraic equation which is true for all values of the variable (s).

Important Formulae

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b)^2 = (a - b)^2 + 4ab$
4. $(a - b)^2 = (a + b)^2 - 4ab$
5. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
6. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
7. $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$
8. $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$
9. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
10. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

GRAPHIC REPRESENTATION OF STRAIGHT LINES

Ordered Pair : A pair of numbers a and b listed in a specific order with a at the first place and b at the second place is called an ordered pair (a, b).

Note that $(a, b) \neq (b, a)$.

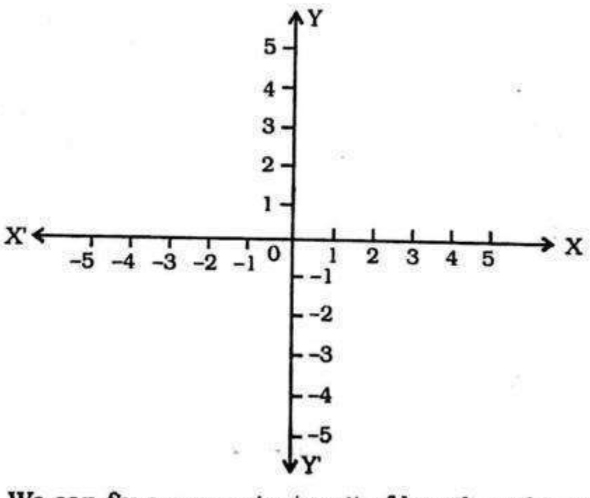
Thus, (2, 3) is one ordered pair and (3, 2) is another ordered pair.

COORDINATE SYSTEM

Coordinate Axes : The position of a point in a plane is determined with reference to two fixed mutually perpendicular lines, called the coordinate axes.

Let us draw two lines X'OX and YOY', which are perpendicular to each other and intersect at the point O.

- These lines are called the coordinate axes or the axes of reference.
- The horizontal line X'OX is called the x-axis.
- The vertical line YOY' is called the y-axis.
- The point O is called the origin.
- The distance of a point from y-axis is called its x-coordinate or abscissa and the distance of the point from x-axis is called its y-co ordinate or ordinate.
- If x and y, denote respectively the abscissa and ordinate of a point P, then (x, y) are called the coordinates of the point P.
- The y-coordinate of every point on x-axis is zero. i.e. when a straight line intersects at x-axis, its y-coordinate is zero. So, the co-ordinates of any point on the x-axis are of the form $(x, 0)$.
- The x-coordinate of every point on y-axis is zero. So, the coordinates of any point on y-axis are of the form $(0, y)$.
- The coordinates of the origin are (0, 0).
- $y = a$ where a is constant denotes a straight line parallel to x-axis.
- $x = a$ where a is constant, denotes a straight line parallel to y-axis.
- $x = 0$ denotes y-axis.
- $y = 0$ denotes x-axis.



We can fix a convenient unit of length and taking the origin as zero, mark equal distances on the x-axis as well as on the y-axis.

Convention of Signs : The distances measured along OX and OY are taken as positive and those along OX' and OY' are taken as negative, as shown in the figure given above.

COORDINATES OF A POINT IN A PLANE

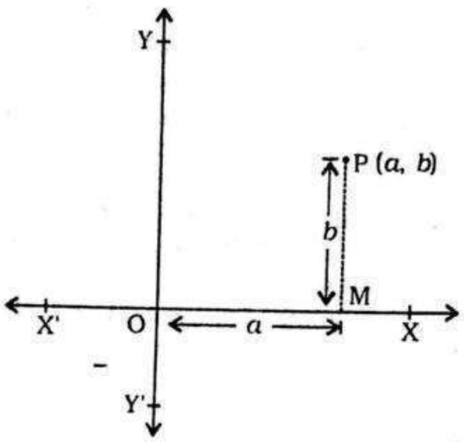
Let P be a point in a plane.

Let the distance of P from the y-axis = a units.

And, the distance of P from the x-axis = b units.

Then, we say that the coordinates of P are (a, b).

a is called the x-coordinate, or abscissa of P.
b is called the y coordinate, or ordinate of P.

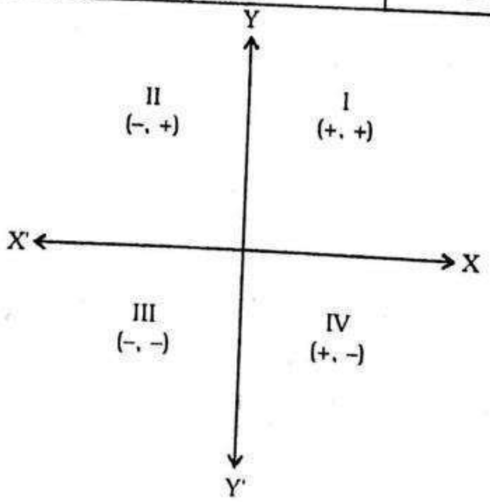


Quadrants : Let X' OX and YOY' be the coordinate axes.

These axes divide the plane of the paper into four regions, called quadrants. The regions XOY, YOX',X'OY' and Y'OX are respectively known as the first, second, third and fourth quadrants.

Using the convention of signs, we have the signs of the coordinates in various quadrants as given below.

Region	Quadrant	Nature of x and y	Signs of co-ordinates
XOY	I	$x > 0, y > 0$	(+, +)
XOX'	II	$x < 0, y > 0$	(–, +)
X'OY'	III	$x < 0, y < 0$	(–, –)
Y'OX	IV	$x > 0, y < 0$	(+, –)



Note : Any point lying on x-axis or y-axis does not lie in any quadrant.

Example : In which quadrants do the given points lie?

(i) (4, -2) (ii) (-3, 7)
(iii) (-1, -2) (iv) (3, 6)

Sol. :

(i) Points of the type (+, –) lie in the 4th quadrant.
Hence, the point (4, –2) lies in quadrant IV.

(ii) Points of the type (–, +) lie in the 2nd quadrant.
Hence, the point (–3, 7) lies in quadrant II.

(iii) Points of the type (–, –) lie in the 3rd quadrant.
Hence, the point (–1, –2) lies in quadrant III.

(iv) Points of the type (+, +) lie in the 1st quadrant.
Hence, the point (3, 6) lies in quadrant I.

Coordinates of a Point on the x-axis : Every point on the x-axis is at a distance of 0 unit from the x-axis. So, its ordinate is 0.

Thus, the coordinates of every point on the x-axis are of the form $(x, 0)$.

Coordinates of a Point on the y-axis : Every point on the y-axis is at a distance of 0 unit from the y-axis. So, its abscissa is 0.

Thus, the coordinates of every point on the y-axis are of the form (0, y).

Remark : The coordinates of the origin are (0, 0).

Example : On which axes do the given points lie?

(i) (7, 0) (ii) (0, -3) (iii) (0, 6) (iv) (-5, 0)

Sol. (i) In (7, 0), we have the ordinate = 0.
∴ (7, 0) lies on the x-axis.

(ii) In (0, –3), we have the abscissa = 0.
∴ (0, –3) lies on the y-axis.

(iii) In (0, 6), we have the abscissa = 0.
∴ (0, 6) lies on the y-axis.

(iv) In (–5, 0), we have the ordinate = 0.
∴ (-5, 0) lies on the x-axis.

MIRROR IMAGES

We may take x-axis or y-axis as the mirror. Then the images of different points are given below.

Point	Mirror-image in x-axis	Mirror-image in y-axis
(i) (x, y)	$(x, -y)$	$(-x, y)$
(ii) $(-x, y)$	$(-x, -y)$	(x, y)
(iii) $(-x, -y)$	$(-x, y)$	$(x, -y)$
(iv) $(x, -y)$	(x, y)	$(-x, -y)$

Graph of $y = mx + c$

Example : Draw the graph of the equation $y = 3x + 2$.

Sol. The given equation is $y = 3x + 2$.

Putting $x = 0$, we get: $y = (3 \times 0) + 2 = 2$.

Putting $x = 1$, we get: $y = (3 \times 1) + 2 = 5$.

Thus, we have the following table.

x	0	1
y	2	5

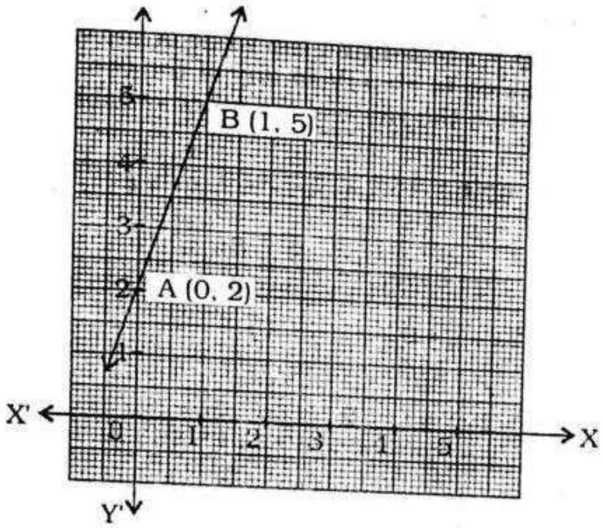
IMPORTANT POINTS : AT A GLANCE

On a graph paper, draw the lines X'OX and YOY' as the x-axis and y-axis respectively.

Now, plot the points A (0, 2) and B (1, 5) on the graph paper.

Join AB and extend it in both the directions.

Thus, line AB is the required graph of the equation, $y = 3x + 2$.



Pair of Linear Equations in Two Variables

Let's consider the following situation :

Ram's age is 3 years more than double the age of his son Abhimanyu.

i.e. $y = 2x + 3$ ......(i)

Where x = the age of Abhimanyu and y = the age of Ram.

Here, equation (i) represents a linear equation in two variables x and y.

Linear Equation in Two Variables : An equation of the form $ax + by + c = 0$ or $ax + by = c$, where a, b, c are real numbers and $a \neq 0$, $b \neq 0$, is called a linear equation in two variables x and y.

Examples : Each of the following equations is a linear equation :

(i) $2x + 3y = 6$, (ii) $x - 3y = 5$, (iii) $\sqrt{5}x - \sqrt{2}y = 0$

The condition $a \neq 0$, $b \neq 0$, is often denoted by $a^2 + b^2 \neq 0$

Solution of a Linear Equation : Any pair of values of x and y is said to be the solution of a linear equation $ax + by + c = 0$, where a, b, c are real numbers and $a \neq 0$, $b \neq 0$ if it satisfies the equation.

i.e.,. $x = \alpha$ (alpha) and $y = \beta$ (beta) is an solution of $ax + by + c = 0$ if $a\alpha + b\beta + c = 0$

Example : Show that $x = 1$ and $y = 2$ is a solution of $3x + y = 5$

Sol. : Substituting $x = 1$ and $y = 2$ in the given equation, we get,

LHS $= 3 \times 1 + 2 = 5 =$ RHS

$\therefore$ $x = 1$ and $y = 2$ is a solution of $3x + y = 5$.

Example : Show that $x = -3$ and $y = 5$ is not a solution of $5x - 2y = 8$

Sol. : Substituting $x = -3$ and $y = 5$ in the equation, we get,

LHS $= 5 \times (-3) - 2 \times 5 = -15 - 10 = -25 \neq$ RHS

$\therefore$ $x = 3$ and $y = 5$ is not a solution of $5x - 2y = 8$

Simultaneous Linear Equations in Two Variables

A pair of linear equations in two variables is said to form a system of simultaneous linear equation.

Thus, a pair of linear equations in x and y. $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ where a_1, b_1, c_1, a_2, b_2, c_2 are all real numbers and $a_1^2 + b_1^2 \neq 0$, $a_2^2 + b_2^2 \neq 0$

Solution of System of Simultaneous Linear Equations in Two Variables

A pair of values of the variables x and y satisfying each of the equations in a given system of two linear equations in x and y is called a solution of the system.

Example : Show that $x = 2$, $y = 1$ is a solution of the system of linear equations $3x + 2y = 8$ and $5x - y = 9$

Sol. :

$3x + 2y = 8$ ...(i)

$5x - y = 9$ ...(ii)

Substituting $x = 2$, $y = 1$ in equation (i), we get,

LHS $= 3 \times 2 + 2 \times 1 = 6 + 2 = 8 =$ RHS

Substituting $x = 2$ and $y = 1$ in equation (ii), we get,

LHS $= 5 \times 2 - 1 = 9 =$ RHS

Hence, $x = 2$ and $y = 1$ is a solution of the given system of linear equations.

Consistency and Inconsistency

- A system of a pair of linear equations in two variables is said to be consistent if it has at least one solution.
- A system of a pair of linear equations in two variables is said to be inconsistent if it has no solution.
- The system of a pair of linear equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ has :

(i) a unique solution (i.e. consistent) if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

The graph of the linear equations intersect at only one point.

(ii) no solution (i.e. inconsistent) if $\frac{a_1}{b_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

The graph of the two linear equations are parallel to each other i.e.e the lines do not intersect.

(iii) an infinite number of solution if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

The graph of the linear equations are coincident.

- Homogeneous equation of the form $ax + by = 0$ is a line passing through the origin. Therefore, this system is always consistent.