Chapter 1

MATRICES AND DETERMINANTS

Learning Objectives

After studying this chapter, the students will be able to understand

- the definition of matrices and determinants
- the properties of determinants
- the concept of inverse matrix
- the concept of adjoint matrix
- the solving simultaneous linear equations
- the input-output analysis

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1.1 Determinants

Introduction

The idea of a determinant was believed to be originated from a Japanese Mathematician Seki Kowa (1683) while systematizing the old Chinese method of solving simultaneous equations whose coefficients were represented by calculating bamboos or sticks. Later the German Mathematician Gottfried Wilhelm Von Leibnitz formally developed determinants. The present vertical notation was given in 1841 by Arthur Cayley. Determinant was invented independently by Crammer whose well known rule for solving simultaneous equations was published in 1750.

KF2PD

In class X, we have studied matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. We know that the area of a triangle with vertices (x_1, y_1) (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

To minimize the difficulty in remembering this type of expression, Mathematicians developed the idea of representing the expression in determinant form and the above expression can be represented in the form

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus a determinant is a particular type of expression written in a special concise form. Note that the quantities are arranged in the form of a square between two vertical lines. This arrangement is called a determinant.

In this chapter, we study determinants up to order three only with real entries. Also we shall study various properties of determinant (without proof), minors, cofactors, adjoint, inverse of a square matrix and business applications of determinants.

We have studied matrices in the previous class. Let us recall the basic concepts and operations on matrices.

1.1.1 Recall

Matrix

Definition 1.1

A matrix is a rectangular arrangement of numbers in horizontal lines (rows) and vertical lines (columns). Numbers are enclosed in square brackets or a open brackets or pair of double bars. It is denoted by A, B, C, …

For example , $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 8 & 6 \end{bmatrix}$

Order of a matrix

If a matrix A has m rows and n columns, then A is called a matrix of order $m \times n$.

For example, If , $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 8 & 6 \end{bmatrix}$ then order of A is 2×3

General form of a Matrix

Matrix of order $m \times n$ is represented as

$$\begin{bmatrix} a_{11} \; a_{12} \; a_{13} & \cdots \; \cdots \; a_{1i} & \cdots \; \cdots \; a_{1n} \\ a_{21} \; a_{22} \; a_{23} & \cdots \; \cdots \; a_{2i} & \cdots \; \cdots \; a_{2n} \\ \cdots \; \cdots \; \cdots & \cdots \; \cdots \; \cdots & \cdots \; \cdots \; \cdots \\ \cdots \; \cdots \; \cdots & \cdots \; \cdots \; \cdots & \cdots \; \cdots \; \cdots \\ a_{m1} \; a_{m2} \; a_{m3} & \cdots \; \cdots \; a_{mi} & \cdots \; \cdots \; a_{mn} \end{bmatrix}$$

It is shortly written as $[a_{ij}]_{mxn}$ $i = 1,2,......m$; $j = 1,2,....n$. Here a_{ij} is the element in the i^{th} row and j^{th} column of the matrix.

Types of matrices

Row matrix

A matrix having only one row is called a row matrix.

For example

$$A = [\; a_{11} \; a_{12} \; a_{13} ....a_{1i} ....a_{1n}]_{1 \times n};$$

$$B = [1 \quad 2]_{1 \, X \, 2}$$

Column matrix

A matrix having only one column is called a column matrix

For examples,

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix}_{m \times 1}, \; B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}_{2 \times 1}$$

Zero matrix (or) Null matrix

If all the elements of a matrix are zero, then it is called a zero matrix. It is represented by an English alphabet 'O'

For examples,

$$\mathrm{O} = [0], \; \mathrm{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \mathrm{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are all zero matrices.

NOTE

Zero matrix can be of any order.

Square matrix

If the number of rows and number of columns of a matrix are equal then it is called a square matrix.

For examples, $A = \begin{bmatrix} 8 & 2 \\ 4 & 2 \end{bmatrix}$ is a square matrix of order 2

$$B = \begin{bmatrix} 3 & -1 & 0 \\ \sqrt{2} & 0 & 5 \\ 2 & \frac{3}{5} & -4 \end{bmatrix}$$ is a square matrix of order 3.

NOTE

The sum of the diagonal elements of a square matrix is called the **trace** of matrix.

Triangular matrix

A square matrix, whose elements above or below the main diagonal (leading diagonal) are all zero is called a triangular matrix.

For examples,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 7 \end{bmatrix}$$

Diagonal matrix

A square matrix in which all the elements other than the main diagonal elements are zero is called the diagonal matrix.

For example,

$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix of order 3

Scalar matrix

A diagonal matrix with all diagonal elements are equal to K (a scalar) is called a scalar matrix.

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 3.

Unit matrix (or) Identity matrix

A scalar matrix having each diagonal element equal to one (unity) is called a Unit matrix.

For examples, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Multiplication of a matrix by a scalar

If $A = [a_{ij}]$ is a matrix of any order and if k is a scalar, then the scalar multiplication of A by the scalar k is defined as $kA = [ka_{ij}]$ for all i, j.

In other words to multiply a matrix A by a scalar k, multiply every element of A by k.

For example, if $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -2 & 8 \end{bmatrix}$ then,

$$2A = \begin{bmatrix} 2 & 4 & 8 \\ 6 & -4 & 16 \end{bmatrix}$$

Negative of a matrix

The negative of a matrix $A = [a_{ij}]_{m \times n}$ is defined as $-A = [-a_{ij}]_{m \times n}$ for all i, j and is obtained by changing the sign of every element.

For example, if $A = \begin{bmatrix} 2 & -5 & 7 \\ 0 & 5 & 6 \end{bmatrix}$

then $-A = \begin{bmatrix} -2 & 5 & -7 \\ 0 & -5 & -6 \end{bmatrix}$

Equality of matrices

Two matrices A and B are said to be equal if

(i) they have the same order and (ii) their corresponding elements are equal.

Addition and subtraction of matrices

Two matrices A and B can be added, provided both the matrices are of the same order. Their sum $A+B$ is obtained by adding the corresponding entries of both the matrices A and B.

Symbolically if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

Similarly $A - B = A + (-B) = [a_{ij}]_{m\times n} + [-b_{ij}]_{m\times n} = [a_{ij} - b_{ij}]_{m\times n}$

Multiplication of matrices

Multiplication of two matrices is possible only when the number of columns of the first matrix is equal to the number of rows of the second matrix. The product of matrices A and B is obtained by multiplying every row of matrix A with the corresponding elements of every column of matrix B element-wise and add the results.

Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [b_{ij}]$ be a $p \times n$ matrix, then the product AB is a matrix $C = [c_{ij}]$ of order $m \times n$.

Transpose of a matrix

Let $A = [a_{ij}]$ be a matrix of order $m \times n$. The transpose of A, denoted by A^T of order $n \times m$ is obtained by interchanging either rows into columns or columns into rows of A.

For example,

$$\text{if } A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$$

NOTE

It is believed that the students might be familiar with the above concepts and our present syllabus continues from the following.

Definition 1.2

To every square matrix A of order n with entries as real or complex numbers, we can associate a number called ***determinant*** of matrix A and it is denoted by $|A|$ or det (A) or Δ.

Thus determinant can be formed by the elements of the matrix A.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then its}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1.1

Evaluate: $\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix} = (2)(4) - (-1)4$$

$$= 8 + 4 = 12$$

To evaluate the determinant of order 3 or more, we define minors and cofactors.

1.1.2 Minors:

Let $|A| = |[a_{ij}]|$ be a determinant of order n. The minor of an arbitrary element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij}.

1.1.3 Cofactors:

The cofactor is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The minors and cofactors of a_{11}, a_{12}, a_{13} in a third order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ are as follows

(i) Minor of a_{11} is M_{11}

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Cofactor of a_{11} is A_{11}

$$= (-1)^{1+1}M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

(ii) Minor of a_{12} is M_{12}

$$= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$$

Cofactor of a_{12} is A_{12}

$$= (-1)^{1+2}M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= -(a_{21}a_{33} - a_{31}a_{23})$$

(iii) Minor of a_{13} is M_{13}

$$= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

Cofactor of is A_{13}

$$= (-1)^{1+3}M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

Example 1.2

Find the minor and cofactor of all the elements in the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Solution

Minor of 1 $= M_{11} = 3$

Minor of -2 $= M_{12} = 4$

Minor of 4 $= M_{21} = -2$

Minor of 3 $= M_{22} = 1$

Cofactor of 1 $= A_{11} = (-1)^{1+1}M_{11} = 3$

Cofactor of -2 $= A_{12} = (-1)^{1+2}M_{12} = -4$

Cofactor of 4 $= A_{21} = (-1)^{2+1}M_{21} = 2$

Cofactor of 3 $= A_{22} = (-1)^{2+2}M_{22} = 1$

Example 1.3

Find the minor and cofactor of each element of the determinant. $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 2 & 5 \\ 4 & 1 & 0 \end{vmatrix}$

Solution

Minor of 3 is $M_{11} = \begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = 0 - 5 = -5$

Minor of 1 is $M_{12} = \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} = 0 - 20 = -20$

Minor of 2 is $M_{13} = \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 2 - 8 = -6$

Minor of 2 is $M_{21} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 0 - 2 = -2$

Minor of 2 is $M_{22} = \begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} = 0 - 8 = -8$

Minor of 5 is $M_{23} = \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = 3 - 4 = -1$

Minor of 4 is $M_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$

Minor of 1 is $M_{32} = \begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} = 15 - 4 = 11$

Minor of 0 is $M_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 6 - 2 = 4$

Cofactor of 3 is A_{11}

$= (-1)^{1+1}M_{11} = M_{11} = -5$

Cofactor of 1 is A_{12}

$= (-1)^{1+2}M_{12} = -M_{12} = 20$

Cofactor of 2 is A_{13}

$= (-1)^{1+3}M_{13} = M_{13} = -6$

Cofactor of 2 is A_{21}

$= (-1)^{2+1}M_{21} = -M_{21} = 2$

Cofactor of 2 is A_{22}

$= (-1)^{2+2}M_{22} = M_{22} = -8$

Cofactor of 5 is A_{23}

$= (-1)^{2+3}M_{23} = -M_{23} = 1$

Cofactor of 4 is $A_{31} = (-1)^{3+1} M_{31} = M_{31} = 1$

Cofactor of 1 is A_{32}

$= (-1)^{3+2} M_{32} = -M_{32} = -11$

Cofactor of 0 is A_{33}

$= (-1)^{3+3} M_{33} = M_{33} = 4$

NOTE

Value of a determinant can be obtained by using any row or column

For example

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ (or)
$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$
(expanding along R_1)

$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$ (or)
$a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}$
(expanding along C_1)

Example 1.4

Evaluate: $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

Solution

$\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$ = (Minor of 1) –2 (Minor of 2) + 4 (Minor of 4)

$= 1\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 4 & 0 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$

$= 0 - 0 - 52 = -52.$

1.1.4 Properties of determinants (without proof)

1. The value of a determinant is unaltered when its rows and columns are interchanged
2. If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes only in sign.
3. If the determinant has two identical rows (columns), then the value of the determinant is zero.
4. If all the elements in a row (column) of a determinant are multiplied by constant k, then the value of the determinant is multiplied by k.
5. If any two rows (columns) of a determinant are proportional, then the value of the determinant is zero.
6. If each element in a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants of the same order.
7. The value of the determinant is unaltered when a constant multiple of the elements of any row (column) is added to the corresponding elements of a different row (column) in a determinant.

Example 1.5

Show that $\begin{vmatrix} x & y & z \\ 2x+2a & 2y+2b & 2z+2c \\ a & b & c \end{vmatrix} = 0$

Solution

$\begin{vmatrix} x & y & z \\ 2x+2a & 2y+2b & 2z+2c \\ a & b & c \end{vmatrix}$

$= \begin{vmatrix} x & y & z \\ 2x & 2y & 2z \\ a & b & c \end{vmatrix} + \begin{vmatrix} x & y & z \\ 2a & 2b & 2c \\ a & b & c \end{vmatrix}$

$= 0 + 0$

$= 0$

DO YOU KNOW?

$$\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 11 & 13 \end{pmatrix}$$

$$\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ 7 & 8 \end{vmatrix} \neq \begin{vmatrix} 4 & 8 \\ 11 & 13 \end{vmatrix}$$

Example 1.6

Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x^2-(x-1)(x+1)$$

$$= x^2-(x^2-1)$$

$$= x^2-x^2+1 = 1$$

NOTE

The value of the determinant of triangular matrix is equal to the product of the main diagonal elements.

Example 1.7

Solve $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

Solution

$$\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(x-2)(x-3) = 0$$

$$x = 1, x = 2, x = 3$$

Example 1.8

Evaluate $\begin{vmatrix} 1 & 3 & 4 \\ 102 & 18 & 36 \\ 17 & 3 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 3 & 4 \\ 102 & 18 & 36 \\ 17 & 3 & 6 \end{vmatrix} = 6\begin{vmatrix} 1 & 3 & 4 \\ 17 & 3 & 6 \\ 17 & 3 & 6 \end{vmatrix}$$

$$= 0 \quad (\text{since } R_2 \equiv R_3)$$

Example 1.9

Evaluate $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)[0-0+\{b+c-(a+b)\}]$$

$$= (a-b)(b-c)(c-a).$$

Exercise 1.1

1. Find the minors and cofactors of all the elements of the following determinants.

   (i) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

3. Solve: $\begin{vmatrix} 2 & x & 3 \\ 4 & 1 & 6 \\ 1 & 2 & 7 \end{vmatrix} = 0$

4. Find $|AB|$ if $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$

5. Solve: $\begin{vmatrix} 7 & 4 & 11 \\ -3 & 5 & x \\ -x & 3 & 1 \end{vmatrix} = 0$

6. Evaluate: $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

7. Prove that $\begin{vmatrix} \frac{1}{a} & bc & b+c \\ \frac{1}{b} & ca & c+a \\ \frac{1}{c} & ab & a+b \end{vmatrix} = 0$

8. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

1.2 Inverse of a Matrix

1.2.1 Singular matrix:

A Square matrix A is said to be singular, if $|A| = 0$

1.2.2 Non – singular matrix:

A square matrix A is said to be non – singular, if $|A| \neq 0$

Example 1.10

Show that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a singular matrix.

Solution

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

$= 4 - 4 = 0$

$\therefore$ A is a singular matrix.

Example 1.11

Show that $\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$ is non – singular.

Solution

Let $A = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 8 & 2 \\ 4 & 3 \end{vmatrix}$

$= 24 - 8 = 16 \neq 0$

$\therefore$ A is a non-singular matrix

DO YOU KNOW?

If A and B are non – singular matrices of the same order then AB and BA are also non – singular matrices of the same order.

1.2.3 Adjoint of a matrix

The adjoint of a square matrix A is defined as the transpose of a cofactor matrix. Adjoint of the matrix A is denoted by *adj A*

i.e., adj $A = [A_{ij}]^T$ where $[A_{ij}]$ is the cofactor matrix of A.

Example 1.12

Find adj A for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution

$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

adj $A = [A_{ij}]^T$

$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

NOTE

(i) $|adjA| = |A|^{n-1}$ n is the order of the matrix A

(ii) $|kA| = k^n|A|$ n is the order of the matrix A

(iii) $adj(kA) = k^{n-1} adjA$ n is the order of the matrix A

(iv) $A(adj\ A) = (adj\ A)A = |A|I$

(v) $AdjI = I, I$ is the unit matrix.

(vi) adj (AB) = (adj B)(adj A)

(vii) $|AB| = |A||B|$

Example 1.13

Find adjoint of $A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ -4 & 1 & 0 \end{bmatrix}$

Solution

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 0 & 0 \\ -4 & 0 \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 0 & 1 \\ -4 & 1 \end{vmatrix} = 0-(-4) = 4$$

$$A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix}$$

$$= -(0-(-3)) = -3$$

$$A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & -3 \\ -4 & 0 \end{vmatrix}$$

$$= 0-12 = -12$$

$$A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 1 & -2 \\ -4 & 1 \end{vmatrix}$$

$$= -(1-8) = 7$$

$$A_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix}$$

$$= 0-(-3) = 3$$

$$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix}$$

$$= 0-0 = 0$$

$$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$$

$$= 1-0 = 1$$

$$[A_{ij}] = \begin{bmatrix} 0 & 0 & 4 \\ -3 & -12 & 7 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\text{Adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 0 & -3 & 3 \\ 0 & -12 & 0 \\ 4 & 7 & 1 \end{bmatrix}$$

1.2.4 Inverse of a matrix

Let A be any non-singular matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I$ then, B is called the inverse of A and is denoted by A^{-1}

We have $A\,(\text{adj } A) = (\text{adj } A)A$

$$= |A|\,\text{I}$$

$$\Longrightarrow A\left(\frac{1}{|A|} adj\ A\right) = \left(\frac{1}{|A|} adj\ A\right)A$$

$$= I$$

$$\Longrightarrow AB = BA = I \text{ where } B = \frac{1}{|A|} adjA$$

$$A^{-1} = \frac{1}{|A|} adjA$$

NOTE

(i) If B is the inverse of A then, A is the inverse of B

i.e., $B = A^{-1} \Rightarrow A = B^{-1}$

(ii) $AA^{-1} = A^{-1}A = I$

(iii) The inverse of a matrix, if it exists, is unique.

(iv) Order of inverse of A will be the same as that of A.

(v) $I^{-1} = I$, where I is the identity matrix.

(vi) $(AB)^{-1} = B^{-1}A^{-1}$ (vii) $|A^{-1}| = \frac{1}{|A|}$

DO YOU KNOW?

Let A be a non-singular matrix, then

$$A^2 = I \Leftrightarrow A = A^{-1}$$

Example 1.14

If $A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$ then, find A^{-1}.

Solution

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 4 \\ -3 & 2 \end{vmatrix} = 16 \neq 0$$

Since A is a nonsingular matrix, A^{-1} exists

Now adj A $= \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{16}\begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$$

Example 1.15

If A $= \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$ then, find A^{-1}

Solution

$$|A| = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = 0$$

Since A is a singular matrix, A^{-1} does not exist.

Example 1.16

If A $= \begin{bmatrix} 2 & 4 & 4 \\ 2 & 5 & 4 \\ 2 & 5 & 3 \end{bmatrix}$ then find A^{-1}

Solution

$$|A| = \begin{vmatrix} 2 & 4 & 4 \\ 2 & 5 & 4 \\ 2 & 5 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 5 & 4 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix}$$

$$= 2(15-20) - 4(6-8) + 4(10-10)$$

$$= 2(-5) - 4(-2) + 4(0)$$

$$= -10 + 8 + 0 = -2 \neq 0$$

Since A is a nonsingular matrix, A^{-1} exists.

$A_{11} = -5;\ A_{21} = 8;\ A_{31} = -4$

$A_{12} = 2;\ A_{22} = -2;\ A_{32} = 0$

$A_{13} = 0;\ A_{23} = -2;\ A_{33} = 2$

$$[A_{ij}] = \begin{bmatrix} -5 & 2 & 0 \\ 8 & -2 & -2 \\ -4 & 0 & 2 \end{bmatrix}$$

$$\text{adj A} = \begin{bmatrix} -5 & 8 & -4 \\ 2 & -2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{-2}\begin{bmatrix} -5 & 8 & -4 \\ 2 & -2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -4 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Example 1.17

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - kA + I_2 = O$ then, find k and also A^{-1}.

Solution

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^2 = \text{A.A}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

Given $A^2 - kA + I_2 = O$

$$\Rightarrow \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - k\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7-2k+1 & 12-3k+0 \\ 4-k+0 & 7-2k+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8-2k & 12-3k \\ 4-k & 8-2k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 4 - k = 0$$

$$\Rightarrow \text{k} = 4.$$

Also $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$

$$= 4 - 3 = 1 \neq 0$$

$$\text{adj } A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Example 1.18

If $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ then, show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution

$$|A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1 \neq 0$$

$\therefore$ A^{-1} exists.

$$|B| = \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 0 + 1 = 1 \neq 0$$

B^{-1} also exists.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$$

$(AB)^{-1}$ exists.

$$\text{adj}(AB) = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} adj(AB)$$

$$= \frac{1}{-1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \quad \ldots (1)$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} adjB$$

$$= \frac{1}{1}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$. Hence proved.

Example 1.19

Show that the matrices

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix}$$

are inverses of each other.

Solution

$$AB = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \frac{35}{35}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

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$$BA = \begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \frac{35}{35}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}$$

Thus $AB = BA = \mathrm{I}$

Therefore A and B are inverses of each other.

Exercise 1.2

1. Find the adjoint of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then verify that $A(adj\ A) = |A|\ \mathrm{I}$ and also find A^{-1}

3. Find the inverse of each of the following matrices

   (i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

   (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$, then verify $adj\ (AB) = (adj\ B)(adj\ A)$

5. If $A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$ then, show that $(adj\ A)A = O$

6. If $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$ then, show that the inverse of A is A itself.

7. If $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ then, find A.

8. Show that the matrices $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$ are inverses of each other.

9. If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ then, verify that $(AB)^{-1} = B^{-1}A^{-1}$

10. Find λ if the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{bmatrix}$ has no inverse.

11. If $X = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix}$ then, find p, q if $Y = X^{-1}$

1.2.5 Solution of a system of linear equations

Consider a system of n linear non – homogeneous equations with n unknowns $x_1, x_2, ...x_n$ as

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

This can be written in matrix form as

$$\begin{bmatrix} a_{11}\, a_{12} \ldots. a_{1n} \\ a_{21}\, a_{22} \ldots a_{2n} \\ \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \\ a_{m1}\, a_{m2} \ldots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

Thus we get the matrix equation

$AX = B$ ... (1)

$$\text{where } A = \begin{bmatrix} a_{11}\, a_{12} \ldots. a_{1n} \\ a_{21}\, a_{22} \ldots. a_{2n} \\ \ldots \ldots \ldots \ldots \\ \ldots \ldots \ldots \ldots \\ a_{m1}\, a_{m2} \ldots. a_{mn} \end{bmatrix};$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

From (1) solution is

$$\boxed{X = A^{-1}B}$$

Example 1.20

Solve by using matrix inversion method:

$2x + 5y = 1$

$3x + 2y = 7$

Solution

The given system can be written as

$$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e., $AX = B$

$X = A^{-1}B$

where $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix}$$

$$= -11 \neq 0$$

A^{-1} exists.

$$adj\, A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{11} \begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$x = 3$ and $y = -1$.

Example 1.21

Solve by using matrix inversion method:

$3x - 2y + 3z = 8;\ 2x + y - z = 1;$

$4x - 3y + 2z = 4$

Solution

The given system can be written as

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e., $AX = B$

$X = A^{-1}B$

Here $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

$$B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= -17 \neq 0$$

A^{-1} exists

$A_{11} = -1 \quad A_{12} = -8 \quad A_{13} = -10$

$A_{21} = -5 \quad A_{22} = -6 \quad A_{23} = 1$

$A_{31} = -1 \quad A_{32} = 9 \quad A_{33} = 7$

$$[A_{ij}] = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$$adj\, A = [A_{ij}]^T$$

$$= \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$= -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1$, $y = 2$ and $z = 3$.

Example 1.22

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹320. The cost of 2kg onion, 4 kg wheat and 6 kg rice is ₹560. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹380. Find the cost of each item per kg by matrix inversion method.

Solution

Let x, y and z be the cost of onion, wheat and rice per kg respectively.

Then, $4x + 3y + 2z = 320$

$2x + 4y + 6z = 560$

$6x + 2y + 3z = 380.$

It can be written as

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$AX = B$$

where $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 50 \neq 0$$

A^{-1} exist

$A_{11} = 0 \quad A_{12} = 30 \quad A_{13} = -20$

$A_{21} = -5 \quad A_{22} = 0 \quad A_{23} = 10$

$A_{31} = 10 \quad A_{32} = -20 \quad A_{33} = 10$

$$[A_{ij}] = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 32 \\ 56 \\ 38 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0-280+380 \\ 960+0-760 \\ -640+560+380 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$$

$x = 20$, $y = 40$ and $z = 60$.

Cost of 1 kg onion is ₹20, cost of 1 kg wheat is ₹40 and cost of 1 kg rice is ₹60.

Exercise 1.3

1. Solve by matrix inversion method:
   $2x + 3y - 5 = 0$; $x - 2y + 1 = 0$
2. Solve by matrix inversion method:
   (i) $3x - y + 2z = 13$; $2x + y - z = 3$; $x + 3y - 5z = -8$
   (ii) $x - y + 2z = 3$; $2x + z = 1$; $3x + 2y + z = 4$.
   (iii) $2x - z = 0$; $5x + y = 4$; $y + 3z = 5$
3. A sales person Ravi has the following record of sales for the month of January, February and March 2009 for three products *A*, *B* and *C* He has been paid a commission at fixed rate per unit but at varying rates for products *A*, *B* and *C*

Months	Sales in Units			Commission
	A	*B*	*C*	
January	9	10	2	800
February	15	5	4	900
March	6	10	3	850

   Find the rate of commission payable on *A*, *B* and *C* per unit sold using matrix inversion method.

4. The prices of three commodities *A*, *B* and *C* are ₹x, ₹y and ₹z per unit respectively. *P* purchases 4 units of *C* and sells 3 units of *A* and 5 units of *B*. *Q* purchases 3 units of *B* and sells 2 units of *A* and 1 unit of *C*. *R* purchases 1 unit of *A* and sells 4 units of *B* and 6 units of *C*. In the process *P*, *Q* and *R* earn ₹6,000, ₹5,000 and ₹13,000 respectively. By using matrix inversion method, find the prices per unit of *A*, *B* and *C*.
5. The sum of three numbers is 20. If we multiply the first by 2 and add the second number and subtract the third we get 23. If we multiply the first by 3 and add second and third to it, we get 46. By using matrix inversion method find the numbers.
6. Weekly expenditure in an office for three weeks is given as follows. Assuming that the salary in all the three weeks of different categories of staff did not vary, calculate the salary for each type of staff, using matrix inversion method.

Week	Number of employees			Total weekly salary (in ₹)
	A	B	C	
1st week	4	2	3	4900
2nd week	3	3	2	4500
3rd week	4	3	4	5800

1.3 Input–Output Analysis

Input–Output analysis is a technique which was invented by Prof. Wassily W. Leontief. Input–Output analysis is a form of economic analysis based on the interdependencies between economic sectors. The method is most commonly used for estimating the impacts of positive or negative economic shocks and analyzing the ripple effects throughout an economy.

The foundation of Input–Output analysis involves input–output tables. Such tables include a series of rows and columns of data that quantify the supply chain for sectors of the economy. Industries are listed in the heads of each row and each column. The data in each column corresponds to the level of inputs used in that industry's production function. For example the column for auto manufacturing shows the resources required for building automobiles (ie., requirement of steel, aluminum, plastic, electronic etc.,). Input–Output models typically includes separate tables showing the amount of labour required per rupee unit of investment or production.

Consider a simple economic model consisting of two industries A_1 and A_2 where each produces only one type of product. Assume that each industry consumes part of its own output and rest from the other industry for its operation. The industries are thus interdependent. Further assume that whatever is produced that is consumed. That is the total output of each industry must be such as to meet its own demand, the demand of the other industry and the external demand (final demand).

Our aim is to determine the output levels of each of the two industries in order to meet a change in final demand, based on knowledge of the current outputs of the two industries, of course under the assumption that the structure of the economy does not change.

Let a_{ij} be the rupee value of the output of A_i consumed by A_j, $i, j = 1, 2$

Let x_1 and x_2 be the rupee value of the current outputs of A_1 and A_2 respectively.

Let d_1 and d_2 be the rupee value of the final demands for the outputs of A_1 and A_2 respectively.

The assumptions lead us to frame the two equations

$$a_{11}+a_{12}+d_1 = x_1; \quad a_{21}+a_{22}+d_2 = x_2 \qquad \ldots (1)$$

Let $\quad b_{ij} = \frac{a_{ij}}{x_j} \; i, j = 1, 2$

$$b_{11} = \frac{a_{11}}{x_1}; \quad b_{12} = \frac{a_{12}}{x_2};$$

$$b_{21} = \frac{a_{21}}{x_1}; \quad b_{22} = \frac{a_{22}}{x_2}$$

The equations (1) take the form

$$b_{11}x_1+b_{12}x_2+d_1 = x_1$$

$$b_{21}x_1+b_{22}x_2+d_2 = x_2$$

The above equations can be rearranged as

$$(1-b_{11})x_1 - b_{12}x_2 = d_1$$

$$-b_{21}x_1 + (1-b_{22})x_2 = d_2$$

The matrix form of the above equations is

$$\begin{bmatrix} 1-b_{11} & -b_{12} \\ -b_{21} & 1-b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$(1-B)X = D$$

where $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving we get

$$\boxed{X=(I-B)^{-1}D}$$

The matrix B is known as the ***Technology matrix***.

1.3.1 The Hawkins – Simon conditions

Hawkins – Simon conditions ensure the viability of the system.

If B is the technology matrix then Hawkins – Simon conditions are

(i) the main diagonal elements in $I - B$ must be positive and

(ii) $|I - B|$ must be positive.

Example 1.23

The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$. Test whether the system is viable as per Hawkins – Simon conditions.

Solution

$$B = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{vmatrix}$$

$$= (0.2)(0.3)-(-0.2)(-0.9)$$

$$= 0.06 - 0.18$$

$$= -0.12 < 0$$

Since $|I - B|$ is negative, Hawkins – Simon conditions are not satisfied.

Therefore, the given system is not viable.

Example 1.24

The following inter – industry transactions table was constructed for an economy of the year 2016.

Industry	1	2	Final consumption	Total output
1	500	1,600	400	2,500
2	1,750	1,600	4,650	8,000
Labours	250	4,800	-	-

Construct technology co-efficient matrix showing direct requirements. Does a solution exist for this system.

Solution

$a_{11} = 500 \quad a_{12} = 1600 \quad x_1 = 2500$

$a_{21} = 1750 \quad a_{22} = 1600 \quad x_2 = 8000$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{500}{2500} = 0.20;$$

$$b_{12} = \frac{a_{12}}{x_2} = \frac{1600}{8000} = 0.20$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{1750}{2500} = 0.70;$$

$$b_{22} = \frac{a_{22}}{x_2} = \frac{1600}{8000} = 0.20$$

The technology matrix is $= \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix}$$

$= \begin{bmatrix} 0.8 & -0.2 \\ -0.7 & 0.8 \end{bmatrix}$, elements of main diagonal are positive

Now, $|I - B| = \begin{vmatrix} 0.8 & -0.2 \\ -0.7 & 0.8 \end{vmatrix}$

$$= (0.8)(0.8)-(-0.7)(-0.2)$$

$$= 0.64 - 0.14$$

$$= 0.50 > 0$$

Since diagonal elements of *I–B* are positive and $|I - B|$ is positive,

Hawkins – Simon conditions are satisfied

Hence this system has a solution.

Example 1.25

In an economy there are two industries P_1 and P_2 and the following table gives the supply and the demand position in crores of rupees.

Production sector	Consumption sector P_1	Consumption sector P_2	Final demand	Gross output
P_1	10	25	15	50
P_2	20	30	10	60

Determine the outputs when the final demand changes to 35 for P_1 and 42 for P_2.

Solution

$$a_{11} = 10 \qquad a_{12} = 25 \qquad x_1 = 50$$

$$a_{21} = 20 \qquad a_{22} = 30 \qquad x_2 = 60$$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{10}{50} = \frac{1}{5};$$

$$b_{12} = \frac{a_{12}}{x_2} = \frac{25}{60} = \frac{5}{12}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{20}{50} = \frac{2}{5};$$

$$b_{22} = \frac{a_{22}}{x_2} = \frac{30}{60} = \frac{1}{2}$$

The technology matrix is $B = \begin{bmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{bmatrix}$$

$= \begin{bmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{bmatrix}$, elements of main diagonal are positive

$$\text{Now, } |I - B| = \begin{vmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{7}{30} > 0$$

Main diagonal elements of $|I - B|$ are positive and $|I - B|$ is positive

$\therefore$ The problem has a solution

$$adj\,(I–B) = \begin{bmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

$$(I–B)^{-1} = \frac{1}{|I - B|} adj(I - B)$$

(Since $|I - B| \neq 0$, $(I - B)^{-1}$ exist)

$$= \frac{1}{\frac{7}{30}} adj(I - B)$$

$$= \frac{30}{7}\begin{bmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{bmatrix}$$

$X = (I–B)^{-1}D$, where $D = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$

$$= \frac{1}{7}\begin{bmatrix} 15 & \frac{25}{2} \\ 12 & 24 \end{bmatrix}\begin{bmatrix} 35 \\ 42 \end{bmatrix} = \begin{bmatrix} 150 \\ 204 \end{bmatrix}$$

The output of industry P_1 should be ₹150 crores and P_2 should be ₹204 crores.

Example 1.26

An economy produces only coal and steel. These two commodities serve as intermediate inputs in each other's production. 0.4 tonne of steel and 0.7 tonne of coal are needed to produce a tonne of steel. Similarly 0.1 tonne of steel and 0.6 tonne of coal are required to produce a tonne of coal. No capital inputs are needed. Do you think that the system is viable? 2 and 5 labour days are required to produce a tonnes of coal and steel respectively. If economy needs 100 tonnes of coal and 50 tonnes of steel, calculate the gross output of the two commodities and the total labour days required.

Solution

Here the technology matrix is given under

	Steel	Coal	Final demand
Steel	0.4	0.1	50
Coal	0.7	0.6	100
Labour days	5	2	-

The technology matrix is $B=\begin{bmatrix}0.4 & 0.1\\0.7 & 0.6\end{bmatrix}$

$$I-B=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}0.4 & 0.1\\0.7 & 0.6\end{bmatrix}$$

$$=\begin{bmatrix}0.6 & -0.1\\-0.7 & 0.4\end{bmatrix}$$

$$|I-B|=\begin{vmatrix}0.6 & -0.1\\-0.7 & 0.4\end{vmatrix}$$

$$=(0.6)(0.4)-(-0.7)(-0.1)$$

$$=0.24-0.07=0.17$$

Since the diagonal elements of $I-B$ are positive and value of $|I-B|$ is positive, the system is viable.

$$\text{adj }(I-B)=\begin{bmatrix}0.4 & 0.1\\0.7 & 0.6\end{bmatrix}$$

$$(I-B)^{-1}=\frac{1}{|I-B|}adj(I-B)$$

$$=\frac{1}{0.17}\begin{bmatrix}0.4 & 0.1\\0.7 & 0.6\end{bmatrix}$$

$$X=(I-B)^{-1}D,\text{ where } D=\begin{bmatrix}50\\100\end{bmatrix}$$

$$=\frac{1}{0.17}\begin{bmatrix}0.4 & 0.1\\0.7 & 0.6\end{bmatrix}\begin{bmatrix}50\\100\end{bmatrix}$$

$$=\frac{1}{0.17}\begin{bmatrix}30\\95\end{bmatrix}$$

$$=\begin{bmatrix}176.5\\558.8\end{bmatrix}$$

Steel output = 176.5 tonnes

Coal output = 558.8 tonnes

Total labour days required

= 5(steel output) + 2(coal output)

= 5(176.5) + 2(558.8)

= 882.5 + 1117.6 = 2000.1

$\simeq$ 2000 labour days.

Exercise 1.4

1. The technology matrix of an economic system of two industries is $\begin{bmatrix}0.50 & 0.30\\0.41 & 0.33\end{bmatrix}$. Test whether the system is viable as per Hawkins Simon conditions.
2. The technology matrix of an economic system of two industries is $\begin{bmatrix}0.6 & 0.9\\0.20 & 0.80\end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions.
3. The technology matrix of an economic system of two industries is $\begin{bmatrix}0.50 & 0.25\\0.40 & 0.67\end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions.
4. Two commodities A and B are produced such that 0.4 tonne of A and 0.7 tonne of B are required to produce a tonne of A. Similarly 0.1 tonne of A and 0.7 tonne of B are needed to produce a tonne of B. Write down the technology matrix. If 68 tonnes of A and 10.2 tonnes of B are required, find the gross production of both of them.

5. Suppose the inter-industry flow of the product of two industries are given as under.

Production sector	Consumption sector		Domestic demand	Total output
	X	Y		
X	30	40	50	120
Y	20	10	30	60

Determine the technology matrix and test Hawkin's -Simon conditions for the viability of the system. If the domestic demand changes to 80 and 40 units respectively, what should be the gross output of each sector in order to meet the new demands.

6. You are given the following transaction matrix for a two sector economy.

Sector	Sales		Final demand	Gross output
	1	2		
1	4	3	13	20
2	5	4	3	12

(i) Write the technology matrix

(ii) Determine the output when the final demand for the output sector 1 alone increases to 23 units.

7. Suppose the inter-industry flow of the product of two sectors X and Y are given as under.

Production sector	Consumption Sector		Domestic demand	Gross output
	X	Y		
X	15	10	10	35
Y	20	30	15	65

Find the gross output when the domestic demand changes to 12 for X and 18 for Y.

Exercise 1.5

Choose the correct answer

1. The value of x if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$ is

(a) 0, −1 (b) 0, 1

(c) −1, 1 (d) −1, −1

2. The value of $\begin{vmatrix} 2x+y & x & y \\ 2y+z & y & z \\ 2z+x & z & x \end{vmatrix}$ is

(a) $x\,y\,z$ (b) $x + y + z$

(c) $2x + 2y + 2z$ (d) 0

3. The cofactor of −7 in the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ is

08XU4

(a) −18 (b) 18

(c) −7 (d) 7

4. If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$ then $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$ is

(a) Δ (b) $-\Delta$

(c) 3Δ (d) -3Δ

5. The value of the determinant $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2$ is

(a) abc (b) 0

(c) $a^2\,b^2\,c^2$ (d) $-abc$

6. If A is square matrix of order 3 then $|kA|$ is

(a) $k|A|$ (b) $-k|A|$

(c) $k^3|A|$ (d) $-k^3|A|$

7. $adj(AB)$ is equal to

(a) $adjA\ adjB$ (b) $adjA^T\ adjB^T$

(c) $adjB\ adjA$ (d) $adjB^T\ adjA^T$

8. The inverse matrix of $\begin{pmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{pmatrix}$ is

(a) $\frac{7}{30}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ (b) $\frac{7}{30}\begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix}$

(c) $\frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ (d) $\frac{30}{7}\begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{4}{5} \end{pmatrix}$

9. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$ then A^{-1} is

(a) $\frac{1}{ad-bc}\begin{pmatrix} d & b \\ -c & a \end{pmatrix}$

(b) $\frac{1}{ad-bc}\begin{pmatrix} d & b \\ c & a \end{pmatrix}$

(c) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(d) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ c & a \end{pmatrix}$

10. The number of Hawkins-Simon conditions for the viability of an input-output analysis is

(a) 1 (b) 3

(c) 4 (d) 2

11. The inventor of input - output analysis is

(a) Sir Francis Galton

(b) Fisher

(c) Prof. Wassily W. Leontief

(d) Arthur Caylay

12. Which of the following matrix has no inverse

(a) $\begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}$

(d) $\begin{pmatrix} \sin a & \sin a \\ -\cos a & \cos a \end{pmatrix}$

13. The inverse matrix of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is

(a) $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ -5 & -3 \end{pmatrix}$ (d) $\begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix}$

14. If $A = \begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}$ then $A(adjA)$ is

(a) $\begin{pmatrix} -4 & -2 \\ -1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

15. If A and B non-singular matrix then, which of the following is incorrect?

(a) $A^2 = I$ implies $A^{-1} = A$

(b) $I^{-1} = I$

(c) If $AX = B$ then $X = B^{-1}A$

(d) If A is square matrix of order 3 then $|adj\ A| = |A|^2$

16. The value of $\begin{vmatrix} 5 & 5 & 5 \\ 4x & 4y & 4z \\ -3x & -3y & -3z \end{vmatrix}$ is

(a) 5 (b) 4 (c) 0 (d) –3

17. If A is an invertible matrix of order 2 then $det(A^{-1})$ be equal to

(a) $det(A)$ (b) $\frac{1}{det(A)}$

(c) 1 (d) 0

18. If A is 3×3 matrix and $|A| = 4$ then $|A^{-1}|$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{16}$ (c) 2 (d) 4

19. If A is a square matrix of order 3 and $|A| = 3$ then $|adjA|$ is equal to

(a) 81 (b) 27 (c) 3 (d) 9

20. The value of $\begin{vmatrix} x & x^2 - yz & 1 \\ y & y^2 - zx & 1 \\ z & z^2 - xy & 1 \end{vmatrix}$ is

(a) 1 (b) 0

(c) –1 (d) $-xyz$

21. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $|2A|$ is equal to

(a) $4\cos 2\theta$ (b) 4

(c) 2 (d) 1

22. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is cofactor of a_{ij}, then value of Δ is given by

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$

(d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

23. If $\begin{vmatrix} x & 2 \\ 8 & 5 \end{vmatrix} = 0$ then the value of x is

(a) $\frac{-5}{6}$ (b) $\frac{5}{6}$

(c) $\frac{-16}{5}$ (d) $\frac{16}{5}$

24. If $\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = -5$ then the value of $\begin{vmatrix} 20 & 15 \\ 15 & 5 \end{vmatrix}$ is

(a) –5 (b) –125

(c) –25 (d) 0

25. If any three rows or columns of a determinant are identical then the value of the determinant is

(a) 0 (b) 2 (c) 1 (d) 3

Miscellaneous Problems

1. Solve: $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

2. Evaluate $\begin{vmatrix} 10041 & 10042 & 10043 \\ 10045 & 10046 & 10047 \\ 10049 & 10050 & 10051 \end{vmatrix}$

3. Without actual expansion show that the value of the determinant $\begin{vmatrix} 5 & 5^2 & 5^3 \\ 5^2 & 5^3 & 5^4 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is zero.

4. Show that $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$.

5. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 7 \\ 1 & -1 & 1 \end{bmatrix}$ verify that $A(adjA) = (adjA)(A) = |A|I_3$.

6. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ then, find the Inverse of A.

7. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ show that $A^2 - 4A + 5I_2 = O$ and also find A^{-1}

8. Solve by using matrix inversion method: $x - y + z = 2,\ 2x - y = 0,\ 2y - z = 1$.

9. The cost of 2 Kg of Wheat and 1 Kg of Sugar is ₹70. The cost of 1 Kg of Wheat and 1 Kg of Rice is ₹70. The

cost of 3 Kg of Wheat, 2 Kg of Sugar and 1 Kg of rice is ₹170. Find the cost of per kg each item using matrix inversion method.

10. The data are about an economy of two industries A and B. The values are in crores of rupees.

Producer	User		Final demand	Total output
	A	B		
A	50	75	75	200
B	100	50	50	200

Find the output when the final demand changes to 300 for A and 600 for B

Summary

- Determinant is a number associated to a square matrix.
- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|A| = a_{11}a_{22} - a_{21}a_{12}$
- Minor of an arbitrary element a_{ij} of the determinant of the matrix is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij}.
- The cofactor is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j}M_{ij}$.
- Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

  $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ or $a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$
- adj A = $[A_{ij}]^T$ where $[A_{ij}]$ is the cofactor matrix of the given matrix.
- $|adjA| = |A|^{n-1}$, where n is the order of the matrix A.
- $A(adjA) = (adjA)(A) = |A|I$.
- $adjI = I$, I is the unit Matrix.
- adj$(AB) = (adj\ B)(adj\ A)$.
- A square matrix A is said to be singular, if $|A| = 0$.
- A square matrix A is said to be non-singular if $|A| \neq 0$
- Let A be any square matrix of order n and I be the identity matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I$ then, B is called the inverse of A and is denoted by A^{-1}
- Inverse of A (if it exists) is $A^{-1} = \frac{1}{|A|} adjA$.
- Hawkins - Simon conditions ensure the viability of the system. If B is the technology matrix then Hawkins-Simon conditions are

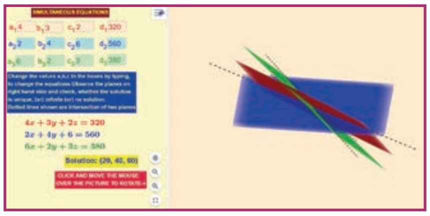
  (i) the main diagonal elements in $I - B$ must be positive and

  (ii) $|I - B|$ must be positive.

GLOSSARY (கலைச்சொற்கள்)	
Adjoint Matrix	சேர்ப்பு அணி
Analysis	பகுப்பாய்வு
Cofactor	இணைக் காரணி
Determinant	அணிக்கோவை
Digonal Matrix	மூலை விட்ட அணி
Input	உள்ளீடு
Inverse Matrix	நேர்மாறு அணி
Minors	சிற்றணிக் கோவைகள்
Non-Singular Matrix	பூச்சியமற்றக் கோவை அணி
Output	வெளியீடு
Scalar	திசையிலி
Singular Matrix	பூச்சியக் கோவை அணி
Transpose of a Matrix	நிரை நிரல் மாற்று அணி
Triangular Matrix	முக்கோண அணி

ICT Corner

Expected final outcomes



Step – 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called "**11th BUSINESS MATHEMATICS and STATISTICS**" will appear. In this several work sheets for Business Maths are given, Open the worksheet named "**Simultaneous Equations**"

Step - 2

Type the parameters for three simultaneous equations in the respective boxes. You will get the solutions for the equations. Also, you can see the planes for the equations and the intersection point(Solution) on the right-hand side. Solve yourself and check the answer. If the planes do not intersect then there is no solution.

Browse in the link

11th Business Mathematics and Statistics: https://ggbm.at/qKj9gSTG (or) scan the QR Code

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